Board of Governors of the Federal Reserve System

International Finance Discussion Papers

Number 622

September 1998

TWO EQUIVALENCE THEOREMS FOR GOVERNMENT FINANCE

Dahai Yu

TWO EQUIVALENCE THEOREMS FOR GOVERNMENT FINANCE

Dahai Yu *

Abstract: This paper studies the effects of a path change in government debt composition and aggregate transfers on allocations and prices. It is shown that the effects are zero under *some* agent-specific transfer scheme even when markets are incomplete. If markets are complete, then the effects are zero under *any* transfer scheme that leaves each agent's lifetime resource unchanged *if and only if* agents are always collectively compensated for next period's return change. The infinite-horizon framework used has an arbitrary number of assets with arbitrary returns and an arbitrary mixture of finitely and infinitely lived agents.

Keywords: debt composition, Ricardian equivalence, general equilibrium.

---

* The author, who is assistant professor of economics, Tufts University, Medford, MA 02155, wrote this paper while he was a consultant in the Division of International Finance, Board of Governors of the Federal Reserve System. The views in this paper are solely the responsibility of the author and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or of any other person associated with the Federal Reserve System. The author benefited enormously from discussions with David Bowman, who suggested the term equivalence in place of relevance, Jon Faust, who helped the author see the connection between the results in this paper and the Ricardian equivalence literature, and Dale Henderson and Michael Woodford, who provided invaluable encouragement and guidance. Email: yud@msn.com.

1. Introduction

Since the celebrated work of Modigliani and Miller (1958), economists have developed a host of equivalence theorems. These include refinements of the original Modigliani-Miller (henceforth MM) Theorem, most notably by Stiglitz (1969, 1974) and DeMarzo (1988), and the Ricardian equivalence results starting with Barro (1974). Wallace (1981) and Chamley and Polemarchakis (1984) study the change of government debt composition through open market exchanges. Their equivalence results are reviewed by Sargent (1987).

Changes in government debt composition through open market exchanges are not directly covered by the MM Theorem because government bonds are outside assets. In the standard MM setting, the equivalence result obtains because total asset payoffs do not change. Open market exchanges cause asset supplies to undergo real changes. With all assets held, the total payoffs agents receive will change. The payoff changes show up in the government budget. If the government keeps its spending unchanged, then by an accounting identity, total transfers (taxes if negative) must change to exactly cancel the total payoff changes and the revenues from open market exchanges in any state of nature. In this context, a change in government finance consists of a path change in debt composition and the accompanying changes in total transfers, and the equivalence question is whether some new, agent-specific transfer scheme that implies the required total transfers can (1) keep the optimal consumption of each agent unchanged and (2) cause the asset markets to clear. For a representative agent economy, the transfer scheme that implies a given set of total transfers is unique, and the equivalence issue is the same as the relevance of government finance issue. In a more general setting, the equivalence issue is related to the relevance of government finance issue in a narrower sense: there is equivalence when a change in government finance is irrelevant under a specific transfer scheme. By covering changes of debt composition (with perhaps zero accompanying changes in transfers), our equivalence concept may be regarded as a generalization of the Ricardian equivalence concept, which focuses on the equivalence between debt and taxes (transfers).

Our study is based on the assumption that the government has the power to impose agent specific, state specific lump-sum transfers. While such a government does not exist in the real world, it still has to rely on private incentives and is far less powerful than the idealized central planner. It is this limit to governmental power that keeps the equivalence issue non-trivial.

In an interesting special case, Wallace (1981) shows that *any* transfer scheme that keeps each agent's lifetime resource unchanged and satisfies another condition can cancel a change in government finance and leave allocations as well as asset prices unchanged. Wallace's complete market framework has two-date lives and a special asset structure (fiat money plus one-period assets). This paper extends the Wallace result by allowing an arbitrary number of assets with arbitrary returns and an arbitrary mixture of finitely and infinitely lived agents.

In Chamley and Polemarchakis (1984), there is no need for transfer changes because payoff changes are exactly canceled by the revenues from open market exchanges. The flexible price level changes in such a way that the real returns of the sole nominal asset become a weighted average of its initial returns and the returns of the real asset whose supply is changed, and the effects of open market exchanges are thereby perfectly canceled. Elegant as it is, their result, which is valid even with incomplete markets, is subject to a number of qualifications and seems difficult to generalize. The qualifications have to do with the side effects of a price level change. The situation is reminiscent of Stiglitz (1974) and DeMarzo (1988), who establish the validity of the MM Theorem with incomplete markets under the condition that one asset's payoffs are independent of another asset's price. This condition, whose importance is demonstrated by Gottardi (1994, 1995) and Detemple, Gottardi and Polemarchakis (1995), can remain satisfied even if bankruptcies are allowed, but is usually unsatisfied in the presence of instruments such as stock options. The parallel condition for the validity of the Chamley-Polemarchakis result, about which we will have little more to say, is that changes in price level do not affect the equilibrium in undesirable ways. Under normal circumstances, this requires that (1) there is a single nominal asset, (2) there are no nominal transfers, and (3) asset payoffs, which can be contingent contracts, do not depend on the price level. The lengthiness of this list is not surprising; in Chamley and Polemarchakis (1984) a single scalar process (the price level) bears all the burden of restoring equilibrium.

The “money” in both Wallace (1981) and Chamley and Polemarchakis (1984) is the Samuelson-Wallace variety (Samuelson, 1958; Wallace, 1980): it is held for returns, not for transactional services. Their “open market operations” are not the same as the open market operations conducted by the Federal Reserve System, namely the open market exchanges between government bonds and transactional money. In this paper, we will briefly discuss how our formal results fare when transactional money is introduced.

Here is a summary of the paper. Section 2 introduces a general framework of asset markets and individual optimization. A key ingredient of the framework is the government budget constraint, which (under the assumption that there is no government spending) unambiguously identifies the aggregate transfers under a given path for government debt.

Section 3 establishes two equivalence theorems. Theorem 1 shows that the effects of a change in government finance on allocations and prices are zero under *some* transfer scheme even if markets are incomplete. The proposed transfer scheme is shown to leave each agent’s lifetime resource unchanged under any state price process. For a representative agent economy, the proposed transfer scheme is the only one possible, and Theorem 1 becomes a generalized Ricardian equivalence result: it implies that neither the level nor the composition of debt matters. It is observed that Theorem 1 remains valid in the presence of transactional money as long as the path of money supply does not change.

Section 3 then moves to a complete market setting and explains why it is *not* generally true that *any* transfer scheme that leaves the lifetime resource of each agent unchanged can cancel a change in government finance. Theorem 2 shows that a sufficient and necessary condition for *any* transfer scheme that leaves the lifetime resource of each agent unchanged to be able to cancel a change in government finance is that agents are always collectively compensated for next period's return change. This condition is closely related to the intergenerational links that normally underlie the validity of the Ricardian equivalence. In Theorem 2, intergenerational links are not imposed, but the changes in government finance are restricted to those that do not imply intergenerational transfers. For an economy consisting of infinitely lived "dynasties", the condition is automatically satisfied. It is observed that Theorem 2 is not valid in the presence of transactional money. Section 4 concludes.

2. The Framework

This section introduces a general framework similar to that of Santos and Woodford (1997). Consider an infinite-horizon economy with sequential trading. Agents have homogeneous information and beliefs about the possible states on each date in the future. Let s^t be a typical state, or node on the information tree N, on date t. Each s^t has a unique immediate predecessor $s^t - 1$ and a finite number of immediate successors, a typical one of which is denoted by $s^{t+1}|s^t$. The economy begins at s^0, the unique node on date 0. We use $s^r|s^t$ to indicate that s^r

belongs the subtree of N starting at s^t. That is, $s^r|s^t$ means either $s^r = s^t$ or s^r is a (not necessarily immediate) successor of s^t.

There is a single numeraire good at each node. There are $k(s^t)$ assets at s^t with $1\times k(s^t)$ nonnegative price $q(s^t)$. Transactional money is not among the assets. All assets can be sold short. At each node s^t ($t \geq 1$), the broadly defined dividends on assets, which include coupon payments on bonds, are specified by a $1\times k(s^t - 1)$ nonnegative vector $d(s^t)$, and asset transformation is specified by a $k(s^t)\times k(s^t - 1)$ nonnegative matrix $b(s^t)$. The nonnegativity of $d(s^t)$ and $b(s^t)$ may be understood as a consequence of free asset disposal. A $k(s^t - 1)\times 1$ portfolio Z held at the end of trading at node $s^t - 1$ is paid $d(s^t)Z$ in dividend and $b(s^t)Z$ in assets at s^t. The one-period return vector $R(s^t)$ is defined as:

$$R(s^t) = d(s^t) + q(s^t)b(s^t) \tag{1}$$

There are three types of assets, and they do not transform into each other. *Equities* are claims to the capital and the technologically determined dividends of productive processes. Equity supply $Z_1(s^t)$ is nonnegative. The equity quantities reflect the scales of operation. In our analysis, the relevant equilibria all have the same equity quantities. As a result, no constant-return-to-scale or absence of externality assumption is needed for the technology. *Securities* are contingent contracts that agents trade with each other. Their supply $Z_2(s^t)$ is zero. Security dividends may depend on price history. In our analysis, the relevant equilibria all have the same prices, and so no reference to this dependence is needed. *Bonds*, which may include non-transactional (Samuelson-Wallace type) fiat money, are assets issued by the government, which holds neither equity nor security. The policy determined bond supply $Z_3(s^t)$ is nonnegative. The government spending is always zero. The numbers of these assets at s^t are $k_1(s^t)$, $k_2(s^t)$ and $k_3(s^t)$, and their separate price, dividend and return vectors and transformation matrices can be straightforwardly defined.

Let H be the set of agents, who are indexed by h. Let $H(s^t)$ be the set of agents alive at s^t. We assume that $H(s^t)$ is finite at each s^t. Let N^h, a subset of the information tree N, be the collection of nodes where agent h is alive. By these definitions, $h \in H(s^t)$ if and only if $s^t \in N^h$. The preferences of each agent h are strictly increasing: more consumption at any $s^t \in N^h$ is strictly preferred. For agent h, s^t is an initial node if $s^t \in N^h$ but $(s^t - 1) \notin N^h$. We assume that each agent h has a unique initial node $\underline{h}$. If $s^t \in N^h$ but $s^{t+1} \notin N^h$ for any $s^{t+1}|s^t$, we say s^t is a terminal node for h. Let $\overline{N}^h$, a subset of N^h, be the collection of agent h's terminal nodes. We assume that, for any h and any $s^t \in N^h$, agent h either lives at none of the immediate successors of s^t (if $s^t \in \overline{N}^h$), or lives at all of the immediate successors of s^t (if $s^t \notin \overline{N}^h$). Let $N^h/\overline{N}^h$ be the collection of nodes that are in N^h but not in $\overline{N}^h$.

Let $Z^h(s^t)$ be agent h's $k(s^t)\times 1$ asset holding after the trading at s^t. The total asset holding at s^t is $\sum_{h\in H(s^t)} Z^h(s^t)$. To make s^0 just an arbitrary node in an economy that really has neither beginning nor end, we assume that, for $h \in H(s^0)$, there is asset endowment $Z^h(s^0-)$ at s^0. This endowment can be understood as $b(s^0)Z^h(s^0 - 1)$, the result of holding assets at $s^0 - 1$. $Z^h(s^0-)$ can be either positive or negative, but we assume that $Z(s^0-) = \sum_{h\in H(s^0)} Z^h(s^0-)$ is nonnegative. There is no asset endowment at any node other than s^0.

At $s^t \in N^h$, agent h receives nonnegative good endowment $\omega^h(s^t)$. Aggregate good endowment at s^t is $\omega(s^t) = \sum_{h\in H(s^t)} \omega^h(s^t)$. The good supply at s^t, $t \geq 1$, is given by $\tilde{\omega}(s^t) = \omega(s^t) +$

$d_1(s^t)Z_1(s^t - 1)$. For s^0, we can simply write $\tilde{\omega}(s^0) = \omega(s^0)$, because the dividend agent h receives from equities held at $s^0 - 1$ can be absorbed into $\omega^h(s^0)$. The dividend agent h receives from securities held at $s^0 - 1$ is also absorbed into $\omega^h(s^0)$, but such dividend sums to zero and does not affect good supply. The good supply is used for either consumption or equity creation. The amount of good used in equity creation at s^t is given by $q_1(s^t)[Z_1(s^t) - b_1(s^t)Z_1(s^t - 1)]$. Like the introduction of bonds and transfers, allowing equity creation represents an extension of the Santos-Woodford framework.

At s^t, agent $h \in H(s^t)$ receives lump-sum transfer (tax if negative) $L^h(s^t)$. We allow $\omega^h(s^t) + L^h(s^t)$ to be negative, but require that, through asset trading, each agent can pay all the taxes and still keep its consumption nonnegative at all time. Total transfer at s^t is $L(s^t) = \sum_{h \in H(s^t)} L^h(s^t)$. The dividend agent h receives from bonds held at $s^0 - 1$ is absorbed into $L^h(s^0)$. The government budget constraint is:

$$q_3(s^0)Z_3(s^0) = q_3(s^0)Z_3(s^0-) + L(s^0) \tag{2a}$$

$$q_3(s^t)Z_3(s^t) = R_3(s^t)Z_3(s^t - 1) + L(s^t),\ t \geq 1 \tag{2b}$$

For each $s^t \in N^h$, agent h chooses asset holding $Z^h(s^t)$ and nonnegative consumption $c^h(s^t)$. The constraints it faces are:

$$c^h(s^0) + q(s^0)Z^h(s^0) = \omega^h(s^0) + q(s^0)Z^h(s^0-) + L^h(s^0) \tag{3a}$$

$$c^h(s^t) + q(s^t)Z^h(s^t) = \omega^h(s^t) + R(s^t)Z^h(s^t - 1) + L^h(s^t), \quad t \geq 1 \tag{3b}$$

$$q(s^t)Z^h(s^t) \geq -\inf_{\{a\}} \frac{1}{a(s^t)} \sum_{r=t+1}^{\infty} \sum_{s^r|s^t} a(s^r)[\omega^h(s^r) + L^h(s^r)],\ s^t \in N^h / \overline{N}^h \tag{4}$$

$$Z^h(s^t) = 0 \quad \text{if } s^t \in \overline{N}^h \tag{5}$$

We have written (3) as an equality on account of increasing preferences. If s^t is agent h's initial node $\underline{h}$ (so that $s^t - 1$ is not in N^h), the right side of (3b) does not have the middle term. (5) requires that each agent holds the empty portfolio whenever it dies.

The infimum in (4) is agent h's borrowing limit at s^t, assumed to be nonnegative. When infinitely lived agents exist, some borrowing limits are needed in order to rule out the Ponzi scheme. While all kinds of borrowing limits are possible a priori, the "canonical" specification given in (4), which is proposed by Santos and Woodford (1997), among others, seems the most natural. In (4), $\{a(s^r)\}$ is a state price process defined on N, whose existence is guaranteed by the absence of finite-horizon arbitrage (Ross, 1976; Yu, 1998). The infimum is over all possible state price processes on N. (4) implicitly assumes that the infinite sum therein converges. Given the possibility of negative $\omega^h + L^h$ terms, the sum may not always converge. When the sum does not converge, one could modify (4) and replace $\sum_{r=t+1}^{\infty}$ by $\liminf_{T\to\infty} \sum_{r=t+1}^{T}$. The borrowing limits in (4) are the tightest that still allow all finite-horizon borrowing. They do not constrain finitely lived agents at all.

Let $\overline{H}(s^t)$ be the collection of agents for whom s^t is a terminal node, and let $H(s^t)/\overline{H}(s^t)$ be the collection of agents that are in $H(s^t)$ but not in $\overline{H}(s^t)$. If $H(s^t)/\overline{H}(s^t)$ is empty at some s^t, then by (5) all the agents alive at s^t will hold the empty portfolio after the trading at s^t, and there cannot be an equilibrium unless all assets have zero supply at s^t. Even if there is an equilibrium, the economy is in some sense discontinued at s^t. To avoid such strange scenarios, we assume that $H(s^t)/\overline{H}(s^t)$ is non-empty at each s^t.

Consider the processes $\{q(s^t), L^h(s^t), c^h(s^t), Z^h(s^t)\}$. If they represent an Arrow-Radner equilibrium, they must satisfy the following necessary conditions:

(i) For each s^t, the policy variables $\{L^h(s^t), Z_3(s^t)\}$ satisfy (2),

(ii) For each h, $\{c^h(s^t), Z^h(s^t)\}$ are optimal under (3)-(5),

(iii) For each s^t,

$$\sum_{h \in H(s^t)} c^h(s^t) + q_1(s^t)[Z_1(s^t) - b_1(s^t)Z_1(s^t - 1)] = \tilde{\omega}(s^t), \qquad (6)$$

$$\sum_{h \in H(s^t)} Z^h(s^t) = Z(s^t), \text{ with } Z_2(s^t) = 0 \text{ and } Z_3(s^t) \text{ policy determined, and} \qquad (7)$$

(iv) $\{Z_1(s^t)\}$ and $\{b_1(s^t), d_1(s^t)\}$ are technologically compatible.

We will not study the existence of the kind of equilibrium described above. The existence of Arrow-Radner equilibrium in a basic finite-horizon setting is established by Radner (1972). For a discussion of the existence issue in an infinite-horizon economy similar to ours, see Hernandez and Santos (1996).

3. Results

This section presents our results. Let $Z^h_{<3>}$ be agent h's holding of equities and securities. Let $\{q(s^t), L^h(s^t), c^h(s^t), Z^h_{<3>}(s^t), Z^h_3(s^t)\}$ represent an initial equilibrium. Suppose the government changes $\{Z_3(s^t)\}$ to $\{\hat{Z}_3(s^t)\}$ through open market exchanges. By (2), the required total transfer changes are given by:

$$\hat{L}(s^0) - L(s^0) = q_3(s^0)[\hat{Z}_3(s^0) - Z_3(s^0)] \qquad (8a)$$

$$\hat{L}(s^t) - L(s^t) = q_3(s^t)[\hat{Z}_3(s^t) - Z_3(s^t)] - R_3(s^t)[\hat{Z}_3(s^t - 1) - Z_3(s^t - 1)] \qquad (8b)$$

Let $\{\hat{L}^h(s^t), \hat{Z}^h_3(s^t)\}$ satisfy, for each s^t,

$$\sum_{h \in H(s^t)} \hat{L}^h(s^t) = \hat{L}(s^t), \quad \sum_{h \in H(s^t)} \hat{Z}^h_3(s^t) = \hat{Z}_3(s^t) \qquad (9)$$

The equivalence question is whether there exist $\{\hat{L}^h(s^t), \hat{Z}^h_3(s^t)\}$ satisfying (9) and (8) and some $\{\hat{Z}^h_{<3>}(s^t)\}$, which may or may not be the same as $\{Z^h_{<3>}(s^t)\}$, such that $\{q(s^t), \hat{L}^h(s^t), c^h(s^t),$

$\hat{Z}^h_{<3>}(s^t)$, $\hat{Z}^h_3(s^t)\}$ represent a new equilibrium. Without using a complete markets assumption, the following theorem shows there indeed exists such a new equilibrium.

Theorem 1: Let $\{q(s^t), L^h(s^t), c^h(s^t), Z^h_{<3>}(s^t), Z^h_3(s^t)\}$ represent an initial equilibrium. Then there exist $\{\hat{L}^h(s^t), \hat{Z}^h_3(s^t)\}$ satisfying (9) and (8) such that $\{q(s^t), \hat{L}^h(s^t), c^h(s^t), Z^h_{<3>}(s^t)$, $\hat{Z}^h_3(s^t)\}$ represent a new equilibrium.

Proof: At each s^t, pick a change absorbing agent $h(s^t) \in H(s^t)/\overline{H}(s^t)$. Let $\tilde{H}$ be the set of agents so picked. Note that an agent may be picked more than once (even an infinite number of times), and so the same agent in $\tilde{H}$ can have more than one picking-node designated names.

We change the transfers according to the following scheme. For each s^t, change $h(s^t)$'s transfer at s^t from $L^{h(s^t)}(s^t)$ to

$$\hat{L}^{h(s^t)}(s^t) = L^{h(s^t)}(s^t) + q_3(s^t)[\hat{Z}_3(s^t) - Z_3(s^t)] \tag{10}$$

Also, change $h(s^t)$'s transfer at each $s^{t+1}|s^t$ from $L^{h(s^t)}(s^{t+1})$ to

$$\hat{L}^{h(s^t)}(s^{t+1}) = L^{h(s^t)}(s^{t+1}) - R_3(s^{t+1})[\hat{Z}_3(s^t) - Z_3(s^t)] \tag{11}$$

The transfer of each agent in $\tilde{H}$ undergoes only the prescribed change, and the transfer of any agent not in $\tilde{H}$ does not change.

We change the bond holdings according to the following scheme. For each s^t, change $h(s^t)$'s bond holding at s^t from $Z^{h(s^t)}_3(s^t)$ to

$$\hat{Z}^{h(s^t)}_3(s^t) = Z^{h(s^t)}_3(s^t) + \hat{Z}_3(s^t) - Z_3(s^t) \tag{12}$$

Also, leave $h(s^t)$'s bond holding at each $s^{t+1}|s^t$ unchanged. The asset holding of each agent in $\tilde{H}$ undergoes only the prescribed change, and the asset holding of any agent not in $\tilde{H}$ does not change.

It is then obvious that the prescribed transfers and asset holdings satisfy (9) and (8), and that the asset holdings clear the markets, are optimal for each $h \notin \tilde{H}$ and imply the initial consumption for each $h \in \tilde{H}$. We now verify that the asset holdings are also optimal for each $h \in \tilde{H}$. At any s^t and under any state price process $\{a(s^t)\}$, we have the basic state price relation

$$a(s^t)q_3(s^t) = \sum_{s^{t+1}|s^t} a(s^{t+1})R_3(s^{t+1})$$

(13)

Multiplying (13) by $\hat{Z}_3(s^t) - Z_3(s^t)$ and using (11), we get

$$q_3(s^t)[\hat{Z}_3(s^t) - Z_3(s^t)] = \frac{1}{a(s^t)} \sum_{s^{t+1}|s^t} a(s^{t+1})R_3(s^{t+1})[\hat{Z}_3(s^t) - Z_3(s^t)]$$

$$= -\frac{1}{a(s^t)} \sum_{s^{t+1}|s^t} a(s^{t+1})[\hat{L}^{h(s^t)}(s^{t+1}) - L^{h(s^t)}(s^{t+1})] \tag{14}$$

(14) implies that $h(s^t)$'s borrowing limit is not violated at s^t as a result of its transfer and asset holding changes prescribed above. Also, by using (10) in (14), we get:

$$a(s^t)[\hat{L}^{h(s^t)}(s^t) - L^{h(s^t)}(s^t)] + \sum_{s^{t+1}|s^t} a(s^{t+1})[\hat{L}^{h(s^t)}(s^{t+1}) - L^{h(s^t)}(s^{t+1})] \tag{15}$$

(15) implies that the future resource as viewed at any predecessor of s^t does not change, and so $h(s^t)$'s borrowing limit is not violated at any predecessor of s^t. It is also easy to see that $h(s^t)$ can attain under the initial transfer any consumption that it can attain under the new transfer without violating any borrowing limit. Therefore, the prescribed asset holdings are also optimal for each $h \in \tilde{H}$ under the prescribed transfers. **Q.E.D.**

Since the agent picking is arbitrary, the same open market operations can usually be canceled by many different transfer schemes. For a representative agent economy, the proposed transfer scheme is the only one possible, and Theorem 1 becomes a generalized Ricardian equivalence result: it implies that neither the level nor the composition of debt matters. An implication of this observation is that the effects of a change in government finance are necessarily based on the heterogeneity of agents.

What happens if one of the bonds is transactional money of either Sidrauski-Brock or Clower-Lucas-Stokey variety? (Sidrauski, 1967; Brock, 1974; Clower, 1967; Lucas and Stokey, 1987.) It is clear from the proof that Theorem 1 will remain valid as long as the path of money supply does not change; in the new equilibrium, the money holding of each agent stays unchanged.

Agent h's lifetime resource as valued by state price process $\{a(s^r)\}$ is given by:

$$W^h(\underline{h}, a) = \frac{1}{a(\underline{h})} \sum_{s^r|\underline{h}} a(s^r)[\,\omega^h(s^r) + L^h(s^r)] \qquad (16)$$

Recall that $\underline{h}$ is agent h's unique initial node. By (15), the transfer changes prescribed in the proof of Theorem 1 leave $W^h(\underline{h}, a)$ unchanged under any $\{a(s^r)\}$ for any $h \in \tilde{H}$.

We have shown that *some* transfer scheme that leaves each agent's lifetime resource unchanged (under any $\{a(s^r)\}$) can cancel a change in government finance. The question now is if and when *any* such transfer scheme can do it. An obvious condition needed for establishing an "any transfer" result is that markets are complete; without this condition we have no reason to believe that unchanging lifetime resource implies unchanging attainable consumption set.

To see that the complete markets condition is not enough, consider an open market exchange that changes the asset supply but not total asset value at s^0 in an economy with two-date lives. Suppose the total dividend change at each $s^1|s^0$, $d(s^1)[\hat{Z}(s^0) - Z(s^0)]$, is absorbed entirely by transfer changes for the agents born at s^1, and the transfer of any agent in $H(s^0)$ does not change either at s^0 or at any $s^1|s^0$. It is then clear that the asset demand by agents in $H(s^0)$ does not change, and so given the changed asset supply at s^0, markets usually cannot clear at s^0 at the initial prices. On the other hand, it is possible for all the agents born on t = 1 or later to also have unchanged lifetime resource; an example is a deterministic economy with two-date lives, with each agent born on t = 1 or later having an old age transfer change that cancels its youth transfer change. Therefore, without further restriction, *some* transfer scheme that leaves each agent's lifetime resource unchanged *cannot* cancel the open market exchange.

The difficulty arises because agents in $H(s^0)$ must hold the changed asset supply at s^0 in any equilibrium, but they are not compensated for the return changes on the t = 1 nodes that the supply change at s^0 causes. The remedy is then to introduce transfers that compensate the return changes. When all the agents in $H(s^0)$ have two-date lives, the compensating transfers have to be made at the t = 1 nodes. In the more general case, some of the compensating transfers can be made at the t > 1 nodes. The compensating transfers are needed not only for s^0, but also for every other node. As the theorem below shows, given complete markets, these compensating transfers and both sufficient and necessary for an "any transfer" result to obtain. When markets are complete, we let $\{a(s^t)\}$ be the unique state price process with $a(s^0) = 1$.

Theorem 2: Let $\{q(s^t), L^h(s^t), c^h(s^t), Z^h(s^t)\}$ represent the initial equilibrium. Suppose markets are complete. Let $\{\hat{L}^h(s^t)\}$ be a set of transfers such that

$$\sum_{s^r \in N^h} a(s^r)\,\hat{L}^h(s^r) = \sum_{s^r \in N^h} a(s^r)\,L^h(s^r) \quad \text{for each } h \tag{17}$$

Then there exists $\{\hat{Z}^h(s^t)\}$ satisfying (9) and (8) such that $\{q(s^t), \hat{L}^h(s^t), c^h(s^t), \hat{Z}^h(s^t)\}$ represent a new equilibrium if and only if

$$R(s^{t+1})[\hat{Z}(s^t) - Z(s^t)] = -\frac{1}{a(s^{t+1})}\sum_{h\in H(s^t)}\sum_{s^r|s^{t+1}} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)] \quad \text{for each } s^t \text{ and each } s^{t+1}|s^t$$

(18)

By (18), the total transfer change from each $s^{t+1}|s^t$ on for all the agents alive at s^t is equal to the total return change at s^{t+1}. If all the agents in $H(s^t)$ have two-date lives, (18) becomes, for each $s^{t+1}|s^t$,

$$R(s^{t+1})[\hat{Z}(s^t) - Z(s^t)] = -\sum_{h\in H(s^t)} [\hat{L}^h(s^{t+1}) - L^h(s^{t+1})] \tag{19}$$

(19) is a slightly more general form of Wallace (1981)'s second requirement for the new transfer scheme.

Multiplying (18) by $a(s^{t+1})$, summing the result over $s^{t+1}|s^t$ and using the basic state price relation (13), we get a useful relation:

$$q(s^t)[\hat{Z}(s^t) - Z(s^t)] = -\frac{1}{a(s^t)}\sum_{h\in H(s^t)}\sum_{s^r|s^t, r>t} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)] \tag{20}$$

Proof of Theorem 2: We first prove the sufficiency part. It is well-known that, with complete markets, the set of attainable consumption for a finitely lived agent is determined entirely by the lifetime resource. The same is true for an infinitely lived agent if borrowing limits are given by (4), as our arguments below will verify. By (17), each agent has the same lifetime resource and so can attain the initial consumption, which remains optimal. The question is whether markets can clear. We will construct a complete set of market clearing asset holdings and show that they imply the initial consumption for each agent and also satisfy all the borrowing limits.

At each node s^t, pick a change absorbing agent $h(s^t) \in H(s^t)/\overline{H}(s^t)$. An agent may be picked more than once.

For each $h \in H(s^0)$ other than $h(s^0)$, choose $\hat{Z}^h(s^0)$ such that

$$q(s^0)[\hat{Z}^h(s^0) - Z^h(s^0)] = \hat{L}^h(s^0) - L^h(s^0)$$

(21a)

$$R(s^1)[\hat{Z}^h(s^0) - Z^h(s^0)] = -\frac{1}{a(s^1)}\sum_{s^r|s^1} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)] \quad \text{for each } s^1|s^0 \qquad (21b)$$

For each $h \in \overline{H}(s^0)$, only (21a) is relevant. By (5a), any $\hat{Z}^h(s^0)$ satisfying (21a) will leave h's consumption at s^0 unchanged. Because markets are complete, (17) implies that a $\hat{Z}^h(s^0)$ satisfying (21) can be found. The "correct" returns given by (21b) means $\hat{Z}^h(s^0)$ can be the first component of a trading plan that keeps agent h's later consumption unchanged as well.

Let $h(s^0)$'s asset holding at s^0 be

$$\hat{Z}^{h(s^0)}(s^0) = \hat{Z}(s^0) - \sum_{h \in H(s^0), h \neq h(s^0)} \hat{Z}^h(s^0)$$

(22)

This asset holding guarantees market clearing at s^0. Define $c(s^t) = \sum_{h \in H(s^t)} c^h(s^t)$. Summing (3a) over all agents in $H(s^0)$ for both the initial and the new transfers and asset holdings, we get:

$$c(s^0) + q(s^0)Z(s^0) = \omega(s^0) + q(s^0)Z(s^0-) + L(s^0)$$

(23a)

$$\hat{c}(s^0) + q(s^0)\hat{Z}(s^0) = \omega(s^0) + q(s^0)Z(s^0-) + \hat{L}(s^0) \qquad (23b)$$

By (8a), (23) implies $\hat{c}(s^0) = c(s^0)$. Because consumption at s^0 is not changed for any agent except perhaps $h(s^0)$, $\hat{c}(s^0) = c(s^0)$ implies $\hat{c}^{h(s^0)}(s^0) = c^{h(s^0)}(s^0)$.

Summing (21b) over all agents in $H(s^0)$ except $h(s^0)$ and subtracting the result from (18) as it is applied to $s^1|s^0$, we get:

$$R(s^1)[\hat{Z}^{h(s^0)}(s^0) - Z^{h(s^0)}(s^0)] = -\frac{1}{a(s^1)}\sum_{s^r|s^1} a(s^r)[\hat{L}^{h(s^0)}(s^r) - L^{h(s^0)}(s^r)] \tag{24}$$

By (24), $\hat{Z}^{h(s^0)}(s^0)$ can be the first component of a trading plan that keeps agent $h(s^0)$'s consumption unchanged.

Suppose that, for some $s^t - 1$ and all its predecessors, each agent in $H(s^t - 1)/\overline{H}(s^t - 1)$ has chosen asset holdings that imply the initial consumption at these nodes and also leave just enough resource at s^t so that it can attain the initial consumption at s^t and all its successors. That is, for each h,

$$R(s^t)[\hat{Z}^h(s^t - 1) - Z^h(s^t - 1)] = -\frac{1}{a(s^t)}\sum_{s^r|s^t} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)] \tag{25}$$

We will show that there then exist a set of market clearing asset holdings at s^t that imply the initial consumption at s^t for each agent in $H(s^t)$ and also leave just enough resources at each $s^{t+1}|s^t$ so that each agent in $H(s^t)$ can attain the initial consumption at $s^{t+1}|s^t$ and all its successors. Since we have found the $\hat{Z}^h(s^0)$ satisfying (21b), which is (25) as applied to $s^1|s^0$, for each agent in $H(s^0)$ (including $h(s^0)$), by mathematical induction we will have constructed a complete set of market clearing asset holdings that imply the initial consumption for each agent.

For each $h \in H(s^t)$ other than $h(s^t)$, choose $\hat{Z}^h(s^t)$ such that

$$q(s^t)[\hat{Z}^h(s^t) - Z^h(s^t)] = R(s^t)[\hat{Z}^h(s^t - 1) - Z^h(s^t - 1)] + \hat{L}^h(s^t) - L^h(s^t) \tag{26a}$$

$$R(s^{t+1})[\hat{Z}^h(s^t) - Z^h(s^t)] = -\frac{1}{a(s^{t+1})}\sum_{s^r|s^{t+1}} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)] \quad \text{for each } s^{t+1}|s^t \tag{26b}$$

By (26) and (25), which is valid for agents born at s^t if the left side is set to zero, we have:

$$a(s^t)q(s^t)[\hat{Z}^h(s^t) - Z^h(s^t)] - \sum_{s^{t+1}|s^t} a(s^{t+1})R(s^t)[\hat{Z}^h(s^t) - Z^h(s^t)] = 0 \tag{27}$$

Therefore, with complete markets, such a $\hat{Z}^h(s^t)$ can be found. By (26), $\hat{Z}^h(s^t)$ implies the initial consumption at s^t for h and also leaves just enough resource at each $s^{t+1}|s^t$ so that h can attain the initial consumption at $s^{t+1}|s^t$ and all its successors.

Let $h(s^t)$'s asset holding at s^t be

$$\hat{Z}^{h(s^t)}(s^t) = \hat{Z}(s^t) - \sum_{h\in H(s^t), h\neq h(s^t)} \hat{Z}^h(s^t) \tag{28}$$

Summing (25) over all agents in $H(s^t)$ and using (20), we get:

$$\sum_{h\in H(s^t)} R(s^t)[\hat{Z}^h(s^t - 1) - Z^h(s^t - 1)] = -\frac{1}{a(s^t)} \sum_{h\in H(s^t)} \sum_{s^r|s^t} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)]$$

$$= -\frac{1}{a(s^t)} \sum_{h\in H(s^t)} \sum_{s^r|s^t, r>t} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)] - \sum_{h\in H(s^t)} [\hat{L}^h(s^t) - L^h(s^t)]$$

$$= q(s^t)[\hat{Z}(s^t) - Z(s^t)] - [\hat{L}(s^t) - L(s^t)], \quad \text{or}$$

$$q(s^t)[\hat{Z}(s^t) - Z(s^t)] = \sum_{h\in H(s^t)} R(s^t)[\hat{Z}^h(s^t - 1) - Z^h(s^t - 1)] + \hat{L}(s^t) - L(s^t) \tag{29}$$

Summing (26a) over all agents in $H(s^t)$ other than $h(s^t)$ and subtracting the result from (29), we get:

$$q(s^t)[\hat{Z}^{h(s^t)}(s^t) - Z^{h(s^t)}(s^t)] = R(s^t)[\hat{Z}^{h(s^t)}(s^t - 1) - Z^{h(s^t)}(s^t - 1)] + \hat{L}^{h(s^t)}(s^t) - L^{h(s^t)}(s^t) \tag{30}$$

(30) and (3b) imply $\hat{c}^{h(s^t)}(s^t) = c^{h(s^t)}(s^t)$. Summing (26b) over all agents in $H(s^t)$ other than $h(s^t)$ and subtracting the result from (18) as it is applied to $s^{t+1}|s^t$, we get:

$$R(s^{t+1})[\hat{Z}^{h(s^t)}(s^t) - Z^{h(s^t)}(s^t)] = -\frac{1}{a(s^{t+1})} \sum_{s^r|s^{t+1}} a(s^r)[\hat{L}^{h(s^t)}(s^r) - L^{h(s^t)}(s^r)] \text{ for each } s^{t+1}|s^t$$

(31)

(31) implies $h(s^t)$ has just enough resource at each $s^{t+1}|s^t$ to attain the initial consumption at $s^{t+1}|s^t$ and all its successors. This completes our construction of a complete set of market clearing asset holdings that imply the initial consumption for each agent.

It is easy to see that the change of asset value by any h at any s^t is exactly matched by the change of the future resource viewed at s^t. For example, by combining (25) with (26a), we get:

$$q(s^t)[\hat{Z}^h(s^t) - Z^h(s^t)] = -\frac{1}{a(s^t)}\sum_{s^r|s^t,r>t} a(s^r)[\hat{L}^h(s^r) - L^h(s^r)], \qquad (32)$$

which shows the change of h's asset value at s^t is equal to the negative of the change of its future transfers. Therefore, no borrowing limit for any h is violated at any s^t.

It is also easy to see that each agent can attain under the initial transfer any consumption it can attain under the new transfer. This establishes the optimality of the new asset holdings and concludes the proof of sufficiency.

The proof of sufficiency has made it clear that, faced with unchanged lifetime resource, a necessary condition for agent h to attain the initial consumption, which remains optimal, is for (26b) to be satisfied at each $s^t \in N^h$. The necessity of (18) follows from summing (26b) over $h \in H(s^t)$. **Q.E.D.**

The key condition of Theorem 2 is (18), which requires that the total return change at any $s^{t+1}|s^t$ be equal to the sum of resource change from s^{t+1} on over agents *alive at* s^t. Put differently, (18) requires that, for each s^t, agents alive at s^t are collectively compensated for next period's return change (in each state of nature) by transfer changes from the next period on. Note that (18) is an aggregate condition; it does not require that an individual agent be compensated in a particular way.

(18) is closely related to the intergenerational links that normally underlie the validity of the Ricardian equivalence. In Theorem 2, intergenerational links are not imposed, but the changes in government finance are restricted to those that do not imply intergenerational transfers: at each s^t, "generation" $H(s^t)$ faces future transfer changes that exactly cancel future return changes. For an economy consisting of infinitely lived "dynasties", (18) is automatically satisfied. (To see this, multiply (8b) as it is applied to some s^r by $a(s^r)$ and sum the result from s^{t+1} on.) Based on these observations, Theorem 2 may also be regarded as a generalized Ricardian equivalence result.

What happens if one of the bonds is transactional money? First, note that redundancy of money cannot be consistent with an equilibrium in which some agent is not satiated with transactional service, because it implies that any agent can costlessly hold an arbitrarily large quantity of money by simultaneously holding a bundle of other assets. If money is not a redundant asset, keeping agent h's future resource unchanged at each $s^t \in N^h$ generally requires that agent h changes its money holdings, and this calls for overall re-optimization by agent h, including changes in consumption. We therefore cannot expect that there exists a new equilibrium with the same consumption patterns and asset prices. Our conclusion, then, is that Theorem 2 is not valid in the presence of transactional money.

4. Conclusion

Wallace (1981) regards his paper as "providing only a suggestion for a general Modigliani-Miller theorem for open-market operations." The present paper picks up on Wallace's suggestion and establishes two general equivalence theorems for government finance.

The theorems are applicable to the kind of open market exchanges found in Wallace (1981). They may also be regarded as generalized Ricardian equivalence results.

References

Barro, R. (1974): Are government bonds net wealth? *Journal of Political Economy*, **84**, 1095-1117.

Brock, W. (1974): Money and growth: the use of long run perfect foresight. *International Economic Review*, **15**, 750-77.

Bryant, J. and N. Wallace (1984): A price discrimination analysis of monetary policy. *Review of Economic Studies*, **51**, 279-88.

Chamley, C and H. Polemarchakis (1984): Assets, general equilibrium and the neutrality of money. *Review of Economic Studies,* **51**, 129-38.

Clower (1967): A reconsideration of the microfoundations of monetary theory. *Western Economic Journal*, **6**, 1-8.

DeMarzo, P. (1988): An extension of the Modigliani-Miller Theorem to stochastic economies with incomplete markets and interdependent securities. *Journal of Economic Theory*, **45**, 353-69.

Detemple, J., P. Gottardi and H. Polemarchakis (1995): The relevance of financial policy. *European Economic Review*, **39**, 1133-54.

Gottardi, J. (1994): On the non-neutrality of money with incomplete markets. *Journal of Economic Theory,* **62**, 209-20.

Gottardi, J. (1995): An analysis of the conditions for the validity of Modigliani-Miller Theorem with incomplete markets. *Economic Theory,* **5**, 191-207.

Hernandez, A. and M. Santos (1996): Competitive equilibria for infinite-horizon economies with incomplete markets. *Journal of Economic Theory*, **71**, 102-30.

Lucas, R. and N. Stokey (1987): Money and interest in a cash-in-advance economy, *Econometrica*, **55**, 491-513.

Modigliani, F. and M. Miller (1958): The cost of capital, corporate finance, and the theory of investment. *American Economic Review*, **48**, 261-97.

Radner, R. (1972): Existence of equilibrium of plans, prices, and price expectations in a sequence of markets. *Econometrica*, **40**, 289-303.

Ross, S. (1976): The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, **13**, 341-60.

Samuelson, P. (1958): An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, **66**, 467-82.

Santos, M. and M. Woodford (1997): Rational asset pricing bubbles. *Econometrica*, **65**, 19-57.

Sargent, T. (1987): *Dynamic Macroeconomic Theory*.

Sidrauski, M. (1967): Rational choice and patterns of growth in a monetary economy. *American Economic Review*, **57**, 534-44.

Stiglitz, J. (1969): A re-examination of the Modigliani-Miller Theorem. *American Economic Review*, **59**, 784-93.

Stiglitz, J. (1974): On the irrelevance of corporate financial policy. *American Economic Review*, **64**, 851-66.

Wallace, N. (1980): The overlapping generations model of fiat money. In *Models of Monetary Economies* edited by J. Kareken and N. Wallace.

Wallace, N. (1981): A Modigliani-Miller Theorem for open-market operations. *American Economic Review*, **71**, 267-74.

Yu, D. (1998): *Frictional Arbitrage Valuation*. Ph.D. thesis, Princeton University.