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# EQUILIBRIUM IN A PRODUCTION ECONOMY WITH AN INCOME TAX

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## ABSTRACT

A state–dependent income tax is incorporated into an intertemporal production economy. Methods are developed for establishing the existence and uniqueness of an equilibrium, and for explicitly constructing this equilibrium. Some tax–policy experiments are suggested, the results of which may have important implications in quantifying the effects of various tax policies.

# Equilibrium in a Production Economy with an Income Tax

Wilbur John Coleman II[1]

## 1. Introduction

This paper develops a monotone–map approach for studying the equilibrium in a production economy in which income is taxed. By transforming the study of the equilibrium into the study of a monotone map, one can prove an existence theorem for the economy's equilibrium and develop an algorithm to construct this equilibrium. With an additional restriction on utility, it can further be proven that this monotone map satisfies a definition of concavity that ensures the equilibrium is unique. These proofs require surprisingly few restrictions on the tax function. In particular, the tax rate can depend on both exogenous and endogenous state variables, which allows one to carry out a wide variety of meaningful tax–policy experiments. For example, with this dependence one can analyze the consequences of tax–rate smoothing verses tax–revenue smoothing, or the consequences of raising distortionary tax rates during periods of relatively high output and lowering them during periods of relatively low output. As these examples suggest, allowing tax rates to depend on endogenous state variables allows a deeper examination of the dependence of an economy's dynamics on tax policy than is possible with either constant tax rates or exogenously varying tax rates.

The approach to studying the model concentrates directly on the first–order and envelope equations that the equilibrium must satisfy, which can be reduced to studying an equation in consumption as a function of the state variables. Most of the difficulty in handling this functional equation stems from the endogeneity of one of the state variables, the capital–labor ratio. This paper deals with this difficulty by setting up a fixed–point equation in the consumption function that is continuous, monotone, concave (as defined here), and that maps a rather natural compact set of consumption functions into itself. These features are sufficient to prove that iterations of this fixed–point equation uniformly converge to the unique equilibrium.

A monotone–map approach for studying an endowment economy was developed by Lucas and Stokey (1987). Stokey, Lucas, and Prescott (1989) apply similar methods to those developed here to study a production economy in a deterministic setting with a constant tax rate. As they note, however, their proof that some fixed point is nontrivial, and thus that an equilibrium exists, does not extend to a stochastic setting. Also in a deterministic setting, but with a constant tax rate on wealth, Becker (1985) showed that the competitive equilibrium coincided with the unique solution to a concave optimization problem. Danthine and Donaldson (1986) showed that Becker's reformulation also works when production is stochastic.[2] Extending the classical approach on the existence of a competitive equilibrium (Arrow and Debreu, 1954, McKenzie, 1954, and Nikaido, 1956) to an economy with taxes, as McKenzie (1981) has noted, has not been successful. To summarize, in the literature just cited it has not been proven that there exists an equilibrium to a stochastic production economy with an income tax, which is the problem addressed in this paper.

Many authors have studied the effects of taxes with models similar to the one developed here. Brock and Turnovsky (1981) and Abel and Blanchard (1983) perform various theoretical comparitive static and dynamic exercises in a perfect foresight model with optimizing agents and with a rich array of taxes. Goulder and Summers (1987) employ a perfect foresight model to focus on the transitional path of the economy when tax rates change over time. Dotsey (1988), in a model with log utility which allows him to find a closed form expression for the equilibrium, attempts to match observed measures of the US economy's response to changes in tax rates. Greenwood and Huffman (1988) simulate the effects of tax policies designed to stabilize output fluctuations. Hopefully, the methods developed in this paper can advance these types of studies in quantifying the dominant intertemporal effects of taxes.

## 2. The Model

The model[3] consists of identical households and firms in an aggregate economy in which time is discrete and the horizon is infinite. Firms employ capital and labor in a constant returns to scale production process that reproduces the single capital good. With constant returns to scale in both factor inputs, the production process can be defined relative to a per–labor production function $f$ with the capital–labor ratio and an exogenous shock as arguments.

**ASSUMPTION 1:** *The production function* $f:\mathbb{R}_+ \times Z \to \mathbb{R}_+$ *is, in its first argument, continuously differentiable, strictly increasing, strictly concave, and* $f(0,z) = 0$, *all* $z$.[4] *For any* $0 \leq \delta < 1$ *there exists some capital–labor ratio* $0 < \bar{x} < \infty$ *such that* $f(\bar{x},z) + \delta\bar{x} \leq \bar{x}$, *all* $z$, *and* $f(\bar{x},z) + \delta\bar{x} = \bar{x}$, *some* $z$.

For a particular choice of $\delta$ in Assumption 1, which will correspond to a depreciation parameter, define $K = [0,\bar{x}]$. Clearly $(f + \delta):K\times Z \to K$, which will mean that $\bar{x}$ is the (unique) maximum maintainable capital–labor ratio.

The state variables for the aggregate economy at time $t$ consist of the aggregate capital–labor ratio $X_t \in K$ and the exogenous shock $z_t \in Z$ (the supply of labor will be constant). The law of motion for $z_t$, which is known by all households and firms, satisfies the following assumption.

**ASSUMPTION 2:** *$Z$ is finite. The shocks evolve according to the Markovian probabilities* $\pi(z' \mid z) = \Pr\{z_{t+1} = z' \mid z_t = z\}$.

Each household assumes that the aggregate capital–labor ratio recursively evolves according to $X_{t+1} = g(X_t, z_t)$, $g: K \times Z \to K$. Solving for the function $g$ which is consistent with the households' behavior will be part of the equilibrium problem.

Firms engage in perfect competition and thus they pay each factor input its marginal product. In terms of the per–labor production function $f$, the factor payment to each unit of capital is $f'(X,z)$, and the factor payment to each unit of labor is $f(X,z) - Xf'(X,z)$. Note that due to constant returns to scale a firm's total factor payment exhausts its output, so it does not matter who owns the firms.

Households are equally endowed with a constant supply of labor each period. No direct utility is derived from labor withheld from the market, so from now on I will simply assume that their entire endowment of labor is inelastically supplied. Households also own all the capital which, along with their endowment of labor, they rent to the firms. With a household's capital–labor ratio denoted by $x_t$, its rental income per unit of labor is $f(X_t, z_t) + (x_t - X_t)f'(X_t, z_t)$. Each household also retains the undepreciated portion of its capital–labor ratio $\delta x_t$, where $0 \leq \delta < 1$.

The government provides each household with the lump–sum transfer $d_t = d(X_t, z_t)$ per endowment of labor, $d: K \times Z \to K$, and taxes all income at the rate $1 - \tau_t = 1 - \tau(X_t, z_t)$.

**ASSUMPTION 3:** $\tau: K \times Z \to (0,1]$ *is continuous in its first argument, and* $\tau(x,z)f'(x,z)$ *is a strictly decreasing function of* $x$.

As with the aggregate investment function $g$, solving for the government transfer function $d$ that equals the amount of taxes collected will be part of the equilibrium problem. Note that while the tax rate depends on the aggregate state variables, which in equilibrium equals the representative household's state variables, the households do not perceive the tax rate as dependent on its actions. This setup thus does not correspond to one with a progressive tax rate.[5]

Beginning a period with after–tax income, undepreciated capital, and the government transfer, a household must decide on an amount $c_t$ per endowment of labor to consume, where the remainder $x_{t+1}$ is the capital–labor ratio carried over to the next period:

$$(2.1) \qquad x_{t+1} = \tau(X_t,z_t)[f(X_t,z_t) + (x_t - X_t)f'(X_t,z_t)] + \delta x_t + d(X_t,z_t) - c_t.$$

Each household's preferences are defined with respect to consumption per endowment of labor. Expected discounted utility, defined over (feasible) consumption plans $C:\mathbb{R}_+ \times K \times Z \to \mathbb{R}_+$, is

$$\mathrm{E}\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\}, \quad c_t = C(x_t,X_t,z_t),$$

where $0 < \beta < 1$, $(x_0,X_0,z_0)$ is known, the expectation is over sequences $\{z_t\}$, and the associated sequences $\{x_t\}$ and $\{X_t\}$ are given by (2.1) and $g$ respectively.

**Assumption 4:** *The single–period utility function* $u:\mathbb{R}_+ \to \mathbb{R}$ *is bounded, continuously differentiable, strictly increasing, strictly concave, and* $u'(0) = \infty$.

Denote a household's state variables at the beginning of a period by $(x,X,z)$. The constraint set for the choice of consumption during the period is

$$M(x,X,z) = [0,\tau(X,z)(f(X,z) + (x - X)f'(X,z)) + \delta x + d(X,z)],$$

and the value function $V$ for a household's problem of choosing an optimal level of consumption satisfies the functional equation

$$(2.2) \qquad V(x,X,z) = \sup_{c \in M(x,X,z)} \Big\{ u(c) +$$

$$\beta \Sigma V\{\tau(X,z)[f(X,z) + (x - X)f'(X,z)] + \delta x + d(X,z) - c, g(X,z), z'\}\pi(z' \mid z)\Big\}.$$

Consider the Banach space of bounded, continuous, real–valued functions $v: \mathbb{R}_+ \times K \times Z \to \mathbb{R}$ equipped with the sup norm, and let $\mathscr{V}$ denote the subset that is increasing and concave in its first argument. For the remainder of this paper, reference to properties such as continuity, differentiability, and so on, will implicitly refer to a function with respect to its first argument.

**Proposition 1**: *Under Assumptions 1–4, given any continuous aggregate investment function $g: K \times Z \to K$ and any continuous transfer function $d: K \times Z \to K$, there exists a unique $V \in \mathscr{V}$ that satisfies (2.2). Moreover, this $V$ is a strictly increasing and strictly concave function. For each $(x,X,z) \in \mathbb{R}_+ \times K \times Z$, the supremum in (2.2) is attained by a unique value $C(x,X,z)$, and the policy function $C: \mathbb{R}_+ \times K \times Z \to \mathbb{R}_+$ is continuous.*

**Proof**: Define the operator $T$ on $\mathscr{V}$ by

$$(Tv)(x,X,z) = \sup_{c \in M(x,X,z)} \Big\{ u(c) +$$

$$\beta \Sigma v\{\tau(X,z)[f(X,z) + (x - X)f'(X,z)] + \delta x + d(X,z) - c, g(X,z), z'\}\pi(z' \mid z)\Big\}.$$

Banach's fixed–point theorem ensures the existence of a unique fixed point $V \in \mathscr{V}$ that satisfies (2.2) if $T$ is a contraction that maps $\mathscr{V}$ into itself. To show that $T(\mathscr{V}) \subset \mathscr{V}$, note that since $u$ and $v$ are bounded, so is $T(v)$. Clearly $T(v)$ is increasing, and since $u$ and $v$ are concave and $M$ is convex, $T(v)$ is concave. To show that $T(v)$ is continuous, note that since $u$ is a strictly concave function, a unique policy function $C_v$ satisfies the supremum. Since $u$ and $v$ are continuous and $M$ is convex, this policy function is continuous, and thus $T(v)$ is continuous. Hence $T(\mathcal{V}) \subset \mathcal{V}$, and since $T$ is monotone with $T(v + k) = T(v) + \beta k$ for constants $k$, by Blackwell's theorem (1969, Theorem 5) $T$ is a contraction mapping. A unique fixed point $V \in \mathcal{V}$ thus exists. Since $u$ is a strictly increasing and strictly concave function, so is $V$. Since, as just mentioned, $C_v$ is unique and continuous for any $v \in \mathcal{V}$, it has these properties under $V$. *Q.E.D.*

Proposition 1 establishes that a household's problem is well posed for any continuous functions $g$ and $d$. Consider, then, the following definition of what it means for $g$ and $d$ to be equilibrium functions.

**Definition:** A *stationary equilibrium* consists of continuous functions $(w,c,g,d)$, $w$ maps $K \times Z$ into $\mathbb{R}$ and $c$, $g$, and $d$ map $K \times Z$ into $K$, such that (i) $w(x,z) = V(x,x,z)$ and $c(x,z) = C(x,x,z)$ where, for the aggregate investment function $g$ and transfer function $d$, the value function $V \in \mathcal{V}$ satisfies (2.2) and $C$ is the associated policy function, (ii) all tax revenues are redistributed as the lump–sum transfer $d = (1 - \tau)f$ (note that this equality could have been imposed in (2.1)), and (iii) the aggregate investment function $g$ is such that households choose to invest according to the same rule:

$$f(x,z) + \delta x - C(x,x,z) = g(x,z).$$

At the equilibrium, some additional results concerning the solution to (2.2) can be obtained.

**PROPOSITION 2:** *If $(w,c,g,d)$ is an equilibrium with the associated policy function $C$ and value function $V$, then $C(x,x,z)$, all $x > 0$, all $z$, lies in the nonempty interior $\mathring{M}(x,x,z)$ of $M(x,x,z)$, and $V$ is a continuously differentiable function at $(x,x,z)$, all $x > 0$, all $z$.*

**PROOF:** Since $u'(0) = \infty$ and $f(0,z) = 0$, all $z$, $C(x,x,z) \in \mathring{M}(x,x,z)$, all $x > 0$, all $z$. By Benveniste and Scheinkman's theorem (1979, Theorem 1) the value function $V$ is differentiable at $(x,x,z)$, all $x > 0$, all $z$, with the derivative $V_1(x,x,z) = u'[C(x,x,z)]$. This derivative is continuous since $u'$ and $C$ are continuous. ***Q.E.D.***

With the results of Proposition 2, any equilibrium consumption function is a strictly positive solution ($c(x,z) > 0$, all $x > 0$, all $z$) to

$$(2.3) \qquad u'[c(x,z)] = \beta\Sigma u'\{c[f(x,z) + \delta x - c(x,z),z']\}$$

$$\times\{\delta + \tau[f(x,z) + \delta x - c(x,z),z']f'[f(x,z) + \delta x - c(x,z),z']\}\pi(z' \mid z).$$

Suppose, on the other hand, that $c$ is a strictly positive solution to (2.3), $g = f + \delta - c$, $d = (1 - \tau)f$, and $w$ is defined by

$$w(x,z) = u[c(x,z)] + \beta\Sigma w[f(x,z) + \delta x - c(x,z),z']\pi(z' \mid z).$$

Is this $(w,c,g,d)$ an equilibrium? For this $g$ and $d$, define $V$ and $C$ as the unique solution to (2.2). By construction, $C(x,x,z) = c(x,z)$ and $V(x,x,z) = w(x,z)$, so this quadruple is an equilibrium. Note, though, that the zero consumption function $c = 0$ is a solution to (2.3), but it is not an equilibrium, so it is important to explicitly deal with strictly positive solutions. The remainder of this paper proves the existence and uniqueness of the strictly positive solution to (2.3), and develops a method by which to construct this solution.

## 3. Existence

In a stochastic setting, since there does not exist a stationary point of consumption (which could be shown to be strictly positive) it is difficult to rule out solutions to (2.3) that are arbitrarily close to zero. Since zero is a solution, any fixed–point equation representing (2.3) that has zero in its domain had better have two solutions, one at zero and one that is strictly positive. Since the contraction–mapping theorem delivers a unique solution, it thus is not well–suited for this problem. Most other existence theorems rely on a notion of compactness for the set of candidate solutions, which, for infinite–dimensional spaces, is a considerably more difficult concept to deal with than the completeness requirement of a contraction–mapping fixed–point theorem. It turns out, though, that for the economy considered here a rather natural compact set of consumption functions can be constructed. This compactness property is also preserved under a fixed–point equation that has a close association with the contraction mapping of a dynamic program. While this fixed–point equation does not have the contraction property, it does have two properties mentioned previously—monotonicity and concavity—that are almost as good.

Define the set of consumption functions

$$C_f(K\times Z) = \left\{ c\colon \begin{array}{l} c\colon K\times Z \to K \text{ is continuous}, \\ 0 \le c(x,z) \le f(x,z) + \delta x, \\ 0 \le c(y,z) - c(x,z) \le f(y,z) - f(x,z) + \delta(y-x) \text{ for } y \ge x. \end{array} \right.$$

The third condition defining $C_f(K\times Z)$ is equivalent to requiring that consumption $c$ and net investment $f + \delta - c$ are increasing functions of the capital–labor ratio $x$.

**Proposition 3:** *With Assumptions* 1–2, $C_f(K\times Z)$ *is convex and compact.*[6]

**PROOF:** Clearly $C_f(K\times Z)$ is convex. To show that $C_f(K\times Z)$ is compact, note that since $|c(x,z) - c(y,z)| \leq |f(x,z) - f(y,z) + \delta(x - y)|$, by the mean value theorem $C_f(K\times Z)$ is equicontinuous at every point $x > 0$, all $z$. At $x = 0$ and each $z$, for any $\epsilon > 0$ choose $\Delta$ such that $f(\Delta,z) + \delta\Delta = \epsilon$, so that $|c(0,z) - c(y,z)| \leq f(y,z) + \delta y < \epsilon$ whenever $y < \Delta$, $c \in C_f(K\times Z)$. $C_f(K\times Z)$ is thus equicontinuous, so by the Arzela–Ascoli theorem[7] $C_f(K\times Z)$ is compact. *Q.E.D.*

Define the fixed–point equation $A$ by

$$u'[(Ac)(x,z)] = \beta\Sigma u'\{c[f(x,z) + \delta x - (Ac)(x,z),z']\} \tag{3.1}$$

$$\times\{\delta + \tau[f(x,z) + \delta x - (Ac)(x,z),z']f'[f(x,z) + \delta x - (Ac)(x,z),z']\}\pi(z'\,|\,z).$$

Clearly any strictly positive fixed point of $A$ can define an equilibrium. Note that $(Ac)(x,z)$ enters an argument of $c$, and that $A(c)$ is defined pointwise. The function $A$ has a well–defined meaning: for the finite–time–horizon version of this economy with the terminal consumption function fixed at $c$, $A^n(c)$ corresponds to the optimal consumption function $n$ steps away from the terminal date. One implication of what follows is thus the equivalence between the limit of the finite–horizon economy and the infinite–horizon economy. This result also establishes the close connection between $A$ and the dynamic programming problem on which $A$ is based.

The following proposition establishes that $A$ is well defined.

**PROPOSITION 4:** *For any $c \in C_f(K\times Z)$, a unique $A(c) \in C_f(K\times Z)$ exists.*

**PROOF:** Define $A(c)$ pointwise as the $y$ for which

$$\zeta(y,x,z) =$$

$$\beta\Sigma u'\{c[f(x,z) + \delta x - y,z']\}\{\delta + \tau[f(x,z) + \delta x - y,z']f'[f(x,z) + \delta x - y,z']\}\pi(z' \mid z) - u'(y)$$

equals zero. Unless $y = 0$ is the root, $\zeta$ is negative for $y$ close to $0$, positive for $y$ close to $f(x,z) + \delta x$, and strictly increases as $y$ increases. This proves the existence of a unique $A(c)$, which also proves that $A(c)$ is continuous. Since $\zeta$ increases with $y$ and decreases with $x$, $A(c)$ is an increasing function of $x$, and by (3.1) $f + \delta - A(c)$ is an increasing function of $x$. Hence $A[C_f(K\times Z)] \subset C_f(K\times Z)$. *Q.E.D.*

At this stage, proving the existence of a fixed point $c = A(c)$, $c \in C_f(K\times Z)$, is vacuous since we already know that $c = 0$ is a fixed point.[8] For this reason, Schauder's fixed–point theorem is inappropriate relative to the set $C_f(K\times Z)$. Tarski's fixed–point theorem, on the other hand, contributes something in that it asserts the existence of a maximal fixed point and provides a particular sequence that converges to it: a continuous, monotone self–map of a non–empty, partially–ordered, compact set $M$ in which some element $m \in M$ is mapped inwards, $A(m) \leq m$, has a maximal fixed point in the set $\{m^\dagger: m^\dagger \leq m, m^\dagger \in M\}$, and $A^n(m)$ converges to this maximal fixed point.[9] For the economy considered here, $f + \delta$ will take the place of $m$ so this fixed point is maximal in the entire set $C_f(K\times Z)$, and thus I will simply refer to it as the maximal fixed point in $C_f(K\times Z)$.

To apply Tarski's fixed–point theorem, a partial ordering on $C_f(K\times Z)$ is required. Define the usual partial ordering by $\hat{c} \leq \tilde{c}$ if $\hat{c}(x,z) \leq \tilde{c}(x,z)$, all $(x,z)$, with which the monotonicity of $A$ can be established.

**Proposition 5:** $A$ *is continuous and monotone.*

**Proof:** Clearly $A(c_n) \to A(c)$ pointwise as $c_n \to c$. Since $C_f(K\times Z)$ is equicontinuous and $K\times Z$ is compact, this convergence is uniform,[10] which establishes that

$A$ is continuous. $(A,\leq)$ is monotone since $\hat{c} \leq \tilde{c}$ implies

$$u'[(A\hat{c})(x,z)] \geq \beta\Sigma u'\{\tilde{c}[f(x,z) + \delta x - (A\hat{c})(x,z),z']\}$$

$$\times\{\delta + \tau[f(x,z) + \delta x - (A\hat{c})(x,z),z']f'[f(x,z) + \delta x - (A\hat{c})(x,z),z']\}\pi(z' \mid z),$$

and thus $A(\hat{c}) \leq A(\tilde{c})$. ***Q.E.D.***

The existence of the maximal fixed point in $C_f(K\times Z)$ as the limit of $A^n(f)$ can now be established.

**Theorem 6:** *Among the set of fixed points of $A$ there exists one that is maximal in $C_f(K\times Z)$, and $A^n(f)$ converges uniformly to this maximal fixed point.*

**Proof:** By Proposition 3 $C_f(K\times Z)$ is a compact set, and by Proposition 4 $A$ maps $C_f(K\times Z)$ into itself. By Proposition 5 $A$ is continuous and $(A,\leq)$ is monotone. Clearly $A(f) \leq f$, so Tarski's theorem can be applied. Since $C_f(K\times Z)$ is a set of equicontinuous functions defined an a compact set, the convergence is uniform. ***Q.E.D.***

Note that Tarski's fixed–point theorem does not assert that the maximal fixed point in $C_f(K\times Z)$ is not $c = 0$, but it does assert that if $A^n(f)$ converges to $0$, then no equilibrium in $C_f(K\times Z)$ exists. To prove that a strictly positive fixed point exists, all that needs to be shown is that $A^n(f)$ converges to a strictly positive function. For the following restriction on the production function, this result can be proven.

**Assumption 5:** *There exists an $x_0$ such that, for all $z$, $f(x_0,z) + \delta x_0 > x_0$ and $\beta\Sigma[\delta + f'(x_0,z')]\pi(z' \mid z) \leq 1$.*

In the deterministic setting, Assumption 1 ensures that Assumption 5 holds. In the stochastic setting, Assumption 5 requires that productivity under a particular shock is not too high relative to productivity under other shocks, which will ensure that consumption under these other shocks is not too low in anticipation of a high productivity shock.

**Proposition 7:** *With Assumptions* 1–5, *a strictly positive fixed point exists.*

**Proof:** I will first prove that the maximal fixed point is not zero, and then that it is strictly positive. Choose an $x_0$ that satisfies the conditions in Assumption 5, and choose an $\alpha$ such that $0 < \alpha < f(x_0,z) + \delta x_0 - x_0$, all $z$. To prove that a nonzero fixed point exists, it is sufficient to show that if $c(x_0,z) \geq \alpha$, all $z$, then $(Ac)(x_0,z) \geq \alpha$, all $z$. A sufficient condition for $(Ac)(x_0,z) \geq \alpha$, all $z$, is

$$u'(\alpha) \geq \beta\Sigma u'\{c[f(x_0,z) + \delta x_0 - \alpha,z']\}$$

$$\times\{\delta + \tau[f(x_0,z) + \delta x_0 - \alpha,z']f'[f(x_0,z) + \delta x_0 - \alpha,z']\}\pi(z' \mid z),$$

all $z$. By construction $f(x_0,z) + \delta x_0 - \alpha > x_0$ and $c(x_0,z) \geq \alpha$, all $z$, so a sufficient condition for this inequality is

$$u'(\alpha) \geq \beta\Sigma u'(\alpha)[\delta + \tau(x_0,z')f'(x_0,z')]\pi(z' \mid z),$$

all $z$. Since $\tau \leq 1$, this inequality is true by hypothesis. A nonzero fixed point thus exists.

To prove that the above nonzero fixed point must be strictly positive, define $x_1$ as the largest $x$ for which $c(x,z) = 0$ for some $z$, say $z^{\dagger}$. Suppose that $x_1 > 0$, for then $f(x_1,z) + \delta x_1 > x_1$, all $z$ (note that $x_1 < x_0$). The left side of

$$u'[c(x_1,z^{\dagger})] = \beta\Sigma u'\{c[f(x_1,z^{\dagger}) + \delta x_1 - c(x_1,z^{\dagger}),z']\}$$

$$\times\{\delta + \tau[f(x_1,z^{\dagger}) + \delta x_1 - c(x_1,z^{\dagger}),z']f'[f(x_1,z^{\dagger}) + \delta x_1 - c(x_1,z^{\dagger}),z']\}\pi(z' \mid z)$$

is then unbounded while the right side is bounded, so it must be true that $x_1 = 0$. *Q.E.D.*

Since $A^n(f)$ converges to the maximal fixed point in $C_f(K\times Z)$, with Assumptions 1–5 it will converge to a strictly positive fixed point.

## 4. Uniqueness

To prove that an equilibrium is unique, a notion of the concavity of $A$ is developed that is similar to the type of concavity needed to ensure, for example, that a production function has a unique strictly positive fixed point. To get some idea as to where this argument is headed, consider the following argument to establish that a strictly positive fixed point of $A$ is unique. Suppose $A$ satisfied the following property: for any strictly positive fixed point $c$ of $A$ and any $0 < t < 1$, $(Atc)(x,z) > t(Ac)(x,z)$, all $x > 0$, all $z$. Suppose, then, that two strictly positive fixed points $c_1$ and $c_2$ exist such that, for some $0 < t < 1$, $c_1 \geq tc_2$ and $c_1(x,z) = tc_2(x,z)$, some $x > 0$, some $z$. This would lead to the following contradiction: $c_1(x,z) = (Ac_1)(x,z) \geq (Atc_2)(x,z) > t(Ac_2)(x,z) = tc_2(x,z)$, all $x > 0$, all $z$. In general, however, no such $t > 0$ exists, as it cannot be ensured that a strictly positive lower bound for $c_1(x,z)/c_2(x,z)$ exists. The above property of $A$ is thus not sufficient to ensure that a strictly positive fixed point is unique. For any $x_0 > 0$, however, a lower bound for $c_1(x,z)/c_2(x,z)$ does exist over the region $x \geq x_0$. The idea pursued here is to somehow strengthen the concept of concavity so that it pertains to this smaller domain.

**Definition**: The monotone function $A: C_f(K \times Z) \to C_f(K \times Z)$ is *concave* if (i) for any strictly positive $c \in C_f(K \times Z)$ and any $0 < t < 1$,

$$(4.1) \qquad (Atc)(x,z) > t(Ac)(x,z), \text{ all } x > 0, \text{ all } z,$$

and (ii) for any strictly positive fixed point $c_1$ of $A$ there exists some $x_0 > 0$ such that the following is true: for any $0 \leq x_1 \leq x_0$ and any $c_2 \in C_f(K \times Z)$ such that $c_1(x,z) \geq c_2(x,z)$, all $x \geq x_1$, all $z$,

$$(4.2) \qquad c_1(x,z) \geq (Ac_2)(x,z), \text{ all } x \geq x_1, \text{ all } z.$$

With this definition of concavity, the following uniqueness theorem can be proven.

**Theorem 8**: *A monotone, concave function* $A: C_f(K \times Z) \to C_f(K \times Z)$ *has at most one strictly positive fixed point* $c \in C_f(K \times Z)$.

**Proof**: Suppose that two distinct strictly positive fixed points $c_1$ and $c_2$ exist. Without loss of generality, assume that $c_1(x,z) < c_2(x,z)$, some $x > 0$, some $z$. Clearly for some small $x_0 > 0$, $c_1(x,z) < c_2(x,z)$, some $x \geq x_0$, some $z$, and for any such $x_0$ a $0 < t < 1$ can be chosen such that $c_1(x,z) \geq tc_2(x,z)$, all $x \geq x_0$, all $z$, and $c_1(x,z) = tc_2(x,z)$, some $x \geq x_0$, some $z$. Since $A$ is concave, this $x_0$ can be chosen such that $c_1(x,z) \geq (Atc_2)(x,z)$, all $x \geq x_0$, all $z$. This leads to the contradiction

$$\begin{aligned} c_1(x,z) &\geq (Atc_2)(x,z) \\ &> t(Ac_2)(x,z) \\ &= tc_2(x,z), \end{aligned}$$

all $x \geq x_0$, all $z$. *Q.E.D.*

Theorem 8 is similar to one proven by Krasnosel'skiĭ and Zabreĭko (1984). They define concavity such that a $0 < t < 1$ can be chosen to satisfy the required properties when $x_0 = 0$. The function $A$ does not appear to satisfy their definition of concavity.

The following two lemmas establish the property needed for (4.2) to hold. This requires the following assumption.

**Assumption 6:** $\beta[\delta + \tau(0,z')f'(0,z')]\pi(z' \mid z) > 1$, *all* $z$, *all* $z'$.

As shown in the following lemma, this assumption ensures that net investment adds to the capital–labor ratio when the capital–labor ratio is low.

**Lemma 9:** *With Assumptions* 1–4 *and* 6, *for any strictly positive fixed point* $c$ *of* $A$ *there exists some* $x_0 > 0$ *such that* $f(x,z) + \delta x - c(x,z) \geq x$, *all* $x \leq x_0$, *all* $z$.

**Proof:** Suppose that $c$ is a strictly positive fixed point of $A$ and that for every $x_0 > 0$ there exists some $x \leq x_0$ and some $z$ such that $f(x,z) + \delta x - c(x,z) < x$. For this $c$, $x$, and $z$, substitute $x$ for $f(x,z) + \delta x - c(x,z)$ in (2.3) to show

$$u'[c(x,z)] > \beta \Sigma u'[c(x,z')][\delta + \tau(x,z')f'(x,z')]\pi(z' \mid z).$$

With Assumption 6, $x_0$ can be chosen sufficiently small so that

$$u'[c(x,z)] > \Sigma u'[c(x,z')],$$

which is a contradiction. *Q.E.D.*

Using Lemma 9, a monotonicity–type result over a region $x \geq x_0 > 0$ can be derived.

**LEMMA 10:** *Allow Assumptions* 1–4 *and* 6 *to hold, suppose* $c_1$ *is a strictly positive fixed point of* $A$, *and define an* $x_0 > 0$, *as in Lemma* 9, *such that* $f(x,z) + \delta x - c_1(x,z) \geq x$, *all* $x \leq x_0$, *all* $z$. *For any* $0 \leq x_1 \leq x_0$, *if* $c_2 \in C_f(K \times Z)$ *is such that* $c_1(x,z) \geq c_2(x,z)$, *all* $x \geq x_1$, *all* $z$, *then* $c_1(x,z) \geq (Ac_2)(x,z)$, *all* $x \geq x_1$, *all* $z$.

**PROOF:** $f(x,z) + \delta x - c_1(x,z) \geq x_1$, all $x \geq x_1$, all $z$, so $c_1[f(x,z) + \delta x - c_1(x,z),z'] \geq c_2[f(x,z) + \delta x - c_1(x,z),z']$, all $x \geq x_1$, all $z$, all $z'$, and thus

$$u'[c_1(x,z)] \leq \beta\Sigma u'\{c_2[f(x,z) + \delta x - c_1(x,z),z']\}$$

$$\times\{\delta + \tau[f(x,z) + \delta x - c_1(x,z),z']f'[f(x,z) + \delta x - c_1(x,z),z']\}\pi(z' \mid z),$$

all $x \geq x_1$, all $z$. This proves that $c_1(x,z) \geq (Ac_2)(x,z)$, all $x \geq x_1$, all $z$. *Q.E.D.*

To establish the property needed for (4.1) to hold, the following assumption is sufficient.

**ASSUMPTION 7:** *Utility exhibits constant relative risk aversion; that is,* $u'(xy) = u'(x)u'(y)$.

**THEOREM 11:** *With Assumptions* 1–4 *and* 6–7, *a strictly positive fixed point of* $A$ *is unique.*

**PROOF:** Proposition 3 establishes that $A: C_f(K \times Z) \to C_f(K \times Z)$, Proposition 4 establishes that $A$ is monotone, and Lemmas 9 and 10 establish the property needed for (4.2) to hold. To show that (4.1) holds, in the functional equation determining $A(tc)$, substitute $tA(c)$ for $A(tc)$ to show that a sufficient condition for (4.1) is

$$u'[t(Ac)(x,z)] > \beta\Sigma u'\{tc[f(x,z) + \delta x - t(Ac)(x,z),z']\}$$

$$\times\{\delta + \tau[f(x,z) + \delta x - t(Ac)(x,z),z']f'[f(x,z) + \delta x - t(Ac)(x,z),z']\}\pi(z' \mid z),$$

all $x > 0$, all $z$. Use Assumption 7 to cancel the first $t$ in the argument of $u'(\cdot)$ on both sides of the inequality, and then use Assumption 3 to show that the strict inequality holds. $A$ is thus concave, so by Theorem 8 a strictly positive fixed point is unique. *Q.E.D.*

## 5. Construction of the Equilibrium

Proposition 12 summarizes and slightly strengthens the results obtained so far.

**PROPOSITION 12:** *Under Assumptions* 1–7, *for any strictly positive* $c_0 \in C_f(K \times Z)$, *the sequence* $\{c_n\}$ *defined recursively by* $c_{n+1} = A(c_n)$, $n \geq 1$, *converges uniformly to the unique strictly positive fixed point* $c^*$.

**PROOF:** We already know that $A^n(f)$ converges uniformly to the unique strictly positive fixed point, so the proof is complete if for any strictly positive $c_0$ there exists a $\underline{c} \leq c_0$ that also converges uniformly to this fixed point. Define an $x_0 > 0$ as in Assumption 5, and an $0 < \alpha < f(x_0,z) + \delta x_0 - x_0$, all $z$ as in the proof of Proposition 6. Also, choose $\alpha$ small enough such that $c_0(x_0,z) \geq \alpha$, all $z$. Define $\underline{c}$ by

$$\underline{c} = \inf_c \{c\colon\ c(x_0,z) = \alpha, \text{ all } z\}.$$

Since $C_f(K \times Z)$ is compact, such a $\underline{c}$ exists. Clearly $\underline{c} \leq c_0$, and since $A$ is monotone

$$A^n(\underline{c}) \leq A^n(c_0) \leq A^n(f).$$

$A(\underline{c}) \geq \underline{c}$ is proven in Proposition 7, and a similar application of Tarski's fixed–point theorem guarantees that $A^n(\underline{c})$ converges uniformly to the fixed point $c^*$. *Q.E.D.*

As shown by Lucas and Stokey (1987, Theorem 3) in a different context, even without the additional assumptions guaranteeing uniqueness, if $A^n(\underline{c})$ and $A^n(f)$ converge to the same solution $c^*$, where $\underline{c} \leq c^* \leq f$, then the solution $c^*$ is unique in the subset $\{c\colon \underline{c} \leq c \leq f\}$ of $C_f(K \times Z)$.

## 6. Concluding Remarks

This paper proves the existence and uniqueness of the equilibrium to a stochastic production economy in which the income tax rate can be a somewhat arbitrary function of the entire state vector. The model and solution methodology developed in this paper can be used to address a variety of tax–policy experiments. In the remainder of this paper I will sketch out how this paper's approach may be used to conduct two tax–policy experiments.

It has often been suggested that via a "Laffer Curve" effect decreasing a relatively high income tax may actually increase tax revenues by stimulating a sufficiently large increase in investment. Since uncertainty is not an important element of this concept, and income fully adjusts to a particular tax rate at the steady state, this suggests focusing on tax revenue at the deterministic steady state. For a constant tax rate $1 - \tau$, tax revenue at this state in an economy with a production function $f(x,z) = x^\alpha$ is

$$(1 - \tau)\left[\frac{\tau\alpha\beta}{1 - \delta\beta}\right]^{\frac{\alpha}{1 - \alpha}}.$$

Note that this definition of the Laffer Curve does not depend on preferences other than on the time preference parameter $\beta$, and the revenue maximizing rate $1 - \tau^* = 1 - \alpha$ depends only on the production function parameter $\alpha$. An $\alpha$ of .33, which is a common setting for this parameter in the stochastic growth literature, leads to $1 - \tau^* = .67$. This value is probably well above tax rates in the US economy.

To address the issue just discussed requires neither uncertainty nor that taxes be a function of the capital–labor ratio. Suppose, however, that we want to compare the equilibrium under tax–rate smoothing, in which case tax revenue varies, versus one with tax–revenue smoothing, in which case the tax rate varies. The tax–rate smoothing case is captured by setting the tax rate $1 - \tau(x,z)$ to some constant. The tax–revenue smoothing case is captured by setting tax revenue $[1 - \tau(x,z)]f(x,z)$ to some constant, say $k$. To bound the implied tax rate to be less than one, some cap $1 - \overline{\tau} \leq 1$ must be imposed so that

$$1 - \tau(x,z) = \min\{1 - \overline{\tau},\ k/f(x,z)\}.$$

Assumption 3 is satisfied if

$$[1 - \min\{1 - \overline{\tau},\ k/f(x,z)\}]f'(x,z)$$

is a strictly decreasing function of $x$, which, for the production function $zx^{\alpha}$, is satisfied for any $k$ whenever $1 - \overline{\tau} \leq 1 - \alpha$. Clearly to carry out this experiment, and many others like it, one needs to rely on the full analysis of the problem laid out in this paper, including the dependence of the tax function on the capital–labor ratio.

## ENDNOTES

1. This paper is based on my dissertation, for which Robert E. Lucas, Jr., Lars P. Hansen, and Robert M. Townsend provided many helpful comments. I would also like to thank two anonymous referees, Andreu Mas–Colell, Edward J. Green, Ross E. Levine, and David B. Gordon for very useful criticisms. This paper should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or members of its staff.

2. Current work by Kehoe, Levine, and Romer (1989) attempts to extend this approach, but they are unable to reformulate the type of problem considered here without using side conditions.

3. I wish to thank an anonymous referee for suggesting that I motivate the model as a competitive equilibrium in the way done here.

4. Note that no upper bound on the derivative at $0$ is imposed, so it is possible to specify $f'(0,z) < \infty$, all $z$, or $f'(0,z) = \infty$, all $z$. A sufficient condition for $\bar{x} > 0$ is if $f'(0,z) \geq 1$, all $z$, and a sufficient condition for $\bar{x} < \infty$ is if $\lim_{x \to \infty} f'(x,z) = 0$, all $z$.

5. An extension which incorporates a progressive tax rate is straightforward. With an income $y_t$, let the tax rate be $1 - \phi(y_t,X_t,z_t)$. Modify equation (2.1) to include $\phi[f(X_t,z_t) + (x_t - X_t)f'(X_t,z_t),X_t,z_t][f(X_t,z_t) + (x_t - X_t)f'(X_t,z_t)]$, and modify Assumption 3 to assume that $y\phi(y,X,z)$ is a strictly increasing and strictly concave function of $y$ and that $\{\phi[f(x,z),x,z] + \phi_1[f(x,z),x,z]f(x,z)\}f'(x,z)$ is a strictly decreasing function of $x$. At the equilibrium, define $\tau(x,z)$ as the above term in braces, and proceed as in Section 3.

6. A set $M$ is sequentially compact if every sequence in $M$ contains a convergent subsequence; it is compact if this limit lies in $M$. See Royden (1968, p. 160).

7. The Arzela–Ascoli theorem states that a subset of continuous functions defined on a compact set is sequentially compact if it is bounded and equicontinuous. A set of real–valued functions $C(K)$ defined on a metric space $K$ is equicontinuous at $x \in K$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|c(x) - c(y)| < \epsilon$ whenever $d(x,y) < \delta$, $y \in K$, and $c \in C(K)$. $C(K)$ is equicontinuous if it is equicontinuous at every point, and uniformly equicontinous if $\delta$ can be chosen independent of $x$. Also, a set is compact if it is sequentially compact and closed. See Royden (1968, pp. 177–179) for the proof of the Arzela–Ascoli theorem and for the definition of equicontinuity.

8. Bizer and Judd (1989) prove the existence of a fixed point at this stage, which is why their proof does not establish the existence of an equilibrium.

9. See Dugundji and Granas (1982, Theorem 1.4.2). The theorem only requires that every countable chain (i.e., every totally–ordered subset) has an infimum, which is satisfied if the set is compact. I wish to thank Darrell Duffie for bringing this theorem to my attention, and an anonymous referee for suggesting that I apply it in this case. In a previous version of this paper, I applied Schauder's theorem on a set which excluded $c = 0$, where I used monotonicity to show that $A$ mapped this set into itself.

10. See Royden (1968, p. 178, Lemma 32).

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