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GAINS FROM TRADE UNDER UNCERTAINTY, ONCE AGAIN

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In "Gains from Trade under Uncertainty" [AER, December 1974], Batra and Russell have examined the situation of a small country which must make its domestic production decisions prior to the realization of the world terms of trade. Once this realization occurs, home production can be traded internationally for preferable bundles of the same value.

Batra and Russell assert that profit-maximizing, "risk neutral" producers would select a different production point than a planner conscious of a "representative" consumer's aversion to risk. The authors conclude, therefore, that "free trade is not the optimal policy" for a small country under uncertainty.

This conclusion is, we believe, misleading. Producers in their model are unresponsive to the preferences of consumers (over consumption lotteries) because Batra and Russell omit the market through which these preferences would be communicated: the commodity futures market. Even if an international market in such contracts is prohibited, nothing would prevent home residents from exchanging forward contracts among themselves. Since, in the context of this small country model, every domestic agent would benefit from exchanging claims on future output at known prices in anticipation of exchanging goods with the rest of the world at prices as yet unknown, it is difficult to see why a domestic futures market would fail to develop.

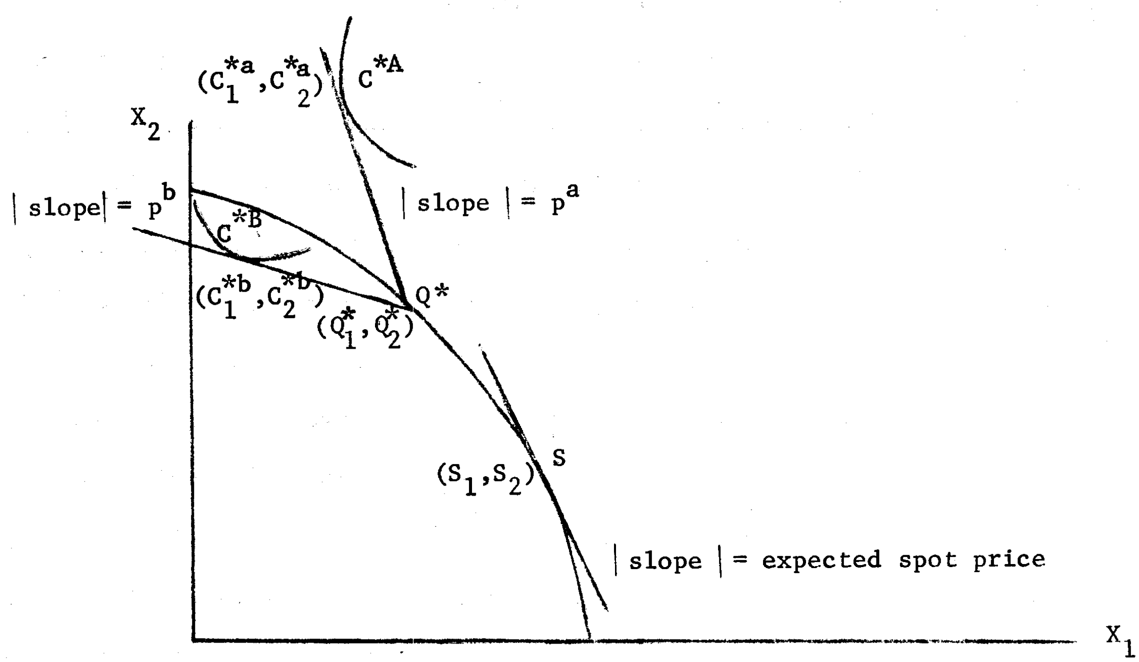
This comment explores the effects of adding a futures market to the international trade model of Batra and Russell. It will be shown that introducing such a market: 1) permits a separation of production decisions from the subjective attitudes toward risk and probability estimates of producers; 2) improves the

*/ Federal Reserve Board and Harvard University, respectively. The views expressed herein are solely those of the authors and do not necessarily represent the views of the Federal Reserve System. The authors wish to express their intellectual debt to Robert Townsend, whose "Incomplete Forward Markets in a Pure Exchange Economy with Stochastic Endowments" (University of Minnesota Discussion Paper 74-47, November 1974) has greatly influenced them.

market allocation of each domestic consumer; and 3) leads to the conclusion that free trade is once again optimal[1/] for a small country.

The Planning Problem

As portrayed below, the planning problem posed by Batra and Russell is to choose a production point (Q) and a consumption point (C^A or C^B)[2/] for each pos-



1/ This third result requires qualification. If states of nature outnumber goods, the addition of merely a futures market does not lead to full optimality. However, in such cases, the addition of complete contingent markets would restore the optimality of free trade; moreover, all home residents would benefit from adding such markets to the world of Batra and Russell. Once again, therefore, the non-optimality they have discovered arises from their exclusion of some markets and not from any distortion in those markets which they happen to include.

2/ In addition, they constrain the consumer to choose the same quantity of the first good in either state ($C_1^a = C_1^b$). Since neither their result nor our comment about it depends on this inessential complication, we have dropped the constraint.

sible subsequent event so as to maximize the expected utility of the "representative domestic consumer". The production decision must be made before the world terms of trade become known. Afterwards, the consumer can engage in balanced international trade.

Given any feasible production point, the representative consumer should move along whichever budget line is realized until he reaches his highest indifference curve. Batra and Russell show that the optimal production point (Q^*) is the one where the marginal rate of transformation is equated to the ratio of the expected marginal utilities evaluated at the optimal consumption points. As they show, the optimal production point will differ--except in the one case discussed in our Appendix I--from the point (S) where the expected value of output is maximized. They assert that profit-maximizing, risk neutral producers[3/] will produce this sub-optimal output.

Their analysis can be simplified by considering how a consumer would rank various "forward positions".[4/] This ranking exists since any forward position today results in an optimal consumption choice in either state tomorrow and, hence, in a maximized expected utility. As we prove in Appendix I, the locus of forward positions resulting in the same maximized expected utility has the following properties:

1. Its slope at forward position (F_1,F_2) is negative and equal in magnitude to the ratio of the expected marginal utilities evaluated at the optimal consumption choices which would result from that forward position. This slope, the rate at which the consumer is willing to substitute one forward position for another, is referred to as the "rate of forward substitution" (RFS).

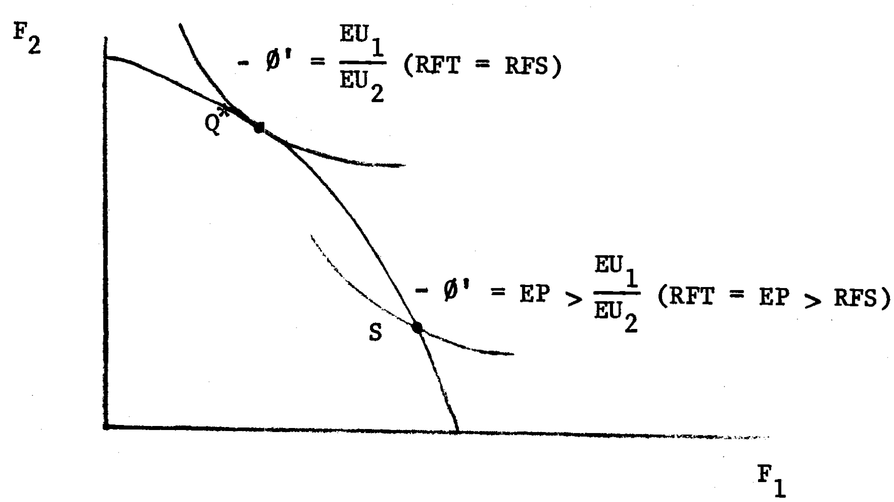
3/ Batra and Russell assume that "risk neutral" producers maximize the expected value of income in terms of good 2. Since they would choose an entirely different production point if they maximized expected income in terms of the other good instead, we find this assumption arbitrary. In addition, the meaning of risk neutrality in connection with producers in unclear since risk aversion is a characteristic of consumer preferences.

4/ A forward position is defined as the endowment point from which next period's spot trading takes place (a person's production point plus his net forward claims on the production of others).

2. The locus is (strictly) convex if the underlying utility function is (strictly) concave.

3. If the consumer is risk neutral, (his indirect utility function is linear in income), each contour will be linear with a slope depending on the subjective probability distribution of the exogenous terms of trade.

Forward positions north-east of each contour are preferred to those on or below it. Two members of the field of iso-maximized expected utility curves are portrayed below. One contour cuts the transformation curve ($Q_2 = \phi(Q_1)$) at S, indicating that forward positions superior to S can be attained with the technology. Batra and Russell contend that, in the absence of a futures market, risk neutral producers would nevertheless produce at S, since the slope of the transformation curve there is equal to the "expected" terms of trade. The second contour is tangent to the transformation curve at Q^*, indicating that no forward position superior to Q^* can be attained with the technology. At Q^*, the rate at which the technology can transform one forward position into another (RFT) is equal to the rate at which the "representative domestic consumer" is willing to substitute one forward position for another (RFS).



Once the terms of trade is realized, the consumer treats his forward position as an endowment which can be exchanged for preferable consumption bundles of equal value. The optimal consumption choice occurs where his rate of commodity substitution (RCS) is equal to the realized world price. These marginal conditions and constraints[5/] define the solution to the planning problem for the case of a "representative" consumer. If domestic consumers differed, the planning problem would be to choose the best consumption lottery for one person given feasible, prescribed utility levels for the others. The planner would again be constrained first to use the domestic technology and then to engage in balanced international trade after the world price is realized.[6/]

In the next section, optimizing behavior in a competitive futures market is described. The allocational effects of adding such a market to the small country model of Batra and Russell is then examined for the case of a "representative" consumer. In Appendix II these results are extended to the case of different consumers.

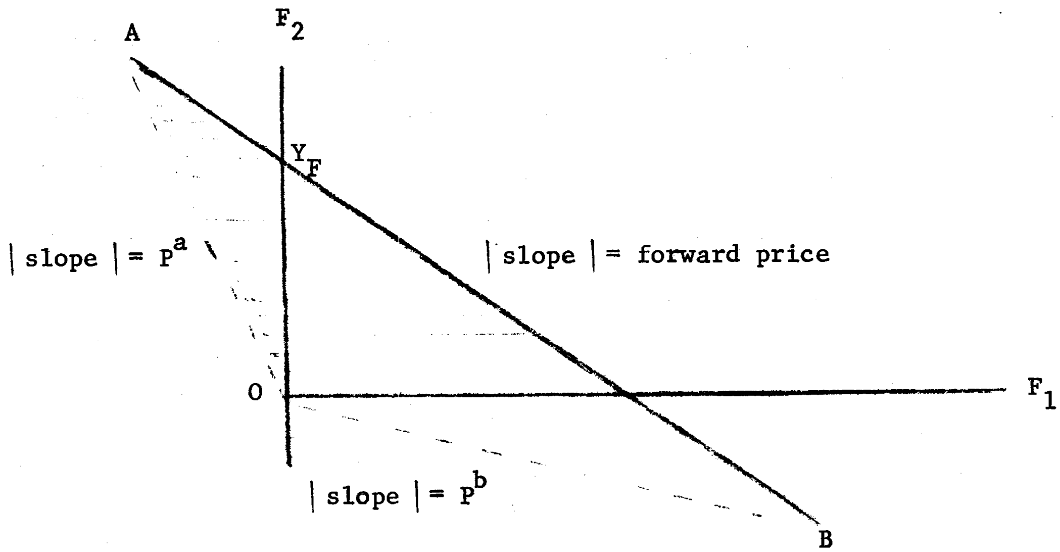
The Forward Market

With a forward market, forward claims can be traded at a fixed rate of exchange. The consumer can choose any position equal in value to his forward income provided he can fulfill his contracts no matter which state occurs. In the diagram below triangle OAB contains the feasible set of forward positions for a consumer with forward income Y_f:

5/ The six decision variables (C_1^A, C_1^B, C_2^A, C_2^B, Q_1, Q_2) are determined optimally when they satisfy the following six conditions:

1) Domestic production is efficient (on the P.P.F.)
2) Trade balances in state A
3) Trade balances in state B
4) $RCS^A = P^A$
5) $RCS^B = P^B$
6) RFS = RFT

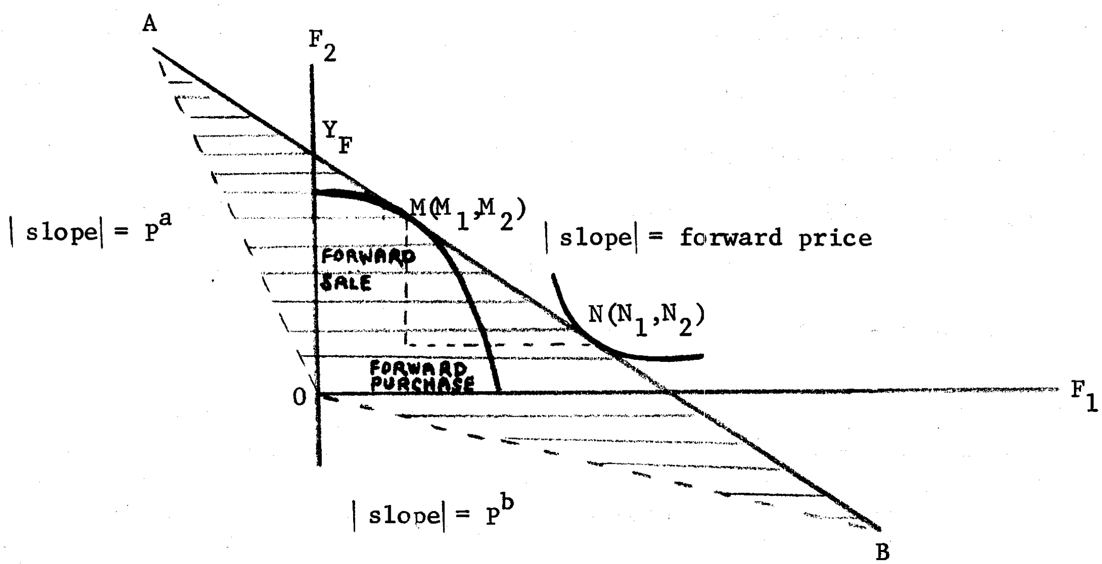
6/ See Appendix II.



At point B (point A), the consumer is short so much of commodity 2 (commodity 1) that the price realization making it expensive would require him to sell spot all his claims to commodity 1 (commodity 2) in order to make good his obligation to deliver the other good. Beyond point B (point A), the consumer could not fulfill his contracts in an adverse state and would, therefore, be prohibited from making them.

The best attainable forward position for a consumer is where he reaches the highest iso-maximized expected utility contour within his budget set. Such a position will occur on line segment $\overline{AB}$, either at a point of tangency, or else at one of the corners (A or B). At a point of tangency, the rate at which a consumer is willing to substitute one forward position for another (RFS) is equal to the forward price.

With a forward market, a consumer ceases to be restricted to forward positions attainable with his own technology--an obvious improvement. He now can attain <u>any</u> forward position equal in value to the forward value of his output by trading on the forward market. In the diagram below, the value of his output is maximized at M, where the rate of forward transformation (RFT) is equal to the forward price. Production at such a point allows each consumer to attain a higher forward position contour than production at any other point. The optimal choice of outputs is determined entirely by the forward price and the technology. Differences in risk aversion, in beliefs about the odds of tomorrow's terms of trade or in any other element imbedded in the preference contours will not affect the optimal production point.7/



Such differences will, of course, determine how much of this forthcoming output is sold forward. In the diagram above, the forward value of profits is maximized by producing at M. The particular consumer portrayed then buys N_1-M_1 additional forward claims on the first good, paying for them with promises to deliver M_2-N_2 units of his output of the second good. If, instead, the individual were risk neutral or

7/ Regardless of the number of goods or states, this separation theorem holds if there is a futures market and state independent technology.

risk-loving, he would still produce at M but would use claims on this output to put himself in a position (corner A or B) where he risked bankruptcy in one state in return for high winnings in the other.

If there were no international market in forward contracts, the forward price would adjust until the domestic supply and demand for forward contracts on the first commodity balanced; by Walras' Law, the other forward market would then clear. Specifically, in the "Robinson Crusoe" world of Batra and Russell, the forward price would adjust until it equalled the slope of the line through Q^*, separating the transformation curve at that point from the iso-maximized expected utility contour through Q^*. When faced with this forward price, a rational producer would produce at Q^*.[8/] Similarly, a rational consumer would choose Q^* as the best forward position attainable given the forward price and his income. At that price, therefore, there would be no excess demand for, or supply of, forward claims. Since the market does then solve the domestic planning problem,[9/] free trade emerges once again as the optimal policy for a small country.

8/ Contrary to the assertion of Batra and Russell, a lone "risk neutral" agent would produce where his own rates of forward substitution and transformation were equal, at a point of tangency like Q^* rather than S--even with no forward market. However, when preferences or technologies differ among individuals, the rates of forward substitution and transformation for each agent will be different in the absence of a forward market. The function of a forward market is then to allow these rates to be equalized across individuals. Without such equalization, free trade cannot be Pareto-optimal.

9/ In Appendix II, this result is extended to the case of two domestic consumers with different preferences and technologies.

Appendix I

The Iso-Maximized Expected Utility Contours and the Exceptional Case Noted by Batra and Russell

Define

$$V^a(F_1,F_2) = \max_{C_1^a,C_2^a \geq 0} U(C_1^a,C_2^a) \text{ s.t. } P^a C_1^a + C_2^a \leq P^a F_1 + F_2.$$

Then

$$V_1^a = U_1(C_1^{*a}, C_2^{*a}), \tag{1}$$

$$V_2^a = U_2(C_1^{*a}, C_2^{*a}), \tag{2}$$

where

$$RCS^a = \frac{U_1^a}{U_2^a} = P^a \text{ and } P^a C_1^{*a} + C_2^{*a} = P^a F_1 + F_2 \geq 0. \tag{3}$$

Similar relation hold for the maximized utility if the other state occurs. Since V( ) is the maximized value of a (strictly) concave function subject to convex constraints, it is (strictly) concave.

Define $H(F_1,F_2)$ to be the expected utility associated with the forward position F_1,F_2:

$$H(F_1, F_2)= \pi V^a(F_1, F_2) + (1-\pi)V^b(F_1, F_2) \tag{4}$$

Since $H(F_1,F_2)$ is the sum of (strictly) concave functions, it is (strictly) concave. Hence, its contours are (strictly) convex.

The slope (RFS) along any contour is $-\frac{dF_2}{dF_1} = \frac{H_1}{H_2}$. From (4),

$$H_1/H_2 = \frac{\pi V_1^a(F_1, F_2) + (1-\pi)V_1^b(F_1, F_2)}{\pi V_2^a(F_1, F_2) + (1-\pi)V_2^b(F_1, F_2)}.$$

Substituting (2) and (3) into (4),

$$-\frac{dF_2}{dF_1} = \frac{\pi U_1(C_1^{*a}, C_2^{*a}) + (1-\pi)U_1(C_1^{*b}, C_2^{*b})}{\pi U_2(C_1^{*a}, C_2^{*a}) + (1-\pi)U_2(C_1^{*b}, C_2^{*b})} = \frac{EU_1}{EU_2} . \quad (5)$$

Batra and Russell indicate[10/] that, in one exceptional case, producers maximizing expected profits will solve the planning problem. The exceptional case arises when consumer preferences contain a constant marginal utility good:

$$U = g(C_1) + \alpha C_2.$$

In this case, the iso-maximized expected utility contours are linear, with a slope equal to the expected terms of trade. Since the optimal consumption choice satisfies $U_1 = U_2P$ and $U_2 = \alpha$,

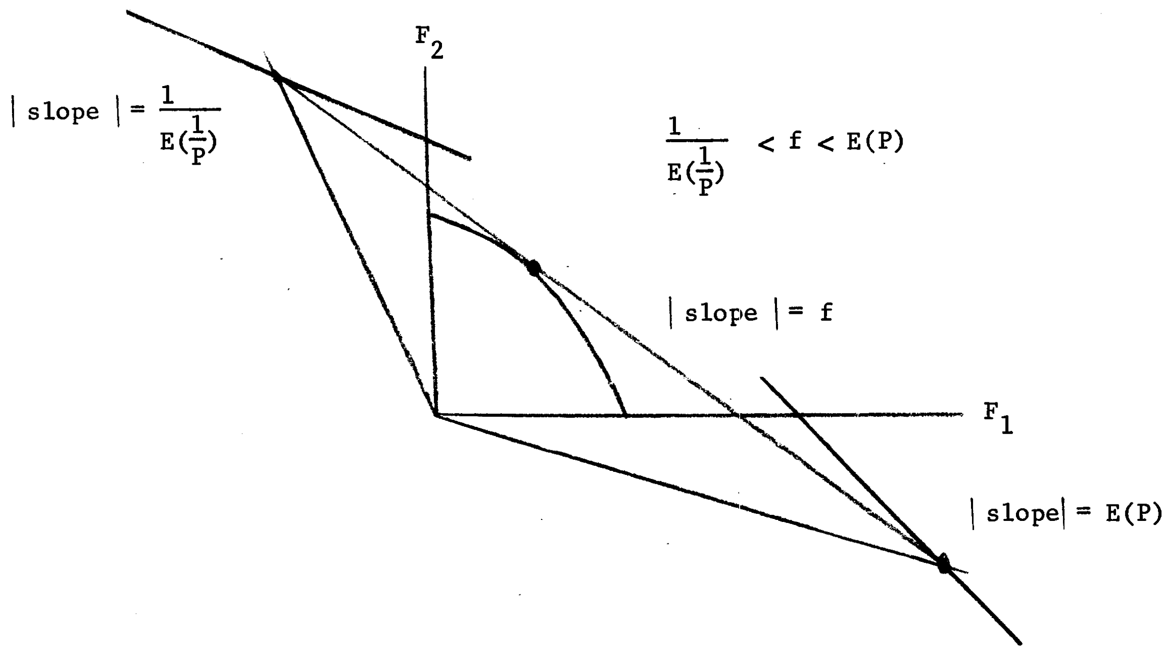
$$-\frac{dF_2}{dF_1} = \frac{EU_1}{EU_2} = \frac{EU_2P}{EU_2} = EP.$$

In the "Robinson Crusoe" world of Batra and Russell, the equilibrium forward price would in this case be driven equal to the constant slope of the linear contours: the expected world price of good 1.

Then, of course, maximization of expected profits in terms of good 2 or the forward value of output (in terms of either good) would result in the same production choice (S). Because, in this exceptional case, the two objectives happen to coincide, maximizing profits valued at expected spot--instead of forward--prices would then solve the planning problem.

10/ See p. 1041, 1047 of their article.

If two consumers have identical technologies and subjective probability distributions but differ as to which commodity has the constant marginal utility, (5) implies that the iso-maximized expected utility contours will be linear with slopes of E(P) and $\frac{1}{E(1/P)}$. The equilibrium forward price then will lie between these two different numbers as illustrated below.



That the expected terms of trade and the reciprocal of its expected reciprocal differ is an example of Jensen's inequality. The case portrayed above illustrates the "Siegel Paradox" which has troubled many analysts.[11/] The paradox is based on the mistaken assumption that risk neutral agents will always drive the forward price equal to the expected spot price. Without mentioning the resource constraints, Siegel pointed out that this cannot happen simultaneously to both the forward price and its reciprocal, since if

$$f = E(P),$$

Jensen's inequality implies that

$$\frac{1}{f} \neq E(\frac{1}{P}).$$

11/ QJE, May 1972, February 1975.

A consequence of Siegel's observation appeared to be the possibility of infinite gain in a market of risk neutral agents evaluating utility in terms of different commodities.

Once proper account is taken of the resource constraints of each agent, however, the paradox is resolved. In the case illustrated above, neither agent has a sufficient advantage in wealth to drive the forward price to the expected terms of trade, evaluated in terms of either commodity. Instead, the equilibrium forward price is pinned below the expected spot price in terms of either good. That is,

$$f < E(\dot{P})$$

and

$$\frac{1}{f} < E(\frac{1}{P}).$$

Appendix II

The Planning Problem and the Market Solution When Domestic Consumers Differ

The planning problem is to:

$$\max W(C_1^a, C_2^a, C_1^b, C_2^b)$$

$$\text{s.t. } \bar{w} = w(c_1^a, c_2^a, c_1^b, c_2^b)$$

$$P^a(C_1^a + c_1^a) + (C_2^a + c_2^a) = P^a(Q_1 + q_1) + (Q_2 + q_2)$$

$$P^b(C_1^b + c_1^b) + (C_2^b + c_2^b) = P^b(Q_1 + q_1) + (Q_2 + q_2)$$

$$Q_2 = \emptyset(Q_1)$$

$$q_2 = \phi(q_1)$$

The optimum values for the 12 variables (4 consumption and 2 production variables for each person) are determined by:

(1) $\dfrac{W_1^a}{W_2^a} = P^a$

(2) $\dfrac{W_1^b}{W_2^b} = P^b$

(3) $\dfrac{w_1^a}{w_2^a} = P^a$

(4) $\dfrac{w_1^b}{w_2^b} = P^b$

(5) $\dfrac{W_1^a}{W_1^b} = \dfrac{w_1^a}{w_1^b}$

(6) $\dfrac{W_1^a + W_1^b}{W_2^a + W_2^b} = \emptyset'$

(7) $\dfrac{w_1^a + w_1^b}{w_2^a + w_2^b} = \phi'$

and the five budget constraints.

(1) - (4) require the planner to equate the rate of commodity substitution (RCS) of each individual to the world terms of trade in each state. (5) indicates that, at the optimum, the rate of substitution across states for the first good must be the same for each individual. The same relationship will then apply to the other good. (6) and (7) indicate that each person equates his own rate of forward transformation ($\emptyset'$) to his rate of forward substitution[12/] $\left(\frac{W_1^a + W_1^b}{W_2^a + W_2^b}\right)$--an implication of the equations is that these rates will be equal for different individuals. Finally, the planning solution must a) conform to technological limitations; b) conform to conditions of balanced trade; and c) assign a specified level of utility to one individual.

Unless states outnumber goods, the allocation under free trade with prior exchange of forward claims has these characteristics. For example, in the two good-two state world, (1) - (4) result from the subsequent international trade and (6) and (7) from prior domestic trade in forward contracts. In addition, since each consumer equates the rate of forward substitution to the forward price, these rates are set equal to each other. But this implies that equation (5) holds. Finally, all the budget constraints are satisfied. Hence, the competitive solution solves the planning problem.

12/ If the Von-Neumann Morgenstern axioms hold for the first individual, his preferences over consumption lotteries (W) may be decomposed:

$$W(C_1^a, C_2^a, C_1^b, C_2^b) = \pi U(C_1^a, C_2^a) + (1-\pi)U(C_1^b, C_2^b).$$

It follows then that

$$\frac{W_1^a + W_1^b}{W_2^a + W_2^b} = \frac{EU_1}{EU_2} = RFS,$$

where the expectation is taken over the subjective probability distribution of that individual. A similar decomposition may be performed for the other person provided his preferences over consumption lotteries also satisfy the VN axioms.

Unless states outnumber goods, futures prices--together with the set of possible world prices--imply fixed rates at which claims on each good in different states can be exchanged. This leads all people to have the same marginal rates of substitution between states for each good. However, for the case of more states than goods, the introduction of mere futures markets will no longer imply fixed rates of exchange. Consequently, the marginal rates of substitution between states will differ among people and equalities similar to (5) will not hold. Hence, free trade will not be fully optimal without the introduction of additional markets (see footnote 1).