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Capital taxation with heterogeneous discounting and collateralized borrowing*

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Abstract

We study optimal long-run capital taxation in a closed economy with heterogeneity in agents' time-discount factors where borrowing is allowed but restricted by a collateral constraint. Financial frictions distort intertemporal optimization margins and the tax system serves a dual role: first, it is used to finance government consumption; second, it serves to alleviate the distortions arising from the binding collateral constraint. The discrepancy between the private and the social discount factors pushes for a subsidy on capital, while the discrepancy introduced by the collateral constraint pushes for a tax in the long-run. When consumption smoothing motives are muted, the two effects counter-balance each other and the tax is zero. With finite elasticity of intertemporal substitution, the second discrepancy dominates and the tax on capital income is positive in the long-run.

JEL classification: E60, E61, E62, H21

Keywords: Ramsey taxation, tax on capital, collateral constraint, heterogeneous discount factors

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1 Introduction

What should the tax on capital income be in the long-run? Chamley (1986) and Judd (1985), working in somewhat different settings, found that in a dynamic Ramsey model with infinitely lived agents and no distortions in the economy capital should be untaxed given that the economy converges to a steady state (Straub and Werning, 2014). The result is based upon the intuition that capital income taxation induces differentiated consumption taxes on present and future consumption. In other words, taxing capital income distorts individuals' intertemporal consumption behavior as they substitute the more heavily taxed future consumption with current consumption (see also Chari et al., 1994; Atkeson et al., 1999).

This paper recasts the optimal taxation problem in an economy with a collateral constraint. In particular, we examine whether the Chamley-Judd result of a zero tax on capital income in the long-run survives in an economy where agents face borrowing constraints akin to those present in Kiyotaki and Moore (1997) and Iacoviello (2005). We retain the environment in Judd (1985), which consists of two-classes of agents, workers and capitalists, but we allow them to discount the future differently. Capitalists are relatively more impatient and want to borrow from patient workers. Moreover, we modify the bond market structure by having capitalists' borrowing be limited by a collateral requirement. The consideration of a collateral constraint is important since one of the key assumptions in Chamley (1986) and Judd (1985) is the ability of private agents to freely shift consumption intertemporally, whereas the presence of a collateral constraint precludes it. Given that collateral constrained economies featuring patient and impatient agents have been extensively used in macroeconomic analysis, it is of interest to understand how the presence of the collateral constraint and of heterogeneous discounting influence the key results on long-run capital taxation.

Our analysis identifies two forces that drive the decision of a Ramsey planner to levy distortionary taxes on capital income in the long-run. On the one hand, the discrepancy in the discount rates between workers and capitalists induces a discrepancy between social discounting and that of capitalists. Without assuming an exogenous discount rate for the planner, we derive that the planner will discount the future at the rate of the workers in the long-run. This difference between the planner's and capitalists' evaluation of intertemporal consumption pushes for a subsidy on capital income in the long-term, which is a general result also applying to other economies (see De Bonis and Spataro, 2005; Reis, 2012).

On the other hand, the presence of binding collateral constraints pushes for a tax on capital income in the long-run. In economies with collateral constraints of the class considered here, agents evaluate the investment in capital not only as an input in the production function, but also on the grounds that it relaxes the collateral constraint. Therefore, as argued by Fostel and Geanakoplos (2008) and Geanakoplos and Zame (2014), capital embeds a collateral premium, which pushes the marginal product of capital down. A Ramsey planner who is endowed only with linear taxation tools cannot undo the financial friction resulting in binding collateral constraints, but could use these tools to affect the distorted intertemporal margins.

The sign of the tax on capital income in the long-run depends on the relative magnitude of the discrepancies in discounting and in the shadow value on the collateral constraint between the planner and the private agents. If capitalists do not care about smoothing consumption intertemporally, i.e. they have linear utilities, then the two discrepancies counter-balance each other and the tax on capital income is zero in the long-run. The Chamley-Judd result survives in collateral constrained economies in this special case. Nevertheless, for finite elasticity of intertemporal substitution the discrepancy in shadow values of collateral dominates the discepancy in discounting and the Ramsey planner levies a strictly positive tax on capital income in the long-run. Moreover, the tax on capital income increases as the elasticity of intertemporal substitution decreases (or equivalently the coefficient of realtive risk aversion for CRRA utility increases). Hence, our positive capital tax result in the presence of collateral constraints does not obtain as a special case for certain parameters governing the motive of capitalists to smooth consumption intertemporally. For example, Lansing (1999) shows that the zero tax result in Judd (1985) can be invalidated, but only when capitalists have logarithmic utilities.

Finally, the Ramsey planner plays a dual role. The first goal is well-known to the Ramsey literature, which is to minimize distortions resulting from linear taxation used to finance exogenous government expenditure. The second goal is to address distortions in optimization margins. In particular, we show that a tax on labor income is used to finance the expenditures of the government and it is zero if lump-sum transfers are available. On the contrary, a tax on capital income serves to correct for the inefficiencies induced by the binding collateral constraint and is generally non-zero even if lump-sum transfers are allowed.

The paper is organized as follows. Section 2 describes the economic environment. Section 3 derives the Ramsey problem, computes the optimal tax policy and discusses the role of taxes in the economy. Finally, section 4 concludes. All proofs are relegated to the Appendix.

Related Literature. Park (2014) studies an optimal Ramsey taxation problem in an environment where agents face a limited commitment problem as in Alvarez and Jermann (2000). She shows that the Ramsey government faces two conflicting objectives: first, to finance government expenditure; second, to internalize the externality of labor and capital to improve risk sharing. Thus, she also argues that the Ramsey planner has a dual role. The steady state tax on capital income is levied to correct for the pecuniary externalities induced by the binding borrowing limits; whereas the tax on labor income is used to finance the remaining budgetary needs of the government. Our paper differs because of the nature of the collateral constraint as well as the absence of idiosyncratic uncertainty. It also shows that pecuniary externalities are not the sole reason to levy distortionary capital taxation.

Another paper that is related to ours is Reis (2012). She finds that the tax on capital income is positive at the deterministic steady state as long as the benevolent government is more impatient than the private agents, accumulates debt and is not able to commit to future policies. She finds that these conditions need to hold for a positive tax on capital income to emerge in the long-run. In our paper, contrary to hers, there are three agents: workers, capitalists and a benevolent government which does not face a commitment problem. The positive capital taxation in the long-run is due to the collateral premium and the difference in discount factors rather due to government impatience and its inability to commit. Aguiar and Amador (2016) model a small open economy with impatient agents compared to the rest of the world and a government with limited commitment. They show that labor income taxes can go to zero in the long-run, while capital income taxes may not be zero. Our paper is different because we model a closed economy with heterogeneous agents where the role of capital as collateral distorts intertemporal margins and calls for positive capital taxation.

Biljanovska (forthcoming) studies a variety of corrective policy tools in the presence of collateral constraints and their ability to replicate first- and second-best allocations. Itskhoki and Moll (2015) also study optimal dynamic Ramsey policies when borrowing is constrained, but tax proceeds are rebated back to the same agents and, thus, policy is Pigouvian. Our analysis differs, importantly, because we do not allow for corrective Pigouvian taxation or other transfers, but rather study distortionary taxation used to fund government expenditure as is the norm in the Ramsey literature. Positive capital taxation would arguably be more difficult to obtain in our framework, since it would remove resources for resource-constrainted capitalists.

Our paper is more broadly related to the literature studying optimal capital income taxation. Chamley (1986) and Judd (1985) established the result of zero capital taxation in the long-run, which rests critically on the possibility of shifting consumption across periods through perfect capital markets (see for e.g. Chamley, 2001). More recently, Chari et al. (2016) revisit the Chamley-Judd result in an environment that allows for a richer tax system (i.e. without caps on linear taxation, unlike Straub and Werning, 2014, who impose limits on taxation), and find that the tax on capital income is still zero in the long-run. However, the environment they consider does not involve any (financial) market imperfections. In our setup, a collateral constraint yields intertemporal wedges that, as we will see, call for a positive tax on capital income in the long-run.

A positive tax on capital income has also been found in models with uninsurable idiosyncratic risk and/or borrowing limits (see Aiyagari, 1995, and subsequent literature), as well as in life-cycle model frameworks (Erosa and Gervais, 2002; Conesa et al., 2009).

2 The economy

This section presents the economy and characterizes the set of attainable allocations. The economy is populated by two types of infinitely-lived agents that behave competitively, workers and capitalists. Workers provide labor hours, but they also lend to capitalists. Capitalists invest in capital and own a production technology that employs both capital and labor to produce a homogeneous good. The investment and production activity is financed by earnings and by issuing bonds sold to workers. Since capitalists cannot commit to repay their debt, their borrowing capacity is limited by a collateral constraint as in Kiyotaki and Moore (1997) and Iacoviello (2005).

2.1 Workers

There is a continuum of identical workers, whose objective is to maximize the sum of future utilities

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - l_t\right),$$

where c_t is consumption, $1 - l_t$ is leisure with l_t denoting labor hours and β is the subjective discount factor taking values $0 < \beta < 1$.

Assumption 1. *The utility function $u(c_t, 1 - l_t)$ is separable in consumption and labor, it is continuous, twice differentiable, increasing in consumption and decreasing in labor, is globally concave and satisfies the Inada conditions.*

For every unit of work supplied, workers get wages, net of tax, $\left(1 - \tau_t^l\right) w_t l_t$. They can also shift consumption across periods by investing in one period corporate bonds, b_{t+1}^w, issued by capitalists, for which they receive a return of r_t. Taking market prices (the price of the consumption good, set to be a numeraire, the wage rate and the lending rate) as given, workers make optimal consumption, labor, and investment in corporate bonds decisions to maximize the present value of the utility subject to the following budget constraint

$$c_t + \frac{b_{t+1}^w}{1 + r_t} \leq b_t^w + \left(1 - \tau_t^l\right) w_t l_t. \tag{1}$$

The equilibrium conditions that characterize the solution to the workers' problem are given by the Euler condition

$$u_{c,t} = \beta u_{c,t+1} \left(1 + r_t\right), \tag{2}$$

and the optimal labor supply decision

$$w_t(1 - \tau_t^l) = -\frac{u_{l,t}}{u_{c,t}}. \tag{3}$$

The subscripts on the utility function in (2) and (3) denote the respective partial derivative at time period t. We will preserve the same notation throughout the paper.

2.2 Capitalists

There is a continuum of identical capitalists, who maximize the following sum of discounted utilities

$$\sum_{t=0}^{\infty} \gamma^t u^c \left(c_t^c\right),$$

where c_t^c denotes capitalists' consumption and γ is the subjective discount factor, with $0 < \gamma < 1$. The utility function has the same properties as those detailed in

Assumption 1.

Assumption 2. *Capitalists discount the future more heavily than workers, $\gamma < \beta$.*

This assumption is introduced to ensure that capitalists are not completely self-financed and that the collateral constraint binds in equilibrium.

Capitalists invest in capital, which accumulates following the law of motion

$$k_{t+1} = i_t + (1-\delta)\, k_t, \tag{4}$$

where i_t denotes investment and δ denotes the depreciation rate. They employ capital and labor, n_t, using Cobb-Douglas production technology to produce a homogeneous good

$$F(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha},$$

where α denotes the share of capital in the production process.

Assumption 3. *The production function, $F(k_t, n_t)$, has a constant returns to scale and is increasing and concave in both its arguments. The following conditions hold: i. $F(0,n) = F(k,0) = 0$; ii. $\lim_{k\to 0} F_k(k,n), \lim_{n\to 0} F_n(k,n) \to \infty$; iii. $\lim_{n\to\infty} F_n(k,n), \lim_{k\to\infty} F_k(k,n) \to 0$. The function is defined for positive values of k and n.*

Capitalists issue a one period corporate bond, b_{t+1}^c, which is repaid at rate r_t. They also pay tax to the government on their capital gains, τ_t^k. Thus, capitalists must meet the following budget constraint

$$c_t^c + b_t^c + i_t \leq \left(1-\tau_t^k\right)\left[F(k_t, n_t) - w_t n_t\right] + \frac{b_{t+1}^c}{1+r_t}. \tag{5}$$

However, due to their inability to commit to repay the debt, capitalists' capacity to borrow is bounded by a collateral constraint of the form

$$\bar{\varepsilon} k_{t+1} \geq \frac{b_{t+1}^c}{1+r_t}, \tag{6}$$

where $\bar{\varepsilon} < 1$ is the liquidity value of capital and is kept constant for simplicity.[1]

Capitalists choose optimally consumption, labor, capital investment and borrowing. The equilibrium conditions that characterize the solution to capitalists'

[1]The collateral constraint can be microfounded from a renegotiation process between the borrowers and the lenders, as shown in Hart and Moore (1994).

problem are given by

$$w_t = F_{n,t}, \tag{7}$$

$$u^c_{c,t}\left(1-\mu_t\right) = \gamma u^c_{c,t+1}\left(1+r_t\right), \tag{8}$$

$$u^c_{c,t}\left(1-\bar{\varepsilon}\mu_t\right) = \gamma u^c_{c,t+1}\left[\left(1-\tau^k_{t+1}\right)F_{k,t+1}+1-\delta\right], \tag{9}$$

where μ_t denotes the Lagrange multiplier on the collateral constraint, and $F_{n,t}$ and $F_{k,t}$ denote the marginal products of labor and capital, respectively at time t. The complementarity slackness condition is given by

$$\mu_t\left(\bar{\varepsilon}k_{t+1}-\frac{b^c_{t+1}}{1+r_t}\right)=0, \qquad \mu_t \geq 0.$$

Lemma 1. *If $\gamma < \beta$, then the collateral constraint always binds at the steady state, i.e. $\mu = (\beta-\gamma)/\beta > 0$.*

Lemma 1 shows that the tightness of the collateral constraint at the steady state is determined by the difference in agents' discount factors. The less patient the capitalists, the tighter the constraint; capitalists borrow as much as possible since the borrowing rate is lower than their discount rate.

The presence of a binding collateral constraint affects capitalists' intertemporal optimal choices with respect to borrowing and capital investment. In particular, eq. (9) suggests that foregoing one unit of consumption at time period t does not only bring an additional unit of capital at $t+1$, but it also relaxes the collateral constraint at t. Hence, capital entails a collateral premium.

2.3 Government

Following Judd (1985), the government cannot issue bonds and runs a balanced budget. It receives revenues from linear taxes on labor, τ^l_t, and capital income, τ^k_t, each period, and consumes an exogenous amount of $g_t \geq 0$. Given this, the government's budget constraint reads

$$g_t = \tau^l_t w_t l_t + \tau^k_t\left[F\left(k_t,n_t\right)-w_t n_t\right]. \tag{10}$$

Now we can define the competitive equilibrium of the outlined economy.

2.4 Competitive Equilibrium

Definition Given initial values for capital, k_0, and borrowing $b_0^w = b_0^c = b_0$, and an exogenous value for government spending, g_t, a competitive equilibrium with a collateral constraint is a sequence of allocations $\left\{c_t, l_t, b_{t+1}^w, c_t^c, n_t, b_{t+1}^c, k_{t+1}\right\}_{t=0}^{\infty}$, prices $\{r_t, w_t\}_{t=0}^{\infty}$, and government policies $\left\{\tau_t^k, \tau_t^l\right\}_{t=0}^{\infty}$ such that

(i) given prices and policies, the allocations solve workers' and capitalists' maximization problems;

(ii) the government budget constraint (10) is satisfied;

(iii) labor, bonds, and goods markets clear:

$l_t = n_t,$

$b_{t+1}^w = b_{t+1}^c = b_{t+1},$

$c_t + c_t^e + i_t + g_t = F\left(k_t, l_t\right), \forall t.$

3 Ramsey Optimal Policy

This section derives the tax on capital and labor income as a solution to the problem of the Ramsey government. We formulate the problem following the so-called "primal approach," consisting of the planner choosing allocations directly instead of policy instruments among the set of allocations attainable by the competitive economy with a collateral constraint.

3.1 Implementable Allocations

Following the approach developed in Lucas and Stokey (1983), the initial step of the primal approach consists of finding the set of allocations that can be supported as a competitive equilibrium with a collateral constraint for some sequence of prices and policy instruments. To this end, we derive the conditions that these allocations need to satisfy such that they can be decentralized as a competitive equilibrium with a collateral constraint. Then, introducing these conditions as constraints in the Ramsey planner's maximization problem makes sure that any allocations chosen by the planner can also be sustained in the competitive economy with a collateral constraint.

Lucas and Stokey (1983) showed that for frictionless markets, competitive equilibrium imposes a single period-zero implementability constraint on allocations (for each agents). However, in presence of borrowing limits, competitive equilibrium allocations must satisfy the same restrictions as in Lucas and Stokey (1983), as well as additional ones that impose that capitalists do not exceed their borrowing limit at any point in time.

Lemma 2. *Given initial values for capital, k_0, borrowing $b_0^w = b_0^c = b_0$, as well as an exogenous value for government spending, g_t, a sequence of allocations $\left\{c_t, l_t, b_{t+1}^w, c_t^c, n_t, b_{t+1}^c, k_{t+1}\right\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium with a collateral constraint if and only if all markets clear and the following conditions are met:*

(i) *Resource constraint,* $c_t + c_t^c + i_t + g = F\left(k_t, l_t\right), \forall t.$

(ii) *Period-zero implementability constraint of workers*

$$\sum_{t=0}^{\infty} \beta^t m\left(c_t, l_t\right) = u_{c,0} b_0, \tag{11}$$

where $m\left(c_t, l_t\right) \equiv u_{c,t} c_t + u_{l,t} l_t.$

(iii) *Period-zero implementability constraint of capitalists*

$$\sum_{t=0}^{\infty} \gamma^t u_{c,t}^c c_t^c = u_{c,0}^c \left[F_{k,0}\left(1 - \tau_0^k\right) k_0 + \left(1 - \delta\right) k_0 - b_0\right] \tag{12}$$

(iv) *(Per-period) Implementability constraints of capitalists*

$$\sum_{t=s}^{\infty} \gamma^{t+1-s} u_{c,t+1}^c c_{t+1}^c = u_{c,s}^c \left(k_{s+1} - \frac{b_{s+1}}{1 + r\left(c_s, c_{s+1}\right)}\right), \forall s \geq 0, \tag{13}$$

where $1 + r\left(c_s, c_{s+1}\right) = \frac{u_{c,s}}{\beta u_{c,s+1}}.$

(v) *Collateral constraint,*

$$\bar{\varepsilon} k_{t+1} \geq \frac{b_{t+1}}{1 + r\left(c_t, c_{t+1}\right)}, \forall t. \tag{14}$$

In a frictionless economy such as the one presented in Judd (1985), the allocations characterizing the competitive equilibrium can be summarized by the economy's resource constraint and the period-zero implementability constraints of workers and capitalists, i.e. (11) and (12), respectively. The presence of the collateral constraint, limiting agents' ability to shift consumption intertemporally, leaves these restrictions intact, but adds two other *sequences* of constraints. Constraint

(13) requires capitalists to adhere to a per-period implementability constraint in addition to their period-zero implementability constraint. This condition, relative to the frictionless market case, requires that the next period allocations lie in a subspace determined by the limit on borrowing. In particular, it requires that the allocations at each date must be such that the present discounted value of future consumption does not exceed the net asset value.[2] Finally, condition (14) requires that the debt limit is respected.

As we show later, in an economy where workers and capitalists have the same discount factors such as the one in Judd (1985), the per-period implementability constraint does not bind due to non-binding (financial) frictions in the economy. The reason is, in absence of (financial) market frictions, capitalists' consumption choices are not necessarily restricted to a fraction of the net wealth; instead agents can freely shift consumption intertemporally. To the contrary, the per-period implementability constraint binds in the economy with a collateral constraint. This is the key difference in the formulation of the problem for the competitive economy with a collateral constraint and the frictionless economy in Judd (1985), which, as shown later, is the source of a positive capital income tax.

The final remark regarding the formalization of the Ramsey problem is concerning τ_0^k. While other papers have shown that the tax on capital in the initial period should not be a choice variable in order to avoid the uninteresting result of the tax rate at $t = 0$ having the role of a lump-sum tax, financing all government expenditure, this assumption is redundant here. Namely, in our setup, letting τ_0^k to be a choice variable for the planner at time period $t = 0$ or imposing additional restrictions on it would not affect the tax on capital income in the long-run.

3.2 The Ramsey Problem

Given Lemma 2 the problem of the Ramsey planner can be formulated as a problem of choosing *allocations* (among the implementable set), and not policy instruments and prices, in order to maximize a given social welfare function subject to the constraints constituting a competitive equilibrium. The social welfare in this economy is given by the utilitarian welfare function where $\omega \geq 0$ is the exogenously assigned weight on capitalists. Then, the problem of the Ramsey government is

[2] For a more detailed study of this case, see for e.g. Aiyagari et al. (2002).

given by

$$\max_{\{c_t,c_t^c,l_t,k_{t+1},b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left\{\beta^t u\left(c_t, 1-l_t\right) + \omega\gamma^t u^c\left(c_t^c\right)\right\}$$

subject to conditions (i)–(v) in Lemma 2.

We analyze optimal taxation by solving the Ramsey optimization problem. We start by composing a Lagrangian. We attach the following Lagrange multipliers, $\lambda_t, \lambda^w, \lambda^c, \lambda_t^{cc}, \mu_t^{RP}$, to constraints (i)–(v) in Lemma 2, respectively. Then, the Lagrangian of the Ramsey government can be written as[3]

$$\begin{aligned}
\mathcal{L} \;=\; & \sum_{t=0}^{\infty} \left\{\beta^t u\left(c_t, 1-l_t\right) + \omega\gamma^t u^c\left(c_t^c\right)\right\} \\
& + \sum_{t=0}^{\infty} \lambda_t \left[F\left(k_t, l_t\right) + \left(1-\delta\right) k_t - c_t - c_t^c - k_{t+1} - g_t\right] \\
& + \lambda^w \sum_{t=0}^{\infty} \beta^t \left\{u_{c,t} c_t + u_{l,t} l_t - u_{c,o} b_0\right\} \\
& + \lambda^c \sum_{t=0}^{\infty} \gamma^t \left\{u_{c,t}^c c_t^c - u_{c,0}^c \left[F_{k,0}\left(1-\tau_0^k\right) k_0 + \left(1-\delta\right) k_0 - b_0\right]\right\} \\
& + \sum_{s=0}^{\infty} \lambda_s^{cc} \sum_{t=s}^{\infty} \gamma^{t+1-s} u_{c,t+1}^c c_{t+1}^c - \sum_{t=0}^{\infty} \lambda_t^{cc} u_{c,t}^c \left(k_{t+1} - \frac{b_{t+1}}{1+r\left(c_t, c_{t+1}\right)}\right) \\
& + \sum_{t=0}^{\infty} \mu_t^{RP} \left(\bar{\varepsilon} k_{t+1} - \frac{b_{t+1}}{1+r\left(c_t, c_{t+1}\right)}\right)
\end{aligned}$$

We assume that the government has the ability to commit to the contingent policy rules it announces at date 0. To derive optimal policy, we exploit the first order necessary conditions of the Ramsey problem $\forall t \geq 1$:[4]

$$c_t : \beta^t u_{c,t} + \beta^t \lambda^w m_{c,t} = \lambda_t, \tag{15}$$

[3]Note that we do not make any assumptions regarding the growth rate of any of the Lagrange multiplier in the planner's optimization problem. Instead, in section 3.3 we derive the converegence rates of all the multipliers following Reis (2012).

[4]In the interest of space, we omit the first order conditions of the Ramsey government at $t = 0$. However, as mentioned earlier, this does not matter for our results on the long-run tax rate on capital income.

$$c_t^c : \omega\gamma^t u_{c,t}^c + \gamma^t \lambda^c \left(u_{c,t}^c + u_{cc,t}^c c_t^c\right) + \left(u_{c,t}^c + u_{cc,t}^c c_t^c\right) \sum_{s=0}^{t-1} \lambda_s^{cc} \gamma^{t-s} - \lambda_t^{cc} u_{cc,t}^c \left(k_{t+1} - \frac{b_{t+1}}{1 + r\left(c_t, c_{t+1}\right)}\right) = \lambda_t, \quad (16)$$

$$b_{t+1} : \lambda_t^{cc} u_{c,t}^c = \mu_t^{RP}, \quad (17)$$

$$k_{t+1} : \lambda_t - \left(\bar{\varepsilon} - 1\right) \mu_t^{RP} = \lambda_{t+1} \left(F_{k,t+1} + 1 - \delta\right), \quad (18)$$

$$l_t : -\beta^t u_{l,t} - \beta^t \lambda^w m_{l,t} = \lambda_t F_{l,t}. \quad (19)$$

These conditions are necessary for an optimal solution to the planner's problem.[5] The key difference between the Ramsey planner's problem subject to a collateral constraint and the one without arises from eq. (16), (17) and (18). In particular, eq. (16) that also feeds in eq. (17) and (18) incorporates some additional terms absent from its frictionless counterpart. Assuming a CRRA utility function for capitalists of the form $u^c = c^{c^{1-\sigma^c}}/(1-\sigma^c)$ with elasticity of intertemporal substitution $EIS \equiv 1/\sigma^c$, we can rewrite eq. (16) by substituting in eq. (17)

$$\omega\gamma^t u_{c,t}^c + \gamma^t \lambda^c \left(u_{c,t}^c + u_{cc,t}^c c_t^c\right) + \Psi_t = \lambda_t, \quad (20)$$

where

$$\Psi_t \equiv \left(1 - \frac{1}{EIS}\right) \sum_{s=0}^{t-1} \gamma^{t-s} \frac{u_{c,t}^c}{u_{c,s}^c} \mu_s^{RP} + \frac{1}{EIS} \frac{k_{t+1} - b_{t+1}/(1 + r(\cdot))}{c_t^c} \mu_t^{RP}. \quad (21)$$

This equation suggests that the planner internalizes that the social benefits from consumption include the direct effect on utility, $\omega\gamma^t u_{c,t}^c$; the cost of distortions due to the revenue burden with distortionary income taxes, $\gamma^t\lambda^c(u_{c,t}^c + u_{cc,t}^c c_t^c)$; and the cost arising from the binding collateral constraint, Ψ_t.

The first two terms in (20) are the familiar ones from the Ramsey literature without financial frictions. The last term, Ψ_t, is novel and depends on the shadow value of the collateral constraint, i.e. the collateral premium, meaning that the planner explicitly accounts how the consumption choice of capitalists is affected by the tightness of the collateral constraint. Notice that Ψ_t is a weighted average of two effects: On one hand, the presence of the collateral constraint restricts the

[5] It is well known that these conditions are necessary, but not sufficient because the implementability constraints are not convex. This issue is generally ignored in the literature and the planner's first order optimality conditions are simply assumed to be both necessary and sufficient for a global optimum.

intertemporal smoothing of consumption; on the other hand, lower consumption allows for capital accumulation, which relaxes the collateral constraint. We will refer to the first as the *intertemporal consumption effect* and to the latter as the *collateral effect*. The two effects are averaged out through capitalists' EIS, or alternatively, their willigness to smooth consumption intertemporally, which will play an important role in determining the sign on the tax on capital income at the steady state.

Moreover, note that the Lagrange multiplier on the collateral constraint, μ^{RP}, in the Ramsey problem does not simply capture a collateral premium on capital investment as is the case with μ in the competitive economy. This can be seen from the left-hand side in condition (18) where the effect of capital investment on future implementable consumption allocations through constraint (13) has been substituted in.[6] Given that $\bar{\varepsilon} < 1$, μ^{RP} enters (18) as an effective "cost" for the Ramsey planner of increasing investment and relaxing the collateral constraint. The intertemporal consumption and collateral effects will determine the level of this "cost" relative to the collateral premium in the competitive economy.

Finally, from equations (6) and (17), respectively, it can easily be shown that when $\mu_t = 0$ *and* $\mu_t^{RP} = 0$, the planner's problem collapses to the familiar one in Judd (1985). This is the case for $\gamma \to \beta$ as it will be apparent from the analytical characterization of the tax on capital income in the following section.

3.3 Optimal Taxes

This section derives analytical formulas for the tax rates on capital and labor income at the steady state using the above derived optimality conditions. Let c, l, c^c, b and k be the steady state values for workers' consumption, labor, capitalists' consumption, borrowing and capital, respectively, which are constant. Furthermore, we show that the multipliers at the steady state converge as well. This is particularly important because the behaviour of the multipliers in the steady state determines the tax on capital income in two ways; first, through the rate at which the planner evaluates intertemporal consumption, and, second, through the multiplier on the collateral constraint in the Ramsey problem.

Lemma 3. *In the steady state, the shadow value of agents' consumption in the Ramsey problem, λ_t, grows at β, i.e.* $\lim_{t\to\infty} \lambda_{t+1}/\lambda_t = \beta$.

[6]The Euler equation (18) is obtained by substituting in (17), which incorporates the Lagrange multiplier on constraint (13), i.e. the limit on future implementable consumption allocations.

Then, in the steady state the consumption–capital investment decision of the Ramsey planner becomes

$$1 - (\bar{\varepsilon} - 1) \lim_{t\to\infty} \frac{\mu_t^{RP}}{\lambda_t} = \beta \left(F_k + 1 - \delta \right). \tag{22}$$

Hence, to determine the tax on capital income, we need to find the value at which the ratio $\lim_{t\to\infty} \mu_t^{RP}/\lambda_t$ converges.

Lemma 4. *In the steady state, the shadow value on the colalteral constraint in the Ramsey problem coverges to* $\lim_{t\to\infty} \mu_t^{RP}/\lambda_t = (\beta-\gamma)u_c^c/(\gamma A - u_{cc}^c(\beta-\gamma)B)$*, where* $A \equiv (u_c^c + u_{cc}^c c^c)$ *and* $B \equiv (k - b/(1+r))$*. Moreover, it is strictly positive and higher than* μ *for* $\sigma^c < (1-\gamma)/(1-\beta)$.[7]

Finally, since the Lagrange multipliers on the resource constraint, λ_t, and the Lagrange multiplier on the intertemporal budget constraint of workers, λ^w, as shown later, enter in the equations of the tax rates, it will be important at this point to highlight that they are finite. This can be shown by using equations (15) and (19), i.e. the first order conditions in the planner's problem with respect to workers' consumption and labor respectively, and solving for λ_t and λ^w as functions of steady state variables.

3.3.1 Tax on capital income

Using equation (9) and (22), the tax on capital income in the steady state is

$$\tau^k = (1 - \bar{\varepsilon}) \left(\gamma \lim_{t\to\infty} \frac{\mu_t^{RP}}{\lambda_t} - \beta\mu \right) \frac{1}{F_k \beta \gamma}, \tag{23}$$

where μ and $\lim_{t\to\infty} \mu_t^{RP}/\lambda_t$ are derived in Lemmas 1 and 4. In words, the tax rate at the steady state is given by the difference in the social versus the private collateral premium, as well as the discrepancy in the discounting that exists between the planner and capitalists. While the discrepancy in the discounting between households/planner and capitalists, governing the tightness of the collateral constraint in the agents' economy calls for a subsidy on capital, the presence of a binding collateral constraint in the Ramsey planner's economy calls for a positive tax on capital income.

[7] This condition is satisfied for a broad and reasonable set of parameter calibration from the literature. For example, $\beta = 0.99$ and $\gamma = 0.97$ require a relative risk aversion coefficient less that 3.

Consider the case when $\lim_{t\to\infty} \mu_t^{RP}/\lambda_t = 0$, i.e. the collateral constraint in the planner's economy does not bind in the long run. In this case, it is easy to see that the planner would opt for a subsidy on capital income equal to $\tau^k = -(1-\bar{\varepsilon})(\beta-\gamma)//(F_k\beta\gamma)$. Intuitively this means that when the planner is not constrained (for whatever reason that could be, but it could be due to access to additional policy instruments for example), it can acculmulate enough resources such that it can subsidize capitalists so that they can accumulate more capital (to relax the constraint). Our analysis reconfirms–yet within the context of heterogenous agent economies with a collateral constraint–the result that having a more patient planner that the private agent investing in capital pushes the tax on capital income to be negative (De Bonis and Spataro, 2005; Reis, 2012).[8]

However, when the collateral constraint binds for the Ramsey planner, it cannot unconditionally levy a subsidy, but takes into account how its evaluation of μ^{RP} weighs into the investment choice. Recall that μ^{RP} does not simply depend on the discount factors unlike μ, but also on strength of the intertemporal consumption and the collateral effects embedded in (21). Contrary to the difference in the discount factors, the difference between μ^{RP} and μ weighs positively in the tax on capital income as expression (23) indicates. In other words, a positive tax on capital income would serve to close the–positive from Lemma 4–gap that exists between the Lagrange multipliers on the collateral constraint in the Ramsey problem and the competitive economy, respectively, weighted by the discount factors. Proposition 1 establishes the sign and the level of the tax on capital incomes when the collateral constraint is binding in the Ramsey solution.

Proposition 1. *A Ramsey planner that faces a binding collateral constraint, sets a positive tax on capital income in the steady state equal to*

$$\tau^k = (1-\bar{\varepsilon})(\beta-\gamma)\,\sigma^c \cdot \frac{1}{F_k\beta\gamma} \cdot \frac{1-\beta}{1-\gamma-\sigma^c(1-\beta)}. \tag{24}$$

Moreover, τ^k is strcitly positive if the EIS of capitalists is finite ($\sigma^c > 0$) and collateralized borrowing matters for investment ($\bar{\varepsilon} < 1$).[9]

The collateral constraint does not only restrict the ability of entrepreneurs to

[8]Indeed, the negative tax result accrues from the difference in discount factors between the planner and capitalists. If the planner was discounting the future as capitalists (something that we prove that is not the case), then the optimal policy would be a tax on capital income to eliminate the effect of the private collateral premium on investment even when the planner can relax the constraint.

[9]The proof of the Proposition is straightforward and omitted. To get (24), substitute μ from Lemma 1 and $\lim_{t\to\infty} \mu_t^{RP}/\lambda_t$ from Lemma 4 in (23), and rearrage terms.

borrow and invest, but also the set of implementable allocations that a planner can choose. Technically, as argued in Chari and Kehoe (1999), the presence of additional restrictions in the Ramsey problem may yield non-zero capital income taxation optimal in the long-run. More specifically, these additional constraints need to be both binding and to depend on the capital stock, as it is the case with the per-period implementability constraint (13). Yet, if investment choices using borrowed funds are effectively not restricted, the additional restrictions imposed by the (binding) collateral constraint have no bearing on implementable allocations and the tax on capital income is zero. This can be easily seen by taking $\bar{\varepsilon} \to 1$ in (24).

For $\bar{\varepsilon} < 1$, the value of τ^k depends on the elasticity of intertemporal substitution, $EIS \equiv 1/\sigma^c$. This is because the EIS determines the strength of the intertemporal consumption and collateral effects. As previously mentioned, the intertemporal consumption effect captures the cost of distortions to consumption due to binding collateral constraint. When capitalists' EIS is low, increasing investment has a higher cost for intertemporal consumption smoothing and μ^{RP} is higher. On the contrary, a stronger collateral effect favors investment and pushes μ^{RP} down. This is the case when the desire for consumption smoothing is stronger, i.e. EIS is lower.

For linear utility, i.e. $\sigma^c = 0$, the collateral effect is muted, and the intertemporal consumption effect is at its weakest level, since capitalists are not concerned with consumption smoothing. Hence, from Proposition 2 the tax on capital income is zero. The reason lies in the way that the discrepancies in the discount factors and the multipliers on the collateral constraint between the planner and the private agents balance each other. From Lemma 4, we get that $\mu^{RP} = (\beta - \gamma)/\gamma$, while from Lemma 1, $\mu = (\beta - \gamma)/\beta$. The ratio of $\mu^{RP}/\mu = \beta/\gamma$ and it is equal to the ratio of the planner's and capitalists discount factors, β/γ. The discount factor discrepancy, which pushes for a subsidy, is exactly outweighed by the discrepancy in the multipliers on the collateral contraint, which pushes for a tax.

For $\sigma^c > 0$, the interplay between the intertemporal consumption effect and the collateral effect will determine whether the ratio of μ^{RP}/μ is higher or lower than the ratio of the discount factors. As shown in Proposition 2, the intertemporal consumption effect is stronger than the collateral effect and the planner set a strictly positive tax on capital income. The tax serves to reduce the demand for capital, which is optimal for the planner given that its evaluation of the overall "shadow costs" arising from the binding collateral constraint (μ^{RP}) over the collateral premium in the competitive economy (μ) is higher than the ratio of the discount factors. Moreover, we show in the Corollary below that the intertemporal consumption effect becomes even stronger compared to the collateral effect as σ^c increases resulting in a higher tax on capital income.

Corollary 1. *The tax on capital income in the steady state is decreasing in capitalists' EIS when the Ramsey planner faces a binding collateral constraint.*

3.3.2 Tax on labor income

Proposition 2. *In the steady state, the tax on labor income is given by*

$$\tau^l = \lambda^w \frac{m_l/u_l - m_c/u_c}{1 + \lambda^w m_l/u_l}. \tag{25}$$

Moreover, for logarithmic utility in labor, standard in the literature, and $\sigma^w \equiv -c \cdot u_{cc}/u_c \geq 1$, *the tax rate is always positive.*

Expression (25) for the tax on labor income is easily derived by combinging the optimal consumption and labor conditions of workers with the corresponding conditions of the planner.

Note that the tax on labor income in the steady state is unaffected by the presence of the binding collateral constraint since the intra-temporal margin is not affected by the presence of the collateral constraint.[10]

3.4 The Role of Distortionary Taxes

The main argument against capital income taxation in the long-run is that it disproportionally taxes present and future consumption and, hence, distorts originally undistorted intertemporal optimization margins. The financial frictions, resulting in binding collateral constraints, introduce intertemporal distortions in the competitive equilibrium. A planner may, then, want to use distortionary taxation to restore efficiency in intertemporal margins on top of funding government expenditure. Therefore, the goal of the Ramsey planner in this environment is twofold: first, it minimizes the distortions from taxation; second, it alleviates the inefficiencies induced by the binding collateral constraint.

To examine what portions of the tax rates is used to alleviate the frictions from the binding collateral constraint, consider the case when lump-sum taxes are

[10]If a working capital loan were also collateralized, as in Jermann and Quadrini (2012) for example, then the collateral constraint would also affect the intra-temporal margin and would make the tax rate on labor income sensitive with respect to the collateral constraint. See for example Biljanovska (forthcoming).

available to the Ramsey planner. Then, the Ramsey government does not need to rely on distortionary taxes to finance its expenditures since it can always choose lump-sum taxes.[11] Corollary 2 below summarizes the properties of the tax system in the steady state when lump-sum taxes are available.

Corollary 2. *When lump-sum taxes are available, the optimal tax system in the steady state is* $\tau^k = (1-\bar{\varepsilon})\left(\gamma \lim_{t\to\infty} \mu_t^{RP}/\lambda_t - \beta\mu\right) / \left(F_k \beta\gamma\right)$ *and* $\tau^l = 0$.

In this case, the Ramsey government can use lump-sum taxation to fully finance government expenditures, but it cannot use them to correct for the inefficiency induced by the binding collateral constraint. The reason why lump-sum taxes cannot affect the bindness of the collateral constraint is because they do not affect agents' marginal decisions. Indeed, the tax on capital income remains intact regardless of whether lump-sum taxes are available or not; on the other hand, the tax on labor income becomes zero. Then, one can conclude that the tax on labor income is used to fully finance the government expenditures, whereas the tax on capital income is there to alleviate the inefficiencies induced by the binding collateral constraint.

An alternative way to see the differential role of labour and capital income taxes is to remove the effect of financial frictions. This could be done, for example, by taking the discount factor of capitalists arbitrarily close to the one of workers, i.e., $\gamma \to \beta$, or by allowing capitalists to borrow against the total value of their capital investment, i.e., $\epsilon \to 1$. In both cases, the tax on capital income goes to zero, while the tax on labour income is given by (25). This result reverts back to the standard Chamley-Judd result, i.e. when the economy is frictionless the tax on capital income is zero and the tax on labor income is positive in the steady state to finance the expenditures of the government.

4 Conclusions

Chamley (1986) and Judd (1985) established the result on the optimality of zero-tax on capital income in the long-run, which rests on the assumption that private agents can freely shift consumption intertemporally in absence of any market frictions. This paper recasts the Ramsey policy problem in a Judd-type economy with two classes of agents, households and capitalists, in which the latter face a

[11] Even though, the assumption of lump-sum taxes may appear unnatural, its purpose is to help in analyzing the direction in which the second goal of the government (alleviating the inefficiencies) drives the tax rates.

collateral constraint. The planner has access to capital and labor income distortionary taxes. The tax on capital income is strictly positive at the steady state for finite elasticity of intertemporal substitution, whereas it is zero if capitalists have linear utilities. The labor income tax is used to fully finance government expenditure; where as the tax on capital income is used by the planner to affect the inefficiencies induced by the binding collateral constraint.

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Appendix

4.1 Proof of Lemma 1

The steady state version of eq. (8) can be obtained by eliminating the time subscripts, yielding $(1-\mu)/(1+r)=\gamma$. Substituting in the definition for $1+r$, obtained from the steady state version of eq. (2), yields $1-\gamma/\beta=\mu$. It follows that if $\gamma<\beta$, μ is constant and positive at the steady state. *Q.E.D.*

4.2 Proof of Lemma 2

To prove necessity, we show that the competitive equilibrium conditions outlined in section 2 imply the conditions outlined in Lemma 2, (i)–(v). The key step in this part of the proof is the derivation of the set of implementability constraints characterizing the competitive equilibrium with a collateral constraint, (11)–(13). To derive (11), multiply the budget constraint of workers, (1), with their marginal utility of consumption. Plugging in the optimality conditions of workers, imposing market clearing conditions, summing over t, and imposing the trasnsversality condition yields the following intertemporal condition

$$\sum_{t=0}^{\infty}\beta^t\left[u_{c,t}c_t+u_{l,t}l_t\right]=\sum_{t=0}^{\infty}\beta^t\left[\beta u_{c,t+1}\left(1+r\left(c_t,c_{t+1}\right)\right)-u_{c,t}\right]b_{t+1}+u_{c,0}b_0.$$

The first term on the right hand side is zero because it equals workers' Euler condition; then we arrive to eq. (11).

Subsequently, to derive (12), multiply the budget constraint of capitalists with their marginal utility of consumption. Plugging in the optimality conditions of capitalists, imposing market clearing conditions, and summing over all t periods, yields the following intertemporal condition

$$\begin{aligned}\sum_{t=0}^{\infty}\gamma^t u_{c,t}^c c_t^c \quad = \quad & \sum_{t=0}^{\infty}\gamma^t\left\{\gamma u_{c,t+1}^c\left[F_{k,t+1}\left(1-\tau_{t+1}^k\right)+(1-\delta)\right]-u_{c,t}^c\right\}k_{t+1}\\ & -\sum_{t=0}^{\infty}\gamma^t\left\{\gamma u_{c,t+1}^c-\frac{u_{c,t}^c}{1+r\left(c_t,c_{t+1}\right)}\right\}b_{t+1}\\ & +u_{c,0}^c\left[F_{k,0}\left(1-\tau_0^k\right)k_0+(1-\delta)k_0-b_0\right].\end{aligned}$$

After substituting in capitalists' consumption-capital investment and consumption-borrowing Euler conditions, i.e. (9) and (8), respectively, the equation reduces to

$$\sum_{t=0}^{\infty}\gamma^t u_{c,t}^c c_t^c = -\sum_{t=0}^{\infty}\gamma^t u_{c,t}^c \mu_t \left(\bar{\varepsilon} k_{t+1} - \frac{b_{t+1}}{1+r\left(c_t, c_{t+1}\right)}\right) + u_{c,0}^c \left[F_{k,0}\left(1-\tau_0^k\right)k_0 + \left(1-\delta\right)k_0 - b_0\right].$$

The first term on the right hand side of the equation above equals the complementarity slackness condition of capitalists, which is zero and it drops out; then we get eq. (12).

Finally, to derive (13), consider any period $s \geq 0$. Then, multiply the budget constraint of the capitalists by their marginal utility of consumption and plug in their optimality conditions. Iterate one period forward and sum over all $t \geq s$ periods to get

$$\begin{aligned}\sum_{t=s}^{\infty}\gamma^{t+1-s}u_{c,t+1}^c c_{t+1}^c = & \sum_{t=s}^{\infty}\gamma^{t+1-s}u_{c,t+1}^c \left[F_{k,t+1}\left(1-\tau_{t+1}^k\right) + \left(1-\delta\right)\right]k_{t+1} \\ & -\sum_{t=s}^{\infty}\gamma^{t+1-s}u_{c,t+1}^c k_{t+2} + \sum_{t=s}^{\infty}\gamma^{t+1-s}u_{c,t+1}^c \frac{b_{t+2}}{1+r\left(c_{t+1}, c_{t+2}\right)} \\ & -\sum_{t=s}^{\infty}\gamma^{t+1-s}u_{c,t+1}^c b_{t+1}.\end{aligned}$$

Add $u_{c,s}^c k_{s+1}$ and $u_{c,s}^c b_{s+1}/\left(\left(1+r\left(c_s, c_{s+1}\right)\right)\right)$ on both sides of the equation above. Following the same steps as in the derivation of the period-zero implementability condition of the capitalists, yields eq. (13). The remaining of the necessity proof is obvious since the resource constraint and the collateral constraint exactly correspond to the definition of the equilibrium as stated, after imposing the market clearing conditions for bonds and labor.

To prove sufficiency, show that given allocations $\left\{c_t, l_t, b_{t+1}^w, c_t^c, n_t, b_{t+1}^c, k_{t+1}\right\}_{t=0}^{\infty}$ that satisfy conditions (i)–(v) in Lemma 2, competitive prices and policy instruments can be constructed such that all the conditions for a competitive equilibrium with a collateral constraint are met. The optimal labor demand decision of capitalists, (3), can be satisfied by choosing the labor factor price; the optimal labor supply decision of workers, (7), can be satisfied by choosing the labor tax rate. The Euler condition of workers, (2), can be satisfied by choosing the return rate on borrowing. Using the allocations for consumption of workers and capitalists, a Lagrange multiplier on the collateral constraint, μ_t, can be derived such that capitalists' Euler condition (8) is satisfied. The tax on capital income can be chosen to satisfy the consumption-capital investment Euler condition of capitalists, (9). The set of implementability conditions, all holding with equality, guarantee that the budget constraints of workers and capitalists are satisfied. Finally, since the budget constraints and the resource constraint hold, the government budget constraint is also met. *Q.E.D.*

4.3 Proof of Lemma 3

This section shows that when allocations c, l, c^c, b and k are constant in the steady state, then the Lagrange multiplier on the resource constraint grows by factor of $\lim_{t\to\infty} \lambda_{t+1}/\lambda_t = \beta$ in the long-run.

Rewrite the optimal labor decision, (19), as

$$\lambda_t = \frac{\beta^t u_{l,t} - \beta^t \lambda^w m_{l,t}}{F_{l,t}}.$$

Iterating it for one period forward, and taking the limit of the ratio between period $t+1$ and t, yields

$$\lim_{t\to\infty} \frac{\lambda_{t+1}}{\lambda_t} = \frac{-\frac{\beta(\beta^t u_{l,t+1} + \beta^t \lambda^w m_{l,t+1})}{F_{l,t+1}}}{-\frac{\beta^t u_{l,t} + \beta^t \lambda^w m_{l,t}}{F_{l,t}}}$$

Since at the steady state, allocations are constant, it follows $\lim_{t\to\infty} \lambda_{t+1}/\lambda_t = \beta$. *Q.E.D.*

4.4 Proof of Lemma 4

Here we show that when allocations c, l, c^c, b and k are constant in the steady state, then

$$\lim_{t\to\infty} \frac{\mu_t^{RP}}{\lambda_t} = \frac{u_c^c}{\frac{(u_c^c + u_{cc}^c c^c)\gamma}{\beta - \gamma} - u_{cc}^c \left(k - \frac{b}{1+r}\right)}.$$

The proof merely consists of combining the first order conditions with capitalists' consumption and borrowing, i.e. eq. (16) and (17), respectively. Start from the first order condition with respect to capitalists' consumption in the planner's problem, (16), which after rearranging can be written as

$$\sum_{s=0}^{t-1} \gamma^{t-s} \lambda_s^{cc} = \frac{\lambda_t - \omega \gamma^t u_{c,t}^c + \lambda_t^{cc} u_{cc,t}^c B_t}{A_t} - \gamma^t \lambda^c,$$

where $B_t \equiv \left(k_{t+1} - \frac{b_{t+1}}{1+r_t}\right)$ and $A_t \equiv \left(u^c_{c,t} + u^c_{cc,t} c^c_t\right)$. For the following period, the same condition becomes

$$\begin{aligned}\sum_{s=0}^{t} \gamma^{t+1-s} \lambda^{cc}_s &= \frac{\lambda_{t+1} - \omega \gamma^{t+1} u^c_{c,t+1} + \lambda^{cc}_{t+1} u^c_{cc,t+1} B_{t+1}}{A_{t+1}} - \gamma^{t+1} \lambda^c \\ &= \gamma \left[\sum_{s=0}^{t-1} \gamma^{t-s} \lambda^{cc}_s + \lambda^{cc}_t\right]\end{aligned}$$

Note that in the steady state all allocations are constant, hence $u^c_{c,t} = u^c_{c,t+1} = u^c_c$ and so on. Moreover, from Lemma 3 that $\lim_{t\to\infty} \lambda_{t+1}/\lambda_t = \beta$. Then, using the two conditions above to solve for λ^{cc}_t, dividing and multiplying each side by λ_t and $u^c_{c,t}$, and taking the limit as time goes to infinity, we get:

$$\lim_{t\to\infty} \frac{\mu^{RP}_t}{\lambda_t} = \frac{\beta u^c_{c,t} + \beta \lim_{t\to\infty} \frac{\mu^{RP}_{t+1}}{\lambda_{t+1}} u^c_{cc} B}{\gamma A} - \frac{u^c_c + \lim_{t\to\infty} \frac{\mu^{RP}_t}{\lambda_t} u^c_{cc} B}{A},$$

where $B \equiv \left(k - \frac{b}{1+r}\right)$ and $A \equiv \left(u^c_c + u^c_{cc} c^c\right)$.

Using the consumption-capital investment Euler equation at the steady state, (22), one can infer the rate that $\left(\mu^{RP}_{t+1}/\lambda_{t+1}\right) / \left(\mu^{RP}_t/\lambda_t\right)$ grows at. Omitting the time subscripts and taking the limit when time goes to infinity, the Euler equation becomes $\lim_{t\to\infty} \mu^{RP}_t/\lambda_t = (1 - \beta [F_k + 1 - \delta])/(\bar{\varepsilon} - 1) = const$. This means that $\lim_{t\to\infty} \mu^{RP}_{t+1}/\lambda_{t+1} = (1 - \beta [F_k + 1 - \delta] / (\bar{\varepsilon} - 1) = const$. Hence $\lim_{t\to\infty} \mu^{RP}_t/\lambda_t = \lim_{t\to\infty} \mu^{RP}_{t+1}/\lambda_{t+1}$. Then, after re-arranging, the ratio of the two multipliers at the limit converges to

$$\lim_{t\to\infty} \frac{\mu^{RP}_t}{\lambda_t} = \frac{(\beta - \gamma) u^c_c}{A\gamma - u^c_{cc} (\beta - \gamma) B}.$$

In the steady state $A = u^c_c(1 - \sigma^c)$ and $B = (1 - \bar{\varepsilon})k$ using the collateral constraint. Moreover, from the budget contraint of capitalists together with their Euler condition we can determine the ratio of capital to consumption in the steady state, $k/c^c = \gamma/((1 - \bar{\varepsilon})(1 - \gamma))$. Substituting we get that

$$\lim_{t\to\infty} \frac{\mu^{RP}_t}{\lambda_t} = \frac{\beta - \gamma}{\gamma} \frac{1 - \gamma}{1 - \gamma - \sigma^c(1 - \beta)},$$

which is positive for $\sigma^c < (1 - \gamma)/(1 - \beta)$ and higher than $\mu = (\beta - \gamma)/\beta$. *Q.E.D.*

4.5 Proof of Corollary 1

Using the expression for the tax on capital income (24), compute

$$\frac{d\left(\tau^k F_k\right)}{d\sigma^c} = \frac{(1-\bar{\varepsilon})(\beta-\gamma)(1-\beta)}{\beta-\gamma} \frac{1-\gamma}{\left[1-\gamma-\sigma^c(1-\beta)\right]^2} > 0,$$

i.e. the multiple $\tau^k F_k$ is increasing in σ^c. Next, we need to determine how τ^k and F_k individually depend on σ^c. Consider the steady state version of the consumption-capital investment Euler equation in the competitive economy (9), which after some algebraic manipulations becomes $(1-\bar{\varepsilon}\mu)/\gamma-(1-\delta) = F_k-\tau^k F_k$, with $\mu = (\beta-\gamma)/\beta$. Taking the total derivative with respect to σ^c, yields

$$\frac{d\left[\frac{1}{\gamma}-\frac{\bar{\varepsilon}}{\gamma}\mu-(1-\delta)\right]}{d\sigma^c} = \frac{dF_k}{d\sigma^c} - \frac{d\left(\tau^k F_k\right)}{d\sigma^c} \Rightarrow \frac{dF_k}{d\sigma^c} = \frac{d\left(\tau^k F_k\right)}{d\sigma^c} > 0.$$

This suggests that F_k increases in σ^c. Moreover, rewriting the same Euler equation and taking again a derivative with respect to σ^c, yields

$$\frac{d\left(1-\tau^k\right)}{d\sigma^c}\frac{d\left(F_k\right)}{d\sigma^c} = \frac{d\left[\frac{1}{\gamma}-\frac{\bar{\varepsilon}}{\gamma}\mu-(1-\delta)\right]}{d\sigma^c} = 0.$$

Since we know that $\frac{d(F_k)}{d\sigma^c} > 0$, then $\frac{d\left(1-\tau^k\right)}{d\sigma^c} < 0$; hence $\frac{d\tau^k}{d\sigma^c} > 0$. *Q.E.D.*

4.6 Proof of Corollary 2

Introducing lump-sum transfers into workers' budget constraint, (1), and government's budget constraint, (10), changes the time period-zero implementability constraint of workers to $\sum_{t=0}^{\infty} \beta^t \left\{m\left(c_t, l_t\right) - u_{c,t}T_t\right\} = u_{c,0}b_0$, where T_t denotes the lump-sum transfer. Optimizing with respect to T_t, requires that the Lagrange multiplier on the time period-zero implementability constraint of workers, (11), to be $\lambda^w = 0$. Substituting this in (25), results in $\tau^l = 0$. *Q.E.D.*