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Staving Off the Backstop: Dynamic Limit-Pricing with a Kinked Demand Curve

by

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I. Introduction

The successful manipulation of the world oil price by the OPEC cartel has stimulated a search among consuming nations for substitute energy sources and technologies. At the current real price of oil, few of these "backstops" can successfully compete. But should the oil price be raised sufficiently, other alternatives to oil might be profitably supplied. At prices where a new substitute could be profitably produced, demand for OPEC oil would be substantially more sensitive to price changes than in the current situation where little substitution is possible.

The question naturally arises as to how -- in the presence of an existing, high-cost, backstop technology -- OPEC should manipulate the price over time to maximize the discounted stream of profits generated by the extraction and sale of its oil reserves. Should it, for example, sell all of its oil at prices strictly below the "limit price" at which the backstop would enter? Or should it sell for a stretch of time at the limit price? In the latter case, should it satisfy the entire market demand at that price or should it co-exist with the backstop technology?

The solution to such pricing problems is, of course, of interest to OPEC, and presumably the cartel's advisors have studied them in depth. But also, if one can justifiably assume that OPEC acts in an economically rational way, the solution to such problems will enable importing nations to forecast more accurately both the time when a backstop will take over and the pricing strategy which OPEC will follow in the interim.

Several years ago, Robert Solow [7, p. 3-5] opened the discussion by analyzing the effect of a high-cost backstop on the price of a low-cost exhaustible resource sold on a competitive market. Solow considered two cases. In the first, the backstop technology was assumed to be inexhaustible -- his examples being solar energy and nuclear fusion. In the second case, the backstop itself was assumed exhaustible -- his example being synthetic crude oil produced from ("liquified") coal. In each case, Solow established that at first the low-cost, competitive extractors would supply the entire market while the price grew just enough to compensate them for not exchanging all of their underground stocks for interest-earning assets; once the limit price was reached, however, the backstop would enter and replace the low-cost extractors whose supplies would just, in equilibrium, be exhausted.

More recently, Stiglitz and Dasgupta [2, 8] have attempted to extend Solow's analysis to the case where the low-cost resource is owned by a monopolist. The backstop is assumed by them to be inexhaustible and the monopolist is assumed, in the tradition of von Stackelberg,[1] to take account of the supply responses of the backstop when formulating his optimal pricing strategy.

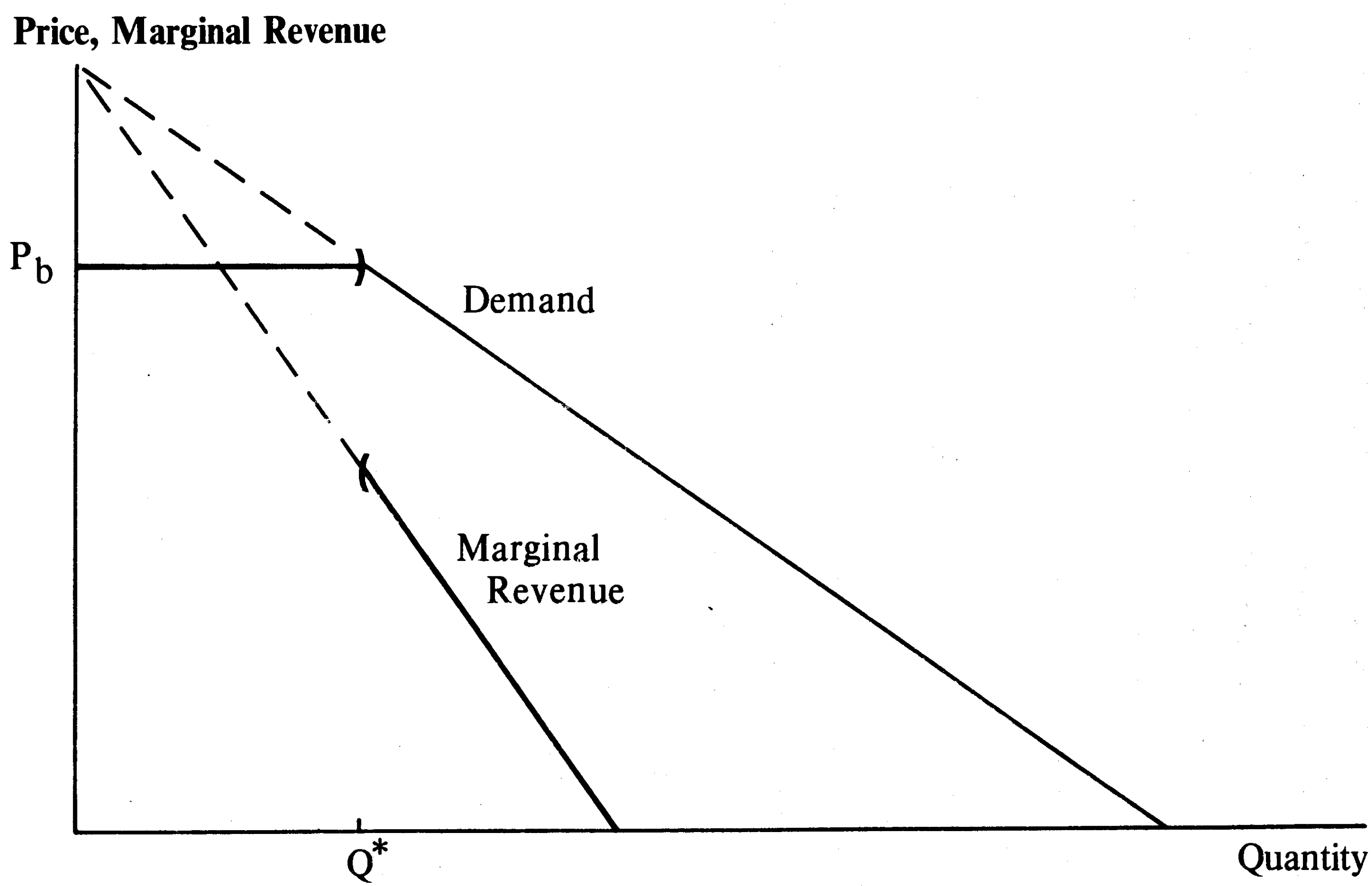
As I will show, there is a subtle but important error in the Stiglitz-Dasgupta analysis. Whether the backstop is inexhaustible or exhaustible, the optimal pricing strategy of the monopolist includes a phase where the monopolist charges the limit price but prevents entry by supplying the entire market himself. Neglect of this phase has resulted in underestimates of (a) the monopoly price which is currently optimal and (b) the time before the backstop will enter.

To simplify the exposition, the bulk of this paper concerns the case of an inexhaustible backstop. Section II introduces the monopolist's pricing problem and presents an intuitive solution to it. Section III verifies formally that the proposed solution is optimal. Section IV discusses the extension of the results to the case of an exhaustible backstop.

II. An Intuitive Analysis When the Backstop is Inexhaustible

For simplicity, assume that an inexhaustible backstop can supply the entire market if the oil price reaches P_b but that it cannot compete at lower prices. By selling oil faster than the rate (Q^*) at which energy users wish to consume when charged the price P_b, the monopolistic extractor can create excess supply and depress the price below P_b. In equilibrium, the demand would then be exactly what it would have been with no backstop. By restricting sales to a lower rate than Q^*, however, the monopolist can not raise the price above P_b as he could in the absence of a backstop; for the excess demand which would have developed will now be satisfied at the price P_b by the backstop suppliers. Figures 1 and 2 illustrate the situation. Figure 1 shows how the instantaneous demand and marginal revenue curves are altered by the presence of the backstop. Figure 2 shows how the associated total revenue function of the monopolist is affected.

If the original revenue function is continuous and strictly concave, the modified revenue function will be continuous and concave. Unlike the original function, however, the modified function will have a linear segment and a point where the derivative (marginal revenue) does not exist. The left-hand derivative at Q^* exceeds the right-hand derivative at that point. The total revenue lost by a unit reduction in sales below Q^* is equal to P_b while



Price, Marginal Revenue
P_b
Demand
Marginal
Revenue
Q^*
Quantity

Figure 1

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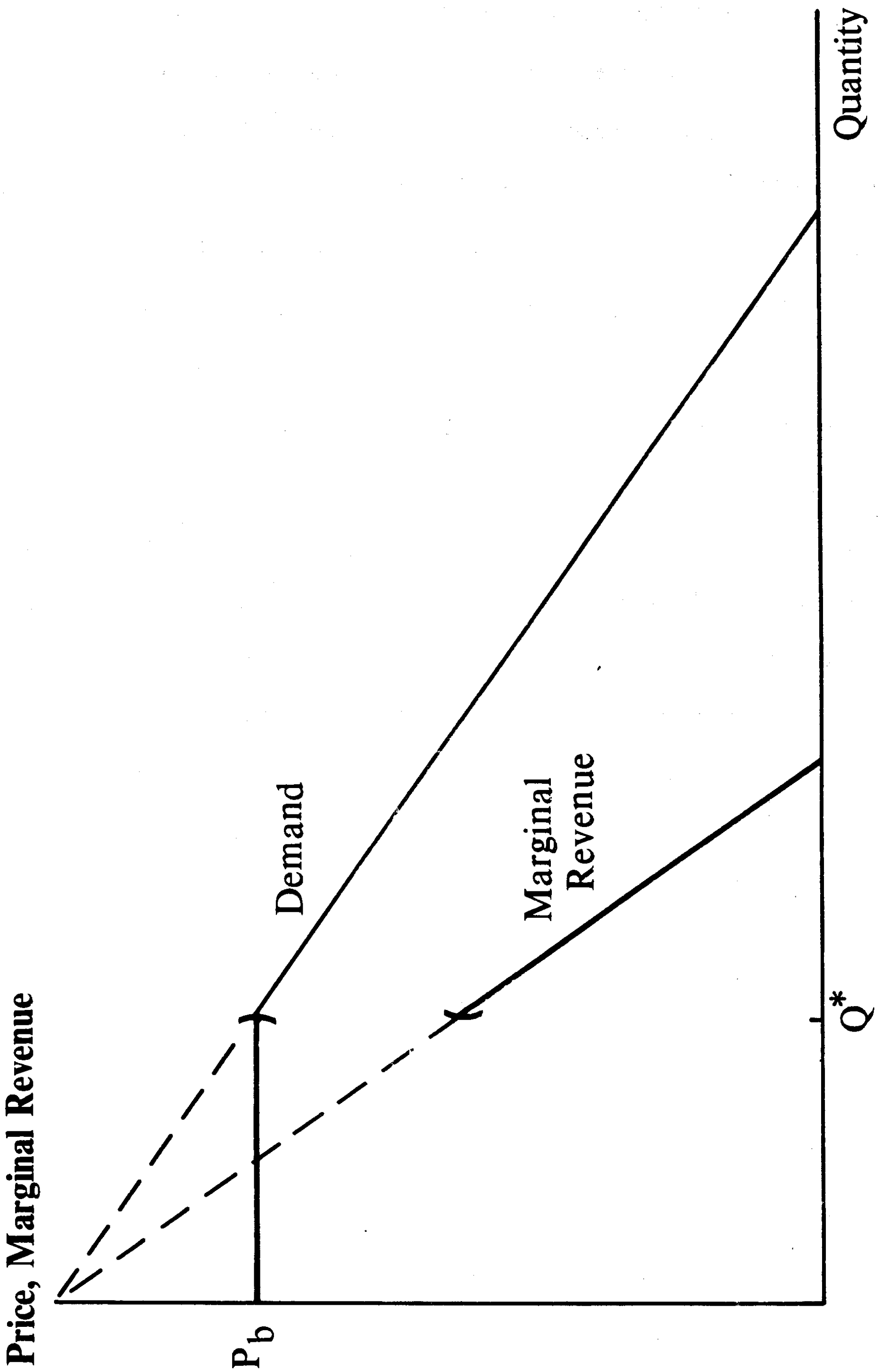
Price, Marginal Revenue
P_b
Demand
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Quantity

Figure 1

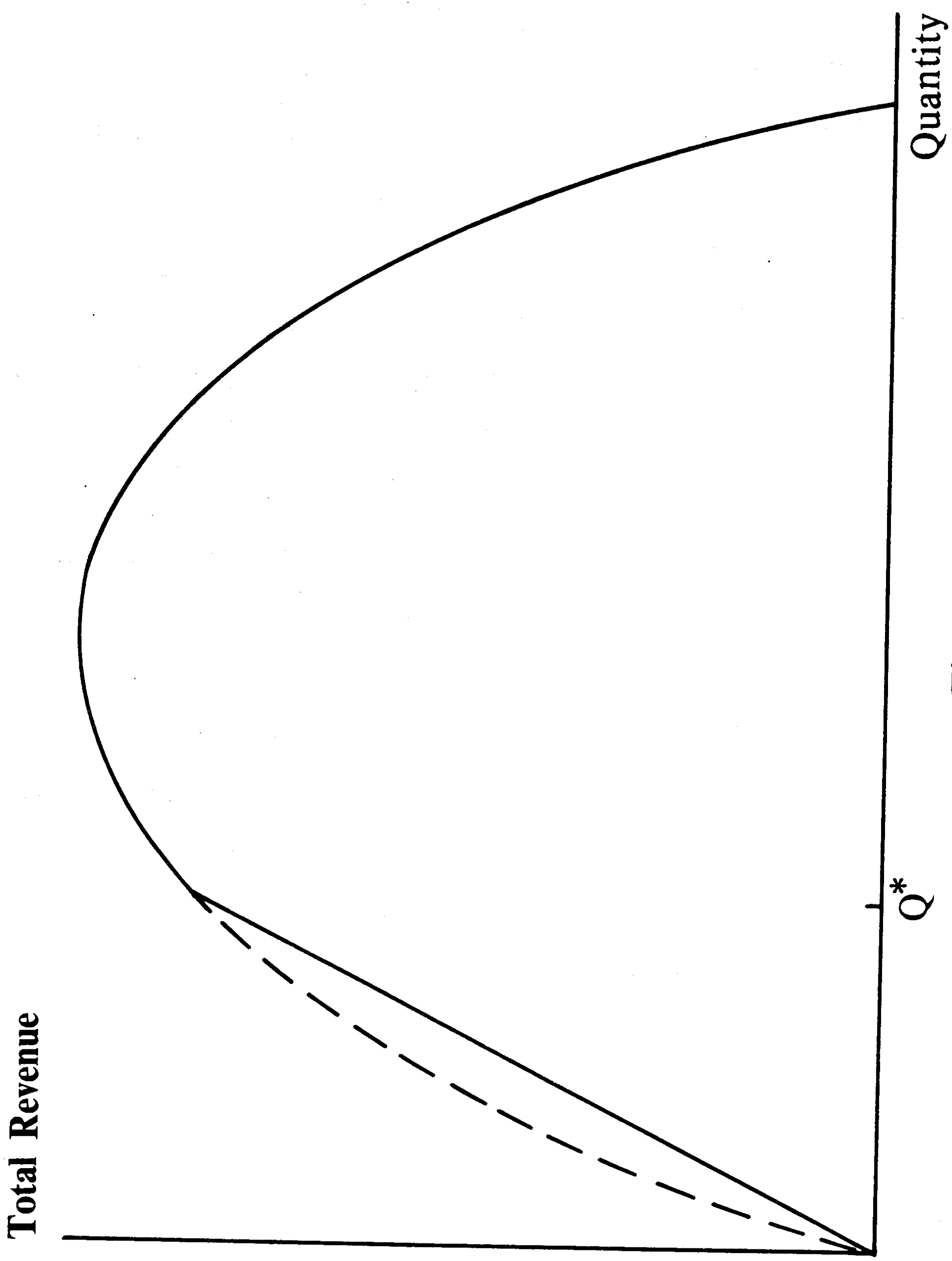


Figure 2

the total revenue gained by a unit increase in sales above Q* is smaller than P_b. The asymmetry arises because the increase in sales, unlike the reduction, alters the price slightly and this alteration affects the revenue earned on the inframarginal units (Q*).

If it is assumed that the monopolist has no costs of extraction, his profit function would be identical to the revenue function of Figure 2. Since the profit function would have a similar shape (a linear segment and a kink) if marginal costs are instead assumed to be a positive constant ($< P_b$), such a case would provide little to distinguish it from the case of zero costs. Hence we will not consider it further. If the monopolist's cost are assumed to be a smooth increasing, strictly-convex function of the rate of extraction, his profit function would no longer contain a linear segment; instead it would be strictly concave. Even in such a case, however, the profit function would still have a kink. Indeed such a kink would persist even if the supply response of the backstop were less extreme than we have assumed. Suppose, for example, that the backstop supply were zero for prices below P_b and increased gradually (rather than suddenly) at higher prices. Provided the backstop supply curve intersected the price axis at less than a vertical slope, the kink in the revenue function would still persist.[2/] Since -- in problems with a price-setting extractor and substitute technologies -- non-differentiable points[3/] on the profit function appear unavoidable, their consequence for optimal extraction policies should be understood.

In the standard monopoly problem studied by Hotelling [4], the extractor begins with an initial inventory $(\bar{I})$, faces a stationary, strictly-concave profit function $\left(\pi(Q)\right)$, and picks a non-negative sales path $\left(Q(t)\right)$ to maximize the integral of discounted profits:

$$\max_{\{Q\}} \int_0^\infty \pi(Q)e^{-rt}dt, \qquad \text{P1}$$

subject to constraints that:

$$Q \geq 0, \; S \geq 0,$$

$$\dot{S} = -Q,$$

$$S(0) = \bar{I}.$$

$S(t)$ denotes the stock remaining underground at time t.

The profit function is assumed in the standard case to be differentiable everywhere. Provided the initial slope of the profit function $\left(\pi'(0)\right)$ is finite, the optimal solution to this problem involves extracting the entire stock in finite time. Furthermore, for the program to be optimal, the discounted marginal profit must have a common extraction occurs. Any program violating these conditions can be dominated. For suppose, in a feasible program, that the discounted marginal profit from extracting positive amounts differed between any two moments.[4/] Then the program under consideration could be dominated by an alternative feasible program where slightly more was sold at the moment with the higher discounted marginal profit and slightly less at the moment with the lower discounted marginal profit. Such an arbitrage could, of course, also be conducted if no extraction occurred at the moment with the higher discounted marginal profit.

In the presence of a backstop technology, the problem faced by the monopolist is similar to P1. The difference is that the instantaneous profit function has a kink at Q* and may have a linear segment to the left of Q*. At Q* the marginal profit is undefined. Denote the modified problem as P2.[5/] Two cases might arise. If the profit function has a linear segment with a positive slope to the left of Q* but a negative slope for <u>all</u> extraction rates to the right of Q*, it is intuitively plausible that the monopolist should simply extract at the rate Q* for $\bar{I}$/Q* years and sell it at P_b.[6/] If, however, marginal profit is positive for some extraction rates above Q*, it is less clear what extraction path is optimal.

It has been suggested[7/] that the monopolist should set the value of marginal profit at a specified[8/] initial level and should reduce extraction over time so as to maintain the same discounted value of marginal profit until the moment extraction falls to Q* and the price rises to P_b -- at which point the monopolist should immediately bow out to the competition.

This proposed program is not optimal for P2, however, since it can be dominated. It involves a moment with positive extraction followed <u>immediately</u> by a moment of zero extraction but higher marginal profit. The lower marginal profit occurs the moment before extraction declines to Q* and the higher marginal profit occurs the moment after extraction falls below Q*.

If the proposed program is not optimal, what is?[9/] No program where extraction exceeds Q* at one point and falls short of Q* moments later can be optimal since the marginal profit would jump up between the two moments, creating the opportunity for profitable arbitrage. The <u>optimal</u> program must avoid this situation by separating these two moments in time. If extraction is ever to exceed Q*, it should decline to Q* <u>linger</u> there for an interval of time (θ) just long enough so that the left-hand derivative at Q* $\left(\pi'(Q_-^*)\right)$ discounted back θ periods is equal to the right-hand derivative at Q* $\left(\pi'(Q_+^*)\right)$, and then decline below Q*.

The optimal length to linger at Q* can be computed from the parameters of the problem. In the absence of extraction costs, $\pi'(Q_-^*) = P_b$ while $\pi'(Q_+^*) = P_b\{1 - \frac{1}{|\eta(Q^*)|}\}$, where η is the price elasticity of consumer demand. Hence, provided the extractor has sufficient reserves ($\bar{I} \geq \theta Q^*$), he should linger at P_b while supplying the entire market for θ years, where θ is defined by the following equation:

$$P_b e^{-r\theta} = P_b\{1 - \frac{1}{|\eta(Q^*)|}\} \quad \underline{10/}$$

$$\Rightarrow \theta = -\frac{1}{r}\ln(1 - \frac{1}{|\eta(Q^*)|}).$$

In the absence of sufficient reserves, the optimal strategy is to extract at the rate Q* for as long as possible and then shut down. The optimal length to linger is, therefore, min (θ, $\bar{I}$/Q*).

To illustrate, suppose that the elasticity of demand for oil at the limit price were 2.0 and the real rate of interest were .05 per year. Then, provided the monopolist has sufficient reserves, he should gradually raise the price to P_b, supply the entire market at that price for approximately fourteen years, and then shut down. If a lower interest rate or elasticity of demand were assumed, the optimal length to linger would be longer.

Suppose, instead, that the cost of extraction were a positive smooth, increasing, strictly-convex function of Q. In that case, the monopolist's profit function would still be kinked at Q* but it would instead be strictly concave. Since the marginal cost $\left(C'(Q^*)\right)$ is continuous at Q*, both the left and the right-hand derivative of the revenue function at that point would be reduced by the same amount. However, as a percentage of the right-hand derivative, the gap between the left and right-hand derivatives at the kink would be larger than in the case without extraction costs. How would these modifications affect the optimal extraction policy? Once again, extraction should decline gradually to Q* so as to maintain the discounted value of marginal profit at some specified level; but then, extraction should linger at Q* longer[11/] than in the previous case ($\hat{\theta} > \theta$) before resuming its decline. Furthermore, the abrupt decline from Q* to zero -- which was optimal when the profit function had the linear segment -- is no longer optimal since such a reduction would result in a jump in marginal profit and the opportunity for profitable arbitrage. Instead, after lingering for $\hat{\theta}$ moments, the monopolist should begin again to reduce the rate of extraction so as to maintain the same level of discounted marginal profit until reserves are depleted. In this last segment of the program, the extraction and the backstop co-exist, each supplying part of the market.

In neither of the cases analyzed is it optimal to cease extraction the moment Q* is reached. Since the monopolist needs some reserves for the phase which begins when Q* is reached, less is available for the earlier phase of more rapid extraction. Consequently, the neglect of the final phase in each case has led to underestimates of the current profit-maximizing monopoly price. Furthermore, when the cost function is linear (constant or zero marginal costs), the neglect of the final phase has led to underestimates of the length of time before the backstop begins producing.[12/] These points are illustrated in Figure 3. The arrowhead at the end of each path indicates the point when the monopolist exhausts his supplies and the backstop replaces him.

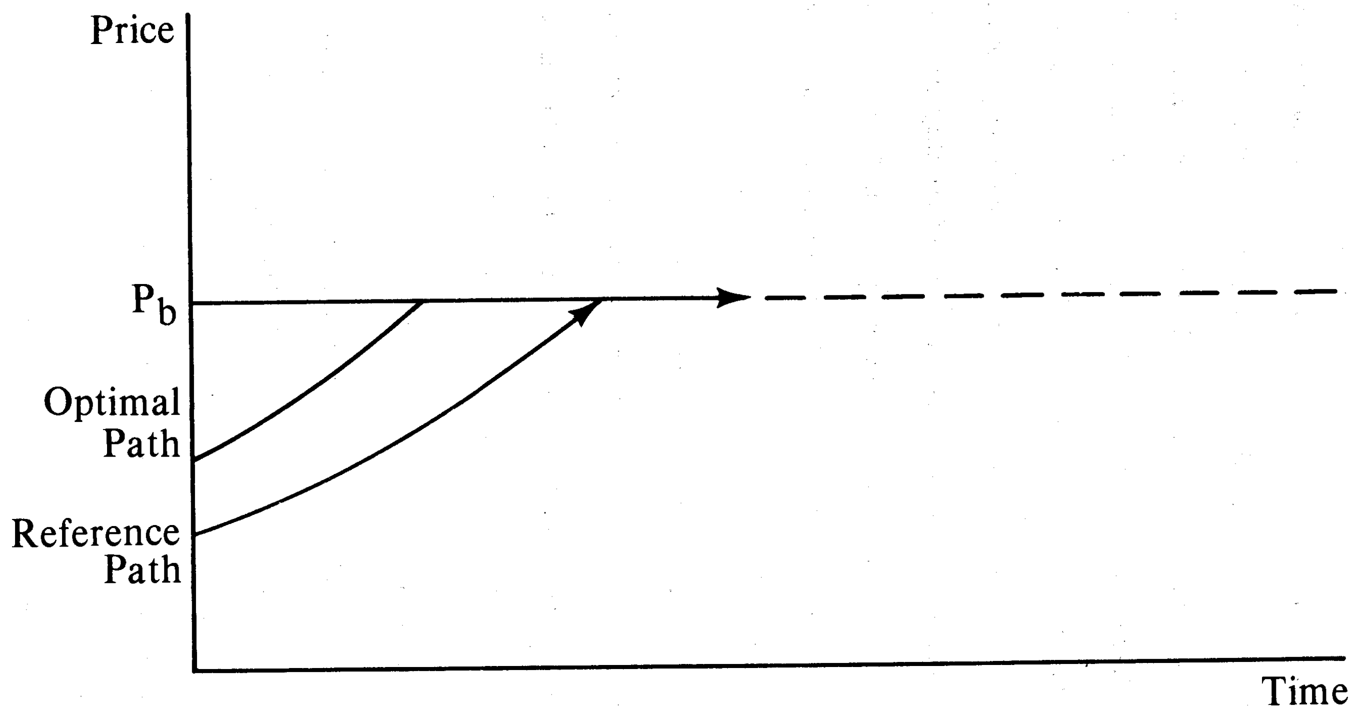


Figure 3

To understand why the backstop will not enter until later than was previous supposed optimal (the "reference path" of Figure 3) consider the following. Since the monopolist should set the initial price higher than was previously thought optimal, the subsequent price path should also be higher. The optimal price path will, therefore, reach the limit price and remain there for a while before the reference path catches up. At that time, it was thought that the monopolist would be out of reserves and that the backstop would, therefore, replace him. But since the monopolist will consistently have charged higher prices, less will have been demanded and he will in fact have some reserves left to extract. Hence, he can (and will) continue to stave off the backstop until his supplies are exhausted.

III. Formal Verification of the Intuitive Analysis

To demonstrate formally that the strategies described above are indeed optimal, we verify that they satisfy conditions known to be sufficient for maximizing a functional. Define the Hamiltonian as

$$H = e^{-rt}\{\pi(Q) + \lambda(t)\,\dot{S}\} = e^{-rt}\{\pi(Q) - \lambda Q\},$$

where S is the state variable, λ the co-state variable, and Q the control. Any program optimal for P2 satisfies the following conditions for some continuous function $\lambda(t)$:

(1) $Q(\geq 0)$ maximizes the Hamiltonian at each instant (except for a set of measure zero).

(2) $-\dfrac{\partial H}{\partial S} = \dfrac{\partial}{\partial t}\left[\lambda e^{-rt}\right] \Rightarrow \dot{\lambda} = r\lambda, \quad (\lambda \geq 0).$

(3) $\dot{S} = -Q, \qquad (S \geq 0)$

(4) $S(0) = \bar{I}.$

Furthermore, it has been shown[13/] that if the maximized Hamiltonian is concave in S, <u>any</u> program satisfying conditions (1) - (4) above and the following transversality condition must be optimal

$$(5)\quad \lim_{t \to \infty} \lambda(t)e^{-rt} \geq 0 \text{ and } \lim_{t \to \infty} \lambda(t)e^{-rt} S(t) = 0.$$

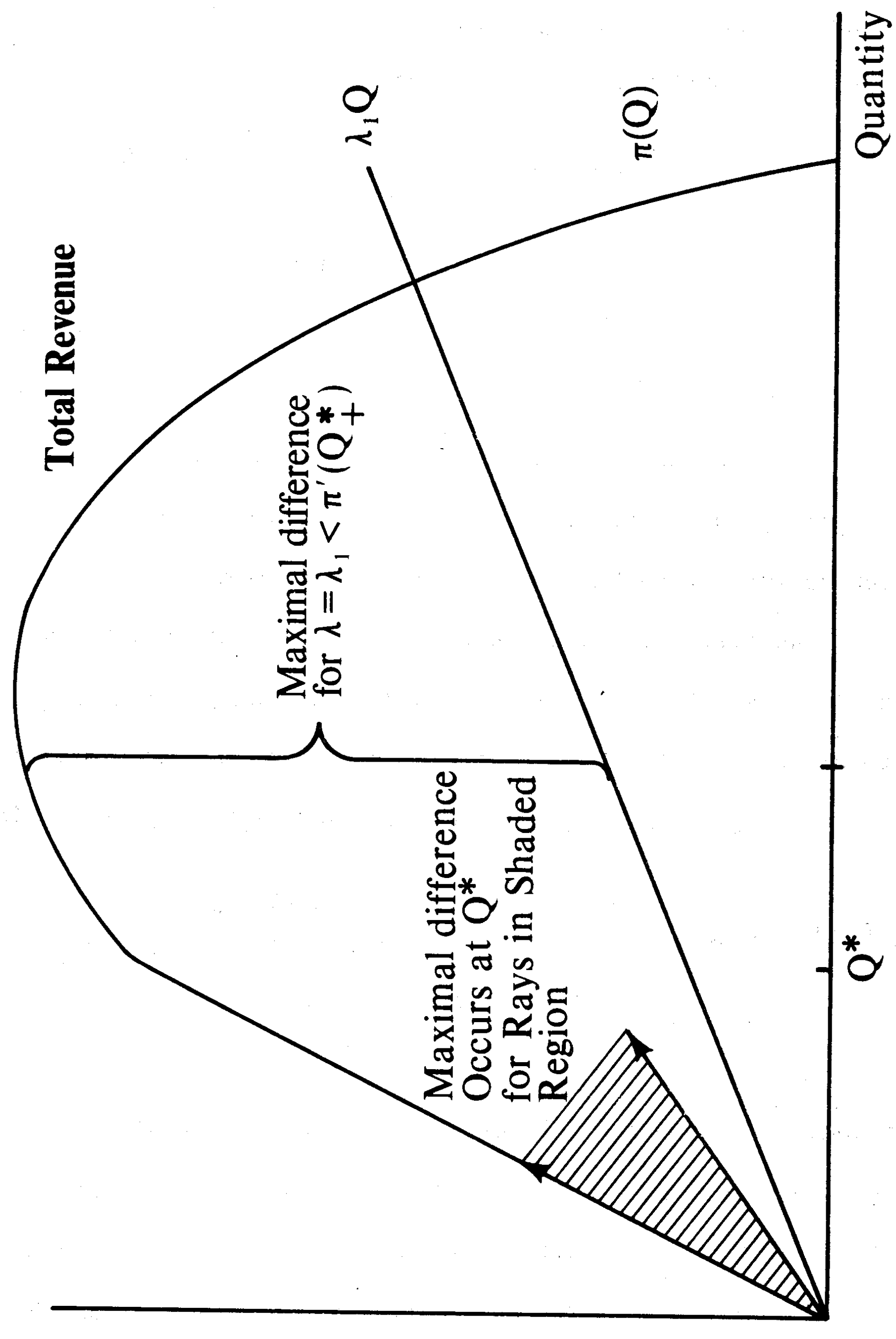
Since, in our case, the maximized Hamiltonian is independent of S, it is (trivially) concave in S. Hence, to prove that a program is optimal, it is sufficient to establish that it satisfies conditions (1) - (5).

Even though the Hamiltonian is not differentiable everywhere with respect to Q, condition (1) can still be applied. That is, for each λ, the Q which maximizes the Hamiltonian can be determined.

We first consider the case with zero extraction costs.[14/] In Figure 4 below, the functions $\pi(Q)$ and λQ -- whose difference at each instant is proportional to the Hamiltonian -- are graphed. The maximizing extraction rate can be expressed analytically. Assuming that $\pi'(Q_+^*) > 0$,[15/] we obtain:

$$(6)\quad Q(\lambda) = \begin{cases} X, \text{ where } \pi'(X) = \lambda, & \text{for } \pi'(Q_+^*) > \lambda \geq 0 \\ Q^* & \text{for } \pi'(Q_-^*) > \lambda \geq \pi'(Q_+^*) \\ Y, \text{ where } 0 \leq Y \leq Q^*, & \text{for } \lambda = \pi'(Q_-^*) \\ 0 & \text{for } \lambda > \pi'(Q_-^*) \end{cases}$$

Conditions (1) - (4) and (6) determine future values of λ and S for any initial assignment of λ. If we can find an assignment which also leads to satisfaction of the transversality condition, the associated program must be optimal. The search is facilitated by constructing the phase portrait sketched in Figure 5.



Total Revenue
$\lambda_1 Q$
$\pi(Q)$
Quantity
Maximal difference for $\lambda = \lambda_1 < \pi'(Q^*_+)$
Maximal difference Occurs at Q^* for Rays in Shaded Region
Q^*

Figure 4

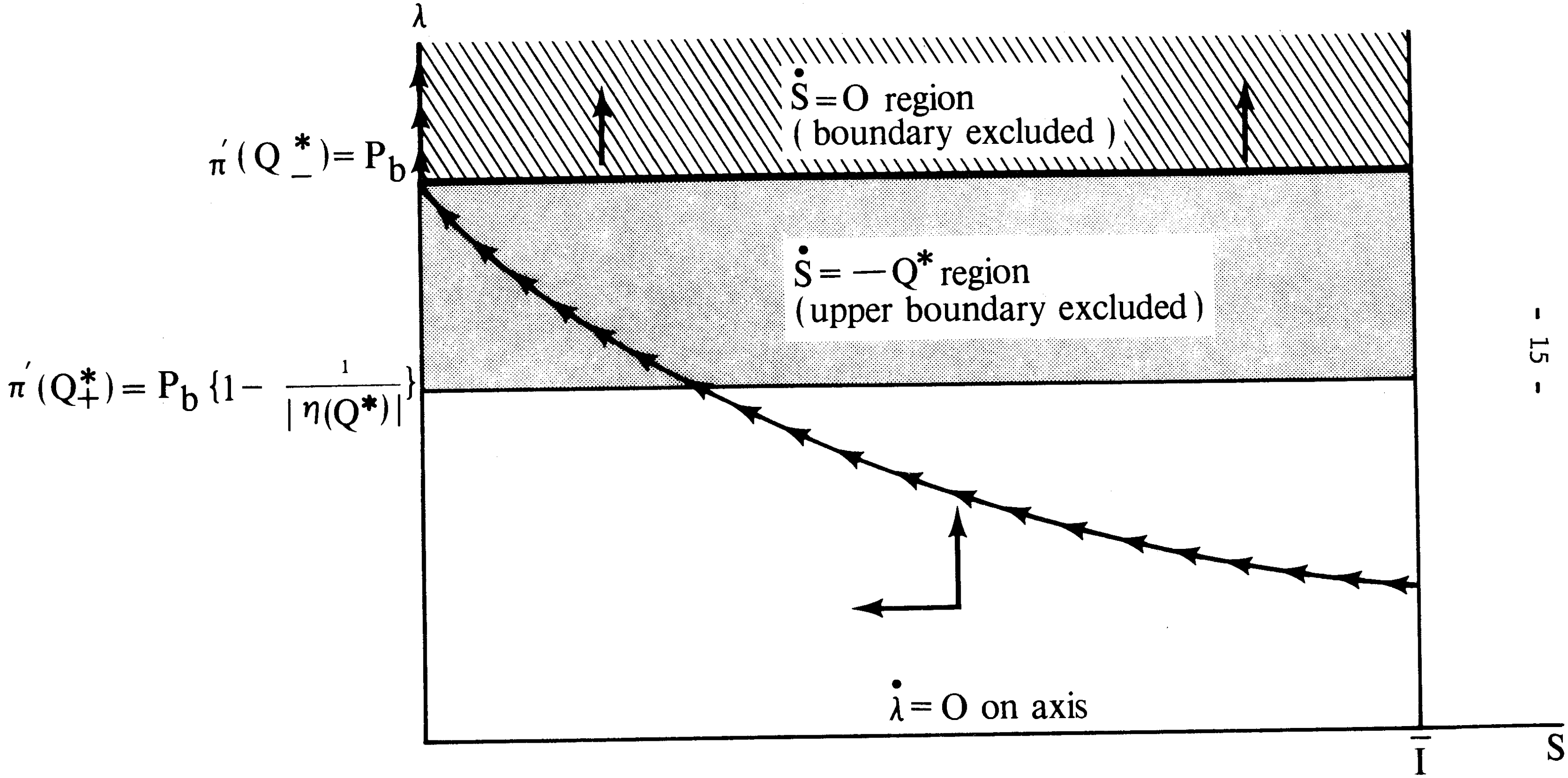

λ
$\pi'(Q_-^*) = P_b$
$\pi'(Q_+^*) = P_b\{1 - \frac{1}{|\eta(Q^*)|}\}$
$\dot{S} = O$ region
(boundary excluded)
$\dot{S} = -Q^*$ region
(upper boundary excluded)
$\dot{\lambda} = O$ on axis
$\bar{I}$
S

Figure 5

Equation (2) implies that there will be upward motion in the portrait above the horizontal axis but none on the axis. As for leftward motion, conditions (3) and (6) imply $\dot{S} = 0$ for the region where $\lambda > P_b\left(= \pi'(Q_-^*)\right)$ and that $\dot{S} = - Q^*$ for the region where $P_b > \lambda \geq \pi'(Q_+^*)$. Since $\dot{S}$ can take on any of a set of values for $\lambda = P_b$, the horizontal motion is not well defined at the boundary of the two regions.

The initial value of the state variable is given by equation (4). Our only choice is in assigning an initial value to the co-state variable. What assignment will satisfy the transverality conditions (equation (5)) as well as the remaining conditions, which we have incorporated in the phase portrait? Since $\dot{\lambda} = r\lambda$, $\lambda(t) = \lambda(0)e^{rt}$; therefore, the transversality condition can be rewritten as $\lim_{t \to \infty} \lambda(0)\ S(t) = \lambda(0) \lim_{t \to \infty} S(t)$. If we intialize the co-state variable at zero, conditions (2) and (6) require that extraction continue forever at a constant (positive) rate. However, such a program violates condition (3) which requires the stock remaining to be non-negative. Therefore, if the transversality condition is to be satisfied, $\lim_{t \to \infty} S(t) = 0$. However, for a trajectory to have this characteristic, it must pass into the $\dot{S} = 0$ region precisely when $S = 0$. Such a trajectory in fact exists and, since it satisfies each condition of the sufficiency theorem, it must be optimal.

We can determine the optimal trajectory by beginning at the point $(0, P_b)$ and working backwards until the stock $\bar{I}$ is reached. If $\bar{I}$ is small, the value of the co-state variable when $S = \bar{I}$ may still exceed $\pi'(Q_+^*)$. This will be the case if $\bar{I} \leq \theta Q^*$. For larger initial stocks (the case illustrated in Figure 5), the initial value of the co-state variable for the trajectory passing through the point $(0, P_b)$ will be less than $\pi'(Q_+^*)$.

In either case, however, we have constructed an extraction program satisfying conditions (1) - (5). In the former case $\left(\lambda(0) \geq \pi'(Q_+^*)\right)$, the optimal policy is to extract at the rate Q* for $\bar{I}/Q^*$ moments, extract at any rate up to Q* for the next instant, and then shut down. In the latter case $\left(\lambda(0) < \pi'(Q_+^*)\right)$, the optimal policy is to extract at a gradually declining rate which initially exceeds Q*; when Q* is reached, extraction lingers at that rate for θ years, becomes momentarily indeterminate,[16] and then ceases altogether.

Few changes need be made to analyze the optimal strategy when cost is a smooth, increasing, strictly-convex function of the rate of extraction. The strictly-concave profit function $(\hat{\pi} = \pi - c)$ replaces the concave revenue (π) function in equations (1) - (5). The rate of extraction $(\hat{Q})$ which maximizes the new Hamiltonian $(\hat{H})$ for given values of the co-state variable $(\hat{\lambda})$ has the following form:

$$(\hat{6}) \qquad \hat{Q}(\hat{\lambda}) = \begin{cases} X, \text{ where } \hat{\pi}'(X) = \hat{\lambda}, \text{ for } \hat{\pi}'(Q_+^*) > \hat{\lambda} \geq 0 \\ Q^*, \text{ for } \hat{\pi}(Q_-^*) > \hat{\lambda} \geq \hat{\pi}'(Q_+^*) \\ Y, \text{ where } \hat{\pi}'(Y) = \hat{\lambda}, \text{ for } \hat{\pi}'(0) > \hat{\lambda} \geq \hat{\pi}'(Q_-^*) \\ 0 \text{ for } \hat{\lambda} \geq \hat{\pi}'(0). \end{cases}$$

Since the profit function is strictly concave and kinked at Q*, $\hat{\pi}'(Q_+^*) < \hat{\pi}'\ (Q_-^*) < \hat{\pi}'(0)$. Depending on the nature of the cost function, various cases might arise. For example, if $P_b < C'(0)$ $\left(\hat{\pi}'(0) < 0\right)$, the monopolist's optimal strategy would be never to extract. If, instead, $C'(0) < P_b < C'(Q^*)$, the monopolist should always extract less than the amount demanded at the backstop price. The phase portrait of Figure 6 is drawn on the assumption that, in the (right-hand) neighborhood of Q*, marginal extraction costs are smaller than marginal revenue $\left(\hat{\pi}'(Q_+^*) > 0\right)$. In that case, a monopolist with an infinite stock to extract would find it optimal to produce at a rate greater than Q*.

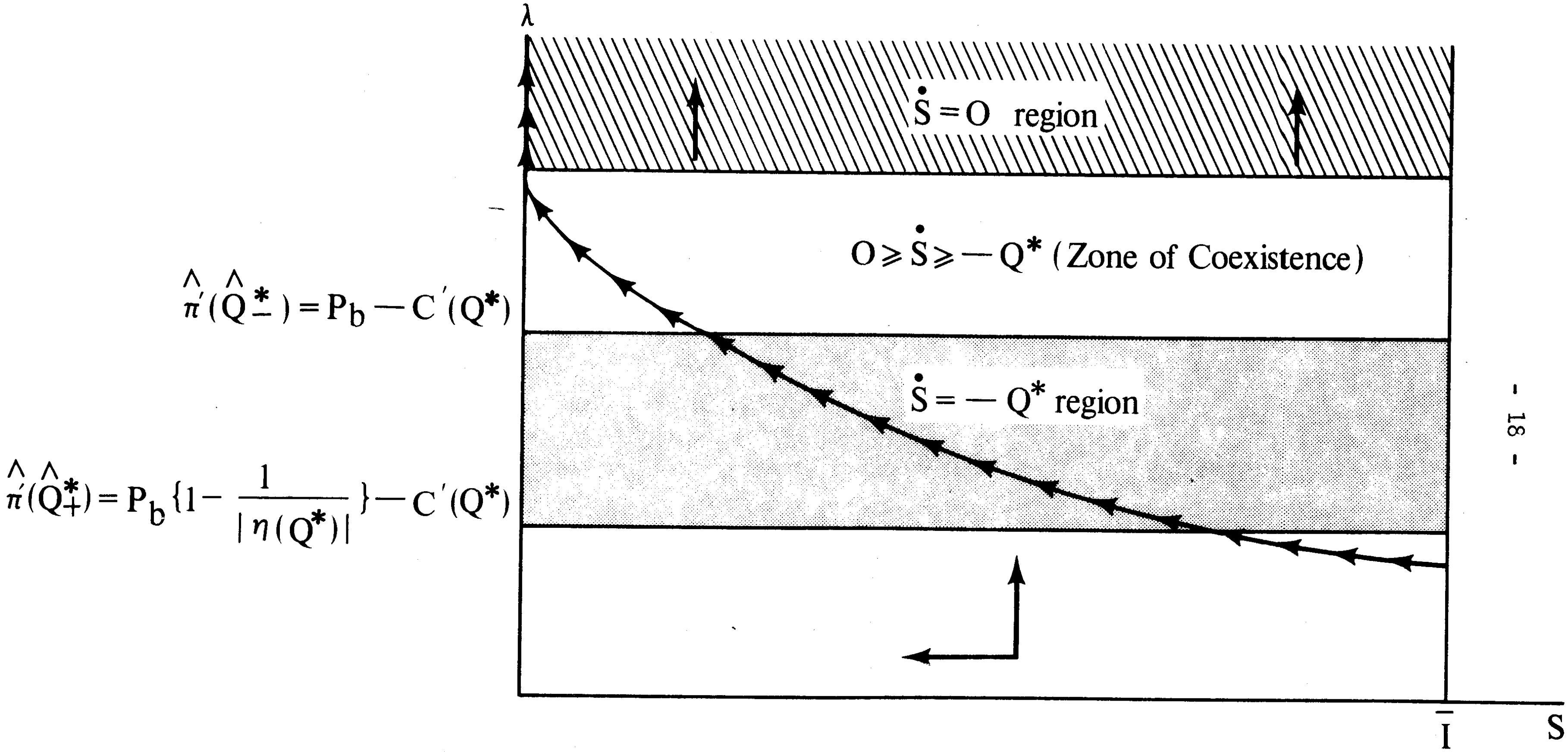

λ
$\hat{\pi}'(\hat{Q}^*_-) = P_b - C'(Q^*)$
$\hat{\pi}'(\hat{Q}^*_+) = P_b\{1 - \frac{1}{|\eta(Q^*)|}\} - C'(Q^*)$
$\dot{S} = O$ region
$O \geqslant \dot{S} \geqslant -Q^*$ (Zone of Coexistence)
$\dot{S} = -Q^*$ region
$\bar{I}$
S

Figure 6

In the case portrayed, the monopolist's initial stock ($\bar{I}$) is assumed to be large ($\bar{I} > \hat{\theta}Q^*$). The optimal trajectory passes through four regions. In the first, the monopolist raises the price over time until the backstop price is reached. In the second, he supplies the entire market at the backstop price for $\hat{\theta}$ periods. In the third, he gradually reduces his output and allows the backstop to take over. In the fourth, he shuts down. The coexistence phase occurs during the time when $\hat{\pi}'(0) > \hat{\lambda} > \hat{\pi}'(Q^*_{-})$.

IV. Extension to the Case of an Exhaustible Backstop

If the resource supplied by the competitive sector is itself exhaustible, a complete characterization of the optimal path is more difficult. For, as Stiglitz [8, p. 659] has shown, the monopolist may then find it optimal to wait for the competitors to exhaust their supplies before beginning to extract his. We can hardly call the competitive sector a "backstop" in such a case; "forestop" might be a more appropriate label. To be a "backstop," the competitive sector must, in equilibrium, be the last to sell its resource. In the analysis below, we will restrict ourselves to "backstop" equilibria; in addition, we will assume that both resources are extracted at constant marginal costs. The reader can readily verify that cases with these characteristics in fact exist by considering the optimal strategy of a monopolist with zero extraction costs, facing a demand curve of constant elasticity. It can be shown [17/] that if the competitive resource is costly to extract, the competitors will wait for the monopolist to exhaust his resource before beginning to extract theirs.

If the competitors find it optimal to go last, they will enter at a real price ($\bar{P}$) which enables them to sell their entire stock ($\bar{I}^c$) over time along a price path which rises enough to keep their discounted marginal profit constant. $\bar{P}$, therefore, must satisfy the following equation:

$$\bar{I}^c = \int_0^\infty Q^d\left\{(\bar{P} - k)e^{-rx} + k\right\}dx,$$

where k is the constant marginal cost of the competitive extractors.

For the competitors to refrain from selling before $\bar{P}$ is reached, their discounted marginal profit must not then exceed its common value during the latter phase when they do sell. This limits the price the monopolist can charge in the earlier phase. For example, X moments before the entry of the competitors at $\bar{P}$, the monopolist cannot charge more than $(\bar{P} - k)e^{-rx} + k$. The ceiling can be viewed as a limit price for the case where the competitive backstop is exhaustible. In contrast to the inexhaustible case, this limit price rises over time. But as long as the marginal cost of the competitive extractors is positive, the limit price must rise by less than the rate of interest.

In what follows we will prove -- for an equilibrium where the competitive extractors go last -- that the monopolist will always find it optimal to linger for a while at this dynamic limit price while supplying the entire market. This contrasts with the proposition of Solow [7] that if the low-cost resource is competitively owned, the competitive extractors will shut down the moment the limit price is reached. The result of this section extends the conclusions of the previous sections by showing that limit-pricing remains the optimal strategy for a monopolist, even when the competitive backstop is exhaustible.

The demonstration utilizes Figure 7. In Figure 7, various paths terminating at $\bar{P}$ are drawn. Path I is the price path which would result if the

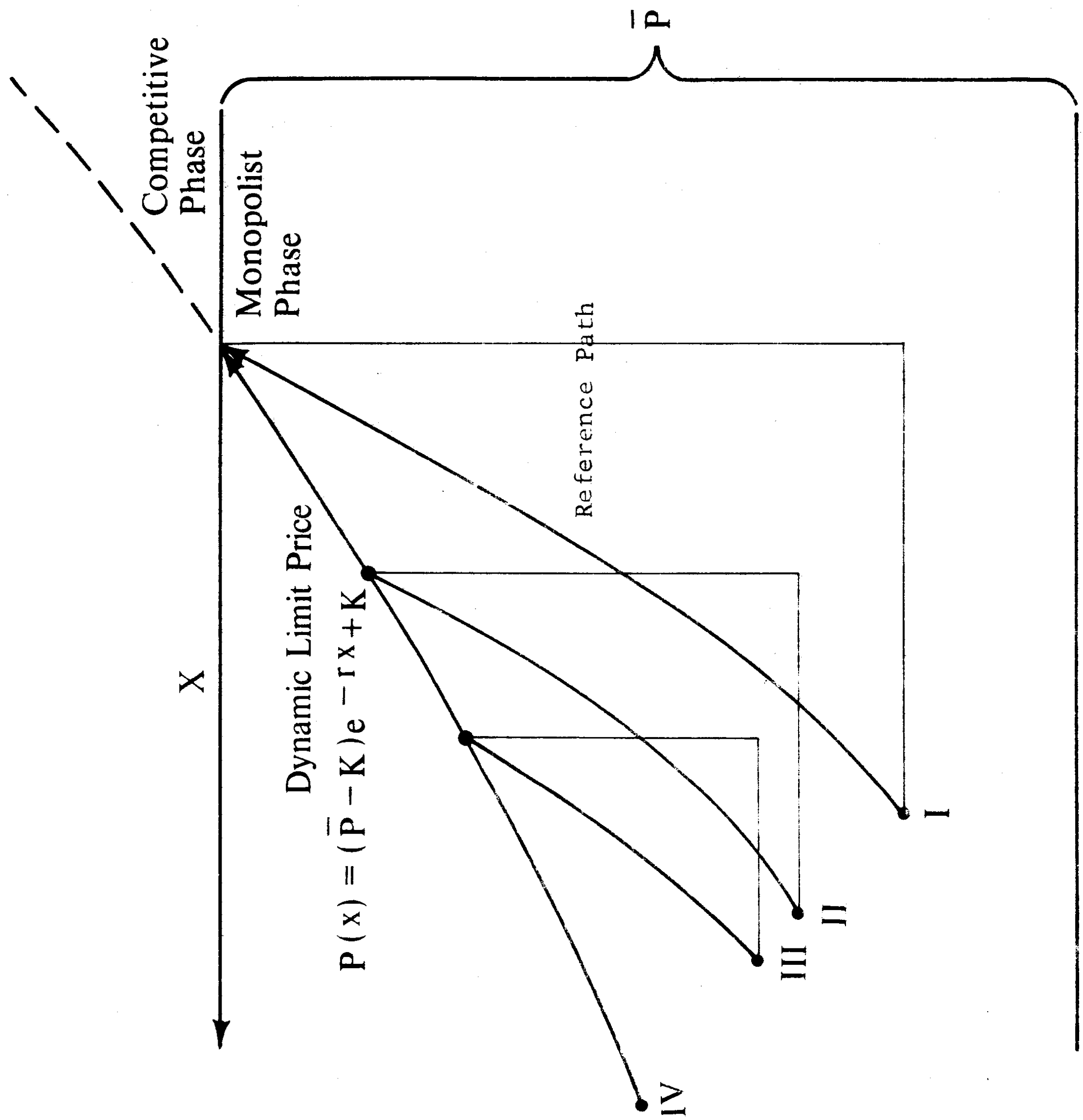

$\bar{P}$
Competitive Phase
Monopolist Phase
Reference Path
X
Dynamic Limit Price
$P(x) = (\bar{P} - K)e^{-rx} + K$
I
II
III
IV

Figure 7

monopolist equated discounted marginal revenue at every moment before $\overline{P}$ is reached. (In order for the proof to be nontrivial, path I is assumed to lie strictly below path IV to the left of their intersection at $\overline{P}$.) Path I is designated the "reference path."

To show that the optimal price path must coincide for at least some interval with the dynamic limit price, we show that all other price paths can be dominated. Suppose, for example, that the optimal price path were among the set of paths lying strictly below the dynamic limit price. This set contains not only the reference path but also paths along which the discounted marginal profit fluctuates. If the optimal path is anywhere in this set, it must be the reference path since all the other paths can -- by a familiar argument -- be dominated.

However, we will show that the reference path itself can be dominated. Consider the family of segmented price paths which result if the monopolist equates discounted marginal revenue at every moment until a particular point on the limit price path is reached, but which then coincide with the limit price path. Each path is assumed to extend back from $\overline{P}$ far enough so that cumulative demand along it exactly equals the monopolist's initial stock ($\overline{I}$).

This implies that the initial price on any upper path must exceed the initial price on any lower path (as is illustrated in Figure 7). For, suppose the contrary. Suppose some upper path extended to the left so far that its left-hand endpoint lay below the left-hand endpoint of some lower path. Then the upper path could be shifted laterally to the right so that the shifted path partially coincided with the lower path, extended beyond it to the left, and lay beneath it to the right before both paths reached $\bar{P}$. But it is then clear that cumulative demand along the shifted path exceeds cumulative demand along the lower path -- contradicting the assumption on which Figure 7 is based.

Each member of the family of segmented price paths in Figure 7 can be characterized by a single number -- the cumulative amount of reserves which the monopolist extracts after the limit price is reached. Path I can be seen as a limiting case where the cumulative amount extracted in the second segment of the price path is zero.

Denote the stock allocated to the second segment as S_2. For any allocation of total stock between the two segments, the monopolist's discounted revenue along the path may be written as:

$$V(S_2) = \{R_1(\bar{I} - S_2) + R_2(S_2)\}e^{-rT(S_2)}$$

where $T(S_2)$ is the length of the first segment of the program,

$R_1(X)$ is the revenue (capatilized to the end of the first segment) of extracting X units of stock in the first segment,

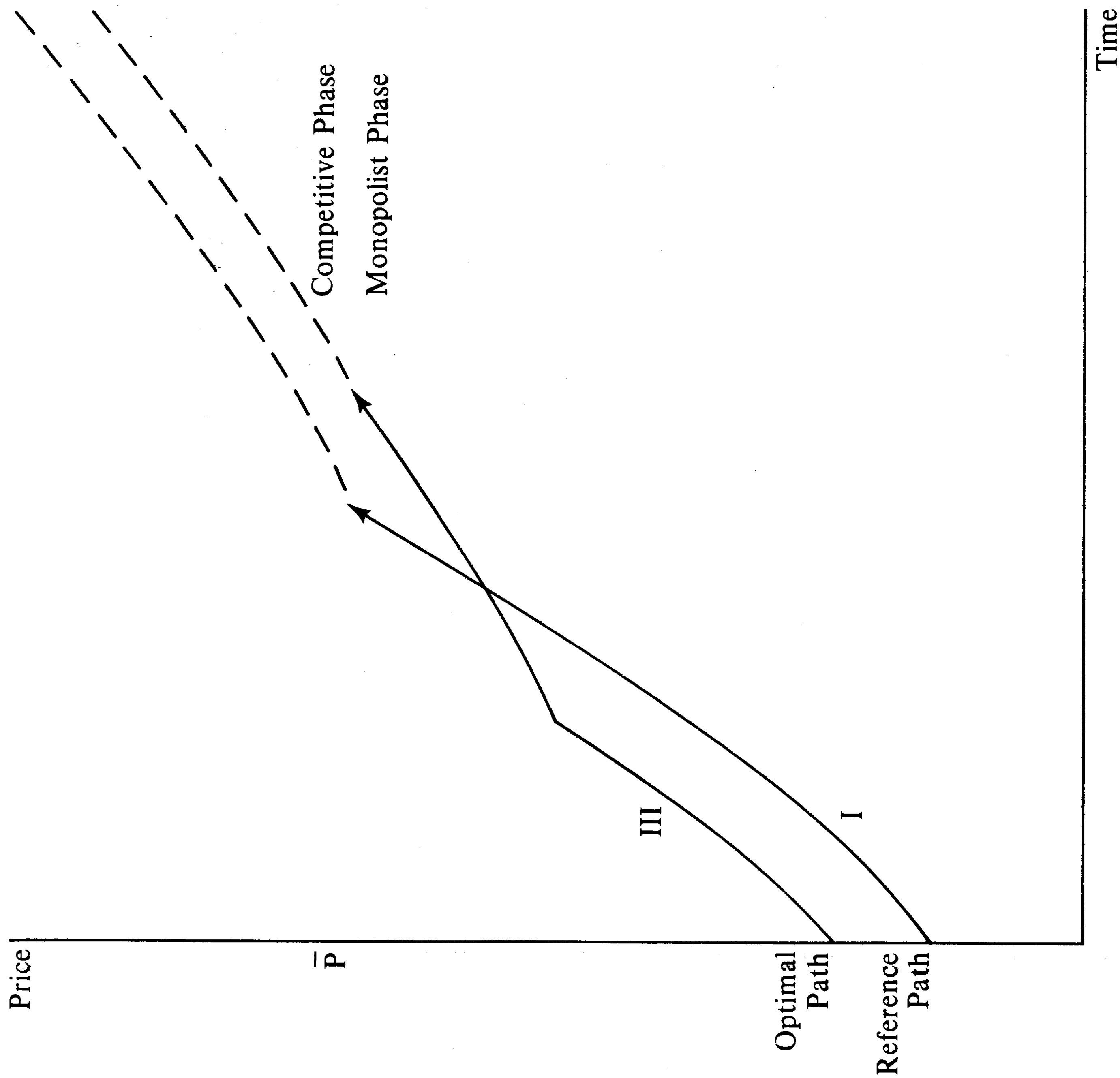
and $R_2(X)$ is the revenue (discounted to the beginning of the second segment) of extracting X units of stock in the second segment.

To prove that the reference path can be dominated, we simply verify that $V'(0) > 0$.[18/] Since an optimal strategy exists but is not contained in the set of paths which lie strictly below the limit price to the left of $\bar{P}$, the optimal strategy must include a phase where the monopolist charges the (dynamic) limit price but prevents the backstop from entering by supplying the entire market himself.

Neglect of the lingering phase will once again result in <u>under-estimates</u> of both the optimal current monopoly price and the date when the backstop will enter. Since the left-hand endpoint of each upper path must be higher than the left-hand endpoint of any lower path, the initial price along the optimal path must exceed the initial price along the reference path. Furthermore, since each path with a lingering phase lies above the reference path, the flow of demand will be less -- x moments before $\bar{P}$ is reached -- than it would be at the same point on the reference path. Therefore, in order for cumulative demand along these two paths to be equal, the optimal path must stretch back from $\bar{P}$ further than the reference path does. Hence, the backstop will enter later along the optimal path than it would along the reference path.

In Figure 8, the optimal path (assumed to be path III) and the reference path are plotted against time. Figure 8 is derived from Figure 7 by shifting path III to the right until its left-hand endpoint is aligned with the left-hand endpoint of the reference path. Figure 8, which is analogous to Figure 3, illustrates the two prediction biases which result if the reference path is mistakenly thought to be optimal.

In addition to the two biases discussed above, which have their counterpart if the backstop is inexhaustible, an additional bias will result when the backstop is exhaustible. Since the backstop enters later along the optimal path than it does along the reference path, the competitive rents accruing to owners of the backstop will be smaller than they would be if the reference path were optimal.



Price
$\bar{P}$
Optimal Path
Reference Path
Time
Competitive Phase
Monopolist Phase
III
I

Figure 8

Figure 8, which is analogous to Figure 3, illustrates the prediction biases which result if the reference path is mistakenly thought to be optimal. In addition to the two biases discussed above, which have their counterpart if the backstop is inexhaustible, an additional bias will result if the backstop is exhaustible. Since the backstop enters later along the optimal path than it does along the reference path, the competitive rents accruing to owners of the high-cost resource will be smaller than might be supposed.

Footnotes

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1/ An analysis of an exhaustible resource industry which utilizes the Cournot solution concept may be found in reference [6].

2/ That is, if

$$S(P) = \begin{cases} 0, \text{ for } P \leq P_b \\ f(P), \text{ for } P > P_b \end{cases}$$

(where $f(P)$ is a smooth, monotonically-increasing function with $f(P_b) = 0$ and $f'(P_b) > 0$), then the excess demand curve would have a kink at P_b. Its left-hand derivative would be $D'(P_b)$ while its right-hand derivative would be $D'(P_b) - f'(P_b)$. The associated revenue function would, therefore, also have a kink.

3/ Gaskins [3] considered dynamic limit pricing when the monopolist controlled an inexhaustible, but since he assumed the profit function of the monopolist had no kinks, the problem discussed here did not arise.

4/ In continuous time, of course, momentary flow alterations in strategies have no consequences; each operation in the arbitrage must take place over a measurable interval of time.

5/ In P2, $\pi(Q) = \min \{P_b Q, R(Q)\} - C(Q)$, where $R(Q)$ is the unconstrained, strictly-concave revenue function and $C(Q)$ is a convex, extraction-cost function. It is assumed that a $Q^* > 0$ exists such that $\frac{R(Q^*)}{Q^*} = P_b$.

6/ I am indebted to Heywood Fleisig for calling this case to my attention several years ago.

7/ See, for example, ref. [2] (p. 48). An allusion to the same analysis also appears in ref. [8] (p. 657).

8/ The initial level of marginal profit is specified so that the initial stock is exhausted at the moment Q^* is reached.

9/ In the problem at hand, an optimum exists if the discount rate is positive and the objective function is continuous. It is assumed that these conditions are satisfied.

10/ In the case being considered, $|\eta(Q^*)| > 1$. Since for $|\eta(Q^*)| \leq 1$ extraction at a positive rate different from Q^* is suboptimal, the formula does not apply.

11/ If the monopolist has positive extraction costs and sufficient reserves, the optimal length to linger is $\hat{\theta}$, where $\hat{\theta} = \ln\left(1 - \frac{P_b}{\left(P_b - C'(Q^*)\right)|\eta(Q^*)|}\right)$.

Since the logarithm of a smaller fraction has a larger magnitude, $\hat{\theta} > 0$. With a positive marginal cost of extraction, the marginal profit to the left and right of Q^* is reduced by the amount $C'(Q^*)$. Since the marginal profit to the right of Q^* is smaller, it takes longer to grow at the rate r by the same absolute amount. It should be noted that the result above ($\hat{\theta} > \theta$) is a consequence of assuming a positive marginal cost of extraction and would hold even if the marginal cost were constant. Morever, the result generalizes in the following way: the higher the marginal cost of extraction at Q^*, the longer the monopolist should linger there.

12/ When the cost function is strictly convex, neglect of the lingering and coexistence phases will result in an underestimate of the time before the backstop supplies the entire market.

13/ Arrow and Kurz [1] Proposition 8, p. 49.

14/ As has been noted, the case with marginal costs a positive constant requires few changes.

15/ If instead, $\pi(Q^*_+) \leq 0$ with π still concave, $Q(\lambda) = Q^*$ for $\pi'(Q^*_-) > \lambda \geq 0$, $0 \leq Q(\lambda) \leq Q^*$ for $\lambda = \pi'(Q^*)$, and $Q(\lambda) = 0$ for $\lambda > \pi'(Q^*)$. In this case, only the inelastic region of the demand curve is available to the monopolist and his best strategy is to sell at the rate Q^* until his supplies are exhausted.

16/ The momentary "non-uniqueness of flow" in this problem has no consequence since trajectories which differ only for an instant will all result in the same integral of discounted profits. Such an indeterminacy arises when the objective function has a linear segment, one familiar example being the so-called "bang-bang" solution of optimal control.

17/ For, suppose the monopolist were the last to extract. If the monopolist could neglect the competitive response, he would enter at some point and supply the entire market while gradually reducing his extraction so as to raise marginal revenue at the rate of interest. Because the elasticity of demand is constant, however, this would cause the price to rise at the rate of interest. Foreseeing this rapid increase in prices, competitive extractors would calculate that by deferring extraction until later, they could increase their discounted profits net of extraction costs. Hence, their response to the proposed price path would be an unwillingness to extract anything before the phase of rapidly rising prices. But the monopolist is no fool and would foresee their response. In light of it, his optimal strategy -- given that he goes last -- is to sell along a price path where the competitors would just be willing to supply their resource first.

But even the best strategy when the monopolist goes last can be dominated by alternatives where the monopolist goes first. Consider the situation where the price path is identical to the one just described but now suppose the competitors extract after the monopolist. The competitors would be willing to delay extraction since they are just indifferent as to when they sell. However, since the monopolist has zero extraction costs, he gains from selling earlier.

Footnote 16 continued

Therefore, the optimal price path must lie somewhere in this set of paths where the competitors are induced to operate last, as was asserted.

In fact, the optimal strategy in this case is for the monopolist to sell first -- along a path which rises at the rate of interest for a while but then coincides with a less steeply rising limit price. As we will show, the monopolist's optimal strategy will always involve continuing to sell after the limit price is reached.

18/ Differentiating, we obtain:

$$V'(0) = \left\{R'_2(0) - R'_1(\bar{I})\right\}e^{-rT(0)} - rT'(0)\left\{R_1(\bar{I}) + R_2(0)\right\}e^{-rT(0)}$$

Selling a little less stock in the first segment and a little more in the second has two effects, which may be visualized by comparing path II to path I. As the first term in the expression above indicates, the reallocation raises the discounted profitability of the second segment and reduces that of the first, evaluated at the time when the two segments of the price path intersect. In addition, as is reflected in the second term, the reallocation of stock alters the time when the limit price is reached. To determine the sign of $V'(0)$, these various changes must be compared.

Since the left-hand endpoint of any upper path begins above the left-hand endpoint of any lower path, the upper path will reach the limit price sooner. This implies that $T'(S_2) < 0$. Since the revenue terms in the expression for $V'(0)$ are non-negative, $V'(0) > 0$ if $R'_2(0) > R'_1(\bar{I})$. The transfer of stock from the first to the second segment of the price path reduces R_1 and increases R_2. Since discounted marginal revenue is equated along the first segment of the price path, $R'_1(\bar{I}) = MR\left(Q^d(\bar{P})\right)$. Furthermore, the following argument establishes that $R'_2(0) = \bar{P}$.

By definition, the discounted profits of the monopolist during the second segment may be written as:

$$R_2(S_2) = \int_{x=0}^{\theta(S_2)} \left\{(\bar{P} - k)e^{-rx} + k\right\}Q^d\left\{(\bar{P} - k)e^{-rx} + k\right\}e^{-r\left(\theta(S_2) - x\right)}dx,$$

where $\theta(S_2)$ is implicity defined by

$$S_2 = \int_{x=0}^{\theta} Q^d\left\{(\bar{P} - k)e^{-rx} + k\right\}dx.$$

From the second equation, it is evident that $\theta(0) = 0$. Differentiating the first equation in the neighborhood of $\theta = 0$, $R'_2(0) = \theta'(0)\bar{P}Q^d(\bar{P})$. Since $S'_2(\theta) = Q^d\left\{(\bar{P} - k)e^{-r\theta} + k\right\}$, $\theta'(0) = \dfrac{1}{Q^d\{\bar{P}\}}$. Hence, $R'_2(0) = \bar{P}$. Since $\bar{P} > MR\left(Q^d(\bar{P})\right)$, $V'(0) > 0$ as was to be shown.

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