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**Collateralized Debt Networks with Lender Default**

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# Collateralized Debt Networks with Lender Default \*

Jin-Wook Chang <sup>†</sup>

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## Abstract

The Lehman Brothers' bankruptcy triggered the failure of the collateralized debt markets, which was a major contributor of the financial crisis in 2008. Such collateralized debt markets have both collateral price channel and counterparty (borrower and lender) channel of contagion. I propose a general equilibrium network model, which incorporates the two channels of contagion by endogenizing leverage (margin), asset prices, and network formation. Agents face a tradeoff between leverage and counterparty risk. Diversification of counterparty risk generates positive externalities by reducing systemic risk, but comes at the cost of lower leverage. Thus, any decentralized equilibrium is inefficient due to under-diversification. I use this framework to show that the loss coverage by a central counterparty (CCP) exacerbates the externality problem and the introduction of CCP can rather increase systemic risk.

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# 1. Introduction

The failure of the collateralized debt market was a major contributor to the financial crisis in 2008 (Gorton and Metrick, 2012; Copeland et al., 2014; Martin et al., 2014).<sup>1</sup> A typical collateralized debt takes the form of one-on-one interaction between two counterparties – a borrower and a lender – because of customization of contract terms. Thus, a collateralized debt network, the collection of such one-on-one relationships, has two transmission channels of shocks – the price channel and the counterparty channel. The goal of this paper is to analyze such a collateralized debt market with both price channel and counterparty channel of contagion through a hybrid model that combines general equilibrium and network frameworks.

For example, the collapse in the prices of subprime mortgages in 2007 had a direct effect on many financial institutions that held related assets. But the initial shock was exacerbated by the resulting bankruptcy of the Lehman Brothers, which spread the losses to Lehman's counterparties (De Haas and Van Horen, 2012; Singh, 2017). These counterparty losses triggered fire sales of assets which made prices to decline even further (Demange, 2016; Duarte and Eisenbach, 2018; Duarte and Jones, 2017). Therefore, a model that incorporates the interaction between price and counterparty channels is necessary to capture the full picture of the crisis in collateralized debt markets (Glasserman and Young, 2016).

Lender default can also be a counterparty channel of shocks (Eren, 2014; Infante et al., 2018). All of Lehman Brothers' assets including borrowers' collateral were frozen under the bankruptcy procedure in 2008. Many borrowers had to over-collateralize<sup>2</sup> their positions to protect the lender (Lehman) in case of borrower default (Scott, 2014). While over-collateralization secured the lender's position, it exposed the borrowers to losses when they could not recover their collateral. The borrowers did not know when their collateral would be returned to them, nor did they know how much they would recover from the bankruptcy process (Fleming and Sarkar, 2014) and paid a sizable cost throughout the recovery.<sup>3</sup>

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<sup>1</sup>Asset-backed commercial papers (ABCP), derivatives, and repurchase agreements (repo) are typical examples of collateralized debt and the most common form of short-term financing among banks and funds.

<sup>2</sup>The financial market has evolved to insulate lenders from borrower counterparty risk over the past few decades. The evolution of securitization has made the cash flows of contracts remote from borrower bankruptcies and significantly eliminated tangible losses that could follow from borrower default (Gorton et al., 2010). However, borrowers are now exposed to the risk of lender default.

<sup>3</sup>Even if the borrowers recovered their assets over the long term, the inability to recover funds in the short term caused disruption. MKM Longboat Capital Advisors closed its \$1.5 billion fund partly because of frozen assets, and the COO of Olivant Ltd. committed suicide, because the fund had \$1.4 billion value of assets, which was believed to be unlikely to recover from Lehman Brothers (Scott, 2014). Another good example is the case of MF Global, a prominent broker-dealer that went bankrupt in 2011. The bankruptcy procedure took nearly five years to resolve all the claims for their borrowers such as hedge funds. The borrowers had to go through the lengthy process of a bankruptcy procedure with considerable costs to stay

The main research questions are the following. First, how do agents spread losses to each other through a given network of counterparties in a collateralized debt market? Second, how do counterparties form an endogenous debt network (borrow and lend to each other) when they account for this contagion channel? Third, how does regulation change the systemic risk when accounting for endogenous network responses of the market?

I propose a general equilibrium model with multiplex network interaction featuring endogenous leverage (margin), endogenous price, and endogenous network formation to study this problem. The model has interaction between the price channel (fire sales) and the counterparty channel (both borrower and lender defaults) that affect endogenous network formation. This paper is the first attempt to endogenize leverage, asset prices, and network formation simultaneously, to the best of my knowledge.

The model has six main features. First, agents trade an asset that can be used as collateral in a competitive market. Price changes in the asset market affect agents' nominal wealth as a *price channel*. Second, there is a multiplex network of collateralized borrowing and lending. Agents enter bilateral customized contracts specifying the face value of the debt and the amount of collateral. Third, agents disagree on the fair value of the asset *ex ante*. Agents trade the asset and use it as collateral to borrow because of the *belief disagreement*. Fourth, the lender of a debt contract can *reuse (rehypothecate)* the collateral to borrow money from someone else. An agent can be a lender as well as a borrower at the same time. Fifth, agents are subject to *liquidity shocks* before paying back their debt. Because of the liquidity shock, agents may go bankrupt. Sixth, both borrower and lender defaults are considered. Borrowers' failure to pay results in a costless transfer<sup>4</sup> of collateral to the lender. When the lender fails to return the collateral, the borrower has to go through a costly process to recover the collateral from the lender. This lender default cost<sup>5</sup> generates propagation through the *counterparty channel*.

The first implication of the model is that there is a tradeoff between counterparty risk and (contract-level) leverage which affects network formation. If there is no lender default cost, then a single intermediation chain is formed endogenously. Borrowers prefer to maximize their contract leverage (or minimize margin) by borrowing from the most favorable lender.

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involved and also could not access the assets that were used as collateral (SIPC, 2016).

<sup>4</sup>For example, typical repo contracts are exempt from automatic stay of bankruptcy provisions.

<sup>5</sup>The lender default cost is similar to the borrower default cost, which is prevalent in the literature. If there is a bankruptcy of a counterparty, then that will incur additional cost in terms of time, effort, and litigation costs, which are a deadweight loss to the economy. For example, there were over 100 hedge funds that had prime brokerage accounts or debt obligations under Lehman Brothers, and these accounts were frozen during the bankruptcy of Lehman Brothers. These positions, valued at more than \$400 billion, were frozen, which further exacerbated the liquidity shortage of the market (Lleo and Ziemba, 2014). The lender default occurred not only because of rehypothecation but also because of Lehman not holding the collateral in a segregated account (Fleming and Sarkar, 2014).

The most optimistic agent borrows from the second-most optimistic agent, who borrows from the third-most optimistic agent, and so on.

However, if there is a lender default cost, borrowers diversify their lenders because of the possibility of counterparty default losses.<sup>6</sup> The tradeoff between counterparty risk and leverage (margin) exists because borrowers have to deal with more pessimistic lenders who lend less for the same collateral. Therefore, an increase in counterparty risk leads to an increase in the number of counterparties and a decrease in reuse of collateral and average leverage, since the optimists borrow directly from the pessimists rather than indirectly through intermediation.

The second implication of the model is that there are positive externalities from diversification. Diversification of counterparties reduces not only individual counterparty risk but also systemic risk by limiting the propagation of shocks and price volatility. If an intermediary becomes safer, then its borrowers become safer as well, so the aggregate counterparty risk becomes smaller. In addition, a lower level of debt leads to lower price volatility, making each agent's balance sheet more stable. Because agents do not fully internalize these externalities, any decentralized equilibrium is inefficient because of under-diversification.

The third implication of the model is that the loss coverage by a central counterparty (CCP)—one of the key elements of the financial system reforms addressed by central banks and financial authorities after the financial crisis in 2008 (Singh, 2010)—exacerbates the externality problems. A CCP *novates* a contract between two counterparties—that is, replaces a contract between a borrower and a lender with two different contracts: a contract between the borrower and the CCP and a contract between the lender and the CCP. Novation procedure acts as a pooling of individual counterparty risks as the CCP handles and absorbs any losses from default. However, the tradeoff between counterparty risk and leverage disappears as individual counterparty risk is covered. Each agent will concentrate all of her borrowing with the single most favorable lender. The endogenous response to the introduction of CCP will transform the implicit network structure into a single-chain network, which arises in a decentralized equilibrium only if there is no default cost. Such reckless borrowing behavior increases systemic risk.<sup>7</sup> This result shows up only if leverage, asset prices, and network formation are all endogenous at the same time, because there is no tradeoff between counterparty risk and leverage otherwise.

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<sup>6</sup>This diversification of lender behavior is similar to firms hedging against bank lending channels by having multiple banks as lenders as in Khwaja and Mian (2008).

<sup>7</sup>However, netting conducted by the CCP decreases systemic risk so the overall effect to systemic risk is ambiguous. A CCP can perform *netting* of counterparty exposures. If agent A owes \$100 to agent B who owes \$100 to agent C, then the CCP can net out the obligations between the two contracts. As a result, agent A owes \$100 to agent C and agent B has no obligation at all.

The fourth implication of the model is that every agent holds a positive amount of cash in any equilibrium. If there is a crash in the asset price, the marginal utility of cash will become very high and surviving agents can enjoy huge return from cash by buying up all the remaining assets at a cheap price. Thus, every agent, including the most optimistic agent, holds a positive amount of cash. This competitive cash holding, due to general equilibrium effect from the asset market counteracts the risk-stacking behavior (increasing correlation with others) of agents that is common in typical financial network models.

The main results of this paper align well with empirical observations in the literature. After the bankruptcy of Lehman Brothers, the velocity (reuse) of collateral decreased from 3 to 2.4 and the average leverage in the over-the-counter (OTC) market also went down (Singh, 2011). Also the average number of linkages between financial institutions have increased about 30 percent over the four years after Lehman Bankruptcy and hedge funds diversified their portfolio of counterparties after the crisis (Craig and Von Peter, 2014; Eren, 2015).<sup>8</sup> Finally, even hedge funds, the most aggressive borrowers, tend to hold large amounts of daily liquidity that are almost equivalent to cash as much as 34 percent of their assets (Aragon et al., 2017), whereas money market mutual funds, which are the ultimate lenders in collateralized debt markets, hold less than 20 percent of their assets as daily liquid assets (Aftab and Varotto, 2017).<sup>9</sup>

### 1.1. Related Literature

“No major institution failed because of losses on its direct exposures to Lehman” (Upper, 2011). Glasserman and Young (2016) also recognize this criticism on financial contagion literature and suggest developing a model that combines the three typical shock transmission channels in financial networks— default cascades, price-mediated losses, and withdrawal of funds. This paper incorporates the first two channels (counterparty and price) by developing a model that combines financial networks with general equilibrium framework. The interaction of the two channels leads to very different incentives for network formation as well as different patterns of cascades.

The contagion through financial networks in this paper is based on the insights from the payment equilibrium literature following Eisenberg and Noe (2001) and Acemoglu et al. (2015). This paper also incorporates discontinuous jumps in the payoffs in case of bankruptcy

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<sup>8</sup>The opposite result happened in unsecured debt markets in which the banks reduced their number of counterparties (Afonso et al., 2011; Beltran et al., 2015). This stark comparison shows the importance of the role of collateral in network formation.

<sup>9</sup>In the online appendix, I show that the amount of cash held by the borrowers can exceed the amount of cash held by lenders. This somewhat counterintuitive result comes from the fact that the potential degree of underpricing is higher for the optimistic borrowers than that of the pessimistic lenders.

as in Elliott et al. (2014). The endogenous network formation is based on portfolio decisions similar to Allen et al. (2012).

The general equilibrium framework of this paper is closely related to the literature on general equilibrium with collateralized debt. The literature started from Geanakoplos (1997) and developed through Geanakoplos (2003), Geanakoplos (2010), Simsek (2013), Fostel and Geanakoplos (2015), and Fostel and Geanakoplos (2016), which introduce models with collateral, how heterogeneity can generate collateralized debt and trade, and how endogenous (contract-level) leverage is determined. In particular, Geerolf (2018) introduces pyramiding, using a contract backed by collateral as collateral. This paper extends the number of possible reuse of collateral to any number.

Many financial network models have an equilibrium in which agents have overlapping asset and counterparty portfolios or common correlation structure (Allen et al., 2012; Cabrales et al., 2017; Elliott et al., 2018; Erol, 2018; Jackson and Pernoud, 2019). In such models, agents have strong incentives to correlate their payoffs with those of their counterparties, because they can enjoy better payments from their counterparties when they are solvent while being insolvent when they expect lower potential payments from their counterparties. However, this paper introduces an opposing force to such incentives which is marginal utility of cash coming from general equilibrium effect. Agents do not hold correlated portfolio because, if everyone in the economy collapses, then the one who does not can make a huge return in such a state where the marginal utility of cash is enormous.

There are other papers considering both counterparty channel and price channel for asset holdings such as Cifuentes et al. (2005), Gai et al. (2011), and Rochet and Tirole (1996). This paper differs by incorporating the price channel for the underlying collateral and endogenous network formation. While having an endogenous market price for the collateral, this paper does not incorporate the moral hazard problem as in Di Maggio and Tahbaz-Salehi (2015).

Allen and Gale (2000), Babus (2016), Babus and Hu (2017), Babus and Kondor (2018), Brusco and Castiglionesi (2007), Chang and Zhang (2019), Elliott et al. (2018), Erol and Vohra (2018), Farboodi (2017), and Freixas et al. (2000) studied endogenous network formation in financial networks. They consider the endogenous network structure and possible inefficiencies and systemic risks. Unlike the models in these papers, this paper allows for endogenous contracts as well as endogenous asset price and reuse of collateral.

Also this paper incorporates lender default and how collateral, which is supposed to insulate counterparty risk, can still remain as a contagion channel. Eren (2014), Gottardi et al. (2017), Infante and Vardoulakis (2018), Infante (2019), and Park and Kahn (2019) investigated the lender default problem in collateralized lending and relevant deadweight loss, in addition to contract and intermediation dynamics. This paper incorporates the

lender default feature into the endogenous network structure. Also the same collateral can be reused for an arbitrary number of times in contrast to other models of rehypothecation.

Finally, this paper is also related to the literature about central clearing. Duffie and Zhu (2011) started the formal discussion, which is extended by Duffie et al. (2015), Arnold (2017), Frei et al. (2017), Paddrik and Young (2017), and Paddrik et al. (2019) analyzing the effect on systemic risk and margin dynamics under a CCP. Biais et al. (2012) uses the search cost as the moral hazard problem of clearing members. This paper has endogenous network changes with a CCP instead of moral hazard problem, and the externality arises from the leverage and counterparty decisions which are absent in the literature.

The rest of the paper is organized as the following. Section 2 introduces the model. Section 3 develops results in decentralized OTC market equilibrium. Section 4 illustrates the equilibrium under CCP. Section 5 concludes. The Appendix contains all the proofs.

## 2. Model

There are three periods  $t = 0, 1, 2$ . There are two goods – cash and an asset denoted as  $e$  and  $h$ , respectively. Cash is the only consumption good, and it is storable — one unit of cash at  $t$  becomes one unit of cash at  $t + 1$ . The asset yields  $s$  amount of cash at  $t = 2$ , and agents gain no utility from just holding the asset. The true  $s$  is publicly revealed to everyone at the beginning of  $t = 1$ . Each agent's preference is risk-neutral and determined by how much cash she consumes at  $t = 2$ . Therefore, agents are facing an investment problem. Each agent is endowed with  $e_0$  amount of cash and  $h_0$  amount of asset at  $t = 0$ , and zero amount of cash and asset at  $t = 1$  and  $t = 2$ .

There are  $n$  types of agents, and the set of all agents is  $N = \{1, 2, \dots, n\}$ . Agent  $j$  believes  $s = s^j$  with probability one.<sup>10</sup> Agents are ordered by subjective beliefs on the payoff of the asset as  $s^1 > s^2 > \dots > s^n > 0$ . This belief disagreement is the reason why agents trade, borrow, or lend in  $t = 0$ . All agents agree upon the true value of asset payoff  $s \in S \equiv \{s^1, \dots, s^n\}$  after this information is publicly revealed at the beginning of  $t = 1$ . However, the asset payoff is realized at  $t = 2$ , so there is a time gap between uncertainty resolution and payoff realization. Also assume that  $ne_0 > nh_0s^1$ , so the cash in the market is sufficient to buy up all the assets at its fundamental value even when  $s = s^1$ .

For each agent  $j \in N$ , there can be a negative liquidity shock  $\epsilon_j$  at  $t = 1$ .<sup>11</sup> The size

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<sup>10</sup>This concentrated belief assumption is used in Geerolf (2018), and the assumption is for tractability.

<sup>11</sup>This  $\epsilon_j$  can be interpreted as senior debt or withdrawal of deposit that precedes debt obligations in the flavor of Diamond and Dybvig (1983). Another interpretation can be that  $\epsilon_j$  is a productivity shock among the projects agent  $j$  has and the agent has to pay certain amount of senior debt before paying the inter-agent debt. Such shocks are very commonly used in the financial network literature as in Acemoglu et al. (2015)

of the shock  $\epsilon_j$  is independent and identically distributed across  $j \in N$ , and the common distribution function is denoted as  $G$  where the support of  $G$  is  $[0, \bar{\epsilon}]$  and differentiable in the support for  $j \in N$  with  $g$  as its density function. Denote agent  $i$ 's density function as  $g_i$  for index purpose for each  $i \in N$ , and define the convolution of the density functions as  $g_\Sigma = g_1 * g_2 * \dots * g_n$  and its distribution function as  $G_\Sigma$ . Suppose that the upper bound of liquidity shock is large enough that  $\bar{\epsilon} > e_0 + h_0 s^1$ . The probability of arrival of liquidity shock is  $0 \leq \theta_j < 1$  for any  $j \in N$ . Assume that  $\theta_j = \theta$  for all  $j \in N$  for now. Denote  $\epsilon_j = 0$  if  $j$  did not receive liquidity shock at  $t = 1$ .<sup>12</sup>

Agents are fully competitive and know each other's type. Every agent believes that she is a price-taker.<sup>13</sup> Also, agents agree to disagree over the payoff  $s$  of the asset. The markets for both goods are competitive Walrasian markets for  $t = 0, 1, 2$ . The price of cash is normalized to 1 at any period, and the price of the asset is  $p_t$  for  $t = 0, 1, 2$ .

At  $t = 0$ , agents can borrow or lend cash using an asset as collateral. All borrowing contracts are 1-period contract between  $t = 0$  and  $t = 1$ .<sup>14</sup> A borrowing contract consists of: the amount of collateral posted  $c_{ij}$ , the amount of promised cash per 1 unit of collateral  $y_{ij}$ , and the identities of the lender and the borrower  $i, j$ . All borrowing contracts are non-recourse, so the actual payment from  $j$  to  $i$  is  $x_{ij} = \min\{y_{ij}, \tilde{p}_1\}$  per unit of collateral. Denote  $q_{ij}(y_{ij})$  as the amount of cash  $i$  lends to  $j$  in  $t = 0$  per unit of collateral. The second index of the subscript of  $q_{ij}$  will be omitted from now on, since the identity of the borrower becomes irrelevant because of competition and non-recourseness. This borrowing amount can be considered as the price of the contract and  $q_i$  is a function of the promise. The gross interest rate is  $1 + r_i(y_{ij}) \equiv y_{ij}/q_i(y_{ij})$ .

Denote  $c_{ij}$  as the amount of collateral posted by the borrower  $j$  to the lender  $i$ . This  $c_{ij}$  amount of asset is held by the lender until  $t = 1$ . If the borrower  $j$  pays back the full amount of promise  $c_{ij}y_{ij}$ , then the lender returns the collateral. Otherwise, the lender keeps the collateral and the cash value of the collateral is  $c_{ij}p_1$ . The lender who is holding the collateral can reuse the collateral to borrow cash from someone else. Let  $h_{i,1}$  denote the amount of asset agent  $i$  holds that has not been used as collateral at  $t = 0$ .

A (collateralized) *debt network* is a weighted directed multiplex (multilayer) graph formed

and Elliott et al. (2018), to see how external shocks propagate through the network.

<sup>12</sup>This  $\epsilon_j = 0$  is a measure zero event if  $j$  received a shock.

<sup>13</sup>This assumption is following the tradition of general equilibrium literature and abstracting out from market power and bargaining problem. One way to interpret this assumption is to consider that each agent  $j$  consists of a continuum (or hundreds) of homogeneous agents within the same type of  $j$  with perfectly correlated uncertainties. Since there is no asymmetric information, the model abstracts out from any adverse selection problem.

<sup>14</sup>Even if 1-period contracts between  $t = 1$  and  $t = 2$  are allowed, agents will only trade borrowing contracts between  $t = 0$  and  $t = 1$  endogenously. This is because there is no belief disagreement at  $t = 1$ .

by nodes  $N$  and links with 2 layers  $\alpha = 1, 2$  defined as  $\vec{\mathcal{G}} = (\mathcal{G}^{[1]}, \mathcal{G}^{[2]})$ , where  $\mathcal{G}^{[\alpha]} = (N, L^{[\alpha]})$ ,  $L_{ij}^{[1]} = c_{ij}$ , and  $L_{ij}^{[2]} = y_{ij}$ . Define the adjacency matrices  $C = [c_{ij}]$  and  $Y = [y_{ij}]$  as collateral matrix and contract (promise) matrix, respectively. A debt network can be represented by a double of  $(C, Y)$  and describes how much each agent borrows from or lends to other agents. Following the convention, set  $c_{ii} = 0$ .

The lenders are obliged to return the collateral when the borrower pays the promise in full. However, if a lender has negative wealth at  $t = 1$ , then the lender goes bankrupt and defaults on the contract. There will be deadweight loss from the lender default in terms of *cash* cost.<sup>15</sup> If agent  $j$  is borrowing from agent  $i$  and the lender  $i$  goes bankrupt, the borrower  $j$  has to pay  $\zeta(c_{ij})[p_1 - y_{ij}]^+$  amount of cash as the lender default cost, where  $\zeta(c)$  is a function of the amount of collateral posted  $c$ ,  $y$  is the promised cash amount, and  $[\cdot]^+ \equiv \max\{\cdot, 0\}$ . The lender default cost coefficient function is twice-continuously differentiable and  $\zeta(0) = 0$ ,  $\zeta'(0) = 0$ ,  $\zeta'(c) > 0$ ,  $\zeta''(c) > 0$ ,  $\zeta(c) \leq c$ ,  $\forall 0 < c \leq nh_0$ . Hence, the cost increases convexly as the total exposure to bankrupt lender increases and it is multiplied by the amount of liquidity shortage from lender default  $[p_1 - y_{ij}]^+$ , that is the excess payoff for borrower  $j$  is supposed to make.

## 2.1. Discussion and Examples

**Timeline.** The timeline of the model, which is depicted in figure 1, can be summarized as the following. Agents are endowed with cash and asset at the beginning of  $t = 0$ . Agents buy or sell the asset and also form a debt network at  $t = 0$ . At the beginning of  $t = 1$ , asset payoff  $s$  becomes publicly known and liquidity shocks  $\epsilon \equiv (\epsilon_1, \dots, \epsilon_n)$  are realized. Some agents may have  $\epsilon_j$  greater than the total cash value of their wealth and go bankrupt. All the debt is paid back during  $t = 1$ , either by the promise amount or by giving up the collateral. The collateral is returned to the borrower (if not defaulted) by the lender, but some borrowers may have to pay additional lender default costs if their counterparties went bankrupt. At the end of  $t = 1$ , all agents' final asset holdings are determined. At  $t = 2$ , the payoff of the asset is realized, and agents consume all the cash they have and enjoy utility.

**Uncertainties.** The model has two sources of uncertainty, the revelation of the payoff of the asset  $\tilde{s}$  and the realization of negative liquidity shocks for each agent. At the beginning of  $t = 1$ , both uncertainties are resolved. Therefore, everyone agrees upon the asset payoff in  $t = 1$ . However, the actual payoff realization of the asset occurs in  $t = 2$ , while they still

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<sup>15</sup>I assume that there is no collateral lost during the bankruptcy process, so all of the collateral will be eventually returned to the borrower. This assumption resembles the Lehman bankruptcy case in which all the collateral returned to the original borrowers. In the case of MF Global, the minority of the borrowers lost a fraction of their collateral. We abstract from the loss of collateral under the bankruptcy procedure. This assumption is justified by the endogenously arising intermediation order in the next section.

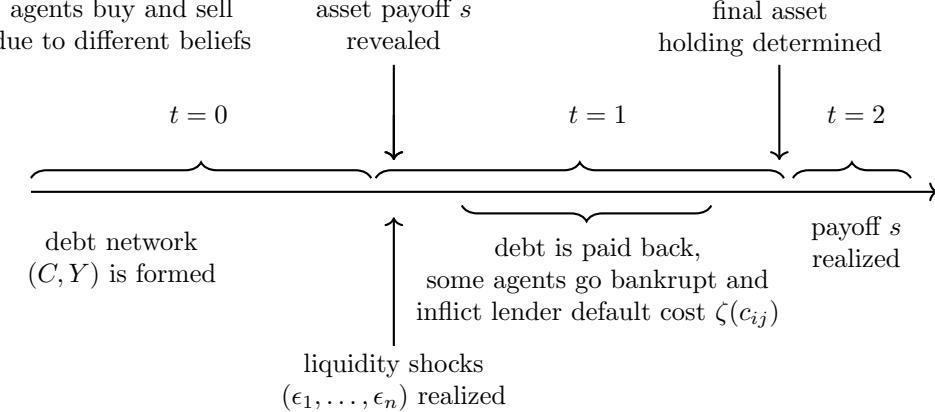


Figure 1: Timeline of the Model

have to pay back the debt they promised and toward the liquidity shocks. If the market is not under distress, then there will be no reason that  $p_1$  is different from the commonly known cash payoff of  $s$ . However, because of the liquidity (cash) shortage in the market, there may not be enough cash in the market to buy all the assets at the fair price of  $s$ .

Figure 2 is an example tree that depicts the underlying states and price realizations. There is a finite number of different  $s$  realization and continuum of different  $\epsilon$  shocks. Agent 1 believes that only the first set of states in  $t = 1$  occurs with positive probability. Agents 2 and 3 believe that only the second and the third set of states in  $t = 1$  occur with positive probability, respectively. The asset price  $p_1$  depends on the state realization  $s$  and liquidity shock realization  $\epsilon$ . Thus, each agent has their own distribution of prices as depicted in figure 2. Given the subjective distributions, each agent buys or sells, borrows or lends for different promises, and the equilibrium prices at  $t = 0$  for the asset and for all the promises will be determined. Note that agents agree upon the distribution of shocks, but each agent's subjective belief puts different upper bounds on price, which is  $s^j$  for agent  $j \in N$ .

**Example of a Collateralized Debt Contract.** Figure 3 visualizes the flow of cash and collateral for a collateralized debt contract. The top-left figure visualizes the transaction at  $t = 0$ , where agent  $j$  posts collateral to the lender  $i$  in the amount of  $c_{ij}$  and  $i$  lends cash in the amount of  $c_{ij}q_i(y_{ij})$  to agent  $j$ . If the price of the asset  $p_1$  is greater than the promise  $y_{ij}$  at  $t = 1$ , then the borrower  $j$  pays the promise and the lender  $i$  returns the collateral as seen in the top right figure. The bottom two sides visualize the other case. The bottom-left figure has the same transaction at  $t = 0$  as in the top case. However, the price of the asset  $p_1$  is now lower than the original promise  $y_{ij}$ , and the borrower defaults on the promise at  $t = 1$  as seen in the bottom right figure. Thus, the lender  $i$  just keeps the collateral.

**Borrower Default.** Because contracts are nonrecourse debt secured by collateral, every

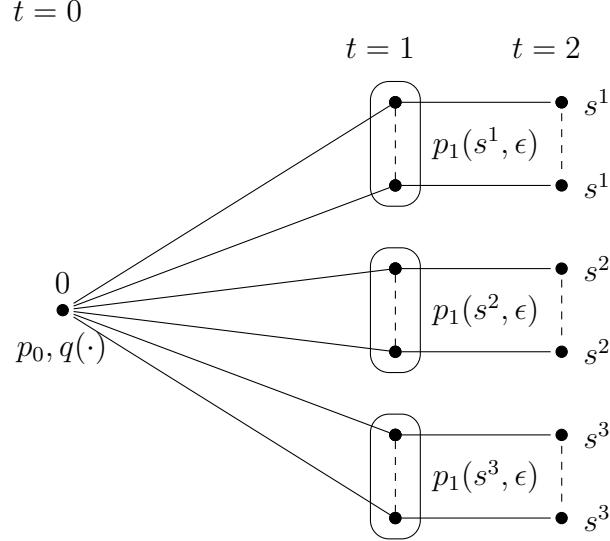


Figure 2: Tree of States and Price Realizations

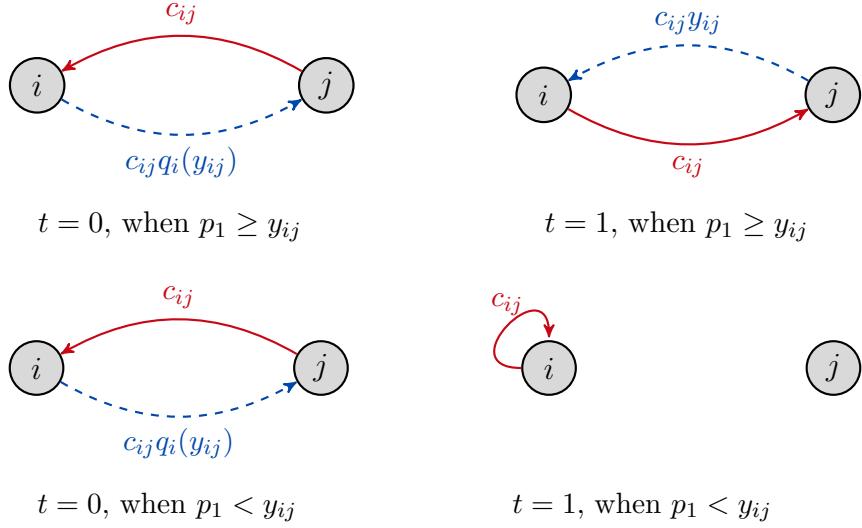


Figure 3: Flows of Cash and Collateral for Two Cases

Note: Blue dashed arrows represent flows of cash and red arrows represent flows of collateral. The top two figures represent the case without borrower default, and the bottom two figures represent the case with borrower default.

borrower with the same promise and collateral makes the same delivery. Shocks to the borrower's wealth do not get to transfer to deliveries toward lenders. Therefore, the lenders are insulated from the borrower's bankruptcy risk. In reality, this is precisely the reason why lenders require collateral from the borrowers. The lenders are protected by the exemption

from automatic stay for repos under borrower bankruptcy (Antinolfi et al., 2015).

**Lender Default.** The lender default cost includes the opportunity cost of time and effort caused by involvement into a costly and lengthy bankruptcy procedure, immediate liquidity needs caused by a depositor run on an agent (bank) with large exposure to the bankrupt agent, legal costs, opportunity cost of investment, reputation cost and so on.<sup>16</sup> Because of this lender default cost, borrowers may want to diversify their counterparties.

The convexly increasing cost structure in the model is not only a tractable alternative to assuming risk-aversion of the agents, which induces diversification behavior, but also a representation of realistic implications. If a hedge fund posted one Treasury bond as collateral to the Lehman Brothers, then they might find out where the original collateral went and retrieve it easily. However, if the hedge fund posted one thousand different bonds as collateral to the Lehman Brothers, then this may take much more time and cost to identify and retrieve all of the collateral of the hedge fund (Scott, 2014).

The slope of  $\zeta$  can proxy for how risk-averse agents are. For example, risk-averse agents would worry more about lender bankruptcy when their risk-aversion goes up, and they would diversify their lenders, or even reduce the amount of the total debt. Similarly, as  $\zeta'$  increases faster, the agents are more willing to diversify lenders or even reduce the amount of the total collateral exposure. Therefore, this convexly increasing cost assumption represents the risk-aversion and aligns with the institutional facts of the bankruptcy related costs.<sup>17</sup> Other specifications, such as concave or constant cost structure, miss the key mechanism of network formation which is the tradeoff between counterparty risk and leverage. This property will be examined in Section 3.

**Rehypothecation.** The model allows reuse of the collateral held by the lender. Such reuse of collateral is called rehypothecation in the financial market, and rehypothecation is prevalent in a wide variety of collateralizable assets (Singh, 2017). In reality, borrowers prefer to allow rehypothecation of their collateral. Even after the fall of the Lehman Brothers, most borrowers continued to allow rehypothecation of their collateral (Singh, 2017). The reason for the prevalent use of rehypothecation is that reuse of collateral generates more funding and market liquidity for the borrowers themselves. Since the lender can reuse the collateral to borrow money from someone else, the lender can provide even more cash to the borrower for the same collateral, and this increases funding liquidity. Furthermore, since the collateral can be used multiple times, the price of the collateral also goes up. This price effect can be thought of as the velocity of capital (Singh, 2010) or the collateral multiplier (Gottardi

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<sup>16</sup>The costs here are similar to bankruptcy cost in Elliott et al. (2014).

<sup>17</sup>Note that this cost could have been symmetrically applied to the borrower default as well. The results in this paper mostly hold for the case with the borrower default cost with a similar structure of convexly increasing cost. The only difference it makes is the difference in leverage (contract price) determination.

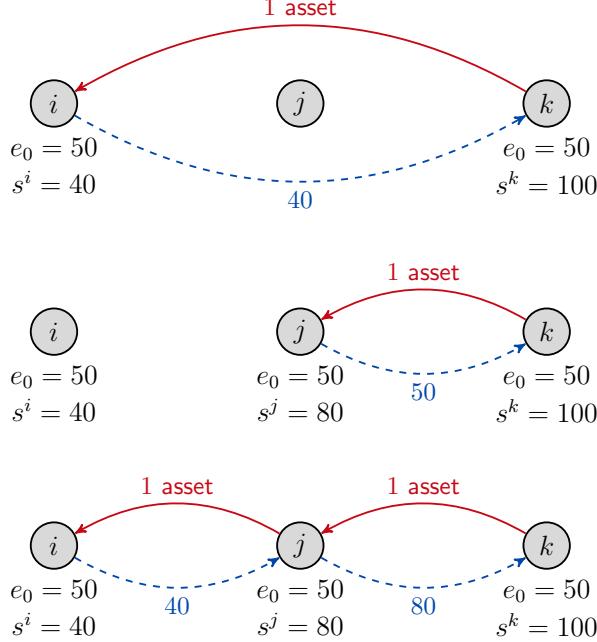


Figure 4: Example of Rehypothecation Effect

Note: Blue dashed arrows represent flows of cash and red arrows represent flows of collateral. The top figure represents the case of borrowing 40 directly from  $i$ , the middle figure represents the case of borrowing 50 from  $j$ , and the bottom figure represents the case of borrowing 80 from  $j$  who rehypothecates and borrows 40 from  $i$  again.

et al., 2017), which contributes to higher market liquidity of the collateralizable asset.

Figure 4 shows an example of borrowing without rehypothecation and borrowing with rehypothecation. Suppose agents  $i$ ,  $j$ , and  $k$  all have the same cash endowment of 50, and they have different beliefs as  $s^i = 40$ ,  $s^j = 80$ ,  $s^k = 100$ . Also suppose that there is no risk in  $t = 1$ , the asset price in  $t = 0$  is  $p_0 = 100$ , and the interest rate is zero. Agent  $k$  is the most optimistic agent and would like to buy as much of the asset as possible. Agent  $k$  can increase the amount of asset purchase by leveraging more. If agent  $k$  wants to borrow from agent  $i$ , any promise above 40 will not be made by  $k$ . This is because agent  $i$  believes the payoff of the asset is 40, and any promise above 40 will just be the same as 40 because of borrower default under agent  $i$ 's perspective. Then, the maximum amount of cash that  $k$  can borrow from  $i$  is 40. If agent  $k$  wants to borrow from agent  $j$ , then  $k$  will promise up to 80, which provides  $k$  a higher leverage than the leverage of borrowing from  $i$ . However, since agent  $j$ 's endowment of cash is only 50,  $k$  cannot borrow more than 50 from  $i$  if there is no rehypothecation allowed. In contrast, if  $j$  is allowed to reuse the collateral, then  $j$  can borrow 40 from  $i$ . Now the effective cash available for  $j$  becomes  $50 + 40 = 90$ , and  $k$  can borrow

80 from  $j$  which is greater than the borrowing amount of 50 under no rehypothecation. The leverage of  $k$  with no rehypothecation is  $100/(100 - 50) = 2$ , while the leverage of  $k$  with rehypothecation is  $100/(100 - 80) = 5$ . Therefore, agent  $k$  can increase leverage by 150 percent by allowing rehypothecation and would prefer to do so to increase her return.

## 2.2. Optimization Problem and Equilibrium Concept

Now that all the model structure is defined, an agent's optimization problem can be defined. Each agent maximizes their expected payoff in  $t = 2$  at the beginning of  $t = 0$  by choosing her investment portfolio. Each agent  $j \in N$  can

- (1) hold cash, in the amount denoted as  $e_1^j$ ,
- (2) can purchase the asset and carry it to the next period, in the amount denoted as  $h_{j,1}$ ,
- (3) borrow from agent  $i \in N$ , posting collateral in the amount of  $c_{ij}$  and promise cash per collateral as  $y_{ij}$ , or
- (4) lend to agent  $k \in N$ , receiving collateral in the amount of  $c_{jk}$  and promised cash per collateral as  $y_{jk}$ .

Note that the portfolio decision does not affect the macro variables, such as contract prices  $q(\cdot)$  and asset price  $p_0$ , under agent  $j$ 's perspective, because each agent is a price-taker. For a given portfolio, the agent's expected wealth (cash equivalent of total cash and asset holding) in  $t = 1$  is determined. However, wealth should be evaluated by the marginal utility (value) of cash for each state, which is  $s/p_1$ . The marginal utility of cash could be greater than 1 if the asset price  $p_1$  is under the fundamental value of the asset  $s$ . This underpricing can happen if the economy does not have enough aggregate cash in  $t = 1$  due to liquidity shocks and lender default costs. Thus, agent  $j$ 's nominal wealth and marginal utility of cash depend on realization of liquidity shocks  $\epsilon$ . Agent  $j$ 's maximization problem becomes

$$\begin{aligned}
& \max_{\substack{e_1^j, \{c_{ij}, y_{ij}\}_{i \in N}, \\ h_{j,1}, \{c_{jk}, y_{jk}\}_{k \in N}}} E_j \left[ \left( e_1^j - \epsilon_j + h_{j,1}p_1 + \sum_{k \in N \setminus \{j\}} c_{jk} \min \{y_{jk}, p_1\} \right. \right. \\
& \quad \left. \left. - \sum_{i \in N \setminus \{j\}} c_{ij} \min \{y_{ij}, p_1\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) [p_1 - y_{ij}]^+ \right) \frac{s}{p_1} \right]^+
\end{aligned} \tag{1}$$

s.t.

$$\begin{aligned}
h_{j,1} + \sum_{k \in N \setminus \{j\}} c_{jk} &\geq \sum_{i \in N \setminus \{j\}} c_{ij}, \\
e_0 + h_0 p_0 &= e_1^j - \sum_{i \in N \setminus \{j\}} c_{ij} q_i(y_{ij}) + \sum_{k \in N \setminus \{j\}} c_{jk} q_j(y_{jk}) + h_{j,1} p_0,
\end{aligned}$$

where  $B(\epsilon)$  is the set of bankrupt agents for given liquidity shock realization  $\epsilon$ , and  $\mathbb{1}[\cdot]$  is an indicator function. The first constraint is the collateral constraint, and the second constraint is the budget constraint. The collateral constraint implies that agent  $j$  should have enough assets, either from direct purchase or from collateral posted by  $k$  who is borrowing from  $j$  to post collateral. The underlying implication of the collateral constraint is the same as in Geanakoplos (1997), but this model keeps track of the identity of borrowers and lenders to analyze the network effect and rehypothecation structure.

The equilibrium concept that will be used throughout the paper is a hybrid version of general equilibrium with price functions that are affected by the network structure as follows.

**Definition 1.** For a given economy  $(N, (s^j, \theta_j, e_0, h_0)_{j \in N}, \zeta, G)$ , a septuple  $(C, Y, e_1, h_1, p_0, \tilde{p}_1, q)$  where  $C, Y \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ ,  $e_1, h_1 \in \mathbb{R}_+^n$ ,  $p_0 \in \mathbb{R}_+$ , and functions  $p_1 : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  and  $q_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  where  $q \equiv (q_1, \dots, q_n)$  is a **network equilibrium** if  $(C, Y, e_1)$  solves the agent maximization problem while satisfying budget and collateral constraints, markets are cleared as  $c_{ij}$  for the solution of agent  $j$  is the same as  $c_{ij}$  for the solution of agent  $i$  for all  $i, j \in N$ , asset market clears as  $\sum_{j \in N} h_{j,1} = H \equiv \sum_{k \in N} h_0$ , and  $p_0, \tilde{p}_1$  realized at  $t = 1$  and  $q$  are determined by no arbitrage conditions for the given network structure in  $t = 1$ .

The network dynamic is essentially occurring in  $t = 1$  through repayment and default costs from bankruptcy. This  $t = 1$  network effect also feeds back into  $t = 0$  optimization decisions which lead to network formation.

### 3. Network Equilibrium

This section characterizes the network equilibrium, the general equilibrium with collateralized debt network formation, and payment realization after network propagation. The payment realization in  $t = 1$  shows how the network structure and shocks affect the market price and the final wealth (and equivalently payoffs) of the agents. The endogenous network of collateralized debt contracts in  $t = 0$  is formed based on the consideration of the properties of a network and how agents clear markets of the asset and contracts. This section will solve for the equilibrium backwards: First, analyze the network contagion, fire sales, and price determination properties in  $t = 1$  and then derive the optimal contract decisions and network formation in  $t = 0$  for the given expected price distribution.

#### 3.1. Payment Equilibrium in Period 1

Since  $t = 2$  is merely the realization of the payoff of the asset and utility, we move to  $t = 1$  and solve for the equilibrium prices and wealth for a given debt network  $(C, Y)$ , cash holdings  $e_1$ , shock realization  $\epsilon$ , and payoff revelation of the asset  $s$ . Each agent  $j \in N$  pays back their promised amount of cash to her lender  $i$  in the amount denoted as  $x_{ij}$ , which follows the payment rule,  $x_{ij} = \min\{y_{ij}, p_1\}$ . Each agent's total nominal wealth (evaluated by cash), denoted as  $m_j$ , could be negative after the payments subtracted by liquidity shock  $\epsilon_j$  for all  $j \in N$ . An agent with negative wealth goes bankrupt, and their wealth does not enter into the demand side of the market. Only agents with positive post-payment wealth can enter the asset market at  $t = 1$  and affect the market price. If the asset is underpriced ( $p_1 < s$ ), then all the agents will spend all of their wealth to buy the asset, because the asset return is greater than the cash return. The price that makes the aggregate wealth equal to  $nh_0p_1$  will be the market clearing price. Thus, for given debt network  $(C, Y)$ , cash holdings vector  $e_1 \equiv (e_1^1, e_1^2, \dots, e_1^n)'$ , asset holdings vector  $h_1 \equiv (h_{1,1}, h_{2,1}, \dots, h_{n,1})'$ , uncertainty realizations of liquidity shocks  $\epsilon \equiv (\epsilon_1, \dots, \epsilon_n)'$  and asset payoff  $s$ , and given lender default cost function  $\zeta$ , we can obtain the vector  $M \equiv (m_1, \dots, m_n)$  of nominal wealth of each agent and the resulting market price of the asset as well as asset holdings. This market clearing price and allocation can be defined as payment equilibrium<sup>18</sup>, which is an intermediate equilibrium of  $t = 1$  as follows.

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<sup>18</sup>In the exogenous debt network literature stemming from Eisenberg and Noe (2001) and to papers such as Acemoglu et al. (2015), the main equilibrium concept is almost the same as the payment equilibrium (the name which I coined from this literature) in this paper. This intermediate step also provides a comparison between the model in this paper and the literature of exogenous financial networks and propagation dynamics. The crucial difference of the model in this paper is that the model here has an additional market for the asset used as collateral which induces the network propagation and the asset price feedback to each other.

**Definition 2.** For a given period-1 economy of  $(N, C, Y, e_1, h_1, \epsilon, s, \zeta)$ , a **payment equilibrium** is  $(M, h_2, p_1)$ , where  $M$  is the wealth vector,  $h_2$  is the asset holding vector, and  $p_1$  is the price of the asset such that  $M$  satisfies the payment rule,  $h_2$  is determined after the bankruptcy and default costs, and  $p_1$  makes the asset market clear.

From the payment rule  $x_{ij} = \min\{y_{ij}, p_1\}$ , contracts with promise of  $y_{ij} > p_1$  will be paid less than the face value—that is, just the price of the asset—and the contracts with promise of  $y_{kl} \leq p_1$  will be paid in full for any  $i, j, k, l \in N$ . If an agent  $j$ 's total wealth  $m_j$  is negative, then the agent cannot even fulfill its obligations to the senior outside debtors (that is, the liquidity shock of  $\epsilon_j$ ), and the agent will go bankrupt. The model considers any event or cost related to the bankruptcy as outside of the collateral debt network, other than the counterparty (lender) default cost. As defined before,  $B(\epsilon)$  is the set of agents who go bankrupt under the shock vector  $\epsilon$ . The market clearing price will indirectly determine this set because, in some cases, an agent could have survived in high  $p_1$  but would go bankrupt in low  $p_1$ . Thus, this set might not be well defined as there could be multiple sets that constitute payment equilibria. Among multiple  $B(\epsilon)$ 's, selecting the smallest set of  $B(\epsilon)$  that holds as payment equilibrium implies selecting the maximum price payment equilibrium. This equilibrium selection rule is well defined, which will be shown later in this subsection. Omit the subscript of  $p_1$  from now on throughout this subsection since this subsection only focuses on  $t = 1$ .

The total nominal wealth of agent  $j$  after all the payment is

$$m_j(p) = e_1^j - \epsilon_j + h_{j,1}p + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\} - \sum_{i \in N} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+,$$

where  $e_1^j - \epsilon_j$  is the remaining cash you have from  $t = 0$  subtracted by (possibly zero) liquidity shock  $\epsilon_j$ . To consider the wealth that is actually effective in demand when we compute the equilibrium, define the *effective nominal wealth* of each agent as  $[m_j(p)]^+$ . If  $m_j(p) < 0$ , then agent  $j$  goes bankrupt, that is  $j \in B(\epsilon)$ , and agent  $j$  will liquidate all of their holdings to pay  $\epsilon_j$ . Thus, the equilibrium asset holding  $h_{j,2}$  is determined by

$$h_{j,2} = \frac{[m_j(p)]^+}{p},$$

when  $p < s$ . If  $p = s$ ,  $h_{j,2} \leq \frac{[m_j(p)]^+}{p}$  but the asset holding cannot be pinned down and also is irrelevant to pin down due to invariance in final utility at  $t = 2$  between holding the asset by paying the fair price and holding the equivalent amount of cash.

The aggregate cash value of the supply of the asset should equal to the aggregate cash

value of the demand of the asset. As long as  $p \leq s$ , there will be an agent who would spend all the excess cash they have to buy the asset. The cash value of the aggregate supply is  $\sum_{j \in N} h_{j,1}p = nh_0p$ . The equality is coming from the market clearing from  $t = 0$ . The cash value of the aggregate demand is a function of the asset price as well. If the price reaches  $s$  and there is enough money to buy up all the supply, then that is an equilibrium. The aggregate effective cash value of demand in the market is  $\sum_{j \in N}[m_j(p)]^+$ , and therefore, the market clearing condition that determines the price becomes

$$\sum_{i \in N}[m_i(p)]^+ = \sum_{j \in N} h_{j,1}p \quad \text{if } 0 \leq p < s, \quad (2)$$

$$\sum_{i \in N}[m_i(p)]^+ \geq \sum_{j \in N} h_{j,1}p \quad \text{if } p = s. \quad (3)$$

The aggregate effective nominal wealth increases as the price increases (see Lemma 5 in the appendix) and the lender default cost decreases. But, then again there is a feedback from the nominal wealth to the price. Note that the equality should hold unless  $p = s$ . We can interpret this as the price is going to be the level that makes the aggregate amount of liquidity that can cover both all available assets and the costs from defaults, cash-in-the-market pricing. Another case to consider is when  $p = 0$ . If there is any extra cash left in the economy, then  $p = 0$  cannot be true. However, if there is no cash left in the economy after paying out the liquidity shocks and default costs, then  $p = 0$  can occur, the market is broken down, and all the asset holdings become indeterminate as in the case of  $p = s$ .

The class of possible debt networks for the double  $(C, Y)$  is very large. In order to make the analysis plausible, I restrict attention to the networks in which collateral from the borrower is enough for the lender to pay her own promises. Define the class of such networks as the networks under *intermediation order*. For a given level of payment  $\hat{y}$ , agent  $j$  should hold enough collateral (either held directly as  $h_{j,1}$  or indirectly by lending as  $c_{jk}$ ) that promises greater than or equal to  $\hat{y}$  to cover all the debt promised to pay  $\hat{y}$  or greater value. Thus, a network is under *intermediation order* if

$$\sum_{\substack{i \in N \\ y_{ij} \geq \hat{y}}} c_{ij} \leq h_{j,1} + \sum_{\substack{k \in N \\ y_{jk} \geq \hat{y}}} c_{jk} \quad \text{for any } \hat{y} \in \mathbb{R}^+ \text{ and } j \in N. \quad (4)$$

In fact, the endogenous network formation in  $t = 0$  in the next subsection will show that the equilibrium networks should be under intermediation order. This intermediation order is equivalent to only allowing pyramiding of contracts – promising a delivery using another contract as a collateral introduced by Geanakoplos (1997). If agent  $j$  uses the contract by

agent  $k$  with promise of  $y_{jk}$  as collateral to promise  $y_{ij}$  to agent  $i$ , the actual delivery becomes  $\min\{y_{ij}, \min\{p, y_{jk}\}\} = \min\{p, \min\{y_{ij}, y_{jk}\}\}$ , so  $y_{ij} \leq y_{jk}$  to be a non-trivial pyramiding of the contract.

For example, if agent  $k$  promises 20 to  $j$  but  $j$  reuses the collateral and promises 30 to  $i$ , then, agent  $j$  violates the intermediation order. In this case, agent  $j$  might not have enough cash to repay  $i$ , if the price is between 20 and 30, so he cannot receive the collateral from  $i$  and then return it to  $k$ . The intermediation order guarantees that if the ultimate borrower (collateral provider) fulfills her promise, then the intermediary (who reuses the collateral) also has enough cash to fulfill his promise to the ultimate lender (cash provider). Note that intermediation order implies collateral constraints.

Under the intermediation order, we can interpret the market clearing condition in a more intuitive way. The negative liquidity shocks  $\epsilon$  destroy the aggregate available cash. The destruction of cash for the demand can be decomposed into three factors:

- (1) each agent's liquidity shock  $\epsilon_j$ ,
- (2) lender default costs from bankrupt lenders  $\sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+$ , and
- (3) second-order bankruptcy from the first two effects which amplifies (2).

The second and third factors create a feedback loop in the market through the price channel and the counterparty channel similar to debt network models without collateral. Note that the first factor is bounded above by the nominal wealth of the agent before the negative liquidity shock,  $\mu_j(p) \equiv m_j(p) + \epsilon_j$ , because any excess liquidity shock still makes the same effective nominal wealth of zero. For a given price  $p$ , the actual destruction of cash from liquidity shock to  $j$  relevant to the demand of the asset market is  $\min\{\epsilon_j, \mu_j(p)\}$ .

For a given price  $p$ , the excess cash of the network can be computed as the *cash savings from  $t = 0$*  subtracted by the *destruction of cash*. Hence, the *remaining cash* becomes

$$RM \equiv \sum_{j \in N} e_1^j - \sum_{j \in N} \min\{\epsilon_j, \mu_j(p)\} - \sum_{j \in N} \sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+,$$

which is the cash saved from  $t = 0$  subtracted by the liquidity shocks and lender default costs. The remaining cash can be considered as the demand side. For the supply side, the amount of collateral that are sold in the market is the amount of collateral from bankrupt agents' balance sheets, that is the *total fire sales of the assets* denoted as

$$FS \equiv \sum_{j \in B(\epsilon)} \sum_{\substack{k \in N \\ p < y_{jk}}} (c_{jk} + h_{j,1}) - \sum_{j \in B(\epsilon)} \sum_{\substack{i \neq j \\ p < y_{ij}}} c_{ij}.$$

Suppose that the price  $p$  is neither 0 or  $s$ . Then, the market clearing condition, equation (2) becomes

$$\pi(p) = \frac{RM}{FS}, \quad (5)$$

which is the *remaining cash* divided by the *total fire sales of the assets* that are under bankrupt agents' balance sheets.

By the intermediation order, the denominator is always nonnegative. However, if there are no assets to be bought (that is, denominator of  $\pi(p)$  is zero), then the price of the asset will be trivially its fair value price  $s$ . If there is no asset to be sold by the bankrupt agent, then there is no reason to lower the price of the asset. If there are enough cash in the market to cover the extra supply (fire sales) with the fair price (i.e.  $\pi(p) \geq s$ ) then the price is also set as fair value price  $s$ . If there are some leftover cash after the payments and costs that is not sufficient to buy all of the assets in fair price, then the market price will be  $\pi(p) < s$  which we define as *liquidity constrained price* of the asset.

The post-shock market clearing condition, equations (2) and (3), can be rearranged to obtain the price equation as follows.

$$p = \begin{cases} 0 & \text{if } \sum_{j \notin B(\epsilon)} e_1^j \leq \sum_{i \notin B(\epsilon)} \epsilon_i \text{ for any } p \in [0, s] \\ s & \text{if } \pi(p) > s \text{ or } FS = 0 \\ \pi(p) & \text{otherwise.} \end{cases} \quad (6)$$

Under the intermediation order, I can show that a payment equilibrium always exists and the set of equilibrium prices is a complete lattice.

**Proposition 1 (Existence and Lattice Equilibrium Prices).** *For any given collateralized debt network  $(N, C, Y, e_1, h_1, \epsilon, s, \zeta)$  with  $C > 0$  that is under intermediation order, there exists a payment equilibrium  $(M, h_2, p_1)$ . Furthermore, among the possible equilibria, there always exists a maximum equilibrium that is  $(\overline{M}, \overline{h_2}, \overline{p_1})$ , where  $\overline{p_1}$  is the highest price among all possible equilibrium prices.*

All the proofs are relegated to the appendix. The intuition of the proof is the following. The delivery  $x_{ij} = \min\{y_{ij}, p\}$  toward the lender increases as price  $p$  increases. By intermediation order, every borrower or intermediary payoff also increases as  $p$  increases. Thus, every individual nominal wealth increases in  $p$ . Since individual nominal wealth  $m_j$  is increasing in  $p$ , as shown by lemma 5 in the proof, the aggregate nominal wealth is increasing in asset price  $p$  and decreasing in lender default cost  $\zeta$ . Since increase in wealth also means

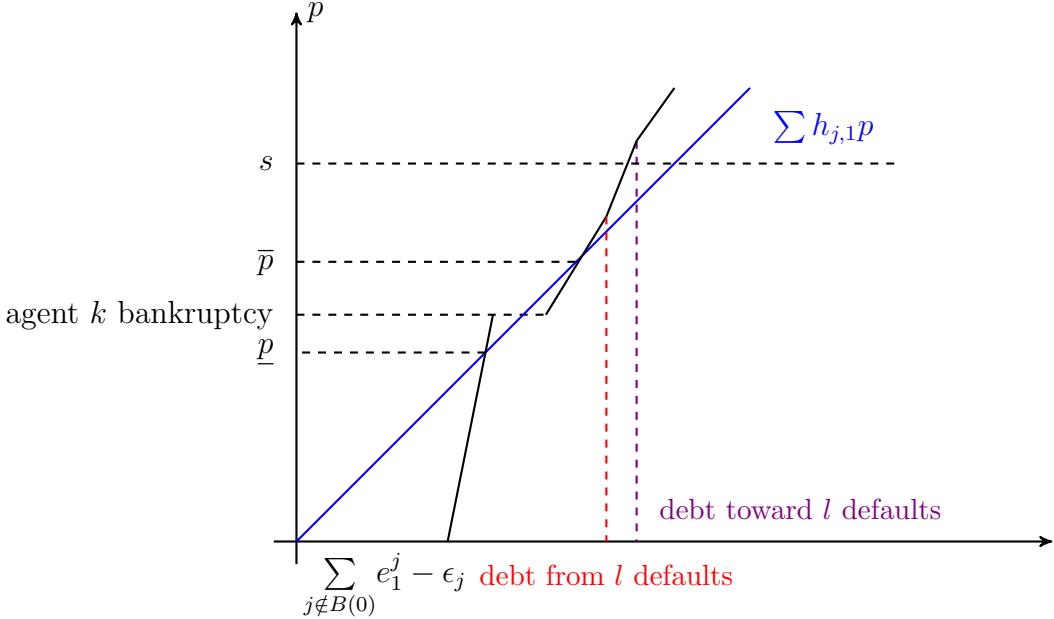


Figure 5: Multiple Payment Equilibria for Market Clearing

bankruptcy is less likely, the lender default cost also decreases when  $p$  increases. Therefore, every single variable that is included in the market clearing condition (weakly) increases in price  $p$ . Although the equilibrium is determined by the vector of all the wealth, we can summarize each equilibrium by price level  $p$ , and there exists a fixed point price that clears the market.

However, the payment equilibrium is not unique. This multiplicity is mainly due to the jumps in  $m_j(p)$  at the point of bankruptcy of other agents. The actual bankruptcy set may also depend on the market clearing price as  $B(\epsilon|p)$ . An agent may have  $m_j(p) > 0$  for given price  $p$  and bankruptcy set  $B(\epsilon|p)$ , but her wealth may be negative at a lower price  $p'$  and given bankruptcy set  $B(\epsilon|p')$  so  $m_j(p') < 0$ . Her bankruptcy will generate even more second-order bankruptcy costs and make  $p'$  to be true. The following proposition summarizes this relation between multiplicity and bankruptcy.

**Proposition 2 (Multiplicity and Bankruptcy).** *For any given collateralized debt network  $(N, C, Y, e_1, \epsilon, s, \zeta)$  with  $C > 0$  that is under intermediation order, there may be multiple equilibria. If  $p$  and  $p'$  are two distinct prices from the two different payment equilibria, then  $B(\epsilon|p) \neq B(\epsilon|p')$ .*

Figure 5 depicts an example of multiple equilibria. There are kinks at prices in which each contract defaults and discontinuous jumps at prices in which each agent goes bankrupt. The first type of kink occurs for  $p < y_{ij}$  which affects both  $m_i(p)$  and  $m_j(p)$ , and the second type of jump occurs at the point where  $m_j(p) = 0$ . From the second statement of proposition

2 and equation (11) in the proof, the existence of lender default cost plays a significant role in generating multiplicity and also the counterparty contagion effect through the second-order bankruptcy. Due to the multiplicity<sup>19</sup> and the lattice property, we assume  $B(\epsilon)$  to be the bankruptcy set from the maximum equilibrium price—that is,  $B(\epsilon) \equiv B(\epsilon, \bar{p})$ , from now on. Also, a *maximum equilibrium selection rule* means choosing the equilibrium with the maximum equilibrium price. We will focus on the results on equilibrium with the maximum equilibrium selection rule as in Elliott et al. (2014).

Trivially, if there is no default cost—that is  $\zeta(c) = 0$  for any  $c$ —then the payment equilibrium is unique because there will be no jumps in the aggregate wealth. Also without a default cost, change in counterparty connections does not matter as long as the total borrowing and lending amount remain the same. The following proposition states this property.

**Proposition 3 (Counterparty Irrelevance).** *If there is no lender default cost—that is,  $\zeta(c) = 0$  for all  $c \geq 0$ —then the payment equilibrium is unique for any given network. Furthermore, two networks  $(C, X)$  and  $(\hat{C}, \hat{X})$  with the same indegrees and outdegrees—that is,  $\mathbf{1}(C \circ X) = \mathbf{1}(\hat{C} \circ \hat{X})$  and  $(C \circ X)\mathbf{1} = (\hat{C} \circ \hat{X})\mathbf{1}$ —will have the same payment equilibrium.*

This proposition shows the necessity of assuming the existence of a lender default cost (or any counterparty risk) in order to generate interesting interaction among agents. Because of the absence of a default cost, agent's individual connection does not matter as long as the total borrowing and lending for each agent are the same. The two networks will have the same equilibrium price and allocation. The result is not so surprising since the main reason for using collateral is to insulate the lender from the counterparty risk. A collateralized debt network has no counterparty risk as in the anonymous market.

From now on, define *systemic risk* under the belief of agent  $j$  as the expected difference between the ex post fair value of the asset and the actual price of the asset,  $\int(s^j - p)d\tilde{G}(\epsilon)$ , where  $\tilde{G}$  is the joint distribution of  $G_i$ 's for all  $i \in N$ . This notion of systemic risk is following the definition of *systemic loss* in value defined in Glasserman and Young (2016). Even though the fundamental value of the asset is  $s$ , the underpricing of the asset,  $s - p$ , comes from the liquidity shocks and lender default costs which vary by the network connections. The systemic risk definition here is taking the ex ante expected value of the systemic losses for each subjective belief. The sum of all the systemic risks for each ex ante belief will be closely related to the social welfare computation, which will be in the next subsection. The aggregate default costs after the revelation of  $s$  and realization of  $\epsilon$  will determine the difference between the two values, and the difference represents how severe the mispricing is due to the total sum of deadweight losses.

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<sup>19</sup>The existence of multiple equilibria implies that there could be even more instability than looking at just the maximum result (Roukny et al., 2018).

The following algorithm is how to solve the equilibrium in quantitative analysis under the maximum equilibrium selection rule.

**Equilibrium Search Algorithm:** Consider the following algorithm of finding the maximum payment equilibrium.

0. Set  $B^{(0)}(\epsilon) = \emptyset$ . Start with step 1.
1. For any step  $k$ , given  $B^{(k-1)}$ , compute  $p^{(k)}$  that satisfies equation (6).
2. For given  $p^{(k)}$ , compute  $m_j(p^{(k)})$  with given  $B^{(k-1)}$  and update  $B^{(k)}$ .
3. If  $B^{(k-1)} = B^{(k)}$ , then we have the maximum equilibrium. Otherwise, move to the next step  $k + 1$  and repeat procedures 1 and 2.

This algorithm guarantees to find the maximum payment equilibrium price of the given network. Also, the algorithm finishes within  $n$  steps because the second-order bankruptcy (cascades) could only occur at the maximum of  $n - 1$  times.

### 3.2. Prices and Interest Rates in Period 0

In this subsection, the prices and interest rates are pinned down by agents' investment decisions and no-arbitrage conditions. Given the results from  $t = 1$ , agents form beliefs on the distribution of  $p_1$  and  $B(\epsilon)$  under shock realizations. As discussed in Section 2, agent  $j$  solves the maximization problem (1). From now on, substitute the probability measure superscript and denote  $E_j[\cdot]$  as nonnegative expected nominal wealth. Any negative wealth will be counted as zero in agent  $j$ 's perspective. Denote the implied *expected lender default cost from agent  $i$  under agent  $j$ 's subjective belief* as

$$\omega_{ij}(y) \equiv E_j \left[ [1 - y/p_1]^+ \mathbb{1}[i \in B(\epsilon)] \right].$$

Then, the *marginal increase in counterparty risk of borrowing from agent  $i$  for agent  $j$*  becomes  $\zeta'(c_{ij})s^j\omega_{ij}(y)$ .

An agent has five different ways to use her budget: holding cash, buying the asset, buying the asset with leverage, lending cash to others, and lending cash with leverage. For each additional unit of cash, an agent should compare the five options for marginal returns. This return comparison will determine the interest rates.

Agent  $j$ 's return on holding cash is  $E_j[s/p_1]$ . The cash return goes up as  $j$  holds less cash because there would be even more underpricing if all other agents go bankrupt. From the market pricing equation (6) in  $t = 1$ , price of the asset in  $t = 1$  can become  $p_1 = 0$  if all the

agents who are holding cash in the economy at  $t = 0$  go bankrupt. Even if the probability of liquidity shock  $\theta_j$  is small for everyone, if there is a positive probability of bankruptcy for each agent, then there is a positive probability of  $p_1$  being zero, and the return on cash holding becomes infinity. Therefore, every agent in the equilibrium should hold a positive amount of cash. This pins down all the returns from borrowing and lending to the return of holding cash  $E_j[s/p_1]$ . The cash return becomes the benchmark return for any other investment decision the agent makes.

**Lemma 1** (Positive Cash Holdings). *If  $\bar{\epsilon} > ne_0 + h_0 s^1$ , then  $e_1^j > 0$  for every  $j \in N$  in any network equilibrium.*

This lemma implies that the model is distinctive from existing models in both financial networks and general equilibrium literature. Financial network models often have an equilibrium in which agents have strongly correlated payoffs (Allen et al., 2012; Cabrales et al., 2017; Elliott et al., 2018; Erol, 2018; Jackson and Pernoud, 2019). Marginal utility of cash in this paper acts as an opposing force and makes agents to decorrelate their payoffs from each other by holding cash. General equilibrium models with linear utility have borrowers holding only the assets and zero amount of cash (Geanakoplos, 2010; Simsek, 2013; Geerolf, 2018). In reality, the ultimate borrowers such as hedge funds usually hold a significant proportion of their portfolio as cash equivalent assets (Aragon et al., 2017). The hybrid network general equilibrium model here replicates this observed phenomenon by adding this liquidity shock in the intermediate period and the possibility of high marginal utility of cash because of liquidity constrained-price, which is below the fair value of the asset.

The (marginal) return on lending depends on how much agents lend but does not depend on which agent they lend to. This irrelevance comes from the fact that lenders do not have counterparty risk due to collateralization and nonrecourse contracts. Therefore, the contract price  $q_i$  does not depend on the identity of the borrower. Suppose  $j$  is lending a positive amount of cash without leverage—that is,  $j$  is a pure lender. The return of lending for  $j$  is

$$\frac{1}{q_j(y)} E_j \left[ \min \left\{ s, y \frac{s}{p_1} \right\} \right] = E_j \left[ \frac{s}{p_1} \right].$$

The return of lending should equal the return of cash for no arbitrage. This equation also represents how the price of a contract (or interest rate) is determined if agent  $j$  does not

leverage their position as

$$q_j(y) = \frac{E_j \left[ \min \left\{ s, y \frac{s}{p_1} \right\} \right]}{E_j \left[ \frac{s}{p_1} \right]} = \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} \right]}{E_j \left[ \frac{1}{p_1} \right]}.$$

If the realization of the asset payoff increases, the asset is more likely to be underpriced than its fundamental value because of more exposure to liquidity shortage and lender default costs. Thus, for the agents with the same wealth, the order of return of holding cash also follows the order of optimism over asset payoffs—that is,  $E_j [s^j/p_1] > E_k [s^k/p_1]$  for any  $j < k$ . In fact, the inequality should always hold in an equilibrium as in lemma 2.

**Lemma 2** (Cash Return Ordering). *For any two agents in a network equilibrium, the cash return from the more optimistic agent is always greater than the cash return from the less optimistic agent—that is,  $E_j \left[ \frac{s^j}{p_1} \right] > E_k \left[ \frac{s^k}{p_1} \right]$  for any  $j < k$  and  $j, k \in N$ .*

The main intuition of the proof is as follows: If agent  $k$ , who is more pessimistic than agent  $j$ , has higher (subjective) return from cash holdings, then any other investment she is making should also have that same return by lemma 1. Suppose agent  $j$  mimics agent  $k$ 's entire investment portfolio. Then the same investment cannot have a return greater than the return from cash holdings from agent  $j$ 's original portfolio, because otherwise it violates the optimality of his own portfolio decision. But, if agent  $j$  and  $k$  face exactly the same cashflow and counterparty risks, then the return from that investment should always be higher from agent  $j$ 's perspective because he is more optimistic about the asset return and the degree of underpricing (and marginal utility of cash) is higher under more optimistic belief. This implies that agent  $j$  can have higher return than agent  $k$  by mimicking a strategy that violates the original assumption of agent  $k$  having higher return from cash holding.

This cash return ordering from lemma 2 implies that interest rates of the same contract increases over an agent's optimism—that is, optimistic agents demand a higher interest rate than pessimistic agents do. This property will be verified again by the contract pricing formula.

Return on buying the asset without leverage is  $E_j [s/p_0]$ , where  $p_0$  is the asset price at  $t = 0$ . Since the return does not depend on  $p_1$ , this return is not (directly) influenced by counterparty risk. Hence, this return on asset is ordered directly by the agent's optimism—

that is,  $E_j[s/p_0] > E_k[s/p_0]$  for all  $j < k$ . Return on asset purchase with leverage is

$$\frac{s^j}{p_0 - q_i(y)} E_j \left[ \left[ 1 - \frac{y}{p_1} \right]^+ - \zeta'(c_{ij}) \left[ 1 - \frac{y}{p_1} \right]^+ \mathbb{1}\{i \in B(\epsilon)\} \right],$$

where agent  $j$  is borrowing cash from agent  $i$  with  $c_{ij}$  amount and promises  $y$ . Similarly, return on lending with leverage is

$$\frac{s^j}{q_j(y') - q_i(y)} E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ij}) \left[ 1 - \frac{y}{p_1} \right]^+ \mathbb{1}\{i \in B(\epsilon)\} \right], \quad (7)$$

where  $j$  buys (lends money) a contract with promise  $y'$ . From the return comparisons and pure lender's no arbitrage condition, an agent's individual leverage decision could be derived, and the following lemma summarizes the result of leverage maximization.

**Lemma 3** (Maximum Leverage). *Suppose that agent  $j$  lends a positive amount of money to an agent (or buys the asset—that is, lends money to herself) and borrows a positive amount of money from agent  $i$  in a network equilibrium. Then, the following statements are true:*

1. *Agent  $j$  maximizes her contract leverage by borrowing the maximum amount of money she can borrow from agent  $i$  which is  $s^i$ .*
2. *If  $j$  is posting the same amount of collateral to agent  $i$  and  $k$  who have the same probability of bankruptcy with  $i < k$ , then  $j$  marginally prefers to borrow from  $i$ .*

The intuition of the proof is as follows: If borrower  $j$  and lender  $i$  agree on the distribution of prices below  $s^i$ , which only depends on liquidity shocks that both agents agree upon, then  $j$  and  $i$  agree upon the expected delivery. Since  $j$  has higher marginal utility of cash in  $t = 0$  than  $i$ , agent  $j$  would like to increase borrowing at any point below  $s^i$ . At the point of  $s^i$ , agents disagree with the promised delivery above  $s^i$ . Agent  $j$  believes the price of the asset  $p_1$  can be greater than  $s^i$  if the aggregate liquidity shock is not large enough, but  $i$  believes the price is bounded above by  $s^i$  even if there is zero liquidity shock. Therefore, the endogenous leverage is determined by the promise of  $y = s^i$  and its price  $q_i(s^i)$ .<sup>20</sup> The logic can be considered as an extension of the three state case in Geanakoplos (2003).

With lemma 3, we can focus only on networks following intermediation order.

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<sup>20</sup>This intuition also brings light to how complicated the model would be if concentrated beliefs are not assumed. For example, if agent's optimism is ordered by first-order stochastic dominance, then endogenous leverage depends not only on the relative hazard ratio, but also on the difference in marginal utilities of cash which also changes endogenously and is extremely intractable to pin down. Moreover, they differ with the distribution of liquidity shocks.

**Corollary 1.** *Any debt network from a network equilibrium is under intermediation order.*

Also by lemma 3 we can pin down  $q_j(y)$  for agents who both borrow and lend. If agent  $j$  borrows  $y$  from agent  $i$  and lends  $y'$  to some other agent (or herself if she buys the asset directly) then her no-arbitrage contract price becomes

$$q_j(y') = q_i(y) + \frac{E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ij}) \left[ 1 - \frac{y}{p_1} \right]^+ \mathbf{1}_{\{i \in B(\epsilon)\}} \right]}{E_j \left[ \frac{1}{p_1} \right]}.$$

By lemma 3 we only need to focus on kink points for borrowing. Hence, any agent who is willing to borrow from agent  $j$  will face the willingness to pay as

$$q_j(y) = q_i(s^i) + \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \zeta'(c_{ij}) \left[ 1 - \frac{s^i}{p_1} \right]^+ \mathbf{1}_{\{i \in B(\epsilon)\}} \right]}{E_j \left[ \frac{1}{p_1} \right]}. \quad (8)$$

Now with the given contract prices, the asset price in  $t = 0$  can be determined. The first step is about who is buying the asset in the equilibrium. Not surprisingly, the agent with the most optimistic belief on the asset payoff, agent 1, always buys the asset.

**Lemma 4** (Natural Buyers and Asset Price). *In a network equilibrium, the most optimists, agent 1, buys the asset with positive quantity, thus  $c_{11} > 0$  and  $p_0 = q_1(s^1)$ . Similarly, agent  $i$  borrows from agent  $i + 1$  with positive amount,  $c_{i+1,i} > 0$  for any  $i < n$ .*

The intuition of the lemma is that if any agent  $j > 1$  is buying the asset, then agent 1 will have even higher return than  $j$  by using the same leverage decisions as  $j$  unless agent 1's cash holding is huge enough to make her required return low. However, when agent 1's cash holding is large, then  $j$ 's return of cash is enormous in case agent 1 goes bankrupt, and agent  $j$  should either increase his cash holding or increase the return on asset purchase—that is,  $p_0$  goes down. But either of them should make agent 1's perceived return on asset purchase (with leverage) increase even faster because of  $s^1/p_0 > s^j/p_0$ . Thus, agent 1 should be a natural buyer of the asset (but not necessarily the only buyer). Similar logic can be applied to any subsequent contracts  $\min\{p_1, s^i\}$  and by induction, we can show that a natural buyer of any contract with a promise of  $s^i$  is agent  $i$ .

Agents other than agent 1, say agent  $j$ , can also hold some amount of assets. In this case, agent 1 holds more cash than agent  $j$  so that the possible underpricing from larger support

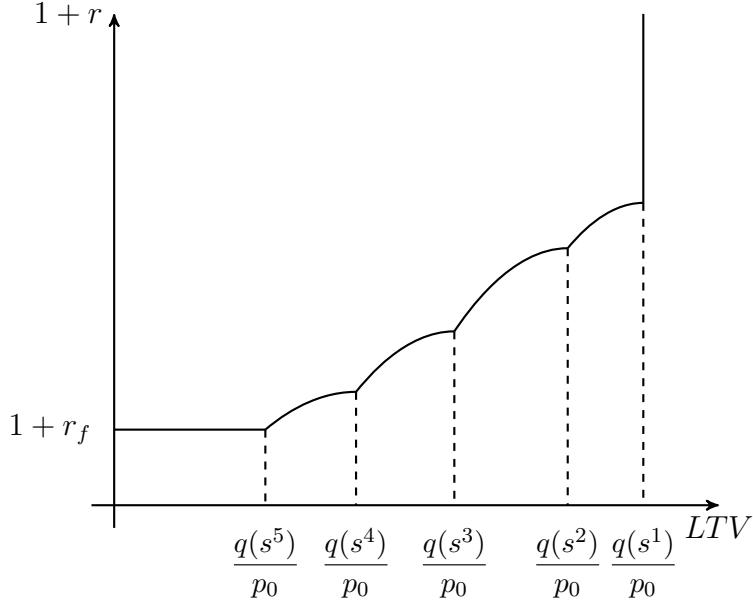


Figure 6: Credit Surface of Collateralized Debt

for agent 1 is mitigated by being less vulnerable to liquidity shocks than others such as agent  $j$ . Thus,  $e_1^1 > e_1^j$  in such cases. This property of optimists holding more cash than pessimists can be formalized for a certain parametric region as shown in the online appendix.<sup>21</sup>

By lemma 4, the asset price and natural buyers for each contract is determined. Finally, the function of contract prices for the market,  $q(y)$ , and the relationship between interest rate and leverage can be summarized as the following proposition and figure 6.

**Proposition 4** (Concave Credit Surface). *In any network equilibrium, the market contract price function  $q(y)$  is piece-wise concave in the amount of promise  $y$  and has kinks and jumps at each payoff points  $s^1, s^2, \dots, s^{n-1}, s^n$ . Furthermore, the credit surface of the equilibrium (the graph between leverage  $q(y)/p_0$  and interest rate  $y/q(y)$ ) is piece-wise concave and continuous in the amount of leverage  $q(y)$  and has kinks at each corresponding payoff points  $q(s^1), q(s^2), \dots, q(s^{n-1}), q(s^n)$  and right derivative of each kink point is greater than the left derivative. Also, the interest rate goes to infinity at the point  $q(s^1)$ .*

The intuition is that an increase in leverage results in higher interest rate due to greater

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<sup>21</sup>This cash holding result may seem unrealistic. However, the empirical facts support this result. On average, 34 percent of a hedge fund's assets can be liquidated within one day (without fire sale discounting) according to Aragon et al. (2017). This proportion is well above the proportion of money market mutual funds (MMMFs) in the SEC reformed regulation by 10 percentage points. Before the regulation, the daily liquid assets for MMMF were on average less than 20 percent, and even after the regulation, the daily liquidity in the portfolio is still below 31 percent (Aftab and Varotto, 2017). Because hedge funds are the ultimate asset buyers (as agent 1) in a collateralized debt market, and money market mutual funds are pure lenders in the market (as agent  $n$ ), the empirical findings are consistent with the result.

risk of borrower default. For each agent  $j$ ,  $s^j$  is the maximum amount of promise agent  $j$  lends to a borrower in equilibrium. Any promise above that will be offered to more optimistic natural buyer such as  $j - 1$ , thus, there will be kinks at each belief points.

### 3.3. Network Formation in Decentralized OTC Market

Given the prices in  $t = 0$ , the remaining parts of the network equilibrium are the amount of cash holdings and the amount traded for each contract. The next result and important step for the proof of existence is that individual agent's diversification behavior generates positive externalities through amplification and feedback effects in both asset price channel and counterparty channel in  $t = 1$ . If agent  $j$  diversifies more and lowers her own return because of counterparty risk concerns, then it will lower the leverage through  $q(y)$  and also decrease price volatility in  $t = 1$ . Furthermore, this risk reducing behavior makes agent  $j$ 's balance sheet  $m_j$  more stable and decreases the probability of  $j$ 's bankruptcy. Thus, second-order bankruptcy contagion decreases even further.

Before stating the proposition, we have to define directions of lowering the aggregate debt level. Since any equilibrium network is under intermediation order, we can restrict our attention to directions that go across such a class of networks. First, for a given collateral matrix  $C$ , a collateral matrix  $C^*$  is *uniformly less indebted* if  $c_{ij} \geq c_{ij}^*$  for any  $i, j$  and  $c_{ij} > c_{ij}^*$  for at least one pair  $ij$ . The second direction comes from diversification. Define  $L_j$  as the largest holder of  $j$ 's collateral, thus,  $\max_{i \in N} c_{ij} = c_{L_j j}$ . For a given network equilibrium and its collateral matrix  $C$ ,  $C^*$  is a *diversification of agent  $j$*  from  $C$ , if

1.  $c_{L_j j} > c_{L_j j}^* \geq \max_{i \in N} c_{ij}^*$ ,  $c_{ij} \leq c_{ij}^*$  for all  $i > L_j$ ,
2.  $\zeta(c_{L_j j}^*)\omega_{L_j j} \geq \zeta(c_{ij}^*)\omega_{ij}$  for any  $i > L_j$ ,
3.  $\sum_{i \in N} c_{ij} \geq \sum_{i \in N} c_{ij}^*$ ,
4.  $c_{ik} \geq c_{ik}^*$  for all  $i, k \in N$  with  $k \neq j$ , and
5.  $(C^*, Y)$  is under intermediation order.

This diversification of agent  $j$  from an equilibrium collateral matrix implies that agent  $j$  has her counterparties more diversified than the original network in either intensive or extensive margins while still maintaining the perceived counterparty risk not exceeding the original largest holder of collateral.

**Proposition 5** (Diversification Externality). *Suppose that  $(C, Y, e_1, h_1, p_0, \tilde{p}_1, q)$  is a network equilibrium. Suppose there is a collateral matrix  $C^*$  and either of the two conditions holds:*

1.  $C^*$  is uniformly less indebted than  $C$ .
2.  $C^*$  is a diversification of agent  $j$  from  $C$  for  $j < n$ .

Then, ex ante expected payment equilibrium price  $p_1$  under  $(N, C^*, Y, e_1, h_1, \cdot, \cdot, \zeta)$  is greater than that under  $(N, C, Y, e_1, h_1, \cdot, \cdot, \zeta)$  and ex ante volatility of  $p_1$  under  $(N, C^*, Y, e_1, h_1, \cdot, \cdot, \zeta)$  is lower than that under  $(N, C, Y, e_1, h_1, \cdot, \cdot, \zeta)$  for each subjective belief of  $j \in N$ .

This proposition implies that the higher the debt level is, either because uniformly more indebted or less diversification, the more the underpricing occurs both in terms of likelihood and intensity. This is because of the increase in lender default cost, as  $\zeta(c)$  increases convexly in  $c$ , and also the contagion intensifies through both second-order bankruptcy of counterparty channel and asset price channel.<sup>22</sup> If a borrower is more indebted, the expected sum of lender default costs is higher. Also if a borrower is less diversified, the expected sum of lender default costs is higher because of convexity of  $\zeta$ . The second-order bankruptcy contagion only makes it even worse in expected sense because that only increases the probability of bankruptcy even more. Thus, diversification generates benefits to all of the agents.

Given all of the tools from  $t = 1$  payment equilibrium and  $t = 0$  borrowing and lending behavior, we can prove existence of a network equilibrium as well as the properties of it.

**Theorem 1** (Existence and Characterization of Network Equilibrium).

For a given economy  $(N, (s^j, \theta_j, e_0, h_0)_{j \in N}, \zeta, G)$  and maximum equilibrium selection rule, there **exists** a network equilibrium  $(C, Y, e_1, h_1, p_0, \tilde{p}_1, q)$ , and any network equilibrium is characterized as follows:

1. For any  $y \in [s^{j+1}, s^j]$

$$q(y) = q(s^{j+1}) + \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \zeta'(c_{(j+1)j}) \left[ 1 - \frac{s^{j+1}}{p_1} \right] \mathbb{1}_{\{j+1 \in B(\epsilon)\}} \right]}{E_j \left[ \frac{1}{p_1} \right]},$$

where we set  $q(s^{n+1}) = s^{n+1} = 0$  and  $\max_j E_j [\mathbb{1}\{n+1 \in B(\epsilon)\}] = 0$ .

2. For any  $i, j \in N, i \neq j$ ,  $y_{ij} = s^i$  and  $(C, Y)$  is under intermediation order.

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<sup>22</sup>Ibragimov et al. (2011) suggests a model with diversification of risk classes leading to systemic risk through commonality. This force is restricted by the competition in the asset market and high marginal utility of cash under crisis states in my model.

3. For any counterparties  $i, k$  of  $j$  with  $c_{ij} > 0, c_{kj} > 0$ ,

$$\begin{aligned} & \frac{s^j}{q(s^j) - q(s^i)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \zeta'(c_{ij}) \left[ 1 - \frac{s^i}{p_1} \right]^+ \mathbb{1}_{\{i \in B(\epsilon)\}} \right] \\ &= \frac{s^j}{q(s^j) - q(s^k)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \zeta'(c_{kj}) \left[ 1 - \frac{s^k}{p_1} \right]^+ \mathbb{1}_{\{k \in B(\epsilon)\}} \right]. \end{aligned}$$

4. For any  $j, i \in N$  and  $j \leq i$ ,  $c_{ji} = 0$ .

5. Cash holdings of each agent is determined by

$$e_1^j = e_0^j + h_0^j p_0 + \sum_{i \in N \setminus \{j\}} c_{ij} q(s^i) - \sum_{k \in N \setminus \{j\}} c_{jk} q(s^j) - h_{j,1} p_0.$$

6. The price of the asset at  $t = 0$  is determined by  $p_0 = q(s^1)$ .

7. The price of the asset at  $t = 1$ ,  $\tilde{p}_1$  is determined by payment equilibrium for  $(C, Y)$ .

Also the set of network equilibria forms a complete lattice, and there exists a maximum leverage network equilibrium that is the equilibrium with the collateral matrix which has the highest aggregate debt among all other equilibria.

The theorem contains several implications. First of all, the theorem suggests the network structure change for the given economy, in particular the mechanism of network formation—tradeoff between leverage and counterparty risk. Any equilibrium collateral matrix should be an acyclical network as an agent only borrows from more pessimistic agents, and the network follows intermediation order due to lemma 3. Each agent can be both borrower and lender because of return differences under subject beliefs and intermediation rents.

For negligible default cost (small  $\zeta$  and  $\theta_j$ ), a single-chain network is formed that is agent  $j$  borrows from agent  $j + 1$  only for all  $j < n - 1$ . This is because even if  $c_{j+1,j} = \sum_{k \in N} c_{jk}$ , the return from borrowing from  $j + 1$  is still greater than return from borrowing from  $l > j + 1$  as the counterparty risk increase is small. Figure 7 is an example of such a network. This resulting intermediation chain resembles the intermediation pattern in Glode and Opp (2016), because the agents with the closest beliefs trade with each other which maximizes the gains of trade. Agents are not concerned about diversifying their counterparty and choose the most profitable counterparty—that is, the most optimistic agent after herself—and concentrates all the borrowing.

However, if the default cost  $\zeta$  is non-negligible, then a multi-chain network is formed in equilibrium. Figure 8 is an example of such a network. Agent  $j$  borrows not only from  $j + 1$

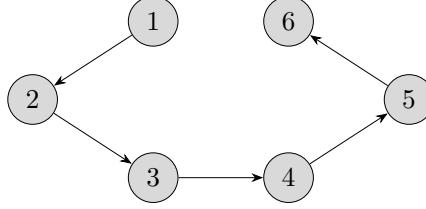


Figure 7: Single-Chain Network

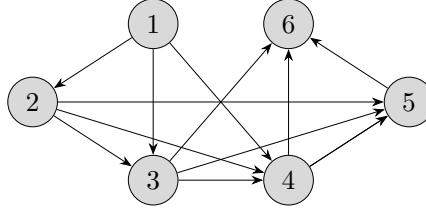


Figure 8: Multi-Chain Network

but also from  $j + 2$  as well. Agents would diversify their counterparties and would like to link with several levels down of optimism. However, this comes at the cost of lower leverage (higher haircut). This network formation mechanism, the tradeoff between leverage and counterparty risk, makes the intermediation pattern distinct from Glode and Opp (2016).

The second implication is the lack of diversification. As shown in proposition 5, diversification of lenders create positive externalities to other agents by making the overall network safer. However, such positive externalities from diversification are not included in individual agent  $j$ 's concern. Therefore, the degree of diversification is always less than the optimal degree in the economy, and the equilibrium is constrained inefficient. Define the social welfare of the economy as the sum of ex ante expected utilities of all the agents as

$$\sum_{j \in N} E_j \left[ m_j(\epsilon) \frac{s^j}{p_1(\epsilon)} \right].$$

An equilibrium is (constrained) inefficient if a social planner can generate higher social welfare by adjusting the allocation while the resource constraints and collateral constraints are satisfied and leaving the  $t = 1$  market interaction decentralized.

**Theorem 2.** *Any network equilibrium under OTC market is constrained inefficient due to under-diversification if  $\zeta$  is non-negligible.*

The third implication of theorem 1 is leverage stacking through the lending chain. Increase in  $q(s^n)$  increases all the subsequent contract prices, which imply that the lending

amount increases. Therefore, lending or leverage at any point in the lending chain has a multiplier effect on the economy. This leverage multiplier effect due to reuse of collateral has been examined in Gottardi et al. (2017) as well. A distinct feature from theorem 1 is that different level in the lending chain has different multiplier effects. An increase in  $s^n$  will have a larger effect than an increase in  $s^2$  as agent  $n$ 's lending stacks  $n - 1$  times through the lending chain through equation (8). A real world implication could be that the increase in the confidence of the ultimate lender (cash providers such as MMMFs) can lead to a huge increase in asset prices through this multiplier effect.

The fourth implication is the dispersion of gains of trade. Unlike the result in one link of borrowing and lending in Simsek (2013) and Geerolf (2018), where the gains of trade are fully concentrated to the borrower (agent 1), the gains of trade are dispersed across all agents through competition across different agents and also varying degree of liquidity shortage. This is due to positive cash holdings for either side, which is coming from a liquidity shock and differential marginal utility of cash. Also in the network or intermediation chain context, this feature implies bargaining power between borrowers and lenders is determined endogenously in contrast with the papers such as Farboodi (2017) and Hugonnier et al. (2018), where they assume exogenous bargaining power.

The fifth implication of theorem 1 is the endogenous market reaction to the change in counterparty risk. As the counterparty risk increases, agents diversify their counterparties more and the overall leverage and debt decrease. The intuition for this result is the following. Agents prefer to hold cash in case of severe liquidity-constrained price and are also willing to lend less for the same promise as a lender. Then, contract price for a borrower would also decrease as the return from the leverage decreases. So the overall debt level decreases not only by decrease in leverage from lender diversification, but also by decrease in asset and contract prices. Similarly, change in the asset payoff belief  $s^j$  would affect both the amount of debt as well as contract prices. The comparative statics results are summarized as the next proposition.

Before stating the proposition, define the velocity<sup>23</sup> of collateral in a network  $C$  as the volume of total collateral posted divided by the stock of source collateral<sup>24</sup>

$$Velocity(C) \equiv \frac{\sum_{i \in N} \sum_{j \neq i} c_{ij}}{\sum_{j \in N} h_{j,1}}.$$

This velocity of collateral represents volume of the reuse of collateral within the network.

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<sup>23</sup>Since this model is not dynamic, the “velocity” means how much a collateral moves around in the market.

<sup>24</sup>This definition is similar to the definition of the velocity of collateral in Singh (2017)—that is, the volume of secured transactions divided by the stock of source collateral.

For example, if the network  $C$  is a single-chain network using all of the source collateral repeatedly, then the velocity of  $C$  is  $n - 1$  because  $c_{21} = c_{32} = \dots = c_{n,n-1} = h_{1,1}$  and  $\text{Velocity}(C) = (c_{21} + c_{32} + \dots + c_{n,n-1})/h_{1,1} = n - 1$ . The velocity of collateral is also an approximate measure of the average length of the lending chain in the network (Singh, 2017).

**Proposition 6** (Comparative Statics on Borrowing Pattern). *For a given network equilibrium with maximum equilibrium selection rule, the following statements are true.*

1. *If  $s^j$  increases (decreases) by the same amount for every  $j \in N$ , then the equilibrium contract prices, leverage for each agent, and the velocity of collateral increases (decreases), while the number of links between agents weakly decreases (increases).*
2. *If  $\theta_j$  increases (decreases) by the same amount for every  $j \in N$ , then the equilibrium contract prices, leverage for each agent, and the velocity of collateral decreases (increases), while the number of links between agents weakly increases (decreases).*

The results above can be summarized in the following theorem.

**Theorem 3** (Network Change under Crisis). *If the economy is under financial distress and the counterparty risks become greater as  $s^j$  decreases or  $\theta_j$  increases, then agents diversify more, the asset price decreases, the average leverage decreases, the velocity of collateral decreases, and the average number of counterparties (weakly) increases.*

The results of theorem 3 are consistent with the empirical facts. As Singh (2017) documented, the velocity (reuse) of collateral decreased from 3 to 2.4 right after the bankruptcy of the Lehman Brothers<sup>25</sup> and the average leverage in the OTC market also went down. Also Craig and Von Peter (2014) shows that the average number of linkages between financial institutions have increased about 30 percent over the four years after the Lehman bankruptcy.<sup>26</sup> After the Lehman's bankruptcy, hedge funds increased the number of prime brokers they work with even further and the prime brokerage market became much more competitive (which translates into lower intermediation rents under theorem 3) after the crisis (Eren, 2015). On the contrary, the opposite result happened in unsecured debt markets. Afonso et al. (2011) and Beltran et al. (2015) find that the banks in the federal funds market reduced their number of counterparties after the Lehman bankruptcy. This stark comparison shows the importance of collateral in network formation.

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<sup>25</sup>The velocity went further down to 1.8 as of 2015. Singh (2017) argues that the collateral landscape has changed further because of central banks' quantitative-easing policies and new regulations which are beyond the scope of this paper.

<sup>26</sup>The dynamics of theorem 3 has occurred even before the Lehman bankruptcy. In the wake of Bear Stearns' demise, hedge funds had increasingly used multiple prime brokers to mitigate counterparty risk. In fact, despite the traditionally concentrated structure of the prime brokerage business, as far back as 2006, about 75 percent of hedge funds with at least \$1 billion in assets under management relied on the services of more than one prime broker (Scott, 2014).

### 3.4. Discussion

The social welfare is comprised of two major parts: the allocative efficiency and financial stability (systemic risk). The allocative efficiency is maximized under a single-chain network because each agent effectively buys (bets) the tranche of the asset that she believes in. However, a single-chain network also minimizes financial stability (maximizes systemic risk) by the concentration of network and maximized leverage. The overall social welfare should depend on the balance between the two (Gofman, 2017). The sources of externalities are fire sales spillover or collateral externalities as in Duarte and Eisenbach (2018) and Dávila and Korinek (2017), and cascades through networks as in Elliott et al. (2014).

The shape of  $\zeta$  is important. We can consider many different cost specifications such as concave or constant costs. These cost functions will fail to replicate the risk-aversion behavior and fail to generate the main mechanism – the tradeoff between leverage and counterparty risk. One possible interesting cost structure can be a function that is concave (or constant) at the beginning and then later becomes convex—that is,  $\zeta''(c) \leq 0$  for  $c \in (0, \bar{c}]$  and  $\zeta''(c) > 0$  for  $c \in (\bar{c}, \infty)$ . This shape will make each and every agent exposed to the agent who is the next most optimistic to her at least of  $\bar{c}$  amount.<sup>27</sup> Even more degree of freedom is possible by allowing heterogenous costs for each and every pair as  $\zeta_{ij}(c)$ . Such heterogenous cost structure would be crucial in estimating the parameters empirically and replicating the core-periphery structure in OTC markets as in Craig and Ma (2019).

The haircut for a hedge fund’s contract is typically greater than the haircut for a dealer’s contract when they borrow from money market mutual funds in the actual markets. Introducing size and cost heterogeneity can attain the haircut differences. If a dealer is much larger than its counterparties (as observed from the data) then the dealer may be able to trade with other agents under much lower haircut. The heterogeneity of size and cost can also help recover the commonly observed network structure – core-periphery networks. More formal analysis on the possible heterogeneity is left for future extensions.

## 4. Central Clearing

As discussed in the introduction, central clearing and the introduction of a central counterparty (CCP) is one of the major issues in market structure regulations. In this section, I define a theoretical way of introducing CCP and perform a counterfactual analysis on the

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<sup>27</sup>This cost structure also makes sense in terms of institutional details since most of the Chapter 11 bankruptcy problems for small financial institutions are straightforward. This cost structure would make even more sense if other agents’ exposure to the same lender also affects the borrower such as  $\zeta(c_{ij}/(c_{i1} + c_{i2} + \dots + c_{in}))$ .

impact of introducing CCP to a decentralized OTC market.

CCP novates one contract between a borrower and a lender into two contracts – a contract between the borrower and the CCP and a contract between the lender and the CCP. This implies the CCP can be considered as a new agent, defined as agent 0, and the CCP simply duplicates the already existing debt network  $C, Y$  into its balance sheet. This can be done by first adding all the columns of  $C$ , and each column sum will be  $c_{0i}$  for all  $i \in N$ . Then, add all the rows of  $C$ , and each row sum will be  $c_{i0}$  for all  $i \in N$ . The contract matrix  $Y$  can also be modified by adding the new row and column for 0 with all the relevant promises of  $s^j$  for each  $j - 1$  row and column. CCP also does pooling, which is buffering the counterparty risk with its own balance sheet. The CCP's cash holdings  $e_1^0$  can be considered as a cash buffer, as CCP guarantee funds that are coming from  $n$  client agents with  $\gamma$  amount of contribution, so  $e_1^0 = n\gamma$ .<sup>28</sup> Define the new debt network with CCP as  $(C_{ccp}, Y_{ccp})$ .

CCP also nets out obligations between two counterparties. We can consider netting of borrower obligations as a transformation of the debt matrix  $C \circ Y$  that is  $\hat{C} \circ \hat{Y}$  s.t.

$$\hat{c}_{ij}\hat{y}_{ij} = [c_{ij}y_{ij} - c_{ji}y_{ji}]^+$$

for all  $i, j \in N$ . This can be considered by a transformation of matrix as  $[C \circ Y - C' \circ Y']^+$ . If this netting procedure is done for the original debt network, then this is a *bilateral netting procedure*. If we run the netting transformation procedure after the inclusion of CCP—that is,  $[C_{ccp} \circ Y_{ccp} - C'_{ccp} \circ Y'_{ccp}]^+$ —then it becomes the *multilateral netting*,  $\hat{C}_{ccp} \circ \hat{Y}_{ccp}$ , which is relatively straightforward operation equivalent to the operation in Duffie and Zhu (2011).

The netting should be considered more carefully when it comes to lender obligations since the lender obligation may not be relevant under certain prices when the borrower defaults on their promises. The netting procedure works as follows.

1. For the given price  $p_1$ , compute the entry-by-entry indicator matrix  $\Gamma \equiv \mathbf{1}(Y = X)$ .
2. Compute the effective collateral matrix  $C' \equiv C \circ \Gamma$ .
3. Perform the CCP netting procedure above to derive  $\hat{C}'_{ccp}$ .
4. Redistribute the relevant collateral obligations from the updated  $\hat{C}'_{ccp}$ .

This redistribution is done by whoever is the final holder of the asset. Under acyclical networks which arise endogenously in theorem 1, there is no indeterminacy of redistribution,

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<sup>28</sup>This structure of participation fee is in fact, how the actual CCP manages its guarantee funds in the CCP's "default waterfall" (Paddrik et al., 2019).

so the new network is well defined. Also any leftover wealth of the CCP is equally distributed to the surviving agents. Thus, the CCP's nominal wealth after payments becomes

$$m_0(\epsilon|p_1) = n\gamma - \sum_{j \in N} \sum_{k \in N} \zeta(c_{jk})[p_1 - y_{jk}]^+ \mathbb{1}\{j \in B(\epsilon)\},$$

and the CCP goes bankrupt when  $m_0(\epsilon|p_1) = 0$ . Note that an economy under CCP has the debt network that is still under intermediation order and there exists an equilibrium.

There are many important properties of a CCP in reality, such as enhanced transparency<sup>29</sup> and collateral management<sup>30</sup>, that are abstracted out from the model. Other than the pooling and netting of the contracts, I assume that the CCP is exactly the same as the other agents in the economy. Also CCPs often do not allow rehypothecation or are sometimes themselves restricted in rehypothecation. But, this restriction comes with a cost of worse flow of collateral and liquidity (Singh, 2017). The main point of this analysis is to focus on the understudied property of endogenous reaction of the market, change in network formation. Any other properties are subject to further studies.

#### 4.1. CCP without Netting

First, consider the effect of novation and pooling only. Since agents are protected from direct counterparty risk when the CCP survives, agent  $j$ 's optimization problem becomes

$$\begin{aligned} & \max_{\substack{e_1^j, \{c_{ij}, y_{ij}\}_{i \in N}, \\ \{c_{jk}, y_{jk}\}_{k \in N}}} E_j \left[ \left( e_1^j - \epsilon_j + h_{j,1}p_1 + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{y_{jk}, p_1\} + \frac{m_0(\epsilon|p_1)}{\sum_{i \in N} \mathbb{1}\{i \notin B(\epsilon)\}} \right. \right. \\ & \quad \left. \left. - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{y_{ij}, p_1\} - \sum_{0 \in B(\epsilon)} \zeta(c_{ij})[p_1 - y_{ij}]^+ \mathbb{1}\{i \in B(\epsilon)\} \right) \frac{s}{p_1} \right]^+ \end{aligned} \quad (9)$$

s.t.

$$\begin{aligned} h_{j,1} + \sum_{k \in N \setminus \{j\}} c_{jk} & \geq \sum_{i \in N \setminus \{j\}} c_{ij}, \\ e_0 + h_0 p_0 & = e_1^j + h_{j,1} p_0 + \gamma - \sum_{i \in N \setminus \{j\}} c_{ij} q(y_{ij}) + \sum_{k \in N \setminus \{j\}} c_{jk} q(y_{jk}). \end{aligned}$$

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<sup>29</sup>The model abstracted away from trading friction and strategic behavior due to information asymmetry. However, opaqueness can provide benefits in allocative efficiency as in Dang et al. (2017).

<sup>30</sup>The CCP pays the same  $\zeta$  cost in the model but the actual deadweight loss may vary by different counterparties. For example, the vast majority of Lehman's clients who went through CCPs obtained access to their accounts within weeks of Lehman's bankruptcy (Fleming and Sarkar, 2014). This implies the  $\zeta$  when the CCP is the borrower could be lower than  $\zeta$  of the other agents. But, the cost of retrieving collateral after the CCP went bankrupt could be much higher than the lender default costs from the OTC markets.

From proposition 5 and theorems 1 and 2, the following proposition holds.

**Proposition 7.** *For a given network equilibrium with maximum equilibrium selection rule under OTC market with collateral matrix  $C$ , suppose that a CCP without netting is introduced to the market.*

1. *If the CCP never goes bankrupt either because of (implicit) guarantee by the government or the large enough agents' contribution  $\gamma$ , then the new network with collateral matrix  $C_{ccp}$  has the highest systemic risk across all collateral matrices that satisfy intermediation order.*
2. *If agents' contribution  $\gamma$  is not large enough and the CCP can go bankrupt in some states, then the new network with collateral matrix  $C_{ccp}$  has higher systemic risk than the original network with collateral matrix  $C$ .*

The CCP's pooling eliminates direct counterparty risk concern from agents and eliminates the tradeoff between counterparty risk and leverage. Thus, agents connect for the most favorable contracts with the most concentrated counterparties. The CCP's pooling rather exacerbates the problem of positive externalities from diversification. The systemic risk rather increases when the economy-wide insurance, pooling, is introduced.

If  $\gamma$  is too high, then some agents may not even participate in the market (if they had the choice) since their return from borrowing or lending in the market does not justify paying the participation fee  $\gamma$ . This non-participation further decreases allocation efficiency, while the individual incentives of the participants are still the same, since marginal incentives are the same. Even though the lending chain leverage may decrease, the network they have is going to maximize the systemic risk for the given component of the network.

The graphical dynamics of the above result is described in figure 9. The top graph is the decentralized OTC network where each agent diversifies their counterparties. The bottom graph is the new network after introducing a CCP in the middle. The notional link in the new network looks like the black links, which are only the contracts between the CCP and the other agents. However, the actual contract flows are the single-chain network in red links, which is different from the OTC network in the top graph. If the endogenous change in the network, from a multi-chain network to a single-chain network, is not taken into account, then the impact of introducing a CCP on systemic risk could be under-evaluated.

## 4.2. CCP with Netting

A CCP also provides benefits in reducing systemic risk through netting. Bilateral netting does not reduce systemic risk at all, because there is no cycle in an endogenously formed

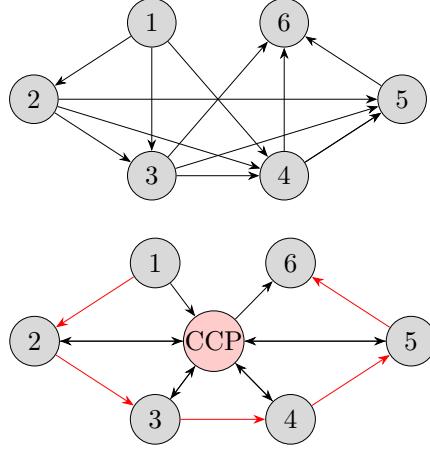


Figure 9: OTC Network and CCP Network

network. However, multi-lateral netting does reduce counterparty exposure.

**Proposition 8.** *Bilateral netting does not affect systemic risk. Multi-lateral netting always decreases systemic risk.*

Multi-lateral netting can reduce risk even if there is no cycle. For example, if agent 1 is borrowing from 2 who is borrowing from 3 and agent 2 goes bankrupt, then agent 1 suffers from default cost. However, if CCP nets out the contracts, then agent 1 can pay 3 to retrieve her collateral and not suffer from default cost because of going through the additional chain of agent 2. Hence, the introduction of a CCP has the cost of systemic risk caused by the change in network structure (higher leverage and concentration) because of pooling and the benefit of reducing net counterparty exposure by multilateral netting.

Exogenous leverage models completely miss all these cost and benefit features. If there is an exogenously given leverage that is fixed as  $y$  and its market clearing price is fixed as  $q(y)$ , then agents will be divided into two groups, buyers (borrowers) and sellers (lenders) of the asset. Then, there is no tradeoff between leverage and counterparty risk since there is only one contract. Agents will fully diversify their counterparties. Thus, a complete bi-partite network as in figure 10 is the equilibrium network under exogenous leverage. Since agents are already diversifying fully, pooling has zero effect on network formation. On the other hand, since all the paths in the network have length of 1 and there is no cycle, netting has zero effect as well.

**Proposition 9** (Irrelevance of CCP). *If there is only one contract  $y$  that is available in the market, then the decentralized OTC equilibrium network is a complete bi-partite network. Furthermore, introduction of a CCP (with or without netting) to such market has no impact on leverage and endogenous network formation.*

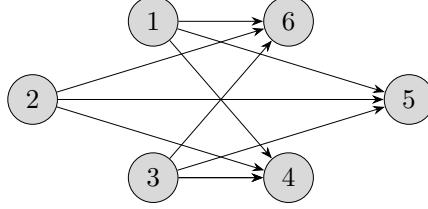


Figure 10: Single Leverage Complete Bi-partite Network

### 4.3. Numerical Examples

In this subsection, I perform a quantitative analysis of the model to provide for numerical examples. There are four agents, each with endowments of 5000 cash and 25 assets, where  $\zeta(c) = c^3$ , and  $S = \{10, 9, 8, 7\}$ . The common shock distribution is a log-normal distribution with a mean of 5 and a standard deviation of 5. For 500 samples of this distribution and the given seed of random number generation, the average shock size is 2406957 and the median shock size is 347.1644. The equilibrium selection rule is the maximum equilibrium selection rule. The algorithm is the following:

#### Quantitative Algorithm.

1. Guess the initial equilibrium collateral matrix  $C_0$ .
2. Compute the payment equilibrium prices  $\tilde{p}_1$  and bankruptcy sets  $B(\epsilon)$  for each simulated state  $\epsilon$  out of  $k$  different states and for each subject beliefs  $s^j$  of agents.
3. Compute each agent's expected returns on each investment decision in  $t = 0$ .
4. Compute the market prices of the asset  $p_0$  and contracts  $q(y)$ .
5. Derive agent's optimal portfolio decisions and set the new collateral matrix as  $C_1$ .
6. Compare  $C_0$  and  $C_1$ . Repeat steps 2-6 until it converges.

First, suppose that the CCP never defaults. Under this case, we compare three different cases of the market structure: decentralized OTC market, CCP without netting, and CCP with netting. For each market structure, we change the values of  $\theta$ , which is the common arrival rate of liquidity shock, and compare the three cases for each  $\theta$  value. In the graphs in figure 11 and 12, the blue solid lines represent the numbers from a decentralized OTC market, the red dashed lines represent the numbers from a market under a CCP without netting, and the black dotted lines represent the numbers from a market under a CCP with netting.

As in the top-left graph in figure 11, the leverage of the three cases starts with 10. In the OTC market, leverage drops around 2 and stays low as the increase in counterparty risk concern reduces the leverage. On the other hand, two cases with CCP have almost the maximum leverage because agents are not concerned with lender default costs, which is fully covered by the CCP. The top-right graph in figure 11 shows the sum of ex ante social welfare for each case. All of the cases have lower social welfare as the arrival rate of shock increases. However, the OTC market has the highest social welfare compared with the two CCP cases. This is due to agents' diversification in the OTC market, which is absent from the CCP markets. Also netting has an important impact as it limits the duplication of lender default costs from bankruptcies which makes a noticeable differences between the two CCP cases. However, the probability of bankruptcy is still the highest in the OTC market as can be seen in bottom-left of figure 11. The reason is that there exists a contagion channel in the OTC market which is nonexistent in CCP cases because the counterparty channel is insulated by the CCP. As predicted by the theory, the velocity of collateral in the network for the OTC market goes down as  $\theta$  increases, while the velocity remains the same for two CCP cases.

Now, suppose that the CCP does not have the government guarantee and only covers its losses by the member contribution for the default guarantee fund  $\gamma$ . The size of  $\gamma$  is set as 1000. Under this case, the CCP can actually go bankrupt if the sum of the lender default costs is too large. The leverage graph in the top left of figure 12 shows an interesting shape. In the market with CCP without netting, the leverage rather increases almost to 30 and then start to revert back to 10, which is still much larger than the OTC market case. These dynamics come from the interaction between the counterparty channel and the price channel through the leverage. As  $\theta = 0.2$  is still a small number, agents are willing to borrow and lend still very aggressively, however, when the CCP goes bankrupt with the low probability then it will make a huge crash in this case. Agents are gambling for the CCP to survive which is very costly for the agents. Also, since the CCP failure implies total market failure, agents are much less concerned about the event of market failure, because that implies the agents themselves are also out of the market as well. In the meantime, they can have large return from cash holdings if they survive. All of these features contributes to the enormous leverage. This colossal leverage also results in lower social welfare as can be seen in the top right of figure 12. The leverage for the case of CCP with netting is much lower than the case without netting. The first reason is, of course, the reduction of counterparty exposure due to netting and much lower likelihood of market breakdown. The agents do not expect the total market break down, but they do care about having more cash in case of a CCP failure, but they still survive. Another reason for the moderate leverage is the diversification behavior of agent 1. As the netting cancels out all the exposures between the intermediaries,

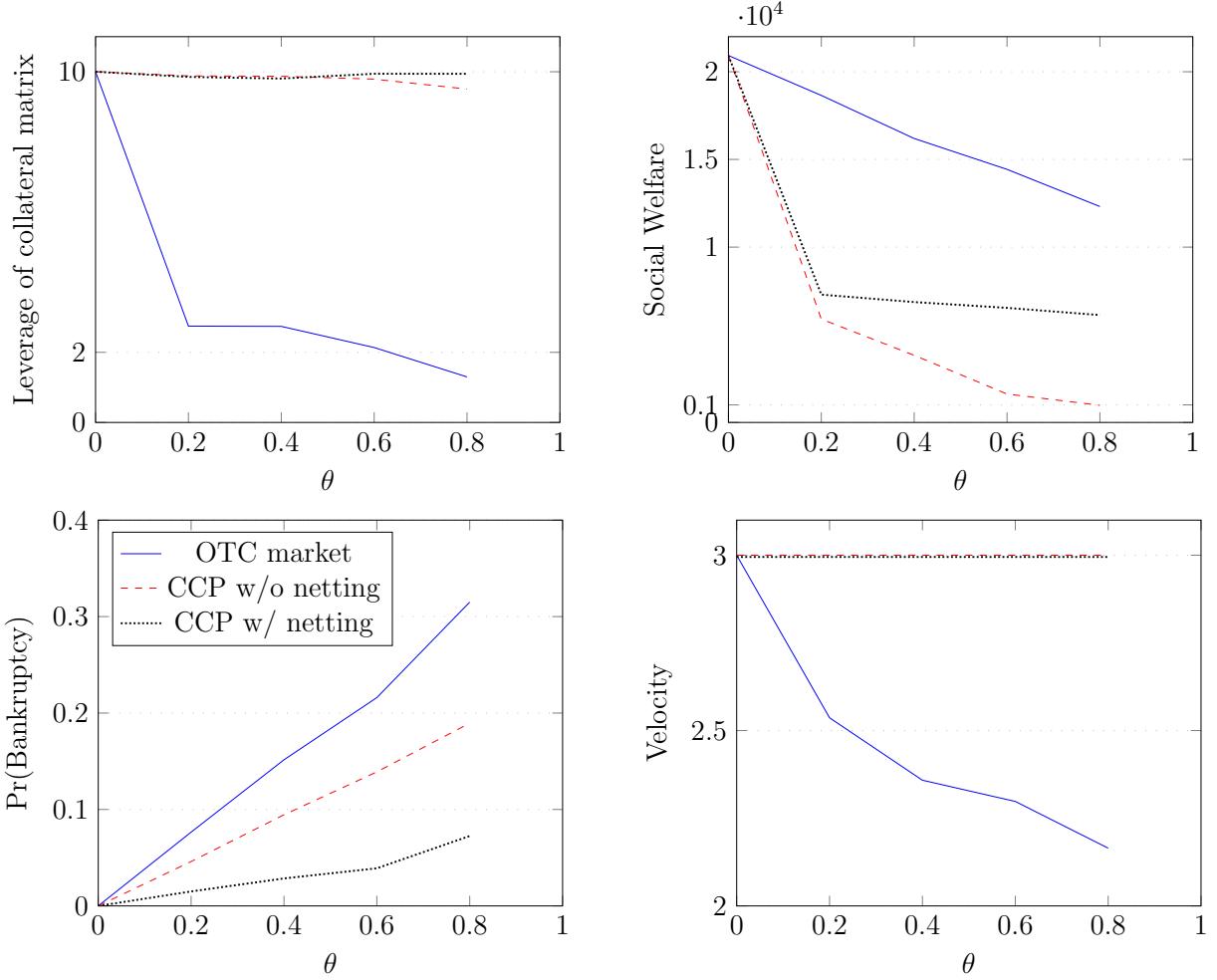


Figure 11: Numerical Results when CCP Never Defaults

agent 1 is still exposed to agent  $n$ 's counterparty risk even after the netting. Therefore, agent 1 wants to diversify and reduces leverage. Since agents are internalizing some of the lender default costs and the netting reduces the total expected lender default costs for a given network, the social welfare under CCP with netting is greater than the social welfare under the OTC market. The bottom left of figure 12 also shows the similar pattern for bankruptcy probabilities. Because agents are recklessly borrowing and lending under CCP without netting, the probability of bankruptcy is very high. The OTC market case is much lower due to diversification but still the CCP with netting has the lowest bankruptcy rate. The velocity of collateral also follows a similar pattern.

I also test the effect of a CCP when the network is exogenously fixed as the decentralized OTC market equilibrium. Suppose that even after the introduction of a CCP, agents still maintain the same links as before. Figure 13 plots social welfare of the three cases –

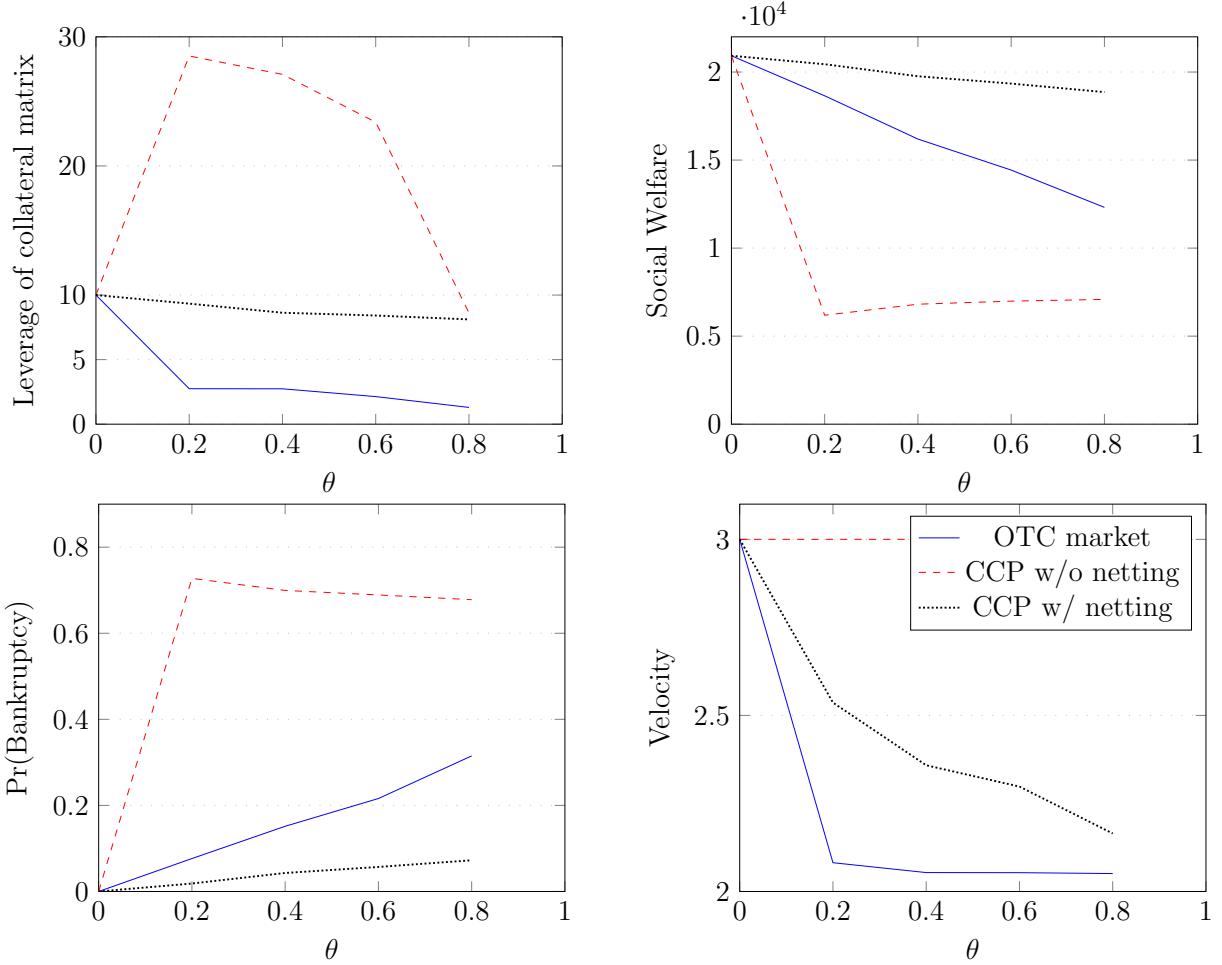


Figure 12: Numerical results when CCP can default

OTC market, CCP without netting, and CCP with netting. Numerical results imply that CCP always increases social welfare if the network remains the same. Since netting reduces counterparty exposure, social welfare under CCP with netting is the highest as seen from the previous results. Figure 13 shows that the reversal of social welfare between the OTC market and the market under a CCP without netting in figure 11 and 12 comes from the endogenous change in network formation.

#### 4.4. Policy Implications

The results in the previous subsection do not necessarily imply normative implication such as “introduction of a CCP is always bad.” As we can see clearly from figure 12, the social welfare under CCP can be higher than the social welfare under the OTC market depending on the parameter values. The correct way to interpret the results is that there can be an

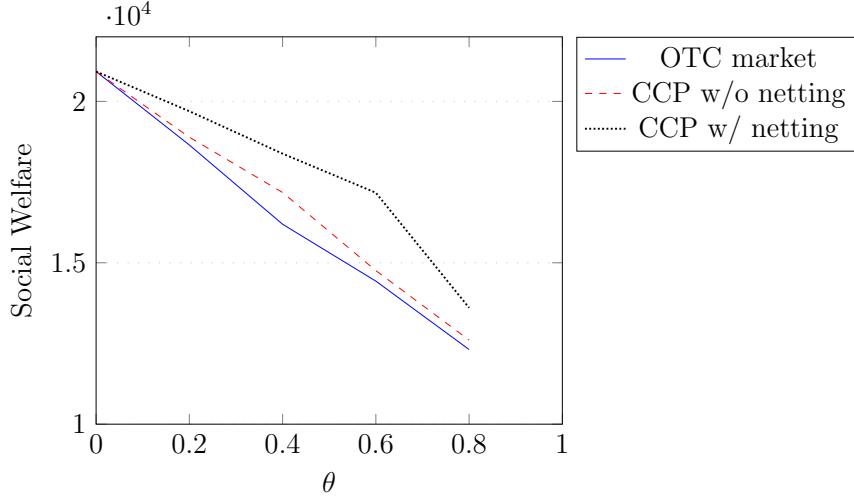


Figure 13: Social Welfare with Exogenous Network

understudied or rather neglected cost (side-effect) of introducing a mandatory CCP. This new cost channel, which is a classic moral hazard problem under insurance, is amplified by the network contagion channels (price and counterparty channels), and the increased correlation of payoffs<sup>31</sup> creates a rather exacerbated externality problem. Therefore, introducing a CCP should be done after the cost and benefit analysis from pooling and netting. For example, the CDS market was already highly centralized, and the cost of CCP for such market could be less than the cost of CCP for well diversified markets.

Another more direct regulation to solve for the diversification externality problem could be introducing a relevant leverage ratio restriction. In Basel III, there is Supplementary Leverage Ratio (SLR), which is effectively a tax on intermediation activity that is proportional to the size of an intermediary's balance sheet, defined as follows.

$$\frac{\text{Tier 1 Capital}}{\text{Total Leverage Exposure}} \geq 3\%$$

A slight modification of this ratio, Network Supplementary Leverage Ratio, can be used as

$$\frac{\text{Tier 1 Capital}}{(c_{11}^2 + c_{21}^2 + \dots + c_{n1}^2) \times \text{Total Leverage Exposure}},$$

and risk externality is included as weights of counterparty exposure in the denominator. Such restrictions provide marginal incentives to diversify and internalize second-order default and

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<sup>31</sup>Note that the correlation problem was mitigated by liquidity holding incentives of each agent in the OTC market. If there is additional frictional period of liquidity resolution as in Gale and Yorulmazer (2013), then there could be even more problem.

maintain borrower or lender discipline of agents and more effective than a crude measure of single counterparty credit limit.

A supplementary policy is liquidity injection to the agent under distress according to its impact to the system as in Demange (2016). This injection or bail-out idea also faces side-effects from moral hazard in terms of network formation (Erol, 2018; Leitner, 2005). Markets under CCP will have even less ambiguity and uncertainty of such bail-out possibility and the resulting degree of concentration can be even greater as in the difference between the figures 11 and 12.

## 5. Conclusion

I constructed a general equilibrium model with collateral featuring endogenous leverage, endogenous price, and endogenous network formation. The model bridges the theory of financial networks and the theory of general equilibrium with collateral. Collateral generates an additional channel of contagion through asset price risk, the price channel, on top of the balance sheet risk through the debt network, the counterparty channel. Borrowers diversify their portfolios of lenders because of the possibility of lender defaults. However, lower counterparty risk comes at the cost of lower leverage. There are positive externalities from diversification because it reduces not only the individual counterparty risk, but also the systemic risk, by limiting the propagation of shocks and resulting price volatility. Because agents do not internalize these externalities, any decentralized equilibrium is inefficient. The key externalities here, arising from the tradeoff between counterparty risk and leverage, are absent in models with exogenous leverage or exogenous networks. The model also predicts the empirically observed changes in network structure, leverage (haircuts), asset price, and velocity of collateral during the financial crisis. Greater counterparty risk induces agents to diversify more, which lowers leverage and the velocity of collateral and increases the number of links. I performed a counterfactual analysis on the introduction of a CCP with this model. The loss coverage by CCP exacerbates the externality problems by eliminating individual agents' incentives to diversify. Thus, the endogenous network change after the introduction of a CCP creates additional systemic risk that exogenous leverage or exogenous network models do not capture.

## A. Appendix: Omitted Proofs

### A.1. Preliminaries

The following lemma is useful for the proofs of the next two results.

**Lemma 5.** *For a given financial network that satisfies collateral constraints, the effective demand  $[m_j(p)]^+$  is increasing in  $p$  for any  $j \in N$ .*

**Proof of Lemma 5.** It is enough to show that  $m_j(p)$ , which is

$$e_1^j - \epsilon_j + h_{j,1} + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{p, y_{jk}\} - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+,$$

is increasing in  $p$ . Since  $\min\{y_{ij}, p\} \leq p$ , both  $\min\{p, y_{ij}\}$  and  $\min\{y_{jk}, p\}$  are increasing in  $p$ . For any value of promise  $\hat{y}$ ,

$$\sum_{\substack{i \in N \setminus \{j\} \\ y_{ij} \geq \hat{y}}} c_{ij} \min\{y_{ij}, p\} \leq \sum_{\substack{k \in N \setminus \{j\} \\ y_{jk} \geq \hat{y}}} c_{jk} \min\{y_{jk}, p\} + h_{j,1}$$

by intermediation order. Therefore, the sum of the payments from other agents will always exceed the sum of payments that  $j$  has to pay to others.<sup>32</sup> Also, by  $\zeta(c) \leq c$ , the total sum of coefficients for  $p$  will always be nonnegative. For fixed  $B(\epsilon)$ , each  $m_j(p)$  is increasing in  $p$ . Therefore, for any  $p' < p$ ,  $B(\epsilon) \subseteq B(\epsilon')$  and the indicator function for the bankruptcy cost is decreasing in  $p$ . ■

### A.2. Properties of Payment Equilibria

**Proof of Proposition 1.** If  $p = s$ , then we automatically have an equilibrium that satisfies inequality (3) or otherwise  $p$  cannot be  $s$ . Now suppose  $p < s$ . The equilibrium equation can be represented as

$$(M, p) = \left( [m_j(p)]_{j \in N}, \frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} h_{j,1}} \right) \equiv \mathcal{M}[(M, p)].$$

Consider an ordering  $\succeq$  such that  $(M, p) \succeq (M', p')$  when  $M \geq M'$  and  $p \geq p'$ . Then an infimum under  $\succeq$  can always be defined for any subset of  $\mathbb{R}^{n+1}$ . By the assumption

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<sup>32</sup>This is, in fact, the reason why there is a collateral constraints. It guarantees the agent to have non-negative amount of cash from all the payments netted out so that they can actually pay the debt.

$(M(s), s) \geq \mathcal{M}[(M(s), s)]$ . Since the denominator of the price equation is constant and  $h_i^2(p)$  and  $[m_i(p)]^+$  are increasing in  $p$  by lemma 5, the function  $\mathcal{M}$  is an order-preserving function. Then, by Knaster-Tarski's fixed point theorem, there exists a fixed point  $(M, p)$ , and the set of  $(M, p)$  that satisfies the equilibrium condition has a maximal point.

Now suppose that the maximal fixed point price  $\bar{p}$  is greater than  $s$ , and we will show that either there exists a price  $0 < p \leq s$  that is also a fixed point or  $p = s$  satisfies equilibrium condition (3). If equation (2) is true when  $p = 0$ , then we already have a fixed point with  $p \leq s$ . If equation (2) is not true when  $p = 0$ , then that implies at least some  $m_j(0)$  is positive for  $j \in N$  after subtracting the counterparty bankruptcy costs. Therefore,  $\frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} h_{j,1}} \geq 0$ . This implies that as  $p$  increases, the difference between the  $p$  and  $\frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} h_{j,1}}$  will be eventually closed out at  $\bar{p}$  by intermediate value theorem. Therefore, the two functions either meet for some  $p \leq s$ , or the gap between them does not close out even when  $p = s$  so equation (3) holds. ■

**Proof of Proposition 2.** For the proof, suppress the  $\epsilon$  term in bankruptcy sets. If no agent is going to bankrupt at any price  $p \in [0, s]$ , then the equilibrium price is trivially and uniquely determined as  $p = s$ . Now suppose some agents go bankrupt at a liquidity constrained price  $\tilde{p}$ —that is,  $B(\tilde{p}) \neq \emptyset$ . Denote  $\mathcal{V}_l$  as the set of agents such that there is a link between  $l$  and  $i$  for any  $i \in \mathcal{V}_l$ . Suppose that  $l \notin B(\tilde{p})$  and  $\mathcal{V}_l \cap B(\tilde{p}) \neq \emptyset$ . Thus, at least at some price close to (or equal to)  $\tilde{p}$ , the agent  $l$  will bear some bankruptcy cost and may go bankrupt. If there is no agent  $l$  that satisfies  $z^l \equiv e_1^l - \epsilon_l < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \zeta(c_{il})[\tilde{p} - y_{il}]^+$  from  $\tilde{p} = 0$  all the way up to  $s$ , then  $B(p) = B(p')$  for any  $p, p' \in [0, s]$  and in fact there is unique equilibrium since there will be no jumps in  $\sum_i [m_i(p)]^+$ .

Now suppose that for some price  $\tilde{p}$  and some agent  $l$ ,  $z^l < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \zeta(c_{il})[\tilde{p} - y_{il}]^+$  is satisfied. Then, there exists  $p^*$  less than  $p$  (due to monotonicity of  $m_l(p)$ ) such that  $\forall p' < p^*$ ,  $m_l(p') < 0$  and suppose  $l$  be the only one who goes bankrupt due to the price decline from  $p$  to  $p' < p^*$  without loss of generality. The left-hand side of the market clearing condition, the sum of effective money, can be decomposed as

$$\begin{aligned} \sum_{j \in N} [m_j(p)]^+ &= \sum_{j \in N} e_1^j + \sum_{j \in N} h_{j,1}p - \sum_{j \in N} \sum_{i \in B(p)} \zeta(c_{ij})[p - y_{ij}]^+ \\ &\quad - \sum_{j \in N} \min \left\{ \epsilon_j, e_1^j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(p)} \zeta(c_{ij})[p - y_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\} \right\}. \end{aligned}$$

Since the term is the same as the supply side of the equation, price is determined by the

remaining cash from  $t = 0$  and the amount of aggregate liquidity shock to the demand, bounded by its entire position, and the counterparty default costs. We can rewrite the market clearing condition into loss-coverage with remaining cash equality as

$$\begin{aligned} \sum_{j \in N} e_1^j &= \sum_{i \in B(p)} \sum_{j \in N} \zeta(c_{ij}) [p - y_{ij}]^+ + \sum_{j \in N} \min \left\{ \epsilon_j, e_1^j + h_{j,1} p \right. \\ &\quad \left. - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(p)} \zeta(c_{ij}) [p - y_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\} \right\} \end{aligned} \quad (10)$$

Then, there can be a price  $\hat{p}$  such that the additional jump in bankruptcy cost  $\beta_l(p) \equiv \sum_{j \in N} \zeta(c_{lj}) [p - y_{lj}]^+$  coincides with the amount of decrease in losses from bankrupt agent's endowments and counterparty costs—that is,

$$\begin{aligned} \beta_l(\hat{p}) &= \epsilon_l + \sum_{j \in B(p)} \left[ \sum_{i \neq j} (c_{ij} - \mathbb{1}\{i \in B(p)\} \zeta(c_{ij})) (\mathbb{1}\{p > \hat{p} \geq y_{ij}\} (p - \hat{p}) \right. \\ &\quad \left. + \mathbb{1}\{p \geq y_{ij} > \hat{p}\} (p - y_{ij})) + \zeta(c_{lj}) [\hat{p} - y_{lj}]^+ \right. \\ &\quad \left. + \sum_{k \in N} c_{jk} (\mathbb{1}\{y_{jk} > p > \hat{p}\} (p - \hat{p}) + \mathbb{1}\{p \geq y_{jk} > \hat{p}\} (y_{jk} - \hat{p})) \right] \\ &\quad - \left[ e_1^l - \sum_{i \neq l} c_{il} \min\{\hat{p}, y_{il}\} - \sum_{i \in B(p)} \zeta(c_{il}) [\hat{p} - y_{il}]^+ + \sum_{k \in N} c_{lk} \min\{\hat{p}, y_{lk}\} \right]. \end{aligned} \quad (11)$$

Therefore,  $\hat{p}$  is also an equilibrium price. ■

**Proof of Proposition 3.** For a fair price, there exists a unique equilibrium price no matter what happens in shocks and bankruptcies. Now focus on liquidity constrained prices. When  $\zeta(c) = 0$  for any  $c \geq 0$ , equation (10), the market clearing condition with loss-coverage, becomes

$$\sum_{j \in N} e_1^j = \sum_{j \in N} \min \left\{ \epsilon_j, e_1^j + h_{j,1} p - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{p, y_{jk}\} \right\},$$

and by intermediation order, the right-hand side is increasing in  $p$ . Also the right-hand side is bounded below by  $\sum_{j \in N} \min\{\epsilon_j, e_1^j\}$ , when  $p = 0$ . By intermediate value theorem, there exists a unique equilibrium price  $p$  between  $[0, s]$  that satisfies the market clearing condition above.

For the second statement of the proposition, first note that the nominal wealth with no lender default cost is

$$m_j(p) = e_1^j + h_{j,1}p - \epsilon_j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{p, y_{jk}\}.$$

The sum of nonnegative nominal wealth is

$$\begin{aligned} \sum_{j \in N} [m_j(p)]^+ &= \sum_{j \in N} e_1^j + \sum_{j \in N} h_{j,1}p \\ &\quad - \sum_{j \in N} \min \left\{ \epsilon_j, e_1^j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\} \right\}, \end{aligned}$$

which can be re-written as the sum of indegrees and outdegrees as below.

$$\sum_{j \in N} [m_j(p)]^+ = \sum_{j \in N} e_1^j + nh_0p - \sum_{j \in N} \min \left\{ \epsilon_j, e_1^j - \sum_{i \in N \setminus \{j\}} c_{ij}x_{ij} + \sum_{k \in N} c_{jk}x_{jk} \right\},$$

which will have the same value with a network with

$$\begin{aligned} \sum_{i \in N \setminus \{j\}} c_{ij}x_{ij} &= \sum_{i \in N \setminus \{j\}} \hat{c}_{ij}\hat{x}_{ij} \\ \sum_{k \in N} c_{jk}x_{jk} &= \sum_{k \in N} \hat{c}_{jk}\hat{x}_{jk}, \end{aligned}$$

so networks  $(C, X)$  and  $(\hat{C}, \hat{X})$  have the same equilibrium price and final asset holdings. ■

### A.3. Properties of Network Equilibrium

**Proof of Lemma 1.** For each agent  $i \in N$ , the maximum cash he can hold for  $t = 1$  is by saving all the cash while not lending any cash because borrowing requires collateral and no arbitrage condition will prevent anyone from making positive cash from borrowing. The price of the asset at  $t = 0$  cannot exceed the most optimistic agent's fair value since there is always a possibility of liquidity constrained underpricing in  $t = 1$ . Thus,  $e_0 + h_0s^1$  is always the upper bound of the maximum amount of cash each agent can hold by selling all the asset endowments and not borrowing from or lending to anyone. Since  $G$  is differentiable with full support of  $[0, \bar{\epsilon}]$ , any agent can go bankrupt regardless of how much cash they hold in  $t = 0$  because  $G([e_0 + h_0s^1, \bar{\epsilon}])$  is positive. Now suppose that agent  $j$  has zero cash holdings—that

is,  $e_1^j = 0$ . Agent  $j$ 's nominal wealth depends on asset price  $p_1$ , which becomes zero if  $p_1 = 0$ . By equation (10), this implies that if every other agent goes bankrupt because of liquidity shocks, which happens with probability greater than  $[G([e_0 + h_0 s^1, \bar{\epsilon}])]^{n-1}$ , while agent  $j$  is not, which happens with positive conditional probability, the price of the asset becomes zero while agent  $j$  is not bankrupt. Marginal utility of cash in such a state becomes  $\lim_{p_1 \rightarrow 0} \frac{s^j}{p_1}$  which is infinity. Hence, expected marginal utility of holding cash in  $t = 0$  becomes infinity as well and agent  $j$  would like to hold a positive amount of cash for any  $j \in N$ . If  $e_1^j > 0$ , then the only state with infinite marginal utility of cash is when  $\epsilon_j = e_1^j$  which happens with zero probability by differentiability of  $G$ . Thus, in an equilibrium,  $e_1^j > 0$  for any  $j \in N$ . ■

**Proof of Lemma 2.** The proof is done by contradiction. Suppose that  $E_j \left[ \frac{s^j}{p_1} \right] \leq E_k \left[ \frac{s^k}{p_1} \right]$  for  $j < k$ . If both  $j$  and  $k$  are simply holding cash exclusively, then they have the same cash holdings and it is trivially  $E_j \left[ \frac{s^j}{p_1} \right] > E_k \left[ \frac{s^k}{p_1} \right]$ . Therefore, at least agent  $k$  should be investing in something other than cash. Suppose that agent  $k$  is borrowing from  $i$  and lending to  $l$ . Then her return from this intermediation is

$$= \frac{E_k \left[ \min \left\{ s^k, y' \frac{s^k}{p_1} \right\} - \min \left\{ s^k, y \frac{s^k}{p_1} \right\} - \zeta'(c_{ik}) \left[ s^k - y \frac{s^k}{p_1} \right] \mathbf{1}_{\{i \in B(\epsilon)\}} \right]}{q_k(y') - q_i(y)} \\ = \frac{s^k E_k \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ik}) \left[ 1 - \frac{y}{p_1} \right] \mathbf{1}_{\{i \in B(\epsilon)\}} \right]}{q_k(y') - q_i(y)} = E_k \left[ \frac{s^k}{p_1} \right].$$

The last equality holds because the return should be equal to the return from holding cash because of positive cash holding by lemma 1. Now consider an agent  $j$  who deviates from her equilibrium portfolio decision. Agent  $j$  can mimic the investment portfolio of agent  $k$  and obtain the return of

$$= \frac{s^j E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ik}) \left[ 1 - \frac{y}{p_1} \right] \mathbf{1}_{\{i \in B(\epsilon)\}} \right]}{q_k(y') - q_i(y)} \leq E_j \left[ \frac{s^j}{p_1} \right],$$

with the last inequality coming from optimality of agent  $j$ 's original portfolio decision. In other words, she would have already done the intermediation more if it exceeded the return from her cash holdings (which is again positive by lemma 1). If agent  $j$  is mimicking  $k$ 's portfolio exactly the same, the two agents will have the same cash holdings and also the

same counterparty risks (or even less if  $j$  was the lender). Then, inequalities

$$\begin{aligned} & E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ik}) \left[ 1 - \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right] \\ & \geq E_k \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ik}) \left[ 1 - \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right], \end{aligned}$$

and  $s^j > s^k$  imply

$$\begin{aligned} E_j \left[ \frac{s^j}{p_1} \right] & \geq \frac{s^j E_j \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ik}) \left[ 1 - \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right]}{q_k(y') - q_i(y)} \\ & > \frac{s^k E_k \left[ \min \left\{ 1, \frac{y'}{p_1} \right\} - \min \left\{ 1, \frac{y}{p_1} \right\} - \zeta'(c_{ik}) \left[ 1 - \frac{y}{p_1} \right] \mathbb{1} \{ i \in B(\epsilon) \} \right]}{q_k(y') - q_i(y)} = E_k \left[ \frac{s^k}{p_1} \right], \end{aligned}$$

that is,  $E_j \left[ \frac{s^j}{p_1} \right] > E_k \left[ \frac{s^k}{p_1} \right]$ , which contradicts the initial assumption  $E_j \left[ \frac{s^j}{p_1} \right] \leq E_k \left[ \frac{s^k}{p_1} \right]$ . The same method could be applied to any other possible investment strategy of agent  $k$  – lending without leverage or buying the asset with or without leverage. Therefore,  $E_j \left[ \frac{s^j}{p_1} \right] > E_k \left[ \frac{s^k}{p_1} \right]$  holds for any equilibrium. ■

The following lemma shows that whenever leveraging is profitable for certain investment, the same leverage makes other investment more profitable than not leveraging.

**Lemma 6.** Suppose  $\frac{a-p}{b-q} = \pi = \frac{c-p}{d-q} = \frac{e}{f}$  and  $\frac{a}{b} < \frac{a-p}{b-q}$  for  $a, b, c, d, e, f, p, q, \pi > 0$ . Then,  $\frac{c}{d} \leq \frac{c-p}{d-q}$  and  $\frac{e}{f} < \frac{e-p}{f-q}$ .

**Proof of Lemma 6.** Since  $\frac{a-p}{b-q} = \pi$ ,  $a-p = b\pi - q\pi$ . By  $\frac{a}{b} < \frac{a-p}{b-q}$ , we obtain  $a < b\pi$ . By combining the previous equation and inequality, we have  $p < q\pi$ . Now suppose that  $\frac{c}{d} > \frac{c-p}{d-q}$ . Then,  $\frac{c-p}{d-q} = \pi$  implies  $c > d\pi$ . Combining this with  $p < q\pi$ , we get  $\frac{c-p}{d-q} > \pi$ , which is a contradiction. Therefore,  $\frac{c}{d} \leq \frac{c-p}{d-q}$ . Similarly, suppose  $\frac{e}{f} \geq \frac{e-p}{f-q}$ . Then, from  $e = f\pi$ , we obtain  $e-p \leq f\pi - q\pi$ , which implies  $q\pi \leq p$ , which is again a contradiction. Thus,  $\frac{e}{f} < \frac{e-p}{f-q}$ . ■

### Proof of Lemma 3.

From the return equation (7), we immediately get  $y' > y$ , and  $q_j(y') > q_i(y)$  should hold for agent  $j$ 's decision optimality and no arbitrage.<sup>33</sup> Similarly, from the positive cash holding

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<sup>33</sup>No arbitrage prevents the case of  $y' < y$  and  $q_j(y') < q_i(y)$ .

and optimality we know that

$$q'_i(y) = \frac{E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y)}{E_i \left[ \frac{1}{p_1} \right]},$$

which is zero for any  $y > s^i$ . The (semi) partial derivative for agent  $j$ 's decision on the contract promise choice  $y$  to agent  $i$  is

$$\begin{aligned} & s^j E_j \left[ -\frac{c_{ij}}{p_1} + \zeta(c_{ij}) \frac{1}{p_1} \mathbf{1}\{i \in B(\epsilon)\} \middle| p_1 > y \right] \Pr_j(p_1 > y) + \lambda c_{ij} q'_i(y) \\ &= s^j E_j \left[ -\frac{c_{ij}}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y) + s^j E_j \left[ \zeta(c_{ij}) \frac{1}{p_1} \mathbf{1}\{i \in B(\epsilon)\} \middle| p_1 > y \right] \Pr_j(p_1 > y) \\ &+ s^j E_j \left[ \frac{1}{p_1} \right] c_{ij} \frac{E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y)}{E_i \left[ \frac{1}{p_1} \right]}, \end{aligned}$$

where  $\lambda$  is the Lagrangian multiplier for the budget constraint, and  $\lambda = s^j E_j [1/p_1]$  from lemma 1 and the envelope theorem. First, if  $y > s^i$ , then the last term is zero. Since  $c_{ij} > \zeta(c_{ij})$ , the first-order derivative (right-semi differential if  $y = s^i$ ) is negative for any  $y > s^i$ . Now consider  $y \leq s^i$ . Even if the counterparty risk is zero, we can show that the above first-order derivative is positive by showing the following inequality for any  $y \leq s^i$ ,

$$\frac{E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y)}{E_j \left[ \frac{1}{p_1} \right]} < \frac{E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y)}{E_i \left[ \frac{1}{p_1} \right]}. \quad (12)$$

Suppose that the above inequality does not hold—that is,

$$\frac{E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y)}{E_j \left[ \frac{1}{p_1} \right]} \geq \frac{E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y)}{E_i \left[ \frac{1}{p_1} \right]}. \quad (13)$$

From lemma 2,

$$\begin{aligned} E_j \left[ \frac{s^j}{p_1} \right] &= s^j \left( E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y) + E_j \left[ \frac{1}{p_1} \middle| p_1 \leq y \right] \Pr_j(p_1 \leq y) \right) \\ &> s^i \left( E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y) + E_i \left[ \frac{1}{p_1} \middle| p_1 \leq y \right] \Pr_i(p_1 \leq y) \right) = E_i \left[ \frac{s^i}{p_1} \right], \end{aligned}$$

which can be rearranged as

$$\begin{aligned} &\frac{1}{s^j \left( E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y) + E_j \left[ \frac{1}{p_1} \middle| p_1 \leq y \right] \Pr_j(p_1 \leq y) \right)} \\ &< \frac{1}{s^i \left( E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y) + E_i \left[ \frac{1}{p_1} \middle| p_1 \leq y \right] \Pr_i(p_1 \leq y) \right)}. \end{aligned} \quad (14)$$

By the assumption (13),

$$\begin{aligned} &\frac{s^j E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y)}{s^j \left( E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y) + E_j \left[ \frac{1}{p_1} \middle| p_1 \leq y \right] \Pr_j(p_1 \leq y) \right)} \\ &\geq \frac{s^i E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y)}{s^i \left( E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y) + E_i \left[ \frac{1}{p_1} \middle| p_1 \leq y \right] \Pr_i(p_1 \leq y) \right)}, \end{aligned}$$

which implies that

$$\frac{s^j E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y)}{s^i E_i \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_i(p_1 > y)} > \frac{s^j E_j \left[ \frac{1}{p_1} \right]}{s^i E_i \left[ \frac{1}{p_1} \right]}.$$

Since the upper bound for price under agent  $j$ 's perspective,  $s^j$ , is higher than that under agent  $i$ 's perspective,  $s^i$ , the previous inequality holds only if  $\Pr_j(p_1 > y)$  is much larger than  $\Pr_i(p_1 > y)$ . However, then  $\Pr_i(p_1 \leq y) > \Pr_j(p_1 \leq y)$  and  $1/p_1$  is larger when  $p_1 \leq y$

than  $1/p_1$  when  $p_1 > y$ . Therefore,

$$\frac{s^j E_j \left[ \frac{1}{p_1} \right]}{s^i E_i \left[ \frac{1}{p_1} \right]} < 1,$$

which violates (14). Therefore, the assumption (13) is false, and (12) holds, which implies the first-order derivative (left-semi differential) is positive for any  $y \leq s^i$ . Hence, agent  $j$  promises  $s^i$  and maximizes her leverage.

In the second part of the lemma, we apply the result from the first part of the lemma and fix the contracts with promises of expected asset payoffs of the lenders. Suppose agent  $j$  is borrowing both from  $i$  and  $k$  with the same probability of bankruptcy and  $i < k$ , for the same amount of contracts—that is,  $c \equiv c_{ij} = c_{ik}$ . The marginal returns from both leveraged positions for  $j$  are

$$R_j^i \equiv \frac{s^j}{q_j(s^j) - q_i(s^i)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \zeta'(c) \left[ 1 - \frac{s^i}{p_1} \right]^+ \mathbb{1}_{\{i \in B(\epsilon)\}} \right]$$

$$R_j^k \equiv \frac{s^j}{q_j(s^j) - q_k(s^k)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \zeta'(c) \left[ 1 - \frac{s^k}{p_1} \right]^+ \mathbb{1}_{\{k \in B(\epsilon)\}} \right]$$

Since  $q'_k$  is increasing at the left limit of  $s^k$  and  $j$  maximizes over  $(s^k, s^i)$  at  $s^i$ , relative increase in the amount of borrowing (or decrease in down payment) should exceed the relative decrease in expected payoff at  $t = 1$  under no arbitrage condition. Therefore, agent  $j$  prefers to borrow more from  $i$  over  $k$ . ■

**Proof of Lemma 4.** Suppose agent  $j > 1$  is buying the asset while agent 1 is not buying, then agent 1 will have an even larger amount of cash holdings. If agent 1's cash holding  $e_1^1$  is large, then  $j$ 's return of cash is large. Return from the asset purchase for agent  $j$  is  $s^j/p_0$ . By lemma 1, agent  $j$  should equate the returns from cash and asset as

$$\frac{s^j}{p_0} = E_j \left[ \frac{s^j}{p_1} \right].$$

But then,  $\frac{s^j}{p_0} < \frac{s^1}{p_0} < E_1 \left[ \frac{s^1}{p_1} \right]$ . Agent  $j$  can sell the asset lower than the market price to agent 1 and accumulate more cash because of the gap between the two returns. This implies agent  $j$  would rather sell her asset to agent 1 and both make profitable trades. The same inference can be done with levered purchases, as both agents can do the same borrowing from

the same set of lenders and simply change the price as the down payment such as  $p_0 - q(s^i)$ . The second statement holds with the similar argument as the problem becomes isomorphic by substituting the asset with the promise of  $s^2$  (which is coming from lemma 3) from agent 1 and so forth. ■

### Proof of Proposition 4.

By lemmas 3 and 4, agents form a chain of intermediation: Agent 1 borrows from 2, who borrows from 3, who borrows from 4, and so on. There will be no missing chain because of lemma 3 and the property of lender cost function  $\zeta$ —that is, at least some positive amount of borrowing occurs through the lending chain linking the agents in the order of optimism. Also, in the equilibrium,  $q_{i+1}(y) > q_i(y)$  for any  $y \leq s^{i+1}$  for any  $i \in N, i < n$ . This is true because, if  $i$  can leverage and maximize return for some other contract such as lending to agent  $i - 1$ , then she can also increase her return from lending at  $y$  by leveraging from agent  $i + 1$  with the same  $y$ . Thus, because of the possible counterparty risk, which is positive due to lemma 3, the marginal return from this intermediation is

$$\frac{-\zeta'(c_{i+1,i})E_j \left[ \left[ 1 - \frac{y}{p_1} \right]^+ \mathbb{1}\{i+1 \in B(\epsilon)\} \right]}{q_i(y) - q_{i+1}(y)},$$

and the sign of  $q_i(y) - q_{i+1}(y)$  should be negative to make the return match agent  $i$ 's other returns<sup>34</sup>. Hence, all the contract prices are determined by the subsequent lender. In other words, competitive contract prices for  $y \in [s^{j+1}, s^j]$  are determined by  $j$ .

From equation (8), we have  $j$ 's contract pricing formula as follows.

$$q_j(y) = q_{j+1}(s^{j+1}) + \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \zeta'(c_{ij}) \left[ 1 - \frac{s^{j+1}}{p_1} \right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\} \right]}{E_j \left[ \frac{1}{p_1} \right]}. \quad (15)$$

Since  $q_{j+1}(s^{j+1})$  is determined by the perspective of  $j + 1$ , the only relevant factor is the second term. As  $y$  increases, the relevant lower bound of price for borrower default increases. Obviously,  $s^j$  is the maximum price in  $j$ 's perspective, and  $q'_j(y) = 0$  at  $y = s^{j+}$ —that is, the right semi-derivative is zero. On the other hand,  $y = s^{j+1}$  provides no additional value and simply becomes  $q_j(s^{j+1}) = q_{j+1}(s^{j+1}) - \zeta'(c_{j+1,j})\omega_{j+1,j}(y)$ , and again we find  $q_j(y) < q_{j+1}(y)$  at  $y = s^{j+1}$ .

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<sup>34</sup>The inequality of  $q_i(y) < q_{i+1}(y)$  will be clear in the contract pricing formula (15) as well.

Now we compute the derivatives. By Leibniz integral rule, for any  $y \in [s^{j+1}, s^j]$ ,

$$q'_j(y) = \frac{E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y)}{E_j \left[ \frac{1}{p_1} \right]} > 0$$

$$q''(y) = -\frac{1}{E_j \left[ \frac{1}{p_1} \right]} \frac{f_j(y)}{y} < 0,$$

where  $f_j$  is the density function of  $F_j$ , which is the distribution function of the asset price in  $t = 1$  that comes from the convolution of shock distributions. Thus,  $q_j(y)$  is concavely increasing in  $y$ . Denote  $\kappa_j$  as the inverse function of  $q_j(y)$  which is well defined in the domain of  $y \in [s^{j+1}, s^j]$  since  $q'_j(y) > 0$  in the domain and  $q'_j(s^j) = 0$ . Suppress the subscript for  $q, \kappa$  for the rest of the proof.

By inverse function theorem of first and second-order derivatives, for any  $q(y)$  in the range of original function, we obtain

$$\kappa'(q(y)) = \frac{1}{q'(y)} > 0$$

$$\kappa''(q(y)) = -\frac{q''(y)}{(q'(y))^3} > 0.$$

Now denote the gross interest rate function as  $\delta(q) \equiv \frac{\kappa(q)}{q}$ , where  $q$  is in the range of  $q(y)$ . The first derivative of the gross interest rate function becomes

$$\delta'(q) = \frac{\kappa'(q)q - \kappa(q)}{q^2} = \frac{\frac{q(y)}{q'(y)} - y}{q(y)^2},$$

where  $\kappa(q) = y$ . The numerator of the term can be rearranged as  $q(y) - yq'(y)$  and this is

positive because

$$\begin{aligned}
q_j(y) &= q_{j+1}(s^{j+1}) + \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \zeta'(c_{ij}) \left[ 1 - \frac{s^{j+1}}{p_1} \right]^+ \mathbb{1}_{\{j+1 \in B(\epsilon)\}} \right]}{E_j \left[ \frac{1}{p_1} \right]} \\
&> \frac{E_j \left[ \frac{y}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y)}{E_j \left[ \frac{1}{p_1} \right]},
\end{aligned}$$

where the last inequality is positive by lemma 6. Therefore, the gross interest rate is increasing in  $y$ . The second derivative of the gross interest rate function becomes

$$\delta''(q) = \frac{1}{q^4} [q^2 (\kappa''(q)q + \kappa'(q) - \kappa'(q)) - 2q (\kappa'(q)q - \kappa(q))],$$

and the numerator is

$$\begin{aligned}
\kappa''(q)q^3 - 2q^2\kappa'(q) + 2q\kappa(q) &= -q''(y) + 2q(y)[y - q(y)\kappa'(q(y))] \\
&= \frac{f_j(y)/y}{E_j \left[ \frac{1}{p_1} \right]} - 2q(y)[q(y)/q'(y) - y] \\
&= \frac{f_j(y)/y}{E_j \left[ \frac{1}{p_1} \right]} - 2q(y) \left[ q(y) \frac{E_j \left[ \frac{1}{p_1} \right]}{E_j \left[ \frac{1}{p_1} \middle| p_1 > y \right] \Pr_j(p_1 > y)} - y \right],
\end{aligned}$$

which is negative because  $q(y) > yq'(y)$  as shown previously. Also  $q(y)/q'(y) - y > 1$  implies the inequality to be trivial, and  $q(y)/q'(y) - y \leq 1$  also means the first term is negligible compared to the conditional expectation in  $q(y)$  of the second term. Thus,  $y/q(y)$  is concavely increasing in the interval of  $q(y) \in [q(s^{j+1}), q(s^j)]$ .

Now we need to check for the kink points and the whole graph. Because  $q'_j(s^j) = 0$ ,  $\delta'_j(q)$  goes to infinity, that is why  $q'_1(s^1)$  is infinity. A unique property of the pricing of equation (8) is that  $y$  close to  $s^{j+1}$  will make  $q_j(y) < q_{j+1}(s^{j+1})$  coming from the left limit of  $q_j(s^{j+1})$ . Therefore, there are intersections around each point of  $s^j$  for  $j \in N$  as can be seen in the figure 14. Since the borrowers would rather prefer to borrow from low  $y$  for higher  $q(y)$ , the market price function for  $q(y)$  will take the upper envelope of the functions  $q$  defined for each interval  $(s^{j+1}, s^j]$  for  $j = 1, 2, \dots, n - 1$ . Hence, the inverse function of  $q$ ,  $\kappa$  will have jumps at each point of  $q(s^j)$  for  $j \neq 1, n$  and the right derivative is greater than the left derivative

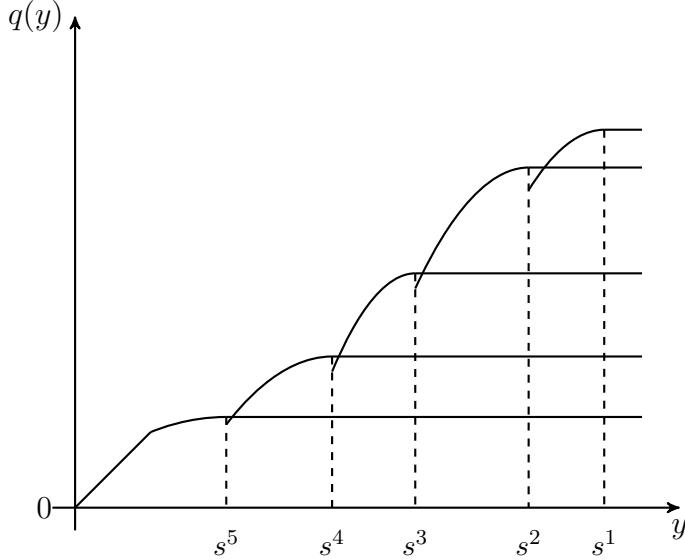


Figure 14: Graph of Contract Prices

of each point. Finally, since the upper envelope of functions  $q$  are continuous because above  $s^j$  there is a point that borrowers prefer to simply borrow from  $j$  at a constant price rate up to the point that  $j - 1$  becomes the preferred lender when  $q(y)$  is greater than or equal to  $q(s^j)$ . Therefore, both the upper envelope function of market price  $q(y)$  is continuous, and the interest rate function is also continuous. ■

The following lemma characterizes the properties of the inverse of equilibrium price, especially with respect to indegree of the bankrupt agents. It will be used to prove proposition 5.

**Lemma 7** (Convexity of Inverse Price). *Consider a class of debt networks  $(N, C, Y, e_1, h_1, \epsilon, s, \zeta)$  with  $C > 0$  that is under intermediation order. Suppose that  $j \in B(\epsilon)$ . Then, the inverse of the asset price  $\frac{1}{p}$  is convexly decreasing in  $c_{ij}$  and convexly increasing in  $c_{jk}$  for any  $i$  and  $k$  in  $N$ . The convexity of inverse of the price with respect to  $c_{ij}$  and  $c_{jk}$  is strict up to the point  $p = y_{ij}$  and  $p = y_{jk}$ , respectively.*

### Proof of Lemma 7.

For prices  $p = s$  and  $p = 0$ , the result is trivially true. Now consider the intermediate case of  $p = \pi(p)$ . Recall that

$$\frac{1}{p} = \frac{\sum_{j \in B(\epsilon)} \sum_{\substack{k \in N \\ p < y_{jk}}} c_{jk} - \sum_{j \in B(\epsilon)} \sum_{\substack{i \neq j \\ p < y_{ij}}} c_{ij}}{\sum_{i \notin B(\epsilon)} (e_1^i - \epsilon_i) + \sum_{j \in B(\epsilon)} \sum_{\substack{i \neq j \\ p \geq y_{ij}}} c_{ij} y_{ij} - \sum_{j \in B(\epsilon)} \sum_{\substack{k \in N \\ p \geq y_{jk}}} c_{jk} y_{jk} - \sum_{j \in B(\epsilon)} \sum_{\substack{k \notin B(\epsilon) \\ p \geq y_{jk}}} \zeta(c_{jk})}.$$

Denote  $\frac{1}{p} = \frac{(num)}{(den)}$ . Suppose  $j \in B(\epsilon)$  and we differentiate the inverse price with respect to  $c_{ij}$ , which will become

$$\frac{\partial(1/p)}{\partial c_{ij}} = \begin{cases} -\frac{1}{(den)} < 0 , & \text{if } p < y_{ij} \text{ and } i \notin B(\epsilon) \\ -\frac{(num)y_{ij}}{(den)^2} < 0 , & \text{if } p \geq y_{ij} \text{ and } i \notin B(\epsilon) \\ 0 , & \text{if } i \in B(\epsilon), \end{cases}$$

and differentiating with respect to  $c_{ij}$  once more gives

$$\frac{\partial^2(1/p)}{\partial c_{ij}^2} = \begin{cases} 0 , & \text{if } p < y_{ij} \text{ or } i \in B(\epsilon) \\ 2\frac{(num)y_{ij}^2}{(den)^3} > 0 , & \text{if } p \geq y_{ij} \text{ and } i \notin B(\epsilon). \end{cases}$$

Thus,  $\frac{1}{p}$  is convexly decreasing in  $c_{ij}$  with strict convexity up to the point  $p = y_{ij}$ . Now differentiate inverse price with respect to lending of bankrupt agent  $j$ ,  $c_{jk}$ .

$$\frac{\partial(1/p)}{\partial c_{jk}} = \begin{cases} \frac{1}{(den)} > 0 , & \text{if } p < y_{jk} \text{ and } i \notin B(\epsilon) \setminus \{j\} \\ \frac{(num)(y_{jk} + \zeta'(c_{jk}))}{(den)^2} > 0 , & \text{if } p \geq y_{jk} \text{ and } i \notin B(\epsilon) \setminus \{j\} \\ 0 , & \text{if } i \in B(\epsilon) \setminus \{j\} \end{cases}$$

The second derivative becomes zero for the case of  $p < y_{jk}$  and  $i \in B(\epsilon) \setminus \{j\}$ . In the case of  $p \geq y_{jk}$  and  $i \notin B(\epsilon) \setminus \{j\}$ , the numerator of the second derivative becomes

$$(den)^2(num)\zeta''(c_{jk}) + 2(den)(y_{jk} + \zeta'(c_{jk}))^2 ,$$

which is again positive. Therefore, the inverse of price is convexly increasing in indegree and strict convexity holds up to the point  $p = y_{jk}$ . ■

The following lemma is also used to prove proposition 5.

**Lemma 8** (Counterparty Risk Order). *For any network equilibrium and any agent  $j \in N$ ,  $\zeta(c_{ij})\omega_{ij} \geq \zeta(c_{kj})\omega_{kj}$  for any  $j < i < k$ .*

**Proof of Lemma 8.** If  $c_{ij} > 0$  and  $c_{kj} = 0$  or  $c_{ij} = c_{kj} = 0$ , then the result holds trivially. Suppose that  $c_{ij} > 0$  and  $c_{kj} > 0$ . Consider the return equations. For  $c_{ij} = c_{kj} = c$ ,  $R_j^i > R_j^k$

as shown in lemma 3 where

$$R_j^i \equiv \frac{s^j}{q_j(s^j) - q_i(s^i)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \zeta'(c) \left[ 1 - \frac{s^i}{p_1} \right]^+ \mathbb{1}_{\{i \in B(\epsilon)\}} \right]$$

$$R_j^k \equiv \frac{s^j}{q_j(s^j) - q_k(s^k)} E_j \left[ \min \left\{ 1, \frac{s^j}{p_1} \right\} - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \zeta'(c) \left[ 1 - \frac{s^k}{p_1} \right]^+ \mathbb{1}_{\{k \in B(\epsilon)\}} \right]$$

and agent  $j$  will borrow more from  $i$  and  $c_{ij}$  will increase. In other words, agent  $j$  has the higher return when she borrows from the more optimistic lender, agent  $i$ . Agent  $k$  should have lower counterparty risk in the perspective of agent  $j$  in order to make the indifference condition  $R_j^i = R_j^k$  hold. Therefore,  $\zeta(c_{ij})\omega_{ij} \geq \zeta(c_{kj})\omega_{kj}$  for  $j < i < k$  in any network equilibrium. ■

**Proof of Proposition 5.** Since every belief is bounded above by  $s^j$  for each  $j \in N$ , a decrease in expectation of  $p_1$  and an increase in the expected sum of default costs implies an increase in volatility. Suppose that  $(s, \epsilon)$  is realized and  $j \in B(\epsilon)$ , which happens with positive probability because of the distribution of  $\epsilon_j$ . By lemma 7,  $c_{ij}$  convexly decreases the inverse price, and  $c_{jk}$  convexly increases the inverse price for  $i, k \in N$ . By lemma 3, the debt network is under intermediation order. Also from the proof of lemma 7 and the intermediation order, the slope from  $c_{jk}$  dominates the slope from  $c_{ij}$ . Thus, any inverse of price  $p_1 \geq y_{ij}$  will be convexly increasing in  $c_{jk}$ .

Suppose that  $C^*$  is uniformly less indebted than  $C$ . Holding the bankruptcy state realizations the same, this decrease in debt decreases volatility directly from the previous argument of lemma 7 for each realization of  $(s, \epsilon)$  involving bankruptcy will have a smaller impact on increase in inverse price and price decline. Also the decrease will generate fewer states of bankruptcy as every agent becomes less susceptible to price as in the wealth equation

$$m_j(p) = e_1^j - \epsilon_j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) [p - y_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\},$$

which has smaller coefficients on prices and also the bankruptcy of lenders have smaller impact and less second-order bankruptcy will occur for the same state realizations.

Now, suppose that  $C^*$  is a diversification of  $j$  from  $C$ . From the application of lemma 7 in the beginning, the direct price effect from diversification is always positive, and the states that incur bankruptcy are fewer by lemma 5. In order to consider the effect from the counterparty channel, consider the simplest case of three agents, 1, 2, and 3, in a network. Suppose agent 1 is borrowing more from agent 2 than from agent 3—that is,  $c_{21} > c_{31}$ . By diversification of agent 1,  $\zeta(c_{21}^*) + \zeta(c_{31}^*) < \zeta(c_{21}) + \zeta(c_{31})$  by convexity of  $\zeta$ . Also, agent 2

has less collateral from agent 1 to reuse. Lower collateral makes agent 2's borrowing from agent 3 less, so  $c_{32} \geq c_{32}^*$  because of collateral constraint. Even though agent 1's promise becomes smaller by  $y_{21} > y_{31}$ , which implies that it is more susceptible to lender bankruptcy, the reduction of rehypothecation means the susceptibility is only replaced by the identity of the agent, from 2 to 1.

The only case left is that diversification happens, and it does not affect any change in intermediation—that is, the rehypothecation constraint is not binding. Now the conditions are

$$\begin{aligned}\zeta(c_{21}) &> \zeta(c_{31}) \\ \zeta(c_{21}^*) &> \zeta(c_{31}^*) \\ \zeta(c_{21})\omega_{21} &> \zeta(c_{31})\omega_{31} \\ \zeta(c_{21}^*)\omega_{21} &> \zeta(c_{31}^*)\omega_{31} \\ \zeta(c_{21}) + \zeta(c_{31}) &> \zeta(c_{21}^*) + \zeta(c_{31}^*) \\ \zeta(c_{21}) &> \zeta(c_{21}^*) \\ \zeta(c_{31}) &< \zeta(c_{31}^*),\end{aligned}$$

with  $0 \leq \omega_{21}, \omega_{31} < 1$ , where the third and fourth inequalities come from lemma 8 and the condition of diversification. By rearranging the inequalities, we obtain

$$\zeta(c_{21})\omega_{21} + \zeta(c_{31})\omega_{31} > \zeta(c_{21}^*)\omega_{21} + \zeta(c_{31}^*)\omega_{31}.$$

Thus, the expected default cost is lower under diversification. Also, even the bankruptcy probability change goes in the same direction. By the distributional assumption on  $G$  and because the second-order bankruptcy of agent 2 is now even more likely when agent 3 is bankrupt,  $\omega_{21|C^*} - \omega_{21|C} > \omega_{31|C^*} - \omega_{31|C}$ . Thus,

$$\zeta(c_{21})\omega_{21|C} + \zeta(c_{31})\omega_{31|C} > \zeta(c_{21}^*)\omega_{21|C^*} + \zeta(c_{31}^*)\omega_{31|C^*},$$

and the increased case of greater default cost from 3 is dominated by the decrease of default cost from a more likely occurrence of agent 2's bankruptcy. Thus, the counterparty channel also decreases the aggregate expected deadweight loss and increases expected price. Therefore, diversification in this case decreases aggregate expected deadweight loss, increases expected price, and decreases volatility.

Finally, we can extend this argument of three agents to any general number of agents. For any  $j \in N$ ,  $c_{L_{jj}} > c_{L_{jj}}^*$  while keeping  $\sum_{i \in N \setminus \{j\}} c_{ij} = \sum_{i \in N \setminus \{j\}} c_{ij}^*$  implies there is an

agent  $i > L_j$  such that  $c_{ij} < c_{ij}^*$ . Using the same argument for agent 1, 2, and 3 on agent  $j$ ,  $L_j$ , and  $i$  will provide the same result. If agent  $j$  is diversifying even further, then that will divide  $c_{L_j j}$  into even further diversification, and convexity will make it an even lower aggregate expected default cost. Thus, any diversification increases expected price and decreases aggregate expected default cost and volatility. ■

**Proof of Theorem 1.** The first and second properties come directly from proposition 4 and lemmas 3 and 4. The third property comes from the indifference equation for borrower  $j$ , who has to be indifferent between borrowing cash from  $i$  and  $k$  if  $j$  is borrowing from the two in a positive amount. The fourth property is again derived from lemma 4, and the fifth property is simply from the budget constraint and contract prices.

Now we show that an equilibrium that satisfies those properties exists. Define  $Z \equiv C \circ Y$ . Consider a class of networks  $\mathcal{Z}$  such that every  $Z \in \mathcal{Z}$  satisfies the intermediation order for fixed  $Y$  s.t.  $y_{ij} = s^i$  for any  $i, j \in N$ . Now use the matrix order to compare the total amount of promises—that is,  $Z > Z'$  implies  $Z_{ij} \geq Z'_{ij}$  for all  $i, j \in N$  and at least one element has strict inequality. Similarly,  $Z \geq Z'$  can be defined allowing equality for every entry. Note that this ordering is only a partial ordering among  $\mathcal{Z}$ . There can be networks  $Z, Z' \in \mathcal{Z}$  with neither  $Z \geq Z'$  nor  $Z' \geq Z$  is true. However,  $(\mathcal{Z}, \geq)$  forms a complete lattice, because for any subset  $\mathcal{Z}' \subseteq \mathcal{Z}$ , the least upper bound  $\bar{Z}$  with  $\bar{Z}_{ij} = \sup_{Z \in \mathcal{Z}'} Z_{ij}$  and the greatest upper bound  $\underline{Z}$  with  $\underline{Z}_{ij} = \inf_{Z \in \mathcal{Z}'} Z_{ij}$  exist because each element is from a subset of Euclidean space. Fix the norm  $\|\cdot\|$  of matrices as the Frobenius norm (or any other  $L_{p,q}$  norm with  $p, q \geq 1$ ). If  $\|Z\|$  increases, then there is more aggregate borrowing in the economy which generates greater probability of bankruptcy and default costs as shown by proposition 5.

Let  $V : \mathcal{Z} \rightarrow \mathcal{Z}$  be a function from network to network—that is, given the price and counterparty risk distribution of the first network in  $t = 1$ ,  $V$  generates the agents' optimal network formation decisions as best responses. Now I show that  $V$  is monotonous in  $(\mathcal{Z}, \geq)$ . Let  $\underline{Z}$  be the network with  $\|\underline{Z}\| = 0$ —that is, no risk of counterparty bankruptcy and dispersion of cash holdings. Under  $\underline{Z}$ , return from cash holding is minimized by lemma 7 in the proof of proposition 5. By intermediation order,  $V(\underline{Z}) \leq \bar{Z}$  where  $\bar{Z}$  denotes the maximum leverage network—that is, the single-chain network with full borrowing as defined in the proof of proposition ???. Similarly,  $V(\bar{Z}) \geq \underline{Z}$  because of the zero lower bound. Therefore, the range of  $V$  is compact.

Since the network is under intermediation order, lemma 3 and proposition 5 imply that  $Z \in \mathcal{Z}$  with a large  $\|Z\|$  has a lower degree of diversification and larger average default costs relative to  $Z' \in \mathcal{Z}$  with lower  $\|Z'\|$ . Then, an increase in  $\|Z\|$  has two effects to the return calculation. First, it increases counterparty exposure  $\omega_{ij}(Z)$  and the default cost,

which implies  $E_j[\mathbb{1}\{i \in B(\epsilon)\}]$ , and  $\beta_i(p_1)$  increase for each  $i, j \in N$ . Second, the state in which the liquidity is constrained is exactly the state in which the optimists are under liquidity shock—that is, when they would have really wanted to have additional liquidity. The marginal utility of cash in such a state is even greater. Thus,  $p_1$  is lower and more volatile under higher  $Z \in \mathcal{Z}$  by lemma 7. Then, the return on cash  $E_j[s^j/p_1]$  becomes greater for each  $j \in N$  under greater  $Z$ ,—that is, the return of cash holding is greater, and the agent's return on leverage goes down. Thus, any increase in  $Z$  (under the two possible directions restricted by intermediation order) will make the optimal response to the given distribution of  $Z$  to be lowering  $\|Z\|$ . In other words, a large  $Z$  makes the agents diversify or reduce borrowing or lending in general. The equilibrium portfolio decision holds as

$$\begin{aligned} E_j[1/p_1] &= \frac{s^j}{q(s^j) - q(s^i)} E_j \left[ 1 - \min \left\{ 1, \frac{s^i}{p_1} \right\} - \frac{\zeta'(c_{ij})}{p_1} \mathbb{1} \left[ 1 > \frac{s^i}{p_1} \right] \mathbb{1} \{i \in B(\epsilon)\} \right] \\ &= \frac{s^j}{q(s^j) - q(s^k)} E_j \left[ 1 - \min \left\{ 1, \frac{s^k}{p_1} \right\} - \frac{\zeta'(c_{kj})}{p_1} \mathbb{1} \left[ 1 > \frac{s^k}{p_1} \right] \mathbb{1} \{k \in B(\epsilon)\} \right], \end{aligned}$$

as in the proof of lemma 3, and the equality condition holds only at a greater diversification or lower overall collateral exposure. Thus,  $V(Z)$  decreases as  $Z$  increases. Then,  $V$  is a monotonic function on a complete lattice, and there exists a fixed point network  $Z^*$  such that  $Z^* = V(Z^*)$  by the Knaster-Tarski fixed point theorem. Therefore, there exists a network equilibrium, and the set of equilibria is also a complete lattice.

Now the rest of the proof is simply applying the results and  $q(y)$  from proposition 4 into market clearing conditions. Combining lemmas 1 and 4 with lemma 3, we can conclude that  $q(s^1) = p_0$ . Also, the nominal wealth are determined by the combination of budget constraints and market clearing conditions. ■

**Proof of Theorem 2.** As discussed in the description

$$\frac{\partial}{\partial c_{ik}} \sum_{j \in N} E_j \left[ m_j(\epsilon) \frac{s^j}{p_1(\epsilon)} \right] \neq \frac{\partial}{\partial c_{ik}} E_j \left[ m_j(\epsilon) \frac{s^j}{p_1(\epsilon)} \right]$$

and due to counterparty externality and price externality being positive coming from the arguments in proposition 5, the direction of inefficiency is coming from under-diversification. ■

### Proof of Proposition 6.

1. Suppose that  $s^i$  increases to  $s^i + \eta$  for every  $i \in N$ . As shown in proposition 4,  $q(y)$  is increasing in  $y$  for any  $y \in [s^i, s^i + \eta]$  and  $q'(y) < 1$  by the lower bound of  $y$  in the numerator.

By equation (8), the function for contract price becomes

$$q(y) = q(s^{j+1}) + \frac{E_j \left[ \min \left\{ 1, \frac{y}{p_1} \right\} - \min \left\{ 1, \frac{s^{j+1}}{p_1} \right\} - \zeta'(c_{j+1,j}) \left[ 1 - \frac{s^{j+1}}{p_1} \right]^+ \mathbb{1}_{\{j+1 \in B(\epsilon)\}} \right]}{E_j \left[ \frac{1}{p_1} \right]}.$$

Any change in the terms related to  $q(s^j)$  has a direct effect of increase in  $q(s^i)$  in linear terms for any  $i < j$  by the recursive equation

$$q(s^i) = q(s^j) + \sum_{k=i+1}^{j-1} \frac{E_k \left[ 1 - \min \left\{ 1, \frac{s^{k+1}}{p_1} \right\} - \zeta'(c_{k+1,k}) \left[ 1 - \frac{s^{k+1}}{p_1} \right]^+ \mathbb{1}_{\{k+1 \in B(\epsilon)\}} \right]}{E_k \left[ \frac{1}{p_1} \right]}.$$

As in the argument in the proof of proposition 4, for any agent  $k < j$ , prices relevant to cashflow of the leveraged contracts are bounded below by the subject belief of the lender  $k+1$ ,  $s^{k+1}$  as in

$$s^k E_k \left[ 1 - \min \left\{ 1, \frac{s^{k+1}}{p_1} \right\} - \zeta'(c_{k+1,k}) \left[ 1 - \frac{s^{k+1}}{p_1} \right]^+ \mathbb{1}_{\{k+1 \in B(\epsilon)\}} \right].$$

However, the return from cash holdings,  $s^k E_k [1/p_1]$  is not bounded by any prices. The ratio between the changes of the two terms is increasing in  $k$  as the lower bound of the price distribution becomes smaller—that is,

$$\begin{aligned} & \frac{\Delta E_k \left[ 1 - \min \left\{ 1, \frac{s^{k+1}}{p_1} \right\} - \zeta'(c_{k+1,k}) \left[ 1 - \frac{s^{k+1}}{p_1} \right]^+ \mathbb{1}_{\{k+1 \in B(\epsilon)\}} \right]}{\Delta E_k \left[ \frac{1}{p_1} \right]} \\ & < \frac{\Delta E_{k+1} \left[ 1 - \min \left\{ 1, \frac{s^{k+2}}{p_1} \right\} - \zeta'(c_{k+2,k+1}) \left[ 1 - \frac{s^{k+2}}{p_1} \right]^+ \mathbb{1}_{\{k+2 \in B(\epsilon)\}} \right]}{\Delta E_{k+1} \left[ \frac{1}{p_1} \right]}. \end{aligned}$$

Thus, a direct increase in  $s^i$  dominates the changes in the denominator and in the expecta-

tions of the return equation

$$R_i^{i+1} \equiv \frac{s^i}{q(s^i) - q(s^{i+1})} E_{i+1} \left[ \min \left\{ 1, \frac{s^i}{p_1} \right\} - \min \left\{ 1, \frac{s^{i+1}}{p_1} \right\} - \zeta'(c) \left[ 1 - \frac{s^{i+1}}{p_1} \right]^+ \mathbf{1}_{\{i+1 \in B(\epsilon)\}} \right].$$

Hence, higher counterparty risk can be justified as the leverage return for agent  $i$  increases. Agent  $i$  will increase  $c_{i+1,i}$  more, which implies fewer links (intensively and extensively), if  $i$  was diversifying. Also, the velocity of collateral (weakly) increases by the increase in  $c_{i+1,i}$  as well as relaxing collateral constraints for the subsequent agents  $i+1, i+2, \dots, n$ .

Also changes in  $q(s^j)$  have indirect effects by the induced borrowing pattern, changing the relative distribution of prices  $F_i$  for given liquidity shocks  $\epsilon$  and the return on cash holdings  $E_i \left[ \frac{s^i}{p_1} \right]$  as well as changing the probability of bankruptcy of the lenders. First, there will be a change in price distribution of  $\tilde{p}_1$ , which influences both the denominator and the numerator of equation (8). The increase in agents' debts will increase the price volatility by proposition 5. The effect from the indirect increase in bankruptcy probability is confined by the increase in  $E_k \left[ \frac{s^k}{p_1} \right]$ , because now the underpricing is more likely due to the increase in  $s^k$ . And the increase in second-order bankruptcy probability  $G_i([\zeta(c_{i+1,i}), \zeta(\hat{c}_{i+1,i})] | i+1 \in B(\epsilon))$  is always lower than the increase in first-order bankruptcy probability, which is taken into account by agent  $i$ . Thus, the direct effect  $E_k \left[ \min\{1, \frac{\eta}{p_1}\} \right] / E_k \left[ \frac{1}{p_1} \right]$  always dominates the indirect effect. Hence,  $q(s^i)$  and leverage increase, and  $R_i^j$  increases for all  $i < j$ , which implies the velocity of collateral increases.

The last thing to check is whether the change will affect the agents with beliefs below agent  $i$ . Note that the increase in  $c_{lk}$  for any  $k, l \leq j$  does not affect the expected sum of lender default costs of each agent in  $\{j+1, j+2, \dots, n\}$ , because any promise between agents  $k, l \leq j$  is going to be defaulted no matter what in their perspective of the upper bound of the asset price  $s^{j+1} > s^{j+2} > \dots > s^n$ . Thus, the debt amount or even the change in price distribution is irrelevant to these pessimistic agents. The only change for them comes from the increase in asset price  $p_0 = q(s^1)$  that increases their nominal value of endowments which incentivizes them to increase borrowing and increase the reuse of collateral—that is, the velocity of collateral.

**2.** Suppose  $\theta_j$  decreases by  $\eta$  for all  $j \in N$ . Then  $R_i^{i+1}$  increases again because of the lower probability of default costs and  $c_{i+1,i}$  increases. The rest of the argument goes the same as in the previous case. In this case, it is even more simple because there is a reduction of counterparty risk in every link that offsets the indirect change. ■

## A.4. Results on Central Clearing

**Proof of Proposition 7.** From equation (9), an individual agent does not care about the terms of  $\gamma$  and  $\frac{m_0(\epsilon|p_1)}{\sum_{i \in N} \mathbb{1}\{i \notin B(\epsilon)\}}$ , since they are determined by the macro variables and the agent considers herself as a price-taker. Under the first case, the expected cost multiplier  $\omega_{ij}$  equals to zero for any  $i, j \in N$ . Therefore, each agent does not have any incentive to diversify and lower leverage and will maximize their leverage. The equilibrium network under CCP has a collateral matrix  $C_{ccp}$ , which has a greater debt than the debt of decentralized equilibrium network  $C$ , by being more indebted (the opposite of less indebted) and less diversified (the opposite of diversification) maximizing concentration of the network. By proposition 5, this equilibrium network maximizes the systemic risk by maximizing the sum of expected default costs. Even if  $\gamma$  is not large and CCP can go bankrupt in some states, agent  $j$ 's perceived risk of borrowing from agent  $i$ ,

$$E_j [[1 - y_{ij}/p_1]^+ \mathbb{1}\{0 \in B(\epsilon) \& i \in B(\epsilon)\}]$$

is always smaller than

$$\omega_{ij} = E_j [[1 - y_{ij}/p_1]^+ \mathbb{1}\{i \in B(\epsilon)\}]$$

under decentralized equilibrium, and the debt of the network becomes larger either by more indebted or less diversification. As argued in the proof of theorem 2, the positive externality becomes even less incorporated into the agent's individual decisionmaking, and the systemic risk is always greater under  $C_{ccp}$  than the systemic risk under  $C$ . ■

**Proof of Proposition 9.** Suppose only one contract  $y$  is available in the market. As in lemma 4, agent 1 will buy the asset and borrow cash from agents who has  $s^j \geq y$  with equal weights as diversification. If agent 1's endowment  $e_0$  is not enough to purchase all the assets with the downpayment, then agent 2 also joins the buyer side and borrows from another pool of lenders. This can be repetitively done for agent 3, 4, and so forth. Similarly, if the demand for cash is too high, then the price of the contract  $q(y)$  will decrease, and even agents with  $s^j < y$  can become a lender, similar to the argument in lemma 4. Since the maximization problem and the budget constraints with down payments are all monotone, there is always an equilibrium. The resulting network becomes a complete bi-partite network for the given component of market participants. Since agents have no tradeoff between choice of counterparties and choice of leverage, they have no incentives to change their network formation behavior even after eliminating the counterparty risk concerns  $\omega_{ij}$  for each  $i, j \in N$ . Since all the walks in the network have a length of 1, there will be no effect

from netting as well. ■

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