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Abstract

Several methods have been proposed to obtain stationarity in open economy models. I find substantial qualitative and quantitative differences between these methods in a two-country framework, in contrast to the results of Schmitt-Grohé and Uribe (2003). In models with a debt elastic interest rate premium or a convex portfolio cost, both the steady state and the equilibrium dynamics are unique if the elasticity of substitution between the domestic and the foreign traded good is high. However, there are three steady states if the elasticity of substitution is sufficiently low. With endogenous discounting, there is always a unique and stable steady state irrespective of the magnitude of the elasticity of substitution. Similar to the model with convex portfolio costs or a debt elastic interest rate premium, though, there can be multiple convergence paths for low values of the elasticity in response to shocks.

Keywords: multiple equilibria, stationarity, incomplete markets

JEL classifications: D51, F41

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1 Introduction

In open economy models with incomplete asset markets the deterministic steady state depends on the initial conditions of the economy and the steady state is compatible with any level of net foreign assets. In a stochastic environment the model generates non-stationary variables as net foreign assets follow a unit root process.[1]

Several modifications of the standard model have been proposed in order to induce stationarity among which are an endogenous discount factor (Uzawa-type preferences), a debt elastic interest rate premium or convex portfolio costs. Schmitt-Grohé and Uribe (2003) present quantitative comparisons of these alternative approaches and find that all of them deliver virtually identical dynamics. However, their analysis is restricted to the case of a small open economy, and therefore further scrutiny is justified. Nevertheless, their work has been cited extensively by others to claim irrelevance of the chosen approach that induces stationarity in a specific model even in multi-country setups.[2]

In this paper, I investigate the theoretical differences between several stationarity inducing approaches in a standard two-country model with limited substitutability between traded goods. If goods are highly substitutable across countries, the stationarity inducing approaches that I investigate have very similar properties. However, for low values of the elasticity of substitution between traded goods there are important nonlinearities which give rise to substantial differences across methods.

Each of the two countries produces one good. These imperfectly substitutable goods are traded in a frictionless goods market. International financial markets are incomplete as the only asset that is traded between countries is one non-state-contingent bond. I consider three approaches to obtain stationarity: an endogenous discount factor, a debt elastic interest rate premium and convex portfolio costs. While I focus on these three most popular approaches, there are other approaches. Ghironi (2003) solves the stationarity problem by introducing an overlapping generations structure.[3] Huggett (1993) solves the stationarity problem by

[1] Obviously, this problem is not unique to international economics. The same issues occur in models with heterogeneous agents and incomplete asset markets.

[2] See also Kim and Kose (2003) for a related study in a small open economy framework. Lubik (2003) analyses some additional approaches that induce stationarity and finds substantial qualitative differences. Hence, the implicit generalization of the results in Schmitt-Grohé and Uribe (2003) by many researchers is not even appropriate for the case of a small open economy. Boileau and Normandin (2005) extend the analysis to a two-country model with one homogeneous good. Interesting quantitative differences can occur in their setup depending on the persistence of technology shocks.

[3] In the technical appendix to this paper, which is available upon request, I study the overlapping generations structure of Ghironi (2003). The mathematical properties of his approach turn out to be closely related to the models with convex portfolio costs or a debt elastic interest rate premium.

introducing explicit limits on the level of asset holdings.[4]

In the standard model with incomplete markets the steady state is undetermined since the growth rate of marginal utility does not depend on the allocation of net foreign assets. Absent arbitrage opportunities, the price of the non-state-contingent bond is equalized across countries implying that expected marginal utility growth is equalized across countries. In the deterministic steady state, this condition contains no information about the steady state values of the system and the system of equilibrium conditions becomes underdetermined. Any level of net foreign asset holdings is a steady state.

If stationarity is induced by convex portfolio costs, there is a unique stable steady state only if the elasticity of substitution between the domestic and foreign traded goods ε is sufficiently large, i.e., ε is above some threshold level $\bar{\varepsilon}$. For lower values of the elasticity of substitution, however, I find three steady states two of which are locally stable, but the third one is not. It is important to note that this multiplicity of steady states is unrelated to the aforementioned indeterminacy in the non-stationary model. I also analyze the dynamic implications of shocks under different values of the elasticity of substitution ε. For a high value of the elasticity of substitution, there is a unique impulse response function for a small technology shock. If $\varepsilon < \bar{\varepsilon}$ this finding no longer holds true. Assume that the economy is in one of the two stable steady states. For a small technology shock there are two paths that lead the economy back into the original steady state. However, for the same shock, the economy can also converge to the other stable steady state. For example, if the shock improves country 1's technology, the real exchange rate may either depreciate on impact by a small, an intermediate or a large amount relative to the original steady state. The model with a debt elastic interest rate shares these features with the model of convex portfolio costs.

If, following Uzawa (1968), the discount factor is assumed to be endogenous, an agent's rate of time preference is strictly decreasing in the agent's utility level.[5] In this setup there is always a unique and stable steady state irrespective of the value of the elasticity of substitution between foreign and domestic goods ε. Ironically, the unique and stable steady state in the model with endogenous discounting features the same allocations as the unstable steady state in the model with portfolio costs for $\varepsilon \leq \bar{\varepsilon}$. In response to a technology shock a high elasticity of substitution ε implies a unique adjustment path of the economy. However, if ε is below the critical value $\bar{\varepsilon}$ I find three different impulse response functions for a given small technology shock. If the shock raises country 1's technology the real exchange rate

[4] Models with occasionally binding constraints, however, cannot be solved reliably using local approximation techniques. This complicates the analysis and explains why Huggett's approach is typically avoided in international macroeconomics.

[5] If the discount factor is increasing in the agent's utility level, the dynamics around any steady state are explosive.

may either appreciate on impact by a small or a large amount relative to the magnitude of the shock or it may depreciate by a large amount on impact.

The reason for the striking differences between the models lies in the nonlinearities that arise for low values of the elasticity of substitution. Absent international financial markets, there are multiple equilibria if ε is below $\bar{\varepsilon}$. Consider an endowment economy with two countries and two traded goods that are imperfect substitutes.[6] Assume that the countries are just mirroring each other with respect to preferences and endowments.[7] Then, there is always one equilibrium with the relative price of the traded goods equal to unity. However, there can be two more equilibria. If the price of the domestic good is very high relative to the price of the foreign good, domestic agents are very wealthy compared to the foreign agents. If the elasticity of substitution is low, foreigners are willing to give up most of their good in order to consume at least some of the domestic good, and domestic agents end up consuming most of the domestic and the foreign good. The reverse is true as well. Foreign agents consume most of the goods, if the foreign good is very expensive in relative terms. Of course, these last two scenarios cannot be an equilibrium for high values of the elasticity of substitution. In the limiting case of perfect substitutability the unique equilibrium features each country consuming its own endowment.

In the dynamic economy with incomplete asset markets, the equilibria of the economy without international financial markets are the candidate steady states. Consider the case of a low elasticity of substitution, i.e., $\varepsilon < \bar{\varepsilon}$. Under the assumption that portfolio costs are zero if and only if net foreign assets are zero, all three candidate steady states are in fact steady states of the bond economy with convex portfolio costs. Similarly, if the debt elastic interest rate premium is zero if and only if net foreign assets are zero, there are three steady states. However, if the discount factor is endogenous, absence of arbitrage requires that the discount factors are equalized across countries in any steady state. As the discount factor is assumed to be strictly decreasing in the agent's utility level, this condition uniquely determines the steady state allocations (provided a strictly concave utility function and a convex technology).

In the simple model presented in this paper, the critical value of the elasticity of substitution $\bar{\varepsilon}$ lies in the range of $0.4 - 0.7$ for reasonable choices of the remaining parameters. However, the value of $\bar{\varepsilon}$ is sensitive to changes in the model. For example, if the model is extended along the lines of Corsetti and Dedola (2005) to allow for non-traded goods and a strong complementarity between traded and non-traded goods, $\bar{\varepsilon}$ can easily assume values

[6] This static example has been subject to numerous studies in general equilibrium theory, see Kehoe (1991) and Mas-Colell et al (1995).

[7] By mirroring I mean, that good 1 (2) enters the utility function of country 1 agents the same way that good 2 (1) enters the utility function of country 2 agents. The same holds for the agents' endowments with goods 1 and 2.

larger than 1 for reasonable parameterizations of the model.

The empirical literature reports a wide range of trade elasticities at the aggregate level from 0 to 1.5.[8] Whalley (1985) reports an elasticity of 1.5. In a recent study, Hooper, Johnson and Marquez (2000) estimate trade elasticities for the G7 countries. They report a short-run trade elasticity of 0.6 for the U.S., and values ranging between 0 and 0.6 for the remaining G7 countries. Earlier studies by Houthakker and Magee (1969) and Marquez (1990) also suggest trade elasticities between 0 and 1. In his study, Taylor (1993) estimates an import demand equation for the US and finds a short-run trade elasticity of 0.22 and a long-run trade elasticity of 0.39.[9]

The remainder of the paper is organized as follows. Section 2 presents the model and analyses it under the assumption that there are no international financial markets. In section 3, agents have access to one non-state-contingent bond. I analyze the characteristics of the steady states under the different stationarity inducing approaches. The impulse response functions for the model with endogenous discounting and convex portfolio costs are investigated in section 4. Finally, section 5 offers concluding remarks.

2 The model

In the remainder of this section, I analyze the simple two-country model under the assumption of balanced trade, i.e., there are no international financial markets. Absent capital accumulation this assumption allows me to present the issues of multiple equilibria without the additional complications that occur in a dynamic model. Furthermore, under financial autarchy the model is stationary.[10]

In Section 3 the simple model is augmented by the assumption that agents have access to international financial markets. I first present the standard model with incomplete markets in order to illustrate the stationarity problem. Three different approaches to induce stationarity are studied: convex portfolio costs, debt elastic interest rate, and endogenous discount factor. The analysis is guided by two questions. How does the number of steady states in the closed

[8] Obviously, these marco estimates are in sharp contrast to the micro evidence where mean estimates vary between 4 and 6. See, e.g., Broda and Weinstein (2005).

[9] One of the most comprehensive empirical study on trade elasticities is Bayoumi (1999), who uses data on 420 bilateral trade flows between 21 industrialized countries. Under the restriction (not rejected statistically) that elasticities are identical for all country pairs, the estimated long-run price elasticity ranges between 0.38 and 0.89 depending on the model specification.

[10] Introducing capital into the model is unlikely to change the results presented in this paper. It is straightforward to show, that in financial autarchy multiple equilibria also occur in a model with capital if the elasticity of substitution is sufficiently low. The nonlinearities underlying this finding are very likely to cause the same differences between the approaches that induce stationarity as in the model without capital.

model relate to the number of equilibria in the model with financial autarchy? How are the dynamic properties of a steady state related to the slope of the excess demand function for the different stationarity inducing approaches?

2.1 Financial autarchy

There are two countries, each populated by an infinite number of households with a total measure of one. Each country produces only one good that can be traded without frictions in the international goods market. The two goods are assumed to be imperfect substitutes in the household's utility function. Labor, which is supplied endogenously, is the sole input into the production process.

Time is discrete and each period the economy experiences one of finitely many events s_t. $s^t = (s_0, ..., s_t)$ denotes the history of events up through and including period t. The probability, as of period 0, of any particular history s^t is $\pi(s^t)$. The initial realization s_0 is given.

Households maximize their expected lifetime utility subject to the budget constraint

$$\max_{\substack{c_{i1}(s^t), c_{i2}(s^t) \\ c_i(s^t), l_i(s^t)}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t U\left(c_i\left(s^t\right), l_i\left(s^t\right)\right) \pi\left(s^t\right) \tag{1}$$

$$s.t.$$

$$\bar{P}_1\left(s^t\right) c_{i1}\left(s^t\right) + \bar{P}_2\left(s^t\right) c_{i2}\left(s^t\right) \leq \bar{P}_i\left(s^t\right) w_i\left(s^t\right) l_i\left(s^t\right) + \bar{P}_i\left(s^t\right) \Pi_i\left(s^t\right) + dW_i\left(s^t\right), \tag{2}$$

where c_i is given by the CES aggregator $c_i(s^t) = \left[\alpha_{i1}^{1-\rho} c_{i1}^{\rho}(s^t) + \alpha_{i2}^{1-\rho} c_{i2}^{\rho}(s^t)\right]^{\frac{1}{\rho}}$ with $\alpha_{ii} \geq \alpha_{ij} > 0$ for $j \neq i$ and $\rho < 1$. $\frac{1}{1-\rho}$ is the elasticity of substitution between traded goods. In appendix A, I generalize the model to allow for any linear-homogeneous aggregator of the form $c_i(s^t) = H_i(c_{i1}(s^t), c_{i2}(s^t))$.

The strictly concave period utility function $U(c, l)$ is assumed to satisfy the following sign conditions:

$$U_c > 0, U_l < 0 \text{ and } U_{cc} < 0, U_{ll} < 0, U_{cl} \leq 0.$$

These assumptions are satisfied by almost all utility functions that are commonly used in macroeconomics. c_i denotes final consumption, l_i labor, c_{ij} is the consumption of good j by a household located in country i, $\bar{P}_i$ is the price at which good i is traded and w_i is the wage in country i denoted in units of country i's traded good. Real profits are Π_i. dW_i is an arbitrary lump sum transfer to agents in country i, with $dW_1(s^t) + dW_2(s^t) = 0$ for all s^t. I introduce this transfer to make the following derivations general enough to be of use for the case with international financial markets.

Agent i chooses consumption of the two traded goods such that

$$\frac{c_{i1}\left(s^t\right)}{c_{i2}\left(s^t\right)} = \frac{\alpha_{i1}}{\alpha_{i2}}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{1}{\rho-1}}, \tag{3}$$

where $\bar{q}$ is the relative price of good 2 to good 1, $\frac{\bar{P}_2}{\bar{P}_1}$. Let P_i denote the price of the final consumption basket, which is related to $\bar{q}$ by

$$\begin{aligned}
\Phi_1\left(\bar{q}\left(s^t\right)\right) &\equiv \frac{\bar{P}_1\left(s^t\right)}{P_1\left(s^t\right)} = \left[\alpha_{11}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{\rho}{\rho-1}} + \alpha_{12}\right]^{\frac{1-\rho}{\rho}} \frac{1}{\bar{q}\left(s^t\right)} \text{ with } \Phi_1'\left(\bar{q}\left(s^t\right)\right) < 0, \\
\Phi_2\left(\bar{q}\left(s^t\right)\right) &\equiv \frac{\bar{P}_2\left(s^t\right)}{P_2\left(s^t\right)} = \left[\alpha_{21}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{\rho}{\rho-1}} + \alpha_{22}\right]^{\frac{1-\rho}{\rho}} \text{ with } \Phi_2'\left(\bar{q}\left(s^t\right)\right) > 0.
\end{aligned}$$

I normalize the price of the consumption basket in country 1 to unity, $P_1 = 1$. Therefore, P_2 is equal to the consumption-based real exchange rate, q. Obviously, q and $\bar{q}$ are related as follows

$$q\left(s^t\right) = \bar{q}\left(s^t\right)\frac{\Phi_1\left(\bar{q}\left(s^t\right)\right)}{\Phi_2\left(\bar{q}\left(s^t\right)\right)}.$$

Using the budget constraint, (2), and equation (3), the demand functions for good 2 are

$$c_{12}\left(s^t\right) = \frac{1}{\left(\frac{\alpha_{11}}{\alpha_{12}}\left(\frac{1}{\bar{q}(s^t)}\right)^{\frac{1}{\rho-1}} + \bar{q}\left(s^t\right)\right)}\left[w_1\left(s^t\right) l_1\left(s^t\right) + \frac{1}{\Phi_1\left(\bar{q}\left(s^t\right)\right)} dW_1\left(s^t\right)\right], \tag{4}$$

$$c_{22}\left(s^t\right) = \frac{1}{\left(\frac{\alpha_{21}}{\alpha_{22}}\left(\frac{1}{\bar{q}(s^t)}\right)^{\frac{\rho}{\rho-1}} + 1\right)}\left[w_2\left(s^t\right) l_2\left(s^t\right) + \frac{1}{\bar{q}\left(s^t\right)\Phi_1\left(\bar{q}\left(s^t\right)\right)} dW_2\left(s^t\right)\right]. \tag{5}$$

Similar expressions can be derived for the demand of good 1. This provides expressions for aggregate consumption c_i, $i = 1, 2$:

$$c_1\left(s^t\right) = \Phi_1\left(\bar{q}\left(s^t\right)\right) w_1\left(s^t\right) l_1\left(s^t\right) + dW_1\left(s^t\right), \tag{6}$$

$$c_2\left(s^t\right) = \Phi_2\left(\bar{q}\left(s^t\right)\right) w_2\left(s^t\right) l_2\left(s^t\right) + \frac{\Phi_2\left(\bar{q}\left(s^t\right)\right)}{\bar{q}\left(s^t\right)\Phi_1\left(\bar{q}\left(s^t\right)\right)} dW_2\left(s^t\right). \tag{7}$$

The optimal allocation of labor relative to consumption is determined from

$$\frac{U_l\left(c_i\left(s^t\right), l_i\left(s^t\right)\right)}{U_c\left(c_i\left(s^t\right), l_i\left(s^t\right)\right)} = -\Phi_i\left(\bar{q}\left(s^t\right)\right) w_i\left(s^t\right). \tag{8}$$

Firms in country i produce the traded good i using a linear production technology, $y_i(s^t) = A_i(s^t)\, l_i(s^t)$. Appendix $A.1$ shows that equations (6) - (8) can be used to express c_i and l_i as functions of the prices w_1, w_2 and $\bar{q}$ (and dW_1) only.

Definition 1 (Competitive Equilibrium in Financial Autarchy) *A competitive equilibrium is a collection of allocations* $c_{i1}(s^t)$, $c_{i2}(s^t)$, $c_i(s^t)$, $l_i(s^t)$, $y_i(s^t)$ *and prices* $\bar{q}(s^t)$, $w_i(s^t)$, $i = 1, 2$, *such that* (i) *for every household the allocations solve the household's maximization problem for given prices,* (ii) *for every firm profits are maximized and* (iii) *the markets for labor and for the two traded goods clear.*

Perfect competition and the linear technology imply that the equilibrium wage equals the productivity parameter, i.e. $w_i(s^t) = A_i(s^t)$. As shown in appendix $A.2$ the equilibrium conditions for this model can be fully summarized by the excess demand function for good 2:

$$
\begin{aligned}
z_2\left(\bar{q}\left(s^t\right), dW_1\left(s^t\right), dW_2\left(s^t\right)\right) &= c_{12}\left(s^t\right) + c_{22}\left(s^t\right) - y_2\left(s^t\right) \\
&= \frac{A_1(s^t)\, l_1(\bar{q}(s^t), dW_1(s^t)) + \frac{1}{\Phi_1(\bar{q}(s^t))} dW_1(s^t)}{\frac{\alpha_{11}}{\alpha_{12}}\left(\frac{1}{\bar{q}(s^t)}\right)^{\frac{1}{\rho-1}} + \bar{q}(s^t)} \\
&+ \frac{A_2(s^t)\, l_2(\bar{q}(s^t), dW_2(s^t)) + \frac{1}{\bar{q}\Phi_1(\bar{q}(s^t))} dW_2(s^t)}{\frac{\alpha_{21}}{\alpha_{22}}\left(\frac{1}{\bar{q}(s^t)}\right)^{\frac{\rho}{\rho-1}} + 1} \\
&- A_2\left(s^t\right) l_2\left(\bar{q}\left(s^t\right), dW_2\left(s^t\right)\right). \qquad (9)
\end{aligned}
$$

An equilibrium is a relative price $\bar{q}^*(s^t)$, s.t. $z_2(\bar{q}^*(s^t), dW_1(s^t), dW_2(s^t)) = 0$. Appendix $A.2$ proves the existence of the competitive equilibrium.

Figure 1 plots the excess demand for good 2 as a function of $\frac{\bar{q}}{1+\bar{q}}$ for different values of the elasticity of substitution ε. In plotting z_2, I assume the following utility function

$$U(c, l) = \frac{c^{1-\gamma}}{1-\gamma} - \chi_0 \frac{l^{1-\chi}}{1-\chi}. \qquad (10)$$

The parameter values are

parameter	value	explanation of the parameter
χ	-2.75	$-\frac{1}{\chi}$ is the Frisch labor supply elasticity
χ_0	7.00	
γ	3.00	coefficient of relative risk aversion
$\alpha_{11} = \alpha_{22}$	0.90	weight on domestic good in CES aggregator
$\alpha_{21} = \alpha_{12}$	0.10	weight on foreign good in CES aggregator
$A_1 = A_2$	1.00	technology level

Table 1: parameterization

Unless noted otherwise these are the parameters for all figures in this paper. Furthermore, I assume $dW_1 = dW_2 = 0$.

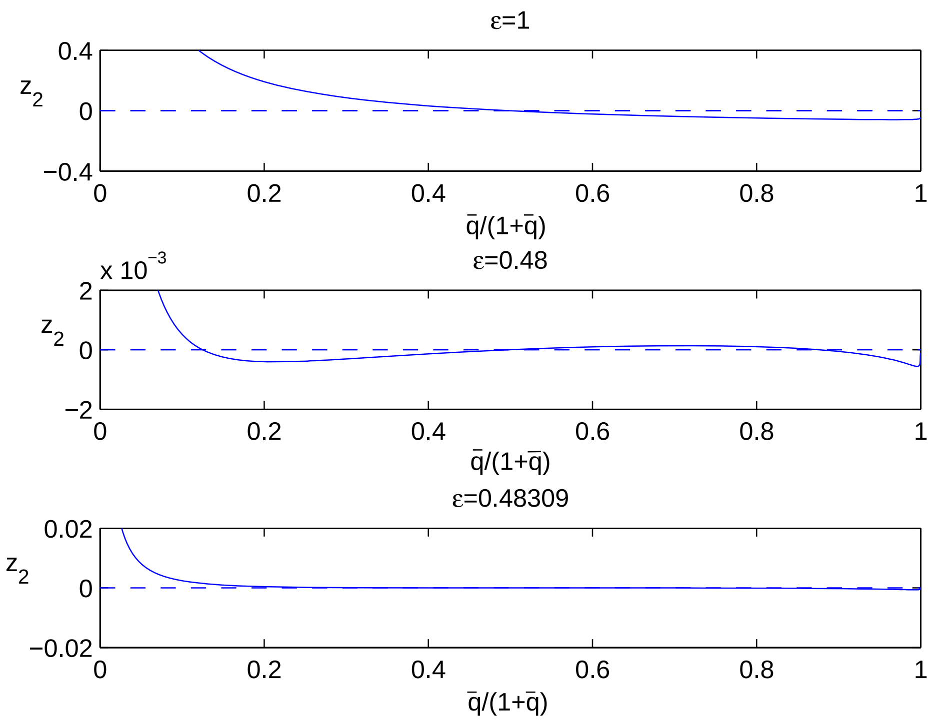

Figure 1: Excess demand for good 2 for different values of ε.

For a given parameterization of the remainder of the model I distinguish three cases for the elasticity of substitution, $\varepsilon = \frac{1}{1-\rho}$:

1. If ε is sufficiently high, as in the first panel, the excess demand function has exactly one zero. Hence, the equilibrium is unique and it features $\frac{\bar{q}}{1+\bar{q}} = \frac{1}{2}$. Furthermore, the slope of the excess demand function at $\frac{\bar{q}}{1+\bar{q}} = \frac{1}{2}$ is negative.

2. If ε is sufficiently low, as in the second panel, the excess demand function has three zeros. While the excess demand function is downward sloping at the first and third equilibrium, it is upward sloping in the second one (counted from left to right).

3. There is a critical value $\bar{\varepsilon}$ (here $\bar{\varepsilon} = 0.48309$) of the elasticity of substitution such that there is a continuum of equilibria because the slope of the excess demand function is zero around $\frac{\bar{q}}{1+\bar{q}} = \frac{1}{2}$.

Multiple equilibria arise at low values of the elasticity of substitution for the following reason. Consider the first equilibrium in the second panel. If the price of good 1 is high relative to the price of good 2,$\frac{\bar{q}}{1+\bar{q}} << \frac{1}{2}$, agents of country 2 produce more of their good than agents of country 1. As the elasticity of substitution between the goods is very low, country 2 agents are willing to pay the high price for good 1 and country 1 ends up consuming most of the two goods. The same logic applied in the third equilibrium, $\frac{\bar{q}}{1+\bar{q}} >> \frac{1}{2}$, with the roles of country 1 and 2 being reversed. The second equilibrium is the symmetric equilibrium featuring $\frac{\bar{q}}{1+\bar{q}} = \frac{1}{2}$. If the elasticity of substitution is high, equilibria 1 and 3 cannot exist. In fact, if the elasticity of substitution is infinite, agents in both countries only consume their own goods and the relative price in the unique equilibrium has to be equal to 1.

2.2 More about multiple equilibria

The above findings are not surprising. The analogous endowment economy with a CES aggregator has been used extensively in general equilibrium theory to study equilibrium multiplicity. For instance see Mas-Colell (1991), Kehoe (1991), Gjerstad (1996) and Bela (1997). In appendix $A.3$, I summarize some of the findings of general equilibrium theory in the context of the model presented here in this paper. In general, the number of equilibria is odd. If the excess demand function is upward sloping in an equilibrium, there have to be at least two more. Unfortunately, nothing can be said with certainty about the number of equilibria unless one can prove that the equilibrium is unique.

To gain an idea about the relationship between the critical value $\bar{\varepsilon}$ and the remaining model parameters, consider the case of $\alpha_{11} = \alpha_{22} \geq \frac{1}{2}$, $\alpha_{12} = \alpha_{21}$ and identical preferences over consumption and leisure in the two countries. Irrespective of the value of the elasticity of substitution, $\bar{q} = 1$ is an equilibrium. As shown more generally in appendix $A.3$, the critical value $\bar{\varepsilon}$ is given by

$$2\alpha_{11}\bar{\varepsilon} + (1 - 2\alpha_{11}) - \frac{\partial l_1}{\partial \bar{q}} \frac{\bar{q}}{l_1} + \frac{\partial l_2}{\partial \bar{q}} \frac{\bar{q}}{l_2} = 0,$$

where $\frac{\partial l_i}{\partial \bar{q}} \frac{\bar{q}}{l_i}$ is the general equilibrium elasticity of labor with respect to $\bar{q}$. With additive separable preferences as in (10), $\frac{\partial l_i}{\partial \bar{q}} \frac{\bar{q}}{l_i} = \frac{1-\gamma}{\gamma-\chi} (1 - \alpha_{ii}) (-1)^i$, $i = 1, 2$, $(\gamma > 0,\ \chi < 0)$ and

$$\bar{\varepsilon} = \frac{2\alpha_{11} - 1}{2\alpha_{11}} + \frac{\gamma - 1}{\gamma - \chi} \frac{1 - \alpha_{11}}{\alpha_{11}}.$$

This implies:

- Even without home bias in consumption, i.e., $\alpha_{11} = \frac{1}{2}$, $\bar{\varepsilon} > 0$ if agents are sufficiently risk averse $(\gamma > 1)$.
- $\bar{\varepsilon}$ is strictly increasing in γ.
- If $\gamma + \chi < 2$, $\bar{\varepsilon}$ is increasing in the home bias. Otherwise, $\bar{\varepsilon}$ is decreasing in α_{11}.
- If $\gamma > 1$, $\bar{\varepsilon}$ is increasing in the Frish labor supply elasticity $\eta = -\frac{1}{\chi}$. Otherwise, $\bar{\varepsilon}$ is decreasing in α_{11}.
- If $\gamma \to \infty$ and $\chi \to 0$, $\bar{\varepsilon} = \frac{1}{2\alpha_{11}}$.

If the household's preferences over consumption and leisure are Cobb-Douglas the appendix shows that $\frac{\partial l_i}{\partial \bar{q}} \frac{\bar{q}}{l_i} = 0$, $i = 1, 2$ and

$$\bar{\varepsilon} = \frac{2\alpha_{11} - 1}{2\alpha_{11}}.$$

In this case, $\bar{\varepsilon}$ does not depend on the preference parameters over c and l.

As these examples show, the critical value of $\bar{\varepsilon}$ (and therefore the presence of multiple equilibria) is greatly affected by certain model choices. In the technical appendix to this paper, which is available upon request, I also show the following:

- In a model with endogenous capital formation, $\bar{\varepsilon}$ also depends on the elasticity of substitution between labor and capital. Relative to the model without capital, $\bar{\varepsilon}$ can be lower or higher for otherwise identical parameters.

- In a model with non-traded goods, $\bar{\varepsilon}$ can be very high (much larger than unity) for reasonable parameter choices if there is some complementarity between traded and non-traded goods as in Corsetti, Dedola and Leduc (2005).

The value of ε also has an important impact on the comparative static properties of the model. Consider a small increase in the productivity level of country 1. Such a change deforms the excess demand function and shifts it upwards. Figure 2 shows the excess demand function for $\varepsilon = 0.5 > \bar{\varepsilon}$ (upper panel) and $\varepsilon = 0.48 < \bar{\varepsilon}$ (lower panel) for two technology shocks of different magnitude. The solid line depicts the original excess demand function with $A_1/A_2 = 1.00$. The dashed line shows the case of $A_1/A_2 = 1.0025$ and the dashed-dotted line is the case of $A_1/A_2 = 1.005$. If the elasticity of substitution is large ($\varepsilon > \bar{\varepsilon}$) the increase in A_1 leads to a small increase in the equilibrium value of the relative price of traded goods irrespective of the magnitude of the shock.

The situation is quite different if the elasticity is low ($\varepsilon < \bar{\varepsilon}$). For a small relative increase in A_1 all three equilibria are preserved. While the price of traded goods rises in the first and third equilibrium relative to the original equilibrium, $A_1/A_2 = 1.00$, $\bar{q}$ drops in the second equilibrium. However, if the technology shock is sufficiently large, the first two equilibria disappear. Only the third equilibrium survives. The dashed-dotted line in panel 2 has only one zero, which occurs around $\frac{\bar{q}}{1+\bar{q}} = 0.92$.

3 Bond economies

In contrast to the last section agents are now assumed to have access to international financial markets. Following the standard assumption in international macroeconomics, financial markets are exogenously incomplete in the sense that the only asset that is traded internationally is one non-state-contingent bond. This bond is in zero net supply, i.e., $B_1(s^t) + B_2(s^t) = 0$.

In order to illustrate the stationarity problem, I begin with the standard incomplete markets setup without stationarity-inducing features.

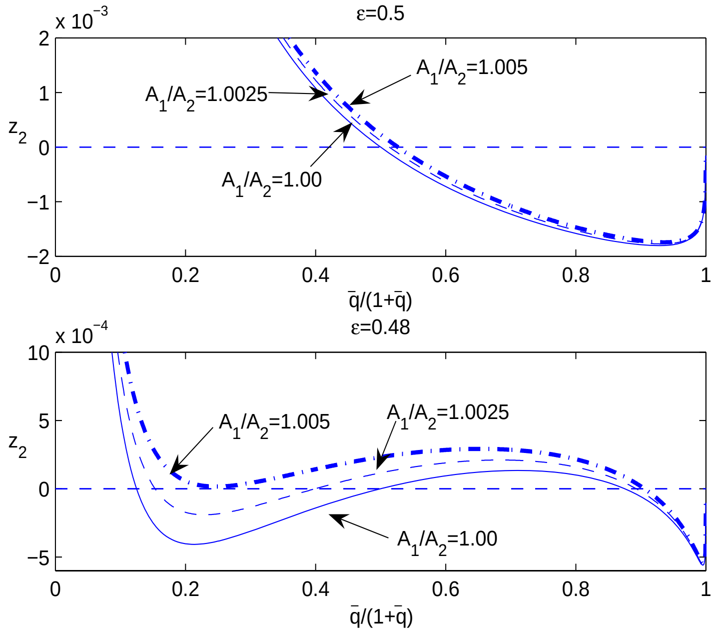

Figure 2: Equilibria for small changes in A_1/A_2.

3.1 The non-stationary model

In the standard two-country model a household faces the following maximization problem

$$\max_{\substack{c_i(s^t),l_i(s^t),\\ c_{i1}(s^t),c_{i2}(s^t),\\ B_i(s^t)}} \sum_{t=0}^{\infty}\sum_{s^t}\beta^t U\left(c_i\left(s^t\right),l_i\left(s^t\right)\right)\pi\left(s^t\right)$$

$$s.t.$$

$$P_i\left(s^t\right)c_i\left(s^t\right) \le \bar{P}_i\left(s^t\right)w_i\left(s^t\right)l_i\left(s^t\right)+\bar{P}_i\left(s^t\right)\Pi_i\left(s^t\right)+B_i\left(s^{t-1}\right)-Q\left(s^t\right)B_i\left(s^t\right).$$

$P_i c_i$ are the household's total consumption expenditures which are equal to $\bar{P}_1 c_{i1}+\bar{P}_2 c_{i2}$. In order to make use of the above derivations, notice that I have replaced the lump-sum transfer between countries $dW_i\left(s^t\right)$ by $B_i\left(s^{t-1}\right)-Q\left(s^t\right)B_i\left(s^t\right)$. $B_i\left(s^{t-1}\right)$ denotes the (nominal) bond holdings that agent i has inherited from period $t-1$. Q is the price of bonds.

Given the assumptions on technology, preferences, and trade stated in the previous section the equilibrium dynamics are fully summarized by

1. the excess demand function for good 2

$$
\begin{aligned}
& z_2\left(\bar{q}\left(s^t\right), B_1\left(s^{t-1}\right)-Q\left(s^t\right) B_1\left(s^t\right)\right) \\
= \; & \frac{A_1 l_1\left(\bar{q}\left(s^t\right), B_1\left(s^{t-1}\right)-Q\left(s^t\right) B_1\left(s^t\right)\right)}{\frac{\alpha_{11}}{\alpha_{12}}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{1}{\rho-1}}+\bar{q}\left(s^t\right)} \\
& -\frac{\frac{\alpha_{21}}{\alpha_{22}}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{\rho}{\rho-1}} A_2 l_2\left(\bar{q}\left(s^t\right),-B_1\left(s^{t-1}\right)-Q\left(s^t\right) B_1\left(s^t\right)\right)}{\frac{\alpha_{21}}{\alpha_{22}}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{\rho}{\rho-1}}+1} \\
& +\left[\frac{1}{\frac{\alpha_{11}}{\alpha_{12}}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{1}{\rho-1}}+\bar{q}\left(s^t\right)}-\frac{1}{\frac{\alpha_{21}}{\alpha_{22}}\left(\frac{1}{\bar{q}\left(s^t\right)}\right)^{\frac{\rho}{\rho-1}}+1}\right] \frac{1}{\Phi_1\left(\bar{q}\left(s^t\right)\right)}\left[B_1\left(s^{t-1}\right)-Q\left(s^t\right) B_1\left(s^t\right)\right],
\end{aligned} \tag{11}
$$

where

$$
Q\left(s^t\right)=\sum_{s^{t+1} \mid s^t} \beta \frac{U_c\left(c_1\left(s^{t+1}\right), l_1\left(s^{t+1}\right)\right)}{U_c\left(c_1\left(s^t\right), l_1\left(s^t\right)\right)} \pi\left(s^{t+1} \mid s^t\right),
$$

and $z_2\left(\bar{q}\left(s^t\right), B_1\left(s^{t-1}\right)-Q\left(s^t\right) B_1\left(s^t\right)\right)=0$ in equilibrium,

2. the familiar risk sharing equation, states that expected marginal utility growth is equalized across countries:

$$
\sum_{s^{t+1} \mid s^t} \beta\left[\frac{U_c\left(c_1\left(s^{t+1}\right), l_1\left(s^{t+1}\right)\right)}{U_c\left(c_1\left(s^t\right), l_1\left(s^t\right)\right)}-\frac{U_c\left(c_2\left(s^{t+1}\right), l_2\left(s^{t+1}\right)\right)}{U_c\left(c_2\left(s^t\right), l_2\left(s^t\right)\right)} \frac{q\left(s^t\right)}{q\left(s^{t+1}\right)}\right] \pi\left(s^{t+1} \mid s^t\right)=0, \tag{12}
$$

where $q\left(s^t\right)=\bar{q}\left(s^t\right) \frac{\Phi_1\left(\bar{q}\left(s^t\right)\right)}{\Phi_2\left(\bar{q}\left(s^t\right)\right)}$.

I have used the assumption that bonds are in zero net supply. As shown in appendix $A.1$, consumption and labor choices can be expressed as functions of the relative price $\bar{q}$. Moreover, this system of difference equations has to satisfy the appropriate initial and transversality conditions.

Unfortunately, the deterministic steady state of this model is not unique. With $c_i=c_i(\bar{q})$ and $l_i=l_i(\bar{q})$ as shown in Appendix $A.1$, equations (11) and (12) have to solve for the steady state values of $\bar{q}$ and B_1. However, in a steady state, equation (12) collapses to an identity and contains no information about the endogenous variables. Hence, there is one equations but two unknowns.

Admittedly, it is possible to choose a particular steady state amongst the set of feasible solutions to (11). It is common practice see, e.g., Baxter and Crucini (1995) to assume $B_1 = B_2 = 0$. Although this choice pins down the original steady state, the dynamic system that describes the behavior of the economy in the neighborhood of the steady state is not stationary. Even a completely temporary shock has long lasting effects on the economy: whatever the level of bond holdings materializes in the period immediately following a shock becomes the new long-run position until a new shock occurs.

This problem is easily seen by looking at the linear approximation of the dynamic system around a candidate steady state with $B_1 = B_2 = 0$. For simplicity assume that preferences are additive-separable in consumption and leisure, i.e.,

$$U(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \chi_0 \frac{l^{1-\chi}}{1-\chi}.$$

Using equations (6) $-$ (8) with $dW_1 = B_1(s^{t-1}) - Q(s^t) B_1(s^t)$, consumption in country 1 can be expressed as

$$c_1(s^t) = \chi_0^{\frac{1}{\chi}} \left[\Phi_1(\bar{q}(s^t)) A_1(s^t)\right]^{\frac{\chi-1}{\chi}} \left[c_1(s^t)\right]^{\frac{\gamma}{\chi}} + B_1(s^{t-1}) - Q(s^t) B_1(s^t).$$

Log-linearization around a steady state delivers

$$\left(1 - \frac{\gamma}{\chi}\right) \hat{c}_{1,t} = \frac{\chi - 1}{\chi} \frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})} \bar{q}_t + \frac{1}{c_1}\left(b_{1,t-1} - \beta b_{1,t}\right),$$

where $\hat{c}_{1,t}$ and $\bar{q}_t$ are the percentage deviation of consumption and the relative price from their respective steady state values. $b_{1,t}$ is the absolute deviations of bond holdings from 0. Notice that the log-linearized excess demand function implies

$$\frac{\partial z_2}{\partial \bar{q}} \bar{q} \bar{q}_t + \frac{\partial z_2}{\partial dW_1}\left[b_{1,t-1} - \beta b_{1,t}\right] = 0.$$

Hence, $\hat{c}_{1,t}$ can simply be expressed as a function of $\bar{q}_t$. Similar reasoning applies for $\hat{c}_{2,t}$, which can also be expressed as a function of $\bar{q}_t$ only.

Finally, the log-linear approximation of equation (12) is given by

$$-\gamma\left(\left(\hat{c}_{1,t+1} - \hat{c}_{2,t+1}\right) - \left(\hat{c}_{1,t} - \hat{c}_{2,t}\right)\right) = \left[1 + \frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})} - \frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})}\right]\left(\hat{q}_t - \hat{q}_{t+1}\right).$$

The findings for $\hat{c}_{i,t}$ (and $\hat{c}_{i,t+1}$) imply

$$\hat{q}_t = \hat{q}_{t+1}.$$

The dynamics of the system can therefore be approximated by

$$\begin{pmatrix} \bar{q}_{t+1} \\ b_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\beta}\frac{\frac{\partial z_2}{\partial \bar{q}}\bar{q}}{\frac{\partial z_2}{\partial dW_1}} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \bar{q}_t \\ b_{t-1} \end{pmatrix}$$

The two eigenvalues that are associated with this system are 1 and $\frac{1}{\beta}$. This implies that a purely transitory shock to technology in country 1 at time t permanently raises the relative price $\bar{q}$, and turns country 1 into a borrower. There is no internal mechanism that leads $\bar{q}$ and B_1 back to the original steady state values. $\bar{q}$ and B_1 only change if new shocks occur.

3.2 Bond economy with convex portfolio costs

Similar to Heathcote and Perri (2002), and Schmitt-Grohé and Uribe (2003) let agents face a convex cost for holding/issuing bonds. The collected fees are reimbursed to the agents by a lump-sum transfer. $\Phi\left(B_i/\bar{P}_i\right)$ denotes the portfolio costs in terms of country i's traded good, where $\Phi'(0) = 0$ and $\Phi' > 0$ otherwise. The representative household in country i solves

$$\max_{\substack{c_i(s^t), l_i(s^t), \\ c_{i1}(s^t), c_{i2}(s^t), \\ B_i(s^t)}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t U\left(c_i\left(s^t\right), l_i\left(s^t\right)\right) \pi\left(s^t\right)$$

$$\begin{aligned} &s.t. \\ &P_i\left(s^t\right) c_i\left(s^t\right) \leq \bar{P}_i\left(s^t\right) w_i\left(s^t\right) l_i\left(s^t\right) + \bar{P}_i\left(s^t\right) \Pi_i\left(s^t\right) + B_i\left(s^{t-1}\right) - Q\left(s^t\right) B_i\left(s^t\right) \\ &\qquad\qquad - \bar{P}_i\left(s^t\right) \Gamma\left(\frac{B_i\left(s^t\right)}{\bar{P}_i\left(s^t\right)}\right) + T_i\left(s^t\right), \end{aligned}$$

where T_i is the lump-sum reimbursement of the portfolio costs.

The equilibrium dynamics are fully summarized by $z_2\left(\bar{q}\left(s^t\right), B_1\left(s^{t-1}\right) - Q\left(s^t\right) B_1\left(s^t\right)\right) = 0$ with z_2 given by equation (11),

$$Q\left(s^t\right) = \sum_{s^{t+1}|s^t} \beta \frac{U_c\left(c_1\left(s^{t+1}\right), l_1\left(s^{t+1}\right)\right)}{U_c\left(c_1\left(s^t\right), l_1\left(s^t\right)\right)} \pi\left(s^{t+1}|s^t\right) - \Gamma'\left(\frac{B_1\left(s^t\right)}{\bar{P}_1\left(s^t\right)}\right),$$

and the risk sharing condition

$$\sum_{s^{t+1}|s^t} \beta \left[\frac{U_c\left(c_1\left(s^{t+1}\right), l_1\left(s^{t+1}\right)\right)}{U_c\left(c_1\left(s^t\right), l_1\left(s^t\right)\right)} - \frac{U_c\left(c_2\left(s^{t+1}\right), l_2\left(s^{t+1}\right)\right)}{U_c\left(c_2\left(s^t\right), l_2\left(s^t\right)\right)} \frac{q\left(s^t\right)}{q\left(s^{t+1}\right)} \right] \pi\left(s^{t+1}|s^t\right) \quad (13)$$

$$= \Gamma'\left(\frac{B_1\left(s^t\right)}{\bar{P}_1\left(s^t\right)}\right) - \Gamma'\left(\frac{B_2\left(s^t\right)}{\bar{P}_2\left(s^t\right)}\right).$$

As $\Gamma' = 0$ for $B_i = 0$ and larger than zero otherwise, equation (13) implies that in a steady state $B_1 = B_2 = 0$. Although the steady state value of bond holdings is uniquely determined, there can still be multiple steady states if $z_2(\bar{q}, 0) = 0$ has multiple solutions in $\bar{q}$. Hence, any steady state of the financial autarchy model is also a steady state of the model with convex portfolio costs. Consequently, for $\varepsilon < \bar{\varepsilon}$ the portfolio cost model has multiple steady states.

The global equilibrium dynamics in an economy with perfect foresight are depicted in figures 3 and 4 in a phase diagram. The dashed lines are the $B_{1,t} - B_{1,t-1} = 0$ locus and the $\bar{q}_{t+1} - \bar{q}_t = 0$ locus of the dynamic system, respectively. Each intersection of the two loci corresponds to a steady state. The manifold which has been computed by a reverse shooting algorithm is depicted by the solid line.[11] If the elasticity of substitution is high ($\varepsilon > \bar{\varepsilon}$), there is a unique and stable steady state, see figure 4.

However, there are three steady states if the elasticity of substitution is low ($\varepsilon < \bar{\varepsilon}$) as depicted in figure 5. As indicated by the arrows the first and the third steady state are locally stable, but the second one $\left(\frac{\bar{q}}{1+\bar{q}} = \frac{1}{2}, B_1 = 0\right)$ is not. For intermediate values of initial bond holdings the economy converges either to the first or to the third steady state. Convergence to a steady state is unique only if the initial bond holdings are sufficiently high in absolute value.

The (local) dynamic properties of the model with convex portfolio costs are summarized in the following theorem.

Theorem 1 *Assume that agents face convex portfolio costs for holding/issuing bonds as described above. If the slope of the excess demand function is negative in a steady state, then this steady state is a saddle point. If the slope of the excess demand function is positive in a steady state, then such a steady state is unstable if* $\Gamma''(0)$ *is sufficiently small, i.e.,* $\Gamma''(0) < \Delta_P$. *Otherwise this steady state is a saddle point.*

Appendix B.3 provides an exact definition of Δ_P and a proof of the above theorem. $\Gamma''(0)$ measures how sensitive the portfolio costs are in the steady state with respect to changes in

[11] See Judd (1998). In plotting the stable manifold I have assumed that the portfolio costs are quadratic, i.e., $\Gamma = \frac{1}{2}\gamma\left(\frac{B_i}{\bar{P}_i}\right)^2$ with $\gamma = 0.05$.

the bond distribution. To keep the model with convex portfolio costs close to the original model, the portfolio costs need to be small and quite insensitive to changes in the allocation of assets. In fact, if Γ is quadratic as in Heathcote and Perri (2002), $\Gamma''(0)$ is sufficiently small for portfolio costs that are of realistic magnitude.

If portfolio costs are chosen to be very large, the model becomes similar to the model with financial autarchy. Under financial autarchy, any steady state is locally stable in this set-up.

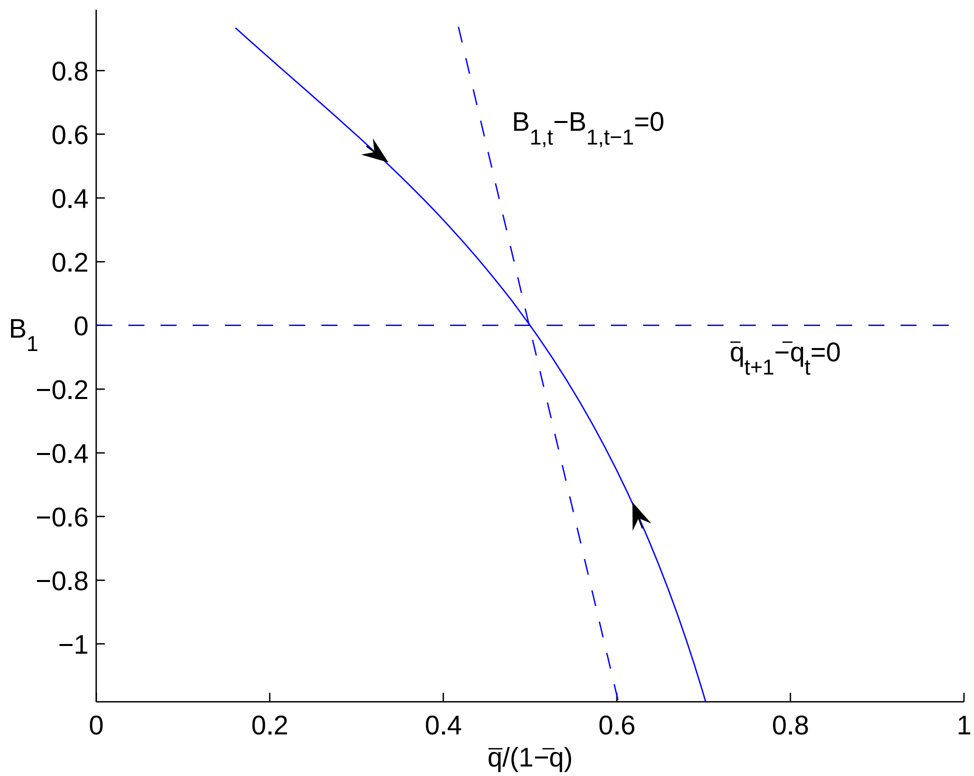

Figure 3: Stability of the steady state for $\varepsilon = 1$ with convex portfolio costs.

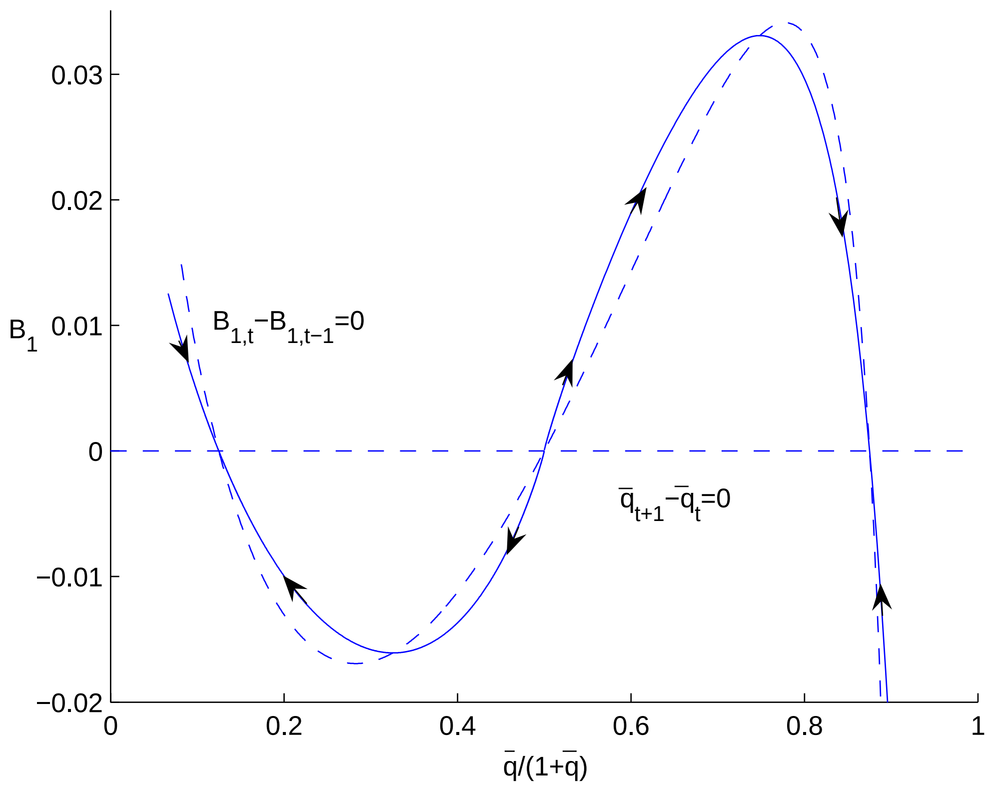

Figure 4: Stability of the steady state for $\varepsilon = 0.48$ with convex portfolio costs.

3.3 Bond economy with debt elastic interest rate

In the setup of this paper the portfolio cost approach is very similar to a model with a debt elastic interest rate. The latter approach assumes that the consumers in countries 1 and 2 face different prices for the bond, and that the spread between the prices is a function of the net foreign asset position. This approach appears among others in Boileau and Normandin (2005), Devereux and Smith (2003), and Schmitt-Grohé and Uribe (2003). The households budget constraint is given by

$$P_i\left(s^t\right) c_i\left(s^t\right) \leq \bar{P}_i\left(s^t\right) w_i\left(s^t\right) l_i\left(s^t\right) + \bar{P}_i\left(s^t\right) \Pi_i\left(s^t\right) + B_i\left(s^{t-1}\right) - Q_i\left(s^t\right) B_i\left(s^t\right),$$

where $Q_i(s^t)$ is the price of the bond in country i. Similar to Devereux and Smith (2003), the interest rate differential is of the form

$$R_1\left(s^t\right) = R_2\left(s^t\right)\Psi\left(B_{1,t+1} - \bar{B}_1\right), \tag{14}$$

where the function $\Psi(B_{1,t+1})$ satisfies $\Psi(0) = 1$ and $\Psi' < 0$. $\bar{B}_1$ is a reference level of debt for country 1. For simplicity, I assume $\bar{B}_1 = 0$. When country 1 is a net borrower, it faces an interest rate that is higher than the interest rate in country 2. When country 1 is a lender, it receives an interest rate that is lower. In equilibrium, interest rates and bond prices satisfy

$$\begin{aligned}
\frac{1}{R_1(s^t)} &= Q_1\left(s^t\right) = \sum_{s^{t+1}|s^t} \beta \frac{U_c\left(c_1\left(s^{t+1}\right), l_1\left(s^{t+1}\right)\right)}{U_c\left(c_1\left(s^t\right), l_1\left(s^t\right)\right)} \pi\left(s^{t+1}|s^t\right), \\
\frac{1}{R_2(s^t)} &= Q_2\left(s^t\right) = \sum_{s^{t+1}|s^t} \beta \frac{U_c\left(c_2\left(s^{t+1}\right), l_2\left(s^{t+1}\right)\right)}{U_c\left(c_2\left(s^t\right), l_2\left(s^t\right)\right)} \frac{q\left(s^t\right)}{q\left(s^{t+1}\right)} \pi\left(s^{t+1}|s^t\right).
\end{aligned}$$

Furthermore, equation (14) implies

$$\frac{R_1(s^t)}{R_2(s^t)} = \frac{\sum_{s^{t+1}|s^t} \beta \frac{U_c\left(c_2\left(s^{t+1}\right), l_2\left(s^{t+1}\right)\right)}{U_c\left(c_2(s^t), l_2(s^t)\right)} \frac{q\left(s^t\right)}{q(s^{t+1})} \pi\left(s^{t+1}|s^t\right)}{\sum_{s^{t+1}|s^t} \beta \frac{U_c\left(c_1(s^{t+1}), l_1(s^{t+1})\right)}{U_c\left(c_1(s^t), l_1(s^t)\right)} \pi\left(s^{t+1}|s^t\right)} = \Psi\left(B_{1,t+1} - \bar{B}_1\right). \tag{15}$$

As c_i, l_i and q can be expressed as functions of $\bar{q}$ only, the dynamics of the economy are fully described by (15) and the condition that the excess demand for good 2 has to be zero, i.e., $z_2\left(\bar{q}, B_1\left(s^{t-1}\right) - Q_1\left(s^t\right) B_1\left(s^t\right), B_2\left(s^{t-1}\right) - Q_2\left(s^t\right) B_2\left(s^t\right)\right) = 0$.

In a steady state equation (15) implies $B_1 = B_2 = 0$ given the assumption $\Psi(0) = 1$. Hence, in the model with a debt elastic interest rate all the steady states of the financial autarchy model are preserved.

As in the model with convex portfolio costs the stability of a steady state can be linked to the slope of the excess demand function.

Theorem 2 *Assume that the interest rate differential between the two countries is debt elastic as described above. If the slope of the excess demand function is negative in a steady state, then this steady state is a saddle point. If the slope of the excess demand function is positive in a steady state, then this steady state is unstable if $\Psi'(0)$ is sufficiently large, i.e., $0 > \Psi'(0) > \Delta_D$. Otherwise this steady state is a saddle point.*

Appendix $B.3$ provides an exact definition of Δ_D. Similar to the model with convex portfolio costs, the condition $\Psi'(0) > \Delta_D$ implies that the interest rate does not react too strongly to changes in the bond holdings. Hence, to the extent that the model with a debt elastic interest rate is supposed to behave close to the original model, any steady state with an upward-sloping excess demand function is unstable.[12]

3.4 Bond economy with endogenous discounting

In this section agents' discount factors are assumed to be endogenous as in Mendoza (1991), Corsetti, Dedola and Leduc (2005), and Schmitt-Grohé and Uribe (2003).[13] This concept of preferences with intertemporal dependences was introduced by Uzawa (1968) and it has been extended and clarified by Epstein (1983, 1987). Uzawa-Epstein preferences fall into the broader class of recursive preferences. The subjective discount factor is assumed to be a decreasing function of the period utility level, i.e., agents become more impatient as current utility rises. For most of the analysis, I assume that agents do not internalize the effect that their current consumption and labor choices have on their discount factor. As the model is solved by a backward shooting algorithm, this assumption simplifies the analysis drastically because it reduces the number of state variables from three to just one.

The problem of the representative household is given by

$$\max_{\substack{c_i\left(s^t\right),l_i\left(s^t\right),\\ c_{i1}\left(s^t\right),c_{i2}\left(s^t\right),\\ B_i\left(s^t\right)}} \sum_{t=0}^{\infty}\sum_{s^t}\theta_i\left(s^t\right)U\left(c_i\left(s^t\right),l_i\left(s^t\right)\right)\pi\left(s^t\right)$$

$$\begin{aligned}
&s.t.\\
&\theta_i\left(s^{t+1}\right)=\beta_i\left[U\left(c_i\left(s^t\right),l_i\left(s^t\right)\right)\right]\theta_i\left(s^t\right)\\
&P_i\left(s^t\right)c_i\left(s^t\right)\le\bar{P}_i\left(s^t\right)w_i\left(s^t\right)l_i\left(s^t\right)+\bar{P}_i\left(s^t\right)\Pi_i\left(s^t\right)+B_i\left(s^{t-1}\right)-Q\left(s^t\right)B_i\left(s^t\right).
\end{aligned}$$

The equilibrium dynamics are fully summarized by $z_2\left(\bar{q}\left(s^t\right),B_1\left(s^{t-1}\right)-Q\left(s^t\right)B_1\left(s^t\right)\right)=0$ with z_2 given by equation (11), where

$$Q\left(s^t\right)=\sum_{s^{t+1}|s^t}\beta_1\left[U\left(c_1\left(s^t\right),l_1\left(s^t\right)\right)\right]\frac{U_{c,1}\left(c_1\left(s^{t+1}\right),l_1\left(s^{t+1}\right)\right)}{U_{c,1}\left(c_1\left(s^t\right),l_1\left(s^t\right)\right)}\pi\left(s^{t+1}|s^t\right)$$

[12] Before I move to the case of a bond economy with endogenous discounting, a short comment about Ghironi's (2003) overlapping generations structure seems to be appropriate. Like the model with a debt elastic interest rate, Ghironi's framework preserves all the candidate steady states that occur in the model without international financial markets. Therefore, it is not surprising, that in his framework a steady state is dynamically unstable (saddle-path stable) if the excess demand function is upward (downward) sloping in the steady state.

[13] Mendoza (1991) uses the endogenous discount factor approach in a small open economy model. Corsetti, Dedola and Leduc (2005) employ this technique in a two country model of the international business cycle.

and the risk sharing condition

$$\sum_{s^{t+1}|s^t}\left[\beta_1\left[U\left(s^t\right)\right]\frac{U_c\left(c_1\left(s^{t+1}\right),l_1\left(s^{t+1}\right)\right)}{U_c\left(c_1\left(s^t\right),l_1\left(s^t\right)\right)}-\beta_2\left[U\left(s^t\right)\right]\frac{U_c\left(c_2\left(s^{t+1}\right),l_2\left(s^{t+1}\right)\right)}{U_c\left(c_2\left(s^t\right),l_2\left(s^t\right)\right)}\frac{q\left(s^t\right)}{q\left(s^{t+1}\right)}\right]\pi\left(s^{t+1}|s^t\right)=0, \tag{16}$$

where $c_i = c_i(\bar{q})$ and $l_i = l_i(\bar{q})$ as shown earlier. This system of difference equations has to satisfy the appropriate initial and transversality conditions.

Equation (16) implies that in a steady state the discount factors are equalized across countries

$$\beta_1\left[U\left(c_1\left(s^t\right),l_1\left(s^t\right)\right)\right]=\beta_2\left[U\left(c_2\left(s^t\right),l_2\left(s^t\right)\right)\right]. \tag{17}$$

As β_i is strictly decreasing in U_i, the utility function is strictly concave, and the technology is concave, there is a unique allocation and a unique price $\bar{q}$ that solves (17). The initial allocation of bond holdings is then determined from the excess demand function. The steady state of the model with endogenous discounting is therefore unique irrespective of the value of the elasticity of substitution between traded goods. Furthermore, this steady state does not necessarily feature zero bond holdings. However, the functional forms of β_1 and β_2 can always be calibrated such that the unique steady state features $B_1 = B_2 = 0$.

The following theorem about the stability of the steady state is proven in the appendix

Theorem 3 *Assume that the agents' discount factors are endogenous and strictly decreasing in the current utility level. Furthermore, assume that agents do not internalize the effects of their choices on their discount factors. Then any steady state is a saddle point irrespective of the sign of the slope of the excess demand function.*

If the discount factor is assumed to be strictly increasing in the utility level, the model dynamics are always explosive irrespective of the slope of the excess demand function.

Using a phase diagram, figures 5 and 6 illustrate the global dynamics for the model with endogenous discounting.[14] The dashed lines are the $B_{1,t} - B_{1,t-1} = 0$ and $\bar{q}_{t+1} - \bar{q}_t = 0$ locus, respectively.

The unique steady state is always saddle-path stable for both high and low values of the elasticity of substitution. However, if ε is low the path to the steady state given an initial wealth distribution is not unique. For initial bond holdings close enough to 0 the initial value of $\bar{q}$ determines the starting point on the stable manifold.

[14] The stable manifold is calculated using a reverse shooting algorithm. The endogenous discount factor is assumed to be of the form $\beta(U) = \frac{1}{1+\psi(U(c,l)-\tilde{U})}$, where the constant ψ is chosen such that $\beta = 0.99$ in the steady state. $\tilde{U}$ is a constant that ensures that $\psi > 0$.

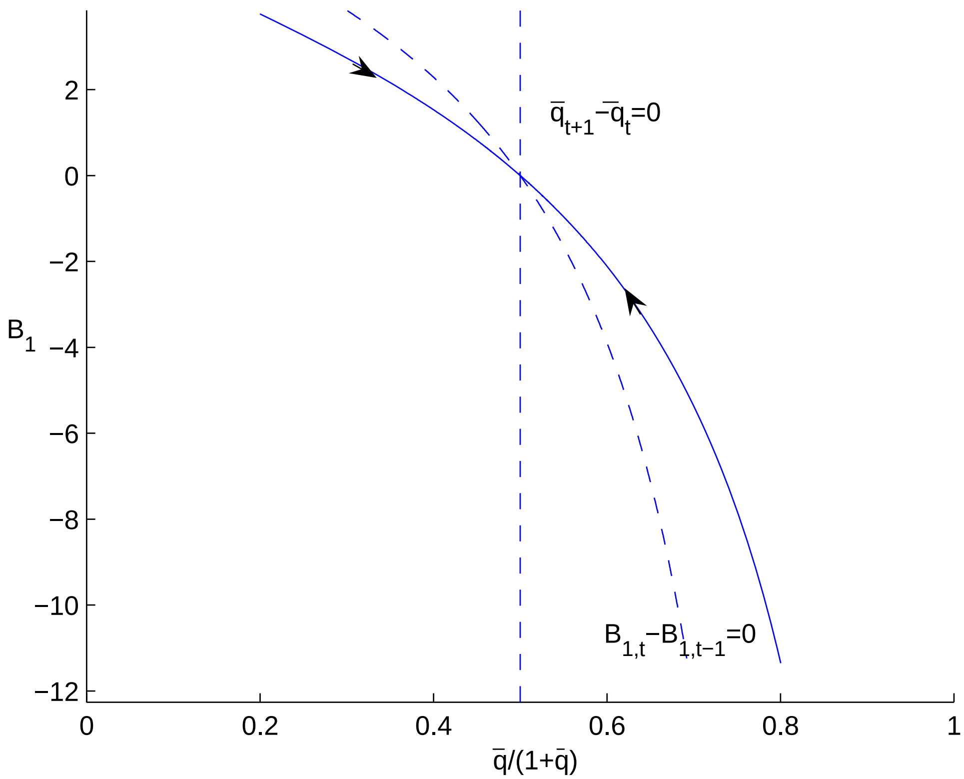

Figure 5: Stability of the steady state for $\varepsilon = 1$ with an endogenous discount factor.

If agents internalize the effects of their consumption and labor decisions on the discount factor, the risk sharing condition is given by

$$\begin{aligned} & \sum_{s^{t+1}|s^t} \left[\beta_1 \left(s^t\right) \frac{U_{c,1}\left(s^{t+1}\right) - \eta_1\left(s^{t+1}\right)\beta_{c,1}\left(s^{t+1}\right)}{U_{c,1}\left(s^t\right) - \eta_1\left(s^t\right)\beta_{c,1}\left(s^t\right)} \right] \pi\left(s^{t+1}|s^t\right) \\ = & \sum_{s^{t+1}|s^t} \left[\beta_2 \left(s^t\right) \frac{U_{c,2}\left(s^{t+1}\right) - \eta_2\left(s^{t+1}\right)\beta_{c,2}\left(s^{t+1}\right)}{U_{c,2}\left(s^t\right) - \eta_2\left(s^t\right)\beta_{c,2}\left(s^t\right)} \frac{\bar{q}\left(s^t\right)}{\bar{q}\left(s^{t+1}\right)} \frac{\Phi_1\left(\bar{q}\left(s^t\right)\right)}{\Phi_1\left(\bar{q}\left(s^{t+1}\right)\right)} \frac{\Phi_2\left(\bar{q}\left(s^{t+1}\right)\right)}{\Phi_2\left(\bar{q}\left(s^t\right)\right)} \right] \pi\left(s^{t+1}|s^t\right). \end{aligned} \tag{18}$$

η_i is the Lagrangian multiplier on the law of motion for the discount factor in country i and

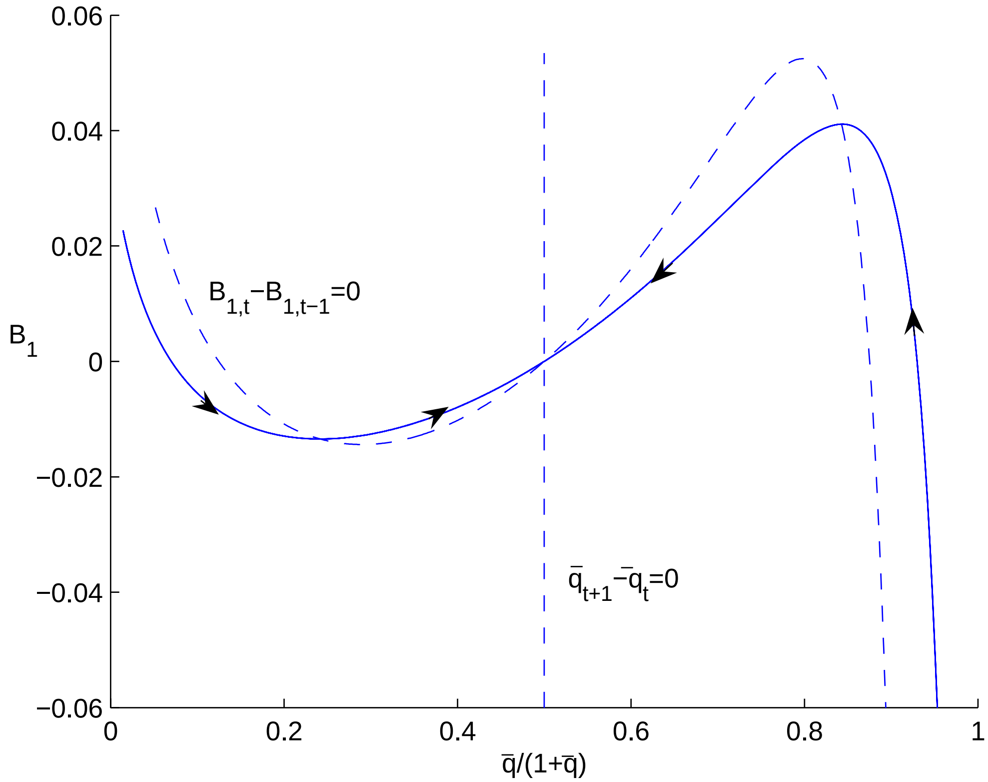

Figure 6: Stability of the steady state for $\varepsilon = 0.48$ with an endogenous discount factor.

it evolves according to

$$\eta_i\left(s^t\right) = \sum_{s^{t+1}|s^t} \left[-U_i\left(s^{t+1}\right) + \beta_i\left(s^{t+1}\right)\eta\left(s^{t+1}\right)\right]\pi\left(s^{t+1}|s^t\right). \tag{19}$$

Again, a steady state requires that the discount factors are equalized across countries, i.e., $\beta_1\left(U\left(s^t\right)\right) = \beta_2\left(U\left(s^t\right)\right)$. Therefore, the model with internalization always has a unique steady state.

A weaker version of theorem 3 applies if agents internalize the effects of their choices on the discount factor.

Theorem 4 *Assume that the agents' discount factors are endogenous and that agents internalize the effects of their choices on their discount factors. Irrespective of the sign of the* *slope of the excess demand function, any steady state is a saddle point if the discount factor does not react too strongly to changes in bond holdings.*

To the extent that the model with endogenous discounting is supposed to be close to the original model, the discount factor should not change excessively as the utility level deviates from its steady state level. Note, that the (in-)stability of the steady state is not at all related to the slope of the excess demand function, but merely to the parameterization of the endogenous discount factor itself.

3.5 Discussion and intuition

Table 2 summarizes the above results:

Model	portfolio cost	debt elastic interest rate	endog. dcf. (no internalization)	endog. dcf. (internalization)
$\varepsilon > \bar{\varepsilon}$				
# steady states	1	1	1	1
dynamics	(saddle) stable	(saddle) stable	(saddle) stable	(saddle) stable
$\varepsilon < \bar{\varepsilon}$				
# steady states	3	3	1	1
dynamics	#1, 3 (saddle) stable, #2 unstable	#1, 3 (saddle) stable, #2 unstable	(saddle) stable	(saddle) stable

Table 2: summary of results

Multiplicity of steady states If there are no international financial markets in the model as in section 2, international bond holdings are zero by definition. In the model with convex portfolio costs, the risk sharing equation imposes the restriction that bond holdings are zero in the steady state. Hence, any equilibrium of the financial autarchy model is a steady state in the bond economy with convex portfolio costs. The same result occurs if the interest rate is assumed to be debt elastic.

With an endogenous discount factor (both with and without internalization), however, the risk sharing condition implies a unique value of the relative price $\bar{q}$ in the steady state. Bond holdings are determined as the residual from the excess demand function. This difference manifests itself in the $\bar{q}_{t+1} - \bar{q}_t = 0$ locus being a horizontal line in the first two cases and a vertical line in the remaining two cases.

To illustrate the intuition behind these differences consider the simplified model with fixed labor supply. Furthermore, focus on the case $\varepsilon < \bar{\varepsilon}$. There are three equilibria in the model without international financial markets. These three equilibria are the candidate equilibria for the bond economies, with case I featuring $\frac{\bar{q}}{1+\bar{q}} << \frac{1}{2}$, case II $\frac{\bar{q}}{1+\bar{q}} = \frac{1}{2}$, and case III $\frac{\bar{q}}{1+\bar{q}} >> \frac{1}{2}$. In situation I, country 1's consumption is much higher than country 2's, and vice versa in case III.

Consider the candidate steady state I for the economy with endogenous discounting. Agents in country 1 consume more and they are substantially less patient, i.e., β_1 is smaller than β_2. Country 1 agents are willing to borrow resources at an interest rate of $\frac{1}{\beta_1}$ while country 2 agents only demand $\frac{1}{\beta_2}$. Hence, country 1 finds it optimal to borrow from country 2. With $B_1 < 0$, case I cannot be an equilibrium. For the same reason case III is not an equilibrium with the roles of the two countries reversed.

In the portfolio cost model the steady state interest rate is independent of the allocations and always equals $\frac{1}{\beta}$. Hence, there are no incentives to borrow and lend in any of the three candidate steady states. All candidate steady states with $B_1 = 0$ are steady states of the model with convex portfolio costs. Again, the same applies under a debt elastic interest rate.

Stability of steady states Theorems 1 and 2 show that under reasonable parameterizations of the convex portfolio cost and the debt elastic interest rate the stability of the dynamic system in the neighborhood of a steady state depends on the sign of the slope of the excess demand function in this steady state. Whenever the excess demand function is upward-sloping in a steady state, the steady state is locally unstable.

Under endogenous discounting (theorems 3 and 4) the stability of the system in the neighborhood of a steady state does not depend on the slope of the excess demand function in the steady state. The stability depends solely on the parameterization of the endogenous discount factor.

The logic behind the stability of the steady state in the model with endogenous discounting is closely related to the argument about its uniqueness. Assume that $\bar{q}$ is below its steady state value. This implies that consumption in country 1 (2) is above (below) its steady state value. Suppose, that the relative price is even lower in the next period, suggesting that the economy moves away from the steady state. This implies an increasing (decreasing) consumption profile in country 1 (2). In addition, the discount factor in country 1 (2) falls (rises). Hence, the price of the non-state-contingent bond falls in country 1 but rises in country 2. Obviously, the opposite movement of bond prices is inconsistent with the absence of arbitrage dictated by the risk sharing condition, equation (16). Hence, if $\bar{q}$ is below its steady state value at time t, $\bar{q}$ must rise in $t+1$ and the economy converges to its unique steady state.

Consider the case of a low elasticity of substitution in the bond economy with convex portfolio costs. All equilibria of the financial autarchy model are also steady states in this setup. However, only cases I and III are stable. The intuition behind the instability of case II is as follows. The price of bonds consists of two pieces: the intertemporal marginal rate of substitution and the derivative of the portfolio costs. Consider the neighborhood of any of the three steady states. If $\bar{q}$ is slightly below its steady state value, consumption in country 1 (2) is above (below) its corresponding steady state value. Stability of a certain steady state requires $\bar{q}$ to rise and c_1 to fall over time. As a result, the intertemporal marginal rate of substitution in country 1 (2) rises (falls), which leads to a divergence of bond prices. However, when $\bar{q}$ rises, bond holdings and, due to the convexity of the portfolio costs, the derivative of the portfolio costs fall. The effect on bond prices is negative in both countries. However, it is stronger in country 2 since portfolio costs are measured in terms of each country's good. This second effect operates towards a rise of the bond price in country 2 relative to country 1. If this effect is strong enough, bond prices can be prevented from drifting apart. In cases I and III, small changes in $\bar{q}$ imply relatively large changes in bond holdings and therefore relatively large changes in the derivative of the portfolio costs. In case II, however, the change in bond holdings is small owing to the fact that the excess demand function is fairly flat around this steady state. Hence, bond prices drift apart and case II is unstable.

Although conceptually different, these results are related to the concept of tâtonnement stability by Samuelson (1947). In the model without international financial markets and $\varepsilon < \bar{\varepsilon}$, the second equilibrium is locally totally unstable as relative prices diverge. The first and third equilibrium are locally stable since for an initial price vector that is sufficiently close to the equilibrium the dynamic trajectory causes relative prices to converge. As in the case of convex portfolio costs or a debt elastic interest rate, stability of a steady state is related to the slope of the excess demand function. However, it is not really clear what is the concept of "time" used in the tâtonnement analysis. In sharp contrast to the models presented in this paper, it cannot be real time as the economy is not in equilibrium along the tâtonnement path and thereby violates feasibility.

4 Technology shocks

4.1 Impulse response functions

This section studies the dynamic response to a technology shock in the model with an endogenous discount factor (without internalization) and with convex portfolio costs, respectively. To keep the discussion simple I do not discuss the cases of a debt elastic interest rate or the endogenous discount factor with internalization. Not surprisingly, though, the results in this section for the debt elastic interest rate are similar to the ones for convex portfolio costs. The case of endogenous discounting with internalization behaves close to the case without internalization.

At time 1 country 1 experiences an unexpected 1% rise in its technology. The shock follows an $AR(1)$ process with a persistence parameter of 0.9. Once the shock is realized, agents perfectly foresee the future path of the economy. The economy is assumed to be in a steady state prior to the shock.

High elasticity of substitution Consider first the case of a large value of the elasticity of substitution, i.e., $\varepsilon > \bar{\varepsilon}$. The steady state is unique and stable for both approaches. Figure 7 plots the impulse response functions in the two economies for different variables of interest. The impulse responses for the endogenous discount factor model are given by the solid lines, and are given by the dashed lines for the portfolio cost model. All variables are shown in levels rather than deviations from the steady state. In the example shown, the steady state values are $\frac{\bar{q}}{1-\bar{q}} = \frac{1}{2}$, $\frac{c_1}{c_2} = \frac{y_1}{y2} = 1$ and $B_1 = 0$. The impulse responses are strikingly similar for the two approaches. Only the dynamics of bond holdings differ, but apparently this has almost no impact on the remaining variables given that bond holdings are small.

The increase in country 1's productivity reduces the costs of production in country 1. Therefore, output in country 1 rises relative to country 2. So does consumption, however, its rise is smaller since country 2 borrows from country 1. The price of good 2 relative to good 1 increases, reflecting the relative abundance of good 1 in the world market. Despite the modelling differences, the impulse response functions look very much alike for the two models. Given the remaining parameters of the models, changes in ε do not substantially change the picture as long as $\varepsilon > \bar{\varepsilon}$.

Low elasticity of substitution However, once ε drops below $\bar{\varepsilon}$, the two models differ drastically owing to the potential multiplicity of steady states. Figures 8 and 9 show the impulse response functions in the two models for $\varepsilon = 0.48$.

In the model with endogenous discounting (figure 8) there are three possible impulse response functions for a 1% technology shock (I dashed, II solid and III dashed-dotted line). As the steady state is unique and globally stable, the economy converges back in each case to the old steady state with $\frac{\bar{q}}{1-\bar{q}} = 0.5$, $\frac{c_1}{c_2} = \frac{y_1}{y_2} = 1$ and $B = 0$.

In the first case (dashed line) the technology shock leads to a fall in the relative price, $\bar{q}$. As the elasticity of substitution between traded goods is low, production in country 2 rises relative to country 1 to generate more income in order to prevent a too strong fall in income. With country 1 increasing its consumption relative to country 2, but lowering its

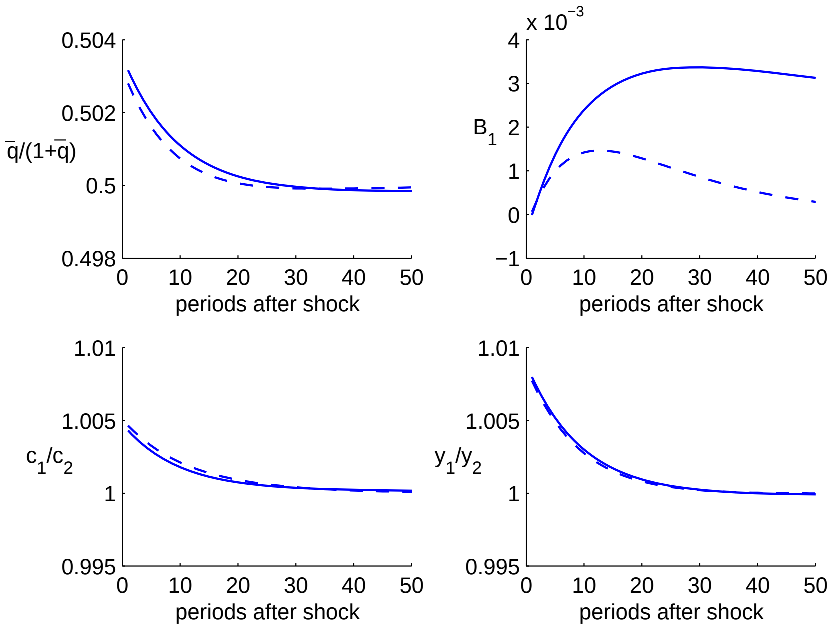

Figure 7: Impulse response functions for a 1% technology innovation in country 1 and $\varepsilon = 1$. All variables are plotted in levels. Solid line shows the response for endogenous discounting, dashed line for convex portfolio costs.

relative use of labor, β_1 falls relative to β_2. The resulting borrowing by country 1 shifts additional resources to country 1 and reinforces the effects. The second case (solid line) behaves qualitatively like the first case. However, the effects are considerably smaller.

However, in the third case (dashed-dotted line) the shock leads to a rise of the relative price of good 2, but to an increase of y_1 relative to y_2. As the price effect is not compensated by the increased production of good 1, consumption in country 1 declines relative to country 2. Furthermore, agents in country 2 become less patient which leads to increased borrowing by country 2 and this reinforces the effects.

If the model is closed by introducing convex portfolio costs, there are two locally stable steady states with $\frac{\bar{q}}{1+\bar{q}} = 0.12$ and 0.88. The third one, featuring $\frac{\bar{q}}{1+\bar{q}} = \frac{1}{2}$, is unstable. In figure 9, I assume that the economy is originally in the steady state with $\frac{\bar{q}}{1+\bar{q}} = 0.12$. In the first two cases (solid and dashed lines) the economy reverts to its original steady state. However, in the third one, the economy moves into a different regime: after an initial massive

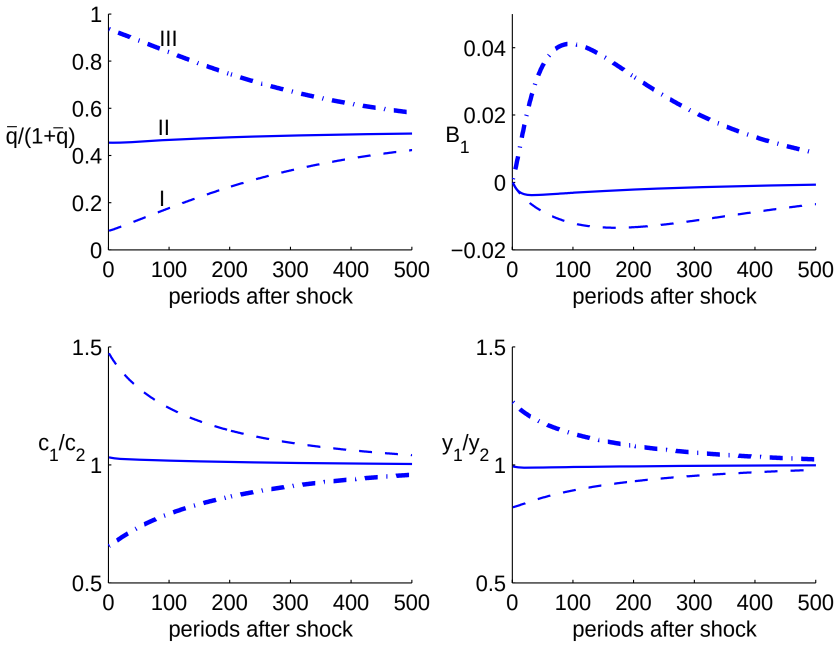

Figure 8: Impulse response functions for a 1% technology innovation in country 1. All variables are plotted in levels. Agents' discount factors are endogenous and $\varepsilon = 0.48$.

depreciation of $\bar{q}$ the economy converges to the other stable steady state with $\bar{q} = 0.88$. All three cases have in common that higher productivity in country 1 leads to an increase in the production of good 1 and a relative decline of its price. Consumption drops below its (new) steady state value. Given the low substitutability between traded goods, the decline in the price of good 1 is too large to be compensated by the additional income that is due to the rise in production of good 1. Even as agents in country 1 borrow from country 2, they cannot prevent their consumption from falling relative to consumption in country 2 starting from the original steady state.

The multiplicity of the impulse response functions is closely related to the multiplicity of steady states in the model of financial autarchy. Consider the model with endogenous discounting and $\varepsilon < \bar{\varepsilon}$. Although, there is a unique steady state in this model, the corresponding model without financial markets has three steady states. For simplicity, consider a permanent shock to technology. This situation is depicted in figure 10. Before the shock, the stable manifold is given by the solid line and the unique steady state features $\frac{\bar{q}}{1-\bar{q}} = 0.5$ and

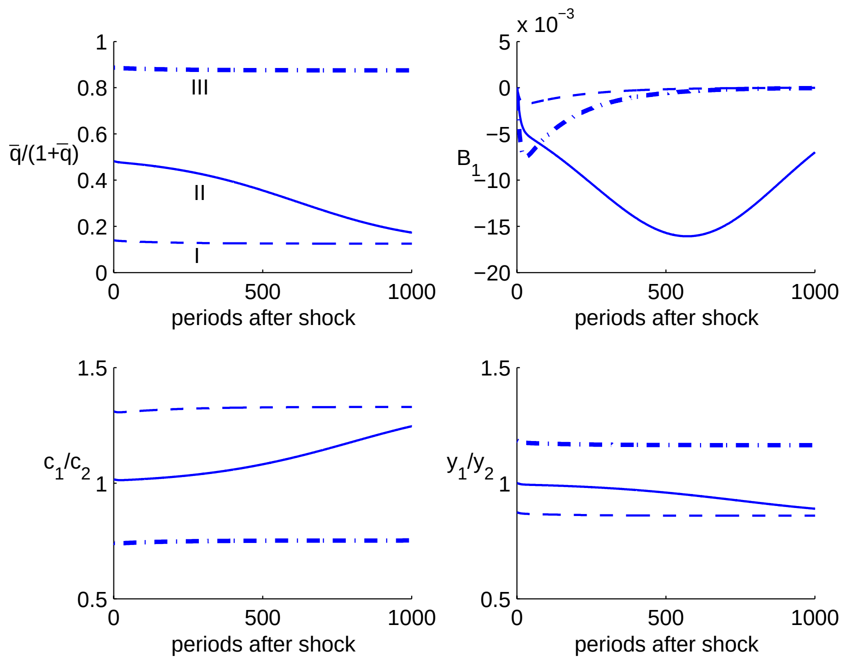

Figure 9: Impulse response functions for a 1% technology innovation in country 1. All variables are plotted in levels. Agents face convex portfolio costs and $\varepsilon = 0.48$.

$B_1 = 0$. In response to the shock, the manifold shifts upwards as indicated by the dashed line. In the new steady state – labeled by the small circle – $\frac{\bar{q}}{1-\bar{q}} = 0.5$ and $B_1 = 0.01$. How does the transition occur? As bond holdings are predetermined, the economy has to start in a point on the new manifold with $B_1 = 0$. There are three points that satisfy this condition, each marked by a little square and an arrow. The arrows indicate the movements of $\bar{q}$ and B_1 along the new manifold. Absent other restrictions on the adjustment path, each of the points I-III can be the starting point of the transition dynamics.

Note, that if the technology shock is too large, there is only one impulse response function. In figure 10 a large enough shock to technology can shift the manifold (and the new steady state) sufficiently up such that points I and II disappear. The unique starting point of the transition dynamics is then point III.

In the case of figures 8 and 9 technology shocks larger than 5%, would imply unique impulse response functions for this very reason. The unique response resembles case III, the dashed-dotted line. The economy with portfolio costs converges to a new steady state after the effects of the shock are foregone. While a 5% rise in technology seems large, this

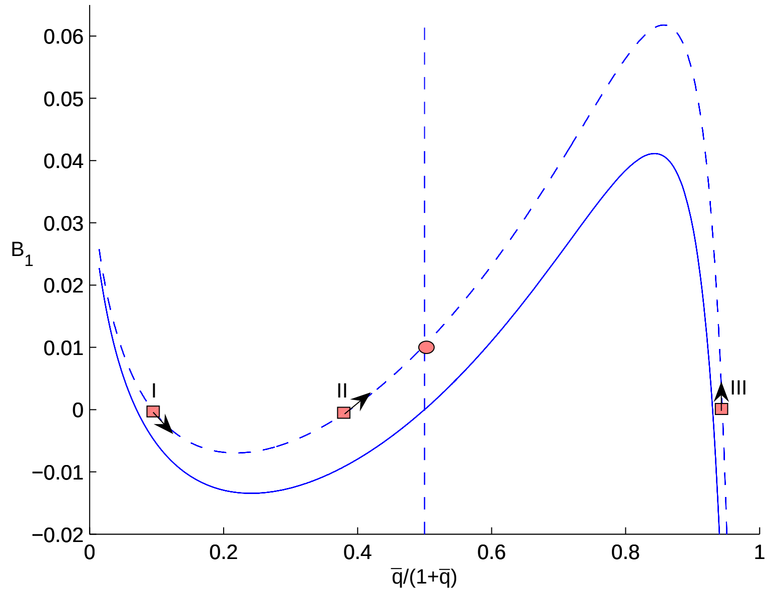

Figure 10: Shift of the stable manifold for a permanent technology shock.

threshold number can be very close to 0 for more persistent shocks or for ε closer to $\bar{\varepsilon}$.

In the above discussion I have presented examples for which the endogenous variables do not oscillate along the equilibrium path. However, if the eigenvalues that go along with the linear approximation of the dynamic system around a steady state are complex there can be other interesting equilibria. For example, it is straightforward to show that in the case of convex portfolio costs there can be parameterizations of the cost function such that for $\varepsilon < \bar{\varepsilon}$ the middle steady state is an unstable focus. In this case, there are most likely more than three impulse response functions. However, to the extent that these additional impulse responses exist, they would not be directly related to the fact that there are multiple steady states under financial autarchy. For this reason, I omit the analysis of such cases.[15]

[15] Only in the model with endogenous discounting without internalization can it be shown that the steady state cannot be a focus. In all other models oscillations can occur for appropriate parameterizations if $\varepsilon < \bar{\varepsilon}$.

4.2 Implications for applied modelling

One lesson from applied general equilibrium modelling is that it is practically impossible to find all the equilibria of a given model unless it can be proven that the equilibrium is unique. As illustrated in this paper, failing to detect equilibrium multiplicity and its associated nonlinearities renders an incomplete description of the model's dynamics.

In particular, equilibrium multiplicity implies that the model dynamics cannot be reliably captured by a log-linear approximation around a given steady state. In the endogenous discount factor model, for example, only the second impulse response function in figure 8 is detected if the model is solved by linearization. For larger shocks, the predicted impulse response function is merely an amplified version of the response under small shocks (case II). Obviously, for larger shocks such a prediction is far off the true impulse responses which resemble case III. This failure to recover the correct dynamics of the model is very different from the typical approximation error for large shocks.

In practice, researchers often calibrate their models to a specific steady state and log-linearize around it, thereby assuming away the issues addressed in this paper. Since different methods of closing open economy models can have diverse dynamic implications – in particular with respect to the stability of certain steady states – applied general equilibrium researchers may find it instructive to solve their models for various stationarity-inducing methods.

However, for the purpose of inducing stationarity into the model with incomplete international financial markets, the endogenous discount factor approach can be misleading. This approach forces the model to have a unique and stable steady state irrespective of the number of steady states that occur under other approaches that can be used to close open economy models. Important nonlinearities may remain undetected. As shown in this paper, convex portfolio costs or a debt elastic interest rate allow for multiple steady states whenever there are multiple solutions to the excess demand function under financial autarchy. These approaches at least do not rule out by construction the ability to detect steady state multiplicity. If the excess demand function is upward-sloping in a steady state – which proves the existence of multiple steady states – these two models turn out to be unstable in the neighborhood of this steady state

5 Conclusions

This paper analyzes different approaches that resolve the stationarity problem in models with incomplete asset markets. If stationarity is induced by an endogenous discount factor, there is always a unique saddle stable steady state. However, despite the uniqueness of the steady state, the equilibrium may not be unique away from the steady state. If the analogous model without international financial markets has multiple equilibria, there are multiple paths that lead the economy back to its steady state in response to a technology shock.

If stationarity is induced by a convex portfolio cost or a debt elastic interest rate, the number of steady states coincides with the number of equilibria in the analogous model without international financial markets. If the excess demand function in the financial autarchy model has multiple zeros not all steady states are saddle stable: a steady state in which the excess demand function is upward sloping is typically unstable. Similar to the model with endogenous discounting, there can be multiple impulse responses to a technology shock. However, in the case of convex portfolio costs or a debt elastic interest rate the economy may or may not converge back to its former steady state but to one of the other steady states of the model.

In the present paper, the differences across stationarity inducing methods hinge on the value of the elasticity of substitution between traded goods, ε, as the elasticity governs the multiplicity of equilibria in the financial autarchy model. Although the critical value $\bar{\varepsilon}$ for which these differences become an issue depends on the specific model, the relevance of the findings in this paper goes further.

In applied macroeconomic studies it is common to chose values between 1 and 1.5 (see e.g. Backus and Smith (1995), Chari, Kehoe and McGrattan (2003), and Heathcote and Perri (2002)). Recently, however, various authors have argued in favor of low values of the trade elasticity. In straightforward extensions of the model presented in this paper Heathcote and Perri (2002) and Collard and Dellas (2004) argue that they improve their models' performance in accounting for features of the international business cycle like the volatility of the terms of trade when moving to elasticities in the range of 0.5. In Benigno and Thoenissen (2006) the model with an elasticity of 0.5 outperforms their baseline calibration with a value of 2.

Other researchers have refrained from assuming such low elasticities directly. Instead, they augment the standard model by distribution costs in nontraded goods to obtain a low implied elasticity of substitution. Representative work is by Burstein, Neves and Rebelo (2003), Burstein, Eichenbaum and Rebelo (2003), Corsetti and Dedola (2005), and Corsetti, Dedola and Leduc (2005). Corsetti, Dedola and Leduc build a two country general equilibrium model with distribution costs in nontraded goods. Only for a low implied value of the elasticity of substitution does their linearized model successfully address two important puzzles in international economics: the high volatility of the real exchange rate relative to fundamentals and the observed negative correlation between the real exchange rate and relative consumption (Backus and Smith (1993)). Corsetti and Dedola (2005) show that this framework admits multiple equilibria in the absence of international borrowing and lending even if the direct elasticity of substitution between traded goods is larger than 1.

As Kollmann (2005) shows, a low elasticity of substitution may also be responsible for the apparent home bias in equity holdings. Rabanal and Tuesta (2005) estimate a DSGE model with sticky prices using a Bayesian approach. Their median estimates for the elasticity of substitution range from 0.01 to 0.91 for different specifications of their model. Lubik and Schorfheide (2005) estimate the elasticity of substitution to be around 0.4.

This list of papers indicates that low values of trade elasticities at the aggregate level may be behind many of the puzzles in international macroeconomics. However, to the extent that the assumption of low (implied) trade elasticities gives rise to multiple equilibria and important nonlinearities, the choice of how to induce stationarity in a model with incomplete international asset markets is no longer innocuous.

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A A formal analysis of existence and multiplicity

This appendix shows the existence of the equilibrium in the model with financial autarchy. In addition, I discuss conditions under which multiple equilibria arise. The appendix ends by discussing different parameterizations for which multiple equilibria occur.

A.1 A more general setup

I assume that c_i is given by the linear homogenous aggregator $c_i = H_i(c_{i1}, c_{i2})$.[16] H_i is assumed to satisfy

$$H_{ij} = \frac{\partial H_i}{\partial c_{ij}} > 0,\ H_{iii} = \frac{\partial^2 H_i}{\partial c_{ii}^2} < 0,\ H_{iji} = \frac{\partial^2 H_i}{\partial c_{ij} \partial c_{ii}} > 0,$$

[16] An aggregator that satisfies the restrictions imposed on H_i is given by the straightforward extension of the CES aggregator which has been suggested by Dotsey and King (2005):

$$\frac{\alpha_{i1}}{(1+\eta)\rho}\left[\frac{(1+\eta)}{\alpha_{i1}}\left(\frac{c_{i1}}{c_i}\right)-\eta\right]^{\rho}+\frac{\alpha_{i2}}{(1+\eta)\rho}\left[\frac{(1+\eta)}{\alpha_{i2}}\left(\frac{c_{i2}}{c_i}\right)-\eta\right]^{\rho}=\frac{1}{(1+\eta)\rho}.$$

This aggregator allows for the elasticity of substitution to be non-constant.

and the Inada conditions

$$\begin{aligned}\lim_{c_{i1}\to 0} H_{i1}\left(c_{i1},c_{i2}\right) &= \lim_{c_{i2}\to 0} H_{i2}\left(c_{i1},c_{i2}\right)=\infty, \\ \lim_{c_{i1}\to \infty} H_{i1}\left(c_{i1},c_{i2}\right) &= \lim_{c_{i2}\to \infty} H_{i2}\left(c_{i1},c_{i2}\right)=0.\end{aligned}$$

The strictly concave period utility function $U\left(c,l\right)$ is assumed to satisfy the following conditions

$$U_c>0, U_l<0 \text{ and } U_{cc}<0, U_{ll}<0, U_{cl}\leq 0.$$

The optimal choices for c_{i1} and c_{i2} can be found from the following optimization program:

$$\begin{aligned}&\max_{c_{i1},c_{i2}} H_i\left(c_{i1},c_{i2}\right)\\ &s.t.\\ &\bar{P}_1 c_{i1}+\bar{P}_2 c_{i2}=\bar{P}_i w_i l_i+dW_i.\end{aligned}$$

Linear homogeneity of H_i implies that the first order conditions can be written as

$$\begin{aligned}H_{i1}\left(\frac{c_{i1}}{c_{i2}},1\right) &= \lambda_i \bar{P}_1, \\ H_{i2}\left(\frac{c_{i1}}{c_{i2}},1\right) &= \lambda_i \bar{P}_2.\end{aligned}$$

Given the properties of H_{i1} and H_{i2}, this can be summarized as

$$\frac{c_{i1}}{c_{i2}}=\tilde{H}_i\left(\frac{1}{\bar{q}}\right),$$

where $\bar{q}$ is the relative price of good 2 to good 1, $\frac{\bar{P}_2}{\bar{P}_1}$. The aggregator H_i is said to allow for home bias in goods if $\tilde{H}_1\left(\frac{1}{\bar{q}}\right)>\tilde{H}_2\left(\frac{1}{\bar{q}}\right)$ for all $\bar{q}$. Let P_i denote the price of the final consumption basket, which turns out to be given by

$$\begin{aligned}\Phi_1\left(\bar{q}\right) &\equiv \frac{\bar{P}_1}{P_1}=\frac{\bar{q}H_1\left(\tilde{H}_1\left(\frac{1}{\bar{q}}\right),1\right)}{\bar{q}\left[\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]} \text{ with } \Phi_1'\left(\bar{q}\right)<0, \\ \Phi_2\left(\bar{q}\right) &\equiv \frac{\bar{P}_2}{P_2}=\frac{\bar{q}H_2\left(\tilde{H}_2\left(\frac{1}{\bar{q}}\right),1\right)}{\left[\tilde{H}_2\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]} \text{ with } \Phi_2'\left(\bar{q}\right)>0.\end{aligned}$$

I normalize the price of the consumption basket in country 1 to unity, $P_1 = 1$, and denote P_2 by q, which is simply the real exchange rate. Obviously, q and $\bar{q}$ are related as follows

$$q = \bar{q}\frac{\Phi_1(\bar{q})}{\Phi_2(\bar{q})}.$$

Using the budget constraint and $\frac{c_{i1}}{c_{i2}} = \tilde{H}_i\left(\frac{1}{\bar{q}}\right)$, the demand functions for good 2 are

$$\begin{aligned} c_{12} &= \frac{1}{\left(\tilde{H}_1\left(\frac{1}{\bar{q}}\right) + \bar{q}\right)}\left[w_1 l_1 + \frac{1}{\Phi_1(\bar{q})}dW_1\right], \\ c_{22} &= \frac{1}{\left(\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}} + 1\right)}\left[w_2 l_2 + \frac{1}{\bar{q}\Phi_1(\bar{q})}dW_2\right], \end{aligned}$$

$$z_2 = \frac{w_1 l_1 + \frac{1}{\Phi_1(\bar{q})}dW_1}{\left(\tilde{H}_1\left(\frac{1}{\bar{q}}\right) + \bar{q}\right)} + \frac{w_2 l_2 + \frac{1}{\bar{q}\Phi_1(\bar{q})}dW_2}{\left(\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}} + 1\right)} - A_2 l_2.$$

Similar expressions can be derived for the demand of good 1 and an expression for $c_i = H_i(c_{i1}, c_{i2})$ can be provided:

$$\begin{aligned} c_1 &= \Phi_1(\bar{q}) w_1 l_1 + dW_1, && (20) \\ c_2 &= \Phi_2(\bar{q}) w_2 l_2 + \frac{\Phi_2(\bar{q})}{\Phi_1(\bar{q})\bar{q}}dW_2. && (21) \end{aligned}$$

Combining c_i with the intratemporal Euler equation for consumption-leisure choices,

$$\frac{U_l(c_i, l_i)}{U_c(c_i, l_i)} = -\Phi_i(\bar{q}) w_i, \tag{22}$$

allows to express l_i and c_i as functions of w_1, w_2 and $\bar{q}$ (and dW_i). l_i and c_i are functions of the price vectors, if each price vector maps into a unique allocation. Keeping prices fixed, (22) and the assumptions on U imply

$$\frac{dc}{dl} = -\frac{[\Phi_i(\bar{q}) A_i U_{lc} + U_{ll}]}{[U_{lc} + \Phi_i(\bar{q}) A_i U_{cc}]} < 0$$

for every price vector. Since the relationship between consumption and labor is strictly positive in equations (20) and (21), the mapping from prices into quantities c_i and l_i, $i = 1, 2$ is unique.

A.2 Existence of the equilibrium

In the following I closely follow Kehoe (1980, 1985 and 1991). Let L_i be the time endowment of agents in country i. Kehoe defines the excess demand for a good as the difference between the demand for a specific good and the aggregate endowment with this good. The economy's endowment with goods 1 and 2 is zero, while the leisure endowments are L_1 and L_2. I denote the excess demand for goods 1 and 2 by $d_{c,i} = c_{1i} + c_{2i}$, $i = 1, 2$. The excess demand for leisure is given by $d_{l,i} = (L_i - l_i) - L_i = -l_i$. Furthermore, let $d = (d_{c,1}, d_{c,2}, d_{l,1}, d_{l,2})$.

The production side of the economy is given by a 4×6 activity analysis matrix A. Each column of A represents an activity, which transforms inputs taken from the vector of aggregate initial endowments or from the outputs of other activities into outputs, which are either consumed or further used as inputs. Positive entries in an activity denote quantities of outputs produced by the activity; negative entries denote quantities of inputs consumed. Aggregate production is denoted by Ay', where y is a 6×1 vector of nonnegative activity levels:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & A_1 & 0 \\ 0 & -1 & 0 & 0 & 0 & A_2 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix},$$

with the first 4 columns of this matrix being free disposal activities.

An equilibrium of this economy is a price vector $\hat{p} = \left(\bar{P}_1, \bar{P}_2, w_1, w_2\right)$ that satisfies the following three properties: first $\hat{p}'A \leq 0$; second there exists a nonnegative vector of activity levels $\hat{y}$ such that $A\hat{y}' = d(\hat{p})$; and third $\bar{P}_1 = 1$. The first condition requires that there be no excess profits available. The second one requires that supply equals demand. The third one is simply a price normalization.

Existence of an equilibrium follows directly from Theorem 1 in Kehoe (1985). Notice, how Kehoe's presentation of the problem can be reduced to the presentation in the main text. Let the activity vector be $y' = (0, 0, 0, 0, l_1, l_2)$. Then

$$d(p) - Ay' = \begin{pmatrix} c_{11} + c_{21} \\ c_{12} + c_{22} \\ -l_1 \\ -l_2 \end{pmatrix} - \begin{pmatrix} A_1 l_1 \\ A_2 l_2 \\ -l_1 \\ -l_2 \end{pmatrix}.$$

Using Walras' Law, an equilibrium is a price vector such that $z_2(p) = c_{12}(p) + c_{22}(p) - A_2 l_2(p) = 0$. As profit maximization implies $w_i = A_i$, all that needs to be found is the relative price $\bar{q} = \bar{P}_2$.

A.3 Multiplicity of equilibria

If all the equilibria of an economy are locally unique, the economy is referred to as regular. Kehoe (1980) provides general conditions that ensure regularity. In addition, he shows that the number of equilibria in a production economy is odd. Let the index of an equilibrium $\hat{p}$ be defined as

$$index\,(\hat{p}) = sgn\left(\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix}\right).$$

$\bar{J}$ is formed by deleting the first row and the first column from $Dd\,(\hat{p})$, the matrix of derivatives of the excess demand functions with respect to each price, if good 1 is the numeraire. $\bar{B}$ is formed by deleting the first row from $B\,(\hat{p})$, where $B\,(\hat{p})$ is the submatrix of A whose columns are all those activities that earn zero profits at $\hat{p}$.

Theorem 2 in Kehoe (1985) states that the sum of the indices across all equilibria equals $+1$, i.e., $\sum_j index\,(\hat{p}^j) = +1$. Hence the number of equilibria in a regular economy is finite and odd. If it cannot be proven that there is a unique equilibrium, this is usually all that can be said about the number of equilibria. Although there has been substantial progress in the development of fixed point algorithms, it is in general impossible to find all the equilibria of an economy if there is no guarantee that there is only one.

What can be said about the equilibria in the model presented in this paper? Using Kehoe's approach,

$$\bar{J} = \begin{bmatrix} \partial d_{c,2}/\partial\bar{q} & 0 & 0 \\ \partial d_{l,1}/\partial\bar{q} & 0 & 0 \\ \partial d_{l,2}/\partial\bar{q} & 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & A_2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

since $w_i = A_i$. It turns out that

$$\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix} = -\frac{\partial d_{c,2}}{\partial\bar{q}} + A_2\frac{\partial d_{l,1}}{\partial\bar{q}} = -\frac{\partial z_2}{\partial\bar{q}}.$$

If the excess demand function as defined in the main text, z_2, is downward sloping in each equilibrium, the equilibrium is unique. However, if an equilibrium with $\frac{\partial z_2}{\partial\bar{q}} > 0$ is found then there must be at least two more equilibria.

In order to find calibrated economies with multiple equilibria for the model presented in this paper, I search for parameters such that the slope of the excess demand function is zero in equilibrium. Totally differentiating equation (9) delivers

$$\begin{aligned}\frac{\partial z_2}{\partial \bar{q}} &= \frac{A_1 l_1}{\bar{q}\left[\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}\left[\frac{\tilde{H}_1'\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}-\bar{q}}{\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}}+\frac{\partial l_1}{\partial \bar{q}}\frac{\bar{q}}{l_1}\right] \\ &\quad +\frac{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)A_2 l_2}{\bar{q}\left[\tilde{H}_2\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}\left[\frac{\frac{\tilde{H}_2'\left(\frac{1}{\bar{q}}\right)}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)}+\bar{q}}{\left[\tilde{H}_2\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}-\frac{\partial l_2}{\partial \bar{q}}\frac{\bar{q}}{l_2}\right].\end{aligned}$$

Let the (possibly variable) elasticity of substitution between traded goods in country i be denoted by $\varepsilon_i(\bar{q})$. Hence, equation (3) implies

$$\varepsilon_i(\bar{q})=\frac{\partial\left(\frac{c_{i1}}{c_{i2}}\right)(\bar{q})}{\partial(\bar{q})\frac{c_{i1}}{c_{i2}}}=-\frac{\tilde{H}_i'\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}}{\tilde{H}_i\left(\frac{1}{\bar{q}}\right)}. \tag{23}$$

The slope of $\tilde{z}_2$ in equilibrium ($\tilde{z}_2(\bar{q}^*)=0$) can then be expressed as

$$\frac{\partial z_2}{\partial \bar{q}}|_{\bar{q}=\bar{q}^*}=-\frac{A_1 l_1}{\bar{q}\left[\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}\left[\frac{\varepsilon_1(\bar{q})\tilde{H}_1\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1}{\tilde{H}_1\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1}+\frac{\varepsilon_2(\bar{q})-1}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1}-\frac{\partial l_1}{\partial \bar{q}}\frac{\bar{q}}{l_1}+\frac{\partial l_2}{\partial \bar{q}}\frac{\bar{q}}{l_2}\right].$$

$\frac{\partial l_i}{\partial \bar{q}}\frac{\bar{q}}{l_i}$ is the general equilibrium elasticity of labor with respect to the relative price $\bar{q}$.

To find an expression for $\frac{\partial l_i}{\partial \bar{q}}\frac{\bar{q}}{l_i}$ notice that equations (4) and (5) together with (3) and the consumption aggregators, $H_i(c_{i1},c_{i2})$, imply $c_i=\Phi_i(\bar{q})A_i l_i$. Total differentiation of $c_i=\Phi_i(\bar{q})A_i l_i$ and (8) yields

$$\frac{\partial l_i}{\partial \bar{q}}\frac{\bar{q}}{l_i}=-\left[\frac{-\eta_i\left[\frac{U_{lc,i}^2}{U_{cc,i}}\frac{U_{c,i}}{U_{l,i}}-U_{ll,i}\frac{U_{c,i}}{U_{l,i}}\right]+\left[U_{lc,i}-\frac{U_{l,i}}{U_{c,i}}U_{cc,i}\right]}{\left[U_{lc,i}-U_{ll,i}\frac{U_{c,i}}{U_{l,i}}\right]+\left[U_{lc,i}-\frac{U_{l,i}}{U_{c,i}}U_{cc,i}\right]}\right]\frac{\Phi_i'(\bar{q})\bar{q}}{\Phi_i(\bar{q})},$$

where η_i is the Frisch labor supply elasticity.[17] From the definition of $\Phi_i(\bar{q})$,

[17] The Frisch (or constant marginal utility of wealth) labor supply elasticity is defined as

$$\eta=\frac{dl}{dw}\frac{w}{l}|_{\lambda}=\frac{U_l}{lU_{ll}-\frac{lU_{lc}^2}{U_{cc}}}.$$

$$\bar{q}\frac{\Phi_1'(\bar{q})}{\Phi_1(\bar{q})} = -\frac{1}{\tilde{H}_1\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1}, \tag{24}$$

$$\bar{q}\frac{\Phi_2'(\bar{q})}{\Phi_2(\bar{q})} = \frac{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1}. \tag{25}$$

To gain additional insights, I define the share of imports in GDP of country 1 to be $1-\alpha_1 = \frac{c_{12}}{\frac{1}{\bar{q}}A_1 l_1}$. Since trade is balanced in the model with financial autarchy, $dW_i = 0$, it is $\frac{c_{11}}{A_1 l_1} = \alpha_1$. From the definition of $\tilde{H}_1$ follows $\tilde{H}_1\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}} = \frac{\alpha_1}{1-\alpha_1}$. Analogously, it is $\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}} = \frac{1-\alpha_2}{\alpha_2}$. Let the relative country size, $\frac{A_1 l_1}{A_2 l_2}$, be denoted by θ. The relative price is then given by $\bar{q} = \theta\frac{1-\alpha_1}{1-\alpha_2}$. With these definitions at hand

$$\frac{\partial z_2}{\partial \bar{q}}|_{\bar{q}=\bar{q}^*} = -\frac{A_1 l_1}{\bar{q}\left[\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}\left[\varepsilon_1(\bar{q})\alpha_1 + \varepsilon_2(\bar{q})\alpha_2 + (1-\alpha_1-\alpha_2) - \frac{\partial l_1}{\partial \bar{q}}\frac{\bar{q}}{l_1} + \frac{\partial l_2}{\partial \bar{q}}\frac{\bar{q}}{l_2}\right],$$

and $\frac{\Phi_1'(\bar{q})\bar{q}}{\Phi_1(\bar{q})} = -(1-\alpha_1)$, $\frac{\Phi_2'(\bar{q})\bar{q}}{\Phi_2(\bar{q})} = 1-\alpha_2$.

I consider the following three classes of utility functions:

1. Additive separable in consumption and leisure (labor)

$$U(c,l) = v_1(c) - v_2(l),$$

and

$$U(c,l) = v_1(c) - v_2(1-l).$$

Since $U_{cl} = 0$,

$$\frac{\partial l_i}{\partial \bar{q}}\frac{\bar{q}}{l_i} = \frac{1-\sigma}{\frac{1}{\eta}+\sigma}\frac{\Phi_i'(\bar{q})\bar{q}}{\Phi_i(\bar{q})},$$

where $\sigma = -\frac{U_{cc}c}{U_c}$, the relative risk aversion, and η is the Frisch labor supply elasticity.

2. Preferences without wealth effects

$$U(c,l) = v_1(c - v_2(l)),$$

and

$$\frac{\partial l_i}{\partial \bar{q}}\frac{\bar{q}}{l_i} = \eta\frac{\Phi_i'(\bar{q})\bar{q}}{\Phi_i(\bar{q})}.$$

3. Cobb-Douglas aggregator

$$U(c,l) = V\left(c^{\xi}(1-l)^{1-\xi}\right),$$

where $V(\cdot)$ is strictly monotone in its argument. In this case

$$\frac{\partial l_i}{\partial \bar{q}}\frac{\bar{q}}{l_i} = 0.$$

To obtain an idea, how likely it is to observe multiple equilibria in a calibrated economy, I proceed as follows. For the case of additive-separable preferences, assume that the two countries share the same constant values for the elasticity of substitution, the Frisch labor supply elasticity and the relative risk aversion. The critical value $\bar{\varepsilon}$ that separates the case of a unique equilibrium from the case of multiple equilibria is then determined by setting $\frac{\partial z_2}{\partial \bar{q}}|_{\bar{q}=\bar{q}^*} = 0$,

$$\bar{\varepsilon} = \frac{\alpha_1+\alpha_2-1}{\alpha_1+\alpha_2} + \frac{\frac{1-\sigma}{\frac{1}{\eta}+\sigma}}{\alpha_1+\alpha_2}\left[\alpha_1+\alpha_2-2\right].$$

α_1 and α_2 are less than unity by definition. Furthermore, with home bias, i.e. $\tilde{H}_1\left(\frac{1}{\bar{q}}\right) > \tilde{H}_2\left(\frac{1}{\bar{q}}\right)$ for all $\bar{q}$, $\alpha_1+\alpha_2-1>0$. It is easy to see that with a completely inelastic labor supply ($\eta=0$), $\bar{\varepsilon} = \frac{\alpha_1+\alpha_2-1}{\alpha_1+\alpha_2}$. If $\eta>0$ and $\sigma>1$, $\bar{\varepsilon} > \frac{\alpha_1+\alpha_2-1}{\alpha_1+\alpha_2}$ and less or equal for $\sigma \leq 1$. $\bar{\varepsilon}$ is increasing in σ and η. An upper bound for $\bar{\varepsilon}$ is given by $\lim_{\sigma\to\infty,\eta\to\infty}\bar{\varepsilon} = \frac{1}{\alpha_1+\alpha_2} < 1$ if $\alpha_1+\alpha_2>1$.

Figure 11 plots $\bar{\varepsilon}$ as a function of η and σ for $\alpha_1=\alpha_2=0.8$. For the most appropriate choices of σ and η, $\bar{\varepsilon}$ lies around 0.5. For example, $\bar{\varepsilon}\approx 0.48$ for $\sigma=3$ and $\eta=\frac{1}{2}$.

Figure 12 shows $\bar{\varepsilon}$ as a function α_1 and α_2 for $\sigma=3$ and $\eta=\frac{1}{2}$. Since $\sigma>1$, $\bar{\varepsilon}$ is increasing in α_1 and α_2 (or decreasing in the import share in GDP).

B Stability and the slope of the excess demand function

This appendix proves theorems 1-4 in the main text. Section $B.1$ provides the algebraic derivations needed for these proofs. Section $B.2$ provides the proofs themselves. All derivations and proofs are based on the more general model described in appendix A.

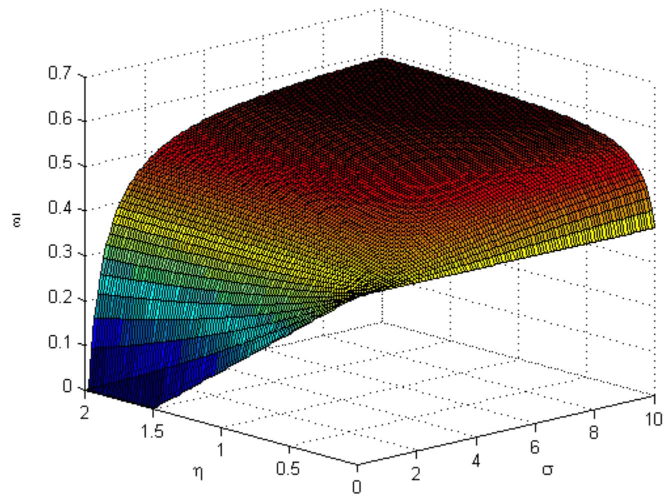

Figure 11: $\bar{\varepsilon}$ as a function of risk aversion, σ, and the Frisch labor supply elasticity, η, for given $\alpha_1 = \alpha_2 = 0.8$.

B.1 Preliminaries

In this section I derive a log-linear approximation of the model's dynamics solely in terms of the relative price $\bar{q}$ and bond holdings B_1. I assume that the model is parameterized such that in any steady state bond holdings are zero.

Consumption and labor With constant technology, equations (20) – (22) imply

$$\hat{l}_{1,t} = \omega_{q,1} \frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})} \bar{q}_t - \omega_{b,1} \frac{1}{c_1} \left[b_{1,t-1} - \beta b_{1,t}\right], \tag{26}$$

$$\hat{l}_{2,t} = \omega_{q,2} \frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})} \bar{q}_t + \omega_{b,2} \frac{\Phi_2(\bar{q})}{\bar{q}\Phi_1(\bar{q})} \frac{1}{c_2} \left[b_{1,t-1} - \beta b_{1,t}\right], \tag{27}$$

$$\hat{c}_{1,t} = \left[1 + \omega_{q,1}\right] \frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})} \bar{q}_t + \left[1 - \omega_{b,1}\right] \frac{1}{c_1} \left[b_{1,t-1} - \beta b_{1,t}\right], \tag{28}$$

$$\hat{c}_{2,t} = \left[1 + \omega_{q,2}\right] \frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})} \bar{q}_t - \left[1 - \omega_{b,2}\right] \frac{\Phi_2(\bar{q})}{\bar{q}\Phi_1(\bar{q})} \frac{1}{c_2} \left[b_{1,t-1} - \beta b_{1,t}\right], \tag{29}$$

where $\bar{q}_t$ denotes the percentage deviation of the relative price $\bar{q}$ from its steady state value at time t. $b_{1,t}$ is the absolute deviation of country 1's bond holdings. If $b_{1,t} > 0$, country 1

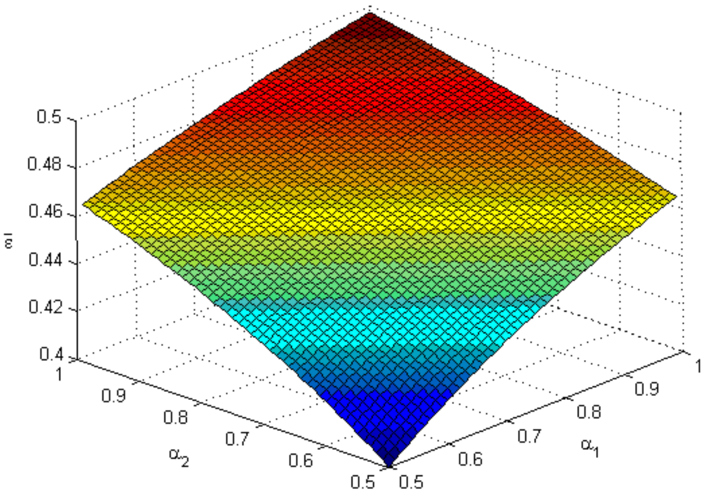

Figure 12: $\bar{\varepsilon}$ as a function of α_1 and α_2 for $\sigma = 3$ and $\eta = \frac{1}{2}$.

is lending to country 2 in period t. $\omega_{b,i}$ and $w_{q,i}$ are given by

$$\begin{aligned}
\omega_{b,i} &= \frac{\left[U_{lc,i} - \frac{U_{l,i}}{U_{c,i}} U_{cc,i}\right]}{\left[U_{lc,i} - \frac{U_{l,i}}{U_{c,i}} U_{cc,i}\right] + \left[U_{cl,i} - \frac{U_{c,i}}{U_{l,i}} U_{ll,i}\right]}, \\
\tau_i &= \frac{-U_{c,i} \frac{1}{l_i}}{\left[U_{lc,i} - \frac{U_{l,i}}{U_{c,i}} U_{cc,i}\right] + \left[U_{cl,i} - \frac{U_{c,i}}{U_{l,i}} U_{ll,i}\right]}, \\
\omega_{q,i} &= -\omega_{b,i} + \tau_i.
\end{aligned}$$

With the assumptions on the utility function $U(c,l)$, which are satisfied by almost all utility functions that are commonly used in macroeconomics, one obtains $0 < \omega_{b,i} < 1$ and $\tau_i > 0$.

Excess demand function Using equations (26) − (29) the log-linear approximation of the excess demand function in equilibrium, $z_2(\bar{q}, dW_1) = 0$, can be written as

$$\frac{\partial z_2}{\partial \bar{q}} \bar{q} \bar{q}_t + \frac{\partial z_2}{\partial dW_1} \left[b_{1,t-1} - \beta b_{1,t}\right] = 0 \tag{30}$$

with

$$
\begin{aligned}
\frac{\partial z_2}{\partial \bar{q}}\bar{q} &= \frac{A_1 l_1}{\left[\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}\left[\frac{\tilde{H}_1'\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}-\bar{q}}{\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}}+\frac{\partial l_1}{\partial \bar{q}}\frac{\bar{q}}{l_1}\right]+\frac{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)A_2 l_2}{\bar{q}\left[\tilde{H}_2\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}\left[\frac{\frac{\tilde{H}_2'\left(\frac{1}{\bar{q}}\right)}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)}+\bar{q}}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)+\bar{q}}-\frac{\partial l_2}{\partial \bar{q}}\frac{\bar{q}}{l_2}\right] \\
&= c_{12}\left\{\left[\varepsilon_1(\bar{q})\,\tilde{H}_1\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1+\omega_{q,1}\right]\frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})}+\left[\frac{1-\varepsilon_2(\bar{q})}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}}-\omega_{q,2}\right]\frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})}\right\}, \\
\frac{\partial z_2}{\partial dW_1} &= \frac{A_1 l_1}{\left[\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}\frac{\partial l_1}{\partial dW_1}\frac{1}{l_1}+\frac{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}A_2 l_2}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1}\frac{\partial l_2}{\partial dW_2}\frac{1}{l_2}+\frac{\frac{1}{\Phi_1(\bar{q})}}{\left[\tilde{H}_1\left(\frac{1}{\bar{q}}\right)+\bar{q}\right]}-\frac{\frac{1}{q\Phi_2(\bar{q})}}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}+1} \\
&= -\frac{1}{\bar{q}\Phi_1(\bar{q})}\left\{1+\left[1-\omega_{b,1}\right]\frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})}-\left[1-\omega_{b,2}\right]\frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})}\right\}.
\end{aligned}
$$

In simplifying the expressions for $\frac{\partial z_2}{\partial \bar{q}}\bar{q}$ and $\frac{\partial z_2}{\partial dW_1}$, I use the definitions of $\frac{\Phi_i'(\bar{q})\bar{q}}{\Phi_i(\bar{q})}$, $i = 1, 2$ (equations (24) and (25)) and the country demand functions for good 2. Furthermore, $\varepsilon_i(\bar{q})$ denotes the elasticity of substitution between the two traded goods in country i as defined in equation (23).

B.2 Linearized models

Convex portfolio costs Using equations (26)−(29) and the log-linearized risk sharing condition, that is derived from equation (13), delivers the following system of linear difference equations

$$
\begin{pmatrix} \bar{q}_{t+1} \\ b_t \end{pmatrix} = \begin{pmatrix} 1+\frac{\frac{\partial z_2}{\partial \bar{q}}\bar{q}}{\frac{\partial z_2}{\partial dW_1}d_q-\frac{\partial z_2}{\partial \bar{q}}\bar{q}d_b}\check{\Gamma} & \frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial dW_1}d_q-\frac{\partial z_2}{\partial \bar{q}}\bar{q}d_b}\check{\Gamma} \\ \frac{1}{\beta}\frac{\frac{\partial z_2}{\partial \bar{q}}\bar{q}}{\frac{\partial z_2}{\partial dW_1}} & \frac{1}{\beta} \end{pmatrix}\begin{pmatrix} \bar{q}_t \\ b_{t-1} \end{pmatrix},
$$

where

$$\begin{aligned}
\check{\Gamma} &= \frac{\Gamma''(0)}{\beta^2 \Phi_1(\bar{q})} \left[1 + \frac{1}{\bar{q}}\right], \\
d_b &= -\left[\frac{U_{cc,1} c_1}{U_{c,1}} - \left(\frac{U_{cc,1} c_1}{U_{c,1}} + \frac{U_{cl,1} l_1}{U_{c,1}}\right) \omega_{b,1}\right] \frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})} \frac{1}{\bar{q}\Phi_1(\bar{q})\, c_{12}} \\
&\quad + \left[\frac{U_{cc,2} c_2}{U_{c,2}} - \left(\frac{U_{cc,2} c_2}{U_{c,2}} + \frac{U_{cl,2} l_2}{U_{c,2}}\right) \omega_{b,2}\right] \frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})} \frac{1}{\bar{q}\Phi_1(\bar{q})\, c_{12}}, \\
d_q &= -\bar{q}\Phi_1(\bar{q})\, c_{12} d_b + 1 + [1 - \omega_{b,1}] \frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})} - [1 - \omega_{b,2}] \frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})}.
\end{aligned}$$

Note that $d_b < 0$ as $-\frac{U_{cc,i} c_i}{U_{c,i}} + \left(\frac{U_{cc,i} c_i}{U_{c,i}} + \frac{U_{cl,i} l_i}{U_{c,i}}\right) \omega_{b,i} = -\frac{U_{ll,i} U_{cc,i} - U_{lc,i}^2}{\left[U_{lc,i} - \frac{U_{l,i}}{U_{c,i}} U_{cc,i}\right] + \left[U_{cl,i} - \frac{U_{c,i}}{U_{l,i}} U_{ll,i}\right]} \frac{l_i}{U_{c,i}} > 0.$

The original (non-stationary) model is obtained for $\Gamma = 0$ $\left(\check{\Gamma} = 0\right)$.

Debt elastic interest rate The model with a debt elastic interest rate is very similar to the model with portfolio costs. Following the standard assumption that agents do not internalize the effects of their decisions on the interest rate, it is

$$\begin{pmatrix} \bar{q}_{t+1} \\ b_t \end{pmatrix} = \begin{pmatrix} 1 - \frac{\Psi'(0)}{\beta} \frac{\frac{\partial z_2}{\partial \bar{q}} \bar{q}}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b} & -\frac{\Psi'(0)}{\beta} \frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b} \\ \frac{1}{\beta} \frac{\frac{\partial z_2}{\partial \bar{q}} \bar{q}}{\frac{\partial z_2}{\partial dW_1}} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \bar{q}_t \\ b_{t-1} \end{pmatrix}.$$

Endogenous discounting without internalization If agents do not internalize the effects of their consumption and leisure choices on the discount factor, the risk sharing condition, equation (16), implies the following system of difference equations

$$\begin{pmatrix} \bar{q}_{t+1} \\ b_t \end{pmatrix} = \begin{pmatrix} 1 + \frac{\frac{\partial z_2}{\partial dW_1} g_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} g_b}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b} & 0 \\ \frac{1}{\beta} \frac{\frac{\partial z_2}{\partial \bar{q}} \bar{q}}{\frac{\partial z_2}{\partial dW_1}} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \bar{q}_t \\ b_{t-1} \end{pmatrix},$$

where

$$\begin{aligned}
g_b &= \left[U_{c,1} c_1 \frac{\beta_1'}{\beta_1} \frac{\Phi_1'(\bar{q})\,\bar{q}}{\Phi_1(\bar{q})} - U_{c,2} c_2 \frac{\beta_2'}{\beta_2} \frac{\Phi_2'(\bar{q})\,\bar{q}}{\Phi_2(\bar{q})}\right] \frac{1}{\bar{q}\Phi_1(\bar{q})\, c_{12}}, \\
g_q &= -\bar{q}\Phi_1(\bar{q})\, c_{12} g_b.
\end{aligned}$$

d_b and d_q are as defined above and $g_b > 0$ as $\beta_i' < 0$ by assumption.

Endogenous discounting with internalization In this last model, agents take into account the effects of their consumption and leisure choices on the discount factor. As equations (18) and (19) reveal this implies two additional state variables. In addition to b_t, $\hat{\eta}_{1,t}$ and $\hat{\eta}_{2,t}$ are also state variables of the linearized system:

$$\begin{pmatrix} \bar{q}_{t+1} \\ \hat{\eta}_{1,t+1} \\ \hat{\eta}_{2,t+1} \\ b_t \end{pmatrix} = \begin{pmatrix} 1 & -zm_1 & zm_2 & 0 \\ \frac{a\phi_1}{m_1} & \left[\frac{1}{\beta_1} - za\phi_1\right] & za\phi_1 \frac{m_2}{m_1} & 0 \\ \frac{a\phi_2}{m_2} & -za\phi_2 \frac{m_1}{m_2} & \left[\frac{1}{\beta_2} + za\phi_2\right] & 0 \\ \frac{1}{\beta} \frac{\frac{\partial z_2}{\partial \bar{q}} \bar{q}}{\frac{\partial z_2}{\partial dW_1}} & 0 & 0 & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \bar{q}_t \\ \hat{\eta}_{1,t} \\ \hat{\eta}_{2,t} \\ b_{t-1} \end{pmatrix},$$

where

$$\begin{aligned} z &= \frac{\frac{\partial z_2}{\partial dW_1}\left(1 - \frac{1}{\beta}\right)}{\frac{\partial z_2}{\partial dW_1}\left(d_q + g_q + h_q\right) - \frac{\partial z_2}{\partial \bar{q}} \bar{q}\left(d_b + g_b + h_b\right)}, \\ a &= \left[1 + \frac{\frac{\partial z_2}{\partial \bar{q}} \bar{q}}{\frac{\partial z_2}{\partial dW_1}} \frac{1}{\bar{q}\Phi_1\left(\bar{q}\right) c_{12}}\right], \\ \phi_i &= U_{c,i} c_i \frac{\beta_i'}{\beta_i} \frac{\Phi_i'\left(\bar{q}\right) \bar{q}}{\Phi_i\left(\bar{q}\right)}, \\ m_i &= \frac{\eta_i \beta_i'}{1 - \eta_i \beta_i'}, \end{aligned}$$

for $i = 1, 2$ and

$$\begin{aligned} h_b &= \left\{U_{c,1} c_1 \frac{\eta_1 \beta_1'}{1 - \eta_1 \beta_1'} \frac{\beta_1''}{\beta_1'} \frac{\Phi_1'\left(\bar{q}\right) \bar{q}}{\Phi_1\left(\bar{q}\right)} - U_{c,2} c_2 \frac{\eta_2 \beta_2'}{1 - \eta_2 \beta_2'} \frac{\beta_2''}{\beta_2'} \frac{\Phi_2'\left(\bar{q}\right) \bar{q}}{\Phi_2\left(\bar{q}\right)}\right\} \frac{1}{\bar{q}\Phi_1\left(\bar{q}\right) c_{12}}, \\ h_q &= -\bar{q}\Phi_1\left(\bar{q}\right) c_{12} h_b. \end{aligned}$$

Important sign restrictions Before I study the local stability in the next section, it is useful to find the signs of the following expressions:

$$\begin{aligned} \frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b &= -\frac{1}{\bar{q}\Phi_1\left(\bar{q}\right)} \left(1 + \left[1 - \omega_{b,1}\right] \frac{\Phi_1'\left(\bar{q}\right) \bar{q}}{\Phi_1\left(\bar{q}\right)} - \left[1 - \omega_{b,2}\right] \frac{\Phi_2'\left(\bar{q}\right) \bar{q}}{\Phi_2\left(\bar{q}\right)}\right)^2 + \Upsilon c_{12} d_b < 0, \\ \frac{\partial z_2}{\partial dW_1} g_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} g_b &= \Upsilon c_{12} g_b > 0, \end{aligned}$$

where $\Upsilon = \left\{ -\tau_1 \frac{\Phi_1'(\bar{q})\bar{q}}{\Phi_1(\bar{q})} + \tau_2 \frac{\Phi_2'(\bar{q})\bar{q}}{\Phi_2(\bar{q})} - \varepsilon_1(\bar{q}) \tilde{H}_1\left(\frac{1}{\bar{q}}\right) \frac{1}{\bar{q}} \frac{\Phi_1'(\bar{q})\bar{q}}{\Phi_1(\bar{q})} + \frac{\varepsilon_2(\bar{q})}{\tilde{H}_2\left(\frac{1}{\bar{q}}\right)\frac{1}{\bar{q}}} \frac{\Phi_2'(\bar{q})\bar{q}}{\Phi_2(\bar{q})} \right\} > 0.$

The sign of

$$\frac{\partial z_2}{\partial dW_1}(g_q + h_q) - \frac{\partial z_2}{\partial \bar{q}} \bar{q}(g_b + h_b) = \Upsilon c_{12}(g_b + h_b)$$

depends on the sign of $\frac{\beta_1'}{\beta_1} + \frac{\eta_1 \beta_1'}{1-\eta_1 \beta_1'} \frac{\beta_1''}{\beta_1'}$. Without imposing more structure on the functional form of the discount factor nothing can be said about the sign of this expression.

B.3 Stability of the steady states

As most of the dynamic systems that are studied in this section are systems with two variables, consider

$$x_{t+1} = Mx_t$$

where M is the 2×2 matrix of coefficients and $x_t' = (\bar{q}_t, b_{t-1})'$. In order to study the dynamic properties of such a system, I study the roots of the characteristic equation that is associated with M,

$$P(\lambda) = \lambda^2 - \lambda \mathrm{tr}(M) + \det(M).$$

For convenience, I summarize the necessary and sufficient conditions such that none, exactly one or both eigenvalues λ lie in the unit circle:

i. if $|\det(M)| < 1$ and $|\mathrm{tr}(M)| < 1 + \det(M)$, the modulus of all eigenvalue is smaller than 1,

ii. if $|\det(M)| > 1$ and $|\mathrm{tr}(M)| < 1 + \det(M)$, the modulus of all eigenvalues is larger than 1,

iii. if $|\det(M)| < 1$ and $|\mathrm{tr}(M)| > 1 + \det(M)$ or $|\det(M)| > 1$ and $|\mathrm{tr}(M)| > 1 + \det(M)$, the modulus of one eigenvalue is larger than 1, while the other one is smaller than 1.

With these results in mind I proof theorems 1-4 in the main text:

Proof of Theorem 1. The coefficient matrix in the model with convex portfolio costs described in section 3.1 is given by

$$M_P = \begin{pmatrix} 1 + \frac{\frac{\partial z_2}{\partial \bar{q}}\bar{q}}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}}\bar{q} d_b} \check{\Gamma} & \frac{\frac{\partial z_2}{\partial dW_1}}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}}\bar{q} d_b} \check{\Gamma} \\ \frac{1}{\beta} \frac{\frac{\partial z_2}{\partial \bar{q}}\bar{q}}{\frac{\partial z_2}{\partial dW_1}} & \frac{1}{\beta} \end{pmatrix}.$$

The determinacy and the trace are $\det(M_P) = \frac{1}{\beta}$ and $\mathrm{tr}(M_P) = \frac{\frac{\partial z_2}{\partial \bar{q}}\bar{q}}{\frac{\partial z_2}{\partial dW_1}d_q - \frac{\partial z_2}{\partial \bar{q}}\bar{q}d_b}\check{\Gamma} + \left(1 + \frac{1}{\beta}\right)$, respectively. Since $\beta < 1$, $|\det(M_P)| > 1$. Furthermore, in any steady state $\frac{\partial z_2}{\partial dW_1}d_q - \frac{\partial z_2}{\partial \bar{q}}\bar{q}d_b < 0$, irrespective of the sign of the excess demand function, $\frac{\partial z_2}{\partial \bar{q}}$.

If $\frac{\partial z_2}{\partial \bar{q}} < 0$, $\mathrm{tr}(M_P) > 1 + \frac{1}{\beta} > 0$ and the modulus of one eigenvalue is larger than 1, while the other one is smaller than 1. Given that bond holdings are the only state variable, the system is saddle-path stable.

If $\frac{\partial z_2}{\partial \bar{q}} > 0$, the modulus of each eigenvalue is larger than 1 for

$$\begin{aligned} |tr(M_P)| &< 1 + \det(M_P) \\ \Delta_P &\equiv -2\beta \frac{(1+\beta)\Phi_1(\bar{q})}{\bar{q}\left[1 + \frac{1}{\bar{q}}\right]} \frac{\frac{\partial z_2}{\partial dW_1}d_q - \frac{\partial z_2}{\partial \bar{q}}\bar{q}d_b}{\frac{\partial z_2}{\partial \bar{q}}} > \Gamma''(0) > 0. \end{aligned}$$

Otherwise, the modulus of exactly one of the eigenvalues is larger than 1, while the other one is smaller than 1. Hence for $\Delta_P > \Gamma''(0)$ the system is unstable whenever $\frac{\partial z_2}{\partial \bar{q}} > 0$. ■

$\Gamma''(0)$ measures the sensitivity of the portfolio costs in the neighborhood of the steady state. In most applications, this sensitivity is low. If $\Gamma''(0)$ is assumed to be very large, the economy is very similar to the economy in financial autarchy. In the latter, any steady state is saddle-path stable. Hence, any steady state can be turned into a saddle point in the model with portfolio costs if the marginal costs of portfolio holdings increase strongly enough as the economy deviates from the steady state.

However, given that the model with convex portfolio costs is supposed to behave closely to the original (non-stationary) model, it is common practice to specify portfolio costs that are small and that do not change dramatically in the neighborhood of the steady state. Such specifications are also in line with actual portfolio costs.

Prrof of Theorem 2. If the interest rate is debt elastic as described in section 3.2, the proof follows the same steps as for theorem 1 with the difference that Δ_P is replaced by Δ_D where

$$\Delta_D \equiv 2(1+\beta)\frac{\frac{\partial z_2}{\partial dW_1}d_q - \frac{\partial z_2}{\partial \bar{q}}\bar{q}d_b}{\frac{\partial z_2}{\partial \bar{q}}\bar{q}} < \Psi'(0) < 0.$$

■

Proof of Theorem 3. The coefficient matrix M_E for the model with endogenous discounting and no internalization (section 3.3) is given by

$$M_E = \begin{pmatrix} 1 + \frac{\frac{\partial z_2}{\partial dW_1} g_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} g_b}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b} & 0 \\ \frac{1}{\beta} \frac{\frac{\partial z_2}{\partial \bar{q}} \bar{q}}{\frac{\partial z_2}{\partial dW_1}} & \frac{1}{\beta} \end{pmatrix}.$$

The determinacy and the trace are $\det(M_E) = \frac{1}{\beta}\left[1 + \frac{\frac{\partial z_2}{\partial dW_1} g_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} g_b}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b}\right]$ and $\mathrm{tr}(M_E) = -\frac{1-\beta}{\beta} \frac{\frac{\partial z_2}{\partial dW_1} g_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} g_b}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b} + 1 + \det(M_E)$, respectively. Since $\frac{\frac{\partial z_2}{\partial dW_1} g_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} g_b}{\frac{\partial z_2}{\partial dW_1} d_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} d_b} < 0$ irrespective of the sign of the slope of the excess demand function, the modulus of exactly one eigenvalue is smaller than 1. With bond holdings being the only state variable, the dynamic system is saddle-path stable. ■

Notice that it is crucial to assume that the endogenous discount factor is decreasing in the utility level. Otherwise $\frac{\partial z_2}{\partial dW_1} g_q - \frac{\partial z_2}{\partial \bar{q}} \bar{q} g_b < 0$ and $|\mathrm{tr}(M_E)| < 1 + \det(M_E)$ irrespective of the slope of the excess demand function. In this case, both eigenvalues would be larger than 1.

If agents internalize the effects of their choices on the endogenous discount factor, only a weaker theorem can be proven since the sign of $d_b + h_b + g_b$ cannot be determined. In preparation for this theorem, consider an increase in the wealth of agents in country 1. I am interested in the change of the intertemporal marginal rate of substitution under the assumption that current and future prices as well as future allocations remain unchanged. The only variables that are allowed to change are current consumption and leisure and therefore also utility in the current period. I refer to this experiment as the direct impact of a wealth increase.

The intertemporal marginal rate of substitution in country 1 is given by

$$IMRS\left(s^{t+1}\right) = \beta\left(U_1\left(s^t\right)\right) \frac{1 - \eta_1\left(s^{t+1}\right)\beta'\left(U_1\left(s^{t+1}\right)\right)}{1 - \eta_1\left(s^t\right)\beta'\left(U_1\left(s^t\right)\right)} \frac{U_{c,1}\left(s^{t+1}\right)}{U_{c,1}\left(s^t\right)}.$$

Equation (19) in the main text reveals that $\eta_1\left(s^t\right)$ is nothing but the negative of the expected discounted lifetime utility of agents of country 1 from $t+1$ onwards. Therefore $\eta_1\left(s^t\right)$ does not depend on any time t variables. Under the assumption that future allocations and prices are held constant, the following equations are relevant for the experiment:

$$\begin{aligned} \frac{U_l\left(c_1\left(s^t\right), l_1\left(s^t\right)\right)}{U_c\left(c_1\left(s^t\right), l_1\left(s^t\right)\right)} &= -A_1 \Phi_1\left(\bar{q}\left(s^t\right)\right) \\ c_1\left(s^t\right) &= \Phi_1\left(\bar{q}\left(s^t\right)\right) A_1 l_1\left(s^t\right) + dW_1\left(s^t\right). \end{aligned}$$

The first equation states the familiar equilibrium condition that the marginal rate of substitution between labor and consumption equals the real wage. As shown earlier, the second equation can be derived straight from the intertemporal budget constraint of the agents. $dW_1(s^t)$ denotes the wealth transfer to country 1. Under the assumption that prices are kept unchanged for all s^{t+j}, $j \geq 0$, the marginal rate of substitution between labor and consumption is held constant.

The direct impact of a wealth increase on the intertemporal marginal rate of substitution is given by

$$\begin{aligned}
&\frac{\partial IMRS(s^{t+1})}{\partial dW_1(s^t)}|_{direct} \\
= &\left[-\frac{U_{cc,1}(s^t)\,c_1(s^t)}{U_{c,1}(s^t)} + \left(\frac{U_{cc,1}(s^t)\,c_1(s^t)}{U_{c,1}(s^t)} + \frac{U_{cl,1}(s^t)\,l_1(s^t)}{U_{c,1}(s^t)}\right)\omega_{1b}(s^t)\right]\frac{1}{c_1(s^t)} IMRS(s^{t+1}) \\
&+\left(\frac{\beta'(U_1(s^t))}{\beta(U_1(s^t))} + \frac{\eta_1(s^t)\,\beta'(U_1(s^t))}{1-\eta_1(s^t)\,\beta'(U_1(s^t))}\frac{\beta''(U_1(s^t))}{\beta'(U_1(s^t))}\right) U_{c,1}(s^t)\, IMRS(s^{t+1}). \qquad (31)
\end{aligned}$$

The first term in equation (31) measures the direct impact of the wealth transfer on the marginal utility of consumption under the assumption that the marginal rate of substitution between leisure and consumption is held constant. Under the assumptions on the utility function the first term is positive. In the experiment the increase in the wealth of the agents of country 1 lowers the labor supply and increases consumption. Consequently, the marginal utility of consumption, $U_{c,1}(s^t)$, rises. This effect operates towards a rise of $IMRS_1$.

The second term measures the effect of the wealth increase on $IMRS_1$ through the endogeneity of the discount factor. There are two effects. First, as consumption and leisure rise in the current period, so does utility $U_1(s^t)$. As the discount factor is decreasing in the utility level this effect operates towards a decline of the $IMRS_1$. Furthermore, the change in the discount factor effects the $IMRS_1$ also through its impact on the discounted future utility summarized in $\eta_1(s^t)$. Absent assumptions on β'' this expression cannot be signed.

If the discount factor is constant, $\frac{\partial IMRS_1(s^{t+1})}{\partial dW_1(s^t)}|_{direct} > 0$. Hence, if the discount factor β_i does not react too strongly to changes in U_i, the effect will still be positive.

Given the original questions this restriction is not too restrictive and $\frac{\partial IMRS_1(s^{t+1})}{\partial dW_1(s^t)}|_{direct}$ is most likely to be positive. Endogenous discounting is introduced to obtain stationarity in the model with incomplete asset markets. To the extent that the stationary model is supposed to behave closely to the original non-stationary model it is desirable that the discount factor does not move around too much.

Under the assumption that $\frac{\partial IMRS_i(s^{t+1})}{\partial dW_i(s^t)}|_{direct} > 0$, $i = 1, 2$, theorem 4 can be proven.

Proof of Theorem 4. The linearized dynamic system for the model with endogenous discounting and internalization is given by $x_{t+1} = M_I x_t$ and $x_t' = (\bar{q}_t, \hat{\eta}_{1,t}, \hat{\eta}_{2,t}, b_{t-1})'$. The

4×4 coefficient matrix M_I is given by

$$M_I = \begin{pmatrix} 1 & -zm_1 & zm_2 & 0 \\ \frac{a\phi_1}{m_1} & \left[\frac{1}{\beta_1} - za\phi_1\right] & za\phi_1 \frac{m_2}{m_1} & 0 \\ \frac{a\phi_2}{m_2} & -za\phi_2 \frac{m_1}{m_2} & \left[\frac{1}{\beta_2} + za\phi_2\right] & 0 \\ \frac{1}{\beta} \frac{\frac{\partial z_2}{\partial \bar{q}} \bar{q}}{\frac{\partial z_2}{\partial dW_1}} & 0 & 0 & \frac{1}{\beta} \end{pmatrix}.$$

The characteristic equations that is associated with M_I simplifies to

$$-\left(\lambda - \frac{1}{\beta}\right)^2 \left(\lambda^2 - \left[\frac{1}{\beta} + 1 + za\left[\phi_2 - \phi_1\right]\right] \lambda + \frac{1}{\beta}\right) = 0,$$

with

$$za\left[\phi_2 - \phi_1\right] = \frac{\frac{1}{\beta}\left(1 - \beta\right) \Upsilon \bar{q} \Phi_1\left(\bar{q}\right) c_{12} g_b}{\left(1 + \left[1 - \omega_{b,1}\right] \frac{\Phi_1'(\bar{q})\bar{q}}{\Phi_1(\bar{q})} - \left[1 - \omega_{b,2}\right] \frac{\Phi_2'(\bar{q})\bar{q}}{\Phi_2(\bar{q})}\right)^2 - \Upsilon \bar{q} \Phi_1\left(\bar{q}\right) c_{12}\left(d_b + h_b + g_b\right)},$$

where $\Upsilon > 0$ irrespective of the sign of the slope of z_2 as shown above. With three state variables, $\hat{\eta}_{1,t}$, $\hat{\eta}_{2,t}$ and b_{t-1}, the dynamic system is saddle-path stable if the modulus of exactly three eigenvalues is larger than 1. Since two of the four eigenvalues are equal to $\frac{1}{\beta}$, stability of the system requires:

$$za\left[\phi_2 - \phi_1\right] > 0$$

or

$$za\left[\phi_2 - \phi_1\right] < -2\left(1 + \frac{1}{\beta}\right).$$

A sufficient condition for stability is $\Phi_1\left(\bar{q}\right) c_{12}\left(d_b + h_b + g_b\right) < 0$ as it implies that $za\left[\phi_2 - \phi_1\right] > 0$:

$$\begin{aligned} & \left(d_b + h_b + g_b\right) \bar{q} \Phi_1\left(\bar{q}\right) c_{12} \\ = \; & \left[-\frac{U_{cc,1}c_1}{U_{c,1}} + \left(\frac{U_{cc,1}c_1}{U_{c,1}} + \frac{U_{cl,1}l_1}{U_{c,1}}\right)\omega_{b,1} + \left(\frac{\beta_1'}{\beta_1} + \frac{\eta_1\beta_1'}{1 - \eta_1\beta_1'}\frac{\beta_1''}{\beta_1'}\right) U_{c,1}c_1\right] \frac{\Phi_1'\left(\bar{q}\right)\bar{q}}{\Phi_1\left(\bar{q}\right)} \\ & - \left[-\frac{U_{cc,2}c_2}{U_{c,2}} + \left(\frac{U_{cc,2}c_2}{U_{c,2}} + \frac{U_{cl,2}l_2}{U_{c,2}}\right)\omega_{b,2} + \left(\frac{\beta_2'}{\beta_2} + \frac{\eta_2\beta_2'}{1 - \eta_2\beta_2'}\frac{\beta_2''}{\beta_2'}\right) U_{c,2}c_2\right] \frac{\Phi_2'\left(\bar{q}\right)\bar{q}}{\Phi_2\left(\bar{q}\right)} \end{aligned}$$

Under the assumption that $\frac{\partial IMRS_i\left(s^{t+1}\right)}{\partial dW_i(s^t)}|_{direct} > 0$, $i = 1, 2$, each of two the expressions in brackets is positive and therefore $\Phi_1\left(\bar{q}\right) c_{12}\left(d_b + h_b + g_b\right) < 0$. ■

Obviously the assumption that $\frac{\partial IMRS_i\left(s^{t+1}\right)}{\partial dW_i(s^t)}|_{direct} > 0$ for each $i = 1, 2$ is unnecessarily strong. However, this expression is somewhat more intuitive than other possible restrictions.

Most notably theorems 3 and 4 show that under endogenous discounting the stability of the system in the neighborhood of a steady state does not depend on the slope of the excess demand function in the steady state. The stability depends solely on the parameterization of the endogenous discount factor. This is very different from the economies studied in theorems 1 and 2. With convex portfolio costs or a debt-elastic interest rate the stability of the system around a steady state depends very much on the slope of the excess demand function in this steady state.[18]

[18] One notable exception to this statement are the cases of very sensitive portfolio costs or debt elastic interest rate. In these cases the model behaves like the economy under financial autarchy and every steady state is stable in the simple model without capital.