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INTERNAL FUNDS AND THE INVESTMENT FUNCTION

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ABSTRACT

An extensive and increasingly persuasive body of empirical evidence has linked a firm's fixed investment expenditure to its supply of internally generated funds. The central concerns of this paper are (1) the theoretical justifiability of such empirically-based investment functions, particularly those where internal funds affect only the speed of adjustment, and (2) the dynamic properties of this latter class of investment functions. A class of models is explored featuring intertemporal profit maximization under conditions of increasing costs of external finance (attributable to bankruptcy or agency costs). The paper shows that, for a major part of the optimal investment path, the function implied by the theory is remarkably close to the most promising variant found empirically: the supply of internal funds affects the speed of adjustment, but not the level of the optimal capital stock. Such investment functions possess the unusual dynamic property that the speed of adjustment increases monotonically along the optimal path.

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Guy V.G. Stevens[1]

I. Introduction

For more than two decades, researchers have discovered repeatedly a statistically significant relationship between a firm's fixed investment expenditures and its cash flow or retained earnings. The cumulative evidence makes it hard to reject an impact of the firm's internal financial situation on its capital spending. (See Coen [5], Gardner and Sheldon [10], Eisner [7], Artus et. al. [3], Fazzari et. al. [9] and, for earlier work, Greenberg [11] and Hochman [14].)

To many who have followed the theory of investment over the years, this development seems like an echo from a bygone era -- the revival of a theory once thought defunct. In the 1960s and early 1970s, work developing stock-adjustment models and the neoclassical model of investment seemed to dominate, if not disprove, theories containing liquidity and financial variables.[2] Thus, for example, in a notable article, Jorgenson and Siebert [17] tested alternative theories on the same body of (microeconomic) data and found liquidity or finance-based theories dominated by the neoclassical theory of investment.

The paradox of this revival can be partially explained by noting that financial variables have returned to the explanation of investment in an eclectic form, often embedded in models which also contain the determinants associated with neoclassical and stock-adjustment models. Thus, in a key article, Coen [5, p. 164], relying on concepts introduced by Greenberg [11] and Hochman [14], proposed and tested an investment function where the <u>speed of adjustment</u> to the neoclassically-determined optimal capital stock was a function of internal funds:

$$I_t = \left\{b_0 + \frac{b_1(F_{t-1} - \delta K_{t-1})}{K_t^* - K_{t-1}}\right\}[K_t^* - K_{t-1}] + \delta K_{t-1} , \qquad (1)$$

where I is the level of capital expenditures; K and K*, the actual and the desired level of the capital stock; F, the firm's cash flow; and δ, the rate of depreciation.

In such models the capital stock adjusts dynamically to K*, the optimum capital stock defined by the neoclassical theory; what is added is a variable speed of adjustment dependent on some measure of the firm's level of internal funds. The empirical work noted above has demonstrated that these financial effects are a statistically significant additional influence on fixed investment, both in the United States and Western Europe.

The central concerns of this paper are the theoretical justification of these eclectic theories of investment and the dynamic implications of the investment functions that result. Plausible heuristic stories have been provided to support the inclusion of cash flow variables in the investment function, but so far no such investment functions have been rigorously derived from an underlying theory of the firm. The results presented below indicate that, under certain conditions, investment functions similar, but not identical to Coen's equation (1) can be justified theoretically. Moreover, the investment functions that result exhibit unusual dynamic properties; in particular, the speed of adjustment increases monotonically until the long-run equilibrium is attained.

II. Issues in Linking Investment to Finance

The idea that has been used most frequently in heuristic justifications for including financial variables in the speed of adjustment is the notion that the rate of interest on outside borrowing will be an increasing function of the level of borrowing or the ratio of debt to assets or equity. Coen [5, p. 150] and Nickell [21] saw this as one of the most promising explanations of the empirical regularity. Such an upward sloping supply function for debt can, in turn, be based on lenders' perceptions of a positive relationship between the debt/assets ratio and the risk and costs of bankruptcy. (See Stiglitz [25], Jensen and Meckling [15], Nickell [20], and Van Horne [26, pp. 260-73].)

Although rigorous derivations of an upward sloping supply schedule for debt are firmly accepted,[3] little has been done to derive its dynamic implications for the firm's investment function. This, along with the associated question of the conditions under which Coen-type investment functions are justified, is, of course, the goal of this paper. To pursue this end, I embed alternative versions of an upward sloping supply schedule for debt into the well-known neoclassical model of the firm.

Increasing Costs of Debt and the Neoclassical Model

To better understand how to proceed, let us initially explore how finance is handled in the original version of the neoclassical model pioneered by Jorgenson [16]. Ignoring tax considerations for the moment, the (present) value, $V(t_0)$, at a given time, t_0, of the neoclassical firm can be written as:

$$V(t_0) = \int_{t_0}^{\infty} e^{-\rho t} DIV(t)dt = \int_{t_0}^{\infty} e^{-\rho t}[pQ(t) - wL(t) - qI(t)]dt, \qquad (2)$$

where DIV(t) is the level of firm dividends at time t; Q, L, I, are output, labor input and real investment expenditures, with p, w, and q their respective prices; ρ is the firm's discount rate.

Given Jorgenson's assumption of a Cobb-Douglas production function, the maximization of the value of this firm leads to the familiar relationship between the optimal capital stock (K*) and other endogenous and exogenous variables (with, in addition to the symbols defined above, $\dot{q}=dq/dt$ and γ equal to the output elasticity of capital):

$$K^*(t) = \frac{\gamma p Q(t)}{q(\delta + \rho - \dot{q}/q)} \,. \qquad (3)$$

So far nothing has been indicated about the <u>financing</u> of the optimal capital stock. The standard approach to finance in the neoclassical model is implied by the substitution of the expression (pQ-wL-qI) on the right hand side of equation (2) for dividends (DIV) on the left hand side. Since the substitution is derived from the abbreviated sources and uses of funds identity, qI = pQ-wL-DIV, and since no debt or interest variables appear in that identity, the implication is that the firm's investment is financed fully by the difference between operating revenues (pQ-wL) and dividends. When dividends are positive, this would be called financing investment out of retained earnings.[4] However, it is important to note that dividends cannot be constrained to be positive in this model. The optimal investment policy may very well imply that for some periods the value of investment, qI, will be greater than operating revenues. During such periods, since debt finance is excluded, the above identity implies that dividends will be required to be <u>negative</u>, i.e., the firm assesses its shareholders for new infusions of capital.

When debt financing is allowed, flotations of debt ($\dot{D}$) and the consequent interest payments at rate r (-rD) are incorporated into the firm's objective function by adding these terms to the right hand side of equation (2). The sources and uses of funds identity becomes:

$$DIV = pQ - wL - qI - rD + \dot{D}. \tag{4}$$

The upward sloping supply curve for debt will be represented by the interest rate, r, being an increasing function of either the level of debt, r(D), or, alternatively, the debt/assets ratio, r(D/qK).

This paper will not attack the question of the optimal mix of external sources of finance, so other types of external finance, such as equity flotations in excess of retained earnings, will be ruled out. This will be accomplished by constraining dividends to be nonnegative and by prohibiting the issuing of new shares of stock. The exclusion of such new equity is meant to mirror the view that it is a high-cost source of finance, requiring the incurring of substantial transaction or other costs. (See, e.g., Duesenberry [6] and Myers and Majluf [19].)

III. A Neoclassical Model with a Linear Interest Rate Function

In the next two sections we consider the paths of capital and debt that maximize the following generalized value function for the neoclassical firm:

$$V(t_0) = \int_{t_0}^{\infty} e^{-\rho t} DIV(t)dt = \int_{t_0}^{\infty} e^{-\rho t}[pQ - wL - qI - r(D,K)D + \dot{D}]dt, \tag{5}$$

subject to $DIV(t) \geq 0$, $D(t) \geq 0$; $I = \dot{K} + \delta K$; $r = r(D)$ or $r(D/qK)$, $\partial r/\partial D > 0$.

The use of equation (5) as the firm's objective function requires that the managers of the firm view the risks of bankruptcy differently from lenders. While the upward sloping supply curve for debt presupposes that lenders envisage both a risk and costs of bankruptcy, the use of equation (5) assumes that the firm either sets its subjective probability of bankruptcy at zero or does not believe that significant costs are associated with this state.[5] A model based on similar differences of opinion between lenders and the firm's managers is analyzed by Stiglitz [25].

The models that follow also make two fairly innocuous simplifications. Assuming a linear homogenous production function, one can express the optimal labor input (L) as a linear function of capital and the ratio of wage and capital costs; as a result, given fixed input prices, the net revenue term above, pQ-wL, can be written as the function paK-bK, where a and b are positive constants.

Further, for a determinate equilibrium to exist, marginal revenue exclusive of investment costs must be a decreasing function of output or capital. Given the assumption of constant returns to scale on the production side, this must come by virtue of the firm's possession of some degree of market power on the demand side; thus, we will assume a downward sloping demand curve, $p = c-dQ$, with c and d positive constants. Making both substitutions, the net revenue term becomes a quadratic function of K ($\alpha,\beta > 0$):

$$pQ - wL = \alpha K - \beta K^2. \tag{6}$$

Finally, for the model examined in this section we make the simplest assumption for the interest rate function consistent with the requirment that it be upward sloping: $r(D,K) = \rho+\psi D$. Since bankruptcy risk is more properly a function of the debt/assets or debt/equity ratio, rather than the level of debt alone, this assumption is relaxed in the next section. Making the indicated substitutions, equation (5) becomes:

$$V(t_0) = \int_{t_0}^{\infty} e^{-\rho t}\, DIV(t)dt = \int_{t_0}^{\infty} e^{-\rho t}[\alpha K - \beta K^2 - (\rho+\psi D)D - qI + \dot{D}]dt, \quad (7)$$

subject to $DIV(t) \geq 0$, $D(t) \geq 0$, where $I = \dot{K} + \delta K$.

The objective becomes to maximize the integral by choosing optimal paths of capital and debt. Following the results in control theory, incorporating constraints on state variables in, for example, Arrow and Kurz [2] or Kamian and Schwartz [18], one determines the optimal solution by forming a Lagrangean expression (L) made up of the normal (current value) Hamiltonian (H) plus multiplier expressions for the inequality constraints.[6] Thus,

$$L = H + \mu_1 DIV + \mu_2 D, \quad (8)$$

where: $H = DIV + \lambda_1(I - \delta K) + \lambda_2\dot{D}$,

$\mu_1(t)$, $\mu_2(t)$ are the multiplier functions for the inequality constraints; $\lambda_1(t)$, $\lambda_2(t)$ are the costate variables; and $\mu_1(t)DIV(t) = 0$, $\mu_2(t)D(t) = 0$, $\mu_1(t)$, $\mu_2(t) \geq 0$.

Substituting for H and DIV:

$$L = (1+\mu_1)[\alpha K - \beta K^2 - qI - (\rho+\psi D)D + \dot{D}] + \lambda_1(I-\delta K) + \lambda_2\dot{D} + \mu_2 D. \quad (9)$$

An optimal path for the firm's capital stock and debt must satisfy the following necessary conditions:

$$\partial L/\partial I = 0 = -q(1 + \mu_1) + \lambda_1 \tag{10}$$

$$\partial L/\partial \dot{D} = 0 = (1 + \mu_1) + \lambda_2 \tag{11}$$

$$-\partial L/\partial K = \dot{\lambda}_1 - \rho\lambda_1 = -(1 + \mu_1)(\alpha - 2\beta K) + \lambda_1\delta \tag{12}$$

$$-\partial L/\partial D = \dot{\lambda}_2 - \rho\lambda_2 = (1 + \mu_1)(\rho + 2\psi D) - \mu_2, \tag{13}$$

with, as noted above, $\mu_1(t)$, $\mu_2(t) \geq 0$, $\mu_1 DIV = 0 = \mu_2 D$.

Irrespective of whether the constraint on dividends is binding, the equations have important implications for the relationship between the optimum capital stock and debt. Equations (10) and (11) imply that $\lambda_2 = -\lambda_1/q$. Although not important, assume for simplicity that q is not a function of time; then $\dot{\lambda}_2 = -\dot{\lambda}_1/q$. In this case, the left hand side of (12) equals -q times the left hand side of (13). From this we have:

$$(1 + \mu_2)(\alpha - 2\beta K) - \lambda_1\delta = q(1 + \mu_1)(\rho + 2\psi D) - q\mu_2. \tag{14}$$

It can be shown, further, that μ_2 is always zero -- i.e., even in the unconstrained problem it turns out that debt cannot violate the nonnegativity constraint.[7] Finally, since it is also true that $(1+\mu_1) = \lambda_1/q$, equation (14) simplifies to:

$$\alpha - 2\beta K = q(\rho + \delta + 2\psi D). \tag{15}$$

Solving for the capital stock, equation (15) becomes:

$$K(t) = \frac{\alpha - q[\rho + \delta + 2\psi D(t)]}{2\beta} \quad . \tag{16}$$

Except for the term containing D(t), this equation is a familiar variant of the neoclassical equation for the firm's optimal capital stock; $q(\rho+\delta)$ is the cost of capital for this latter model. It is also clear from (16) that it must be the case that $\alpha-q(\rho+\delta)>0$, or K(t) can never be positive. From equation (16) we can also see a familiar implication of the Jorgensonian theory of investment where no dynamic adjustment costs are present: depending on the level of debt, equation (16) may imply a jump in the stock of capital. Instead of the usual differential equation in K(t) encountered in control theory, equation (16) is a degenerate equation linking (only) optimal <u>levels</u> of K(t) and D(t).

The system at any point in time is completed by the inequality constraints DIV $\geq$ 0 and D $\geq$ 0, and the equation for dividends (the sources and uses of funds identity). Except when a jump occurs, which is discussed below, the dividend expression is:

$$DIV = \alpha K - \beta K^2 - q(\dot{K}+\delta K) - (\rho+\psi D)D + \dot{D} \geq 0. \tag{17}$$

Of the four possible combinations of dividends and debt greater or equal to zero, two can be ruled out as either impossible or irrelevant. We prove in the appendix (section I) that the two cases D>0, DIV>0 and D=0, DIV=0 are impossible. Thus, if D>0, the only possible case is DIV = 0: the firm pays no dividends as long as it is paying interest on debt -- at an interest rate higher than the firm's discount rate. In this case, unless a jump occurs, the system becomes equation (16) and equation (17) holding with an equality; these are sufficient to determine the optimal path of K(t) and D(t).

The other possible case is D=0 and DIV>0. It turns out that this is the long-run stationary equilibrium (Appendix, section IV). Using equation (16) and setting debt equal to zero, one derives the long-run capital stock, K*: $[\alpha-q(\rho+\delta)]/2\beta$. Given that $D=\dot{D}=\dot{K}=0$ in this state, equation (17) is sufficient to determine equilibrium dividends.

Investment and Debt: the Initial Jump

To examine the dynamic process of investment, debt expansion and reduction, and dividend payments, let us consider the effects of an exogenous shock to the system. Suppose that initially the firm is in a state of long-run equilibrium, with a stationary capital stock and debt equal to zero, as discussed above. Consider the effects of an upward shift in the firm's demand curve -- causing a discrete upward shift in the parameter α to α^* in equation (15) or (16).

Since equation (16) holds at every point in time, whether there is a jump or not, the first point to notice is that either D or K or both will have to undergo a discrete jump in order that the equation be satisfied. The necessary conditions for optimal control with possible jumps in the state variables are discussed at length in the appendix (section III), following work by Vind [27] and Arrow and Kurz [2]. It is shown there that equation (16) still holds in the presence of a jump, and that the levels of the jumps in capital and debt are related by a variant of the firm's sources and uses of funds identity, equation (18), below. At the instant of a jump, the net impact of any <u>flow</u> is zero, so the sources and uses of funds equation (17) becomes particularly simple, linking the value of (jump) changes in liabilities and assets (i.e., the first difference in the firm's balance sheet identity). Incorporating the assumption of no new equity financing and the finding that dividends are zero during the jump (appendix, section III), the sources and uses of funds equation over the jump becomes:

$$D^{+}(0) - D^{-}(0) = q[K^{+}(0) - K^{-}(0)], \tag{18}$$

where the "+" and the "-" superscripts refer to the levels of the indicated variable just after and before the jump. Using (18) to substitute for the variable $D^{+}(0)$, one can then use equation (16) to determine the optimal jump for the capital stock, $K^{+}(0)$. One can show further that, because of the concavity of the relevant functions, there can be a jump only at time zero.[8]

The optimal jump and subsequent developments are depicted in figure 1. In the figure, the capital stock, K(t), is measured along the horizontal axis. The level of debt and various functions of the capital stock are measured along the vertical axis; for simplicity, the price of a unit of capital stock, q, will be set at 1 at every point in time.

The firm is assumed to be initially at a point of long-run equilibrium, point E_0 on the K axis: a point with no debt and an equilibrium capital stock of $K^{-}(0)$.[9] In figure 1 the left hand side of the new equilibrium condition (15), $\alpha^{*}-\beta K$, after the upward shift in α to α^{*}, is represented by the curve MRP (the firm's marginal revenue product); the right hand side of (15), the firm's marginal cost of capital, will be shown <u>at the time of the jump</u> to be the line labeled MCC. Originally the MRP curve passed through the point $[K^{-}(0), q(\rho+\delta)]$, where the MRP was equal to $q(\rho+\delta)$, the marginal cost of capital in long-run equilibrium when debt equals zero. Now, however, the discrete increase in α has shifted the

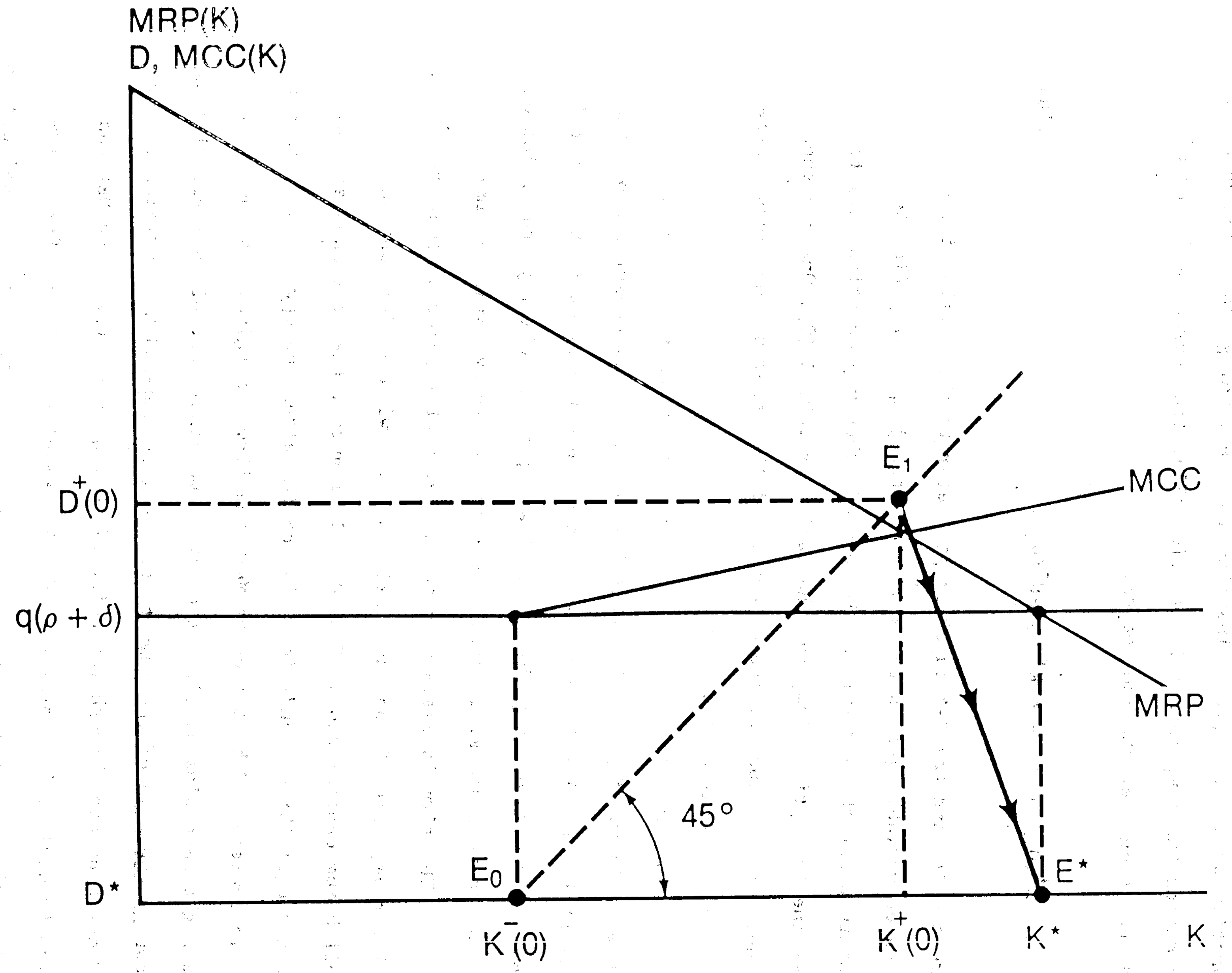

Figure 1

Optimal Paths of Debt and Capital for the Model of Section

MRP upward and to the right, indicating that at the initial level of capital and debt $[K^{-}(0), 0]$, the equilibrium condition (15) no longer holds. Given the discrete shift in the MRP curve, we have seen that only a discrete jump in capital will reestablish equilibrium; in the very short run, this jump in capital can only be financed by an equal (in value) jump in debt. Since equation (18) shows that changes in debt and capital during the jump must be proportional to each other, and given the assumption that q=1, the two jumps in figure 1 must be along the 45 degree line that passes through the original equilibrium point E_0.

With equation (18) satisfied at all points on the 45 degree line, the optimal jump is determined by that combination of debt and capital on the 45 degree line that satisfies the other equilibrium condition, equation (15), i.e., that combination that equates the MRP curve to the marginal cost of capital, $q(\rho+\delta+2\psi D)$. This latter is a function of D, but during the jump, since (18) shows that debt and capital jump in proportion, the MCC can be represented as a function of capital; the MCC curve is shown in figure 1 as a straight line through the old equilibrium point with a slope of 2ψ.[10] Thus the new optimal short-run level of capital is determined in figure 1 by the intersection of the MRP and MCC curves; this is denoted by $K^{+}(0)$, implying an optimal jump of $K^{+}(0)-K^{-}(0)$. One finds the optimal level (and jump) of debt, $D^{+}(0)$, by moving from $K^{+}(0)$ up to the 45 degree line and over to the y axis. Thus, the point E_1 is the new short-run equilibrium.

Determining the Optimal Paths of Capital and Debt after the Jump

Unlike the original Jorgenson model, the determination of the optimal jump is not the end of the story, but only its beginning. The key to the movement to the long-run equilibrium is that the firm's financial resources change over time endogenously as the firm operates and generates profits. These profits are linked to further capital stock growth, changes in debt, and dividend payouts through the flow of funds constraint (17). The dynamic paths of these variables are of course determined by equilibrium condition (16) and the flow of funds constraint (17). [11]

In Stevens [24] considerable space is devoted to solving the system defined by equation (16) and the differential equation (17), and describing the dynamic paths implied for K(t) and D(t) after the jump. However, the two most important points about the path can be demonstrated without the full derivation. First, the equilibrium condition (16), which still holds at every point along the dynamic path, implies a negative <u>linear</u> relationship between capital and debt; thus in (K,D) space in figure 1, the adjustment path appears as a straight line -- between E_1 and the final new long-run equilibrium, E*, where debt equals zero again and capital, K*, equals $[\alpha - q(\rho+\delta)]/2\beta$.

A second important point is that, unlike the "normal" dynamic adjustment path of stock-adjustment models, investment does not gradually decrease over time, allowing an asymptotic "soft landing" at the long-run equilibrium capital stock. Rather, in this model, both capital and debt approach their equilibrium values at ever-increasing rates. This can be verified by examining equation (17) above, while noting that dividends equal zero and that the time derivatives of capital and labor are related linearly by differentiating equation (16): $\dot{K} = -q\psi\dot{D}/\beta$. Substituting for $\dot{D}$ in equation (17), we have along the optimal path:

$$\dot{K}[q+\beta/(q\psi)] = [\alpha K - q\delta K - \beta K^2] - (\rho+\psi D)D \qquad (19)$$

We can show that the rate of change of $\dot{K}$, $d\dot{K}/dt$, is always positive by examining the time derivative of the right hand side of (19). The last term, a function of D, is just the level of interest payments on the debt; since debt is falling all along the optimal path, both components of the interest payment are falling over time, implying a positive effect over time on $\dot{K}$. The remaining terms on the right hand side, in brackets, equal operating profits minus depreciation costs; this will be positive as long as its derivative with respect to K is positive: $\alpha - q\delta - 2\beta K > 0$. This, of course, must hold from equilibrium condition (15), which states that marginal operating profit, $\alpha - 2\beta K$, must cover both depreciation costs and marginal interest costs. Thus the time derivative of the right hand side of (19) must be positive along the optimal path and $\dot{K}$ increases throughout.

The Investment Function and Cash Flow

A key aim of this paper is to examine the firm's investment function in the light of the empirical results of Coen [5], Artus et. al. [3], and others successfully linking the speed of adjustment to cash flow. Using equation (16), it is possible to relate the firm's investment both to cash flow and to the "gap" between actual capital and the long-run equilibrium level, K* (defined above):

$$K^* - K(t) = q\psi D(t)/\beta. \tag{20}$$

We have noted how the differentiation of equation (16) or (20) leads to a linear relationship between $\dot{K}$ and $\dot{D}$ (after the initial jump):

$$\dot{K}(t) = -q\psi\dot{D}(t)/\beta. \tag{21}$$

Combining equation (21) with equation (20) leads to:

$$\dot{K}(t) = -\dot{D}(t)/D(t)[K^{*} - K(t)]. \tag{22}$$

Equation (22) shows that the investment function can be written as a flexible accelerator with a variable speed of adjustment -- the latter equal to the absolute value of the percentage rate of change of debt.

One can go further, however, and, using equation (19), relate the above speed of adjustment to the firm's cash flow.[12] Note that the right hand side of (19) equals net revenues ($\alpha K - \beta K^2$) minus depreciation charges ($q\delta K$), minus interest costs on the debt, $(\rho + \psi D)D$. The right hand side, therefore, is a measure of net profits; denote it by $\Pi(t)$. Further, since $\dot{K}$ and $\dot{D}$ are linearly related as shown in (21), the left hand side of (19) can be expressed as a linear function of $\dot{D}$. Thus, equation (19), a version of the sources and uses of funds constraint, reduces to:

$$-\dot{D}(t)[1 + q^2\psi/\beta] = \Pi(t). \tag{23}$$

Using (23) to substitute for $\dot{D}$ in the investment function (22), we derive a final investment function that features a variable speed of adjustment that is a function of the firm's net profits:[13]

$$\dot{K} = \left\{\frac{\beta\Pi(t)}{(\beta + q^2\psi)D(t)}\right\}[K^{*} - K(t)]. \tag{24}$$

We have in equation (24) an investment function that is remarkably similar to the suggestive, but ad hoc, formulation of equation (1) used successfully in empirical work by the researchers noted above.

For the relevant range, optimal investment can be described by a flexible accelerator with a variable speed of adjustment. Moreover, the speed of adjustment is a function of the firm's level of profits, proportional to the ratio of net profits to its level of debt.

IV. A More Realistic Model: Corporate Taxes and the Borrowing Rate a Function of the Debt/Assets Ratio

In this section we will see that the general lessons of the previous section carry over to more realistic models: a dynamic investment function will again be implied by the upward sloping supply curve for debt, and the speed of adjustment will be a function --albeit more complicated-- of the firm's cash flow.

One of the most appealing modifications of the preceding model is to let the firm's borrowing rate be a function of the debt/equity or the debt/assets ratio, rather than the level of debt alone. We have alluded to the risk of bankruptcy as the preferred theoretical reason supporting the upward sloping supply curve of debt. With this justification, the probability of the firm falling into the bankruptcy state, where it cannot cover its interest costs, can be shown to be a function of the debt/assets ratio and not of the absolute level of debt.[14]

Despite this more realistic supply schedule for debt, the optimal long-run level of debt would still be zero. An obvious modification that assures a positive long-run debt/assets ratio is to introduce corporate income taxation along with the usual deductibility of interest.[15]

The two new assumptions result in predictable changes in the optimal control problem of section III, equations (5) through (16). The borrowing rate, $r(t)$, will now be a function of the debt/assets ratio, ϕ:

$$r(t) = \rho + \psi\phi = \rho + \psi[D(t)/qK(t)]. \qquad (25)$$

Corporate taxes reduce potential dividends payable to stockholders. We shall assume the simplest possible case: a constant tax rate, τ, assessed on total revenue minus interest and depreciation charges:

$$\text{Tax} = \tau[\alpha K - \beta K^2 - (\rho + \psi\phi)D - \delta qK]. \qquad (26)$$

Incorporating these modifications into equation (7), the expression for the value of the firm that is to be maximized, leads to the following modified Lagrangean expression:[16]

$$L = (1+\mu_1)(1-\tau)[\alpha K - \beta K^2 - q\delta K - (\rho+\psi D/qK)D] + \dot{D} - q\dot{K} + \lambda_1\dot{K} + \lambda_2\dot{D}. \qquad (27)$$

subject to: $\dot{K} = I - \delta K$, $\mu_1(t)DIV(t) = 0$, and $\mu_1(t) \geq 0$. The new necessary conditions for an optimum are:

$$\partial L/\partial\dot{K} = 0 = -q(1 + \mu_1) + \lambda_1 \qquad (28)$$

$$\partial L/\partial\dot{D} = 0 = (1 + \mu_1) + \lambda_2 \qquad (29)$$

$$-\partial L/\partial K = \dot{\lambda}_1 - \rho\lambda_1 = -(1+\mu_1)(1-\tau)[\alpha - 2\beta K - q\delta + q\psi D^2/(qK)^2] \qquad (30)$$

$$-\partial L/\partial D = \dot{\lambda}_2 - \rho\lambda_2 = (1+\mu_1)(1-\tau)(\rho + 2\psi D/qK), \qquad (31)$$

and with, $DIV = (1-\tau)[\alpha K - \beta K^2 - q\delta K - (\rho+\psi\phi)D] - q\dot{K} + \dot{D} \geq 0.$ (32)

As was the case for the preceding model, the costate variables can be eliminated from equations (28) to (31) and, irrespective of whether the constraint on dividends is binding, one can solve for the capital stock in terms of the determinants of the firm's marginal revenue product (α,β) and its marginal cost of capital $[q(\rho+\delta+2\psi\phi-\psi\phi^2)]$:

$$K(t) = \frac{\alpha - q[\rho + \delta + 2\psi\phi - \psi\phi^2]}{2\beta}. \tag{33}$$

The solution of this system is qualitatively quite similar to that of the preceding section. Equation (33) shows that shifts in α or any other parameter of the model will lead to jumps in K(t) and D(t) -- and the ratio of the two, ϕ. Moreover, by an analysis similar to that above we can demonstrate that of the various possible combinations of the inequality constraints on D and DIV, only two are possible. (See the appendix, section 2.) Denoting the long-run equilibrium debt/assets ratio as ϕ^* (to be determined below), the two feasible cases become: DIV=0, $\phi\geq\phi^*$ and DIV$\geq$0, $\phi=\phi^*$. In the present model, with a positive long-run debt/assets ratio, when that ratio is above the long-run optimum, dividends fall to zero. When the debt/assets ratio is worked down to ϕ^*, both debt and capital attain their long-run equilibrium values and dividends becomes positive.

Similar to the approach taken in the earlier model, one can determine the long run equilibrium values K*, ϕ^*, and D* by examining the solution to the system when μ_1 becomes zero (dividends become non-negative). Noting that λ_1 and λ_2 are in this case equal to q and -1, respectively, from equations (30) and (31) we derive the following:

$$\phi^* = D^*/qK^* = \tau\rho/(1-\tau)2\psi \tag{34}$$

$$K^* = \frac{\alpha - q[\rho/(1-\tau) + \delta - \psi\phi^{*2}]}{2\beta} \tag{35}$$

We will assume that the parameters ρ, τ, and ψ take on values such that ϕ^* is less than 1.[17]

In Stevens [24] the dynamic paths of the endogenous variables in response to a shock to the system are examined in detail. Since we are primarily interested in the form of the associated investment function, and since the development is similar to that in the previous section, I shall be brief here. Assume that the system is in long-run equilibrium and consider, again, the effects of an upward shift of α to α^*, related to the intercept of the firm's demand curve.

Once again there must be a jump in K and D to satisfy equation (33); the modified sources and uses of funds equation (18) applies during the jump, providing a second equation to determine both K and D. Since D and K increase in proportion, the debt/assets ratio increases above the long-run equilibrium level ϕ^* and dividends fall to zero.

From this point, after the initial jump, the dynamics of the system are governed by equation (33) and the modified dividend equation (32). Since both equations are nonlinear because of the debt/assets ratio, it is impossible in this case to derive an explicit solution for the paths of debt and capital. However, it is shown in the Appendix (section II) that capital increases monotonically from its initial value to the long-run equilibrium K*; as well, the debt/assets ratio, ϕ, decreases monotonically to ϕ^*.

It should be noted that this model, like the previous one, leads to ever-increasing rates of investment, $\dot{K}$, along the optimal path. The monotonic fall of the debt/assets ratio, combined with the increasing generation of after-tax profits, leads to continuously increasing funds for investment.

4.1 <u>A New Relationship between Investment and Financial Variables</u>

Let us once again return to the ultimate question: How do the complications introduced in this particular model affect the validity of the variable speed-of-adjustment investment function developed by Coen and others? As might be suspected, the nonlinearities introduced by making the debt/assets ratio the key variable in the interest rate function severly complicate the form of the investment function. Nevertheless, we still have the elements that link the variables $\dot{K}$, $\dot{D}$, and after-tax profits, Π: the dividend equation (32) and the fact that optimal dividends are zero during the period when net investment is positive. It should be noted again that equation (32) holds only after the initial jump.

In deriving the investment function, I shall take the same approach as in the preceding section: first deducing the level of capital in terms of its deviations from the long-run equilibrium, K*, and then differentiating this expression with respect to time. Equation (33) relates the capital stock at any given time to a single endogenous variable: in this case the debt/assets ratio, ϕ. Subtracting the expression for K(t) in (33) from K*, defined in equation (35), gives:

$$2\beta(K^*-K)/q = 2\psi\phi - \tau\rho/(1-\tau) - \psi(\phi^2 - \phi^{*2}), \qquad (36)$$

From the expression (34) for the optimal debt/assets ratio, ϕ^*, it can be seen that the second term of the right hand side of (36) equals $-2\psi\phi^*$, so K^*-K may be expressed compactly solely as a function of the difference between the actual and the long-run equilibrium debt/assets ratio:

$$K^* - K = q\psi[2(\phi-\phi^*) - (\phi^2-\phi^{*2})]/2\beta. \tag{37}$$

Differentiating this nonlinear function with respect to time leads to one form of the investment function:

$$\dot{K} = -q\psi(1-\phi)\phi/\beta\{\dot{D}/D - \dot{K}/K\} > 0. \tag{38}$$

This is obviously more complicated than the simple relation, $\dot{K} = -q\psi/\beta\dot{D}$, derived for the previous model. However, after the initial jump, the dividend equation still leads to $q\dot{K}-\dot{D} = \Pi$, where Π is now defined as <u>after-tax</u> profits. Substituting $q\dot{K}-\Pi$ for $\dot{D}$, equation (38) becomes:

$$\dot{K} = -q\psi(1-\phi)\phi/\beta[q\dot{K}(1-\phi)/D - \Pi/D] = \frac{(q/\beta)\psi(1-\phi)\phi\Pi}{q(q/\beta)\psi\phi(1-\phi)^2 + D}. \tag{39}$$

Dividing (39) through by the expression for K^*-K, we arrive at a significantly more complicated form of the variable speed-of-adjustment investment function, but one which retains the key property that the speed of adjustment depends on the firm's profits:

$$\dot{K} = \left\{\frac{2\phi(1-\phi)\Pi}{[D + q^2\psi\phi(1-\phi)^2/\beta][2(\phi-\phi^*) - (\phi^2-\phi^{*2})]}\right\}[K^* - K(t)] \tag{40}$$

V. Conclusions and Caveats

This paper demonstrates that it is possible to provide theoretical support for the eclectic investment functions tested successfully by Coen [5], Artus et. al. [3], and others. Models of the firm featuring intertemporal profit maximization subject to increasing costs of external debt lead to investment functions that, in important ways, are close to those estimated empirically. In particular, the theoretically-implied investment functions are a combination of the concepts of a flexible accelerator and a variable speed of adjustment, with the latter dependent on the supply of internal funds.

The dynamic adjustment mechanism implied by investment functions of this class is unusual with respect to both its cause and pattern. Unlike the typical dynamic-cost explanation of investment over time, costs in the above models are static, independent of the rate of change of any variable. However, borrowing costs are increasing functions of variables, debt or the debt/assets ratio, that change dynamically because of the firm's operations. Thus, these static costs also change over time as a result of the firm's actions and, as the costs change, investment is generated.

Not only is the causal mechanism generating investment different, but so is the dynamic pattern of the investment. Rather than having the rate of investment decline monotonically as the firm's capital stock approaches the equilibrium level, investment in this model <u>increases</u> monotonically as an increasing supply of internal funds is divided optimally between debt reduction and investment.

Although the models presented in this paper serve to provide theoretical underpinnings for this important class of empirical investment functions, one must note the caveats. We have pointed out that the investment function does not hold for points where capital and debt jump. Moreover, the underlying structure of the theoretical model suggests extensions to more general models and investment functions. The class of investment functions examined in this paper depends on a model with a <u>single</u> dynamic element determined by the cost of outside debt and the supply of internal funds. Models with additional dynamic factors should imply different investment functions. An illustration is provided by a small modification of the first model studied in section III: letting the demand-curve parameter α be a continuous function of time, instead of a constant. It is easily shown that the same key equations (16) and (17) hold to determine the optimal paths of capital and debt. However, from (16) the capital stock now is a function of two dynamic variables, α and D; and investment, $\dot{K}$, equals $[\dot{\alpha}-2q\psi\dot{D}]/2\beta$, instead of the previous $-2q\psi\dot{D}/2\beta$. Both terms now enter the variable speed of adjustment. Despite the presence of this second term, the key relationship between $\dot{D}$ and profits (Π) is still operational, so the speed of adjustment will continue to be a function of the firm's cash flow. Now, however, cash flow will be only part of the adjustment story.

APPENDIX

I. Feasible and Infeasible Cases in the Model of Section III ($r = \rho+\psi D$)

A. D and DIV cannot be both positive simultaneously

Assume the contrary: D>0 and DIV>0. This implies that both multipliers for the inequalities, μ_1 and μ_2, must both equal 0; the former implies, from (14) that $\lambda_1 = -1$ and $\dot{\lambda}_1 = 0$. Substituting these values into (13), the marginal condition for debt, leads to $\rho=\rho+\psi D$; since ψ is positive, D must be equal to zero -- which leads to the contradiction.

B. If D=0, then DIV>0 (ruling out case D=0, DIV=0)

With D=0, we see from equation (16) that K is a constant equal to $[\alpha - q(\rho+\delta)]/2\beta$. By assumption the numerator is positive; if it were not, the optimal capital stock would never be positive, for below, in section 4, this value is shown to be K*, the long-run equilibrium value and the maximum attained by the capital stock. Moreover, with D=0, the dividend equation becomes: $DIV = [\alpha K - \beta K^2 - q\delta K] - q\dot{K} + \dot{D}$. The first three terms are profits after depreciation costs (Π) and, given the positive level of K, can be shown to be positive. We must rule out the possibility that the sum of the last two terms could offset the first three.

From equation (21) the sign of this sum depends on the sign of $\dot{D}$. Since D=0 is at its lower limit, and given that D must have continuous first derivatives, $\dot{D}$ cannot be negative or the non-negativity of D would be violated; hence $\dot{D}$ must be zero or positive. In both cases DIV > 0 and the assertion is proved.

II. Feasible and Infeasible Cases in the Model of Section IV ($r = \rho+\psi\phi$)

A. If $\phi > \phi^*$, DIV = 0 (where $\phi = D/qK$).

The proof is similar to that of section I.A above. Assume the contrary, that DIV > 0. But this implies that $\mu_1 = 0$, and from equations (28) and (31), ϕ must equal ϕ^* -- leading to a contradiction.

B. After the initial jump, ϕ decreases monotonically.

Suppose $\dot{\phi}>0$. Assume for simplicity that q=1. Given $\dot{\phi}>0$, we have from equation (33) that $\dot{K}<0$. This immediately rules out the sub-case with $\dot{D}>0$, for the combination $\dot{D}>0$, $\dot{K}<0$ violates the sources and uses of funds constraint $\dot{K} - \dot{D} = \Pi>0$ (where Π is profits). Consider the other sub-case with $\dot{D}<0$. From the initial assumption we have $\dot{\phi}=\dot{D}/D-\dot{K}/K>0$, implying $0>\dot{D}>\dot{K}\phi$, and $0>(1-\phi)\dot{K}>\dot{K}-\dot{D}$. But $\dot{K}-\dot{D}<0$ contradicts the implication of the sources and uses constraint that says it must be positive and equal to Π. Thus $\dot{\phi}$ must be ≤ 0 along the optimal path. The case of $\dot{\phi}=0$ can be ruled out by noting that this implies that both $\dot{K}$ and $\dot{D}$ also equal zero.

III. Jump Conditions

Conditions under which state variables may have jumps are developed in detail in Arrow and Kurz [2], pp. 51-57, and Kamien and Schwartz [18], section 18. Both sources exposit the approach originally developed by Vind [27], which fits jumps into the normal optimal control framework by distinguishing between "natural" time, t, which is suspended during jumps, and "artificial" time, w, which is continuous throughout.

The approach is adapted to the modified Jorgenson model of section 3 as follows. Natural time, t, is now assumed to be a state variable whose motion is determined by an on-off control, u_0:

$$dt/dw = u_0(w) = \begin{cases} 0, & \text{during the jump} \\ 1, & \text{otherwise} \end{cases} \tag{A1}$$

Similarly, all state variables that can have jumps have on-off switches incorporated into their laws of motion. Thus, in the problem of section III, the differential equation for the capital stock, $\dot{K}=I-\delta K$, becomes:

$$dK/dw = u_0(I-\delta K) + (1-u_0)u_1 \tag{A2}$$

where $u_1 = dK/dw$ = some constant per unit of w during a jump. Thus, over a jump occurring at t_0, using (A2) and the fact that u_0 equals zero during a jump:

$$K^+(t_0) - K^-(t_0) = \int_{w_1}^{w_2} (dK/dw)\,dw = \int_{w_1}^{w_2} u_1 dw = u_1(w_2 - w_1). \tag{A3}$$

One can let u_1 be an arbitrary constant, for by letting w_2-w_1 increase, the change in K during the jump can attain any positive value. However, since we have a second state variable subject to jumps, dD/dw during a jump is not arbitrary, since the change in D can, in principle, be different from the change in K. Let it equal u_2. We show below that, in the present case, $u_1 = u_2$.

A similar approach can be taken toward the sources and uses of funds identity or dividend equation. During a jump the normal expression for dividends is switched off [the right hand side of equation (7)]; during the jump, sources of funds are changes in debt, $D^+(t_0)-D^-(t_0)= u_2(w_2-w_1)$, and uses are the sum of changes in the value of the capital stock (qu_1 per unit of w) and dividends. Thus, we have:

$$DIV(w) = u_0[\alpha K - \beta K^2 - (\rho+\psi D)D - qI + \dot{D}] + (1-u_0)[u_2 - qu_1] \tag{A4}$$

We can now transform our original problem into this new notation. Modifying equation (7) in section II, we wish to maximize:

$$V(t_0) = \int_{t_0}^{\infty} e^{-\rho t}\left\{u_0[\alpha K - \beta K^2 - (\rho+\psi D)D - qI + \dot{D}] + (1-u_0)[u_2 - qu_1]\right\}dw \tag{A5}$$

subject to $DIV(w) \geq 0$, $D(w) \geq 0$, where $I = \dot{K} + \delta K$, and where t is treated as a state variable such that dt/dw is defined by equation (A1). We set up a Lagrangean expression similar to equations (8) and (9) in the text:

$$\begin{aligned} L = {} & e^{-\rho t}(1+\mu_1)\left\{u_0[\alpha K - \beta K^2 - (\rho+\psi D)D - qI + \dot{D}] + (1-u_0)[u_2 - qu_1]\right\} \\ & + \lambda_1[u_0(I-\delta K)+(1-u_0)u_1] + \lambda_2[u_0\dot{D} + (1-u_0)u_2] + \lambda_0 u_0 \end{aligned} \tag{A6}$$

Besides the presence of the switch variable, this equation differs from the Lagrangean in section III in a few, unimportant ways. Since it turns out that D is never less than zero (see footnote 7), the non-negativity constraint on D has been dropped. Also, because the discount factor $e^{-\rho t}$ does not change when a jump occurs, equation (A6) does not use the current value Hamiltonian. In addition to the original control variables, I and $\dot{D}$, we also have u_0, u_1, and u_2; moreover, besides K and D as state variables, there is also t. The necessary conditions for the optimal paths are:

$$\partial L/\partial I = 0 = -q(1+\mu_1)e^{-\rho t}u_0 + \lambda_1 u_0 \qquad (A7)$$

$$\partial L/\partial \dot{D} = 0 = (1+\mu_1)e^{-\rho t}u_0 + \lambda_2 u_0 \qquad (A8)$$

$$-\partial L/\partial K = \dot{\lambda}_1 = -(1+\mu_1)(\alpha - 2\beta K)e^{-\rho t}u_0 + \lambda_1 \delta u_0 \qquad (A9)$$

$$-\partial L/\partial D = \dot{\lambda}_2 = (1+\mu_1)(\rho + 2\psi D)e^{-\rho t}u_0, \qquad (A10)$$

$$\partial L/\partial u_1 = 0 = (1-u_0)[-q(1+\mu_1)e^{-\rho t} + \lambda_1] \qquad (A11)$$

$$\partial L/\partial u_2 = 0 = (1-u_0)[(1+\mu_1)e^{-\rho t} + \lambda_2] \qquad (A12)$$

$$-\partial L/\partial t = \dot{\lambda}_0 = -\rho(1+\mu_1)[u_0\{\alpha K - \beta K^2 - (\rho+\psi D)D - qI + \dot{D}\} + (1-u_0)\{u_2 - qu_1\}] \qquad (A13)$$

Noting that u_0 can only take the values 0 or 1, the maximum of L with respect to u_0 is either the value of the original Lagrangean expression of section III (when u_0 equals 1) or $u_2-qu_1 + \lambda_1 u_1 + \lambda_2 u_2$ (when $u_0=0$). As noted above, $\mu_1(t) \geq 0$ and $\mu_1 DIV = 0$.

The first point to make is that when no jump is occuring ($u_0=1$), all the findings of section III still hold. For example, one can reduce equations (A7) to (A10) to equation (15) of section III:

$$\alpha - 2\beta K = q(\rho + \delta + 2\psi D). \qquad (A14)$$

Moreover, since the necessary conditions are equivalent to those in section III for $u_0 = 1$, all the propositions about the non-negativity of debt and dividends, as well as those concerning the long-run, also still hold.

Concerning states when jumps occur, the theorem of Arrow and Kurz (Proposition 12, p.57) can be applied to show that given the concavity of the original problem in the state variables, K(t) and D(t), there can be a jump only at time zero. Thus, immediately after the jump, i.e., using the notation "+" and "-" introduced in section 3 of the text to denote post and pre-jump values at time zero, (A14) must hold:

$$\alpha - 2\beta K^+(0) = q[\rho + \delta + 2\psi D^+(0)]. \qquad (A15)$$

We need one more relation to determine the values of $K^+(0)$ and $D^+(0)$. This will be some form of the sources and uses of funds identity. In fact we can prove that capital and debt jump change in proportion to each other, thus validating equation (18) of the paper. The first term of the Lagrangean (A6) is dividends (DIV); given that $u_0 = 0$, during the jump DIV = (u_2-qu_1). We want to prove that DIV equals zero during the jump, thus implying that $u_2=qu_1$. Note first that u_2 cannot be $<qu_1$, for then the non-negativity of dividends would be violated, since u_1 is positive.

Suppose, on the contrary, that u_2 were so high that dividends became positive during the jump -- a one-shot payment to shareholders. One can show that this path for dividends would be suboptimal by examining the costate variable for debt, λ_2, at the end of the jump. Since u_2 must be positive during the jump, at the end of the jump the firm has a positive level of debt. We have shown above (Appendix, section I) that, therefore, dividends must be zero at the end of the jump ($t=0+\epsilon$), if not during the jump. But, given that the constraint is binding at the end of the jump, equation (11) shows that the costate variable for debt must be greater than 1 in absolute value: $\lambda_2=-(1+\mu_1)e^{-\rho t}=-(1+\mu_1)$. Thus, at the end of the jump the marginal value of reducing debt by one unit ($-\lambda_2$) is greater than one. However, the value of one dollar of dividends during the jump is just one, since t=0 and no discounting is involved. Thus, had there been positive dividends during the jump, it would always be in the interest of the shareholder to reduce dividends by a dollar in exchange for a dollar reduction of the debt at the end of the jump. It, therefore, can never by optimal to have positive dividends during the jump, and u_2 must equal qu_1.

We thus have the proof of equation (18) in section III of the paper -- that during the jump the sources and uses of funds identity reduces to:

$$D^+(0) - D^-(0) = q[K^+(0) - K^-(0)]. \tag{A16}$$

Given (A15) and (A16), the optimal jump is determined.

IV. Long Run Equilibrium

In this section we will show that the point determined by μ_1=0 is indeed a long-run equilibrium for both models, in the sense that $\dot{K}(t)$ and $\dot{D}(t)$ are zero at this point. Moreover, it can be show that the equilibrium is actually reached.

Both equilibria can be computed by setting μ_1 equal to zero. For both models this implies that λ_1 equals q and λ_2 equals -1, and that the derivatives of both costate variables equal 0. Thus, for the model in section III, introducing the above values for the costate variables into equation (12) implies that $K(t) = [\alpha - q(\rho+\delta)]/2\beta$ and doing the same in equation (13), noting that μ_2 =0, implies that D(t) = 0. Using the corresponding equations (30) and (31) for the model in section 4, one derives the corresponding values for ϕ^* and K^* shown in equations (33) and (34).

Two major points must be established with respect to these asserted equilibria: first, that the respective systems actually reach them and, second, that the points are stationary equilibria --in the sense that the derivatives are zero at the point.

Considering the system of section III first, we can prove that the proposed equilibrium is attained because it has already been shown that $\dot{K}$ and $\dot{D}$ are monotonic and increasing in <u>absolute</u> value. (See equation (19) and the associated discussion.) Thus, for example, for any feasible path,

we have $K(t) - K(t_0) = \int_{t_0}^{t} \dot{K}dt \geq \dot{K}(t_0)(t-t_0)$, since $\dot{K}$ is increasing. But

since for some t^*, we can make the right hand side of the expression equal to $K^*-K(t_0)$, we know that K(t) will reach K^* at or prior to t^*. A similar argument holds for the model of section IV, since it can be proved that the debt/assets ratio, ϕ, decreases monotonically and, by implication from equation (33), $\dot{K}$ increases monotonically (section II.B of this appendix).

For the first model, the monotonicity of the derivatives also assures that the derivatives are zero at the proposed equilibria. Consider the derivative $\dot{D}$ at the point (K^*, 0). Since D has hit its constraint, the derivative (which is piecewise continuous) cannot be negative at D=0. Further, the Arrow-Kurz theorem shows that there can be no jump here. Given the continuity of $\dot{D}$, if $\dot{D}(0)$ were positive, it would also have to be so for some point $0+\epsilon$; but this is contradicted by the fact that D is monotonically decreasing. Thus, $\dot{D}(0)$ must be zero and the point (K^*, 0) is the (unique) long-run equilibrium for the model of section III.

For the model in section IV the argument must be different because at the proposed long-run equilibrium, (K^*, ϕ^*), the system has not hit the constraint D=0. One argument is simply to note that any values of K and ϕ different from K* and ϕ* imply that μ_1 must be greater than zero; but in such a case no dividends are paid and the value of the firm equals 0. Hence, it is optimal for the firm to settle at K*, ϕ*.

Moreover, it would be sub-optimal for the firm to attain (K*, ϕ*) and then depart from the point; first, since it has been shown that ϕ decreases monotonically, once passing through the point, the system will never return to it, ϕ falling until the constraint D=0 is attained. Second, one can show easily that of any stationary equilibrium, the point (K*, ϕ*) leads to the highest profits or dividend stream. To verify this one can find the stationary policy that maximizes either the integral:

$$\int_{t_0}^{\infty} e^{-\rho t}\{(1-\tau)[\alpha K - \beta K^2 - (\rho+\psi\phi)D] - qI + \dot{D}\}dt,$$

or the equivalent single-period objective function:

$$(1-\tau)[\alpha K - \beta K^2 - (\rho+\psi\phi)D - q\delta K] - \rho(qK - D).$$

The maximum of both is the long-run equilibrium (K*,ϕ*).

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FOOTNOTES

1. The author is a staff economist in the Divison of International Finance. The views presented in this paper of course represent the views of the author only, and should not be interpreted as representing the view of the Board of Governors of the Federal Reserve System or members of its staff. My thanks go to my colleagues David Gordon and Peter Tinsley for invaluable discussions of the material in this paper.

2. See, e.g., Eisner [7], Jorgenson [16], and Bischoff [4].

3. However, the existence of significant costs associated directly or indirectly with the state of bankruptcy is still a matter of debate. (See Haugen and Senbet [12].)

4. One can defend the assumption of all-equity financing by noting that with a perfect market for riskless debt (at a constant rate of interest), the firm's real decisions for investment and production are independent of whether financing is from equity or riskless debt.

5. The explicit incorporation of uncertainty and costs of bankruptcy into the firm's objective function would severely complicate the analysis, particularly in view of the sources and uses of funds constraint introduced into the model. Even if tractable, the analysis of such a model would probably not change the conclusion of this paper that financial variables can be shown to account for some of the variability in the speed of adjustment under some, but not all circumstances.

For discussions of the complications with intertemporal maximization under uncertainty, see Abel [1], Nickell [20], and Stevens [24].

6. Later in this section and in the Appendix we analyze the complications introduced when there can be jumps in the state variables, K and D. It is shown that equations (10) through (16), below, hold whether a jump occurs or not.

7. Probably the easiest way to show that μ_2 always equals zero is to prove that μ_2=0 when D=0; thus the constraint is inactive at the boundary. In the appendix, section I.B, we show that D=0 implies that DIV>0; thus, when D=0, μ_1=0 and, using the arguments developed in the text, equation (13) becomes: $\rho=\rho+2\psi D-\mu_2$. Since D=0 by assumption, $\mu_2=0$.

That there is no tendency for debt to become negative can be seen intuitively as follows: negative levels of debt, like negative borrowing, is equivalent to lending; but because of the linear interest rate function, the rate of return on such lending is less that ρ. Hence, since paying dividends has the implicit return ρ, it will never be optimal to carry on such lending.

8. This is proved by Arrow and Kurz [2], Proposition 12, p.57.

9. See the appendix, section IV, for a proof that this is a point of long-run equilibrium.

10. Given the expression for MCC in the text or as the right hand side of equation (15), using $D^+-D^- = D^+ = q(K^+-K^-)$, and q=1, MCC = $\rho+\delta+2\psi(K^+-K^-)$.

11. It is shown in section I of the appendix that when debt is greater than zero, dividends must equal zero.

12. It should be emphasized that equation (19) and the implied relationship between cash flow and $\dot{D}$ does not hold during the jump.

13. Note that, from equation (19), the measure of profits, $\Pi(t)$, is defined as net of depreciation and interest expenses. Depreciation is subtracted because equations (22) or (24) are functions for net investment, not gross. For an equation for gross investment, one just adds $\delta K(t)$ to equation (24).

14. See, e.g., Nickell [20], Scott [22], and Stiglitz [25].

15. The following treatment of taxation is meant only to be a suggestive way to get a positive long-run debt/assets ratio; a realistic approach

would require, among a number of things, the integration of corporate taxation with the personal taxation faced by stockholders.

16. Since it was shown that $\mu_2(t)$ always equals zero (footnote 7), the term $\mu_2 D$ that appeared in equation (9) is dropped.

17. Realistic values for these parameters in the U.S. context seem to lead to values for ϕ^* well under 1; for example, assume the interest rate (ρ) for a firm with no debt is .10; then, if the firm's borrowing rate were to double to .20 as the debt/assets ratio moved from 0 to 1, the implied value for ψ would be .10. If one assumes, finally, that the tax rate, τ, is .25, the implied value for ϕ^* is 1/6 = 17 percent.

It can be shown that if the firm starts from a long-run equilibrium at $\phi^* < 1$, the debt/assets ratio, ϕ, will be < 1 throughout the optimal path. (See the Appendix, section II.B)

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