Optimal Monetary Policy with Heterogeneous Agents*

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Abstract

Incomplete-markets models with heterogeneous agents are increasingly used for policy analysis. We propose a novel methodology for solving fully dynamic optimal policy problems in models of this kind, both under discretion and commitment, based on optimization techniques in function spaces. We illustrate our methodology by studying optimal monetary policy in an incomplete-markets model with long-term nominal debt and costly inflation. Under discretion, an inflationary bias arises from the central bank's attempt to redistribute wealth from creditors to debtors, who have a higher marginal utility of consumption. Under commitment, this inflationary force is counteracted over time by the incentive to prevent expected future inflation from lowering the price at which

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issuers of new bonds do so; under certain conditions, long-run inflation is zero as both effects cancel out asymptotically. We find numerically that the optimal commitment features first-order initial inflation followed by a gradual decline towards its (near zero) long-run value.

Keywords: optimal monetary policy, commitment and discretion, incomplete markets, Gateaux derivative, nominal debt, inflation, redistributive effects, continuous time

JEL codes: E5, E62, F34.

1 Introduction

Ever since the seminal work of Bewley (1983), Huggett (1993) and Aiyagari (1994), incomplete markets models with uninsurable idiosyncratic risk have become a workhorse for policy analysis in macro models with heterogeneous agents.[1] Among the different areas spawned by this literature, the analysis of the dynamic aggregate effects of fiscal and monetary policy has begun to receive considerable attention in recent years.[2]

As is well known, one difficulty when working with incomplete markets models is that the state of the economy at each point in time includes the cross-household wealth distribution, which is an infinite-dimensional, endogenously-evolving object.[3] The development of numerical methods for computing equilibrium in these models has made it possible to study the effects of aggregate shocks and of particular policy rules. However, the infinite-dimensional nature of the wealth distribution has made it difficult to make progress in the analysis of *optimal* policy problems in this class of models.

In this paper, we propose a novel methodology for solving fully dynamic optimal policy problems in incomplete markets models with uninsurable idiosyncratic risk, both under discretion and commitment. The methodology relies on the use of calculus techniques in infinite-dimensional Hilbert spaces to compute the first order conditions. In particular, we employ a generalized version of the classical derivative known as Gateaux derivative.

[1] For a survey of this literature, see e.g. Heathcote, Storesletten and Violante (2009).
[2] See our discussion of the related literature below.
[3] See e.g. Ríos-Rull (1995).

We illustrate our methodology by analyzing optimal monetary policy in an incomplete markets economy. Our framework is close to Huggett's (1993) standard formulation. As in the latter, households trade non-contingent claims, subject to an exogenous borrowing limit, in order to smooth consumption in the face of idiosyncratic income shocks. We depart from Huggett's real framework by considering *nominal* non-contingent bonds with an arbitrarily long maturity, which allows monetary policy to have an effect on equilibrium allocations. In particular, our model features a classic Fisherian channel (Fisher, 1933), by which realized inflation redistributes wealth from lending to indebted households.[4] In order to have a meaningful trade-off in the choice of the inflation path, we also assume that inflation is costly, which can be rationalized on the basis of price adjustment costs. Moreover, expected *future* inflation lowers the price of the long-term bond through higher inflation premia. We also depart from the standard closed-economy setup by considering a small open economy, with the aforementioned bonds being also held (and priced) by risk-neutral foreign investors; this makes the analysis somewhat more tractable.[5] Finally, we cast the model in continuous time, which offers important computational advantages relative to the (standard) discrete-time specification.[6]

On the analytical front, we show that *discretionary* optimal policy features a 'redistributive inflationary bias', whereby the utilitarian central bank uses current inflation so as to try and redistribute wealth from lenders to indebted households. In particular, we show that optimal discretionary inflation is determined by the following simple expression,

$$\overbrace{x'(\pi_t)}^{\text{marginal inflation cost}} = \mathbb{E}_{f_t(a,y)}\left[\overbrace{Q_t(-a)}^{\text{market value net liabilities}} \quad \overbrace{u'(c_t(a,y))}^{\text{marginal consumption utility}}\right], \tag{1}$$

[4] See Doepke and Schneider (2006a) for an influential study documenting net nominal asset positions across US household groups and estimating the potential for inflation-led redistribution. See Auclert (2016) for a recent analysis of the Fisherian redistributive channel in an incomplete-markets general equilibrium model that allows for additional redistributive mechanisms.

[5] We restrict our attention to equilibria in which the domestic economy is always a net debtor *vis-à-vis* the rest of the World, such that domestic bonds are always in positive net supply. As a result, the usual bond market clearing condition in closed-economy models is replaced by a no-arbitrage condition for foreign investors that effectively prices the nominal bond. This allows us to reduce the number of constraints in the policy-maker's problem featuring the infinite-dimensional wealth distribution.

[6] We show however how our methodology can also be applied in a discrete-time environment.

where $\mathbb{E}_{f_t(a,y)}[\cdot]$ denotes the average across real net wealth (a) and income (y) levels at time t, with joint distribution $f_t(\cdot)$, $u'(x')$ is the marginal (dis)utility of consumption (inflation), with $x'' > 0 > u''$, and Q_t is the price of the long-term bond. That is, optimal discretionary inflation increases with the average cross-household net liability position weighted by each household's marginal utility of consumption. Under market incompleteness and standard concave preferences for consumption, indebted households (those with $a < 0$) have a higher marginal consumption utility than lending ones ($a > 0$). As a result, they receive a higher effective weight in the optimal inflation decision, giving the central bank an incentive to redistribute wealth from lending to indebted households. To the best of our knowledge, this redistributive inflationary bias is a novel insight in the literature on incomplete markets models with uninsurable idiosyncratic risk. Moreover, while our model is deliberately simple –with the aim of illustrating our methodology as transparently as possible–, such inflationary bias would carry over to normative analyses in more fully-fledged models of this kind that incorporate a Fisherian channel.[7]

Under *commitment*, the same redistributive motive to inflate exists, but it is counteracted by an opposing force: the central bank internalizes how expectations of *future* inflation affect the price at which households issue new bonds from the time the optimal commitment plan is formulated ('time zero') onwards. Indeed, optimal inflation under commitment is driven by the same right-hand-side term in equation (1) *plus* a costate with zero initial value that increases with

$$\mathbb{E}_{f_t(a,y)}\left[a_t^{new}(a,y)\, u'(c_t(a,y))\right],$$

i.e. the average purchase of new bonds across households $a_t^{new}(\cdot)$ weighted again by marginal consumption utilities. In the model, the households that *issue* new bonds (those with $a_t^{new} < 0$) have lower net wealth and hence higher marginal utility than bond-purchasing ones ($a_t^{new} > 0$), so they receive a larger weight in the above expression. This gives the central bank an incentive to promise lower and lower inflation

[7] Our assumption that the country is always a net debtor *vis-à-vis* the rest of the World, i.e. $\mathbb{E}_{f_t(a,\cdot)}(-a) \geq 0$, implies an additional (*cross-border*) redistributive motive to inflate: from foreign investors to indebted domestic households. Our simulations show that both motives are quantitatively relevant for optimal inflation. Importantly, the *domestic* redistributive motive to inflate illustrated in equation (1) is preserved even with a zero net foreign asset position, due the concavity of preferences. Hence it would also go through in a closed-economy setup.

in the future so as to prevent bond issuers from doing so at very low prices.[8] This disinflationary force has too a redistributive motive, but unlike the aforementioned inflationary bias –which uses current inflation in order to favor indebted household– it relies on future inflation and bond prices so as to favor bond-issuing households (who largely coincide with the indebted ones). Moreover, we find that under certain conditions both forces –the inflationary and deflationary one– cancel each other out in the long-run, such that steady-state inflation under the optimal commitment is zero.[9]

We then solve numerically for the full transition path under commitment and discretion. We calibrate our model to match a number of features of a prototypical European small open economy, such as the size of gross household debt or the net international position.[10] We find that optimal inflation at time *zero* –which is very similar under commitment and discretion due to the absence of pre-commitments in the former case– is first-order in magnitude, reflecting the above mentioned redistributive motive. From time zero onwards, inflation remains high under discretion due to the redistributive inflationary bias. Under commitment, by contrast, inflation falls gradually towards its long-run level (essentially zero, under our calibration), reflecting the central bank's concern with preventing expectations of future inflation from being priced into new bond issuances; in other words, the central bank front-loads inflation so as to transitorily redistribute existing wealth from lenders to indebted households, but commits to gradually undo such initial inflation.

We also analyze the *redistributive* effects of optimal policy. We show that, relative to a zero-inflation scenario, inflationary policies (whether under discretion or commitment) redistribute consumption from lending to indebted households. A key channel through which this redistribution takes place is the fact that future inflation reduces the price of the long-term bond, which reduces the real market value of bond holdings for lending households and that of liabilities for indebted ones. These effects find an echo in the *welfare* analysis. The discretionary policy implies sizable

[8]This incentive to commit to low future inflation has again an additional, cross-border dimension, because the domestic economy as a whole is a net issuer of new bonds.

[9]In particular, in the limiting case in which households' (and hence the benevolent central bank's) discount rate is arbitrarily close to that of foreign investors, optimal steady-state inflation under commitment is arbitrarily close to zero.

[10]These targets are used to inform the calibration of the gap between the central bank's and foreign investors' discount rates, which as explained before is a key determinant of long-run inflation under commitment.

(first-order) losses relative to the optimal commitment. Such losses are suffered by lending households, but also by *indebted* ones, because the welfare costs of permanent inflation dominate the gains from increased consumption.

Finally, we compute the optimal monetary policy response to an aggregate shock, such as an increase in the World real interest rate.[11] We find that inflation rises slightly on impact, as the central bank tries to partially counteract the negative effect of the shock on household consumption. However, the inflation reaction is an order of magnitude smaller than that of the shock itself. Intuitively, the value of sticking to past commitments to keep inflation near zero weighs more in the central bank's decision than the value of using inflation transitorily so as to stabilize consumption in response to an unforeseen event.

Overall, our findings shed some light on current policy and academic debates regarding the appropriate conduct of monetary policy once household heterogeneity is taken into account. In particular, our results suggest that, while some inflation may be justified in the short-run so as to redistribute resources to households with higher marginal utilities, a central bank with the ability to commit should not sustain such an inflationary stance –as it would if it acted under discretion–, but should instead promise to undo it over time, precisely in order to favor the same households.

Finally, we stress that our results are *not* meant to suggest that monetary policy is the best tool to address redistributive issues, as there are probably more direct policy instruments. What our results indicate is that, in the context of economies with uninsurable idiosyncratic risk, the optimal design of monetary policy will to some extent reflect redistributive motives, the more so the less other policies (e.g. fiscal policy) are able to achieve optimal redistributive outcomes.

Related literature. Our first main contribution is methodological. To the best of our knowledge, ours is the first paper to solve for a fully dynamic optimal policy problem, both under commitment and discretion, in a general equilibrium model with uninsurable idiosyncratic risk in which the cross-sectional net wealth distribution (an infinite-dimensional, endogenously evolving object) is a state in the planner's optimization problem. Different papers have analyzed Ramsey problems in similar setups. Dyrda and Pedroni (2014) study the optimal dynamic Ramsey taxation in an Aiyagari economy. They assume that the paths for the optimal taxes follow splines

[11] In the analysis of aggregate shocks we focus on the commitment case, and in particular on the optimal commitment plan 'from a timeless perspective.'

with nodes set at a few exogenously selected periods, and perform a numerical search of the optimal node values. Acikgoz (2014), instead, follows the work of Dávila et al. (2012) in employing calculus of variations to characterize the optimal Ramsey taxation in a similar setting. However, after having shown that the optimal long-run solution is independent of the initial conditions, he analyzes quantitatively the steady state but does not solve the full dynamic optimal path.[12] Other papers, such as Gottardi, Kajii, and Nakajima (2011), Itskhoki and Moll (2015), Bilbiie and Ragot (2017), Le Grand and Ragot (2017) or Challe (2017), analyze optimal Ramsey policies in incomplete-market models in which the policy-maker does not need to keep track of the wealth distribution.[13] In contrast to these papers, we introduce a methodology for computing the fully dynamic, nonlinear optimal policy under commitment in an incomplete markets setting where the policy-maker needs to keep track of the entire wealth distribution. Regarding discretion, we are not aware of any previous paper that has quantitatively analyzed it in models with uninsurable idiosyncratic risk.

A recent paper by Bhandari et al. (2017), released after the first draft of this paper was circulated, analyzes optimal fiscal and monetary policy with commitment in a heterogeneous agents New Keynesian environment with aggregate uncertainty. Their methodology differs from ours in two main dimensions. First, they employ a local method (second-order perturbations), in contrast to the global method presented here. Second, their methodology cannot address problems with exogenous, occasionally binding borrowing limits such as those used in models *à la* Aiyagari-Bewley-Huggett, which are precisely the focus of our paper.

The use of infinite-dimensional calculus in problems with non-degenerate distributions is employed in Lucas and Moll (2014) and Nuño and Moll (2017) to find the first-best and the constrained-efficient allocation in heterogeneous-agents models. In these papers a social planner directly decides on individual policies in order to control a distribution of agents subject to idiosyncratic shocks. Here, by contrast, we show how these techniques may be extended to game-theoretical settings involving several

[12] Werning (2007) studies optimal fiscal policy in a heterogeneous-agents economy in which agent types are permanently fixed. Park (2014) extends this approach to a setting of complete markets with limited commitment in which agent types are stochastically evolving. Both papers provide a theoretical characterization of the optimal policies based on the primal approach introduced by Lucas and Stokey (1983). Aditionally, Park (2014) analyzes numerically the steady state but not the transitional dynamics, due to the complexity of solving the latter problem with that methodology.

[13] This is due either to particular assumptions that facilitate aggregation or to the fact that the equilibrium net wealth distribution is degenerate at zero.

agents who are moreover forward-looking.[14] Under commitment, as is well known, this requires the policy-maker to internalize how her promised future decisions affect private agents' expectations; the problem is then augmented by introducing costates that reflect the value of deviating from the promises made at time zero.[15]

The second main contribution of the paper relates to our normative insights on monetary policy. A recent literature addresses, from a *positive* perspective, the redistributive channels of monetary policy transmission in the context of general equilibrium models with incomplete markets and household heterogeneity. In terms of modelling, our paper is closest to Auclert (2016), Kaplan, Moll and Violante (2016), Gornemann, Kuester and Nakajima (2012), McKay, Nakamura and Steinsson (2016) or Luetticke (2015), who also employ different versions of the incomplete markets model with uninsurable idiosyncratic risk.[16] Other contributions, such as Doepke and Schneider (2006b), Meh, Ríos-Rull and Terajima (2010), Sheedy (2014), Challe et al. (2017) or Sterk and Tenreyro (2015), analyze the redistributive effects of monetary policy in environments where heterogeneity is kept finite-dimensional. We contribute to this literature by analyzing *optimal* monetary policy, both under commitment and discretion, in an economy with uninsurable idiosyncratic risk.[17]

As explained before, our analysis assigns an important role to the Fisherian redistributive channel of monetary policy, a long-standing topic that has experienced a revival in recent years. Doepke and Schneider (2006a) document net nominal asset positions across US sectors and household groups and estimate empirically the redistributive effects of different inflation scenarios; Adam and Zhu (2014) perform a similar analysis for Euro Area countries. Auclert (2016) analyzes several redistribu-

[14] This relates to the literature on *mean-field games* in mathematics. The name, introduced by Lasry and Lions (2006a,b), is borrowed from the mean-field approximation in statistical physics, in which the effect on any given individual of all the other individuals is approximated by a single averaged effect. In particular, the case under commitment is loosely related to Bensoussan, Chau and Yam (2015), who analyze a model of a major player and a distribution of atomistic agents.

[15] In the commitment case, we construct a Lagrangian in a suitable function space and obtain the corresponding first-order conditions. The resulting optimal policy is time inconsistent (reflecting the effect of investors' inflation expectations on bond pricing), depending only on time and the initial wealth distribution.

[16] For work studying the effects of different aggregate shocks in related environments, see e.g. Guerrieri and Lorenzoni (2017), Ravn and Sterk (2013), and Bayer et al. (2015).

[17] Although this paper focuses on monetary policy, the techniques developed here lend themselves naturally to the analysis of other policy problems, e.g. optimal fiscal policy, in this class of models. Recent work analyzing fiscal policy issues in incomplete-markets, heterogeneous-agent models includes Heathcote (2005), Oh and Reis (2012), Kaplan and Violante (2014) and McKay and Reis (2016).

tive channels, including the Fisherian one, using both a sufficient statistics approach and an incomplete-markets model. We show how, in a model with uninsurable idiosyncratic risk featuring long-term nominal debt and costly inflation, a utilitarian central bank would want to exploit the Fisherian channel to improve aggregate welfare. In doing so, we uncover a 'redistributive inflationary bias', as the central bank attempts to redistribute wealth from lending to indebted households, who have a higher marginal utility of consumption. We also find that, under commitment, such bias is counteracted by a disinflationary force that has too a redistributive motive: the central bank promises lower inflation going forward in order to favor bond-issuing households, who largely coincide with the indebted ones. We argue that these redistributive forces would carry over to more fully-fledged incomplete-markets models that incorporate the above channels.

2 Model

We extend the basic Huggett framework to an open-economy setting with nominal, non-contingent, long-term debt and disutility costs of inflation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space. Time is continuous: $t \in [0, \infty)$. The domestic economy is composed of a measure-one continuum of households. There is a single, freely traded consumption good, the World price of which is normalized to 1. The domestic price (equivalently, the nominal exchange rate) at time t is denoted by P_t and evolves according to

$$dP_t = \pi_t P_t dt, \tag{2}$$

where π_t is the domestic inflation rate (equivalently, the rate of nominal exchange rate depreciation).

2.1 Households

2.1.1 Income and net assets

Household $k \in [0, 1]$ is endowed at time t with y_{kt} units of the good, where y_{kt} follows a two-state Poisson process: $y_{kt} \in \{y_1, y_2\}$, with $y_1 < y_2$. The process jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2.

Households trade nominal, noncontingent, long-term bonds (denominated in domestic currency) with one another and with foreign investors. Following standard practice in the literature, we model long-term debt in a tractable way by assuming that bonds pay exponentially decaying coupons.[18] In particular, a bond issued at time t promises a stream of nominal payments $\left\{\delta e^{-\delta(s-t)}\right\}_{s\in(t,\infty)}$, totalling 1 unit of domestic currency over the (infinite) life of the bond. Thus, from the point of view of time t, a bond issued at $\tilde{t} < t$ is equivalent to $e^{-\delta\left(t-\tilde{t}\right)}$ newly issued bonds. This implies that a household's entire bond portfolio can be summarized by the current total nominal coupon payment, which we denote by δA_{kt}. One can then interpret δ as the 'amortization rate' and A_{kt} as the nominal face value of the bond portfolio. The latter evolves according to

$$dA_{kt} = \left(A_{kt}^{new} - \delta A_{kt}\right) dt,$$

where A_{kt}^{new} represents the face value of the flow of new bonds purchased at time t. For households with a negative net position, $(-)\,A_{kt}$ represents the face value of outstanding net liabilities ('debt' for short). Our formulation also implies that at each t one need only consider the price of one bond cohort, e.g. newly issued bonds. Let Q_t denote the nominal market price of bonds issued at time t. The budget constraint of household k is then

$$Q_t A_{kt}^{new} = P_t\left(y_{kt} - c_{kt}\right) + \delta A_{kt},$$

where c_{kt} is the household's consumption. Combining the last two equations, we obtain the following dynamics for the nominal face value of net wealth,

$$dA_{kt} = \left(\frac{\delta A_{kt} + P_t\left(y_{kt} - c_{kt}\right)}{Q_t} - \delta A_{kt}\right) dt. \tag{3}$$

We define the *real* face value of net wealth as $a_{kt} \equiv A_{kt}/P_t$. Its dynamics are obtained by applying Itô's lemma to equations (2) and (3),

$$da_{kt} = \left[\frac{\delta a_{kt} + y_{kt} - c_{kt}}{Q_t} - \left(\delta + \pi_t\right) a_{kt}\right] dt, \tag{4}$$

where $\frac{\delta a_{kt}+y_{kt}-c_{kt}}{Q_t} = A_{kt}^{new}/P_t \equiv a_{kt}^{new}$ is the real face value of new bonds acquired at

[18] Ever since Woodford (2001), bonds with exponentially decaying coupons have become common as a tractable way of modelling long-term debt in macroeconomic analyses. For a recent example, see e.g. Auclert (2016).

t. We assume that each household faces the following exogenous borrowing limit,

$$a_{kt} \geq \phi. \tag{5}$$

where $\phi \leq 0$.

2.1.2 Preferences

Household have preferences over paths for consumption c_{kt} and domestic inflation π_t discounted at rate $\rho > 0$,

$$\mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[u(c_{kt}) - x\left(\pi_t\right) \right] dt \right\}.$$

The consumption utility function u is bounded and continuous, with $u' > 0$, $u'' < 0$ for $c > 0$. The inflation disutility function x satisfies $x' > 0$ for $\pi > 0$, $x' < 0$ for $\pi < 0$, $x'' > 0$ for all π, and $x(0) = x'(0) = 0$.[19] From now onwards we drop subscripts k for ease of exposition. The household chooses consumption at each point in time in order to maximize its welfare. The *value function* of the household at time t can be expressed as

$$v_t(a, y) = \max_{\{c_s\}_{s \in [t, \infty)}} \mathbb{E}_t \left\{ \int_t^\infty e^{-\rho(s-t)} \left[u(c_s) - x\left(\pi_s\right) \right] ds \right\},$$

subject to the law of motion of net wealth (4) and the borrowing limit (5). We use the shorthand notation $v_{it}(a) \equiv v(a, y_i)$ for the value function when household income is low ($i = 1$) and high ($i = 2$). The *Hamilton-Jacobi-Bellman* (HJB) equation corresponding to the problem above is

$$\rho v_{it}(a) = \frac{\partial v_{it}}{\partial t} + \max_c \left\{ u(c) - x\left(\pi_t\right) + \mathbf{s}_{it}\left(a, c\right) \frac{\partial v_{it}}{\partial a} \right\} + \lambda_i \left[v_{jt}(a) - v_{it}(a) \right], \tag{6}$$

for $i, j = 1, 2$, and $j \neq i$, where $\mathbf{s}_{it}\left(a, c\right)$ is the *drift* function, given by

$$\mathbf{s}_{it}\left(a, c\right) \equiv \frac{\delta a + y_i - c}{Q_t} - \left(\delta + \pi_t\right) a, \tag{7}$$

[19] This specification of disutility costs of inflation nests the case of quadratic costly price adjustments *à la* Rotemberg (1982). See Section 4.1 for further discussion.

$i = 1, 2$. The first order condition for consumption is

$$u'(c_{it}(a)) = \frac{1}{Q_t} \frac{\partial v_{it}(a)}{\partial a}, \tag{8}$$

where $c_{it}(a) \equiv c(a, y_i)$. Therefore, household consumption increases with nominal bond prices and falls with the slope of the value function. Intuitively, a higher bond price (equivalently, a lower yield) gives the household an incentive to save less and consume more. A steeper value function, on the contrary, makes it more attractive to save so as to increase net bond holdings.

We close this section by establishing the following result.

Lemma 1 *The household value function $v_{it}(a)$ is strictly concave.*

The proofs of all lemmas and propositions can be found in Appendix A. Lemma 1, together with equation (8), imply that $\partial u'/\partial a < 0$, i.e. marginal consumption utility falls with net wealth.

2.2 Foreign investors

Households trade bonds with competitive risk-neutral foreign investors that can invest elsewhere at the risk-free real rate $\bar{r}$. As explained before, bonds are amortized at rate δ. Foreign investors also discount future nominal payoffs with the accumulated domestic inflation (i.e. exchange rate depreciation) between the time of the bond purchase and the time such payoffs accrue. Therefore, the nominal price of the bond at time t is given by

$$Q_t = \int_t^\infty \delta e^{-(\bar{r}+\delta)(s-t) - \int_t^s \pi_u du} ds. \tag{9}$$

Taking the derivative with respect to time, we obtain

$$Q_t(\bar{r} + \delta + \pi_t) = \delta + \dot{Q}_t, \tag{10}$$

where $\dot{Q}_t \equiv dQ_t/dt$. The partial differential equation (10) provides the risk-neutral pricing of the nominal bond. The boundary condition is $\lim_{T\to\infty} e^{-(\bar{r}+\delta)T - \int_0^T \pi_u du} Q_T = 0$. The steady state bond price is $Q_\infty = \frac{\delta}{\bar{r}+\delta+\pi_\infty}$, where π_∞ is the inflation level in the steady state.[20]

[20] Given the nominal bond price Q_t, the *bond yield* r_t implicit in that price is defined as the discount rate for which the discounted future promised cash flows equal the bond price. The discounted future

2.3 Central Bank

There is a central bank that chooses monetary policy. We assume that there are no monetary frictions so that the only role of money is that of a unit of account. The monetary authority chooses the inflation rate π_t. This could be done, for example, by setting the nominal interest rate on a lending (or deposit) short-term nominal facility with foreign investors. In Section 3, we will study in detail the optimal inflationary policy of the central bank.

2.4 Competitive equilibrium

The state of the economy at time t is the joint density of net wealth and income, $f_t(a, y) \equiv \{f_t(a, y_i)\}_{i=1}^2 \equiv \{f_{it}(a)\}_{i=1}^2$. Let $\mathbf{s}_{it}(a, c_{it}(a)) \equiv s_{it}(a)$ be the drift of individual real net wealth evaluated at the optimal consumption policy. The dynamics of the net wealth-income density are given by the *Kolmogorov Forward* (KF) equation,

$$\frac{\partial f_{it}(a)}{\partial t} = -\frac{\partial}{\partial a}\left[s_{it}(a) f_{it}(a)\right] - \lambda_i f_{it}(a) + \lambda_j f_{jt}(a), \tag{11}$$

$a \in [\phi, \infty)$, $i, j = 1, 2$, $j \neq i$. The density satisfies the normalization

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} f_{it}(a)\, da = 1. \tag{12}$$

We define a competitive equilibrium in this economy.

Definition 1 (Competitive equilibrium) *Given a sequence of inflation rates π_t and an initial net wealth-income density $f_0(a, y)$, a competitive equilibrium is composed of a household value function $v_t(a, y)$, a consumption policy $c_t(a, y)$, a bond price function Q_t and a density $f_t(a, y)$ such that:*

1. *Given π, the price of bonds in (10) is Q.*

2. *Given Q and π, v is the solution of the households' problem (6) and c is the optimal consumption policy.*

3. *Given Q, π, and c, f is the solution of the KF equation (11).*

promised payments are $\int_0^\infty e^{-(r_t+\delta)s}\delta ds = \delta/(r_t + \delta)$. Therefore, the bond yield is $r_t = \delta/Q_t - \delta$.

Notice that, given π, the problem of foreign investors can be solved independently of that of the household, which in turn only depends on π and Q but not on the aggregate distribution.

We henceforth use the notation

$$\mathbb{E}_{f_t(a,y)}\left[g_t\left(a,y\right)\right] \equiv \sum_{i=1}^{2} \int_{\phi}^{\infty} g_t\left(a,y_i\right) f_t\left(a,y_i\right) da$$

to denote the cross-household average at time t of any function g_t of individual net wealth and income levels, or equivalently the aggregate value of such a function (given that the household population is normalized to 1). We can define some aggregate variables of interest. The aggregate real face value of net wealth in the economy is $\bar{a}_t \equiv \mathbb{E}_{f_t(a,y)}[a]$. Aggregate consumption is $\bar{c}_t \equiv \mathbb{E}_{f_t(a,y)}\left[c_t\left(a,y\right)\right]$, and aggregate income is $\bar{y}_t \equiv \mathbb{E}_{f_t(a,y)}[y]$. These quantities are linked by the current account identity,[21]

$$\frac{d\bar{a}_t}{dt} = \frac{\delta \bar{a}_t + \bar{y}_t - \bar{c}_t}{Q_t} - \left(\delta + \pi_t\right) \bar{a}_t \equiv \bar{a}_t^{new} - \left(\delta + \pi_t\right) \bar{a}_t, \tag{13}$$

For future reference, we may also define the real face value of gross household debt, $\bar{b}_t \equiv \sum_{i=1}^{2} \int_{\phi}^{0} \left(-a\right) f_{it}\left(a\right) da$.

We make the following assumption.

Assumption 1 *The value of parameters is such that in equilibrium the economy is always a net debtor against the rest of the World: $\bar{a}_t \leq 0$ for all t.*

This condition is imposed for tractability. We have restricted domestic households to save only in bonds issued by other domestic households, and this would not be possible if the country was a net creditor *vis-à-vis* the rest of the World. In addition to this, we have assumed that the bonds issued by the households are priced by foreign investors, which requires that there should be a positive net supply of bonds to the rest of the World to be priced. In any case, this assumption is consistent with the experience of the small open economies that we target for calibration purposes, as we explain in Section 4.

[21] The derivation of equation (13) is available upon request.

3 Optimal monetary policy

We now turn to the design of the optimal monetary policy. Following standard practice, we assume that the central bank is utilitarian, i.e. it gives the same Pareto weight to each household. In order to illustrate the role of commitment vs. discretion in our framework, we will consider both the case in which the central bank can credibly commit to a future inflation path (the Ramsey problem), and the time-consistent case in which the central bank decides optimal current inflation given the current state of the economy (the Markov Stackelberg equilibrium).

Before starting the formal analysis, it is worthwhile to emphasize the two key transmission channels of inflation in our model. First, as shown in equation (7), *current* inflation π_t erodes the real face value of households' net bond holdings through a classic Fisherian effect, which benefits currently indebted households (those with $a < 0$ at time t) and vice versa for currently lending ones ($a > 0$). Second, from the bond pricing condition (9), *future* inflation $\{\pi_s\}_{s\in(t,\infty)}$ lowers the nominal price of the long-term bond Q_t. This, from equation (7), allows households with a positive saving flow ($\delta a + y_i - c_t(a, y_i) > 0$) to purchase more *new* bonds, and forces bond-issuing households (those with $\delta a + y_i - c_t(a, y_i) < 0$) to do so at lower prices and thus increase their indebtedness. Crucially, a central bank that is able to credibly commit to a future inflation path will take both effects into account. By contrast, a discretionary central bank will only consider the Fisherian effect.

3.1 Central bank preferences

The central bank is assumed to be benevolent and hence maximizes economy-wide aggregate welfare, defined as

$$W_0 \equiv \mathbb{E}_{f_0(a,y)}\left[v_0(a,y)\right]. \tag{14}$$

It will turn out to be useful to express the above welfare criterion as follows.

Lemma 2 *The welfare criterion (14) can alternatively be expressed as*

$$W_0 = \int_0^\infty e^{-\rho t}\mathbb{E}_{f_t(a,y)}\left[u\left(c_t(a,y)\right) - x(\pi_t)\right]dt.$$

3.2 Commitment

Consider first the case in which the central bank credibly commits at time zero to an inflation path $\{\pi_t\}_{t\in[0,\infty)}$. The optimal inflation path is then a function of the initial distribution $f_0(a,y)$ and of time: $\pi_t \equiv \pi^R[f_0(\cdot),t]$. The value functional of the central bank is given by

$$W^R[f_0(\cdot)] = \max_{\{\pi_t,Q_t,v_t(\cdot),c_t(\cdot),f_t(\cdot)\}_{t\in[0,\infty)}} \int_0^\infty e^{-\rho t}\mathbb{E}_{f_t(a,y)}\left[u\left(c_t(a,y)\right) - x(\pi_t)\right] dt, \quad (15)$$

subject to the law of motion of the distribution (11), the bond pricing equation (10), and households' HJB equation (6) and optimal consumption choice (8). Notice that the optimal value W^R and the optimal policy π^R are not ordinary functions, but *functionals*, as they map the infinite-dimensional initial distribution $f_0(\cdot)$ into $\mathbb{R}$. The central bank maximizes welfare taking into account not only the state dynamics (11), but *also* the households' HJB equation (6) and the investors' bond pricing condition (10), both of which are forward-looking. That is, the central bank understands how it can steer households' and foreign investors' expectations by committing to an inflation path.

Definition 2 (Ramsey problem) *Given an initial distribution f_0, a* Ramsey problem *is composed of a sequence of inflation rates π_t, a household value function $v_t(a,y)$, a consumption policy $c_t(a,y)$, a bond price function Q_t and a distribution $f_t(a,y)$ such that they solve the central bank problem (15).*

If v, f, c and Q are a solution to the problem (15), given π, they constitute a competitive equilibrium, as they satisfy equations (11), (10), (6) and (8). Therefore the Ramsey problem could be redefined as that of finding the π such that v, f, c and Q are a competitive equilibrium and the central bank's welfare criterion is maximized.

The above Ramsey problem is an optimal control problem in a suitable function space. In order to solve this problem, we construct a Lagrangian in such a space. In

Appendix A, we show that the Lagrangian $\mathcal{L}[\pi, Q, f, v, c] \equiv \mathcal{L}_0$ is given by

$$\begin{aligned}
\mathcal{L}_0 \equiv & \int_0^\infty e^{-\rho t} \sum_{i=1}^{2} \int_\phi^\infty \{ [u(c_{it}(a)) - x(\pi_t)] f_{it}(a) \\
& + \zeta_{it}(a) \left[-\frac{\partial f_{it}(a)}{\partial t} - \frac{\partial}{\partial a}[s_{it}(a) f_{it}(a)] - \lambda_i f_{it}(a) + \lambda_j f_{j,t}(a) \right] \\
& + \theta_{it}(a) \left[\frac{\partial v_{it}}{\partial t} + u(c_{it}(a)) - x(\pi_t) + s_{it}(a) \frac{\partial v_{it}}{\partial a} + \lambda_i [v_{jt}(a) - v_{it}(a)] - \rho v_{it}(a) \right] \\
& + \eta_{it}(a) \left[u'(c_{it}(a)) - \frac{1}{Q_t} \frac{\partial v_{it}}{\partial a} \right] \} da dt \\
& + \int_0^\infty e^{-\rho t} \mu_t \left[Q_t (\bar{r} + \pi_t + \delta) - \delta - \dot{Q}_t \right] dt,
\end{aligned}$$

where $j = 1, 2, j \neq i$. We then obtain first-order conditions with respect to the functions π, Q, f, v, c by taking *Gateaux derivatives*, which extend the concept of derivative from $\mathbb{R}^n$ to infinite-dimensional spaces.[22] As an example, the Gateaux derivative with respect to the density $f_t(a, y)$ is

$$\lim_{\alpha \to 0} \frac{\mathcal{L}[f + \alpha h, \cdot] - \mathcal{L}[f, \cdot]}{\alpha} = \left. \frac{d}{d\alpha} \mathcal{L}[f + \alpha h, \cdot] \right|_{\alpha=0},$$

where $h_t(a, y)$ is an arbitrary function in the same function space as $f_t(a, y)$. The first-order conditions require that the Gateaux derivatives should be zero for *any* function $h_t(a, y)$.

In the appendix we show that in equilibrium the Lagrange multiplier $\zeta_{it}(a)$ associated with the KF equation (11), which represents the social value of an individual household, coincides with the private value $v_{it}(a)$.[23] In addition, the Lagrange multipliers $\theta_{it}(a)$ and $\eta_{it}(a)$ associated with the households' HJB equation (6) and first-order condition (8), respectively, are both zero. That is, households' forward-looking optimizing behavior does not represent a source of time-inconsistency, as the monetary authority would choose at all times the same individual consumption and saving policies as the households themselves. Therefore, the only nontrivial Lagrange

[22] The general definition of Gateaux derivative is shown in Appendix A.

[23] One of the advantages of our small-open-economy formulation is that the social value of a household coincides with its private value. In the closed-economy version of the model this would not be the case, making the computations more complex, but still tractable.

multiplier is μ_t, the one associated with the bond pricing equation (10).[24]

The following proposition characterizes the solution to this problem.

Proposition 1 (Optimal inflation - Ramsey) *In addition to equations (11), (10), (6) and (8), if a solution to the Ramsey problem (15) exists, the inflation path π_t must satisfy*

$$x'(\pi_t) = \mathbb{E}_{f_t(a,y)}\left[Q_t(-a)\,u'(c_t(a,y))\right] + \mu_t Q_t, \tag{16}$$

where μ_t is a costate with law of motion

$$\frac{d\mu_t}{dt} = (\rho - \bar{r} - \pi_t - \delta)\,\mu_t - \mathbb{E}_{f_t(a,y)}\left[-a_t^{new}(a,y)\,u'(c_t(a,y))\right], \tag{17}$$

and initial condition $\mu_0 = 0$, where $a_t^{new}(a,y) \equiv \frac{\delta a + y - c_t(a,y)}{Q_t}$.

Equation (16) determines optimal inflation under commitment. According to this equation, marginal inflation disutility x' (which is increasing in inflation) equals the sum of two terms. The first term, $\mathbb{E}_{f_t(\cdot)}\{Q_t(-a)\,u'(c_t(\cdot))\}$, is the average across households of the real market value of net liabilities, $Q_t(-a)$, weighted by each household's marginal utility of consumption, u'. It captures the marginal effect of inflation on social welfare through its impact on the real value of net nominal positions. For indebted households ($a < 0$), the latter effect is positive as inflation erodes the real value of their debt burden, whereas the opposite is true for lending ones ($a > 0$). Crucially for our purposes, this term reflects the central bank's incentive to inflate for *redistributive* purposes, which in our model is double. On the one hand, under Assumption 1 the country is always a net debtor ($\mathbb{E}_{f_t(\cdot)}(-a) \geq 0$), giving the central bank a motive to redistribute wealth from foreign investors to domestic borrowers (*cross-border redistribution*). On other hand, and perhaps more interestingly, the concavity of preferences implies that indebted households have a higher marginal utility of consumption u' than lending ones. Thus, even if the country has a zero net position *vis-à-vis* the rest of the World, as long as there is dispersion in net wealth the central bank has a reason to redistribute from indebted to lending households (*domestic redistribution*).

[24] Importantly, these techniques are not restricted to continuous-time problems. In fact, the equivalent discrete-time model can also be solved using the same techniques at the cost of somewhat less elegant expressions. Appendix E shows how our methodology can be used to solve for the optimal policy under commitment in the discrete-time version of our model.

The second term on the right-hand side of equation (16) captures the value to the central bank of promises about time-t inflation made to foreign investors at time 0. The costate μ_t is zero at the time of announcing the Ramsey plan ($t = 0$), because the central bank is not bound by previous commitments. From then on, it evolves according to equation (17). In the latter equation, the term $\mathbb{E}_{f_t(\cdot)}\left\{-a_t^{new}(\cdot)\, u'(c_t(\cdot))\right\}$ is the cross-household average of the real face value of new bond *issuances* –with $a_t^{new}(\cdot)$ denoting purchases of new bonds–, weighted again by the marginal utility of consumption. Intuitively, the central bank understands that a commitment to higher inflation in the future lowers bond prices today, which reduces welfare for those households that need to *sell* new bonds ($a_t^{new} < 0$) and vice versa for those that purchase new bonds ($a_t^{new} > 0$). If the former households have a higher marginal utility u' than the latter ones, then μ_t should become more and more *negative* over time.[25] From equation (16), this would give the central bank an incentive to lower inflation over time, thus tempering the redistributive motive to inflate discussed above.[26]

We now establish an important result regarding the *long-run* level of optimal inflation under commitment.

Proposition 2 (Optimal long-run inflation under commitment) *In the limit as $\rho \to \bar{r}$, the optimal steady-state inflation rate under commitment tends to zero: $\lim_{\rho \to \bar{r}} \pi_\infty = 0$.*

That is, provided households' discount factor (and hence that of the benevolent central bank) is arbitrarily close to that of foreign investors, then optimal long-run inflation under commitment will be arbitrarily close to zero. The intuition is the following. As explained before, at each point in time the optimal inflation under commitment reflects the tension between two forces: current inflation helps currently indebted households, but past expectations of such inflation hurts past issuers of the long-term bond by lowering the price at which they do so. In the long run, both forces cancel each other out at zero inflation for the case of ρ arbitrarily close to $\bar{r}$.

[25] Indeed this will be the case in our numerical analysis.

[26] Notice that the Ramsey problem is not time-consistent, due precisely to the presence of the (forward-looking) bond pricing condition in that problem. If at some future point in time $t > 0$ the central bank decided to reoptimize given the state at that point, $f_t(\cdot)$, the new path for optimal inflation would not need to coincide with the original path, as the costate at that point would be $\mu_t = 0$ (corresponding to a new commitment formulated at time t), whereas under the original commitment it is $\mu_t \neq 0$.

Proposition 2 is reminiscent of a well-known result from the New Keynesian literature, namely that optimal long-run inflation in the standard New Keynesian framework is exactly zero (see e.g. Benigno and Woodford, 2005). In that framework, the optimality of zero long-run inflation arises from the fact that, at that level, the welfare gains from trying to exploit the short-run output-inflation trade-off (i.e. raising output towards its socially efficient level) exactly cancel out with the welfare losses from permanently worsening that trade-off (through higher inflation expectations). Key to that result is the fact that, in that model, price-setters and the (benevolent) central bank have the same (steady-state) discount factor. Here, the optimality of zero long-run inflation reflects instead the fact that, provided the discount rate of the investors pricing the bonds is arbitrarily close to that of the central bank, the aggregate welfare gains from trying to redistribute wealth from creditors to debtors becomes arbitrarily close to the aggregate welfare losses from lowering the price of new bond issuances.

Assumption 1 restricts us to have $\rho > \bar{r}$, as otherwise households would we able to accumulate enough wealth so that the country would stop being a net debtor to the rest of the World. However, Proposition 2 provides a useful benchmark to understand the long-run properties of optimal policy in our model when ρ is close to $\bar{r}$. This will indeed be the case in our numerical analysis.

3.3 Discretion

Assume now that the central bank cannot commit to any future policy. The inflation rate π at each point in time then depends only on the value at that point in time of the aggregate state variable, the net wealth-income distribution $f_t(a, y)$; that is, $\pi_t \equiv \pi^M [f_t(\cdot)]$. This is a *Markov* (or *feedback*) *Stackelberg* equilibrium in a space of distributions.[27] As explained by Basar and Olsder (1999, pp. 413-417), a continuous-time feedback Stackelberg solution can be defined as the limit as $\Delta t \to 0$ of a sequence of problems in which the central bank chooses policy in each interval $(t, t + \Delta t]$ but not across intervals.[28] Formally, the value functional of the central bank at time t is

[27] Finite-dimensional Markov Stackelberg equilibria have been analyzed in the dynamic game theory literature, both in continuous and discrete time. See e.g. Basar and Olsder (1999) and references therein. In macroeconomics, an example of Markov Stackelberg equilibrium is Klein, Krusell, and Ríos-Rull (2008)

[28] In particular, for any arbitrary $T > 0$, we divide the interval $[0, T]$ in subintervals of the form $[0, \Delta t] \cup (\Delta t, 2\Delta t] \cup ...((N-1)\Delta t, N\Delta t]$, where $N \equiv T/\Delta t$.

given by

$$W^M[f_t(\cdot)] = \lim_{\Delta t \to 0} W^M_{\Delta t}[f_t(\cdot)],$$

where

$$\begin{aligned} W^M_{\Delta t}[f_t(\cdot)] &= \max_{\{\pi_s, Q_s, v_s(\cdot), c_s(\cdot), f_s(\cdot)\}_{s\in(t,t+\Delta t]}} \int_t^{t+\Delta t} e^{-\rho(s-t)} \mathbb{E}_{f_s(a,y)}[u(c_s(a,y)) - x(\pi_s)]\, ds \\ &\quad + e^{-\rho \Delta t} W^M_{\Delta t}[f_{t+\Delta t}(\cdot)], \end{aligned} \tag{18}$$

subject to the law of motion of the distribution (11), the bond pricing equation (10), and household's HJB equation (6) and optimal consumption choice (8). Notice, as in the case with commitment, that the optimal value W^M and the optimal policy π^M are not ordinary functions, but *functionals*, as they map the infinite-dimensional state variable $f_t(a,y)$ into $\mathbb{R}$.

Definition 3 (Markov Stackelberg equilibrium) *Given an initial distribution* f_0*, a* Markov Stackelberg equilibrium *is composed of a sequence of inflation rates* π_t*, a household value function* $v_t(a,y)$*, a consumption policy* $c_t(a,y)$*, a bond price function* Q_t *and a distribution* $f_t(a,y)$ *such that they solve the central bank problem (18).*

The following proposition characterizes the solution to the central bank's problem under discretion.

Proposition 3 (Optimal inflation - Markov Stackelberg) *In addition to equations (11), (10), (6) and (8), if a solution to the Markov Stackelberg problem (18) exists, the inflation rate function* π_t *must satisfy*

$$x'(\pi_t) = \mathbb{E}_{f_t(a,y)}[Q_t(-a)\, u'(c_t(a,y))]. \tag{19}$$

Our approach is to solve the problem in (18) following a similar approach as in the Ramsey problem above but taking into account how the policies in the current time interval affect the continuation value in the next time interval, as represented by the value functional $W^M_{\Delta t}[f_{t+\Delta t}(\cdot)]$ at time $t+\Delta t$. Then we take the limit as $\Delta t \to 0$.

In contrast to the case with commitment, in the Markov Stackelberg equilibrium no promises can be made at any point in time, hence the value of the costate (the term μ_t in equation 16) is zero at all times. Therefore, in equation (19) there is only a static trade-off between the aggregate welfare cost of inflation and the aggregate welfare gain from redistributing wealth. Thus, under discretion inflation is driven exclusively by the redistributive motive to inflate, as captured by the right-hand side of equation (19). In fact, it is possible to establish the existence of an *inflationary bias* under discretionary optimal monetary policy.

Proposition 4 (Redistributive inflationary bias under discretion) *Optimal inflation under discretion is positive at all times: $\pi_t > 0$ for all $t \geq 0$.*

The formal proof can be found in Appendix A, although the result follows quite directly from equation (19). Notice first that, from Assumption 1, the country as a whole is a net debtor: $\mathbb{E}_{f_t(a,y)}(-a) = -\bar{a}_t \geq 0$. Moreover, the strict concavity of preferences implies that indebted households ($a < 0$) have a higher marginal consumption utility u' than lending ones ($a > 0$) and hence effectively receive more weight in the inflation decision. Taking both things together, we have that the right-hand side of equation (19) is strictly positive at all times. Since $x'(\pi) > 0$ only for $\pi > 0$, it follows that inflation must be positive. Notice that, even if the economy as a whole is neither a creditor or a debtor ($\bar{a}_t = 0$), the fact that u' is strictly decreasing in net wealth implies that, as long as there is wealth dispersion, the central bank will have a reason to inflate.

To the best of our knowledge, this redistributive inflationary bias is a novel result in the context of incomplete markets models with uninsurable idiosyncratic risk. It is also different from the classical inflationary bias of discretionary monetary policy originally emphasized by Kydland and Prescott (1977) and Barro and Gordon (1983). In those papers, the source of the inflation bias is a persistent attempt by the monetary authority to raise output above its natural level. Here, by contrast, it arises from the welfare gains that can be achieved for the country as a whole by redistributing wealth towards indebted households. Importantly, while the model analyzed here is deliberately simple with a view to illustrating our methodology, this redistributive motive to inflate would carry over to more fully fledged models with uninsurable idiosyncratic risk that feature a Fisherian channel.

4 Numerical analysis

In the previous section we have characterized the optimal monetary policy in our model. In this section we solve numerically for the dynamic equilibrium under optimal policy, using numerical methods to solve continuous-time models with heterogeneous agents, as in Achdou et al. (2017) or Nuño and Moll (2017). The use of continuous time improves the efficiency of the numerical solution.[29] This computational speed is essential as the computation of the optimal policies requires several iterations along the complete time-path of the distribution.[30]

Before analyzing the dynamic path of this economy under the optimal policy, we first analyze the steady state towards which such path converges asymptotically. The numerical algorithms that we use are described in Appendices B (steady-state) and C (transitional dynamics).

4.1 Calibration

The calibration is intended to be mainly illustrative, given the model's simplicity and parsimoniousness. We calibrate the model to replicate some relevant features of a prototypical European small open economy.[31] Let the time unit be one year. For the calibration, we consider that the economy rests at the steady state implied by a zero inflation policy.[32] When integrating across households, we therefore use the stationary wealth distribution associated to such steady state.[33]

We assume the following specification for preferences,

$$u(c) - x(\pi) = \log(c) - \frac{\psi}{2}\pi^2. \tag{20}$$

[29] First, the HJB equation is a deterministic partial differential equation which can be solved using efficient finite-difference methods. Second, the dynamics of the distribution can be computed relatively quickly as they amount to calculating a matrix adjoint: the operator describing the law of motion of the distribution is the adjoint of the operator employed in the dynamic programming equation and hence the solution of the latter makes straightforward the computation of the former.

[30] In a home PC, the Ramsey problem presented here can be solved in less than five minutes.

[31] We will focus for illustration on the UK, Sweden, and the Baltic countries (Estonia, Latvia, Lithuania). We choose these countries because they (separately) feature desirable properties for the purpose at hand. On the one hand, UK and Sweden are two prominent examples of relatively small open economies that retain an independent monetary policy, like the economy in our framework. This is unlike the Baltic states, who recently joined the euro. However, historically the latter states have been relatively large debtors against the rest of the World, which make them square better with our theoretical restriction that the economy remains a net debtor at all times (UK and Sweden have also remained net debtors in basically each quarter for the last 20 years, but on average their net balance has been much closer to zero).

[32] This squares reasonably well with the experience of our target economies, which have displayed low and stable inflation for most of the recent past.

[33] The wealth dimension is discretized by using 1000 equally-spaced grid points from $a = \phi$ to $a = 10$. The upper bound is needed only for operational purposes but is fully innocuous, because the stationary distribution places essentially zero mass for wealth levels above $a = 8$.

As discussed in Appendix D, our quadratic specification for the inflation utility cost, $\frac{\psi}{2}\pi^2$, can be micro-founded by modelling firms explicitly and allowing them to set prices subject to standard quadratic price adjustment costs *à la* Rotemberg (1982). We set the scale parameter ψ such that the slope of the inflation equation in a Rotemberg pricing setup replicates that in a Calvo pricing setup for reasonable calibrations of price adjustment frequencies and demand curve elasticities.[34]

We jointly set households' discount rate ρ and borrowing limit ϕ such that the steady-state net international investment position (NIIP) over GDP $(\bar{a}/\bar{y})$ and gross household debt to GDP $(\bar{b}/\bar{y})$ replicate those in our target economies.[35]

We target an average bond duration of 4.5 years, as in Auclert (2016). In our model, the Macaulay bond duration equals $1/(\delta+\bar{r})$. We set the world real interest rate $\bar{r}$ to 3 percent. Our duration target then implies an amortization rate of $\delta = 0.19$.

The idiosyncratic income process parameters are calibrated as follows. We follow Huggett (1993) in interpreting states 1 and 2 as 'unemployment' and 'employment', respectively. The transition rates between unemployment and employment (λ_1, λ_2) are chosen such that (i) the unemployment rate $\lambda_2/(\lambda_1+\lambda_2)$ is 10 percent and (ii) the job finding rate is 0.1 at monthly frequency or $\lambda_1 = 0.72$ at annual frequency.[36] These numbers describe the 'European' labor market calibration in Blanchard and Galí (2010). We normalize average income $\bar{y} = \frac{\lambda_2}{\lambda_1+\lambda_2}y_1 + \frac{\lambda_1}{\lambda_1+\lambda_2}y_2$ to 1. We also set y_1 equal to 71 percent of y_2, as in Hall and Milgrom (2008). Both targets allow us to solve for y_1 and y_2. Table 1 summarizes our baseline calibration.

[34] The slope of the continuous-time New Keynesian Phillips curve in the Calvo model can be shown to be given by $\chi(\chi+\rho)$, where χ is the price adjustment rate (the proof is available upon request). As shown in Appendix D, in the Rotemberg model the slope is given by $\frac{\varepsilon-1}{\psi}$, where ε is the elasticity of firms' demand curves and ψ is the scale parameter in the quadratic price adjustment cost function in that model. It follows that, for the slope to be the same in both models, we need $\psi = \frac{\varepsilon-1}{\chi(\chi+\rho)}$.Setting ε to 11 (such that the gross markup $\varepsilon/(\varepsilon-1)$ equals 1.10) and χ to 4/3 (such that price last on average for 3 quarters), and given our calibration for ρ, we obtain $\psi = 5.5$.

[35] According to Eurostat, the NIIP/GDP ratio averaged minus 48.6% across the Baltic states in 2016:Q1, and only minus 3.8% across UK-Sweden. We thus target a NIIP/GDP ratio of minus 25%, which is about the midpoint of both values. Regarding gross household debt, we use BIS data on 'total credit to households', which averaged 85.9% of GDP across Sweden-UK in 2015:Q4 (data for the Baltic countries are not available). We thus target a 90% household debt to GDP ratio.

[36] Analogously to Blanchard and Galí (2010; see their footnote 20), we compute the equivalent annual rate λ_1 as $\lambda_1 = \sum_{i=1}^{12}(1-\lambda_1^m)^{i-1}\lambda_1^m$,where λ_1^m is the monthly job finding rate.

Table 1. Baseline calibration

Parameter	Value	Description	Source/Target
$\bar{r}$	0.03	world real interest rate	standard
ψ	5.5	scale inflation disutility	slope NKPC in Calvo model
δ	0.19	bond amortization rate	Macaulay duration = 4.5 yrs
λ_1	0.72	transition rate unemp-to-employment	monthly job finding rate 0.1
λ_2	0.08	transition rate employment-to-unemployment	unemployment rate 10%
y_1	0.73	income in unemployment state	Hall & Milgrom (2008)
y_2	1.03	income in employment state	$E(y) = 1$
ρ	0.0302	subjective discount rate	NIIP -25% of GDP
ϕ	-3.6	borrowing limit	HH debt/GDP 90%

4.2 Steady state under optimal policy

We start our numerical analysis of optimal policy by computing the steady state equilibrium to which each monetary regime (commitment and discretion) converges. Table 2 displays a number of steady-state objects. Under commitment, the optimal long-run inflation is close to zero (-0.05 percent), consistently with Proposition 2 and the fact ρ and $\bar{r}$ are very close to each other in our calibration.[37] As a result, long-run gross household debt and net total assets (as % of GDP) are very similar to those under zero inflation.

Table 2. Steady-state values under optimal policy

	units	Commitment	Discretion
Inflation, π	%	-0.05	1.68
Bond yield, r	%	2.95	4.68
Net assets, $\bar{a}$	% GDP	-24.1	-0.6
Gross assets (creditors)	% GDP	65.6	80.0
Gross debt (debtors), $\bar{b}$	% GDP	89.8	80.6
Current acc. deficit, $\bar{c} - \bar{y}$	% GDP	-0.63	-0.01

Under discretion, by contrast, long run inflation is 1.68 percent, which reflects the inflationary bias discussed in the previous section. The presence of an inflationary

[37] As explained in section 3, in our baseline calibration we have $\bar{r} = 0.03$ and $\rho = 0.0302$.

bias makes bond yields higher through the Fisher equation: $r_\infty = \frac{\delta}{Q_\infty} - \delta = \bar{r} + \pi_\infty$, where we have used $Q_\infty = \frac{\delta}{\delta+\bar{r}+\pi_\infty}$. The economy's aggregate net liabilities fall substantially relative to the commitment case (0.6% vs 24.1%), mostly reflecting larger asset accumulation by lending households.[38]

4.3 Optimal transitional dynamics

As explained in Section 3, the optimal policy paths depend on the initial (time-0) distribution of net wealth and income across households, $f_0(a, y)$, which is an (infinite-dimensional) primitive in our model. In the interest of isolating the effect of the policy regime (commitment vs discretion) on the equilibrium allocations, we choose a common initial distribution in both cases. For the purpose of illustration, we consider the stationary distribution under zero inflation as the initial distribution.[39] Later we will analyze the robustness of our results to a wide range of alternative initial distributions.

Consider first the case under commitment (Ramsey policy). The optimal paths are shown by the green solid lines in Figure 1.[40] Under our assumed functional form for inflation disutility in (20), it follows from equation (16) and the fact that $\mu_0 = 0$ (no pre-commitments at time zero) that initial optimal inflation is $\pi_0 = \psi^{-1}\mathbb{E}_{f_0(\cdot)}Q_0(-a)\,u'(c_0(\cdot))$. Therefore, the time-0 inflation rate, of about 4.6 percent, reflects exclusively the redistributive motive discussed in Section 3. From time zero onwards, Ramsey inflation follows

$$\pi_t = \frac{1}{\psi}\mathbb{E}_{f_t(a,y)}\left[Q_0(-a)\,u'(c_t(a,y))\right] + \frac{1}{\psi}\mu_t Q_t, \tag{21}$$

where the costate μ_t follows in turn equation (17). As shown in the figure, inflation gradually declines towards its (near) zero long-run level. Panels (b) and (c) show why: while the redistributive motive to inflate (the first right-hand-side term in equation 21)

[38] It is important to remark that the optimal steady-state inflation both under commitment and discretion differs from the inflation rate that maximizes steady-state welfare (subject to the constraint that $\bar{a}_t \leq 0$ holds at all times), equal to 1.8% in our case. This is analogous to the distinction between the steady-state and the "Golden Rule" consumption level in the neoclassical growth model.

[39] We thus assume $f_0(a, y_i) = f^{a|y}_{\pi=0}(a \mid y_i)\, f^y(y_i)$, $i = 1, 2$, where $f^y(y_i) = \lambda_{j\neq i}/(\lambda_1+\lambda_2)$, $i, j = 1, 2$, and $f^{a|y}_{\pi=0}$ is the stationary conditional density of net wealth under zero inflation. Notice that aggregate income is constant at $\bar{y}_t = \frac{\lambda_2}{\lambda_1+\lambda_2}y_1 + \frac{\lambda_1}{\lambda_1+\lambda_2}y_2 = 1$, given our calibration of $\{y_i\}_{i=1,2}$.

[40] We have simulated 800 years of data at monthly frequency.

remains roughly stable, the costate μ_t becomes more and more *negative* over time.[41] The reason for the latter effect is the following. As explained in Section 3, the costate captures the central bank's understanding of the fact that a commitment to lower future inflation raises bond prices today, which redistributes resources towards those households that issue new bonds. The upper-left panel in Figure 3 below shows that the latter households (i.e. those with $a_t^{new}(a,y) < 0$) have lower net wealth and hence higher marginal consumption utility than bond-purchasing households ($a_t^{new}(a,y) > 0$).[42] Therefore, households with $a_t^{new}(\cdot) < 0$ receive more weight in equation (17). This, together with the fact that the country is a net issuer of new bonds at all times ($\mathbb{E}_{f_t(\cdot)}(-a_t^{new}(\cdot)) > 0$), implies that the costate becomes negative immediately after time zero, and more so as time goes by. In summary, under the optimal commitment the central bank *front-loads* inflation in order to redistribute net wealth towards indebted households, but commits to gradually reducing inflation in order to prevent those same households from selling new bonds at excessively low prices.

Under discretion (dashed blue lines in Figure 1), time-zero inflation is 4.3 percent, close to the value under commitment.[43] In contrast to the commitment case, however, from time zero onwards optimal discretionary inflation remains relatively high, declining very slowly to its asymptotic value of 1.7 percent. This reflects the inflationary bias for redistributive purposes explained in Section 3. This inflationary bias produces permanently lower nominal bond prices (due to higher inflation premia) than under commitment.

Finally, panel (g) shows that both inflationary policies succeed at reducing the country's net liabilities with the rest of the World –equivalently, at increasing (the real face value of) its net wealth, which evolves according to equation (13), with $\bar{y}_t = 1$. Even though the fall in bond prices forces the country as a whole to issue more new bonds and thus raise its external debt burden (panel e), this is dominated by the

[41] Panels (b) and (c) in Figure 1 display the two terms on the right-hand side of (21), i.e. the $u'(c)$-weighted average net liabilities and $\mu(t)\,Q(t)$ both rescaled by the inflation disutility parameter ψ. Therefore, the sum of both terms equals optimal inflation under commitment.

[42] Figure 3 displays policy functions for high-income (employed) households, which account for 90% of the population in our calibration. Figure 7 in Appendix H shows the analogous objects for low-income households. Notice that essentially all indebted households ($a < 0$) are also issuers of new bonds ($a_t^{new}(a,y) < 0$). Conversely, most bond-issuing households are also indebted, the only exception being low-income households with $a \in [0, 1.5]$.

[43] Since $\mu_0 = 0$, and given a common initial wealth distribution, time-0 inflation under commitment and discretion differ only insofar as time-0 consumption policy functions in both regimes do. Numerically, the latter functions are similar enough that π_0 is very similar in both regimes.

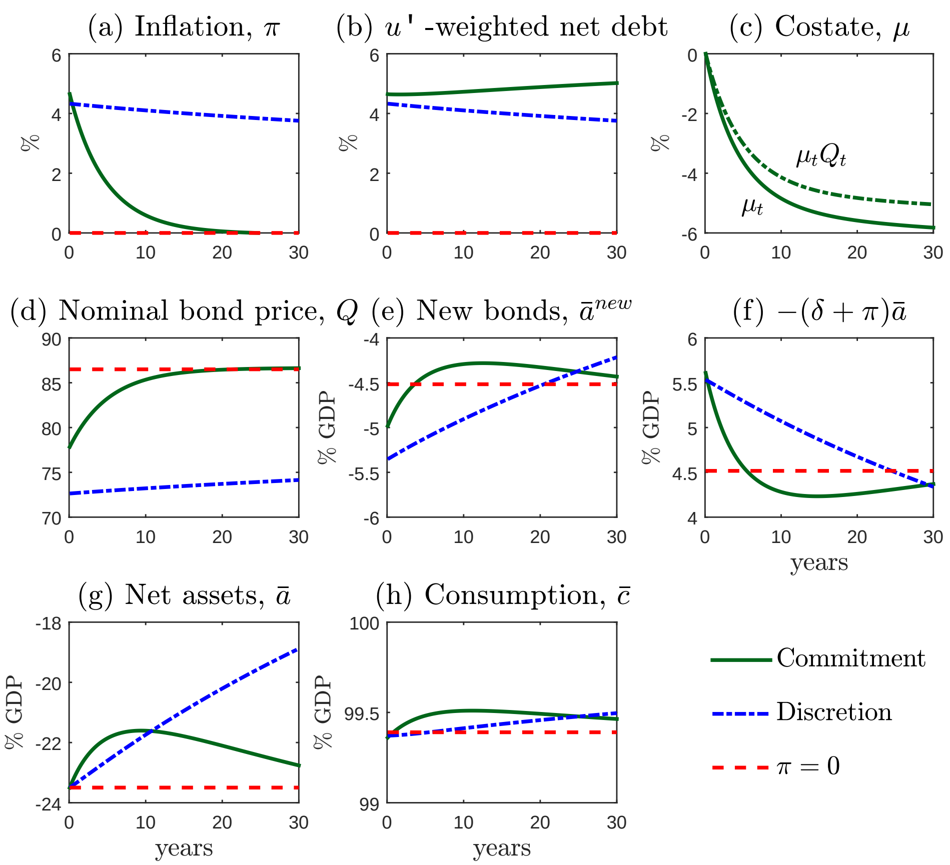

Figure 1: Aggregate dynamics under optimal monetary policy

erosion of such debt burden thanks to inflation (panel f).[44] This aggregate behavior however masks differences between both policy regimes in terms of redistributive effects, to which we turn next.

4.4 Redistributive effects of optimal inflation

We have seen that heterogenous net holdings of nominal assets across households, together with the concavity of preferences, gives the central bank a reason to inflate for redistributive purposes. In other words, the net wealth distribution is a key input of optimal inflation dynamics, both under discretion and commitment. Conversely, inflation plays a role in the evolution of the endogenous net wealth distribution over time. This section investigates the effects of monetary policy on the wealth distribution. We also analyze the redistributive effects on *consumption*, which is a key determinant of household welfare.

Wealth redistribution. Figure 2 displays the evolution over time of the marginal density of the real face value of net wealth $f_t^a(a) \equiv \sum_{i=1}^{2} f_{it}(a)$ under both policy regimes. Panels (a) and (b) display the distribution itself in both cases. In order to make such evolution more visible, panels (c) and (d) show the same densities *net* of the initial one, $f_0^a(a)$, which as explained before is common and assumed to equal the steady-state distribution implied by a zero inflation policy. Thus, panels (c) and (d) illustrate the redistributive effects of both inflationary regimes relative to the zero inflation policy.

Let us start with the commitment case (panel c). The transitory inflation in that regime succeeds at redistributing wealth towards indebted households (those with $a < 0$) and away from lending ones ($a > 0$). This can be seen in the relatively fast decline in the mass of households with negative net wealth, as well as in the more gradual decline in the mass of relatively rich households. This is mirrored by the increase over time in the mass of households with intermediate wealth levels.

Under discretion (panel d), by contrast, the extent of the domestic wealth redistribution is more modest. The reason is that bond prices fall considerably more than under commitment, reflecting expectations of higher inflation in the future. This un-

[44] During the first 3-4 years, the increase in net wealth is somewhat faster under commitment, because in those years inflation is quite similar to that under discretion but the initial fall in bond prices is much smaller –which in turn reflects the fact that foreign investors anticipate the short-lived nature of inflation under commitment and hence require a relatively small inflation premium.

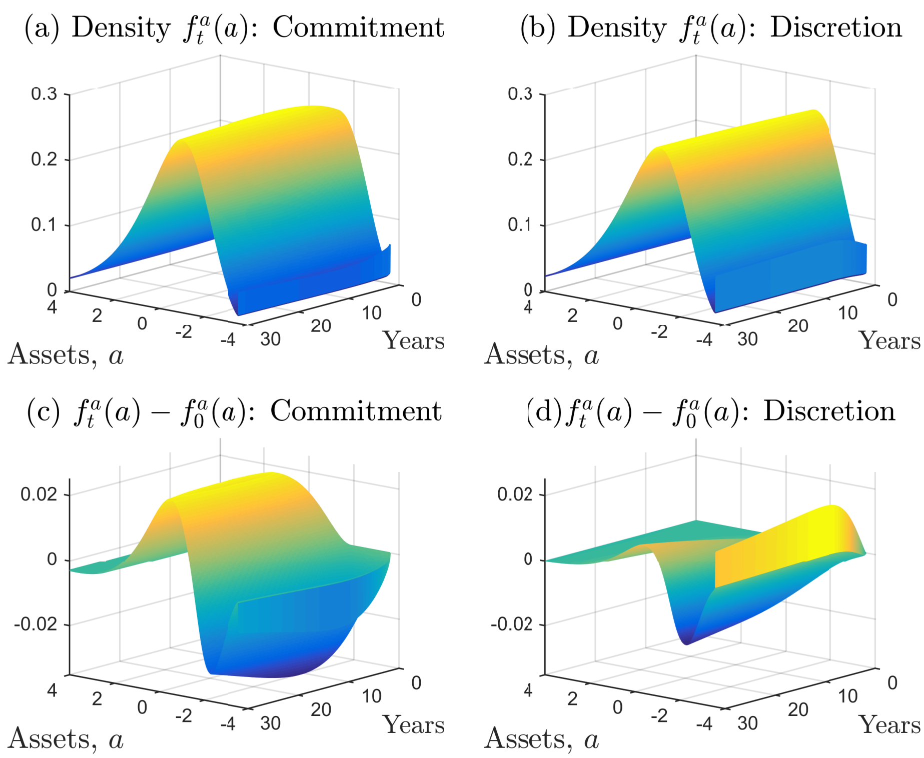

Figure 2: Dynamics of the net wealth distribution

does much of the redistributive effect from current inflation, because indebted households are also issuers of new bonds, and hence suffer from low bond prices. While there is some decline in the mass of poorer households and a corresponding increase in that of households with intermediate wealth levels, this effect is weaker than under commitment. As for rich households, they are barely affected by discretionary inflation.

To summarize, the optimal commitment is more successful than the discretionary policy at redistributing wealth (in face value terms) towards indebted households, by promising to inflate only transitorily and thus preventing such households from having to sell new bonds at very low prices.

Consumption redistribution. One may also ask to what extent the utilitarian central bank succeeds at redistributing *consumption* and hence welfare across households. The center-right panel of Figure 3 shows how the consumption policy function at time 0, $c_0(a, y)$, is affected in each optimal inflationary regime *vis-à-vis* the zero inflation regime.[45] Clearly, both discretionary and Ramsey inflation reduce consumption for lending households ($a > 0$) and increase it for indebted ones ($a < 0$). A key channel through which this consumption redistribution happens is the impact of *future expected* inflation on initial bond prices Q_0, and therefore on the initial real *market* value of household's net wealth, $Q_0 a$.[46] Thus, higher future inflation reduces bond prices, which hurts lending households and favors indebted ones –in the latter case by reducing the real market value of their liabilities, $Q_0(-a)$.

The panels in the first two columns and last two rows in Figure 3 offer a dynamic perspective on consumption redistribution *after* time 0. Most of the action takes place under commitment (first column). On the one hand, the consumption policy tilts over time in detriment (favor) of indebted (lending) households, largely reflecting

[45] Figure 3 shows policy functions for high-income (employed) households only ($y = y_2$), who account for 90% of the population in our calibration. The corresponding policies for low-income (unemployed) households ($y = y_1$) are displayed in Figure 7 in Appendix H. As shown there, the impact of inflationary policies on consumption redistribution is qualitatively similar to that for high-income households.

[46] To illustrate this channel, in Appendix F we consider a simplified version of our model with constant nonfinancial income ($y_1 = y_2 = y$) and the natural borrowing limit replacing the exogenous one (ϕ). There it is shown that, under our assumed log preferences, the consumption policy function equals $c_t(a) = \rho(Q_t a + h_t)$, where h_t is a measure of life-time income. With constant inflation π (which approximates well the discretionary outcome in the full model) the latter function simplifies to $c_t(a) = \rho(Qa + y/\bar{r})$, with $Q = \delta/(\delta + \bar{r} + \pi)$, such that inflation reduces consumption by lowering Q. In our full model, consumption cannot be solved in closed-form. However, bond prices remain a key determinant of household consumption by shifting $c_t(a, y)$ over time.

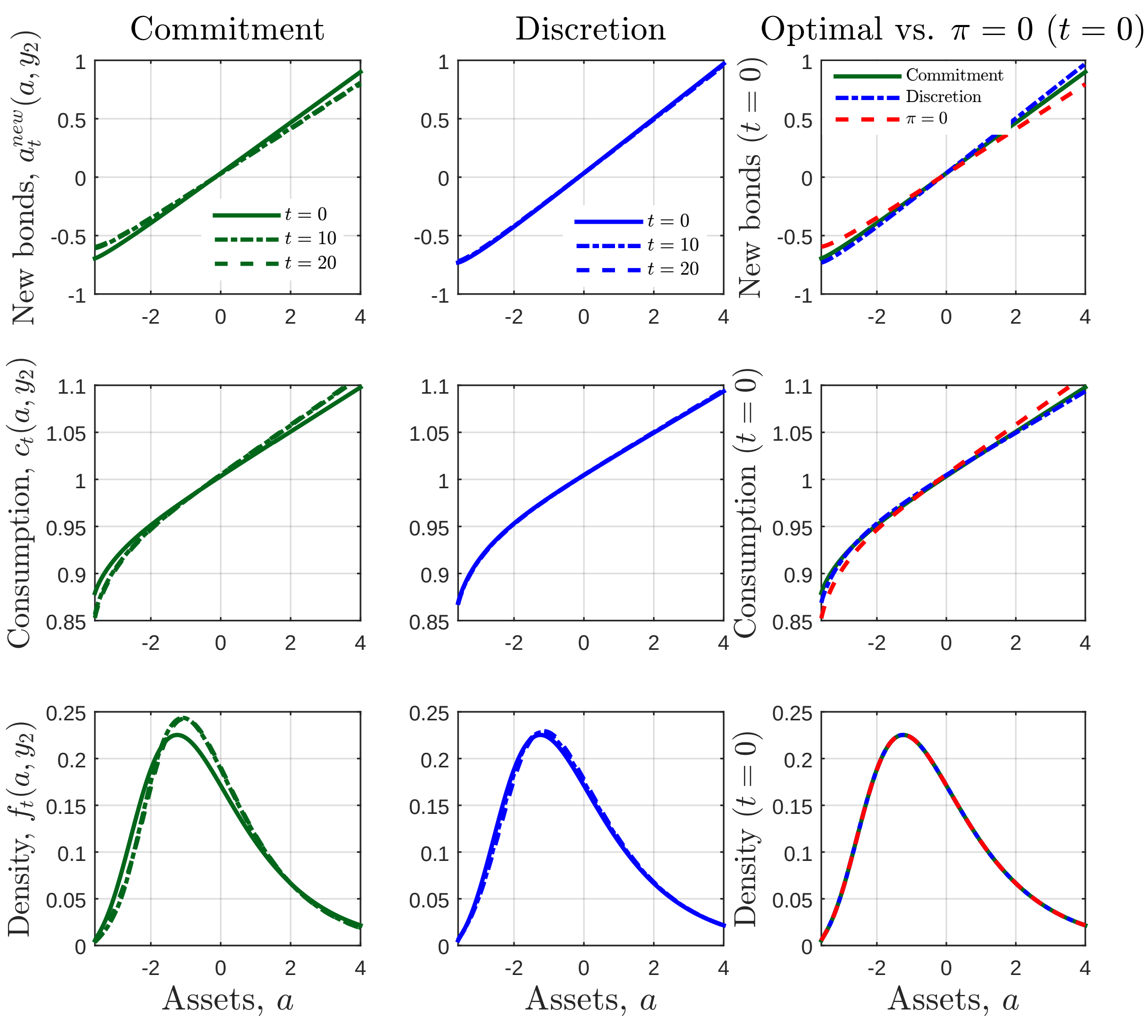

Figure 3: Policy functions and net wealth densities across policies and over time (high-income households, $y = y_2$)

the gradual recovery in bond prices (see panel d in Figure 1). On the other hand, and as mentioned before, the Ramsey policy succeeds at moving some highly indebted households towards the range of intermediate net wealth levels, which favors their consumption over time. These two effects tend to cancel each other out. As regards discretion, in this case neither the consumption policy nor the wealth density show much time variation. To sum up, the time-0 consumption effects discussed in the previous paragraph tend to be the dominant force as far as consumption redistribution is concerned.

4.5 Welfare analysis

We now turn to the welfare analysis of alternative policy regimes. Aggregate welfare is defined as

$$\mathbb{E}_{f_0(a,y)}\left[v_0\left(a,y\right)\right]=\int_0^\infty e^{-\rho t}\mathbb{E}_{f_t(a,y)}\left[u\left(c_t\left(a,y\right)\right)-x\left(\pi_t\right)\right]dt\equiv W\left[c\right],$$

Table 3 displays the welfare losses of suboptimal policies *vis-à-vis* the Ramsey optimal equilibrium. We express welfare losses as a permanent consumption equivalent, i.e. the number Θ (in %) that satisfies in each case $W^R\left[c^R\right]=W\left[\left(1+\Theta\right)c\right]$, where R denotes the Ramsey equilibrium.[47] The table also displays the welfare losses incurred respectively by lending and indebted households.[48] The welfare losses from discretionary policy are of first order: 0.31% of permanent consumption. This welfare loss is suffered not only by lending households (0.23%), *but also by indebted ones* (0.08%). The reason is that, while discretionary inflation succeeds at redistributing consumption to the latter households (as shown in the previous subsection), this beneficial effect is dominated by the direct welfare costs of permanent inflation, which are born by all households alike.

[47] Under our assumed separable preferences with log consumption utility, it is possible to show that $\Theta=\exp\left\{\rho\left(W^R\left[c^R\right]-W\left[c\right]\right)\right\}-1$.

[48] That is, we report $\Theta^{a>0}$ and $\Theta^{a<0}$, where $\Theta^{a>0}=\exp\left[\rho\left(W^{R,a>0}-W^{MPE,a>0}\right)\right]-1$, with $\Theta^{a<0}$ defined analogously, and where for each policy regime we have defined $W^{a>0}\equiv\int_0^\infty\sum_{i=1}^2 v_{0i}\left(a\right)f_{it}(a)da$, $W^{a<0}\equiv\int_\phi^0\sum_{i=1}^2 v_{0i}\left(a\right)f_{it}(a)da$. Notice that $\Theta^{a>0}$ and $\Theta^{a>0}$ do not exactly add up to Θ, as the expontential function is not a linear operator. However, Θ is sufficiently small that $\Theta\approx\Theta^{a>0}+\Theta^{a>0}$.

Table 3. Welfare losses relative to the optimal commitment

	Economy-wide	Lending HHs	Indebted HHs
Discretion	0.31	0.23	0.08
Zero inflation	0.05	-0.17	0.22

Note: welfare losses are expressed as a % of permanent consumption

We also compute the welfare losses from a policy of zero inflation, $\pi_t = 0$ for all $t \geq 0$. As the table shows, the latter policy approximates the aggregate welfare outcome under commitment very closely, for two reasons. First, the welfare losses –relative to commitment– suffered by indebted households due to the lack of inflationary redistribution are largely compensated by the corresponding gains for lending households. Second, zero inflation avoids any direct welfare costs from inflation.

4.6 Robustness

Appendix G contains a number of robustness exercises, including (i) the sensitivity of steady-state inflation under commitment to the gap between domestic households' and foreign investors' discount rates ($\rho - \bar{r}$), and (ii) the sensitivity of initial inflation π_0 (which is very similar under commitment and discretion) to the initial wealth distribution. The results can be summarized as follows. First, Ramsey optimal steady-state inflation decreases approximately linearly with the gap $\rho - \bar{r}$, because the central bank's incentive to protect bond issuing households –by committing to lower future inflation and thus raising bond prices– becomes more and more dominant relative to its incentive to redistribute resources towards currently indebted households –by raising current inflation.

Second, initial inflation increases with the dispersion of the initial net wealth distribution (while holding constant the initial net foreign asset position), reflecting a stronger redistributive motive. This exercises also reveals that both the domestic and cross-border redistributive motives are quantitatively important for explaining initial inflation, with contributions of about one third and two thirds, respectively.

4.7 Aggregate shocks

So far we have restricted our analysis to the transitional dynamics, given the economy's initial state, while abstracting from aggregate shocks. We now extend our analysis to allow for aggregate disturbances. For the purpose of illustration, we consider a one-time, unanticipated increase of 1 percentage point in the World real interest rate, followed by a gradual return to its baseline value of $\bar{r} = 3\%$. After the shock the dynamics of the (time-varying) World real rate $\bar{r}_t$ are given by

$$d\bar{r}_t = \eta_r \left(\bar{r} - \bar{r}_t\right) dt,$$

with $\eta_r = 0.5$. Notice that, up to a first order approximation, this is equivalent to solving the model considering an aggregate stochastic process $d\bar{r}_t = \eta_r \left(\bar{r} - \bar{r}_t\right) dt + \sigma dZ_t$ with $\sigma = 0.01$ and Z_t being a Brownian motion. In fact the impulse responses reported in Figure 4 coincide up to a first order approximation with the ones obtained by considering aggregate fluctuations and solving the model by first-order perturbation around the deterministic steady state, as in the method of Ahn et al. (2017).

The dashed red lines in Figure 4 display the responses to the shock under a strict zero inflation policy, $\pi_t = 0$ for all t. The shock raises nominal (and real) bond yields, which leads households to reduce their consumption on impact. The reduction in consumption induces an increase in assets holdings in the case of creditors and a reduction in debt (i.e. an increase in net assets) in the case of debtors. This allows consumption to slowly recover and to reach levels slightly above the steady state after roughly 5 years from the arrival of the shock.

The solid lines in Figure 4 display the economy's response under the optimal commitment policy. An issue that arises here is how long after 'time zero' (the implementation date of the Ramsey optimal commitment) the aggregate shock is assumed to take place. Since we do not want to take a stand on this dimension, we consider the limiting case in which the Ramsey optimal commitment has been going on for a sufficiently long time that the economy rests at its stationary equilibrium by the time the shock arrives. This can be viewed as an example of optimal policy 'from a timeless perspective', in the sense of Woodford (2003). In practical terms, it requires solving the optimal commitment problem analyzed in Section 3.2 with two modifications (apart of course from the time variation in $\bar{r}_t$): (i) the initial wealth distribution is the stationary distribution implied by the optimal commitment itself,

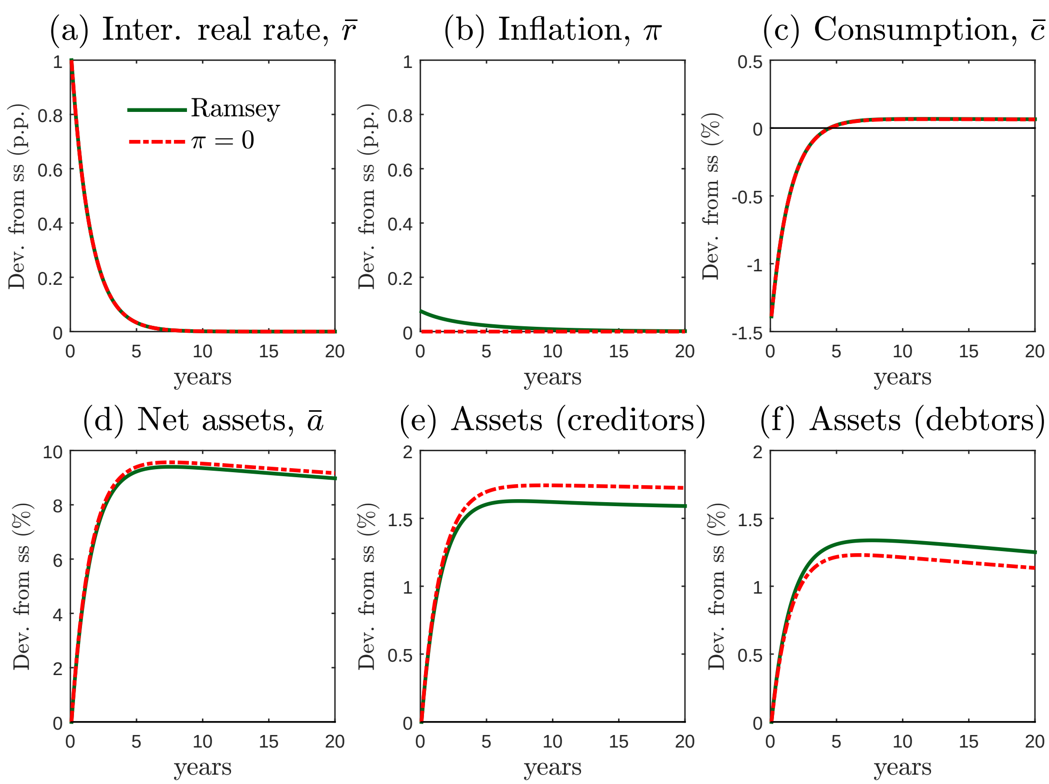

Figure 4: Responses to a World real interest rate shock under commitment (from a timeless perspective).

and (ii) the initial condition $\mu_0 = 0$ (absence of precommitments) is replaced by $\mu_0 = \mu_\infty$, where the latter object is the stationary value of the costate in the commitment case. Both modifications guarantee that the central bank behaves as if it had been following the time-0 optimal commitment for an arbitrarily long time.

As shown by the figure, under commitment inflation rises slightly on impact, as the central bank tries to partially counteract the negative effect of the shock on household consumption. However, the inflation reaction is an order of magnitude smaller than that of the shock itself. Intuitively, the value of sticking to past commitments to keep inflation near zero weighs more in the central bank's decision than the value of using inflation transitorily so as to stabilize consumption in response to an unforeseen event.

5 Conclusion

We have analyzed optimal monetary policy, under commitment and discretion, in a continuous-time, small-open-economy version of a standard incomplete-markets model extended to allow for nominal, long-term claims and costly inflation. Our analysis sheds light on a recent policy and academic debate on the consequences that wealth heterogeneity across households should have for the appropriate conduct of monetary policy.

Our first main contribution is methodological: to the best of our knowledge, our paper is the first to solve for a fully dynamic optimal policy problem, both under commitment and discretion, in an incomplete-markets model with uninsurable idiosyncratic risk. While models of this kind have been established as a workhorse for policy analysis in macro models with heterogeneous agents, the fact that in such models the infinite-dimensional, endogenously-evolving wealth distribution is a state in the policy-maker's problem has made it difficult to make progress in the analysis of fully optimal policy problems. Our analysis proposes a novel methodology, based on infinite dimensional calculus, for dealing with problems of this kind.

Our second main contribution relates to our normative results. Optimal *discretionary* monetary policy features a 'redistributive inflationary bias'. In particular, optimal discretionary inflation depends positively on the average net liabilities across households weighted by their marginal utility of consumption. Under incomplete markets and standard concave preferences, indebted households have a higher marginal utility than lending ones, giving the central bank an incentive to use inflation on a permanent basis in order to redistribute wealth from the latter to the former. Under *commitment*, such redistributive motive to inflate exists as well, but it is counteracted over time by a 'deflationary force' that has too a redistributive motive. By promising lower and lower inflation in the future, the central bank increases the price of the long-term nominal bond (through lower inflation premia). This favors the households that issue new bonds, who also have a higher marginal utility than those that purchase new bonds. In the long run, and under certain parametric conditions, both effects exactly cancel each other out and optimal inflation is zero. Numerically, the optimal commitment policy is found indeed to imply inflation 'front-loading', with an initial inflation very similar to that under discretion, but a gradual undoing of such inflationary stance.

While the model used here is deliberately simple –with a view to illustrating our methodology as transparently as possible–, the above normative insights are likely to carry over to more fully fledged macroeconomics models featuring uninsurable idiosyncratic risk and a Fisherian redistributive channel. More generally, extending the methods developed here for computing fully optimal monetary policy to New Keynesian frameworks with uninsurable idiosyncratic risk and household heterogeneity, of the type constructed e.g. by Auclert (2016), Kaplan et al. (2016), Gornemann et al. (2012) or McKay et al. (2016), is an important task that we leave for future research.

Finally, we stress that our results should not be interpreted as suggesting that monetary policy is the best tool to address redistributive issues, as there are probably more direct policy instruments such as taxes or transfers. What our results indicate is that, in the context of economies with uninsurable idiosyncratic risk, the optimal design of monetary policy will typically reflect redistributive motives, the more so the less other policies (e.g. fiscal policy) are able to achieve optimal redistributive outcomes.

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Online appendix (not for publication)

A. Proofs

Mathematical preliminaries

First we need to introduce some mathematical concepts. An *operator* T is a mapping from one vector space to another. Given the stochastic process a_t in (4), define an operator $\mathcal{A}$,

$$\mathcal{A}v \equiv \begin{pmatrix} s_1(t,a)\frac{\partial v_1(t,a)}{\partial a} + \lambda_1\left[v_2(t,a) - v_1(t,a)\right] \\ s_2(t,a)\frac{\partial v_2(t,a)}{\partial a} + \lambda_2\left[v_1(t,a) - v_2(t,a)\right] \end{pmatrix}, \tag{22}$$

so that the HJB equation (6) can be expressed as

$$\rho v = \frac{\partial v}{\partial t} + \max_c \left\{ u\left(c\right) - x\left(\pi\right) + \mathcal{A}v \right\},$$

where $v \equiv \binom{v_1(t,a)}{v_2(t,a)}$ and $u\left(c\right) - x\left(\pi\right) \equiv \binom{u(c_1)-x(\pi)}{u(c_2)-x(\pi)}$.[49]

Let $\Phi \equiv [\phi, \infty)$ be the valid domain. The space of Lebesgue-integrable functions $L^2\left(\Phi\right)$ with the inner product

$$\langle v, f\rangle_\Phi = \sum_{i=1}^{2} \int_\Phi v_i f_i da = \int_\Phi v^{\mathbf{T}} f da, \; \forall v, f \in L^2\left(\Phi\right), \tag{23}$$

is a Hilbert space.[50] Notice that we could have alternatively worked in $\Phi = \mathbb{R}$ as the density $f(t,a,y) = 0$ for $a < \phi$.

Given an operator $\mathcal{A}$, its *adjoint* is an operator $\mathcal{A}^*$ such that $\langle f, \mathcal{A}v\rangle_\Phi = \langle \mathcal{A}^* f, v\rangle_\Phi$. In the case of the operator defined by (22) its adjoint is the operator

$$\mathcal{A}^* f \equiv \begin{pmatrix} -\frac{\partial(s_1 f_1)}{\partial a} - \lambda_1 f_1 + \lambda_2 f_2 \\ -\frac{\partial(s_2 f_2)}{\partial a} - \lambda_2 f_2 + \lambda_1 f_1 \end{pmatrix}, \tag{24}$$

with boundary conditions

$$s_i\left(t,\phi\right) f_i\left(t,\phi\right) = \lim_{a\to\infty} s_i\left(t,a\right) f_i\left(t,a\right) = 0, \quad i = 1, 2, \tag{25}$$

[49] The *infinitesimal generator* of the process is thus $\frac{\partial v}{\partial t} + \mathcal{A}v$.

[50] See Luenberger (1969) or Brezis (2011) for references.

such that the KF equation (11) results in

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f, \tag{26}$$

for $f \equiv \binom{f_1(t,a)}{f_2(t,a)}$. We can see that $\mathcal{A}$ and $\mathcal{A}^*$ are adjoints as

$$\begin{aligned}
\langle \mathcal{A}v, f\rangle_\Phi &= \int_\Phi (\mathcal{A}v)^{\mathbf{T}} f da = \sum_{i=1}^{2} \int_\Phi \left[s_i \frac{\partial v_i}{\partial a} + \lambda_i \left[v_j - v_i\right]\right] f_i da \\
&= \sum_{i=1}^{2} v_i s_i f_i|_\phi^\infty + \sum_{i=1}^{2} \int_\Phi v_i \left[-\frac{\partial}{\partial a}(s_i f_i) - \lambda_i f_i + \lambda_j j_j \right] da \\
&= \int_\Phi v^{\mathbf{T}} \mathcal{A}^* f da = \langle v, \mathcal{A}^* f\rangle_\Phi .
\end{aligned}$$

We introduce the concept of Gateaux and Frechet derivatives in $L^2(\Phi)$, where $\Phi \subset \mathbb{R}^n$ as generalizations of the standard concept of derivative to infinite-dimensional spaces.[51]

Definition 4 (Gateaux derivative) *Let $W[f]$ be a functional and let h be arbitrary in $L^2(\Phi)$. If the limit*

$$\delta W[f;h] = \lim_{\alpha \to 0} \frac{W[f+\alpha h] - W[f]}{\alpha} \tag{27}$$

exists, it is called the Gateaux derivative *of W at f with increment h. If the limit (27) exists for each $h \in L^2(\Phi)$, the functional W is said to be* Gateaux differentiable *at f.*

If the limit exists, it can be expressed as $\delta W[f;h] = \frac{d}{d\alpha} W[f+\alpha h]|_{\alpha=0}$. A more restricted concept is that of the Fréchet derivative.

Definition 5 (Fréchet derivative) *Let h be arbitrary in $L^2(\Phi)$. If for fixed $f \in L^2(\Phi)$ there exists $\delta W[f;h]$ which is linear and continuous with respect to h such that*

$$\lim_{\|h\|_{L^2(\Phi)} \to 0} \frac{|W[f+h] - W[f] - \delta W[f;h]|}{\|h\|_{L^2(\Phi)}} = 0,$$

then W is said to be Fréchet differentiable *at f and $\delta W[f;h]$ is the* Fréchet derivative *of W at f with increment h.*

[51] See Luenberger (1969), Gelfand and Fomin (1991) or Sagan (1992).

The following proposition links both concepts.

Theorem 1 *If the Fréchet derivative of W exists at f, then the Gateaux derivative exists at f and they are equal.*

Proof. See Luenberger (1969, p. 173). ■

The familiar technique of maximizing a function of a single variable by ordinary calculus can be extended in infinite dimensional spaces to a similar technique based on more general derivatives. We use the term *extremum* to refer to a maximum or a minimum over any set. A a function $f \in L^2(\Phi)$ is a maximum of $W[f]$ if for all functions h, $\|h - f\|_{L^2(\Phi)} < \varepsilon$ then $W[f] \geq W[h]$. The following theorem generalizes the Fundamental Theorem of Calculus.

Theorem 2 *Let W have a Gateaux derivative, a necessary condition for W to have an extremum at f is that $\delta W[f; h] = 0$ for all $h \in L^2(\Phi)$.*

Proof. Luenberger (1969, p. 173), Gelfand and Fomin (1991, pp. 13-14) or Sagan (1992, p. 34). ■

In the case of constrained optimization in an infinite-dimensional Hilbert space, we have the following Theorem.

Theorem 3 (Lagrange multipliers) *Let H be a mapping from $L^2(\Phi)$ into $\mathbb{R}^p$. If W has a continuous Fréchet derivative, a necessary condition for W to have an extremum at f under the constraint $H[f] = 0$ at the function f is that there exists a function $\eta \in L^2(\Phi)$ such that the Lagrangian functional*

$$\mathcal{L}[f] = W[f] + \langle \eta, H[f] \rangle_\Phi \tag{28}$$

is stationary in f, that is., $\delta\mathcal{L}[f; h] = 0$.

Proof. Luenberger (1969, p. 243). ■

Finally, according to Definition 5 above, if the Fréchet derivative $\delta W[f]$ of $W[f]$ exists then it is linear and continuous. We may apply the Riesz representation theorem to express it as an inner product

Theorem 4 (Riesz representation theorem) *Let $\delta W[f;h] : L^2(\Phi) \to \mathbb{R}$ be a linear continuous functional. Then there exists a unique function $w[f] = \frac{\delta W}{\delta f}[f] \in L^2$ (Φ) such that*

$$\delta W[f;h] = \left\langle \frac{\delta W}{\delta f}, h \right\rangle_{\Phi} = \sum_{i=1}^{2} \int_{\Phi} w_i[f](a)\, h_i(a)\, da.$$

Proof. See Brezis (2011, pp. 97-98). ■

Proof of Lemma 1

In order to prove the concavity of the value function we express the model in discrete time for an arbitrarily small Δt. The Bellman equation of a household is

$$\begin{aligned} v_t^{\Delta t}(a,y) &= \max_{a' \in \Gamma(a,y)} \left[u\left(\frac{Q(t)}{\Delta t} \left[\left(1 + \left(\frac{\delta}{Q(t)} - \delta - \pi(t) \right) \Delta t \right) a + \frac{y\Delta t}{Q(t)} - a' \right] \right) - x(\pi(t)) \right] \Delta t \\ &\quad + e^{-\rho \Delta t} \sum_{i=1}^{2} v_{t+\Delta t}^{\Delta t}(a', y_i)\, \mathbb{P}(y' = y_i | y), \end{aligned}$$

where $\Gamma(a,y) = \left[0, \left(1 + \left(\frac{\delta}{Q(t)} - \delta - \pi(t) \right) \Delta t \right) a + \frac{y\Delta t}{Q(t)} \right]$, and $\mathbb{P}(y' = y_i|y)$ are the transition probabilities of a two-state Markov chain. The Markov transition probabilities are given by $\lambda_1 \Delta t$ and $\lambda_2 \Delta t$.

We verify that this problem satisfies the conditions of Theorem 9.8 of Stokey and Lucas (1989): (i) Φ is a convex subset of $\mathbb{R}$; (ii) the Markov chain has a finite number of values; (iii) the correspondence $\Gamma(a,y)$ is nonempty, compact-valued and continuous; (iv) the function u is bounded, concave and continuous and $e^{-\rho\Delta t} \in (0,1)$; and (v) the set $A^y = \{(a,a') \text{ such that } a' \in \Gamma(a,y)\}$ is convex. We conclude that $v_t^{\Delta t}(a,y)$ is strictly concave for any $\Delta t > 0$. Finally, for any a_1, $a_2 \in \Phi$

$$\begin{aligned} v_t^{\Delta t}(\omega a_1 + (1-\omega) a_2, y) &> \omega v_t^{\Delta t}(a_1, y) + (1-\omega) v_t^{\Delta t}(a_2, y), \\ \lim_{\Delta t \to 0} v_t^{\Delta t}(\omega a_1 + (1-\omega) a_2, y) &> \lim_{\Delta t \to 0} \left[\omega v_t^{\Delta t}(a_1, y) + (1-\omega) v_t^{\Delta t}(a_2, y) \right], \\ v(t, \omega a_1 + (1-\omega) a_2, y) &> \omega v(t, a_1, y) + (1-\omega) v_t(t, a_2, y), \end{aligned}$$

so that $v(t,a,y)$ is strictly concave.

Proof of Lemma 2

Given the welfare criterion defined in equation (14), we have

$$
\begin{aligned}
W_0 &= \int_\phi^\infty \sum_{i=1}^{2} v_0(a,y_i) f_0(a,y_i)da \\
&= \int_\phi^\infty \sum_{i=1}^{2} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left[u\left(c_t\right) - x\left(\pi_t\right)\right] dt | a_0 = a,\ y_0 = y_i \right] f_{i0}(a)da \\
&= \int_\phi^\infty \sum_{i=1}^{2} \left[\sum_{j=1}^{2} \int_\phi^\infty \int_0^\infty e^{-\rho t} \left[u(c_{jt}\left(\tilde{a}\right)) - x\left(\pi_t\right)\right] f_t(\tilde{a},\tilde{y}_j;a,y_i)dtd\tilde{a} \right] f_{i0}(a)da \\
&= \int_0^\infty \sum_{j=1}^{2} e^{-\rho t} \int_\phi^\infty \left[u(c_{jt}\left(\tilde{a}\right)) - x\left(\pi_t\right)\right] \left[\sum_{i=1}^{2} \int_\phi^\infty f_t(\tilde{a},\tilde{y}_j;a,y_i) f_{i0}(a)da \right] d\tilde{a}dt \\
&= \int_0^\infty e^{-\rho t} \sum_{j=1}^{2} \int_\phi^\infty \left[u(c_{jt}\left(\tilde{a}\right)) - x\left(\pi_t\right)\right] f_t(\tilde{a},\tilde{y}_j)d\tilde{a}dt,
\end{aligned}
$$

where $f_t(\tilde{a},\tilde{y}_j;a,y_i)$ is the transition probability from $a_0 = a$, $y_0 = y_i$ to $a_t = \tilde{a}$, $y_t = \tilde{y}_j$ and in the last equality we have used the Chapman–Kolmogorov equation,

$$
f_t(\tilde{a},\tilde{y}_j) = \sum_{i=1}^{2} \int_\phi^\infty f_t(\tilde{a},\tilde{y}_j;a,y_i) f_0(a,y_i)da.
$$

Proof of Proposition 1. Solution to the Ramsey problem

The idea of the proof is to construct a Lagragian in a Hilbert function space and to obtain the first-order conditions by taking the Gateaux derivatives.

Step 1: Statement of the problem. The problem of the central bank is given by

$$
W\left[f_0\left(\cdot\right)\right] = \max_{\{\pi_t,Q_t,v_t(\cdot),c_t(\cdot),f_t(\cdot)\}_{t=0}^\infty} \int_0^\infty e^{-\rho t} \left[\sum\nolimits_{i=1}^{2} \int_\Phi \left(u\left(c_t\right) - x\left(\pi_t\right)\right) f_{it}(a)da \right] dt,
$$

subject to the law of motion of the distribution (11), the bond pricing equation (10) and the individual HJB equation (6). This is a problem of constrained optimization in an infinite-dimensional Hilbert space that includes also time, which we denote as

$\hat{\Phi} = [0, \infty) \times \Phi$. We define $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ as the space of functions f that verify

$$\int_{\hat{\Phi}} e^{-\rho t} |f|^2 = \int_0^\infty \int_\Phi e^{-\rho t} |f|^2 \, dt da = \int_0^\infty e^{-\rho t} \|f\|_\Phi^2 \, dt < \infty.$$

We need first to prove that this space, which differs from $L^2\left(\hat{\Phi}\right)$ is also a Hilbert space. This is done in the following lemma, which is proved later on.

Lemma 3 *The space $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ with the inner product*

$$(f, g)_\Phi = \int_{\hat{\Phi}} e^{-\rho t} f g = \int_0^\infty e^{-\rho t} \langle f, g \rangle_\Phi \, dt = \left\langle e^{-\rho t} f, g \right\rangle_{\hat{\Phi}}$$

is a Hilbert space.

Step 2: The Lagragian. From now on, for compactness we use the operator $\mathcal{A}$, its adjoint operator $\mathcal{A}^*$, and the inner product $\langle \cdot, \cdot \rangle$ defined in expressions (22), (24), and (23), respectively. The Lagrangian is defined in $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ as

$$\begin{aligned}
\mathcal{L}[\pi, Q, f, v, c] \equiv & \int_0^\infty e^{-\rho t} \langle u - x, f \rangle_\Phi \, dt + \int_0^\infty \left\langle e^{-\rho t} \zeta(t, a), \mathcal{A}^* f - \frac{\partial f}{\partial t} \right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t} \mu(t) \left(Q(\bar{r} + \pi + \delta) - \delta - \dot{Q} \right) dt \\
& + \int_0^\infty \left\langle e^{-\rho t} \theta(t, a), u - x + \mathcal{A} v + \frac{\partial v}{\partial t} - \rho v \right\rangle_\Phi dt \\
& + \int_0^\infty \left\langle e^{-\rho t} \eta(t, a), u' - \frac{1}{Q} \frac{\partial v}{\partial a} \right\rangle_\Phi dt
\end{aligned}$$

where $e^{-\rho t}\zeta(t, a)$, $e^{-\rho t}\eta(t, a)$, $e^{-\rho t}\theta(t, a) \in L^2\left(\hat{\Phi}\right)$ and $e^{-\rho t}\mu(t) \in L^2[0, \infty)$ are the Lagrange multipliers associated to equations (11), (8), (6) and (10), respectively. The

Lagragian can be expressed as

$$\begin{aligned}
\mathcal{L} &= \int_0^\infty e^{-\rho t}\left\langle u-x+\frac{\partial \zeta}{\partial t}+\mathcal{A}\zeta-\rho\zeta+\mu\left(Q\left(\bar{r}+\pi+\delta\right)-\delta-\dot{Q}\right),f\right\rangle_\Phi dt \\
&+\int_0^\infty e^{-\rho t}\left(\langle\theta,u-x\rangle_\Phi+\left\langle\mathcal{A}^*\theta-\frac{\partial\theta}{\partial t},v\right\rangle_\Phi+\left\langle\eta,u'-\frac{1}{Q}\frac{\partial v}{\partial a}\right\rangle_\Phi\right)dt \\
&+\langle\zeta\left(0,\cdot\right),f\left(0,\cdot\right)\rangle_\Phi-\lim_{T\to\infty}\left\langle e^{-\rho T}\zeta\left(T,\cdot\right),f\left(T,\cdot\right)\right\rangle_\Phi \\
&+\lim_{T\to\infty}\left\langle e^{-\rho T}\theta\left(T,\cdot\right),v\left(T,\cdot\right)\right\rangle_\Phi-\langle\theta\left(0,\cdot\right),v\left(0,\cdot\right)\rangle+\int_0^\infty e^{-\rho t}\sum_{i=1}^2 v_i s_i\theta_i\big|_\phi^\infty dt,
\end{aligned}$$

where we have applied

$$\langle\zeta,\mathcal{A}^*f\rangle=\langle\mathcal{A}\zeta,f\rangle,\langle\theta,\mathcal{A}v\rangle=\langle\mathcal{A}^*\theta,v\rangle_\Phi+\sum_{i=1}^2 v_i s_i\theta_i\big|_\phi^\infty$$

and integrated by parts

$$\begin{aligned}
\int_0^\infty\left\langle e^{-\rho t}\zeta,-\frac{\partial f}{\partial t}\right\rangle_\Phi dt &= -\sum_{i=1}^2\int_0^\infty\int_\Phi e^{-\rho t}\zeta_i\frac{\partial f_i}{\partial t}dadt \\
&= -\sum_{i=1}^2\int_\Phi f_i e^{-\rho t}\zeta_i\big|_0^\infty da+\sum_{i=1}^2\int_0^\infty\int_\Phi f_i\frac{\partial}{\partial t}\left(e^{-\rho t}\zeta_i\right)dadt \\
&= \sum_{i=1}^2\int_\Phi f_i\left(0,a\right)\zeta_i\left(0,a\right)da-\lim_{T\to\infty}\sum_{i=1}^2\int_\Phi e^{-\rho T}f_i\left(T,a\right)\zeta_i\left(T,a\right)da \\
&\quad+\sum_{i=1}^2\int_0^\infty\int_\Phi e^{-\rho t}f_i\left(\frac{\partial\zeta_i}{\partial t}-\rho\zeta_i\right)dadt \\
&= \langle\zeta\left(0,\cdot\right),f\left(0,\cdot\right)\rangle_\Phi-\lim_{T\to\infty}\left\langle e^{-\rho T}\zeta\left(T,\cdot\right),f\left(T,\cdot\right)\right\rangle_\Phi \\
&\quad+\int_0^\infty e^{-\rho t}\left\langle\frac{\partial\zeta}{\partial t}-\rho\zeta,f\right\rangle_\Phi dt,
\end{aligned}$$

and

$$
\begin{aligned}
\int_0^\infty \left\langle e^{-\rho t}\theta, \frac{\partial v}{\partial t} - \rho v \right\rangle dt &= \sum_{i=1}^2 \int_0^\infty \int_\Phi e^{-\rho t}\theta_i \left(\frac{\partial v_i}{\partial t} - \rho v_i \right) dadt \\
&= \sum_{i=1}^2 \int_\Phi \theta_i e_i^{-\rho t} v\Big|_0^\infty da - \sum_{i=1}^2 \int_0^\infty \int_\Phi v_i \left[\frac{\partial}{\partial t}\left(e^{-\rho t}\theta_i\right) + \rho\theta_i \right] dadt \\
&= \lim_{T\to\infty} \sum_{i=1}^2 \int_\Phi e^{-\rho T} v_i(T,a)\,\theta_i(T,a)\,da - \sum_{i=1}^2 \int_\Phi v_i(0,a)\,\theta_i(0,a)\,da \\
&\quad -\sum_{i=1}^2 \int_0^\infty \int_\Phi e^{-\rho t} v_i \left(\frac{\partial \theta_i}{\partial t} \right) dadt \\
&= \lim_{T\to\infty} \left\langle e^{-\rho T}\theta(T,\cdot), v(T,\cdot)\right\rangle_\Phi - \left\langle \theta(0,\cdot), v(0,\cdot)\right\rangle_\Phi \\
&\quad + \int_0^\infty e^{-\rho t} \left\langle -\frac{\partial \theta}{\partial t}, v \right\rangle_\Phi dt,
\end{aligned}
$$

Step 3: Necessary conditions. In order to find the maximum, we need to take the Gateaux derivatives with respect to the controls f, π, Q, v and c.

- The Gateaux derivative with respect to $f(t,a)$ is

$$
\begin{aligned}
\frac{d}{d\alpha}\mathcal{L}[\pi, Q, f+\alpha h, v, c]\,|_{\alpha=0} &= \langle \zeta(0,\cdot), h(0,\cdot)\rangle_\Phi - \lim_{T\to\infty}\left\langle e^{-\rho T}\zeta(T,\cdot), h(T,\cdot)\right\rangle_\Phi \\
&\quad - \int_0^\infty e^{-\rho t}\left\langle u - x + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta - \rho\zeta, h \right\rangle_\Phi dt,
\end{aligned}
$$

which should equal zero for any function $e^{-\rho t}h \in L^2\left(\hat{\Phi}\right)$ such that $h(0,\cdot)=0$, as the initial value of $f(0,\cdot)$. We obtain

$$
\rho\zeta = u - x + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta, \quad \text{for } a > \phi,\ t > 0 \tag{29}
$$

Given that $e^{-\rho t}\zeta(t,a) \in L^2\left(\hat{\Phi}\right)$, we obtain the transversality condition $\lim_{T\to\infty} e^{-\rho T}\zeta(T,a) = 0$. Equation (29) is the same as the individual HJB equation (6). The boundary conditions are also the same (state constraints on the domain Φ) and therefore their solutions should coincide: $\zeta(t,a,y) = v(t,a,y)$, that is, the Lagrange multiplier $\zeta(t,a,y)$ equals the private value $v(\cdot)$.

- In the case of $c(t,a)$, the Gateaux derivative is

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}[\pi,Q,f,v,c+\alpha h]|_{\alpha=0} &= \int_0^\infty e^{-\rho t}\left\langle\left(u'-\frac{1}{Q}\frac{\partial\zeta}{\partial a}\right)h,f\right\rangle_\Phi dt \\ &+\int_0^\infty e^{-\rho t}\left(\left\langle\theta,\left(u'-\frac{1}{Q}\frac{\partial v}{\partial a}\right)h\right\rangle_\Phi+\langle\eta,u''h\rangle_\Phi\right)dt,\end{aligned}$$

where $\frac{\partial}{\partial a}(\mathcal{A}\zeta)=-\frac{1}{Q}\frac{\partial\zeta}{\partial a}$. The Gateaux derivative should be zero for any function $e^{-\rho t}h\in L^2\left(\hat{\Phi}\right)$. Due to the first order conditions (8) and to the fact that $\zeta(\cdot)=v(\cdot)$ this expression reduces to

$$\int_0^\infty e^{-\rho t}\langle\eta(t,a),u''(t,a)h(t,a)\rangle_\Phi dt=0.$$

As u is strictly concave, $u''<0$ and hence $\eta(t,a)=0$ for all $(t,a)\in\hat{\Phi}$, that is, the first order condition (8) is not binding as its associated Lagrange multiplier is zero.

- In the case of $v(t,a)$, the Gateaux derivative is

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}[\pi,Q,f,v+\alpha h,c]|_{\alpha=0} &= \int_0^\infty e^{-\rho t}\left(\left\langle\mathcal{A}^*\theta-\frac{\partial\theta}{\partial t},h\right\rangle_\Phi\right)dt \\ &+\lim_{T\to\infty}\left\langle e^{-\rho T}\theta(T,\cdot),h(T,\cdot)\right\rangle_\Phi-\langle\theta(0,\cdot),h(0,\cdot)\rangle_\Phi \\ &+\sum_{i=1}^2 h_i s_i\theta_i|_\phi^\infty,\end{aligned}$$

where we have already taken into account the fact that $\eta(\cdot)=0$. Given that $e^{-\rho t}\theta(t,a)\in L^2\left(\hat{\Phi}\right)$, we obtain the transversality condition $\lim_{T\to\infty}e^{-\rho T}\theta(T,\cdot)=0$. As the Gateaux derivative should be zero at the maximum for any suitable h, we obtain a Kolmogorov forward equation in θ

$$\frac{\partial\theta}{\partial t}=\mathcal{A}^*\theta,\quad \text{for } a>\phi,\ t>0, \tag{30}$$

with boundary conditions

$$\begin{aligned}s_i(t,\phi)\theta_i(t,\phi) &= \lim_{a\to\infty}s_i(t,a)\theta_i(t,a)=0,\ i=1,2, \\ \theta(0,\cdot) &= 0.\end{aligned}$$

This is a KF equation with an initial density of $\theta(0,\cdot)=0$.[52] Therefore, the distribution at any point in time should be zero $\theta(\cdot)=0$. Both the Lagrange multiplier of the households' HJB equation θ and that of the first-order condition η are zero, reflecting the fact that the HJB equation is slack, that is, that the monetary authority would choose the same consumption as the households. This would not be the case in a closed economy, in which some externalities may arise, as discussed, for instance, in Nuño and Moll (2017).

- The Gateaux derivative in the case of $\pi(t)$ is

$$\frac{d}{d\alpha}\mathcal{L}\left[\pi+\alpha h,Q,f,v,c\right]|_{\alpha=0}=\int_0^\infty e^{-\rho t}\left\langle -x'-a\left(\frac{\partial v}{\partial a}\right)+\mu Q,f\right\rangle_\Phi hdt,$$

where we have already taken into account the fact that $\theta(\cdot)=\eta(\cdot)=0.$ and $\zeta(\cdot)=v(\cdot)$. As the Gateaux derivative should be zero for any $h(t)\in L^2[0,\infty)$, the optimality condition then results in

$$\mu(t)\,Q(t)=\sum_{i=1}^{2}\int_\Phi\left(a\frac{\partial v_i}{\partial a}+x'\right)f_i(t,a)\,da, \tag{31}$$

where we have applied the normalization condition (equation 12): $\langle 1,f\rangle_\Phi=1$.

- In the case of $Q(t)$ the Gateaux derivative is

$$\frac{d}{d\alpha}\mathcal{L}\left[\pi,Q+\alpha h,\cdot\right]|_{\alpha=0}=\int_0^\infty e^{-\rho t}\left\langle -\frac{\delta h}{Q^2}a\frac{\partial v}{\partial a}-\frac{(y-c)\,h}{Q^2}\frac{\partial v}{\partial a}+\mu\left[h\left(\bar r+\pi+\delta\right)-\dot h\right],f\right\rangle_\Phi dt,$$

where we have already taken into account the fact that $\zeta(\cdot)=v(\cdot)$ and $\theta(\cdot)=$

[52] Notice that if we denote $g(t)\equiv\left\langle\mathcal{A}^*\theta-\frac{\partial\theta}{\partial t},1\right\rangle_\Phi$ and $G(t)\equiv\int_t^\infty e^{-\rho s}g(s)ds$ then the fact that $\mathcal{A}^*\theta-\frac{\partial\theta}{\partial t}=0$, for $a>\phi$, $t>0$, implies that $G(t)=0$, for $t>0$. As $G(t)$ is differentiable, then it is continuous and hence $G(0)=0$ so that the condition $G(0)+\langle\theta(0,\cdot),h(0,\cdot)\rangle_\Phi=0$ for any $h(0,\cdot)\in L^2(\Phi)$ requires $\theta(0,\cdot)=0$. A similar argument can be employed to analyzed the boundary conditions in Φ.

$\eta(\cdot) = 0$. Integrating by parts

$$\begin{aligned}
\int_0^\infty e^{-\rho t} \left\langle -\mu \dot{h}, f \right\rangle_\Phi dt &= -\int_0^\infty e^{-\rho t} \mu \dot{h} \langle 1, f \rangle_\Phi dt = -\int_0^\infty e^{-\rho t} \mu \dot{h} dt \\
&= - e^{-\rho t} \mu h \Big|_0^\infty + \int_0^\infty e^{-\rho t} (\dot{\mu} - \rho\mu) h dt \\
&= \mu(0) h(0) + \int_0^\infty e^{-\rho t} \langle (\dot{\mu} - \rho\mu) h, f \rangle_\Phi dt.
\end{aligned}$$

Therefore, the optimality condition in this case is

$$\int_0^\infty e^{-\rho t} \left\langle -\frac{\delta}{Q^2} a \frac{\partial v}{\partial a} - \frac{(y-c)}{Q^2} \frac{\partial v}{\partial a} + \mu(\bar{r} + \pi + \delta - \rho) + \dot{\mu}, f \right\rangle_\Phi h dt + \mu(0) h(0) = 0.$$

The Gateaux derivative should be zero for any $h(t) \in L^2[0, \infty)$. Thus we obtain

$$\begin{aligned}
\left\langle -\frac{\delta}{Q^2} a \frac{\partial v}{\partial a} - \frac{(y-c)}{Q^2} \frac{\partial v}{\partial a}, f \right\rangle_\Phi + \mu(\bar{r} + \pi + \delta - \rho) + \dot{\mu} &= 0, \; t > 0, \\
\mu(0) &= 0.
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\frac{d\mu}{dt} &= (\rho - \bar{r} - \pi - \delta) \mu + \sum_{i=1}^{2} \int_\Phi \frac{\partial v_{it}}{\partial a} \frac{\delta a + (y-c)}{Q(t)^2} f_i(t, a) da, \; t > 0, \quad (32) \\
\mu(0) &= 0.
\end{aligned}$$

Finally, using the household's first order condition $\frac{\partial v_{it}}{\partial a} = Q_t u'(c_{it})$ to substitute for $\frac{\partial v_{it}}{\partial a}$ in equations (31) and (32) yields the expressions in the main text.

Proof of Lemma 3

We need to show that $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ is complete, that is, that given a Cauchy sequence $\{f_n\}$ with limit $f : \lim_{n\to\infty} f_n = f$ then $f \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$. If $\{f_n\}$ is a Cauchy sequence then

$$\|f_n - f_m\|_{(\cdot,\cdot)_\Phi} \to 0, \text{ as } n, m \to \infty,$$

or

$$\begin{aligned}\|f_n - f_m\|^2_{(\cdot,\cdot)_\Phi} &= \int_{\hat{\Phi}} e^{-\rho t} |f_n - f_m|^2 = \left\langle e^{-\frac{\rho}{2}t}(f_n - f_m), e^{-\frac{\rho}{2}t}(f_n - f_m)\right\rangle_{\hat{\Phi}} \\ &= \left\| e^{-\frac{\rho}{2}t}(f_n - f_m)\right\|^2_{\hat{\Phi}} \to 0,\end{aligned}$$

as $n, m \to \infty$. This implies that $\left\{e^{-\frac{\rho}{2}t} f_n\right\}$ is a Cauchy sequence in $L^2\left(\hat{\Phi}\right)$. As $L^2\left(\hat{\Phi}\right)$ is a complete space, then there is a function $\hat{f} \in L^2\left(\hat{\Phi}\right)$ such that

$$\lim_{n\to\infty} e^{-\frac{\rho}{2}t} f_n = \hat{f} \tag{33}$$

under the norm $\|\cdot\|^2_{\hat{\Phi}}$. If we define $f = e^{\frac{\rho}{2}t}\hat{f}$ then

$$\lim_{n\to\infty} f_n = f$$

under the norm $\|\cdot\|_{(\cdot,\cdot)_\Phi}$, that is, for any $\varepsilon > 0$ there is an integer N such that

$$\|f_n - f\|^2_{(\cdot,\cdot)_\Phi} = \left\|e^{-\frac{\rho}{2}t}(f_n - f)\right\|^2_{\hat{\Phi}} = \left\|e^{-\frac{\rho}{2}t} f_n - \hat{f}\right\|^2_{\hat{\Phi}} < \varepsilon,$$

where the last inequality is due to (33). It only remains to prove that $f \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$:

$$\|f\|^2_{(\cdot,\cdot)_\Phi} = \int_{\hat{\Phi}} e^{-\rho t} |f|^2 = \int_{\hat{\Phi}} \left|\hat{f}\right|^2 < \infty,$$

as $\hat{f} \in L^2\left(\hat{\Phi}\right)$.

Proof of Proposition 2: Optimal long-run inflation under commitment in the limit as $\bar{r} \to \rho$

In the steady state, equations (17) and (16) in the main text become

$$(\rho - \bar{r} - \pi - \delta)\mu + \frac{1}{Q^2}\sum_{i=1}^{2}\int_{\phi}^{\infty} \frac{\partial v_i}{\partial a}\left[\delta a + (y_i - c_i)\right] f_i(a)\, da = 0,$$

$$\mu Q = x'(\pi) + \sum_{i=1}^{2}\int_{\phi}^{\infty} a \frac{\partial v_i}{\partial a} f_i(a)\, da,$$

respectively. Notice that we have replaced $Qu'(c_i)$ by $\frac{\partial v_i}{\partial a}$. Consider now the limiting case $\rho \to \bar{r}$, and guess that $\pi \to 0$. The above two equations then become

$$\begin{aligned}
\mu Q &= \frac{1}{\delta Q}\sum_{i=1}^{2}\int_{\phi}^{\infty}\frac{\partial v_i}{\partial a}\left[\delta a+(y_i-c_i)\right]f_i\left(a\right)da, \\
\mu Q &= \sum_{i=1}^{2}\int_{\phi}^{\infty}a\frac{\partial v_i}{\partial a}f_i\left(a\right)da,
\end{aligned}$$

as $x'\left(0\right)=0$ under our assumed preferences in Section 3.4. Combining both equations, and using the fact that in the zero-inflation steady state the bond price equals $Q=\frac{\delta}{\delta+\bar{r}}$, we obtain

$$\sum_{i=1}^{2}\int_{\phi}^{\infty}\frac{\partial v_i}{\partial a}\left(\bar{r}a+\frac{y_i-c_i}{Q}\right)f_i\left(a\right)da=0. \tag{34}$$

In the zero inflation steady state, the value function v satisfies the HJB equation

$$\rho v_i(a)=u(c_i\left(a\right))+\left(\bar{r}a+\frac{y_i-c_i\left(a\right)}{Q}\right)\frac{\partial v_i}{\partial a}+\lambda_i\left[v_j(a)-v_i(a)\right], \quad i=1,2,\ j\neq i, \tag{35}$$

where we have used $x\left(0\right)=0$ under our assumed preferences. We also have the first-order condition

$$u'\left(c_i\left(a\right)\right)=Q\frac{\partial v_i}{\partial a}\Rightarrow c_i\left(a\right)=u'^{-1}\left(Q\frac{\partial v_i}{\partial a}\right).$$

We guess and verify a solution of the form $v_i(a)=\kappa_i a+\vartheta_i$, so that $u'\left(c_i\right)=Q\kappa_i$. Using our guess in (35), and grouping terms that depend on a and those that do not, we have that

$$\begin{aligned}
\rho\kappa_i &= \bar{r}\kappa_i+\lambda_i\left(\kappa_j-\kappa_i\right), \qquad (36)\\
\rho\vartheta_i &= u\left(u'^{-1}\left(Q\kappa_i\right)\right)+\frac{y_i-u'^{-1}\left(Q\kappa_i\right)}{Q}\kappa_i+\lambda_i\left(\vartheta_j-\vartheta_i\right), \qquad (37)
\end{aligned}$$

for $i,j=1,2$ and $j\neq i$. In the limit as $\bar{r}\to\rho$, equation (36) results in $\kappa_j=\kappa_i\equiv\kappa$, so that consumption is the same in both states. The value of the slope κ can be computed from the boundary conditions.[53] We can solve for $\{\vartheta_i\}_{i=1,2}$ from equations (37),

$$\vartheta_i = \frac{1}{\rho} u \left(u'^{-1}(Q\kappa)\right) + \frac{y_i - u'^{-1}(Q\kappa)}{\rho Q}\kappa + \frac{\lambda_i (y_j - y_i)}{\rho(\lambda_i + \lambda_j + \rho) Q}\kappa,$$

for $i, j = 1, 2$ and $j \neq i$. Substituting $\frac{\partial v_i}{\partial a} = \kappa$ in (34), we obtain

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} \left(\bar{r}a + \frac{y_i - c_i}{Q}\right) f_i(a)\, da = 0. \tag{38}$$

Equation (38) is exactly the zero-inflation steady-state limit of equation (13) in the main text (once we use the definitions of $\bar{a}$, $\bar{y}$ and $\bar{c}$), and is therefore satisfied in equilibrium. We have thus verified our guess that $\pi \to 0$.

Proof of Proposition 3. Solution to the Markov Stackelberg equilibrium

The approach is to consider that, given any arbitrary horizon $T > 0$, the interval $[0, T]$ is divided in N subintervals of length $\Delta t := T/N$. In each subinterval $(t, t+\Delta t]$ the central bank solves a Ramsey problem with terminal value $W^M_{\Delta t}[f(t+\Delta t, \cdot)]$, taken as given the initial density $f_t(\cdot)$ and the terminal value $W^M_{\Delta t}[f_{t+\Delta t}(\cdot)]$. Notice that the initial density $f_t(\cdot)$ of a subinterval subinterval $(t, t+\Delta t]$ is the *final* density of the previous subinterval whereas the terminal value $W^M_{\Delta t}[f_{t+\Delta t}(\cdot)]$ is the *initial* value of the next subinterval. A Markov Stackelberg equilibrium is the limit when $N \to \infty$, or equivalently, $\Delta t \to 0$.

Step 1: The discrete-step problem. First we solve the dynamic programming problem in a subinterval $(t, t + \Delta t]$. This is now a Ramsey problem in the Hilbert

[53] The condition that the drift should be positive at the borrowing constraint, $s_i(\phi) \geq 0$, $i = 1, 2$, implies that

$$s_1(\phi) = \bar{r}\phi + \frac{y_1 - u'^{-1}(Q\kappa)}{Q} = 0,$$

and

$$\kappa = \frac{u'(\bar{r}\phi Q + y_1)}{Q}.$$

In the case of state $i = 2$, this guarantees $s_2(\phi) > 0$.

space $L^2\left(\hat{\Phi}_t\right)_{(\cdot,\cdot)_\Phi}$ with $\hat{\Phi}_t = (t, t+\Delta t] \times \Phi$. We define

$$
\begin{aligned}
W_{\Delta t}^{M}\left[f\left(t,\cdot\right)\right] \;=\; & \max_{\{\pi_s, Q_s, v_s(\cdot), c_s(\cdot), f_s(\cdot)\}_{s\in(t,t+\Delta t]}} \int_t^{t+\Delta t} e^{-\rho(s-t)} \sum_{i=1}^{2} \int_\phi^\infty \left(u\left(c_{is}\left(a\right)\right) - x\left(\pi_s\right)\right) f_i(s,a)\,da\,ds \\
& + e^{-\rho\Delta t} W_{\Delta t}^{M}\left[f\left(t+\Delta t,\cdot\right)\right],
\end{aligned}
$$

subject to the law of motion of the distribution (11), the bond pricing equation (10), and household's HJB equation (6) and optimal consumption choice (8). This can be seen as a finite-horizon commitment problem with terminal value $W_{\Delta t}^{M}\left[f\left(t+\Delta t,\cdot\right)\right]$. We proceed as in the proof of Proposition 1 and construct a Lagragian

$$
\begin{aligned}
\mathcal{L}\left[\pi, Q, f, v, c\right] \;\equiv\; & \int_t^{t+\Delta t} e^{-\rho(s-t)} \left\langle u - x, f\right\rangle_\Phi ds + e^{-\rho\Delta t} W_{\Delta t}^{M}\left[f\left(t+\Delta t,\cdot\right)\right] \\
& + \int_t^{t+\Delta t} \left\langle e^{-\rho(s-t)} \zeta\left(t,a\right), \mathcal{A}^* f - \frac{\partial f}{\partial t}\right\rangle_\Phi ds \\
& + \int_t^{t+\Delta t} e^{-\rho(s-t)} \mu\left(s\right)\left(Q\left(\bar{r} + \pi + \delta\right) - \delta - \dot{Q}\right) ds \\
& + \int_t^{t+\Delta t} \left\langle e^{-\rho(s-t)} \theta\left(s,a\right), u - x + \mathcal{A} v + \frac{\partial v}{\partial t} - \rho v\right\rangle_\Phi ds \\
& + \int_t^{t+\Delta t} \left\langle e^{-\rho(s-t)} \eta\left(s,a\right), u' - \frac{1}{Q}\frac{\partial v}{\partial a}\right\rangle_\Phi ds,
\end{aligned}
$$

with $W_{\Delta t}^{M}\left[\cdot\right]$ defined in (18). Proceeding as in the proof of Proposition 1, we can express the Lagragian as

$$
\begin{aligned}
\mathcal{L} \;=\; & \int_t^{t+\Delta t} e^{-\rho(s-t)} \left\langle u - x + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta - \rho\zeta + \mu\left(Q\left(\bar{r} + \pi + \delta\right) - \delta - \dot{Q}\right), f\right\rangle_\Phi ds \\
& + \int_t^{t+\Delta t} e^{-\rho(s-t)} \left(\left\langle \theta, u - x\right\rangle_\Phi + \left\langle \mathcal{A}^*\theta - \frac{\partial \theta}{\partial t}, v\right\rangle_\Phi + \left\langle \eta, u'\right\rangle_\Phi + \left\langle \frac{1}{Q}\frac{\partial \eta}{\partial a}, v\right\rangle_\Phi\right) ds \\
& + \left\langle \zeta\left(t,\cdot\right), f\left(t,\cdot\right)\right\rangle_\Phi - \left\langle e^{-\rho\Delta t}\zeta\left(t+\Delta t,\cdot\right), f\left(t+\Delta t,\cdot\right)\right\rangle_\Phi \\
& + \left\langle e^{-\rho\Delta t}\theta\left(t+\Delta t,\cdot\right), v\left(t+\Delta t,\cdot\right)\right\rangle_\Phi - \left\langle \theta\left(t,\cdot\right), v\left(t,\cdot\right)\right\rangle \\
& + \int_t^{t+\Delta t} e^{-\rho(s'-t)} \left[\sum_{i=1}^{2} v_i s_i \theta_i\big|_\phi^\infty - \frac{1}{Q}\sum_{i=1}^{2} v_i \eta_i\big|_\phi^\infty\right] ds' + e^{-\rho\Delta t} W_{\Delta t}^{M}\left[f\left(t+\Delta t,\cdot\right)\right]
\end{aligned}
$$

- The first order condition with respect to f in this case is

$$
\begin{aligned}
0 &= \langle \zeta(t,\cdot), h(t,\cdot)\rangle_\Phi - \left\langle e^{-\rho\Delta t}\zeta(t+\Delta t,\cdot), h(t+\Delta t,\cdot)\right\rangle_\Phi \\
&- \int_t^{t+\Delta t} e^{-\rho t}\left\langle u - x + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta - \rho\zeta, h\right\rangle_\Phi dt \\
&+ e^{-\rho\Delta t}\frac{d}{d\alpha}\, W^M_{\Delta t}\left[f(t+\Delta t,\cdot) + \alpha h(t+\Delta t,\cdot)\right]\Big|_{\alpha=0}.
\end{aligned}
$$

Given the Riesz representation theorem (Theorem 4), the Gateaux derivative can be expressed as

$$
\frac{d}{d\alpha}\, W^M_{\Delta t}\left[f(t+\Delta t,\cdot) + \alpha h(t+\Delta t,\cdot)\right]\Big|_{\alpha=0} = \langle w(t+\Delta t,\cdot), h(t+\Delta t,\cdot)\rangle_\Phi
$$

where

$$
w(t,\cdot) = \frac{\delta W^M_{\Delta t}}{\delta f}\left[f(t,\cdot)\right] : [0,\infty)\times\Phi \to \mathbb{R}^2.
$$

Notice that, as there is no aggregate uncertainty, the dynamics of the distribution only depend on time. As it will be clear below $w(t,a)$ is the *central bank's value* at time t of a household with net wealth a. As the Gateaux derivative should be zero for any $h \in L^2\left((t,t+\Delta t]\times\Phi\right)$ we obtain

$$
\begin{aligned}
\rho\zeta &= u - x + \frac{\partial\zeta}{\partial t} + \mathcal{A}\zeta, \quad \text{for } a > \phi,\ s \in (t,t+\Delta t), \qquad (39) \\
\zeta(t+\Delta t,\cdot) &= w(t+\Delta t,\cdot).
\end{aligned}
$$

The boundary conditions are state constraints on the domain Φ. Notice that we have employed the fact that $h(t,\cdot) = 0$ as $f(t,\cdot)$ is given. The rest of Gateaux derivatives are obtain by following exatly the same steps as in the proof of Proposition 1 above, but restricted to the interval $(t, t+\Delta t]$ and without simplifying terms.

- In the case of $c(t,a)$, this yields

$$
\left(u' - \frac{1}{Q}\frac{\partial\zeta}{\partial a}\right) f + \eta u'' = 0, \quad \text{for } a \geq \phi,\ s \in (t,t+\Delta t], \qquad (40)
$$

- In the case of $v(t,a)$:

$$\begin{aligned} \mathcal{A}^{*}\theta - \frac{\partial \theta}{\partial t} + \frac{1}{Q}\frac{\partial \eta}{\partial a} &= 0, \quad \text{for } a > \phi,\ s \in (t, t+\Delta t), \\ \theta\left(t+\Delta t, \cdot\right) &= \theta\left(t, \cdot\right) = 0 \\ s_i\left(s,\phi\right)\theta_i\left(s,\phi\right) - \frac{1}{Q\left(s\right)}\eta_i\left(s,\phi\right) &= \lim_{a\to\infty}\left[s_i\left(s,a\right)\theta_i\left(s,a\right) - \frac{1}{Q\left(s\right)}\eta_i\left(s,a\right)\right] = 0,\ i = 1, 2. \end{aligned} \tag{41}$$

- In the case of $\pi(t)$:

$$\left\langle -x' - a\frac{\partial \zeta}{\partial a} + \mu Q, f \right\rangle_{\Phi} + \left\langle -x' - a\frac{\partial v}{\partial a}, \theta \right\rangle_{\Phi} = 0, \tag{42}$$

for $s \in (t, t+\Delta t]$.

- Finally, in the case of $Q(t)$:

$$\begin{aligned} 0 &= \left\langle -\frac{\delta}{Q^2}a\frac{\partial \zeta}{\partial a} - \frac{(y-c)}{Q^2}\frac{\partial \zeta}{\partial a}, f \right\rangle_{\Phi} + \left\langle \left(-\frac{\delta}{Q^2}a - \frac{(y-c)}{Q^2}\right)\frac{\partial v}{\partial a}, \theta \right\rangle_{\Phi} \\ &\quad + \mu\left(\bar{r} + \pi + \delta - \rho\right) + \dot{\mu} + \left\langle \eta, \frac{1}{Q^2}\frac{\partial v}{\partial a} \right\rangle_{\Phi}, \ \text{for } s \in (t, t+\Delta t), \\ \lim_{s\to t}\mu\left(s\right) &= \mu\left(t+\Delta t\right) = 0. \end{aligned} \tag{43}$$

Step 2: Taking the limit. If we take the limit as $N \to \infty$, or equivalently, $\Delta t \to 0$, we obtain that $\zeta(t,\cdot) = w(t,\cdot)$ for all $t \geq 0$ and hence equation (39) results in

$$\rho w = u - x + \frac{\partial w}{\partial t} + \mathcal{A}w, \quad \text{for } t \geq 0, \tag{44}$$

with state constraints on the domain Φ. The transversality condition $\lim_{T\to\infty} e^{-\rho T} w(T,\cdot) = 0$ as $\lim_{T\to\infty} e^{-\rho T} W\left[f(T,\cdot)\right] = 0$. Equation (44) coincides with the individual HJB equation (6) and hence, as in the case with commitment, we obtain that $w(t,\cdot) = v(t,\cdot)$, that is, the social value is the same as the private value.

Proceeding as in the case with commitment, the fact that $\zeta(t,\cdot) = v(t,\cdot)$ and that the utility function is strictly concave in equation (40) yields $\eta(t,\cdot) = 0$. In the limit $\Delta t \to 0$ the transversality conditions (41) and (43) result in $\mu(t) = 0$ and $\theta(t,\cdot) = 0$, for all $t \geq 0$.

Finally, the optimality condition with respect to $\pi(t)$ (42) simplifies to

$$\left\langle -x' - a\left(\frac{\partial v}{\partial a}\right), f\right\rangle_{\Phi} = 0,$$

or equivalently

$$0 = \sum_{i=1}^{2} \int_{\Phi} \left(a\frac{\partial v_{it}}{\partial a} + x'\right) f_i(t,a)\, da.$$

Using the household first order condition $\frac{\partial v_{it}}{\partial a} = Q_t u'(c_{it})$ to substitute for $\frac{\partial v_{it}}{\partial a}$ above yields the expression in the main text.

Proof of Proposition 4: Inflation bias in the Markov Stackelberg equilibrium

As the value function is strictly concave in a by Lemma 1, it satisfies

$$\frac{\partial v_{it}(\tilde{a})}{\partial a} < \frac{\partial v_{it}(0)}{\partial a} < \frac{\partial v_{it}(\hat{a})}{\partial a}, \text{ for all } \tilde{a} \in (0,\infty),\ \hat{a} \in (\phi, 0),\ t \geq 0,\ i = 1, 2. \quad (45)$$

In addition, Assumption 1 (the country is a always a net debtor: $\bar{a}_t \leq 0$) implies

$$\sum_{i=1}^{2} \int_{0}^{\infty} (a)\, f_{it}(a) da \leq \sum_{i=1}^{2} \int_{\phi}^{0} (-a)\, f_{it}(a) da,\ \forall t \geq 0. \quad (46)$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{2} \int_{0}^{\infty} a f_{it}(a) \frac{\partial v_{it}(a)}{\partial a} da &< \frac{\partial v_{it}(0)}{\partial a} \sum_{i=1}^{2} \int_{0}^{\infty} a f_{it}(a)\, da \leq \frac{\partial v_{it}(0)}{\partial a} \sum_{i=1}^{2} \int_{\phi}^{0} (-a)\, f_{it}(a) da \\ &< \sum_{i=1}^{2} \int_{\phi}^{0} (-a)\, f_{it}(a) \frac{\partial v_{it}(a)}{\partial a} da, \end{aligned} \quad (47)$$

where we have applied (45) in the first and last inequalities and (46) in the intermediate one.[54] The optimal inflation in the Markov Stackelberg equilibrium (19)

[54] We have also used the fact that $af(a) > 0$ for all $a > 0$ and $(-a) f(a) > 0$ for all $a < 0$, as well as $\partial v_{it}(0)/\partial a > 0$ (which follows from the household first order condition and the assumption that $u' > 0$).

satisfies

$$\sum_{i=1}^{2}\int_{\phi}^{\infty} af_i\frac{\partial v_i}{\partial a}da+x'=0.$$

Combining this expression with (47) we obtain

$$x'=\sum_{i=1}^{2}\int_{\phi}^{\infty}(-a)\,Q_t u' f_i da=\sum_{i=1}^{2}\int_{\phi}^{\infty}(-a)\,\frac{\partial v_i}{\partial a}f_i da>0.$$

Finally, taking into account the fact that $x'(\pi)>0$ only for $\pi>0$, we have that $\pi(t)>0$.

B. Computational method: the stationary case

B.1 Exogenous monetary policy

We describe the numerical algorithm used to jointly solve for the equilibrium value function, $v(a,y)$, and bond price, Q, given an exogenous inflation rate π. The algorithm proceeds in 3 steps. We describe each step in turn. We assume that there is an upper bound arbitrarily large $\varkappa$ such that $f(t,a,y)=0$ for all $a>\varkappa$. In steady state this can be proved in general following the same reasoning as in Proposition 2 of Achdou et al. (2017). Alternatively, we may assume that there is a maximum constraint in asset holding such that $a\leq\varkappa$, and that this constraint is so large that it does not affect to the results. In any case, let $[\phi,\varkappa]$ be the valid domain.

Step 1: Solution to the Hamilton-Jacobi-Bellman equation Given π, the bond pricing equation (10) is trivially solved in this case:

$$Q=\frac{\delta}{\bar{r}+\pi+\delta}. \tag{48}$$

The HJB equation is solved using an *upwind finite difference* scheme similar to Achdou et al. (2017). It approximates the value function $v(a)$ on a finite grid with step $\Delta a : a\in\{a_1,...,a_W\}$, where $a_j=a_{j-1}+\Delta a=a_1+(j-1)\,\Delta a$ for $2\leq j\leq J$. The bounds are $a_1=\phi$ and $a_I=\varkappa$, such that $\Delta a=(\varkappa-\phi)\,/\,(J-1)$. We use the notation $v_{i,j}\equiv v_i(a_j)$, $i=1,2$, and similarly for the policy function $c_{i,j}$.

Notice first that the HJB equation involves first derivatives of the value function, $v_i'(a)$ and $v_i''(a)$. At each point of the grid, the first derivative can be approximated with a forward (F) or a backward (B) approximation,

$$v_i'(a_j) \approx \partial_F v_{i,j} \equiv \frac{v_{i,j+1} - v_{i,j}}{\Delta a}, \tag{49}$$

$$v_i'(a_j) \approx \partial_B v_{i,j} \equiv \frac{v_{i,j} - v_{i,j-1}}{\Delta a}. \tag{50}$$

In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable, given by

$$s_i(a) \equiv \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{(y_i - c_i(a))}{Q}, \tag{51}$$

for $\phi \leq a \leq 0$, where

$$c_i(a) = \left[\frac{v_i'(a)}{Q}\right]^{-1/\gamma}. \tag{52}$$

Let superscript n denote the iteration counter. The HJB equation is approximated by the following upwind scheme,

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} + \rho v_{i,j}^{n+1} = \frac{(c_{i,j}^{n})^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 + \partial_F v_{i,j}^{n+1} s_{i,j,F}^{n} \mathbf{1}_{s_{i,j,F}^{n} > 0} + \partial_B v_{i,j}^{n+1} s_{i,j,B}^{n} \mathbf{1}_{s_{i,j,B}^{n} < 0} + \lambda_i \left(v_{-i,j}^{n+1} - v_{i,j}^{n+1}\right),$$

for $i = 1, 2$, $j = 1, ..., J$, where $\mathbf{1}(\cdot)$ is the indicator function and

$$s_{i,,jF}^{n} = \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{y_i - \left[\frac{Q}{\partial_F v_{i,j}^{n}}\right]^{1/\gamma}}{Q}, \tag{53}$$

$$s_{i,j,B}^{n} = \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{y_i - \left[\frac{Q}{\partial_B v_{i,j}^{n}}\right]^{1/\gamma}}{Q}. \tag{54}$$

Therefore, when the drift is positive ($s_{i,,jF}^{n} > 0$) we employ a forward approximation of the derivative, $\partial_F v_{i,j}^{n+1}$; when it is negative ($s_{i,j,B}^{n} < 0$) we employ a backward approximation, $\partial_B v_{i,j}^{n+1}$. The term $\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} \to 0$ as $v_{i,j}^{n+1} \to v_{i,j}^{n}$. Moving all terms involving v^{n+1} to the left hand side and the rest to the right hand side, we obtain

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} + \rho v_{i,j}^{n+1} = \frac{(c_{i,j}^{n})^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 + v_{i,j-1}^{n+1}\alpha_{i,j}^{n} + v_{i,j}^{n+1}\beta_{i,j}^{n} + v_{i,j+1}^{n+1}\xi_{i,j}^{n} + \lambda_i v_{-i,j}^{n+1}, \tag{55}$$

where

$$\alpha_{i,j}^{n} \equiv -\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a},$$
$$\beta_{i,j}^{n} \equiv -\frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a}+\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a}-\lambda_{i},$$
$$\xi_{i,j}^{n} \equiv \frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a},$$

for $i=1,2,\ j=1,...,J$. Notice that the state constraints $\phi \leq a \leq 0$ mean that $s_{i,1,B}^{n}=s_{i,J,F}^{n}=0$.

In equation (55), the optimal consumption is set to

$$c_{i,j}^{n}=\left(\frac{\partial v_{i,j}^{n}}{Q}\right)^{-1/\gamma}. \tag{56}$$

where

$$\partial v_{i,j}^{n}=\partial_{F}v_{i,j}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}+\partial_{B}v_{i,j}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}+\partial\bar{v}_{i,j}^{n}\mathbf{1}_{s_{i,F}^{n}\leq 0}\mathbf{1}_{s_{i,B}^{n}\geq 0}.$$

In the above expression, $\partial\bar{v}_{i,j}^{n}=Q(\bar{c}_{i,j}^{n})^{-\gamma}$ where $\bar{c}_{i,j}^{n}$ is the consumption level such that $s\left(a_{i}\right)\equiv s_{i}^{n}=0:$

$$\bar{c}_{i,j}^{n}=\left(\frac{\delta}{Q}-\delta-\pi\right)a_{j}Q+y_{i}.$$

Equation (55) is a system of $2\times J$ linear equations which can be written in matrix notation as:

$$\frac{1}{\Delta}\left(\mathbf{v}^{n+1}-\mathbf{v}^{n}\right)+\rho\mathbf{v}^{n+1}=\mathbf{u}^{n}+\mathbf{A}^{n}\mathbf{v}^{n+1}$$

where the matrix $\mathbf{A}^n$ and the vectors v^{n+1} and $\mathbf{u}^n$ are defined by

$$\mathbf{A}^n = -\begin{bmatrix} \beta_{1,1}^n & \xi_{1,1}^n & 0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 \\ \alpha_{1,2}^n & \beta_{1,2}^n & \xi_{1,2}^n & 0 & \cdots & 0 & 0 & \lambda_1 & \ddots & 0 \\ 0 & \alpha_{1,3}^n & \beta_{1,3}^n & \xi_{1,3}^n & \cdots & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{1,J-1}^n & \beta_{1,J-1}^n & \xi_{1,J-1}^n & 0 & \cdots & \lambda_1 & 0 \\ 0 & 0 & \cdots & 0 & \alpha_{1,J}^n & \beta_{1,J}^n & 0 & 0 & \cdots & \lambda_1 \\ \lambda_2 & 0 & \cdots & 0 & 0 & 0 & \beta_{2,1}^n & \xi_{2,1}^n & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \lambda_2 & 0 & \cdots & \alpha_{2,J}^n & \beta_{2,J}^n \end{bmatrix}, \quad \mathbf{v}^{n+1} = \begin{bmatrix} v_{1,1}^{n+1} \\ v_{1,2}^{n+1} \\ v_{1,3}^{n+1} \\ \vdots \\ v_{1,J-1}^{n+1} \\ v_{1,J}^{n+1} \\ v_{2,1}^{n+1} \\ \vdots \\ v_{2,J}^{n+1} \end{bmatrix} \tag{57}$$

$$\mathbf{u}^n = \begin{bmatrix} \frac{(c_{1,1}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{1,2}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{1,J}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{2,1}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{2,J}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \end{bmatrix}.$$

The system in turn can be written as

$$\mathbf{B}^n \mathbf{v}^{n+1} = \mathbf{d}^n \tag{58}$$

where , $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{1}{\Delta}\mathbf{v}^n$.

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess $\{v_{i,j}^0\}_{j=1}^J$, $i = 1, 2$. Set $n = 0$. Then:

1. Compute $\{\partial_F v_{i,j}^n, \partial_B v_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (49)-(50).
2. Compute $\{c_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (52) as well as $\{s_{i,j,F}^n, s_{i,j,B}^n\}_{j=1}^J$, $i = 1, 2$ using (53) and (54).
3. Find $\{v_{i,j}^{n+1}\}_{j=1}^J$, $i = 1, 2$ solving the linear system of equations (58).
4. If $\{v_{i,j}^{n+1}\}$ is close enough to $\{v_{i,j}^{n+1}\}$, stop. If not set $n := n + 1$ and proceed to 1.

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}^n$.

Step 2: Solution to the Kolmogorov Forward equation The stationary distribution of debt-to-GDP ratio, $f(a)$, satisfies the Kolmogorov Forward equation:

$$\begin{aligned} 0 &= -\frac{d}{da}\left[s_i\left(a\right)f_i(a)\right] - \lambda_i f_i(a) + \lambda_{-i} f_{-i}(a), \; i = 1, 2. && (59) \\ 1 &= \int_{\phi}^{\infty} f(a)da. && (60) \end{aligned}$$

We also solve this equation using an finite difference scheme. We use the notation $f_{i,j} \equiv f_i(a_j)$. The system can be now expressed as

$$\begin{aligned} 0 = & -\frac{f_{i,j}s_{i,j,F}\mathbf{1}_{s^n_{i,j,F}>0} - f_{i,j-1}s_{i,j-1,F}\mathbf{1}_{s^n_{i,j-1,F}>0}}{\Delta a} - \frac{f_{i,j+1}s_{i,j+1,B}\mathbf{1}_{s^n_{i,j+1,B}<0} - f_{i,j}s_{i,,jB}\mathbf{1}_{s^n_{i,,jB}<0}}{\Delta a} \\ & -\lambda_i f_{i,j} + \lambda_{-i} f_{-i,j}, \end{aligned}$$

or equivalently

$$f_{i,j-1}\xi_{i,j-1} + f_{i,j+1}\alpha_{i,j+1} + f_{i,j}\beta_{i,j} + \lambda_{-i}f_{-i,j} = 0, \tag{61}$$

then (61) is also a system of $2 \times J$ linear equations which can be written in matrix notation as:

$$\mathbf{A^T f} = \mathbf{0}, \tag{62}$$

where $\mathbf{A^T}$ is the transpose of $\mathbf{A} = \lim_{n\to\infty} \mathbf{A}^n$. Notice that $\mathbf{A}^n$ is the approximation to the operator $\mathcal{A}$ and $\mathbf{A^T}$ is the approximation of the adjoint operator $\mathcal{A}^*$. In order to impose the normalization constraint (60) we replace one of the entries of the zero vector in equation (62) by a positive constant.[55] We solve the system (62) and obtain a solution $\hat{\mathbf{f}}$. Then we renormalize as

$$f_{i,j} = \frac{\hat{f}_{i,j}}{\sum_{j=1}^{J}\left(\hat{f}_{1,j} + \hat{f}_{2,j}\right)\Delta a}.$$

Complete algorithm The algorithm proceeds as follows.

[55] In particular, we have replaced the entry 2 of the zero vector in (62) by 0.1.

Step 1: Individual economy problem. Given π, compute the bond price Q using (48) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}$ and of the matrix $\mathbf{A}$.

Step 2: Aggregate distribution. Given $\mathbf{A}$ find the aggregate distribution $\mathbf{f}$.

B.2 Optimal monetary policy - Markov Stackelberg equilibrium

In this case we need to find the value of inflation that satisfies condition (19). The algorith proceeds as follows. We consider an initial guess of inflation, $\pi^{(1)} = 0$. Set $m := 1$. Then:

Step 1: Individual economy problem problem. Given $\pi^{(m)}$, compute the bond price $Q^{(m)}$ using (48) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}^{(m)}$ and of the matrix $\mathbf{A}^{(m)}$.

Step 2: Aggregate distribution. Given $\mathbf{A}^{(m)}$ find the aggregate distribution $\mathbf{f}^{(m)}$.

Step 3: Optimal inflation. Given $\mathbf{f}^{(m)}$ and $\mathbf{v}^{(m)}$, iterate steps 1-2 until $\pi^{(m)}$ satisfies[56]

$$\sum_{i=1}^{2}\sum_{j=2}^{J-1} a_j f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)}\right)}{2} + \psi\pi^{(m)} = 0.$$

B.3 Optimal monetary policy - Ramsey

Here we need to find the value of the inflation and of the costate that satisfy conditions (17) and (16) in steady-state. The algorith proceeds as follows. We consider an initial guess of inflation, $\pi^{(1)} = 0$. Set $m := 1$. Then:

Step 1: Individual economy problem problem. Given $\pi^{(m)}$, compute the bond price $Q^{(m)}$ using (48) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}^{(m)}$ and of the matrix $\mathbf{A}^{(m)}$.

Step 2: Aggregate distribution. Given $\mathbf{A}^{(m)}$ find the aggregate distribution $\mathbf{f}^{(m)}$.

[56] This can be done using Matlab's `fzero` function.

Step 3: Costate. Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$,compute the costate $\mu^{(m)}$ using condition (16) as

$$\mu^{(m)} = \frac{1}{Q^{(m)}}\left[\sum_{i=1}^{2}\sum_{j=2}^{J-1} a_j f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)}\right)}{2} + \psi\pi^{(m)}\right].$$

Step 4: Optimal inflation. Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$ and $\mu^{(m)}$, iterate steps 1-3 until $\pi^{(m)}$ satisfies

$$\left(\rho - \bar{r} - \pi^{(m)} - \delta\right)\mu^{(m)} + \frac{1}{\left(Q^{(m)}\right)^2}\left[\sum_{i=1}^{2}\sum_{j=2}^{J-1}\left(\delta a_j + y_i - c_{i,j}^{(m)}\right) f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)}\right)}{2}\right].$$

C. Computational method: the dynamic case

C.1 Exogenous monetary policy

We describe now the numerical algorithm to analyze the transitional dynamics, similar to the one described in Achdou et al. (2017). With an exogenous monetary policy it just amounts to solve the dynamic HJB equation (6) and then the dynamic KFE equation (11). Define T as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and discretize it in N intervals of lenght

$$\Delta t = \frac{T}{N}.$$

The initial distribution $f(0, a, y) = f_0(a, y)$ and the path of inflation $\{\pi_n\}_{n=0}^{N}$ are given. We proceed in three steps.

Step 0: The asymptotic steady-state The asymptotic steady-state distribution of the model can be computed following the steps described in Section B. Given π_N, the result is a stationary destribution $\mathbf{f}_N$, a matrix $\mathbf{A}_N$ and a bond price Q_N defined at the asymptotic time $T = N\Delta t$.

Step 1: Solution to the bond pricing equation The dynamic bond princing equation (10) can be approximated backwards as

$$\left(\bar{r}+\pi_n+\delta\right) Q_n=\delta+\frac{Q_{n+1}-Q_n}{\Delta t}, \Longleftrightarrow Q_n=\frac{Q_{n+1}+\delta \Delta t}{1+\Delta t\left(\bar{r}+\pi_n+\delta\right)}, \quad n=N-1, .., 0, \tag{63}$$

where Q_N is the asymptotic bond price from Step 0.

Step 2: Solution to the Hamilton-Jacobi-Bellman equation The dynamic HJB equation (6) can approximated using an upwind approximation as

$$\rho \mathbf{v}^n=\mathbf{u}^n+\mathbf{A}_n \mathbf{v}^n+\frac{\left(\mathbf{v}^{n+1}-\mathbf{v}^n\right)}{\Delta t},$$

where $\mathbf{A}^n$ is constructing backwards in time using a procedure similar to the one described in Step 1 of Section B. By defining $\mathbf{B}^n=\left(\frac{1}{\Delta t}+\rho\right) \mathbf{I}-\mathbf{A}_n$ and $\mathbf{d}^n=\mathbf{u}^n+\frac{\mathbf{V}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n=\left(\mathbf{B}^n\right)^{-1} \mathbf{d}^n. \tag{64}$$

Step 3: Solution to the Kolmogorov Forward equation Let $\mathbf{A}_n$ defined in (57) be the approximation to the operator $\mathcal{A}$. Using a finite difference scheme similar to the one employed in the Step 2 of Section A, we obtain:

$$\frac{\mathbf{f}_{n+1}-\mathbf{f}_n}{\Delta t}=\mathbf{A}_n^{\mathbf{T}} \mathbf{f}_{n+1}, \Longleftrightarrow \mathbf{f}_{n+1}=\left(\mathbf{I}-\Delta t \mathbf{A}_n^{\mathbf{T}}\right)^{-1} \mathbf{f}_n, \quad n=1, .., N \tag{65}$$

where $\mathbf{f}_0$ is the discretized approximation to the initial distribution $f_0(b)$.

Complete algorithm The algorithm proceeds as follows:

Step 0: Asymptotic steady-state. Given π_N, compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N.

Step 1: Bond pricing. Given $\{\pi_n\}_{n=0}^{N-1}$, compute the bond price path $\{Q_n\}_{n=0}^{N-1}$ using (63).

Step 2: Individual economy problem. Given $\{\pi_n\}_{n=0}^{N-1}$ and $\{Q_n\}_{n=0}^{N-1}$ solve the HJB equation (64) backwards to obtain an estimate of the value function $\{\mathbf{v}_n\}_{n=0}^{N-1}$, and of the matrix $\{\mathbf{A}_n\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\{\mathbf{A}_n\}_{n=0}^{N-1}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (65).

C.2 Optimal monetary policy - Markov Stackelberg equilibrium

In this case we need to find the value of inflation that satisfies condition (19)

Step 0: Asymptotic steady-state. Compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N and inflation rate π_N. Set $\pi^{(0)} \equiv \{\pi_n^{(0)}\}_{n=0}^{N-1} = \pi_N$ and $k := 1$.

Step 1: Bond pricing. Given $\pi^{(k-1)}$, compute the bond price path $Q^{(k)} \equiv \{Q_n^{(k)}\}_{n=0}^{N-1}$ using (63).

Step 2: Individual economy problem. Given $\pi^{(k-1)}$ and $Q^{(k)}$ solve the HJB equation (64) backwards to obtain an estimate of the value function $\mathbf{v}^{(k)} \equiv \{\mathbf{v}_n^{(k)}\}_{n=0}^{N-1}$ and of the matrix $\mathbf{A}^{(k)} \equiv \{\mathbf{A}_n^{(k)}\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\mathbf{A}^{(k)}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (65).

Step 4: Optimal inflation. Given $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$, iterate steps 1-3 until $\pi^{(k)}$ satisfies

$$\Theta_n^{(k)} \equiv \sum_{i=1}^{2} \sum_{j=2}^{J-1} a_j f_{n,i,j}^{(k)} \frac{\left(v_{n,i,j+1}^{(k)} - v_{n,i,j-1}^{(k)}\right)}{2} + \psi \pi_n^{(k)} = 0.$$

This is done by iterating

$$\pi_n^{(k)} = \pi_n^{(k-1)} - \xi \Theta_n^{(k)},$$

with constant $\xi = 0.05$.

C.3 Optimal monetary policy - Ramsey

In this case we need to find the value of the inflation and of the costate that satisfy conditions (17) and (16)

Step 0: Asymptotic steady-state. Compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N and inflation rate π_N. Set $\pi^{(0)} \equiv \{\pi_n^{(0)}\}_{n=0}^{N-1} = \pi_N$ and $k := 1$.

Step 1: Bond pricing. Given $\pi^{(k-1)}$, compute the bond price path $Q^{(k)} \equiv \{Q_n^{(k)}\}_{n=0}^{N-1}$ using (63).

Step 2: Individual economy problem. Given $\pi^{(k-1)}$ and $Q^{(k)}$ solve the HJB equation (64) backwards to obtain an estimate of the value function $\mathbf{v}^{(k)} \equiv \{\mathbf{v}_n^{(k)}\}_{n=0}^{N-1}$ and of the matrix $\mathbf{A}^{(k)} \equiv \{\mathbf{A}_n^{(k)}\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\mathbf{A}^{(k)}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (65).

Step 4: Costate. Given $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$, compute the costate $\mu^{(k)} \equiv \{\mu_n^{(k)}\}_{n=0}^{N-1}$ using (17):

$$\begin{aligned}\mu_{n+1}^{(k)} &= \mu_n^{(k)} \left[1 + \Delta t \left(\rho - \bar{r} - \pi^{(k)} - \delta\right)\right] \\ &\quad + \frac{\Delta t}{\left(Q_n^{(k)}\right)^2} \left[\sum_{i=1}^{2} \sum_{j=2}^{J-1} \left(\delta a_j + y_i - c_{n,i,j}^{(k)}\right) f_{n,i,j}^{(k+1)} \frac{\left(v_{n,i,j+1}^{(k)} - v_{n,i,j-1}^{(k)}\right)}{2}\right],\end{aligned}$$

with $\mu_0^{(k)} = 0$.

Step 5: Optimal inflation. Given $\mathbf{f}^{(k)}$, $\mathbf{v}^{(k)}$ and $\mu^{(k)}$ iterate steps 1-4 until $\pi^{(k)}$ satisfies

$$\Theta_n^{(k)} \equiv \sum_{i=1}^{2} \sum_{j=2}^{J-1} a_j f_{n,i,j}^{(k)} \frac{\left(v_{n,i,j+1}^{(k)} - v_{n,i,j-1}^{(k)}\right)}{2} + \psi \pi_n^{(k)} - Q_n^{(k)} \mu_n^{(k)} = 0.$$

This is done by iterating

$$\pi_n^{(k)} = \pi_n^{(k-1)} - \xi \Theta_n^{(k)}.$$

D. An economy with costly price adjustment

In this appendix, we lay out a model economy with the following characteristics: (i) firms are explicitly modelled, (ii) a subset of them are price-setters but incur a convex cost for changing their nominal price, and (iii) the social welfare function and the equilibrium conditions constraining the central bank's problem are the same as in the model economy in the main text.

Final good producer

In the model laid out in the main text, we assumed that output of the consumption good was exogenous. Consider now an alternative setup in which the consumption good is produced by a representative, perfectly competitive final good producer with the following Dixit-Stiglitz technology,

$$y_t = \left(\int_0^1 y_{jt}^{(\varepsilon-1)/\varepsilon} dj \right)^{\varepsilon/(\varepsilon-1)}, \tag{66}$$

where $\{y_{jt}\}$ is a continuum of intermediate goods and $\varepsilon > 1$. Let P_{jt} denote the nominal price of intermediate good $j \in [0, 1]$. The firm chooses $\{y_{jt}\}$ to maximize profits, $P_t y_t - \int_0^1 P_{jt} y_{jt} dj$, subject to (66). The first order conditions are

$$y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} y_t, \tag{67}$$

for each $j \in [0, 1]$. Assuming free entry, the zero profit condition and equations (67) imply $P_t = (\int_0^1 P_{jt}^{1-\varepsilon} dj)^{1/(1-\varepsilon)}$.

Intermediate goods producers

Each intermediate good j is produced by a monopolistically competitive intermediate-good producer, which we will refer to as 'firm j' henceforth for brevity. Firm j operates a linear production technology,

$$y_{jt} = n_{jt}, \tag{68}$$

where n_{jt} is labor input. At each point in time, firms can change the price of their product but face quadratic price adjustment cost as in Rotemberg (1982). Letting $\dot{P}_{jt} \equiv dP_{jt}/dt$ denote the change in the firm's price, price adjustment costs in units of the final good are given by

$$\Psi_t \left(\frac{\dot{P}_{jt}}{P_{jt}} \right) \equiv \frac{\psi}{2} \left(\frac{\dot{P}_{jt}}{P_{jt}} \right)^2 \tilde{C}_t, \tag{69}$$

where $\tilde{C}_t$ is aggregate consumption. Let $\pi_{jt} \equiv \dot{P}_{jt}/P_{jt}$ denote the rate of increase in the firm's price. The instantaneous profit function in units of the final good is given by

$$\begin{aligned}\Pi_{jt} &= \frac{P_{jt}}{P_t}y_{jt} - w_t n_{jt} - \Psi_t\left(\pi_{jt}\right) \\ &= \left(\frac{P_{jt}}{P_t} - w_t\right)\left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} y_t - \Psi_t\left(\pi_{jt}\right),\end{aligned} \tag{70}$$

where w_t is the perfectly competitive real wage and in the second equality we have used (67) and (68).[57] Without loss of generality, firms are assumed to be risk neutral and have the same discount factor as households, ρ. Then firm j's objective function is

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t}\Pi_{jt}dt,$$

with Π_{jt} given by (70). The state variable specific to firm j, P_{jt}, evolves according to $dP_{jt} = \pi_{jt}P_{jt}dt$. The aggregate state relevant to the firm's decisions is simply time: t. Then firm j's *value function* $V\left(P_{jt}, t\right)$ must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho V\left(P_j, t\right) = \max_{\pi_j}\left\{\left(\frac{P_j}{P_t} - w_t\right)\left(\frac{P_j}{P_t}\right)^{-\varepsilon} y_t - \Psi_t\left(\pi_j\right) + \pi_j P_j \frac{\partial V}{\partial P_j}\left(P_j, t\right)\right\} + \frac{\partial V}{\partial t}\left(P_j, t\right).$$

The first order and envelope conditions of this problem are (we omit the arguments of V to ease the notation),

$$\psi \pi_{jt}\tilde{C}_t = P_j \frac{\partial V}{\partial P_j}, \tag{71}$$

$$\rho \frac{\partial V}{\partial P_j} = \left[\varepsilon w_t - \left(\varepsilon - 1\right)\frac{P_j}{P_t}\right]\left(\frac{P_j}{P_t}\right)^{-\varepsilon}\frac{y_t}{P_j} + \pi_j\left(\frac{\partial V}{\partial P_j} + P_j \frac{\partial^2 V}{\partial P_j^2}\right).$$

In what follows, we will consider a symmetric equilibrium in which all firms choose the same price: $P_j = P, \pi_j = \pi$ for all j. After some algebra, it can be shown that the above conditions imply the following pricing Euler equation,[58]

$$\left[\rho - \frac{d\tilde{C}\left(t\right)}{dt}\frac{1}{\tilde{C}\left(t\right)}\right]\pi\left(t\right) = \frac{\varepsilon - 1}{\psi}\left(\frac{\varepsilon}{\varepsilon - 1}w\left(t\right) - 1\right)\frac{1}{\tilde{C}_t} + \frac{d\pi\left(t\right)}{dt}. \tag{72}$$

[57] In the proofs of Propositions 1 and 3, w has been used to denote the social value of individual households. Nonetheless, there is no possibility of confusion in this section.

[58] The proof is available upon request.

Equation (72) determines the market clearing wage $w(t)$.

Households

The preferences of household $k \in [0,1]$ are given by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \log(\tilde{c}_{kt})\, dt,$$

where $\tilde{c}_{kt}$ is household consumption of the final good. We now define the following object,

$$c_{kt} \equiv \tilde{c}_{kt} + \frac{\tilde{c}_{kt}}{\tilde{C}_t} \int_0^1 \Psi_t(\pi_{jt})\, dj,$$

i.e. household k's consumption plus a fraction of total price adjustment costs ($\int \Psi_t(\cdot)\, dj$) equal to that household's share of total consumption ($\tilde{c}_{kt}/\tilde{C}_t$). Using the definition of Ψ_t (eq. 69) and the symmetry across firms in equilibrium ($\dot{P}_{jt}/P_{jt} = \pi_t, \forall j$), we can write

$$c_{kt} = \tilde{c}_{kt} + \tilde{c}_{kt}\frac{\psi}{2}\pi_t^2 = \tilde{c}_{kt}\left(1 + \frac{\psi}{2}\pi_t^2\right). \tag{73}$$

Therefore, household k's instantaneous utility can be expressed as

$$\begin{aligned} \log(\tilde{c}_{kt}) &= \log(c_{kt}) - \log\left(1 + \frac{\psi}{2}\pi_t^2\right) \\ &= \log(c_{kt}) - \frac{\psi}{2}\pi_t^2 + o\left(\left\|\frac{\psi}{2}\pi_t^2\right\|^2\right), \end{aligned} \tag{74}$$

where $o(\|x\|^2)$ denotes terms of order second and higher in x. Expression (74) is the same as the utility function in the main text (eq. 20), up to a first order approximation of $\log(1+x)$ around $x = 0$, where $x \equiv \frac{\psi}{2}\pi^2$ represents the percentage of aggregate spending that is lost to price adjustment. For our baseline calibration ($\psi = 5.5$), the latter object is relatively small even for relatively high inflation rates, and therefore so is the approximation error in computing the utility losses from price adjustment. Therefore, the utility function used in the main text provides a fairly accurate approximation of the welfare losses caused by inflation in the economy with costly price adjustment described here.

Households can be in one of two idiosyncratic states. Those in state $i = 1$ do not work. Those in state $i = 2$ work and provide z units of labor inelastically. As in the main text, the instantaneous transition rates between both states are given by λ_1 and λ_2, and the share of households in each state is assumed to have reached its ergodic distribution; therefore, the fraction of working and non-working households is $\lambda_1/(\lambda_1+\lambda_2)$ and $\lambda_2/(\lambda_1+\lambda_2)$, respectively. Hours per worker z are such that total labor supply $\frac{\lambda_1}{\lambda_1+\lambda_2}z$ is normalized to 1.

An exogenous government insurance scheme imposes a (total) lump-sum transfer τ_t from working to non-working households. All households receive, in a lump-sum manner, an equal share of aggregate firm profits *gross* of price adjustment costs, which we denote by $\hat{\Pi}_t \equiv P_t^{-1}\int_0^1 P_{jt}y_{jt}dj - w_t\int_0^1 n_{jt}dj$. Therefore, disposable income (gross of price adjustment costs) for non-working and working households are given respectively by

$$I_{1t} \equiv \frac{\tau_t}{\lambda_2/(\lambda_1+\lambda_2)} + \hat{\Pi}_t,$$

$$I_{2t} \equiv w_t z - \frac{\tau_t}{\lambda_1/(\lambda_1+\lambda_2)} + \hat{\Pi}_t.$$

We assume that the transfer τ_t is such that gross disposable income for households in state i equals a constant level y_i, $i=1,2$, with $y_1<y_2$. As in our baseline model, both income levels satisfy the normalization

$$\frac{\lambda_2}{\lambda_1+\lambda_2}y_1 + \frac{\lambda_1}{\lambda_1+\lambda_2}y_2 = 1.$$

Also, later we show that in equilibrium gross income equals one: $\hat{\Pi}_t + w_t\frac{\lambda_1}{\lambda_1+\lambda_2}z = 1$. It is then easy to verify that implementing the gross disposable income allocation $I_{it}=y_i$, $i=1,2$, requires a transfer equal to $\tau_t = \frac{\lambda_2}{\lambda_1+\lambda_2}y_1 - \frac{\lambda_2}{\lambda_1+\lambda_2}\hat{\Pi}_t$. Finally, total price adjustment costs are assumed to be distributed in proportion to each household's share of total consumption, i.e. household k incurs adjustment costs in the amount $(\tilde{c}_{kt}/\tilde{C}_t)(\frac{\psi}{2}\pi_t^2\tilde{C}_t) = \tilde{c}_{kt}\frac{\psi}{2}\pi_t^2$. Letting $I_{kt}\equiv y_{kt}\in\{y_1,y_2\}$ denote household k's gross disposable income, the law of motion of that household's real net wealth is thus given by

$$\begin{aligned} da_{kt} &= \left[\left(\frac{\delta}{Q_t}-\delta-\pi_t\right)a_{kt} + \frac{I_{kt}-\tilde{c}_{kt}-\tilde{c}_{kt}\psi\pi_t/2}{Q_t}\right]dt \\ &= \left[\left(\frac{\delta}{Q_t}-\delta-\pi_t\right)a_{kt} + \frac{y_{kt}-c_{kt}}{Q_t}\right]dt, \end{aligned} \tag{75}$$

where in the second equality we have used (73). Equation (75) is exactly the same as its counterpart in the main text, equation (4). Since household's welfare criterion is also the same, it follows that so is the corresponding maximization problem.

Aggregation and market clearing

In the symmetric equilibrium, each firm's labor demand is $n_{jt} = y_{jt} = \bar{y}_t$. Since labor supply $\frac{\lambda_1}{\lambda_1+\lambda_2} z = 1$ equals one, labor market clearing requires

$$\int_0^1 n_{jt} dj = \bar{y}_t = 1.$$

Therefore, in equilibrium aggregate output is equal to one. Firms' profits gross of price adjustment costs equal

$$\hat{\Pi}_t = \int_0^1 \frac{P_{jt}}{P_t} y_{jt} dj - w_t \int_0^1 n_{jt} dj = \bar{y}_t - w_t,$$

such that gross income equals $\hat{\Pi}_t + w_t = \bar{y}_t = 1$.

Central bank and monetary policy

We have shown that households' welfare criterion and maximization problem are as in our baseline model. Thus the dynamics of the net wealth distribution continue to be given by equation (11). Foreign investors can be modelled exactly as in Section 2.2. Therefore, the central bank's optimal policy problems, both under commitment and discretion, are exactly as in our baseline model.

E. The methodology in discrete time

The aim of this appendix is to illustrate how the methodology can be extended to discrete-time models. We assume again that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a filtered probability space but time is discrete: $t \in \mathbb{N}$.

E.1. Model

Households The domestic price at time t, P_t, evolves according to

$$P_t = (1 + \pi_t) P_{t-1}, \tag{76}$$

where π_t is the domestic inflation rate.

Household $k \in [0,1]$ is endowed with an income y_{kt} per period, where y_{kt} follows a two-state Markov chain: $y_{kt} \in \{y_1, y_2\}$, with $y_1 < y_2$. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

Outstanding bonds are amortized at rate $\delta > 0$ per period. The nominal value of the household's net asset position A_{kt} evolves as follows,

$$A_{kt+1} = A_{kt}^{new} + (1-\delta) A_{kt},$$

where A_{kt}^{new} is the flow of new issuances. The nominal market price of bonds at time t is Q_t and c_{kt} is the household's consumption. The budget constraint of household k is

$$Q_t A_{kt}^{new} = P_t (y_{kt} - c_{kt}) + \delta A_{kt}.$$

The dynamics for net nominal wealth are

$$A_{kt+1} = (1 + r_t) A_{kt} + \frac{P_t (y_{kt} - c_{kt})}{Q_t}. \tag{77}$$

where $r_t \equiv \frac{\delta}{Q_t} - \delta$ is the nominal bond yield. The dynamics of the real net wealth as $a_{kt} \equiv A_{kt}/P_t$ are

$$a_{kt+1} = \frac{1}{1+\pi_t} \left[(1 + r_t) a_{kt} + \frac{y_{kt} - c_{kt}}{Q_t} \right] = s_t (a_{kt}, y_{kt}). \tag{78}$$

From now onwards we drop subscripts k for ease of exposition. For any Borel subset $\tilde{A}$ of Φ we define the transition function associated to the stochastic process a_t as

$$H_t \left[(a, y_i), \left(\tilde{A}, y_j \right) \right] = \mathbb{P}(a_{t+1} \in \tilde{A}, y_{t+1} = y_j | a_t = a, y_t = y_i), \quad i, j = 1, 2.$$

This transition function equals

$$H_t \left[(a, y_i), \left(\tilde{A}, y_j \right) \right] = p_{ij} \mathbf{1}_{\tilde{A}} (s_{t,i}(a)),$$

where $\mathbf{1}_{\tilde{A}}(\cdot)$ is the indicator function of subset $\tilde{A}$ and $s_{t,i}(a) \equiv s_t(a, y_i)$.

Household have preferences over paths for consumption c_{kt} and domestic inflation π_t discounted at rate $\beta > 0$,

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left(u(c_t)-x\left(\pi_t\right)\right)\right]. \tag{79}$$

We use the short-hand notation $v_i(t,a) \equiv v(t,a,y_i)$ for the value function when household income is low ($i=1$) and high ($i=2$). The Bellman equation results in

$$v_i(t,a) = \max_{c_t} u(c_t) - x\left(\pi_t\right) + \beta\left(\mathcal{T}v_i\right)(t+1,a),\ \ i=1,2, \tag{80}$$

where operator $\mathcal{T}$ is the Markov operator associated with (78), defined as[59]

$$\begin{aligned}\left(\mathcal{T}v_i\right)(t+1,a) &= \mathbb{E}_t\left[v(t+1,a_{t+1},y_{t+1})|a_t=a,y_t=y_i\right] \qquad (81)\\ &= \sum_{j=1}^{2}\int v_j\left(t+1,a'\right)H_t\left[\left(a,y_i\right),\left(da',y_j\right)\right] = \sum_{j=1}^{2}p_{ij}v_j(t+1,s_{t,i}\left(a\right)).\end{aligned}$$

The first order condition of the individual problem is

$$u'\left(c_i\right)+\beta\left(\mathcal{T}\frac{\partial v_i}{\partial a}\right)(t+1,a)\frac{\partial s_{t,i}\left(a\right)}{\partial c_i} = u'\left(c_i\right)-\left(\mathcal{T}\frac{\partial v_i}{\partial a}\right)(t+1,a)\frac{\beta}{\left(1+\pi_t\right)Q_t} = 0 \tag{82}$$

Foreign investors The nominal price of the bond at time t is given by

$$Q_t = \frac{\delta+(1-\delta)\,Q_{t+1}}{\left(1+\pi_t\right)\left(1+\bar{r}\right)}.$$

Distribution dynamics The state of the economy at time t is the joint density of net wealth and income, $f(t,a,y_i) \equiv f_i(t,a)$, $i=1,2$. The dynamics of this density are given by the *Chapman–Kolmogorov* (CK) equation,

$$f_i(t,a) = \left(\mathcal{T}^*f_i\right)(t-1,a) \tag{83}$$

[59] Notice that we consider the complete space $\mathbb{R}$ as the borrowing limit affects the dynamics through the admissible consumption paths.

where the adjoint operator $\mathcal{T}_{t-1}^{*}$ is defined as

$$(\mathcal{T}^{*} f_i)\,(t-1,a) = \sum_{j=1}^{2} \int H_{t-1}\left[(a', y_j), (a, y_i)\right] f_j(t-1, a') da' = \sum_{j=1}^{2} p_{ji} \frac{f_j(t-1, s_{t-1,j}^{-1}(a))}{ds_{t-1,j}/da}, \tag{84}$$

where $s_{t,i}^{-1}(a)$ is the inverse function of $s_{t,i}(a)$: if $a' = s_{t,i}(a)$ then $a = s_{t,i}^{-1}(a')$.

The proof of the CK equation is as follows. Let

$$\mathbb{P}(a_t \leq a, y_t = y_i) = \int_{-\infty}^{a} f_i(t, a')\, da',$$

be the joint probability of $a_t \leq a$ and $y_t = y_i$. It is equal to

$$\sum_{j=1}^{2} p_{ji} \int_{-\infty}^{s_{t-1,j}^{-1}(a)} f_j(t-1, a')\, da',$$

and taking derivatives with respect to a:

$$f_i(t,a) = \sum_{j=1}^{2} p_{ji} f_j\left(t-1, s_{t-1,j}^{-1}(a)\right) \frac{ds_{t-1,j}^{-1}(a)}{da} = \sum_{j=1}^{2} p_{ji} \frac{f_j(t-1, s_{t-1,j}^{-1}(a))}{ds_{t-1,j}/da},$$

where we have applied the inverse function theorem.

If we define $\mathcal{T}v(t,\cdot) = [\mathcal{T}v_1(t,\cdot), \mathcal{T}v_2(t,\cdot)]^{\mathbf{T}}$ and $\mathcal{T}^{*}f(t,\cdot) = [\mathcal{T}^{*}f_1(t,\cdot), \mathcal{T}^{*}f_2(t,\cdot)]^{\mathbf{T}}$ the inner product results in

$$\begin{aligned} \langle \mathcal{T}v(t+1,\cdot), f(t,\cdot)\rangle &= \sum_{i=1}^{2} \int (\mathcal{T}v_i)(t+1,a) f_i(t,a) da = \sum_{i=1}^{2} \int \sum_{j=1}^{2} p_{ij} v_j(t+1, s_{t,j}(a)) f_i(t,a) da \\ &= \sum_{j=1}^{2} \int \sum_{i=1}^{2} p_{ij} f_i(t,a) v_j(t+1, s_{t,j}(a)) da. \end{aligned}$$

By changing variable $a' = s_{t,i}(a)$:

$$\begin{aligned}\langle \mathcal{T} v(t+1,\cdot), f(t,\cdot)\rangle &= \sum_{j=1}^{2} \int \sum_{i=1}^{2} p_{ij} f_i(t, s_{t,i}^{-1}(a')) v_j(t+1,a') \frac{da'}{ds_{t,i}/da} \\ &= \sum_{j=1}^{2} \int \left[\sum_{i=1}^{2} p_{ij} \frac{f_i(t, s_{t,i}^{-1}(a'))}{ds_{t,i}/da} \right] v_j(t,a') da' \\ &= \sum_{j=1}^{2} \int \left(\mathcal{T}_t^* f_j\right)(t,a') v_j(t,a') da' = \langle v(t+1,\cdot), \mathcal{T}_t^* f(t,\cdot)\rangle ,\end{aligned}$$

showing that $\mathcal{T}$ and $\mathcal{T}^*$ are adjoint operators with one period lag.[60]

E.2. Optimal monetary policy (Ramsey)

Central bank preferences The central maximizes economy-wide aggregate welfare,

$$W_0 = \sum_{t=0}^{\infty} \beta^t \left[\int_{\phi}^{\infty} \sum\nolimits_{i=1}^{2} \left[u\left(c_i\left(t,a\right)\right) - x\left(\pi\left(t\right)\right)\right] f_i(t,a) da \right]. \tag{85}$$

Lagragian In this case the Lagragian can be written as

$$\begin{aligned}\mathcal{L}\left[\pi, Q, f, v, c\right] &= \sum_{t=0}^{\infty} \beta^t \langle u_t - x_t, f_t\rangle + \sum_{t=0}^{\infty} \left\langle \beta^t \zeta_t, \mathcal{T}^* f_{t-1} - f_t \right\rangle \\ &+ \sum_{t=0}^{\infty} \beta^t \mu_t \left(Q_t - \frac{\delta + (1-\delta) Q_{t+1}}{(1+\pi_t)(1+\bar{r})} \right) \\ &+ \sum_{t=0}^{\infty} \left\langle \beta^t \theta_t, u_t - x_t + \beta \mathcal{T} v_{t+1} - v_t \right\rangle \\ &+ \sum_{t=0}^{\infty} \left\langle \beta^t \eta_t, u_t' - \frac{\beta}{(1+\pi_t) Q_t} \left(\mathcal{T} \frac{\partial v_{t+1}}{\partial a} \right) \right\rangle ,\end{aligned}$$

where $\beta^t \zeta_t(a)$, $\beta^t \eta_t(a)$, $\beta^t \theta_t(a) e^{-\rho t} \mu_t$ are Lagrange multipliers.

The problem of the central bank in this case is

$$\max_{\{\pi_s, Q_s, v_s(\cdot), c_s(\cdot), f_s(\cdot)\}_{s=0}^{\infty}} \mathcal{L}\left[\pi, Q, f, v, c\right]. \tag{86}$$

[60] A general proof for the time-invariant case can be found in theorem 8.3 in Stockey and Lucas (1989).

We can apply the fact that $\mathcal{T}$ and $\mathcal{T}^*$ are adjoint operators to express

$$\begin{aligned}
\left\langle \beta^t \zeta_t, \mathcal{T}^* f_{t-1} - f_t \right\rangle &= \beta^t \left\langle \mathcal{T} \zeta_t, f_t \right\rangle - \beta^t \left\langle \zeta_t, f_t \right\rangle, \\
\left\langle \beta^t \theta_t, u_t - x_t + \beta \mathcal{T} v_{t+1} - v_t \right\rangle &= \beta^t \left\langle \theta_t, u_t - x_t - v_t \right\rangle + \beta^{t+1} \left\langle \mathcal{T}^* \theta_t, v_{t+1} \right\rangle, \\
\left\langle \beta^t \eta_t, u_t' - \frac{\beta}{(1+\pi_t) Q_t} \left(\mathcal{T} \frac{\partial v_{t+1}}{\partial a} \right) \right\rangle &= \beta^t \left\langle \eta_t, u_t' \right\rangle - \frac{\beta^{t+1}}{(1+\pi_t) Q_t} \left\langle \mathcal{T}^* \eta_t, \frac{\partial v_{t+1}}{\partial a} \right\rangle.
\end{aligned}$$

Necessary conditions In order to find the maximum, we need to take the Gateaux derivative with respect to the controls f, π, Q, v and c.

The Gateaux derivative with respect to $f_t(\cdot)$ in the direction h is

$$\beta^t \left\langle u_t - x_t, h_t \right\rangle + \beta^{t+1} \left\langle \mathcal{T} \zeta_{t+1}, h_t \right\rangle - \beta^t \left\langle \zeta_t, h_t \right\rangle = 0. \tag{87}$$

Expression (87) should equal zero for any function $h_{it}(\cdot) \in L^2(\mathbb{R})$, $i = 1, 2$:

$$\zeta_i(t, a) = u(c_{t,i}) - x(\pi_t) + \beta (\mathcal{T} \zeta_i)(t, a),$$

which coincides with the household's Bellman equation (80) and hence $\zeta_i(t, a) = v_i(t, a)$.

In the case of $c_t(a)$, the Gateaux derivative is

$$\begin{aligned}
&\beta^t \left\langle u_t' h_t, f_t \right\rangle - \frac{\beta^{t+1}}{(1+\pi_t) Q_t} \left\langle h_t \mathcal{T} \frac{\partial \zeta_{+1}}{\partial a}, f_t \right\rangle + \beta^t \left\langle \theta_t, u_t' h_t \right\rangle - \frac{\beta^{t+1}}{(1+\pi_t) Q_t} \left\langle \theta_t, h_t \mathcal{T} \frac{\partial v_{t+1}}{\partial a} \right\rangle \\
&+ \beta^t \left\langle \eta_t, u_t'' h_t \right\rangle + \frac{\beta^{t+1}}{(1+\pi_t)^2 Q_t^2} \left\langle \eta_t, h_t \left(\mathcal{T} \frac{\partial^2 v_{t+1}}{\partial a^2} \right) \right\rangle,
\end{aligned}$$

where we have applied the fact that $\frac{\partial}{\partial c} \mathcal{T} \frac{\partial v_{t+1}}{\partial a} = -\frac{1}{(1+\pi_t) Q_t} \mathcal{T} \frac{\partial^2 v_{t+1}}{\partial a^2}$. This expression should be zero for any function $h_{it}(\cdot) \in L^2(\mathbb{R})$, $i = 1, 2$. Notice that

$$\left\langle \theta_t, \left(u_t' - \frac{1}{(1+\pi_t) Q_t} \beta \mathcal{T} \frac{\partial v_{t+1}}{\partial a} \right) h_t \right\rangle = 0$$

due to the first order condition of the individual problem (82). Analogously,

$$\left\langle f_t, \left(u_t' - \frac{1}{(1+\pi_t) Q_t} \beta \mathcal{T} \frac{\partial \zeta_{t+1}}{\partial a} \right) h_t \right\rangle = 0$$

as $\zeta = v$. Therefore the optimality condition with respect to c results in

$$\eta_t \left[u_t'' + \frac{\beta}{(1+\pi_t)^2 Q_t^2} \left(\mathcal{T} \frac{\partial^2 v_t}{\partial a^2} \right) \right] = 0 \tag{88}$$

As the instantaneous utility function is assumed to be strictly concave, $u_t'' < 0$, and the individual value function v is also strictly concave $\frac{\partial^2 v_t}{\partial a^2} < 0$ for all t and a, then

$$u_t'' + \frac{\beta}{(1+\pi_t)^2 Q_t^2} \left(\mathcal{T} \frac{\partial^2 v_t}{\partial a^2} \right) < 0$$

and the equality in equation (88) is only satisfied if $\eta_i(t, \cdot) = 0$, $i = 1, 2$.

In the case of $v_t(a)$, the Gateaux derivative is

$$-\beta^t \langle \theta_t, h_t \rangle + \beta^t \langle \mathcal{T}^* \theta_{t-1}, h_t \rangle,$$

where we have taken into account the fact that $\eta_i(t, \cdot) = 0$. The Gateaux derivative should be zero for any function $h_{it}(\cdot) \in L^2(\mathbb{R})$, $i = 1, 2$ so that we obtain a CK equation that describes the propagation of the "promises" to the individual households:

$$\theta_t = \mathcal{T}^* \theta_{t-1},$$

where $\theta_{-1} = 0$ as there are no precommitments. Hence $\theta_i(t, \cdot) = 0$, $i = 1, 2$..

In the case of Q_t, we compute the standard (finite-dimensional) derivative:

$$\begin{aligned} \beta^{t+1} \left\langle \frac{\partial}{\partial Q_t} \mathcal{T} v_{t+1}, f_t \right\rangle + \beta^t \mu_t - \beta^{t-1} \mu_{t-1} \frac{(1-\delta)}{(1+\pi_{t-1})(1+\bar{r})} &= 0, \\ \beta \left\langle \left[-\frac{\delta}{Q_t^2} a - \frac{(y_t - c_t)}{Q_t^2} \right] \mathcal{T} v_{t+1}, f_t \right\rangle + \mu_t - \beta^{-1} \mu_{t-1} \frac{(1-\delta)}{(1+\pi_{t-1})(1+\bar{r})} &= 0, \end{aligned}$$

and thus

$$\mu_t = \frac{\mu_{t-1}(1-\delta)}{\beta(1+\pi_{t-1})(1+\bar{r})} + \frac{\beta}{Q_t^2} \sum_{i=1}^{2} \int (\delta a + y_i - c_i(t, a)) \left(\mathcal{T} \frac{\partial v_i}{\partial a} \right) (t+1, a) f_i(t, a) \, da.$$

The lack of any precommitment to bondholders implies $\mu_{-1} = 0$. If we take into account the first order condition of households $u'(c_i) = \left(\mathcal{T} \frac{\partial v_i}{\partial a} \right)(t+1, a) \frac{\beta}{(1+\pi_t) Q_t}$, this simplifies to

$$\mu_t = \frac{\mu_{t-1}(1-\delta)}{\beta(1+\pi_{t-1})(1+\bar{r})} + \frac{(1+\pi_t)}{Q_t}\sum_{i=1}^{2}\int(\delta a + y_i - c_i(t,a))\,u_c(c_i(t,a))\,f_i(t,a)\,da.$$

Finally, we compute the standard derivative with respect to π_t :

$$\begin{aligned} \beta^t\langle x_t', f_t\rangle + \beta^{t+1}\left\langle \frac{\partial}{\partial \pi_t}\mathcal{T}v_{t+1}, f_t\right\rangle + \beta^t\mu_t\left(\frac{\delta+(1-\delta)Q_{t+1}}{(1+\pi_t)^2(1+\bar{r})}\right) &= 0, \\ \langle x_t', f_t\rangle - \frac{\beta}{(1+\pi_t)^2}\left\langle \mathcal{T}\left(a_{t+1}\frac{\partial v_{t+1}}{\partial a}\right), f_t\right\rangle + \mu_t\left(\frac{Q_{t+1}}{(1+\pi_t)^2(1+\bar{r})}\right) = 0, \end{aligned}$$

and hence

$$\mu_t Q_{t+1} = (1+\bar{r})\sum_{i=1}^{2}\int\left[\frac{\beta}{(1+\pi_t)^2}\mathcal{T}\left(a\frac{\partial v_i}{\partial a}\right)(t+1,a) - x'(t,a)\right]f_i(t,a)\,da,$$

which, taking into account the first order condition of households, simplifies to

$$\mu_t Q_{t+1} = (1+\bar{r})\sum_{i=1}^{2}\int\left[\frac{Q_t}{(1+\pi_t)}u'(c_i(t,a)) - x'(t,a)\right]f_i(t,a)\,da,$$

The solution to the Ramsey problem in discrete time is given by the following proposition

Proposition 5 (Optimal inflation - Ramsey discrete time) *If a solution to the Ramsey problem (86) exists, the inflation path $\pi(t)$ must satisfy*

$$\mu_t Q_{t+1} = (1+\bar{r})\sum_{i=1}^{2}\int\left[\frac{Q_t}{(1+\pi_t)}u'(c_i(t,a)) - x'(t,a)\right]f_i(t,a)\,da, \tag{89}$$

where $\mu(t)$ is a costate with law of motion

$$\mu_t = \frac{\mu_{t-1}(1-\delta)}{\beta(1+\pi_{t-1})(1+\bar{r})} + (1+\pi_t)\sum_{i=1}^{2}\int u'(c_i(t,a))\frac{\delta a + y_i - c_i(t,a)}{Q_t}f_i(t,a)\,da. \tag{90}$$

and initial condition $\mu_{-1} = 0$.

Notice that this proposition is the the equivalent of Proposition 1 in discrete time.

F. A simplified model version

Consider a simplified model version with no idiosyncratic uncertainty ($y_1 = y_2 \equiv y$). Let also the natural borrowing limit replace the exogenous lower bound for real net wealth (ϕ). The household's problem is otherwise unchanged relative to the one in the main text. As in our numerical implementation, assume log consumption utiliy: $u(c) - x(\pi) = \log(c) - \psi\pi^2/2$. The HJB equation (equation 6 in the main text) simplifies to

$$\rho v_t(a) = \frac{\partial v_t}{\partial t} + \max_c \left\{ \log(c) - \psi\pi_t^2/2 + \mathbf{s}_t(a,c)\frac{\partial v_t}{\partial a} \right\}, \tag{91}$$

where $\mathbf{s}_t(a,c) = \left(\frac{\delta}{Q_t} - \delta - \pi_t\right) a + \frac{y-c}{Q_t}$. The first order condition for consumption is

$$\frac{1}{c_t(a)} = \frac{1}{Q_t}\frac{\partial v_t(a)}{\partial a}. \tag{92}$$

From now on, drop function arguments for ease of notation. The envelope condition is

$$\rho \frac{\partial v_t}{\partial a} = \frac{\partial^2 v_t}{\partial t \partial a} + \left(\frac{\delta}{Q_t} - \delta - \pi_t\right)\frac{\partial v_t}{\partial a} + \mathbf{s}_t \frac{\partial^2 v_t}{\partial a^2}. \tag{93}$$

Differentiating (92) with respect to a and t, we obtain respectively

$$0 = \frac{\partial c_t}{\partial a}\frac{\partial v_t}{\partial a} + c_t \frac{\partial^2 v_t}{\partial a^2}.$$

$$\dot{Q}_t = \frac{\partial c_t}{\partial t}\frac{\partial v_t}{\partial a} + c_t \frac{\partial^2 v_t}{\partial a \partial t}.$$

Using the latter two expressions and (92) to substitute for $\frac{\partial v_t}{\partial a}$, $\frac{\partial^2 v_t}{\partial t \partial a}$ and $\frac{\partial^2 v_t}{\partial a^2}$ in (93), and rearranging,

$$\rho = \frac{\dot{Q}_t}{Q_t} + \frac{\delta}{Q_t} - \delta - \pi_t - \left(\mathbf{s}_t \frac{\partial c_t}{\partial a} + \frac{\partial c_t}{\partial t}\right)\frac{1}{c_t}.$$

Using the fact that $\mathbf{s}_t \frac{\partial c_t}{\partial a} + \frac{\partial c_t}{\partial t} = \frac{dc_t}{dt} \equiv \dot{c}_t$, we obtain the following consumption Euler equation,

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= \frac{\dot{Q}_t}{Q_t} + \frac{\delta}{Q_t} - \delta - \pi_t - \rho \\ &= \bar{r} - \rho,\end{aligned}$$

where the last equality follows from the bond pricing condition, $\frac{\dot{Q}_t}{Q_t} + \frac{\delta}{Q_t} - \delta - \pi_t = \bar{r}$ (see equation 10 in the main text).[61]

We now guess and verify that the value function takes the form

$$v_t(a) = \frac{1}{\rho} \log(a + h_t/Q_t) + \hat{v}_t, \tag{94}$$

where

$$h_t \equiv \int_t^\infty e^{-\int_t^s (r_u - \pi_u) du} y \frac{Q_t}{Q_s} ds.$$

where $r_t = \frac{\delta}{Q_t} - \delta$ and $\hat{v}_t$ is the solution to the ordinary differential equation

$$\frac{d\hat{v}_t}{dt} + \log(\rho Q_t) - \frac{\psi}{2}\pi_t^2 + \frac{r_t - \pi_t - \rho}{\rho} = \rho \hat{v}_t, \tag{95}$$

with the transversality condition $\lim_{t\to\infty} e^{-\rho t}\hat{v}_t = 0$. We then have $\partial v_t(a)/\partial a = \rho^{-1}(a + h_t/Q_t)^{-1}$ and hence, from equation (92), the optimal consumption is

$$c_t(a) = \rho(Q_t a_t + h_t).$$

If we substitute the value function and the consumption in the HJB (91) we obtain

$$\begin{aligned}\log(a + \frac{h_t}{Q_t}) + \rho\hat{v}(t) &= \frac{\partial}{\partial t}\left[\frac{1}{\rho}\log(a + \frac{h_t}{Q_t}) + \hat{v}(t)\right] + \log(\rho Q_t) + \log(a_t + \frac{h_t}{Q_t}) - \frac{\psi}{2}\pi_t^2 \\ &+ \left[(r_t - \pi_t)a + \frac{y - \rho(Q_t a_t + h_t)}{Q_t}\right]\frac{\partial}{\partial a}\left[\frac{1}{\rho}\log(a + \frac{h_t}{Q_t}) + \hat{v}(t)\right].\end{aligned}$$

Notice that

$$\frac{\partial}{\partial t}\left[\frac{1}{\rho}\log(a + \frac{h_t}{Q_t}) + \hat{v}(t)\right] = \frac{1}{\rho}\frac{1}{a + h_t/Q_t}\frac{\partial}{\partial t}(\frac{h_t}{Q_t}) + \frac{d\hat{v}(t)}{dt},$$

[61] In the full model, and with log preferences, the Euler equation generalizes to $\dot{c}_{it}/c_{it} = \bar{r} - \rho + \lambda_i(c_{it}/c_{jt} - 1)$ for $i, j = 1, 2,\ j \neq i$.

and

$$\frac{\partial}{\partial t}(h_t/Q_t) = \int_t^\infty e^{-\int_t^s (r_u - \pi_u)du} y \frac{1}{Q_s} ds = \frac{-y}{Q_t} + \frac{(r_t - \pi_t)\, h_t}{Q_t}.$$

Using the latter in the HJB equation, cancelling terms, and rearranging,

$$\begin{aligned} 0 \;\; = \;\; & \frac{(r_t - \pi_t - \rho)\,(h_t + Q_t a)}{Q_t} \frac{1}{\rho} \frac{1}{a + h_t/Q_t} \\ & + \frac{d\hat{v}(t)}{dt} - \rho \hat{v}(t) + \log(\rho Q_t) - \frac{\psi}{2}\pi_t^2, \end{aligned}$$

We finally obtain

$$\rho \hat{v}(t) = \frac{r_t - \pi_t - \rho}{\rho} + \frac{d\hat{v}(t)}{dt} + \log(\rho Q_t) - \frac{\psi}{2}\pi_t^2,$$

which is satisfied due to equation (95).

G. Robustness

Steady state Ramsey inflation. In Proposition 2, we established that the Ramsey optimal long-run inflation rate converges to zero as the central bank's discount rate ρ converges to that of foreign investors, $\bar{r}$. In our baseline calibration, both discount rates are indeed very close to each other, implying that Ramsey optimal long-run inflation is essentially zero. We now evaluate the sensitivity of Ramsey optimal steady state inflation to the difference between both discount rates. From equation (21), Ramsey optimal steady state inflation is

$$\pi_\infty = \frac{1}{\psi} \sum_{i=1}^{2} \int_\phi^\infty Q_\infty (-a)\, u'(c_{i\infty}(a))\, f_{i\infty}(a)\, da + \frac{1}{\psi} \mu_\infty Q_\infty, \tag{96}$$

where from equation (17)

$$\mu_\infty = -\frac{\mathbb{E}_{f_\infty(a,y)}\left[-a_\infty^{new}(a,y)\, u'(c_\infty(a,y))\right]}{\pi_\infty + \delta - (\rho - \bar{r})}. \tag{97}$$

Figure 5 displays π (left axis), as well as its two determinants (right axis) on the right-hand side of equation (96). Optimal inflation decreases approximately linearly with the gap $\rho - \bar{r}$. As the latter increases, two counteracting effects take place. On the one hand, it can be shown that as the households become more impatient relative to foreign investors, the net asset distribution shifts towards the left, i.e. more and more households become net borrowers and come close to the borrowing limit, where the marginal utility of wealth is highest.[62] As shown in the figure, this increases the central bank's incentive to inflate for the purpose of redistributing wealth towards debtors. On the other hand, higher indebtedness implies also more issuance of new debt. Moreover, a higher gap $\rho - \bar{r}$ increases the extent to which the central bank internalizes the effect of trend inflation on the price of bond issuances. The latter two effects imply that in equation (97), *ceteris paribus*, the numerator increases and the denominator falls, respectively, such that μ_∞ becomes more *negative.* This gives the central bank an incentive to committing to *lower* long-run inflation. As shown by Figure 5, this second 'commitment' effect dominates the redistributive inflationary effect, such that in net terms optimal long-run inflation becomes more negative as the discount rate gap widens.

Initial inflation. As explained before, time-0 optimal inflation and its subsequent path depend on the initial net wealth distribution across households, which is an infinite-dimensional object. In our baseline numerical analysis, we set it equal to the stationary distribution in the case of zero inflation. We now investigate how initial inflation depends on such initial distribution. To make the analysis operational, we restrict our attention to the class of Normal distributions truncated at the borrowing limit ϕ. That is,

$$f_0(a) = \begin{cases} \phi(a;\mu,\sigma) / [1 - \Phi(\phi;\mu,\sigma)], & a \geq \phi \\ 0, & a < \phi \end{cases}, \tag{98}$$

where $\phi(\cdot;\mu,\sigma)$ and $\Phi(\cdot;\mu,\sigma)$ are the Normal pdf and cdf, respectively.[63] The parameters μ and σ allow us to control both (i) the initial net foreign asset position and (ii) the domestic dispersion in household wealth, and hence to isolate the effect of each factor on the optimal inflation path. Notice also that optimal long-run inflation rates do *not* depend on $f_0(a)$ and are therefore exactly the same as in our baseline numerical analysis regardless of μ and σ.[64] This allows us to focus here on inflation

[62] The evolution of the long-run wealth distribution as $\rho - \bar{r}$ increases is available upon request.

[63] In these simulations, we assume that the initial net asset distribution conditional on income is the same for high- and low-income households: $f_0^{a|y}(a \mid y_2) = f_0^{a|y}(a \mid y_1) \equiv \tilde{f}_0(a)$. This implies that the marginal asset density coincides with its conditional density: $f_0^a(a) = \sum_{i=1,2} f_0^{a|y}(a \mid y_i) f^y(y_i) = \tilde{f}_0(a)$.

[64] As shown in Table 2, long-run inflation is -0.05% under commitment, and 1.68% under discre-

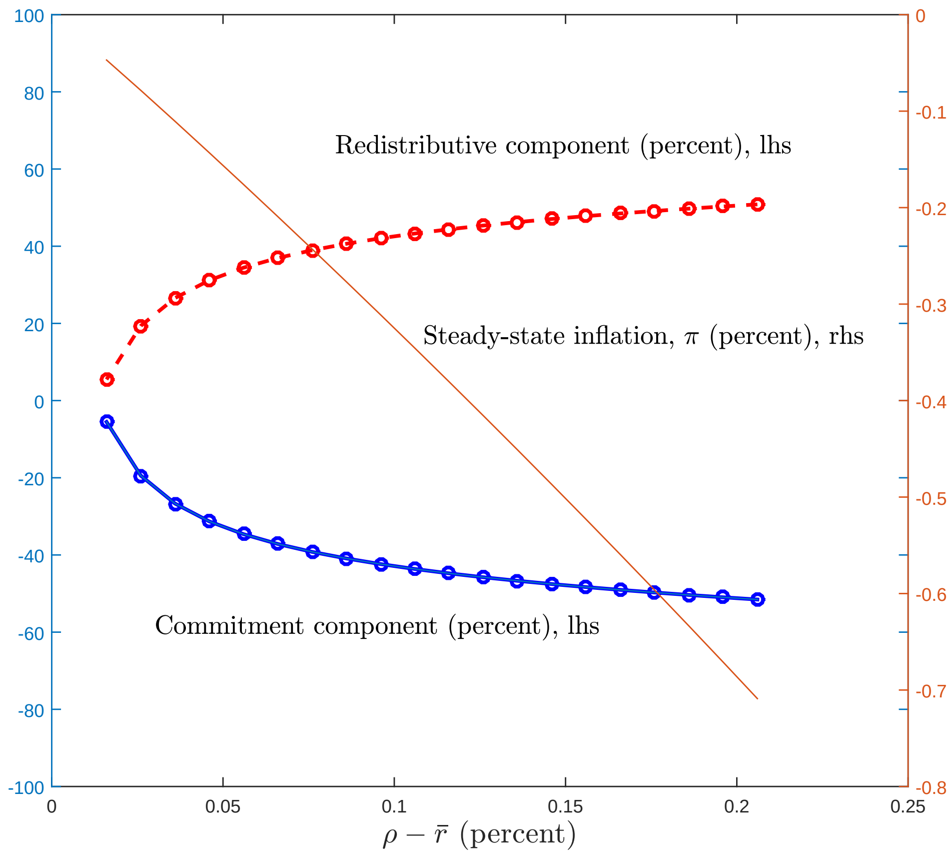

Figure 5: Sensitivity analysis to changes in $\rho - \bar{r}$.

at time 0, while noting that the transition paths towards the respective long-run levels are isomorphic to those displayed in Figure 1.[65] Moreover, we report results for the commitment case, both for brevity and because results for discretion are very similar.[66]

Figure 6 displays optimal initial inflation rates for alternative initial net wealth distributions. In the first row of panels, we show the effect of increasing wealth dispersion while restricting the country to have a zero net position *vis-à-vis* the rest of the World, i.e. we increase σ and simultaneously adjust μ to ensure that $\bar{a}_0 = 0$.[67] In the extreme case of a (quasi) degenerate initial distribution at zero net assets (solid blue line in the upper left panel), the central bank has no incentive to create inflation, and thus optimal initial inflation is zero. As the degree of initial wealth dispersion increases, so does optimal initial inflation.

The bottom row of panels in Figure 6 isolates instead the effect of increasing the liabilities with the rest of the World, while assuming at the same time $\sigma \simeq 0$, i.e. eliminating any wealth dispersion.[68] As shown by the lower right panel, optimal inflation increases fairly quickly with the external indebtedness; for instance, an external debt-to-GDP ratio of 50 percent justifies an initial inflation of over 6 percent.

We can finally use Figure 6 to shed some light on the contribution of each redistributive motive (cross-border and domestic) to the initial optimal inflation rate, $\pi_0 = 4.6\%$, found in our baseline analysis. We may do so in two different ways. First, we note that the initial wealth distribution used in our baseline analysis implies a consolidated net foreign asset position of $\bar{a}_0 = -25\%$ of GDP ($\bar{y} = 1$). Using as initial condition a *degenerate* distribution at exactly that level (i.e. $\mu = -0.205$ and $\sigma \simeq 0$) delivers $\pi_0 = 3.1\%$ (see panel d). Therefore, the pure *cross-border* redistributive motive explains a significant part (about two thirds) but not all of the optimal

tion.

[65] The full dynamic optimal paths under any of the alternative calibrations considered in this section are available upon request.

[66] As explained before, time-0 inflation in both policy regimes differ only insofar as the respective time-0 value functions do, but numerically we found the latter to be always very similar to each other. Results for the discretion case are available upon request.

[67] We verify that for all the calibrations considered here, the path of $\bar{a}_t$ after time 0 satisfies Assumption 1.

[68] That is, we approximate 'Dirac delta' distributions centered at different values of μ. Since such distributions are not affected by the truncation at $a = \phi$, we have $\bar{a}(0) = \mu$, i.e. the net foreign asset position coincides with μ.

time-0 inflation under the Ramsey policy. Alternatively, we may note that our baseline initial distribution has a standard deviation of 1.95. We then find the (σ, μ) pair such that the (truncated) normal distribution has the same standard deviation while ensuring that $\bar{a}_0 = 0$ (thus switching off the cross-border redistributive motive); this requires $\sigma = 2.1$, which delivers $\pi_0 = 1.5\%$ (panel b). We thus find again that the pure *domestic* redistributive motive explains about a third of the baseline optimal initial inflation. We conclude that both the cross-border and the domestic redistributive motives are quantitatively important for explaining the optimal inflation chosen by the monetary authority.

H. Additional figures

Figure 7 is the analogue of Figure 3 in the main text for low-income (unemployed) households, i.e. those with $y = y_1$. Qualitatively, the effects of optimal inflation on consumption redistribution are similar to those for high-income (employed) households. First, relative to zero inflation, optimal inflationary policies favor time-0 consumption for indebted households and vice versa for lending ones (center-right panel). Second, the optimal commitment policy moves some low-wealth households to the range of intermediate wealth levels, which favors their consumption over time (lower-left panel).

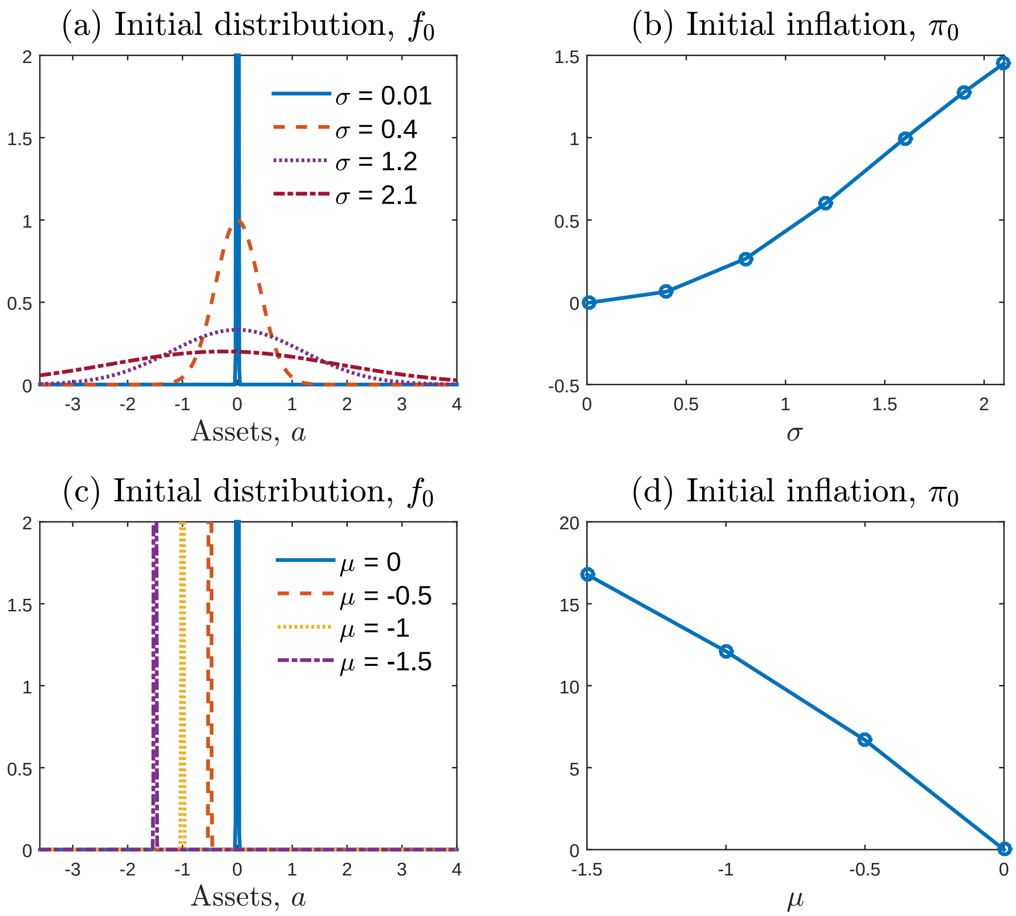

Figure 6: Ramsey optimal initial inflation for different initial net asset distributions.

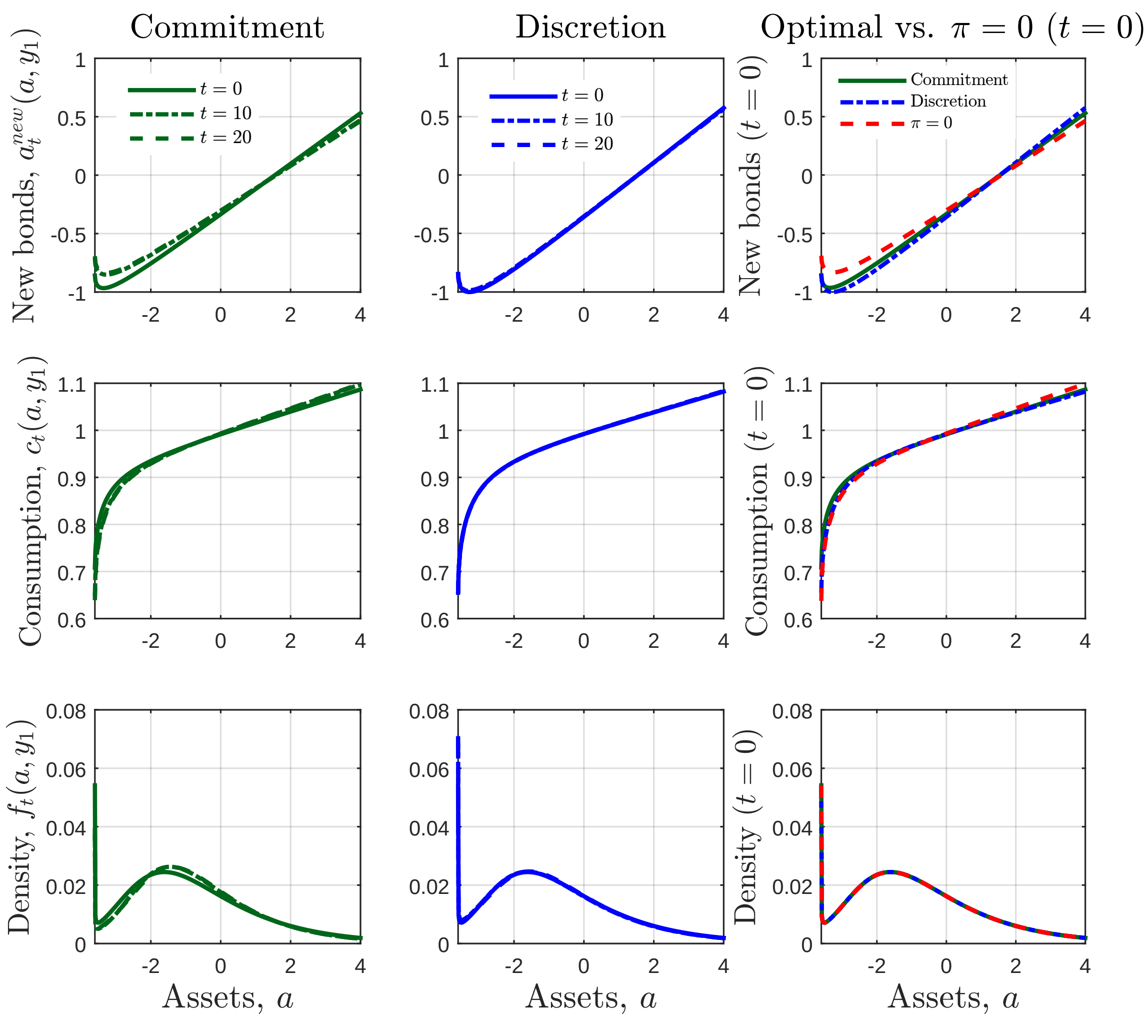

Figure 7: Policy functions and net wealth densities across policies and over time (low-income households, $y = y_1$)