Board of Governors of the Federal Reserve System

International Finance Discussion Papers

Number 615

June 1998

EQUILIBRIUM LIQUIDITY PREMIA

Dahai Yu

EQUILIBRIUM LIQUIDITY PREMIA

Dahai Yu *

Abstract: This paper studies in a general framework the relative prices of perpetuities with identical dividends and different bid-ask spreads. It establishes four sets of conditions under which the liquidity premium is always positive (i.e., an asset with smaller spread always commands a higher price). To show that the liquidity premium is not necessarily positive, the paper presents two examples of general equilibrium in which the liquidity premium is sometimes negative. The paper also establishes four sets of conditions under which the price-spread relation is convex and uses results on asset price bubbles to establish liquidity premium bounds.

Keywords: liquidity premium, bid-ask spread, general equilibrium.

---

* The author is a consultant in the Division of International Finance, Board of Governors of the Federal Reserve System. The views in this paper are solely the responsibility of the author and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or of any other person associated with the Federal Reserve System. The author is deeply indebted to Michael Woodford, who made a number of critical suggestions, and is grateful to Pierluigi Balduzzi, Patrick Bolton, Gregory Chow, Charles Jones, Dimitri Vayanos, Jiang Wang, Paul Willen and Tim Van Zandt for helpful comments. Email: yud@msn.com.

When two assets have the same dividend patterns but different levels of transaction costs, it appears the asset with lower bid-ask spread (greater liquidity) is likely to command a higher price. A general analysis of this issue, however, seems to be lacking. A number of authors establish negative price-spread relations (i.e., positive liquidity premia) under special assumptions. Amihud and Mendelson (1986), who present results in terms of returns rather than prices, in effect prove that the buying (ask) price is a decreasing and convex function of the bid-ask spread. Their assumptions include the following: agents enter the market following a Poisson distribution, the duration of their stay in the market follows an exponential distribution, the asset prices are constant over time, and no investor short sells assets. The empirical studies by Amihud and Mendelson (1986, 1991) generally confirm their theoretical findings. Kane (1994) further specializes Amihud and Mendelson (1986) in order to find a closed form solution. Using the techniques of Lucas (1978), Aiyagari and Gertler (1991) establish a return-spread relation for a Markovian steady state equilibrium in which asset prices are constant over time. Their assumptions (besides the constancy of prices) include: there are only two assets, one asset has no transaction costs, assets can not be sold short, and the dividend of each asset is constant over time. Vayanos (1998) and Vayanos and Vila (1998) study the stationary equilibrium of overlapping generations models in which agents are identical in preferences and endowments and birth and death rates are equal and constant over time.

The special assumptions in these studies make it possible for interesting comparative static-type results to be derived. However, they also make it difficult for us to fathom the robustness of the conclusions. In particular, it remains unclear whether a more liquid asset must always command a higher price under less restrictive assumptions.

In this paper, we analyze the price-spread relation in a general discrete-time, infinite-horizon setting. We model the information structure by a tree, each node of which represents a state of the world. We allow only a finite number of nodes on each date. We also assume that agents have finite lives. Other than these restrictions, our model is very general. We place no restriction on the birth and death patterns of agents. No utility function is used except in examples, and the only restriction on preferences is that they be strictly increasing. The number of assets is arbitrary. The dividends are also arbitrary, except that the assets whose price-spread relations are under investigation are assumed to have identical dividends.

Here is a synopsis of our analysis. We assume there exist two perpetuities, which we call assets A and B, that have identical dividends. Let $\alpha(s^r)$ and $\beta(s^r)$ be their percentage bid-ask spreads at s^r, a generic node on date r. We assume that $0 < \alpha(s^r) < \beta(s^r) < 1$ at every node, or that asset A is unambiguously more liquid than asset B. Asset B has positive (though not necessarily constant) supply. Let $q_A(s^r)$ and $q_B(s^r)$ be the strictly positive buying prices of the two assets at s^r. Their selling (bid) prices at s^r are then $q_A(s^r)[1 - \alpha(s^r)]$ and $q_B(s^r)[1 - \beta(s^r)]$. We will focus on the price ratio $k(s^r) \equiv q_B(s^r)/q_A(s^r)$. A positive (negative) liquidity premium at s^r corresponds to $k(s^r) < 1$ ($k(s^r) > 1$). In Section I, we establish four theorems that ensure the global positivity of liquidity premium under different conditions. Theorem 1A shows that $k < 1$ at every node in any equilibrium in which k is constant. Theorem 2A shows that if $\alpha(s^r)$ and $\beta(s^r)$ satisfy a mild bounding condition, $k(s^r) < 1$ at every node in any equilibrium in which k has a uniform upper bound and the value of k at any node continues to prevail at at least one immediate successor node. This theorem covers many Markovian equilibria. Theorem 3A shows that if $\alpha(s^r)$ and $\beta(s^r)$ satisfy a mild bounding condition and all agents have one-period investment horizons for asset B, $k(s^r) < 1$ at every node in any equilibrium in which k has a uniform upper bound. Theorem 4A shows that if $\alpha(s^r)$ and $\beta(s^r)$ satisfy a mild bounding condition, $k(s^r) < 1$ at every node at which asset B is traded in any equilibrium in which asset B is never sold short (either by choice or due to short-sale constraint) and k has a uniform upper bound. Each of these theorems also has a twin that establishes the convexity of the price-spread relation, for which the existence of a third perpetuity with the same dividends as assets A and B is assumed.

In establishing these theorems, we use a combination of optimality and general equilibrium arguments. To facilitate exposition, we describe a procedure to decompose the asset trades by a finitely lived agent into a finite number of *trade pairs*. Our arguments resemble the familiar arbitrage arguments in that all they require about preferences is that they be strictly increasing. However, out of a reluctance to specify the assets in the economy in more detail than necessary, our arguments are not based on examining economy-wide opportunities. Instead, we focus on only a handful of assets and make full use of the assumption that they have identical dividends.

Section I does not explain why an equilibrium satisfying the conditions of the theorems therein is natural. Section II uses results in Yu (1998, Essay 2) to partially address this issue. According to Yu (1998), in any equilibrium in which the aggregate endowment of the economy has a finite present value under some state price process, the fundamental value of any asset with positive supply under this state price process must fall into the interval delineated by its buying and selling prices. We show that this result implies boundedness of k. It also gives rise to an upper bound for the liquidity premium of asset A.

Having described occasions in which there must be a positive liquidity premium, we give two examples of negative liquidity premium in Section III. In the examples, all the conditions of Theorem 2A are satisfied except that the value of k at some node does not prevail at an immediate successor, and all the conditions of Theorem 3A are satisfied except that there exist agents with two-period investment horizons for asset B. We explain through these examples why behavior that leads to negative liquidity premium can be rational. We also use slightly modified versions of these examples to illustrate Theorem 4A and show that, when asset B is never sold short, the liquidity premium can be negative when asset B is not traded. Section IV concludes.

I. Sufficient Conditions for Positive Liquidity Premium

We consider an infinite-horizon economy with sequential trading and model uncertainty by an information tree, a generic node of which on date r is denoted by s^r. Each node represents a state of the world and has a unique immediate predecessor (with the exception of s^0, which has no predecessor) and a finite number of immediate successors. We use $s^t|s^r$ to indicate that s^t belongs to the subtree starting at s^r. That is, $s^t|s^r$ means either $s^t = s^r$ or s^t is a (not necessarily immediate) successor of s^r. Agents strictly prefer more consumption over less at each node at which they live. An agent that lives at some node s^r must live either at all the immediate successors of s^r (so s^r is not a terminal node) or at none them (so s^r is a terminal node). The life spans of agents have a finite uniform upper bound L. At each of its terminal nodes, an agent must liquidate all its asset holdings (sell the assets held in long positions and buy assets to cover up short positions) and the liquidation proceeds must be nonnegative. Consumption at any node must be nonnegative, but agents can borrow as much as they want at all non-terminal nodes. Agents may receive endowments of goods at the nodes at which they live, but there are asset endowments only at s^0.

Suppose there are two competitively traded perpetuities A and B that pay identical nonnegative dividends. Asset B has positive supply at every node. Let $\alpha(s^r)$ and $\beta(s^r)$ be the percentage bid-ask spreads of assets A and B at s^r. We assume that $0 < \alpha(s^r) < \beta(s^r) < 1$ for any s^r, or that asset A is unambiguously more liquid. Let $q_A(s^r)$ and $q_B(s^r)$ be the strictly positive buying prices of the two assets at s^r. Their selling prices at s^r are then $q_A(s^r)[1 - \alpha(s^r)]$ and $q_B(s^r)[1 - \beta(s^r)]$. Short selling of assets A and B may or may not be allowed.

To facilitate exposition, we describe a "first in, first out" procedure to decompose the trades in asset B by some agent h not endowed with asset B into a finite number of trade pairs. Given agent h's finite life span and the liquidation requirement, its purchase and sale of asset B must cancel each other over its lifetime. Suppose agent h does not trade in asset B at any of the predecessors of some node s^R and makes a purchase at s^R, creating a long position. We associate all its sales of asset B at the successors of s^R with the purchase at s^R until that purchase is canceled. Let s^T be a successor of s^R where a sale completes the canceling of the purchase at s^R. If there are purchases of asset B between s^R and s^T, we associate the sale at s^T not used up to cancel the purchase at s^R and all the sales in asset B at the successors of s^T with the purchase at the earliest node after s^R until that purchase is also canceled, and proceed to cancel the next purchase if there is one. If there is no purchase of asset B between s^R and s^T and the sale at s^T is not completely used up to cancel the purchase at s^R, we associate all the purchases at the successors of s^T with the unused sale at s^T until that sale is canceled. If there is no purchase of asset B between s^R and s^T and the sale at s^T is completely used up to cancel the purchase at s^R, the immediate successors of s^T begin with a zero position in asset B and we can start the association scheme anew. Treating purchases and sales symmetrically, we can use the above procedure to uniquely assign all of agent h's purchases and sales of asset B to a finite number of *trade associations*. Each such association is composed of a purchase (sale) at some node s^r and sales (purchases) at the successors of s^r such that the total sales (purchases) along each branch (string of nodes with one for each date) starting at s^r is equal to the purchase (sale) at s^r. By slicing the initial purchase (sale) of a trade association sufficiently finely, we can further decompose each association into a finite number of trade pairs. Each such pair P is composed of a purchase (sale) at some node s^r and an *equal* sale (purchase) at each of the nodes in a *successor set* X(P). The set X(P) satisfies two conditions: (1) each node in X(P) is a successor of s^r, and (2) each of the terminal nodes for agent h that is a successor of s^r is either an element of X(P) or a successor of exactly one node in X(P).

Trade pairs (as well as trade associations) are of two types: buy-and-then-sell, and sell-and-then-buy. There is some arbitrariness on how a trade association is decomposed into trade pairs, but all the pairs from the decomposition of a buy-sell (sell-buy) association are always of the buy-sell (sell-buy) type. Obviously, an agent not endowed with asset B never holds a short (long) position in asset B *if and only if* all its trade pairs for asset B are of the buy-sell (sell-buy) type. Once the decomposition into trade pairs is complete, we can examine an agent's trades in asset B one trade pair at a time. Forgoing a pair will never cause a violation of the requirement that all assets be liquidated at each of an agent's terminal nodes.

We are ready to present sufficient conditions for $k < 1$ at every node.

A. Constant Price Ratio

Theorem 1A: *In any equilibrium in which* $q_B(s^r) = kq_A(s^r)$ *at any* s^r*, where k is some constant, it must be that* $k < 1$.

Proof: We assert that, with $k \geq 1$, no agent born after date 0 (and is therefore not endowed with assets) will have a buy-sell pair for asset B. Suppose instead some such agent h has a trade pair P that represents buying a unit of asset B at s^r and selling it at each of the nodes in the successor set X(P). Consider P*, an alternative to P. P* represents buying k units of asset A at s^r and selling them at each node in X(P). The cost of P* at s^r is $kq_A(s^r) = q_B(s^r)$, the same as the cost of P. At each node between s^r and X(P) and each node in X(P), P* generates more or equal dividend in comparison with P. Let s^t be a generic node in X(P). The selling proceeds of P at s^t is $[1 - \beta(s^t)]q_B(s^t)$, while the selling proceeds of P* at s^t is $k[1 - \alpha(s^t)]q_A(s^t) = [1 - \alpha(s^t)]q_B(s^t)$, a greater amount. Therefore, with $k \geq 1$, agent h can do strictly better by substituting P* for P, and so P or any other buy-sell pair is inconsistent with optimization, and agent h or any other agent born after date 0 will never hold a long position in asset B. Consequently, given the positive supply of asset B, the market for asset B can not clear at any node on date L or any later date, where all the agents alive are born after date 0. Therefore, only $k < 1$ can be consistent with equilibrium. **Q.E.D.**

Because assets A and B are an arbitrary pair, Theorem 1A, like the theorems we will develop later, actually establishes a general decreasing relation between the buying price and the bid-ask spread. Trivially, $q_A > q_B$ implies $(1 - \alpha)q_A > (1 - \beta)q_B$, or there is also a decreasing relation between the selling price and the bid-ask spread. Note that, like Theorems 1B, 2 and 3 below, Theorem 1A is valid regardless of whether short selling asset A or B is allowed.

We now turn to the curvature of the price-spread relation. Suppose there is another perpetuity C that pays identical dividends as assets A and B. Asset C has positive supply at every node. Its bid-ask spread $\gamma(s^r)$ satisfies $\beta(s^r) < \gamma(s^r) < 1$ at any s^r, and its buying price q_C is strictly positive. At each node, define q and q* as:

$$q = \frac{1}{\gamma - \alpha}[(\gamma - \beta)q_A + (\beta - \alpha)q_C], \quad q^* = \frac{1}{\gamma - \alpha}[(\gamma - \beta)(1 - \alpha)q_A + (\beta - \alpha)(1 - \gamma)q_C] \quad (1)$$

Trivially, $q > q_B$ indicates a convex relation between the buying price and the bid-ask spread, and $q^* > (1 - \beta)q_B$ indicates a convex relation between the selling price and the bid-ask spread.

Theorem 1B: *Suppose α, β and γ are constants. Then $q > q_B$ and $q^* > (1 - \beta)q_B$ at any node in any equilibrium with constant price ratios for assets A, B and C.*

Proof: According to Theorem 1A, $q_A > q_B > q_C$ at any node. Define m by $q_B = mq$. Given the constant price ratios and the constancy of α, β and γ, m is a constant. We assert that, with $m \geq 1$, no agent born after date 0 will have a buy-sell pair for asset B. Suppose instead some such agent h has a trade pair P that represents buying a unit of asset B at s^r and selling it at each of the nodes in a successor set X(P). Consider P**, an alternative to P. P** represents buying $m\frac{\gamma-\beta}{\gamma-\alpha}$ unit of asset A and $m\frac{\beta-\alpha}{\gamma-\alpha}$ unit of asset C at s^r and selling them at each node in X(P). The cost of P** at s^r is $m\frac{\gamma-\beta}{\gamma-\alpha}q_A(s^r) + m\frac{\beta-\alpha}{\gamma-\alpha}q_C(s^r) = mq(s^r) = q_B(s^r)$, the same as the cost of P. The total units in P** is $m\frac{\gamma-\beta}{\gamma-\alpha} + m\frac{\beta-\alpha}{\gamma-\alpha} = m \geq 1$. At each node between s^r and X(P) and each node in X(P), P** generates more or equal dividend in comparison with P. Let s^t be a generic node in X(P). The selling proceeds of P at s^t is $(1 - \beta)q_B(s^t)$. The selling proceeds of P** at s^t is $m\frac{\gamma-\beta}{\gamma-\alpha}(1 - \alpha)q_A(s^t) + m\frac{\beta-\alpha}{\gamma-\alpha}(1 - \gamma)q_C(s^t)$. We have:

$$m\frac{\gamma-\beta}{\gamma-\alpha}(1 - \alpha)q_A(s^t) + m\frac{\beta-\alpha}{\gamma-\alpha}(1 - \gamma)q_C(s^t)$$

$$= \frac{q_B(s^t)}{q(s^t)}\left[\frac{\gamma-\beta}{\gamma-\alpha}(1 - \beta)q_A(s^t) + \frac{\gamma-\beta}{\gamma-\alpha}(\beta - \alpha)q_A(s^t) + \frac{\beta-\alpha}{\gamma-\alpha}(1 - \beta)q_C(s^t) - \frac{\beta-\alpha}{\gamma-\alpha}(\gamma - \beta)q_C(s^t)\right]$$

$$= (1 - \beta)q_B(s^t) + \frac{q_B(s^t)}{q(s^t)}\frac{(\gamma-\beta)(\beta-\alpha)}{\gamma-\alpha}[q_A(s^t) - q_C(s^t)] > (1 - \beta)q_B(s^t) \qquad (2)$$

Therefore, with $m \geq 1$, agent h can do strictly better by substituting P** for P, and so P or any other buy-sell pair is inconsistent with equilibrium, and agent h or any other agent born after date 0 will never hold a long position in asset B. Consequently, given the positive supply of asset B, only $m < 1$ or $q > q_B$ at every node can be consistent with equilibrium. Furthermore, from (1) and the result that $q(s^r) > q_B(s^r)$ for any s^r, we have:

$$q^*(s^r) = (1 - \beta)\frac{1}{\gamma-\alpha}[(\gamma - \beta)\frac{1-\alpha}{1-\beta}q_A(s^r) + (\beta - \alpha)\frac{1-\gamma}{1-\beta}q_C(s^r)]$$

$$= (1 - \beta)\frac{1}{\gamma-\alpha}\ [(\gamma - \beta)q_A(s^r) + (\gamma - \beta)\frac{\beta-\alpha}{1-\beta}q_A(s^r) + (\beta - \alpha)q_C(s^r) - (\beta - \alpha)\frac{\gamma-\beta}{1-\beta}q_C(s^r)]$$

$$= (1 - \beta)q(s^r) + \frac{(\gamma-\beta)(\beta-\alpha)}{\gamma-\alpha}[q_A(s^r) - q_C(s^r)] > (1 - \beta)q_B(s^r)$$

(3)

Q.E.D.

By establishing the negativity and convexity of the price-spread relation with arbitrary preferences, endowments and asset specifications, Theorem 1 shows that the same kind of relation in Amihud and Mendelson (1986) is the direct consequence of their constant price assumption and not the result of other special features.

B. Constant Price Ratio on Branches

Though more general than the assumption of constant prices, the assumption of constant price ratio is also quite special. We now relax this assumption somewhat and assume instead that the price ratio k has a uniform upper bound and that the value of k at any node will continue to prevail at at least one immediate successor node.

Theorem 2A: *Suppose the bid-ask spreads satisfy the uniform bound*

$$\rho[1 - \beta(s^r)] \leq 1 - \alpha(s^r) \quad (4)$$

for any s^r*, where* $\rho > 1$ *is a constant. Then* $k < 1$ *at any node in any equilibrium in which the value of k at any* s^r *will continue to prevail at at least one immediate successor of* s^r *and* $k(s^r) \leq K$ *for any* s^r*, where K is some constant.*

Proof: Suppose instead $q_A(s^r) \leq q_B(s^r)$ or $k(s^r) \geq 1$ at some node s^r. Then by assumption the ratio of q_B over q_A is equal to $k(s^r)$ at all the nodes on some branch starting at s^r. Let s^R be a node on this branch that is at least L dates ahead of s^r. Given the positive supply of asset B, some agent h must hold a long position of asset B at s^R. It must then have a trade pair P that represents buying a unit of asset B at some node s^τ between s^r and s^R and selling it at each node in a successor set X(P). Noting that $k(s^\tau) = k(s^r)$ by construction, it is also easy to show that, if

$$\frac{1-\beta(s^t)}{1-\alpha(s^t)} k(s^t) < k(s^r) \quad (5)$$

at each $s^t \in X(P)$, agent h could do strictly better by replacing P by P*, which represents buying $k(s^r)$ units of asset A at s^τ and selling them at each node in X(P). (Note that, given the variable price ratio, this replacement may lower the portfolio liquidation value of agent h at some nodes in X(P) or between s^τ and X(P). It is for this reason that we allow unlimited borrowing at all non-terminal nodes.) Therefore, there must be some $s^t \in X(P)$ at which

$$\frac{1-\beta(s^t)}{1-\alpha(s^t)} k(s^t) \geq k(s^r) \quad (6a)$$

Using (4) in (6a), we get:

$$k(s^t) \geq \rho\ k(s^r) \tag{6b}$$

(6b) shows that, if $k(s^r) \geq 1$ at some node s^r, there would exist some successor s^t of s^r at which $k(s^t)$ is at least ρ times $k(s^r)$. But then the ratio of q_B over q_A is equal to $k(s^t)$ at all the nodes on some branch starting at s^t. Repeating this argument n times for some sufficiently large n, we can establish that there would exist some successor s^T of s^r at which

$$k(s^T) \geq \rho^n\ k(s^r) > K \tag{6c}$$

But (6c) contradicts the premise that K is the uniform upper bound of k. Therefore, in any equilibrium satisfying the conditions of the theorem, it must be that $k < 1$ or $q_A > q_B$ at all nodes. **Q.E.D.**

Theorem 2A relies on two critical assumptions. The first is the existence of a uniform upper bound for k. In section 4, we will use results in Yu (1998) to explain why the existence of such a bound is quite natural. The second critical assumption is that the value of k at any node continues to prevail at at least one immediate successor. We have no similar defense for this assumption. We do observe, however, that the assumption is satisfied by any Markovian equilibrium in which k can take on only a finite number of values and the diagonal elements of the transition matrix for k are always all positive. (k can take on only a finite number of values if asset prices are quoted in discrete increments, which is usually the case in the real world, and are bounded.)

The theorem below is in parallel with Theorem 1B.

Theorem 2B: *Suppose* α, β *and* γ *are constants. Then* $q > q_B$ *and* $q^* > (1 - \beta)q_B$ *at any node in any equilibrium in which the ratio of* q_B *over* q_A *and the ratio of* q_C *over* q_B

at any s^r will both continue to prevail at at least one immediate successor of s^r, both ratios are uniformly bounded from above, and the ratio between q_C and q_A is uniformly bounded away from 1.

The assumption that the ratio between q_C and q_A is uniformly bounded away from 1 plays a role similar to the bounding condition (4). The asset bundle composed of $\frac{\gamma-\beta}{\gamma-\alpha}$ unit of asset A and $\frac{\beta-\alpha}{\gamma-\alpha}$ unit of asset C is more liquid than asset B. However, the equivalent spread of the asset bundle may not be bounded away from β unless the ratio between q_C and q_A is uniformly bounded away from 1.

Proof of Theorem 2B: According to Theorem 2A, $q_A > q_B > q_C$ at any node. Define $m(s^r)$ by $q_B(s^r) = m(s^r)q(s^r)$. By the definition of $q(s^r)$ in (1), $m(s^r)$ is determined entirely by the ratios among $q_A(s^r)$, $q_B(s^r)$ and $q_C(s^r)$, and boundedness of the price ratios implies boundedness of m. Suppose $q_B(s^r) \geq q(s^r)$ or $m(s^r) \geq 1$ at some s^r. Then by assumption the value of m is equal to $m(s^r)$ at all the nodes on some branch starting at s^r. Let s^R be a node on this branch that is at least L dates ahead of s^r. Given the positive supply of asset B, some agent h must hold a long position of asset B at s^R. It must then have a trade pair P that represents buying a unit of asset B at some node s^τ between s^r and s^R and selling it at each node in a set X(P). Noting that $m(s^\tau) = m(s^r)$ by construction, it is also easy to show that there must be some $s^t \in X(P)$ at which

$$m(s^r)\frac{\gamma-\beta}{\gamma-\alpha}(1-\alpha)q_A(s^t) + m(s^r)\frac{\beta-\alpha}{\gamma-\alpha}(1-\gamma)q_C(s^t) \leq (1-\beta)q_B(s^t) \qquad (7)$$

If (7) were not true for any $s^t \in X(P)$, P would be strictly dominated by P**, which represents buying $m(s^r)\frac{\gamma-\beta}{\gamma-\alpha}$ unit of asset A and $m(s^r)\frac{\beta-\alpha}{\gamma-\alpha}$ unit of asset C at s^τ and selling them at each node in X(P). By the assumption that the ratio between q_C and q_A is uniformly bounded away from 1, there exists some constant $\delta > 0$ such that $q_A/q_c \geq 1 + \delta$ at any node. Noting that $q_B(s^t) = m(s^t)q(s^t)$, from (7) and the definition of q in (1) we get:

$$\frac{m(s^t)}{m(s^r)} \geq \frac{(\gamma-\beta)(1-\alpha)q_A(s^t)+(\beta-\alpha)(1-\gamma)q_C(s^t)}{(1-\beta)[(\gamma-\beta)q_A(s^t)+(\beta-\alpha)q_C(s^t)]}$$

$$= 1 + \frac{(\gamma-\beta)(\beta-\alpha)[q_A(s^t)/q_C(s^t)-1]}{(1-\beta)[(\gamma-\beta)q_A(s^t)/q_C(s^t)+\beta-\alpha]} \geq 1 + \frac{(\gamma-\beta)(\beta-\alpha)\delta}{(1-\beta)(\gamma-\alpha)} \qquad (8)$$

(8) shows that, if $m(s^r) \geq 1$ at some node s^r, there would exist some successor s^t of s^r at which $m(s^t)$ is at least $1 + \frac{(\gamma-\beta)(\beta-\alpha)\delta}{(1-\beta)(\gamma-\alpha)}$ times $m(s^r)$. Repeating this argument n times for some sufficiently large n, we would find some successor s^T of s^r at which m is greater than any pre-assigned bound. **Q.E.D.**

C. Short Investment Horizons

Theorem 3A: *Suppose the bid-ask spreads satisfy the uniform bound in (4) and all agents have one-period investment horizons for asset B. Then k < 1 at any node in any equilibrium in which $k(s^r) \leq K$ at any s^r, where K is some constant.*

Proof: Suppose $k \geq 1$ at s^r. Because asset B has positive supply, some agents must have long positions of asset B at the end of s^r trading. Because they have one-period investment horizons for asset B, these agents must have trade pairs that represent buying asset B at s^r and selling it at the immediate successors of s^r. By arguments similar to those in the proof of Theorem 2A, k must increase by a factor of at least $\rho > 1$ between s^r and some date r + 1 node $s^{r+1}|s^r$. Some other agents may have trade pairs that represent selling asset B short at s^r, but the positive supply of asset B dictates that the sum of the short positions at s^r is smaller than the sum of the long positions. When the agents with long positions sell their holdings at s^{r+1}, therefore, some corresponding purchases at s^{r+1} must belong to trade pairs between s^{r+1} and its immediate successors. The fact that some agents choose to create a long position in asset B at s^{r+1} then implies that k must increase by a factor of at least $\rho > 1$ between s^{r+1} and some date r + 2 node $s^{r+2}|s^{r+1}$. Continuation of this reasoning leads to unboundedness of k. **Q.E.D.**

The theorem below is in parallel with Theorems 1B and 2B. Its proof is obvious and omitted.

Theorem 3B: *Suppose α, β and γ are constants and all agents have one-period investment horizons for asset B. Then $q > q_B$ and $q^* > (1 - \beta)q_B$ at any node in any equilibrium in which both the ratio of q_B over q_A and the ratio of q_C over q_B are uniformly bounded from above and the ratio between q_C and q_A is uniformly bounded away from 1.*

D. No Short Selling of the Less Liquid Asset

Theorem 4A: *Suppose the bid-ask spreads satisfy the uniform bound in (4). Then $k < 1$ at any node at which asset B is traded in any equilibrium in which asset B is never sold short and $k(s^r) \leq K$ for any s^r, where K is some constant.*

Proof: Suppose $k \geq 1$ at some node s^r where asset B is traded. Because asset B is never sold short, the buyer of asset B must have a trade pair P that represents buying a unit of asset B at s^r and selling it at a successor set X(P). By arguments similar to those in the proof of Theorem 2A, k must increase by a factor of at least $\rho > 1$ between s^r and some s^t in X(P). But s^t is another node at which asset B is traded, and so repeating the above arguments leads to the unboundedness of k. **Q.E.D.**

Theorem 4A is valid in any equilibrium in which asset B is never sold short, regardless whether the absence of short sale is by choice or the result of short-sale constraint. The theorem below is in parallel with Theorems 1B, 2B and 3B. Its proof is obvious and omitted.

Theorem 4B: *Suppose* α, β *and* γ *are constants. Then* $q > q_B$ *and* $q^* > (1 - \beta)q_B$ *at any node at which asset B is traded in any equilibrium in which assets B and C are never sold short, both the ratio of* q_B *over* q_A *and the ratio of* q_C *over* q_B *are uniformly bounded from above, and the ratio between* q_C *and* q_A *is uniformly bounded away from 1.*

II. Bounds on the Size of the Liquidity Premium

In this section, we use results in Yu (1998, Essay 2) to explain why the uniform upper bound for k in Theorems 2, 3 and 4 is quite natural and to establish a simple bound for the liquidity premium. Following Santos and Woodford (1997), Yu (1998) defines present and fundamental values in terms of state price processes. Let $v_\omega(s^r, a)$ be the present value of the aggregate endowment ω at s^r under state price process {a}. By Theorem 5 of Yu (1998, Essay 2), in any equilibrium in which $v_\omega(s^r, a)$ is finite, the fundamental value of any asset with positive supply under {a} must fall into the interval delineated by its buying and selling prices. This indicates the absence of bubbles.

Because A and B have identical dividends, they must have the same fundamental value at any s^r under any {a}. In this section, we assume that, like asset B, asset A also has positive supply at every node. Let $f(s^r, a)$ be the common fundamental value at s^r under {a}. In the absence of bubbles on assets A and B, we have:

$$q_B(s^r) \geq f(s^r, a) \geq [1 - \beta(s^r)]q_B(s^r) \tag{9a}$$

$$q_A(s^r) \geq f(s^r, a) \geq [1 - \alpha(s^r)]q_A(s^r) \tag{9b}$$

Combining (9a) and (9b), we get:

$$1 - \alpha(s^r) \leq k(s^r) \equiv \frac{q_B(s^r)}{q_A(s^r)} \leq \frac{1}{1-\beta(s^r)} \tag{10}$$

The second part of (10) is a bound on how negative can the liquidity premium be. If $\beta(s^r)$ is bounded away from 1, then (10) gives a uniform upper bound for k. Thus the existence of a uniform upper bound is linked to the finiteness of $v_\omega(s^r, a)$. In addition, the first part of (10) gives a lower bound for k or an upper bound for a positive liquidity premium. For a more complete discussion, see Yu (1998, Essay 4).

III. Examples of Negative Liquidity Premium

Like the negatively sloped demand curve, positive liquidity premium is the norm. At the same time, as our examples will show, just as Giffen goods are a theoretical possibility in the price theory, so is negative liquidity premium in a finance theory based on optimization and market clearing. In each example, the economy has two perpetuities with constant percentage bid-ask spreads α and β satisfying $0 < \alpha < \beta < 1$. These are the only assets, and they can be sold short. Each asset has a constant supply of 1 and pays a constant dividend $d > 0$ on each node. There is a single numeraire good at each node. Utility functions are twice differentiable, strictly increasing and strictly concave.

A. Example 1: Deterministic Periodicity

Consider a deterministic economy with dates $t = 0, 1, \ldots$. One each even date, an even agent is born. It lives on three dates, and its utility function is:

$$U(c) = U^1(c^1) + U^2(c^2) + U^3(c^3) \tag{11a}$$

On each odd date, an odd three-date agent is born, with utility function

$$U^*(c^*) = U^{*1}(c^{*1}) + U^{*2}(c^{*2}) + U^{*3}(c^{*3}) \tag{11b}$$

The good endowments are ω^1, ω^2, ω^3 and ω^{*1}, ω^{*2}, ω^{*3}. In addition, there are two extra agents alive at $t = 0$. One is endowed with one unit of asset A and two units of asset B and lives only at $t = 0$. The other is endowed with - 1 unit of asset B and lives on dates $t = 0$ and $t = 1$, with utility function and good endowments exactly like that of an odd agent on its last two dates. The buying prices of the assets are q_A and q_B on even dates and q_A^* and q_B^* on odd dates. We assume $q_A > q_B$ and $q_A^* < q_B^*$, or that the liquidity premium is positive on even dates and negative on odd dates.

We consider the following trading pattern. An even agent buys one unit of asset A and two units of asset B on its first date and sells them on its third (last) date. It does not trade assets on its second date. An odd agent does not trade in asset A. It does not trade asset B on its second date. On its first date, it sells one unit of asset B short. On its third date, it buys one unit of asset B to cover up its short position. Obviously, this trading pattern implies market clearing. It also implies that even (odd) agents trade only with even (odd) agents. We use u^i to denote U^i's first derivative. The trading pattern is optimal if and only if the following conditions are satisfied.

$$u^1 q_A = u^2 d + u^3[(1 - \alpha)q_A + d] \quad (12a)$$

$$u^1 q_B = u^2 d + u^3[(1 - \beta)q_B + d] \quad (12b)$$

$$u^1 q_A \geq u^2[(1 - \alpha)q_A^* + d] \quad (12c)$$

$$u^1(1 - \alpha)q_A \leq u^2(q_A^* + d) \quad \text{(redundant given (12a) and (12g))} \quad (12d)$$

$$u^1 q_B \geq u^2[(1 - \beta)q_B^* + d] \quad (12e)$$

$$u^1(1 - \beta)q_B \leq u^2(q_B^* + d) \quad \text{(redundant given (12d))} \quad (12f)$$

$$u^2 q_A^* \geq u^3[(1 - \alpha)q_A + d] \quad (12g)$$

$$u^2(1 - \alpha)q_A^* \leq u^3(q_A + d) \quad \text{(redundant given (12a) and (12c))} \quad (12h)$$

$$u^2 q_B^* \geq u^3[(1 - \beta)q_B + d] \quad \text{(redundant given (12g))} \quad (12i)$$

$$u^2(1 - \beta)q_B^* \leq u^3(q_B + d) \quad \text{(redundant given (12b) and (12e))} \quad (12j)$$

$$c^1 = \omega^1 - q_A - 2q_B, \quad c^2 = \omega^2 + 3d, \quad c^3 = \omega^3 + q_A(1 - \alpha) + 2q_B(1 - \beta) + 3d \quad (12k)$$

$$u^{*1} q_A^* \geq u^{*2} d + u^{*3}[(1 - \alpha)q_A^* + d] \quad (13a)$$

$$u^{*1}(1 - \alpha)q_A^* \leq u^{*2} d + u^{*3}(q_A^* + d) \quad (13b)$$

$$u^{*1}(1 - \beta)q_B^* = u^{*2} d + u^{*3}(q_B^* + d) \quad (13c)$$

$$u^{*1} q_A^* \geq u^{*2}[(1 - \alpha)q_A + d] \quad (13d)$$

$$u^{*1}(1 - \alpha)q_A^* \leq u^{*2}(q_A + d) \quad \text{(redundant given (13b), (13c) and (13g))} \quad (13e)$$

$$u^{*1} q_B^* \geq u^{*2}[(1 - \beta)q_B + d] \quad \text{(redundant given (13d))} \quad (13f)$$

$$u^{*1}(1 - \beta)q_B^* \leq u^{*2}(q_B + d)$$

(13g)

$$u^{*2} q_A \geq u^{*3}[(1 - \alpha)q_A^* + d] \quad (13h)$$

$$u^{*2}(1 - \alpha)q_A \leq u^{*3}(q_A^* + d) \tag{13i}$$

$$u^{*2}q_B \geq u^{*3}[(1 - \beta)q_B^* + d] \quad \text{(redundant given (13c) and (13g))} \tag{13j}$$

$$u^{*2}(1 - \beta)q_B \leq u^{*3}(q_B^* + d) \quad \text{(redundant given (13i))} \tag{13k}$$

$$c^{*1} = \omega^{*1} + (1 - \beta)q_B^*, \quad c^{*2} = \omega^{*2} - d, \quad c^{*3} = \omega^{*3} - q_B^* - d \tag{13l}$$

(12a) and (12b) ensure that buying more or less than one unit of asset A or two units of asset B on the first date and selling it on the third date will not raise an even agent's total utility. (12c) and (12d) ensure that buying or selling asset A on the first date and liquidating it on the second date will not raise an even agent's total utility. (12e) and (12f) are the asset B counterparts of (12c) and (12d). (12g) and (12h) ensure that buying or selling asset A on the second date and liquidating it on the third date will not raise an even agent's total utility. (12i) and (12j) are the asset B counterparts of (12g) and (12h). (13) contains parallel conditions for an odd agent.

By adjusting utility functions and good endowments, we can make the u's attain any positive values that we want. Therefore, to construct a desired equilibrium, all we need to do is to find a set of strictly positive values for α, β, d, the u's and the q's with $\alpha < \beta < 1$, $q_A > q_B$ and $q_A^* < q_B^*$ that satisfy conditions (12a) through (12i) and (13a) through (13k). It is straightforward to verify that $\alpha = 1/5$, $\beta = 1/4$, $d = 1$, $q_A = 21/8$, $q_B = 5/2$, $q_A^* = 21/11$, $q_B^* = 7/3$, $u^1 = 1$, $u^2 = 65/72$, $u^3 = 5/9$, $u^{*1} = 1$, $u^{*2} = 231/452$ and $u^{*3} = 42/113$ represent an equilibrium.

The behavior of all the agents in this example is perfectly rational. An even agent purchases assets on its first date to enhance its consumption later in life, and asset prices are such that an even agent finds buying assets A and B equally attractive at the margin.

On its second date, an even agent has the option of selling the assets. Because the prices of either asset is not very high on the second date and the third date marginal utility is substantial, an even agent finds it preferable to hold on to both assets and sell them on the third date. The behavior of an odd agent can be explained along similar lines.

B. Example 2: Stochastic Single Occurrence

The periodicity in example 1 is not needed for the existence of a negative liquidity premium. In this subsection, we construct a stochastic example in which negative liquidity premium occurs at a single node. Consider an economy that branches only on t = 2. For each date $t \geq 2$, there are two nodes: λ^t and μ^t.

$$
\begin{tikzcd}
 & & \lambda^2 \arrow[r] & \lambda^3 \arrow[r] & \lambda^4 \arrow[r] & \dots \\
t = 0 \arrow[r] & t = 1 \arrow[ur] \arrow[dr] & & & & \\
 & & \mu^2 \arrow[r] & \mu^3 \arrow[r] & \mu^4 \arrow[r] & \dots
\end{tikzcd}
$$

There are three types of *typical* agents. Each typical agent lives over two nodes on two dates, has no asset endowment, purchases a unit of each asset on its first date and sells off assets on its second (last) date. Agents within each type have the same time-separable utility function and good endowments. On t = 0, a type 1 agent is born. On each λ node, a type λ agent is born. On each μ node from t = 3 onward, a type μ agent is born.

Let q_A and q_B be the constant asset buying prices at t = 0 and t = 1. It is easy to verify that the type 1 agent's asset transactions are optimal under some utility function and good endowments if

$$\frac{q_A(1-\alpha)+d}{q_A} = \frac{q_B(1-\beta)+d}{q_B}, \quad \text{or} \quad d(q_A - q_B) = q_A q_B(\beta - \alpha) \tag{14}$$

(14) implies $q_A > q_B$, or that the liquidity premium is positive. Let q_A^λ and q_B^λ (q_A^μ and q_B^μ) be the constant buying prices on the λ nodes (μ nodes from t = 3 onward). It is also easy to verify that the asset transactions of type λ (μ) agents can be optimal if q_A^λ and q_B^λ (q_A^μ and q_B^μ) satisfy (14).

Only the type 1 agent lives between t = 0 and t = 1, only λ agents live at the λ nodes from t = 3 on, and only μ agents live at μ nodes from t = 4 on. An extra agent is born and lives only on t = 0 and is endowed with one unit of each asset. With the three sets of constant prices satisfying (14), the liquidity premium is positive at every node other than μ^2, and, with the utility functions and good endowments that guarantee the optimality of all the typical agents, asset demands and asset supplies are equated on t = 0, λ^3 and any node from date t = 4 onward. In addition, the type 1 agent wants to sell one unit of each asset on t = 1, and the typical agents born on λ^2 and μ^3 want to buy one unit of each asset on their birth nodes.

Three agents (x, y and z) are born on t = 1. Agent x lives on four nodes: t = 1, λ^2, μ^2 and μ^3. Agents y and z live on three nodes: t = 1, λ^2 and μ^2. There is no birth on μ^2. Suppose agent x buys one unit of each asset on t = 1 and sells off assets on λ^2 and μ^3, agent y buys one unit of asset B on t = 1 and sells it off on t = 2 nodes, agent z sells one unit of asset B on t = 1 and buys it back on t = 2 nodes, and agents y and z do not trade in asset A. Clearly, this trading pattern implies market clearing. Let q_A* and q_B* be the asset prices at μ^2. Our task is to show that the trading pattern is consistent with optimization by agents x, y and z under a set of positive values for α, β, d, the marginal utilities and the asset prices with $q_A^* < q_B^*$.

Agent x's behavior is optimal if and only if the following conditions are satisfied:

$$u^{x,1}q_A = u^{x,\lambda 2}[(1 - \alpha)q_A^{\lambda} + d] + u^{x,\mu 2}d + u^{x,\mu 3}[(1 - \alpha)q_A^{\mu} + d] \qquad (15a)$$

$$u^{x,1}q_B = u^{x,\lambda 2}[(1 - \beta)q_B^{\lambda} + d] + u^{x,\mu 2}d + u^{x,\mu 3}[(1 - \beta)q_B^{\mu} + d] \qquad (15b)$$

$$u^{x,1}q_A \geq u^{x,\lambda 2}[(1 - \alpha)q_A^{\lambda} + d] + u^{x,\mu 2}[(1 - \alpha)q_A^* + d] \qquad (15c)$$

$$u^{x,1}(1 - \alpha)q_A \leq u^{x,\lambda 2}(q_A^{\lambda} + d) + u^{x,\mu 2}[q_A^* + d] \qquad (15d)$$

$$u^{x,1}q_B \geq u^{x,\lambda 2}[(1 - \beta)q_B^{\lambda} + d] + u^{x,\mu 2}[(1 - \beta)q_B^* + d]$$

(15e)

$$u^{x,1}(1 - \beta)q_B \leq u^{x,\lambda 2}(q_B^{\lambda} + d) + u^{x,\mu 2}[q_B^* + d] \qquad (15f)$$

$$u^{x,\mu 2}q_A^* \geq u^{x,\mu 3}[(1 - \alpha)q_A^{\mu} + d] \qquad (15g)$$

$$u^{x,\mu 2}(1 - \alpha)q_A^* \leq u^{x,\mu 3}(q_A^{\mu} + d) \qquad (15h)$$

$$u^{x,\mu 2}q_B^* \geq u^{x,\mu 3}[(1 - \beta)q_B^{\mu} + d] \qquad (15i)$$

$$u^{x,\mu 2}(1 - \beta)q_B^* \leq u^{x,\mu 3}(q_B^{\mu} + d) \qquad (15j)$$

The expressions for consumption are not essential and are omitted here. (15a) and (15b) ensure that buying more or less than one unit of either asset at t = 1 and selling it at λ^2 and μ^3 will not raise total utility. (15c) and (15d) ensure that buying or selling asset A at t = 1 and liquidating it at λ^2 and μ^2 will not raise total utility. (15e) and (15f) are the asset B counterparts of (15c) and (15d). (15g) and (15h) ensure that buying or selling asset A at μ^2 and liquidating it at μ^3 will not raise total utility. (15i) and (15j) are the asset B counterparts of (15g) and (15h).

The optimization conditions for agents y and z are:

$$u^{y,1}q_A \geq u^{y,\lambda 2}[(1 - \alpha)q_A^{\lambda} + d] + u^{y,\mu 2}[(1 - \alpha)q_A^* + d] \quad (16a)$$

$$u^{y,1}(1 - \alpha)q_A \leq u^{y,\lambda 2}(q_A^{\lambda} + d) + u^{y,\mu 2}[q_A^* + d] \quad (16b)$$

$$u^{y,1}q_B = u^{y,\lambda 2}[(1 - \beta)q_B^{\lambda} + d] + u^{y,\mu 2}[(1 - \beta)q_B^* + d] \quad (16c)$$

$$u^{z,1}q_A \geq u^{z,\lambda 2}[(1 - \alpha)q_A^{\lambda} + d] + u^{z,\mu 2}[(1 - \alpha)q_A^* + d] \quad (17a)$$

$$u^{z,1}(1 - \alpha)q_A \leq u^{z,\lambda 2}(q_A^{\lambda} + d) + u^{z,\mu 2}[q_A^* + d] \quad (17b)$$

$$u^{z,1}(1 - \beta)q_B = u^{z,\lambda 2}(q_B^{\lambda} + d) + u^{z,\mu 2}[q_B^* + d] \quad (17c)$$

It is straightforward to verify that $\alpha = 1/5$, $\beta = 1/4$, $d = 1$, $q_A = 2$, $q_B = 20/11$, $q_A^{\lambda} = 5$, $q_B^{\lambda} = 4$, $q_A^* = 20/7$, $q_B^* = 3$, $q_A^{\mu} = 20/7$, $q_B^{\mu} = 5/2$, $u^{x,1} = u^{y,1} = u^{z,1} = 1$, $u^{x,\lambda 2} = 1/660$, $u^{x,\mu 2} = 11/20$, $u^{x,\mu 3} = (14/55)(119/69) = 1666/3795$, $u^{y,\lambda 2} = 1/44$, $u^{y,\mu 2} = 76/143$, $u^{z,\lambda 2} = 1/4$ and $u^{z,\mu 2} = 5/176$ represent an equilibrium.

Agent x buys assets at $t = 1$ to enhance later consumption. Its marginal utility at μ^2 is low, so it chooses not the sell assets at μ^2. Agent y buys asset B at $t = 1$ to enhance later consumption. It finds asset A not as attractive to buy as asset B, because it has high marginal utility at μ^2, where asset B can be sold at a higher price. Agent z short sells asset B at $t = 1$ to enhance its consumption there. It finds asset A not as attractive to sell as asset B, because it has low marginal utility at μ^2, where covering asset B's short position is more costly.

C. Modifying the Examples to Illustrate Absence of Short Sales

If we take out all the odd agents in example 1 while keeping the asset prices and the preferences and endowments of the even agents intact, the kind of even agent behavior described in that example continues to be both optimal and market clearing. What we end up with is a new equilibrium in which the liquidity premium is negative on all the odd dates. This does not contradict Theorem 4A, because asset B is not traded on odd dates in our new equilibrium. Similarly, if we take out agents y and z in example 2, we end up with a new equilibrium in which no agent ever short sells asset B and the liquidity premium is positive at every node other than μ^2, where asset B is not traded.

IV. Conclusion

This paper serves several purposes. By establishing four sets of sufficient conditions, it shows that a positive liquidity premium and a convex price-spread relation must occur in a variety of equilibrium settings. In particular, the positive liquidity premium and the convex price-spread relation in Amihud and Mendelson (1986) is the direct consequence of the constant price assumption and not the result of other special features. The proofs of the sufficiency theorems demonstrate the fruitfulness of general equilibrium arguments. The paper's discussions of the implications of finite present value of aggregate endowments establish a linkage between the liquidity premium and the theory of asset price bubbles found in Yu (1998). Finally, through examples, the paper shows how a negative liquidity premium can be consistent with optimization and market clearing.

References

Aiyagari, S. Rao and Mark Gertler, 1991, Asset returns with transactions costs and uninsured individual risk, *Journal of Monetary Economics*, 27, 311-331.

Amihud, Yakov and Haim Mendelson, 1986, Asset pricing and the bid-ask spread, *Journal of Financial Economics*, 17, 223-249.

Amihud, Yakov and Haim Mendelson, 1991, Liquidity, maturity, and the yields on U.S. treasury securities, *Journal of Finance*, 66, 1411-1425.

Kane, Alex, 1994, Trading cost premiums in capital asset returns: A closed form solution, *Journal of Banking and Finance*, 18, 1177-1183.

Lucas, Robert E. Jr., 1978, Asset prices in an exchange economy, *Econometrica*, 46, 1429-1445.

Ross, Stephen A., 1976, The arbitrage theory of capital asset pricing, *Journal of Economic Theory*, 13, 341-360.

Santos, Manuel S. and Michael Woodford, 1997, Rational asset pricing bubbles, *Econometrica*, 65, 19-57.

Vayanos, Dimitri, 1998, Transaction costs and asset prices: A dynamic equilibrium model, *Review of Financial Studies*, 11, 1-58.

Vayanos, Dimitri and Jean-Luc Vila, 1998, Equilibrium interest rate and liquidity premium with transaction costs, forthcoming, *Economic Theory*.

Yu, Dahai, 1998, *Frictional Arbitrage Valuation*, Ph.D. dissertation, Princeton University.