Appendix C to "Housing and Debt over the Life Cycle and over the Business Cycle": A Simple Extension with Default

Matteo Iacoviello*
Federal Reserve Board

Marina Pavan†
Universitat Jaume I & LEE

September 20, 2011

Abstract

This appendix sketches a brief description of an extension of the baseline model in Iacoviello and Pavan (2011) where we allow for mortgage default following housing depreciation shocks.

1. Introduction

The following is a brief outline of an extension of the model in Iacoviello and Pavan (2011), where households are allowed to default on their mortgage debt. At any period, indebted households can decide to default on their debt, in which case they lose their house, are banned from borrowing and must become tenants.[1] Default is triggered by shocks to housing depreciation that are large enough to cause leverage individuals to own on their house more than it is worth. The perfectly competitive financial sector cannot discriminate borrowers, that is, lenders cannot apply different borrowing interest rates to different borrowers, and charge the same interest premium to all their debtors in order to break even.

2. The model with mortgage default

The environment features the same characteristics as in the baseline model, except for the existence of shocks to the depreciation rate of housing and capital. These shocks are assumed to move one-to-one with the technology shocks: $\delta_{H,t} = \delta_H (A_t)$ and $\delta_{K,t} = \delta_K (A_t)$.[2] As in Iacoviello and Pavan (2011), we adopt the approximate aggregation/bounded rationality approach developed

*Matteo Iacoviello, Division of International Finance, Federal Reserve Board, 20th and C St. NW, Washington, DC 20551. E-mail: matteo.iacoviello@frb.gov.

†Marina Pavan, Universitat Jaume I & LEE, Castellón, Spain. E-mail: pavan@eco.uji.es.

[1] In this simple version, the household is banned from borrowing in the default period only, and no credit history is recorded.

[2] In the numerical implementation, capital depreciation is assumed to rise together with housing depreciation to avoid perverse substitution effects between capital and housing investment, which would lead to an increase

by Krusell and Smith (1997, 1998), and solve for the model equilibrium by forecasting future prices through the first moment of aggregate capital and, in this case, aggregate housing as well. The inclusion of the aggregate housing stock into the set of relevant state variables is necessary in this setup given the existence of shocks to the value of houses, and the need to forecast the interest rate premium as well.[3]

2.1. The household's problem

As in the main text, denote $x_t \equiv (z_t, b_{t-1}, h_{t-1}, A_t, H_{t-1}, K_{t-1})$ the vector collecting individual and aggregate state variables. The dynamic problem of an age a household with discount factor β_i can now be stated as:

$$V_a(x_t;\beta_i) = \max_{I^i \in \{I^h, I^r, I^d\}} \left\{ I^h V_a^h(x_t;\beta_i), I^r V_a^r(x_t;\beta_i), I^d V_a^d(x_t;\beta_i) \right\}$$

where V_a^h, V_a^r and V_a^d are the value functions at age a for owning, renting a house and defaulting respectively, and $I^i = 1$ corresponds to the decision to buy/own, rent or default for $i = h, r$ or d. The value of being a homeowner solves:

$$\begin{aligned} V_a^h(x_t;\beta_i) &= \max_{c_t, b_t, h_t, l_t} \left\{ \lambda_a u\left(c_t, h_t, \bar{l} - l_t\right) + \beta_i \chi_{a+1} \textstyle\sum_{z', A'} \pi_{A,A'} \pi_{z,z'} V_{a+1}(x_{t+1};\beta_i) \right\} \\ \text{s.t.} \quad & c_t + h_t + \Psi(h_t, h_{t-1}) = y_{at} + b_t - (R_t + \mathcal{I}\{b_{t-1} > 0\} r_t^p) b_{t-1} + (1 - \delta_{H,t}) h_{t-1} \\ & b_t \leq \min\{m_H h_t, m_Y \Re_t\}, \ c_t \geqslant 0, \ l_t \in \left(0, \bar{l}\right) \end{aligned}$$

where we use the same notation than in the main paper to denote the transaction costs for housing, etc. The function $\mathcal{I}\{b > 0\}$ is equal to 1 if $b > 0$, i.e. if the household is a net debtor at the beginning of the period. We denote with r_t^p the interest rate premium charged to borrowers. The depreciation rate for housing $\delta_{H,t}$ changes over the business cycle, being higher in the worst recession.

As in the benchmark model, the value of renting a house is determined by solving the problem:

$$\begin{aligned} V_a^r(x_t;\beta_i) &= \max_{c_t, b_t, s_t, l_t} \left\{ \lambda_a u\left(c_t, s_t, \bar{l} - l_t\right) + \beta_i \chi_{a+1} \textstyle\sum_{z', A'} \pi_{A,A'} \pi_{z,z'} V_{a+1}(x_{t+1};\beta_i) \right\} \\ \text{s.t.} \quad & c_t + p_t s_t + \Psi(0, h_{t-1}) = y_{at} + b_t - R_t b_{t-1} + (1 - \delta_{H,t}) h_{t-1} \\ & b_t \leq 0, \ c_t \geqslant 0, \ l_t \in \left(0, \bar{l}\right), h_t = 0. \end{aligned}$$

Households that have a net negative asset position ($b_{t-1} > 0$) at the beginning of the period have the option of defaulting on their debt, losing their house and being only able to rent. The

in aggregate capital when a bad shock to housing hits. Moreover, the numerical implementation assumes that the variance of technology shocks in arbitrarily small, so that the only shocks are effectively the two depreciation shocks.

[3] The typical R^2 of the forecasting equations for K, R and the interest premium is 0.99, 0.995 and 0.99 respectively for the regressions including H. It drops to 0.89, 0.99 and 0.98 when we do not include housing in the forecasting regressions.

corresponding value is the following:

$$V_a^d(x_t;\beta_i) = \max_{c_t,b_t,s_t,l_t}\left\{\lambda_a u\left(c_t,s_t,\bar{l}-l_t\right) + \beta_i\chi_{a+1}\textstyle\sum_{z',A'}\pi_{A,A'}\pi_{z,z'}V_{a+1}(x_{t+1};\beta_i)\right\}$$

$$\text{s.t.} \qquad c_t + p_t s_t = y_{at} + b_t$$

$$b_t \le 0,\ c_t \geqslant 0,\ l_t \in \left(0,\bar{l}\right), h_t = 0.$$

At the agent's last age, $V_{T+1}(x_{T+1};\beta) = 0$ for any $(x_{T+1};\beta)$.

At any point in time, the following are the forecasting functions:

$$\text{for aggregate capital:} \quad K_t = F^K(K_{t-1}, H_{t-1}, A_t)$$

$$\text{for aggregate labor:} \quad L_t = F^L(K_{t-1}, H_{t-1}, A_t)$$

$$\text{for aggregate housing:} \quad H_t = F^H(K_{t-1}, H_{t-1}, A_t).$$

Moreover, we assume the agents directly forecast the value of the interest rate premium as a function of aggregate capital, housing stock and total factor productivity, $r_t^p = F^p(K_{t-1}, H_{t-1}, A_t)$.[4]

2.2. The financial sector with the possibility of mortgage default

In the perfectly competitive financial sector with the option to default, the interest rate on loans is higher than the one on deposits, so that the financial intermediaries' profits are zero. We assume that lenders cannot observe (or face a high cost of observing) the default probability of each individual household or, correspondingly, cannot price discriminate among borrowers and must charge the same interest rate premium r_t^p on every loan.[5] When someone defaults, the financial intermediary retrieves the value of the housing collateral, net of depreciation and transaction costs.

Let's denote with D_{t-1} the aggregate debt at the beginning of period t, of which D_{t-1}^N is the total amount re-paid (not defaulted upon) and D_{t-1}^D is the total amount defaulted, so that $D_{t-1} = D_{t-1}^N + D_{t-1}^D$ at any period. Then a zero profit condition holds such that:

$$D_{t-1} = \frac{(R_t + r_t^p)D_{t-1}^D + (1 - \delta_{H,t} - \Psi\left(0, H_{t-1}^D\right))H_{t-1}^D}{R_t}$$

[4] To the best of our knowledge, Nakajima and Rios-Rull (2005) is the only model to include aggregate risk and default (in the form of consumer bankruptcy) in a heterogeneous agents' equilibrium setting. In their model, however, the assumptions on the timing of the default decision ensure that the prices of loans do not depend on the distribution of agents. We take a different approach and adopt a "bounded rationality" technique to forecast borrowing premia, similar to the one used in Krusell and Smith (1997).

[5] We adopted this modeling strategy for the interest rate premium since it is the most consistent with our setting, in which, as in RBC models in general, interest rates are determined "ex-post" as a function of next period' aggregate shock realization.

One alternative could have been to condition the interest rate premium on the characteristics of the borrower. In that case, though, given the timing assumption of our model, we should have kept track of complex multi-dimensional objects dependent on individual and aggregate variables, and the zero-profit condition would not have been a trivial object to define ex-post.

In the default literature with no aggregate volatility, financial intermediaries commit "ex-ante" to being paid a certain interest rate, so that ex-post profits can be different from zero (Athreya, 2008; Chatterjee et al., 2007; Chatterjee and Eyigungor, 2011).

where H^D_{t-1} is the collateral (aggregate value of houses guaranteeing the defaulted debt) repossessed by the financial sector.

Re-arranging, the interest rate premium at any t is then given by

$$r_t^p = \frac{R_t D^D_{t-1} - (1 - \delta_{H,t} - \Psi\left(0, H^D_{t-1}\right))H^D_{t-1}}{D^N_{t-1}}$$

and is charged to all borrowers, households and firms alike.[6,7]

2.3. Definition of Equilibrium:

We are now ready to define the equilibrium for this economy.

Definition 2.1. *A recursive competitive equilibrium consists of value functions* $\{V_a(x_t;\beta)\}$, *policy functions* $\{I^h_a(x_t;\beta),\ I^r_a(x_t;\beta),\ I^d_a(x_t;\beta),\ h_a(x_t;\beta), s_a(x_t;\beta), b_a(x_t;\beta), c_a(x_t;\beta), l_a(x_t;\beta)\}$ *for each* β, *age and period* t, *prices* $\{R_t\}^\infty_{t=1}$, $\{r^p_t\}^\infty_{t=1}$, $\{w_t\}^\infty_{t=1}$ *and* $\{p_t\}^\infty_{t=1}$, *aggregate variables* K_t, L_t, H^o_t *and* H^r_t *for each period* t, *lump-sum taxes* Γ *and pension* P, *and laws of motion* F^K, F^H, F^L *and* F^p *such that at any* t*:*

Agents optimize: Given R_t, w_t, p_t *and* r^p_t *and the laws of motion* F^K, F^H, F^L *and* F^p, *the value functions solve the individual's problem, with the corresponding policy functions.*

Factor prices and rental prices satisfy:

$$R_t + r^p_t - 1 + \delta_K = \alpha A_t \left(K_{t-1}/L_t\right)^{(\alpha-1)}$$

$$w_t = (1-\alpha)\, A_t(K_{t-1}/L_t)^\alpha$$

$$p_t = E_t\left(\frac{R_{t+1} - (1-\delta_H)}{R_{t+1}}\right)$$

and the interest rate premium r^p_t *is determined from the equilibrium condition of the financial sector as above.*

Markets clear:

$$L_t = \int l_a(x_t;\beta)\,\eta_a z_t \partial\Phi_t \quad \textit{(labor market)}$$

$$C_t + H_t - (1-\delta_{H,t})\,H_{t-1} + \Omega_t + K_t - (1-\delta_{K,t})\,K_{t-1} = Y_t \quad \textit{(goods market)}$$

where H_t *and* Ω_t *are defined as*

$$H_t = H^o_t + H^r_t = \int I^h_a(x_t;\beta)\,h_a(x_t;\beta)\,\partial\Phi_t + \int [I^r_a(x_t;\beta) + I^d_a(x_t;\beta)] s_a(x_t;\beta)\,\partial\Phi_t,$$

[6] However, we do not model firms' decision to default. We assume that firms also have to pay the higher interest for borrowing, given that lenders cannot discriminate interest rates on loans.

[7] More precisely, the interest rate premium calculated on the basis of the equilibrium condition is the following:

$$r^p_t = \frac{R_t \int I^d_a(x_t;\beta)\, b_{t-1}\partial\Phi_t - \int I^d_a(x_t;\beta)\,(1-\delta_{H,t} - \Psi\,(0, h_{t-1}))h_{t-1}\partial\Phi_t}{K_{t-1} + \int (1 - I^d_a(x_t;\beta))\mathcal{I}\,(b_{t-1} > 0)\, b_{t-1}\partial\Phi_t}$$

$$\Omega_t = \int \Psi\left(h_a\left(x_t;\beta\right), h_{t-1}\right)\partial\Phi_t$$

and, by Walras' law, the supply of savings equals total capital.
The government budget is balanced:

$$\sum\nolimits_{a=1}^{\widetilde{T}} \Pi_a \Gamma = \sum\nolimits_{a=\widetilde{T}+1}^{T} \Pi_a P.$$

The laws of motion for the aggregate capital, aggregate labor, aggregate housing and interest rate premia are given by

$$\begin{aligned} K_t &= F^K\left(K_{t-1}, H_{t-1}, A_t\right), \; L_t = F^L\left(K_{t-1}, H_{t-1}, A_t\right) \\ H_t &= F^H\left(K_{t-1}, H_{t-1}, A_t\right), \; r_t^p = F^p\left(K_{t-1}, H_{t-1}, A_t\right). \end{aligned}$$

3. Brief outline of numerical implementation

Households perceive that prices depend on the first moment of the aggregate capital and the aggregate housing stock only, and that these variables change over time according to the laws of motion specified above. In particular, agents take their decisions based initially on an arbitrary value of the interest rate premium r^p, and consider the future r^p to be given by a linear function of K, H and A (see Krusell and Smith, 1997).

Given the optimal policy functions solving the individual problem, we simulate the agents' choices and directly compute the interest premium that makes the financial intermediaries' profits to be nul at any period, for a large number of periods.

We then use the obtained time series (of which we discarded the first part) to regress the aggregate variables K_{t+1}, H_{t+1}, L_{t+1} and the premia r_{t+1}^p on constants, K_t and H_t, for each value of the aggregate shock A_t.

We iterate these steps (solution of optimal rules and simulation) until convergence of the parameters in the laws of motion, measuring goodness of fit of the regressions with the implied R^2.

4. Results

The model can be used to see how shocks to housing values interact with the mortgage default rate, interest rate, debt and housing stock. To illustrate the main mechanism at work in the model with default, we assume technology shocks away, and solve the model with depreciation shocks for housing and capital only. We fix the labor supply at unity, so that movements in the aggregate capital stock are the only source of movements in output. We choose the model parameters at the values of Table 2 in Iacoviello and Pavan (2011), except the discount rate gap which is 4 percent, and the loan-to-value which is set at 85 percent. The depreciation shocks for housing and capital are set to $\delta_H = 25\%$ and $\delta_K = 13\%$ respectively in the worst state of the world, and to $\delta_H = 15\%$ and $\delta_K = 11\%$ in the next worst case, while $\delta_H = 5\%$ and $\delta_K = 9\%$ in all other states. Recall that the transaction cost to change housing stock is 5 percent, except in the case of default when the defaulting agent can walk away from the debt at no cost.[8]

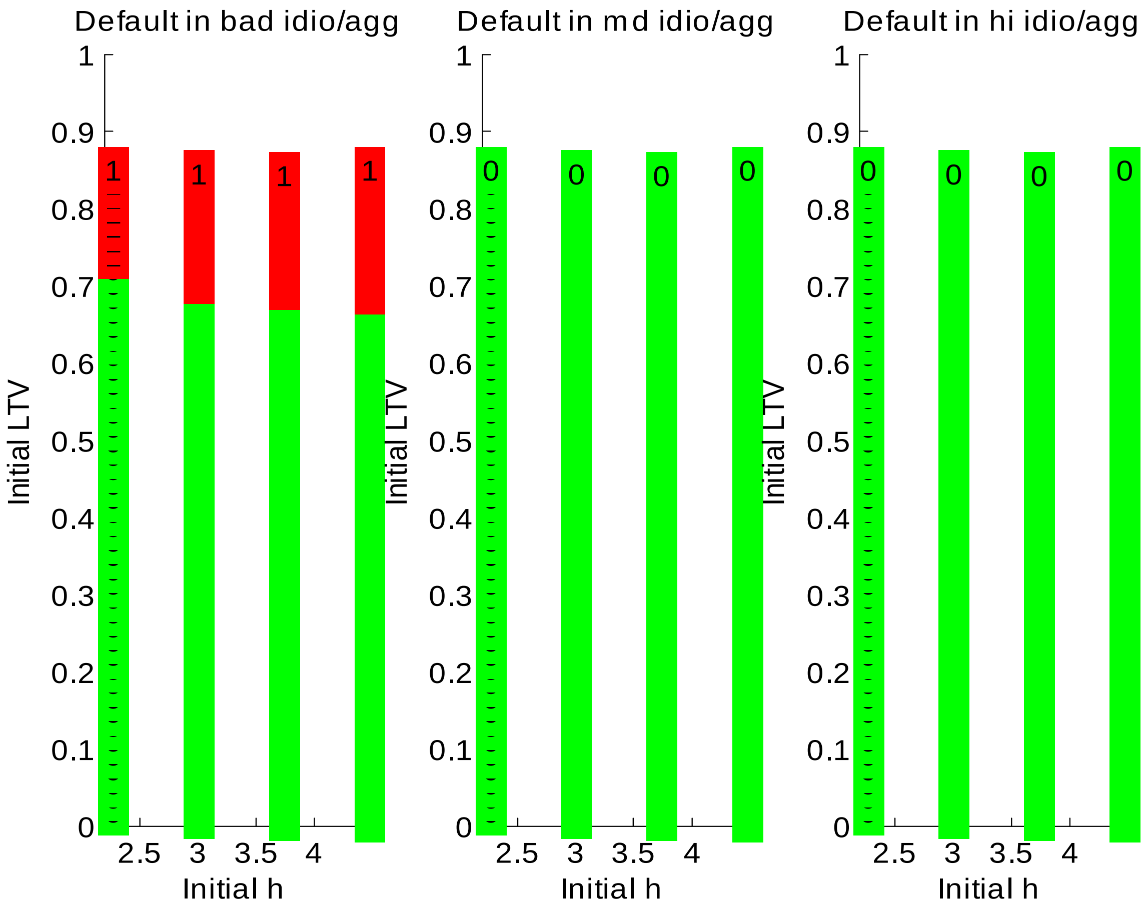
Figure A.1 illustrates the homeowner's optimal default decision for different combinations of initial house, loan-to-value (LTV) ratio and idosyncratic income shock. In response to a housing depreciation shock that wipes 25% of the house value, homeowners who are characterized by a bad idiosyncratic income realization and by an initial leverage ranging from 68 to 73 percent or higher will choose to default. To consider what this means, assume that the house is worth 100, so that the initial mortgage balance in the house is 68 to 73 dollars. The depreciation shock reduces the value of the house to 75, so "poor" agents who own on their house between $68 - 73$ and higher will choose to default. Notice in the Figure that the bigger is the initial house, the lower is the LTV threshold that triggers default: households with a very high housing stock are more far away from their target level of housing, the default option allows them to save the high transaction costs to pay, so they are willing to default even in the case in which they still have some equity left in the house (after the depreciation shock), provided that the equity in the house is less than the transaction cost.

Figure A.2 shows a simulation of the main macroeconomic variables over 100 model periods. In the bad states of the world, when housing depreciation takes on very large values, interest rate premia reach values of about 1.5 percent, the aggregate default rate rises from 0 to about 10 percent, and the aggregate housing and capital stock persistently decline. Further details on computational results can be obtained from the authors.

[8] It would be straightforward to add to the model other penalties for defaulting (income loss, stigma) besides exclusion from the credit market in the current period.

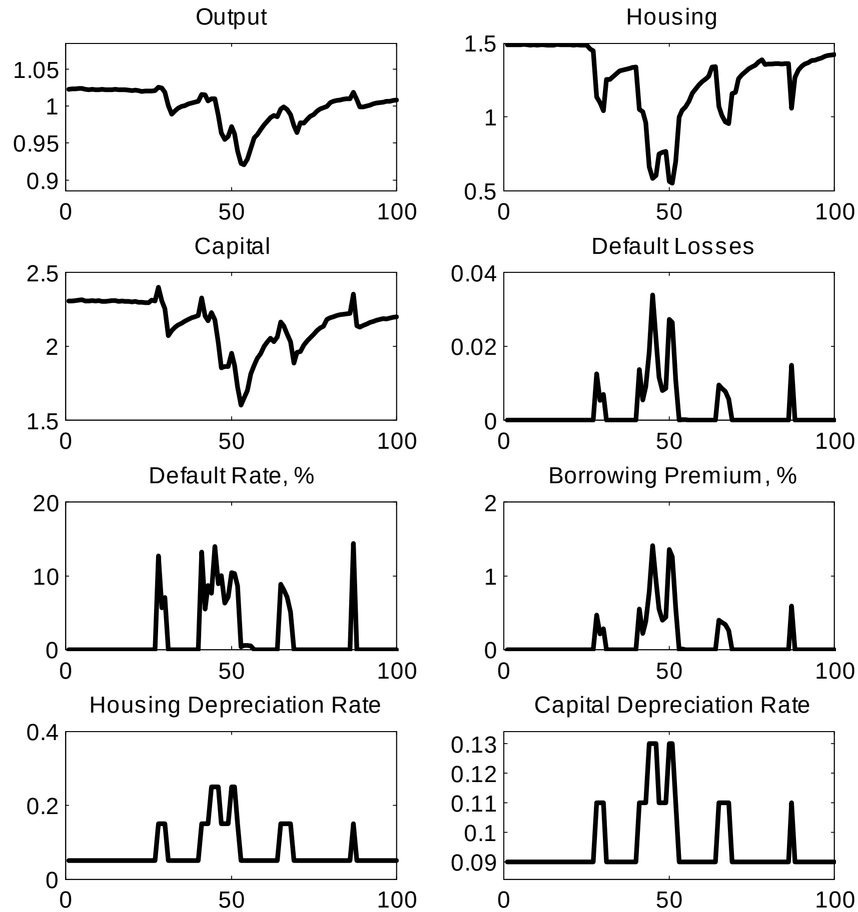
Figures

Figure A.1: Default Policy in different states of the world



Note: The figure illustrates, for each combination of initial house and LTV, the homeowner's default decision. It is plotted for an impatient agent who is 35 years old. From the left to the right: lowest idiosyncratic and lowest aggregate state; median idiosyncratic and lowest aggregate state; highest idiosyncratic and lowest aggregate state.

Figure A.2: Macroeconomic variables in default and no–default periods



Note: The figure illustrates a macroeconomic simulation of 100 periods. Average output is normalized to unity. Housing, Capital and Default Losses are expressed as a ratio to average output. Defaults rise in bad states of the world when the housing and capital stock is subject to depreciation shocks.

References

Athreya, Kartik B. (2008) 'Default, insurance, and debt over the life-cycle.' *Journal of Monetary Economics* 55(4), 752–774

Chatterjee, Satyajit, and Burcu Eyigungor (2011) 'A quantitative analysis of the u.s. housing and mortgage markets and the foreclosure crisis.' Working Papers 11-26, Federal Reserve Bank of Philadelphia

Chatterjee, Satyajit, Dean Corbae, Makoto Nakajima, and José-Víctor Ríos-Rull (2007) 'A quantitative theory of unsecured consumer credit with risk of default.' *Econometrica* 75(6), 1525–1589

Iacoviello, Matteo, and Marina Pavan (2011) 'Housing and debt over the life cycle and over the business cycle.' Unpublished Manuscript

Krusell, Per, and Anthony A. Smith (1997) 'Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns.' *Macroeconomic Dynamics* 1(02), 387–422

Krusell, Per, and Anthony A. Smith (1998) 'Income and wealth heterogeneity in the macroeconomy.' *Journal of Political Economy* 106(5), 867–896

Nakajima, Makoto, and José-Víctor Ríos-Rull (2005) 'Default and aggregate fluctuations in storage economies.' In 'Frontiers in Applied General Equilibrium Modeling' in honor of Herbert Scarf (Cambridge University Press)