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Application to a Financial Crisis**

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# A Price-Differentiation Model of the Interbank Market and Its Application to a Financial Crisis

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## **Abstract**

Rate curves for overnight loans between bank pairs, as functions of loan values, can be used to infer valuation of reserves by banks. The inferred valuation can be used to interpret shifts in rate curves between bank pairs, for example, in response to a financial crisis. This paper proposes a model of lending by a small bank to a large monopolistic bank to generate a tractable rate curve. An explicit calibration procedure for model parameters is developed and applied to a dataset from Mexico around the 2008 financial crisis. During the crisis, relatively small banks were lending to large banks at lower rates than usual, and the calibration suggests that a broad decline in valuation of reserves is responsible for this outcome, rather than a general increase in the supply of lending or compositional effects.

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# 1 Introduction

Banks lend and borrow central bank reserves overnight. An empirical pairwise rate curve, as a function of loan value, between a lender and a borrower can be constructed from repeated transactions over time. Having a framework to interpret such a curve is useful, as it contains information beyond simple mean rates and volumes. In particular, the shape of a rate curve gives information on the implicit private valuation of reserves by each bank pair, as shown by this paper. The value or incentive for holding reserves drives money market transactions, so having a tool for mapping reserve valuation into observable trades is useful in understanding money markets and broader financial markets.

Interpreting a rate curve requires a model generating a relationship between loan rates and values. In this paper, this is simply achieved by modelling the interaction between a bank pair as a price-differentiation problem between a large monopolistic bank and a small bank, in which the large bank offers a rate curve as a function of loan value to the small bank. The uncertainty in reserve holdings and the decreasing marginal value of reserves jointly determine the rate curve in the model. This relationship can be inverted to infer the implicit marginal value of reserves from an observed rate curve.

The model is motivated by a transaction-level database from Mexican interbank call money market. The large bank in the model acts as a monopolist, which is a simplifying assumption, but is reasonable in the case of Mexican interbank call money market for three reasons. First, a few of the largest banks participated in most of the trades in the database, plausibly granting them significant market power. Second, each of the remaining banks was mostly seen as trading prominently (though not exclusively) with one of those largest banks. Lastly, the choice of a trading counterparty did not seem to be driven by a search for the best rate, evidenced by the general lack of any strong correlation between trade volumes and rates across counterparties. These three observations are described in more detail in the next section.

It is difficult to directly estimate the parameters of the model, as they map into observable moments in a highly non-linear way, making off-the-shelf optimizers unreliable. However, this paper develops an alternative calibration procedure, mostly based on matching sample moments with their model counterparts, which can be followed with ease, and with clear understanding of how data map into parameters (as opposed to a black-box approach).

This calibration is applied to the data around 2008 financial crisis. Near the peak of the crisis, small banks lending to large banks experienced worse trading conditions than usual, receiving lower interest rates. The model interprets this as the outcome of a broad decrease in the marginal value of reserves across small banks, which shifted rate curves lower. This

interpretation is consistent with multiple explanations, such as increased risk in investment opportunities or increased risk aversion by small banks. At the same time, the calibration shows that the lower rates were not driven by a general increase in the supply of reserves lending or compositional effects.

In addition, the model-generated rate curve implies a positive correlation between loan values and rates, which is consistent with the data.<sup>1</sup> Also, calibrated model parameters have reasonable magnitudes.

The validity of the model depends on the market power held by a small number of large banks. Three features consistent with such market power have been described for Mexico, but they are not unique and have been documented in other countries as well. For instance, concentration of activity in large banks was observed by Furfine (1999) and Bech and Atalay (2008) for the US, by Cocco et al. (2009) for Portugal, Akram and Christophersen (2010) for Norway, and Sokolov et al. (2012) for Australia. Also, larger banks getting advantageous rates over smaller banks were observed in the US, for example, by Furfine (2001) and Afonso et al. (2011). Therefore, the model and its calibration can be potentially applied to data from countries other than Mexico.

The model and its calibration allow a quantitative interpretation of the rate curve between a pair of banks, which is a novel addition to existing frameworks such as Ho and Saunders (1985), Coleman II et al. (1996), Allen et al. (2012), Gofman (2013) and Afonso and Lagos (2015). Also, the calibration of model parameters around the peak of the 2008 financial crisis complements previous empirical studies on how interbank markets respond to financial crises, such as Furfine (2002) and Afonso et al. (2011) on the US and Acharya and Merrouche (2010) on the UK.

Section 2 describes the Mexican interbank market and presents empirical observations that motivate the model. Section 3 develops the model and discusses its implications. Section 4 tests implications of the model, proposes a calibration procedure for model parameters using the Mexican data, and discusses the impact of the 2008 financial crisis on the Mexican banking sector through the lens of the model. Section 5 concludes.

## 2 Mexican Overnight Interbank Market

This section describes features of Mexican overnight interbank market that motivate the model.

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<sup>1</sup>This result holds for loans that small banks lend to large banks. This particular market segment accounts for a large part of meaningful variations in rates in the data, as explained in the following sections.

## 2.1 Data Description

The dataset contains all transactions in the interbank call money market in Mexico, which is an important source of overnight loans for Mexican banks. A call money operation is an unsecured loan that can be recalled by the lender before it is due. If a lender recalled a loan, it would receive back the principal immediately but earn zero interest. It is not known precisely how frequent recalls are, but they are generally known to be rare.<sup>2</sup> Therefore, these loans are treated as simple unsecured loans in this paper.

Call money operations can have a tenor of 1, 2, or 3 banking days, but almost all of them are overnight. More generally, overnight call money operations constitute most of unsecured interbank loans in Mexico, in both transaction number and volume.

The call money market is over-the-counter (OTC) with no centralized exchange, and thus, rates vary across individual trades, depending on the identity of lenders and borrowers.<sup>3</sup> All subsequent empirical analyses are based on overnight call money operations only.

The time span of the dataset is two years, 2008 and 2009.<sup>4</sup> The total number of transactions is 21,451, with 44 per day on average. The number of banks with at least one transaction during this period is 38. However, nine of them have various data issues, and are dropped in some of the analyses, which results in less than 3% loss in transaction volume in the data.<sup>5</sup>

The mean principal value of a loan is 536 million Mexican pesos, which was roughly 40 million US dollars based on exchange rates that prevailed in 2008 and 2009. The cross-sectional intraday standard deviation of interest rates is 12 basis points in annualized terms, on average (all rates appear as annualized rates in this paper).

## 2.2 Interest Rate and Bank Size

Interest rates vary little between some of the largest banks in the system. Figure 1 shows the mean distance between individual rates and the central bank target rate, for loans between the  $n$  largest banks, in terms of total assets, for  $n$  increasing from 2 to 36.<sup>6</sup> The target rate is used as the base to control for changes in rates over time. The figure shows

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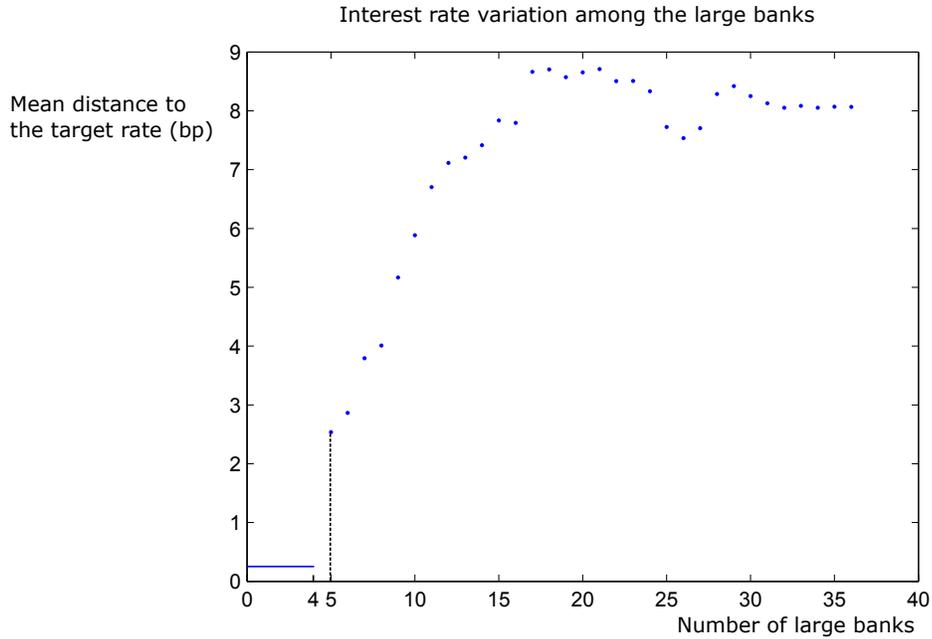
<sup>2</sup>A lender has an incentive to plan its lending properly so that it does not have to recall a loan, as recalled loans earn zero interest. Also, the lack of a register for recalls indirectly suggests their insignificance.

<sup>3</sup>There is no good documentation on why this market has not been centralized. One reason may be that the number of transactions is small, so there is not a strong incentive to establish an exchange. More generally, many financial assets are traded outside exchanges.

<sup>4</sup>More precisely, it is from 01/21/2008 to 12/31/2009. The dataset starts on 01/21/2008, not on 01/01/2008, because there was a significant change in the monetary policy implementation framework on 01/21/2008.

<sup>5</sup>Mostly due to limited data availability.

<sup>6</sup>Two of the banks without available balance sheet information are dropped from all subsequent analyses.



For 4 or fewer banks, the marks are very close to zero and below a quarter of a basis point.

Figure 1: Variation in Interest Rates within Different Subgroups of Banks.

that for  $n \leq 4$ , rates are very close to the target with little variation, with mean distance close to 0.1 bp or smaller. As  $n$  grows beyond 4, however, the standard deviation tends to increase with  $n$ .

In line with this observation, the four largest banks are defined as ‘large banks’ and the remaining banks as ‘small banks.’ This definition is used throughout this paper unless explicitly stated otherwise.

Under this definition, it turns out that most loans (in terms of volume) are either between two large banks, or between a large bank and a small bank. The mean rates (minus the target rate) for different subsets of loans are shown in table 1, based on the size of lenders and borrowers. Large banks lend to other large banks at rates practically identical to the target rate, on average. When small banks borrow from large banks, they pay rates above the target rate, and when small banks lend to large banks, they pay rates below the target rate, on average.<sup>7</sup> Small banks apparently face a disadvantage trading with large banks, which is consistent with the findings from Cocco et al. (2009) and Furfine (2001).

Small banks lending to large banks mostly receive rates lower than the target rate. Figure 2 plots the distribution of rates on two subsets of loans: Large banks lending to other large banks, and small banks lending to large banks. The green line, which is the observed rate

<sup>7</sup>Small banks mostly lend to large banks, rather than borrow from them. Both averages are statistically significant.

		Borrower is:	
		Large	Small
Lender is:	Large	0.000 (0.028)	0.015 (0.114)
	Small	-0.153 (0.129)	-0.010 (0.111)

Note: Numbers in ( ) are standard deviations, not standard errors of the mean.

Table 1: Mean Spreads to the Target Rate for Different Bank Size.

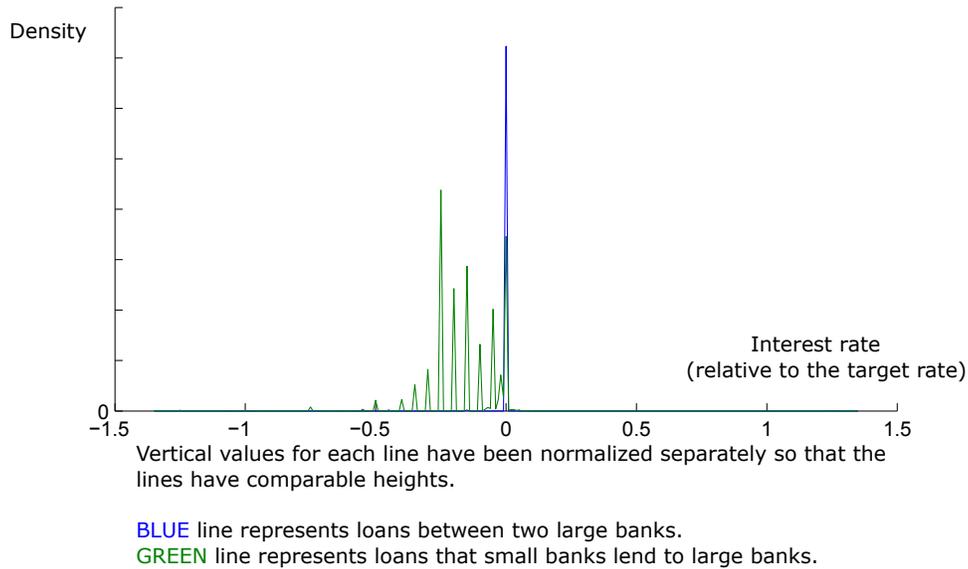


Figure 2: Densities of Interest Rates.

distribution for small banks lending, is almost entirely located to the left of the blue line, which is the observed rate distribution for large banks lending.

### 2.3 Counterparty Choice of Small Banks

Small banks mostly trade with large banks rather than with other small banks. This observation is consistent with the findings from Bech and Atalay (2008) and Furfine (1999).

Figure 3 shows the volume of loans between all banks except  $n$  largest banks, divided by total volume. The ratio decreases rapidly as  $n$  grows from two to four. It decreases much more slowly as the number continues to grow beyond four. The four largest banks act as counterparties to many other banks, and the other banks trade mostly with these four largest banks: 47.9 percent of loans are between large banks and 43.6 percent of loans are between a large bank and a small bank. Loans between two small banks only account for

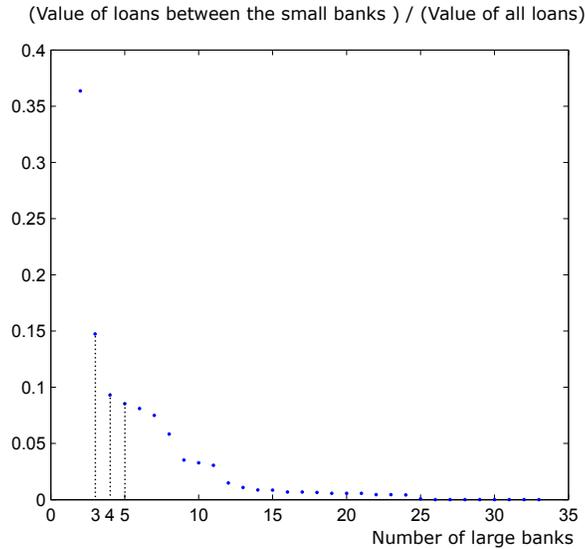


Figure 3: Volume of Loans between Small Banks.

8.5 percent.

A small bank typically trades mostly with one of the large banks. For each small bank, it is easy to identify which large bank it principally trades with because the difference in volume between the most frequently traded large bank and the second most is substantial. On average, 61.8 percent of a small bank’s loans with large banks are concentrated on a single large bank. In contrast, the second most frequently traded large bank accounts for only 19.7 percent.

### 3 Model

#### 3.1 Empirical Observations and Model Assumptions

The model represents the interaction between a large bank and a small bank, in isolation from all the other banks. The large bank acts as a monopolist and offers its profit-maximizing rate curve as a function of loan value to the small bank.

This section mostly describes cases in which the small bank has excess reserves to lend, but the description can be easily extended to allow both lending and borrowing. This simplifies the exposition, and is enough for the calibration exercise which uses only loans that small banks lend to large banks.<sup>8</sup>

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<sup>8</sup>The total volume of borrowing by small banks from large banks is less than 10 percent of the total volume of lending in the data. The model can be extended with little effort into cases where small banks borrow from large banks or switch sides depending on the realization of random variables. Especially, given the assumptions on the form of marginal value for reserves, the cases in which small banks lent could be

Faced with an excess or a deficit of reserves, both the large bank and the small bank assign the same constant marginal value to reserves, excluding the cost of ‘handling’ it. However, by assumption, the large bank can handle it costlessly, while the small bank faces some cost (‘liquidity cost’), increasing with the size of the excess or deficit. Due to this cost, the small bank with excess reserves assigns a lower marginal value to a unit of reserves than the large bank. This cost, in practice, can be interpreted as differential access to potential investment opportunities.

If the model had two large banks, instead of a large bank and a small bank, they would assign the same constant marginal value to a unit of reserves and lend and borrow at a single constant rate. This is indeed observed with the Mexican database, and empirically, the almost constant rates between large banks are practically identical to the central bank target rate. Based on this observation, the target rate is used as the marginal value of reserves to large banks in empirical applications of the model.

These assumptions also rationalize the observation that small banks lend at lower rates and borrow at higher rates than large banks, described in the last section. Especially, the small bank must lend at below the target rate, which is observed from the data.

In the data, a small bank typically has a single large bank with which it trades a majority of loans, and thus the assumption that the large bank acts as a monopolist toward the small bank in the model is a reasonable simplification. In addition, the rate that the most frequent large bank counterparty offers to a small bank is not better than what other banks offer in the data, further supporting the presence of market power.

### 3.2 Formal Setup

There are two banks, a large bank (bank  $L$ ) and a small bank (bank  $S$ ). The model describes events over a banking day. First, a random variable  $x$  is realized, which denotes the amount of excess reserves that the small bank holds before trading. Then, the two banks meet once and determine  $l$ , the amount of reserves that the small bank lends to the large bank.  $x < 0$  means that the small bank has  $-x$  amount of reserve deficit and  $l < 0$  means that the small bank borrows  $-l$  from the large bank.

If  $l = 0$ , the small bank’s profit  $\pi_S$  is

$$\pi_S = \int_0^x (p - c(y)) dy. \tag{1}$$

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considered independently from the cases in which small banks borrowed, even if both lending and borrowing were possible as probabilistic outcomes.

$p - c(y)$  is the marginal value of reserves, where  $p$  is a constant and  $c(y)$  is a strictly increasing function such that  $c(0) = 0$ .  $p$  is the constant marginal value of reserves in the absence of liquidity cost and  $c(y)$  is the marginal liquidity cost.  $c(y)$  represents a cost in the sense that for any  $x$ ,

$$\int_0^{\infty} (p - c(y)) dy < px. \quad (2)$$

$c(y)$  always reduces the profit because it has the same sign as  $y$ .

With  $l \neq 0$ , the small bank lends  $l$  reserves, and its profit is

$$\pi_S = \int_0^{x-l} (p - c(y)) dy + rl. \quad (3)$$

$r$  is the interest rate on the loan of value  $|l|$ . Compared to equation 1, reserve position changes from  $x$  to  $x - l$  because it transfers  $l$  units of reserves to the large bank. In return, the small bank receives interest payment  $rl$ .

The large bank's profit function  $\pi_L$  has the same form as  $\pi_S$ , except that it does not have the cost term,  $c(y)$ :

$$\pi_L = \int_0^l p dy - rl = pl - rl. \quad (4)$$

The large bank may have its own excess or deficit of reserves at the beginning, but it is irrelevant to trading decisions. Since the marginal value of reserves is constant, the starting reserve position does not matter in determining  $r$  and  $l$ .

This setup can be interpreted as the large bank absorbing some of excess or deficit in reserves from the small bank, and handling it at a lower cost.

### 3.3 Trading Mechanism

$x$ , the initial reserve position of the small bank, is a random variable. The small bank knows its exact value, but the large bank only knows its probability distribution. The large bank's problem is to offer a curve, or a menu, of rates as a function of the loan value to maximize its expected profit, taking into account the fact that the small bank would choose a point on the curve that maximized its own profit.

Formally, let  $r(y)$  be the rate curve and  $l$  be the loan value, which can be either positive

or negative. The problem of the small bank is to maximize its own profit given  $x$  and  $r(y)$ :

$$\max_l \int_0^{x-l} (p - c(y)) dy + r(l)l. \quad (5)$$

Since the choice of  $l$  is determined by  $x$  and  $r(y)$ ,  $l$  can be written as  $l(x|r(y))$ .

With this new notation, the expected-profit-maximization problem of the large bank is

$$\max_{r(y)} \int_{-\infty}^{\infty} [p - r(l(x|r(y)))] l(x|r(y)) f(x) dx, \quad (6)$$

where  $f(x)$  is the probability density function of  $x$ .

### 3.4 Characterization of the Solution

From this point,  $x$  is assumed to be a positive continuous random variable and  $r(y)$  is treated as a function defined only over  $y \geq 0$ . This does not result in any loss of generality because in solving the large bank's problem, the region  $x > 0$  can be solved separately from  $x < 0$ . The reason is that the large bank would choose  $r \leq p$  for  $l > 0$  and  $r \geq p$  for  $l < 0$  to avoid making a loss. Then, given the increasing marginal cost  $c(y)$ , the small bank has no incentive to choose  $l < 0$  when  $x > 0$  or to choose  $l > 0$  when  $x < 0$ .<sup>9</sup>

Given any  $r(y)$ ,  $l(x|r(y)) \geq l(x'|r(y))$  if  $x > x'$ . Since the marginal cost  $c(x)$  is increasing in  $x$ , the small bank benefits more from increasing its lending when its reserve position  $x$  is larger. Therefore,  $l$  is a weakly increasing function of  $x$ , for any given  $r(y)$ .

Furthermore, for  $l(x)$  to be chosen by the small bank, the small bank must be indifferent between lending  $l(x)$  at rate  $r(l(x))$  and lending  $l(x + dx)$  at rate  $r(l(x + dx))$ :

$$l'(x)[r(l(x)) + r'(l(x))l(x) - p + c(x - l(x))] = 0. \quad (\text{IC}) \quad (7)$$

A first-order condition for maximizing the large bank's objective function is also needed. Roughly speaking, the first-order condition represents a balance between the extra profit from increasing  $l(x)$  and  $r(l(x))$  while leaving the small bank's pointwise profit at  $x$  unchanged, and the cost of increasing  $r(y)$  for all  $y > l(x)$  to conserve the incentive compatibility

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<sup>9</sup>A further separation result is that for every  $x \geq 0$ , an optimal solution implies  $l \leq x$ . Therefore, there is no need to specify the form of  $c(y)$  for  $y \leq 0$ , except the assumptions that  $c(y)$  is increasing in  $y$  and  $c(0) = 0$ . This is not explicitly proved in the paper because this result is not used anywhere. Roughly, at the smallest  $x$  such that  $l > x$ , both  $l$  and  $r(l)$  can be reduced to leave bank  $S$  at  $x$  indifferent, without violating incentive compatibility for bank  $S$  with other  $x$ .

condition for the small bank. The resulting expression is:

$$f(x)c(x - l(x)) - (1 - F(x))c'(x - l(x)) + \lambda(x) = 0, \text{ (FOC)} \quad (8)$$

where  $F(x)$  is the cumulative distribution function of  $x$  and  $\lambda(x)$  is a shadow cost of the constraints that  $l(0) = 0$ , and that  $l(x)$  is a weakly increasing function of  $x$ .

**Proposition 1.** *Suppose that  $f(x) > 0$  for every  $x \geq 0$ . The solution to the banks' optimization problems is characterized by the following two equations:*

$$l'(x)[r(l(x)) + r'(l(x))l(x) - p + c(x - l(x))] = 0. \text{ (IC)} \quad (9)$$

$$f(x)c(x - l(x)) - (1 - F(x))c'(x - l(x)) + \lambda(x) = 0, \text{ (FOC)} \quad (10)$$

subject to  $l(0) = 0$  and  $l'(x) \geq 0$ .  $\lambda(x)$  represents a shadow cost of these two constraints on  $l(x)$ . The proof is in the appendix. ■

The solutions  $r(y)$  and  $l(x)$  depend on  $c(y)$  and the distribution of  $x$ . However, there is a general tendency for the interest rate to increase as  $l$  increases, at least for large values of  $l$ , as long as the distribution of  $x$  does not have a 'heavy' tail, in the particular sense that either the support of  $x$  is bounded or the inverse of hazard function  $(1 - F(x))/f(x)$  becomes small for large  $x$ . Under such an assumption, the first-order condition (FOC) implies that  $x - l(x)$  should be close to 0 for large  $x$ . Intuitively, if the distribution of  $x$  were bounded or did not have a heavy tail, the large bank would want to lend as much as possible for large values of  $x$ . The reason is that the cost to conserve the incentive compatibility of the small bank,  $(1 - F(x))c'(x - l(x))$ , would become small relative to the profit from lending more, for large  $x$ . Then, as  $c(x - l(x))$  became small for large values of  $x$ ,  $r'(l(x)) = (p - c(x - l(x)) - r(l(x)))/l(x)$  would tend to be positive, given equation (IC).

### 3.5 Additional Assumptions

Additional assumptions are introduced to simplify solutions. The marginal liquidity cost,  $c(y)$ , is assumed to be a power function:  $c(y) = \alpha y^\theta$ , for positive constants  $\alpha$  and  $\theta$ . The two parameters  $\alpha$  and  $\theta$  can be roughly mapped into the level and the slope of the rate curve through the model.

If the hazard rate of  $x$ ,  $f(x)/(1 - F(x))$ , is monotonically weakly increasing in  $x$ , the solution to the optimization problems has a relatively simple form:<sup>10</sup>

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<sup>10</sup>Weibull distribution is an example of such a distribution, which will be used in parameter calibration. A normal distribution, a uniform distribution and an exponential distribution also have monotonically weakly increasing hazard rates.

**Proposition 2.** *Suppose that  $f(x) > 0$  for all  $x \geq 0$ , the hazard rate of  $x$  is weakly increasing, and  $c(y) = \alpha y^\theta$  for some positive  $\alpha$  and  $\theta$ . Then, the solution  $l(x)$  to the optimization problems has the following simple form:*

$$l(x) = [x - \theta \frac{1 - F(x)}{f(x)}]^+, \quad (11)$$

where the notation  $[\cdot]^+$  denotes the maximum of the expression inside the brackets and 0. Also, there exists a unique  $x_0 > 0$  such that  $l(x) > 0$  if and only if  $x > x_0$ .

For any  $x > x_0$ , condition (IC) can be rewritten as:

$$r(l(x)) + r'(l(x))l(x) - p + \alpha(x - l(x))^\theta = 0, \quad (12)$$

or more conveniently,

$$r(l) = p - \frac{1}{l} \int_0^l \alpha \left( \theta \frac{1 - F(l^{-1}(z))}{f(l^{-1}(z))} \right)^\theta dz, \quad (13)$$

where  $l$  denotes both loan value as an argument in  $r(l)$  and loan value  $l(x)$  as a function of  $x$  at the same time.  $l^{-1}(z)$  is a proper function because  $l(x)$  is strictly increasing for  $x \geq x_0$ .

The proof is in the appendix. ■

Since  $(1 - F(x))/f(x)$  is monotonically decreasing in  $x$ ,  $r(l)$  is a monotonically increasing function of  $l$ . Figure 4 shows the solutions for some chosen values of  $\theta$  and distributions of  $x$ .<sup>11</sup>

### 3.6 Discussion

The large bank determines the optimal rate curve, subject to the small bank's own optimizing behavior. This problem is known as a price differentiation problem, but it had not been previously applied to the study of overnight interbank loan markets.

The model does not take into account any default risk in the borrower. For a risk-neutral lender, the additive premium on the interest rate due to default risk would approximately equal default probability. However, default risk is mitigated if a lender can recall a loan, as in Mexico. Also, when a large bank is borrowing from a small bank, the default risk is typically very small, because large banks tend to be relatively safe and can often rely on

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<sup>11</sup>Liquidity cost functions  $c(y) = \alpha y^\theta$  with  $\alpha = 0.3$  and varying  $\theta$  are used. Weibull( $\lambda, k$ ) refers to a distribution with cumulative distribution function  $F(x) = 1 - \exp(-(x/\lambda)^k)$ . Uniform( $a, b$ ) refers to a uniform distribution over  $(a, b)$ .

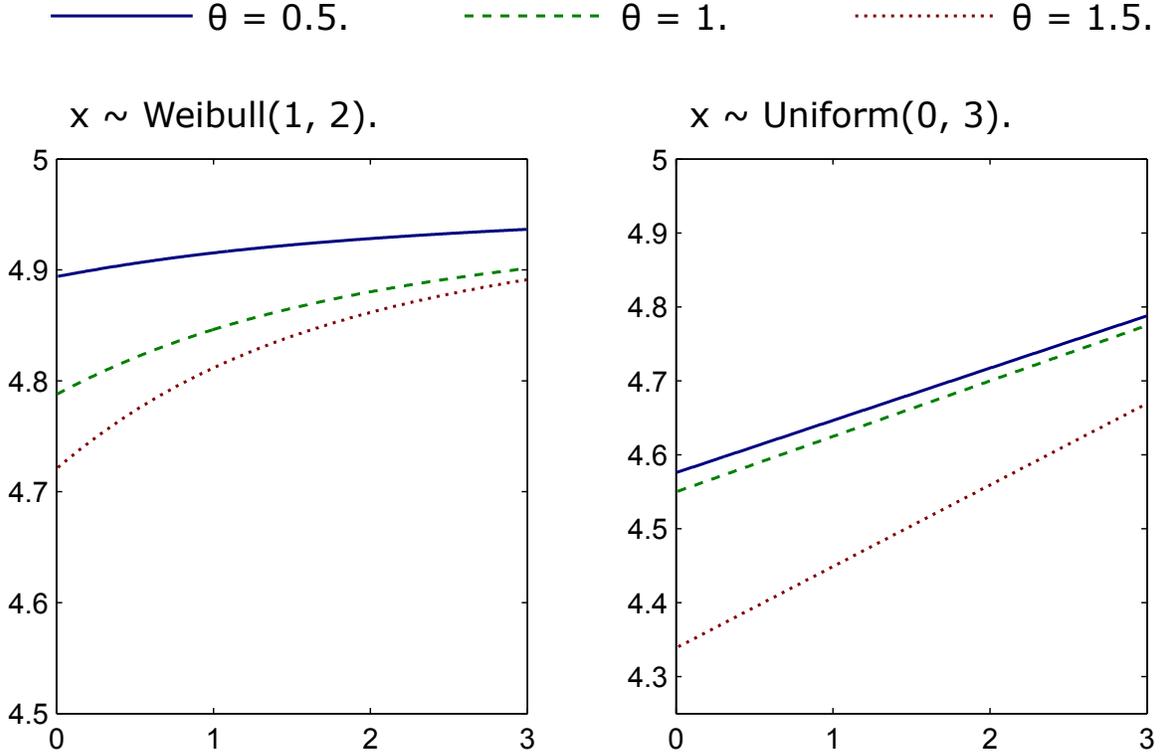


Figure 4: Rate Curves  $r(l)$ .

emergency funding from the government or the central bank.<sup>12</sup>

## 4 Empirical Application of the Model

Linear regression models are estimated with the data to study how the interest rate depends on loan value and the identities of lenders and borrowers. A significantly positive correlation is found between rate and value, which is consistent with the model. For clarity, each observation is indexed by a unique (ordered) triple  $(i, j, t)$ , where  $i$  is the index for the lender,  $j$  is the index for the borrower and  $t$  is the index for the date on which the loan is made. This indexing is valid because there is at most a single loan between a given pair of banks on most dates.<sup>13</sup> Also, it conforms to the model, which allows only a single loan

<sup>12</sup>For example, for the four largest banks in Mexico, their most recent ratings by Moody's as of December 2014 were P-2. The historical default probability within three months for corporations rated in that category was 0.00 percent. Therefore, annualized, the contribution to the interest rate from default risk would be less than 1 basis point.

<sup>13</sup>In principle, a triple  $(i, j, t)$  does not identify a unique observation because there can be multiple transactions between banks  $i$  and  $j$  on date  $t$ . In practice, there is at most a single loan between a given pair of banks on most dates. Even when there are more than one loans, the interest rates on those multiple loans tend to be identical. If there are multiple transactions corresponding to a single triple  $(i, j, t)$ , these loans are consolidated to produce a single observation. For the new consolidated observation, the loan value is the

between two banks.

Additional linear regression models including dummies for a crisis period shows that rate curves between bank pairs shifted to lower levels during the crisis. To systematically interpret these shifts, the model is calibrated with the data allowing for parameter changes during the crisis. A parameter calibration procedure, which is similar to the generalized method of moments (GMM), is introduced.

## 4.1 Linear Models

The following model tests for a positive relationship between interest rates and loan values, controlling for the identity of the lender and the borrower:

$$r_{ijt} - p_t = \alpha_i + \beta_j + \gamma \cdot \log(Value_{ijt}) + \epsilon_{ijt} \quad (14)$$

where  $r$  is the interest rate,  $p$  is the target rate,  $\alpha_i$  is the lender fixed effect,  $\beta_j$  is the borrower fixed effect, and  $Value_{ijt}$  is the principal value of the loan. An expanded model allows different coefficients on the loans that small banks lend to large banks ( $\gamma + \gamma_A$ ), and on the loans that small banks lend to their principal large bank counterparties ( $\gamma + \gamma_B$ ):

$$r_{ijt} - p_t = (\text{Intercepts}) + [\gamma + \gamma_A I_A(i, j, t) + \gamma_B I_B(i, j, t)] \cdot \log(Value_{ijt}) + \eta_{ijt} \quad (15)$$

where  $I_A(i, j, t)$  is 1 if bank  $i$  is small and  $j$  is large, and is 0 otherwise, and  $I_B(i, j, t)$  is 1 if bank  $i$  is small and  $j$  is the ‘principal’ counterparty of  $i$ , and is 0 otherwise.<sup>14</sup> Intercepts also change with  $I_A$  and  $I_B$ :

$$(\text{Intercepts}) = \alpha_i + \alpha_A(i) I_A(i, j, t) + \alpha_B(i) I_B(i, j, t) + \beta_j + \beta_A(j) I_A(i, j, t) + \beta_B(j) I_B(i, j, t). \quad (16)$$

In addition, the two linear models are estimated with  $\gamma$ ,  $\gamma_A$  and  $\gamma_B$  as normal random coefficients at the level of lender-borrower pairs,  $(i, j)$ . Table 2 reports estimated coefficients of these models.

A positive relationship between rates and loan values is found under every specification, which is significant both statistically and economically. In the expanded models (models 3 and 4 in table 2),  $\gamma$  is much smaller than  $\gamma_A$ , suggesting that the positive relationship

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sum of individual loan values, and the interest rate is the average of individual interest rates, weighted by the value of individual loans.

<sup>14</sup>The principal counterparty of a small bank is defined as the large bank counterparty with the largest trade volume with the small bank.

Model	(1)	(2)	(3)	(4)
$\gamma(\times 100)$	1.538* (0.186)	0.996* (0.153)	0.470* (0.163)	0.474* (0.150)
$\gamma_A(\times 100)$	.	.	2.213* (0.480)	1.866* (0.423)
$\gamma_B(\times 100)$	.	.	0.600 (0.669)	0.500 (0.642)
Random Coefficients	No	Yes	No	Yes
Number of $(i, j)$ Groups	418	418	418	418
$R^2$	0.482	.	0.546	.
Number of obs.	18679	18679	18679	18679

\* denotes significance at 5%.

<sup>1</sup> Standard errors are clustered for each  $(i, j)$ .

Table 2: Linear Models of the Relationship between the Loan Size and the Interest Rate.

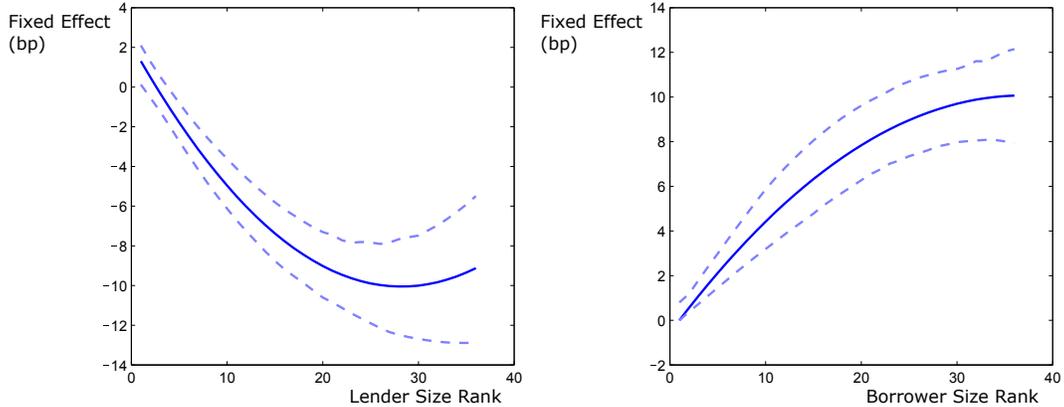


Figure 5: Relationship between Bank Size Rank and Interest Rate.

between interest rates and loan values is much more pronounced when small banks lend to large banks. Small banks receive a 2.5 bp higher rate on a 170 percent larger loan on average, which is substantial relative to 12 bps of intraday cross-sectional standard deviation. This is consistent with the model’s implication that the small bank receives a higher interest rate from the large bank when it lends more.

Figure 5 is a plot of lender fixed effects ( $\alpha_i$ ) and borrower fixed effects ( $\beta_j$ ) from the simple linear model (model 1 in table 2), relative to the largest bank.<sup>15</sup> Larger banks tend to get better interest rates than smaller banks, both when lending and when borrowing. This is broadly consistent with the model’s assumption that the large bank faces no liquidity cost.

Finally, to ensure that the positive relationship between rates and loan values is not driven by trends over time or other factors not captured by bank fixed effects, a linear model is separately estimated on many subsets of the data. Specifically, a linear regression model of the following form is estimated on the loans between small banks and their principal counterparties, separately for each small bank and for each 60-business-day window:

$$r_t - p_t = \alpha + \gamma \cdot \log(Value_t) + \epsilon_t. \quad (17)$$

$\gamma$  is positive in 82% of the 5,656 subsamples generated in this way.

## 4.2 Mexican Interbank Market during the 2008 Financial Crisis

Casual observations indicate that near the peak of the 2008 financial crisis, the interest rate ‘discount’ (the target rate minus interest rates) on the loans that small banks lent to

<sup>15</sup>The estimated coefficients  $\alpha_i$  and  $\beta_j$  are replaced by quadratic polynomials of the bank size rank that are closest to them under the  $L^1$ -norm. The polynomial is a good representation of actual fixed effect coefficients, and the choice of  $L^1$ -norm over  $L^2$  helps it by giving less weight to a few outliers.

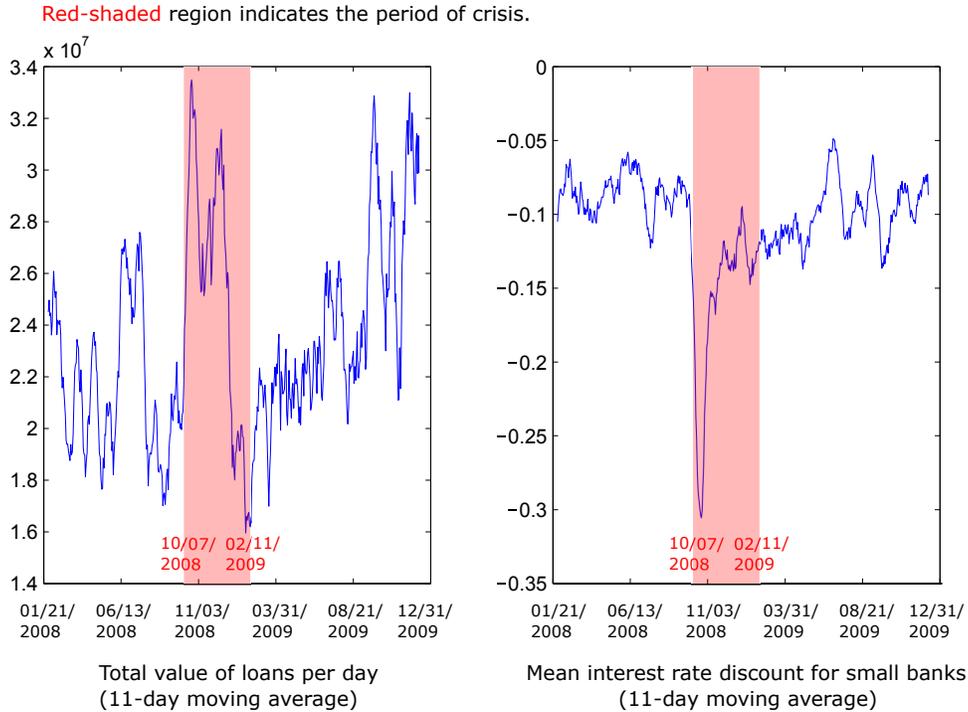


Figure 6: Behavior of the Interbank Market during the Crisis.

large banks increased. The peak of the crisis is defined using VIMEX, an implied stock market volatility index for Mexico.<sup>16</sup> This peak will be simply referred to as the crisis period in the remainder of the paper.

Notably, total volume of loans seems to have increased around the peak, with no sign of market disruption. This is consistent with earlier studies showing that overnight interbank markets have been functioning smoothly through crises in the US, for example.<sup>17</sup> Figure 6 shows interbank volume and rate discounts faced by small banks during the crisis, and figure 7 shows the time-series of the implied stock market volatility index.

Linear models with a time dummy for the crisis period are estimated to formally document changes to the interbank market. In this section, the dataset is narrowed down to include only  $(i, j, t)$  triples with small banks lending to their principal large bank counterparties.

First, there is an increase in loan values during the crisis, but it is not significant. This

<sup>16</sup>Implied stock market volatility is defined, for example, in Bollerslev et al. (2009). The crisis period is defined as the continuous block of dates around the peak of the implied stock market volatility index over which the daily closing level of the index stayed above half of the peak value. This definition results in 87 business days of crisis.

<sup>17</sup>Furfine (2002) documents that the Federal Funds Market worked well during the Russian debt crisis, and Afonso et al. (2011) documents that there was at most a small drop in the total value of loans traded in the Federal Funds Market after the default of Lehman Brothers in 2008.

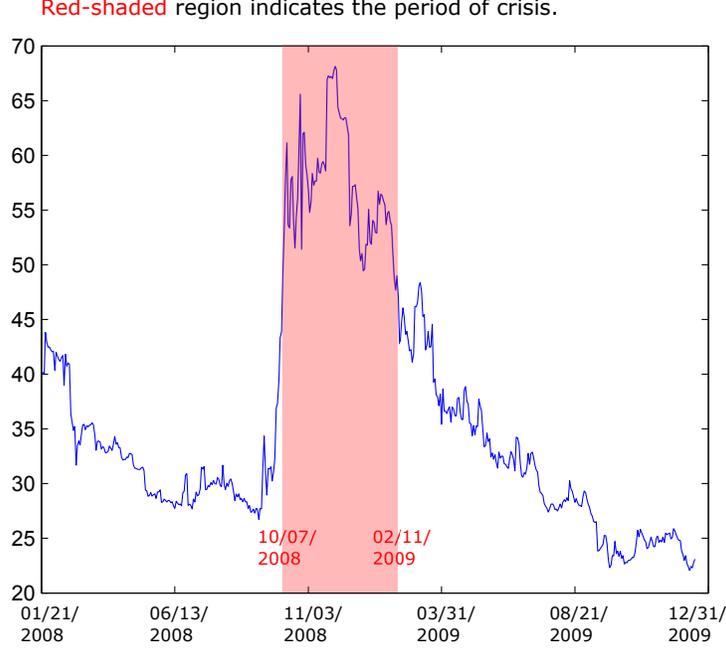


Figure 7: Time-Series of the Implied Stock Market Volatility Index.

result comes from estimating the following linear model:

$$\log(Value_{it}) = \alpha_i + \beta I_C(t) + \epsilon_{it} \quad (18)$$

where  $Value_{it}$  is the principal value of the loan that small bank  $i$  lends to its principal lcounterparty on date  $t$ .  $\alpha_i$  represents the lender fixed effect (borrower fixed effect is not necessary because  $i$  determines the identity of the borrower as well) and  $I_C(t)$  takes the value of 1 if  $t$  is within the crisis period and 0 otherwise.

Next, there is a significant negative shift to rate curves, or equivalently, a significant increase in the rate discounts that small banks faced. To show this, the preceding linear model in equation 18 is estimated with  $r_{it} - p_t$  as the dependent variable.

Finally, a significant positive relationship between loan values and rates exist (as shown already), and there is no significant change in their correlation during the crisis. To show this, the following linear model is estimated:

$$r_{it} - p_t = \alpha_i + \beta_i I_C(t) + [\gamma + \delta I_C(t)] \cdot \log(Value_{it}) + \eta_{it} \quad (19)$$

Table 3 presents estimated coefficients from these models. In addition, random-slope models in which coefficients of interest ( $\beta$  for equation 18, and  $\gamma$  and  $\delta$  for 19) are normal random coefficients at the level of individual small banks are estimated.

Model	(1)	(2)	(3)	(4)	(5)	(6)
Dependent Variable	Log Value	Log Value	Rate	Rate	Rate	Rate
$\beta$ : Level Change during the Crisis	0.1130 (0.1020)	0.0379 (0.1017)	-0.0633* (0.0117)	-0.0549* (0.0089)	.	.
$\gamma$ : Slope on Log Value	.	.	.	.	0.0350* (0.0052)	0.0305* (0.0051)
$\delta$ : Slope Change during the Crisis	.	.	.	.	-0.0034 (0.0053)	-0.0016 (0.0053)
Random Coefficients	No	Yes	No	Yes	No	Yes
$R^2$	0.504	.	0.518	.	0.618	.
Number of Obs.	3736	3736	3736	3736	3736	3736

\* denotes significance at 5%

<sup>1</sup> Standard errors are clustered for each  $i$ .

Table 3: Linear Models of the Impact of the Crisis.

Models 1 and 2 show that the value of loans that small banks lent to large banks increased by 11.3% and 3.8%, respectively, on average, during the crisis. However, the increase in loan values is not statistically significant due to relatively large day-to-day fluctuations and an apparent increasing trend toward the end of the data period.

Models 3 and 4 show that the interest rates that the small banks received were significantly lower during the crisis, by about 5 to 6 basis points. Models 5 and 6 show that the slope of rate curves decreased only a little, with no statistical significance.

The linear models show that interest rate discounts faced by small banks increased during the crisis. The small banks' increased need to lend might have weakened their bargaining power against the large banks. Alternatively, worse outside options during the crisis other than directly lending to large banks could have increased rate discounts.

### 4.3 Mapping the Effects of the Crisis onto the Model

Changes in loan values and shifts in rate curves can be accounted by changes in model parameters. In this section, two forms of parameter changes are discussed, given  $c(y) = \alpha y^\theta$ : A change in  $\alpha$  (cost shift), and a change in the distribution of  $x$ , representing a change in the pattern of reserve holdings by the small bank.

Generally, an increase in the liquidity cost, represented by  $c(y) = C\alpha y^\theta$  for  $C > 1$ , shifts the rate curve downward and makes it steeper. Similarly, an increase in reserve holdings, represented by a change in the distribution of reserves from  $x$  to  $Cx$  for  $C > 1$ , shifts the rate curve downward and makes it steeper. The magnitude of these changes is partly determined by the shape parameter of the cost function,  $\theta$ .

**Proposition 3.** *Suppose that the liquidity cost function is  $c(y) = \alpha y^\theta$  and the small bank's reserve holdings  $x$  has cumulative distribution function  $F(x)$ . Also, suppose that  $(r(l), l(x))$  is a solution to the two banks' optimization problems. Then,*

(i) *With liquidity cost function  $c(y) = C\alpha y^\theta$  for some  $C > 0$ ,  $(r_c(l), l(x))$  is an optimal solution, where*

$$r_c(l) = p - C(p - r(l)). \quad (20)$$

(ii) *With the original  $c(y) = \alpha y^\theta$  and the level of reserves following the distribution of  $Cx$ , so that its cumulative distribution function is  $F(x/C)$ ,  $(r_x(l), Cl(x/C))$  is an optimal solution, where*

$$r_x(l) = p - C^\theta(p - r(\frac{l}{C})). \quad (21)$$

*The proof is in the appendix. ■*

## 4.4 Parameter Calibration

Parameters of the model are calibrated to account for shifts in rate curves during the crisis. Roughly speaking, the change in the shape of the distribution of  $x$  can be measured by comparing loan values during the crisis with those outside the crisis. If this change in distribution cannot explain all of the observed shifts in rate curves, the residual shift can be attributed to a shift in the cost function.

Parameters of the model are calibrated following a clearly defined procedure. They are mostly chosen to set the values of certain moments at zero. Therefore, the calibration is conceptually close to GMM. Indeed, it is likely to dominate a GMM estimator obtained by brute force, resulting in a much smaller objective function.<sup>18</sup>

For calibration, the distribution of  $x$ , the small bank's reserve holdings, is assumed to follow Weibull distribution, with the scale parameter  $\lambda > 0$  and the shape parameter  $k \geq 1$ . Therefore,  $x$  has the following cumulative probability distribution function  $F(x)$ :

$$F(x) = 1 - \exp\left(-\left(\frac{x}{\lambda}\right)^k\right). \quad (22)$$

With  $k \geq 1$ , the hazard rate is weakly increasing.

The model is characterized by four parameters,  $(\theta, k, \lambda, \alpha)$ , where  $\theta$  and  $\alpha$  characterize the liquidity cost function  $c(y) = \alpha y^\theta$ . Two additional parameters,  $C > 0$  and  $\alpha' \geq 0$ , are introduced to capture changes to loan values and rate curves during the crisis.  $C$  shifts the distribution of  $x$  to  $Cx$ , implying that the cumulative distribution function for  $x$  is  $F(x) = 1 - \exp\left(-\left(x/(C\lambda)\right)^k\right)$  during the crisis.  $\alpha'$  characterizes the liquidity cost during the crisis,  $c(y) = \alpha' y^\theta$ .

Bringing the model to the data, each individual small bank is characterized by five parameters,  $(k, \lambda, C, \alpha, \alpha')$ , during both normal times and the crisis.  $\theta$  is treated as a structural parameter applying to all banks at all times. For given  $\theta$  and for each bank, the five parameters are chosen to satisfy the following four conditions on sample moments. For each bank, individual observations are indexed by date  $t$ , and  $l_t$  and  $r_t$  denote the volume and the rate of the bank's lending to its principal large bank counterparty on date  $t$ , respectively.  $D_t$  is the dummy variable for the crisis period.<sup>19</sup>  $E$  denotes sample mean over  $t$  for which a trade is observed,  $l_t > 0$ :

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<sup>18</sup>The nonlinear objective function does not work well with off-the-shelf optimizers for finding a GMM estimator. It is hard to reduce the objective function by a meaningful amount, using the calibrated parameters as the starting value. The appendix describes how the calibration can be technically treated as a GMM problem.

<sup>19</sup> $D_t = 1$  if  $t$  is within the crisis period and  $D_t = 0$  if  $t$  is outside the crisis period.

$$E[(l_t - E_N[l|l > 0])(1 - D_t)] = 0. \text{ (MC1)}$$

$$E[(l_t - E_C[l|l > 0])D_t] = 0. \text{ (MC2)}$$

$$E[(r_t - r_N(l_t))(1 - D_t)] = 0. \text{ (MC3)}$$

$$E[(r_t - r_C(l_t))D_t] = 0. \text{ (MC4)}$$

In the first condition, (MC1),  $E_N$  denotes model-expected-value outside the crisis, which is matched to its sample counterpart. For any given  $(\theta, k)$ , there exists unique  $\lambda$  satisfying the condition. In the second condition, (MC2),  $E_C$  denotes model-expected-value during the crisis, generated by using  $C\lambda$  as the scale parameter for the Weibull distribution for  $x$ , and by using  $c(y) = \alpha'y^\theta$ . The condition matches the model-expected-value to its sample counterpart. For any given  $(\theta, k, \lambda)$ , there is unique  $C$  satisfying the condition.  $C$  represents roughly the ratio of the average loan value during the crisis to that outside the crisis.

In the last two conditions, (MC3) and (MC4),  $r_N(l)$  and  $r_C(l)$  are the interest rates on the loan of value  $l$  implied by the model, outside and during the crisis, respectively. Their means are matched to those of their sample counterparts. For any given  $(\theta, k, \lambda, C)$ , these two conditions uniquely determine  $\alpha$  and  $\alpha'$ . These two parameters roughly represent the distance of the rate curve from the target rate,  $p$ .

The four conditions just described thus uniquely determines  $(\lambda, C, \alpha, \alpha')$  for any given  $(\theta, k)$ . To determine  $k$ , the sample second moment of  $l_t$  is matched with its theoretical counterpart:

$$E[l_t^2 - E_N[l^2|l > 0](1 - D_t) - E_C[l^2|l > 0]D_t] = 0. \text{ (MC5)}$$

(MC5) can be satisfied only if

$$E(l_t^2) \leq 2[E(1 - D_t)E(l_t|D_t = 0)^2 + ED_tE(l_t|D_t = 1)^2], \quad (23)$$

which roughly means that the sample variance of  $l_t$  needs to be smaller than the square of its sample mean. This condition is satisfied for 18 of the 25 small banks. For the other seven banks,  $k$  is chosen to minimize the absolute value of the left-hand side of (MC5), which is achieved by maximizing  $E_N[l^2|l > 0]$  while still satisfying (MC1) and (MC2).<sup>20</sup>

Finally,  $\theta$  is chosen to minimize the sum of the square distance between observed interest rates and model-implied interest rates across all the banks. Mathematically, it minimizes

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<sup>20</sup>It turns out that maximizing  $E_N[l^2|l > 0]$  is equivalent to maximizing  $E_C[l^2|l > 0]$ . This is shown in the proof of proposition 4 in the appendix.

the sum of the following expression across all small banks:

$$\sum_{t, l_t > 0} [r_t - r_N(l_t)(1 - D_t) - r_C(l_t)D_t]^2. \quad (24)$$

The following proposition summarizes the calibration procedure:

**Proposition 4.** *Suppose that*

$$E(l_t^2) \leq 2[E(1 - D_t)E(l_t|D_t = 0)^2 + ED_tE(l_t|D_t = 1)^2]. \quad (25)$$

*For any given  $\theta > 0$ , there exists a vector of parameters  $(k, \lambda, C, \alpha, \alpha')$  such that the following moment conditions are satisfied:*

$$E[(l_t - E_N[l|l > 0])(1 - D_t)] = 0. \quad (MC1)$$

$$E[(l_t - E_C[l|l > 0])D_t] = 0. \quad (MC2)$$

$$E[(r_t - r_N(l_t))(1 - D_t)] = 0. \quad (MC3)$$

$$E[(r_t - r_C(l_t))D_t] = 0. \quad (MC4)$$

$$E[l_t^2 - E_N[l^2|l > 0](1 - D_t) - E_C[l^2|l > 0]D_t] = 0. \quad (MC5)$$

*If the initial inequality does not hold, there exists a vector of parameters  $(k, \lambda, C, \alpha, \alpha')$  minimizing the absolute value of the left-hand side of (MC5) while satisfying (MC1), (MC2), (MC3) and (MC4).*

*Given  $(\theta, k)$ , the parameters  $(\lambda, C, \alpha, \alpha')$  are unique and easy to compute, as described in the proof of the proposition.*

The proof of proposition 4 relies on following analytic expressions:

**Proposition 5.** *Suppose that  $x$  follows Weibull distribution with the scale parameter  $\lambda$  and the shape parameter  $k \geq 1$ , and  $c(y) = \alpha y^\theta$ . Then,*

$$l = \left[ x - \frac{\theta \lambda^k}{k x^{k-1}} \right]^+. \quad (26)$$

$$E[l|l > 0] = \theta x_0 + \lambda(1 - \theta) \exp\left(\frac{\theta}{k}\right) \Gamma\left(1 + \frac{1}{k}, \frac{\theta}{k}\right). \quad (27)$$

$\Gamma(a, z) \equiv \int_z^\infty w^{a-1} e^{-w} dw$  is the upper incomplete gamma function and  $x_0 \equiv \lambda(\theta/k)^{1/k}$  is the maximum value of  $x$  such that  $l = 0$ .

*The rate curve,  $r - p$ , has the following expression:*

$$r - p = -\alpha \frac{x_0^{k\theta}}{l} \left[ \frac{1}{1 + \theta - \theta k} (x^{1+\theta-\theta k} - x_0^{1+\theta-\theta k}) - \frac{x_0^k}{1 + \theta} (x^{(1-k)(1+\theta)} - x_0^{(1-k)(1+\theta)}) \right]. \quad (28)$$

The proofs of propositions 4 and 5 are in the appendix.

## 4.5 Calibration Results

The calibrated value of  $\theta$  is 0.79, which implies marginal cost function of the form  $c(y) = \alpha y^{0.79}$ . The speed of growth in the marginal cost lies somewhere between square root and linear.

The ratio of  $\alpha'$  to  $\alpha$ , measured for each bank, has a median of 1.40, first quartile of 1.01 and third quartile of 2.78. Also, the ratio is greater than one for 19 of the 25 banks. This result shows that at the peak of the crisis, small banks valued reserve holdings at relatively low levels. This is consistent with multiple hypotheses, such as increased perception of risk in investment opportunities or increased risk aversion reducing attractiveness of investment opportunities.

The parameter  $C$  represents a change in the average value of loans for individual banks. Its median across the banks is 0.91, first quartile is 0.72, and third quartile is 1.31. There is a large variation across individual banks on how the size of their loans changed during the crisis, with no clear common direction. This result is consistent with what was found with linear models, which found only an insignificant change in loan values during the crisis.

These calibrated parameters imply that changes in loan volumes account for only 1.0 bp of the observed 6.7 bp increase in the average rate discount that small banks faced while lending to large banks.<sup>21</sup> As expected from the mixed directions of changes in  $C$  and the generally higher level of  $\alpha'$  relative to  $\alpha$ , most of the increase in rate discounts can be attributed to a broad decrease in profitability from reserve holdings.<sup>22</sup>

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<sup>21</sup>The averages reported here weigh each individual loan equally. The contribution of  $C$  is computed by the average change in model-implied rates resulting from replacing  $\lambda$  by  $C\lambda$ . The contribution is calculated in two ways, either with the normal cost function characterized by  $\alpha$ , or with the crisis cost function characterized by  $\alpha'$ . The reported contribution is the average of the two numbers.

<sup>22</sup>However, using a value-weighted average to compute the change in discount substantially increases the contribution of  $C$ , even though it is still smaller than that of  $\alpha'$ :  $C$  contributes 2.6 bps of 6.5 bps increase in discount.

## 5 Conclusion

This paper has developed a model of trading between a pair of banks in the overnight interbank market. The model generates the rate curve between a pair of banks as a function of loan values. The shape of the rate curve is primarily determined by the shape of decreasing marginal returns to reserve holdings.

The model can explain some prominent features of Mexican interbank market, including the general positive relationship between the interest rate and the loan value, controlling for bank fixed effects. Also, the parameters of the model are calibrated for a time period around the 2008 financial crisis to observe the crisis' impact on model parameters. The calibrated parameters point to a general decline in profitability from reserve holdings experienced by small banks.

Even though the model's solutions are in relatively complicated non-linear forms, the paper has developed a straightforward calibration procedure with clear interpretation as matching moments of data with those of the model. Therefore, the model can be easily applied to interbank loan data from other countries and markets.

## References

- Acharya, Viral V. and Ouarda Merrouche (2010) “Precautionary Hoarding of Liquidity and Inter-Bank Markets: Evidence from the Sub-Prime Crisis,” *NBER Working Paper 16395*.
- Afonso, Gara, Anna Kovner, and Antoinette Schoar (2011) “Stressed, Not Frozen: The Federal Funds Market in the Financial Crisis,” *Journal of Finance*, pp. 1109–1139.
- Afonso, Gara and Ricardo Lagos (2015) “Trade Dynamics in the Market for Federal Funds,” *Econometrica*, Vol. 83, pp. 263–313.
- Akram, Q. Farooq and Casper Christophersen (2010) “Interbank Overnight Interest Rates – Gains from Systemic Importance,” *Norges Bank Working Paper*.
- Allen, Jason, James Chapman, Federico Echenique, and Matthew Shum (2012) “Efficiency and Bargaining Power in the Interbank Loan Market,” *Bank of Canada Working Paper 2012-29*.
- Bech, Morten L. and Enghin Atalay (2008) “The Topology of the Federal Funds Market,” *Federal Reserve Bank of New York Staff Report No. 354*.
- Bollerslev, Tim, George Tauchen, and Hao Zhou (2009) “Expected Stock Returns and Variance Risk Premia,” *Review of Financial Studies*, Vol. 22.
- Cocco, João F., Francisco J. Gomes, and Nuno C. Martins (2009) “Lending Relationships in the Interbank Market,” *Journal of Financial Intermediation*, Vol. 18, pp. 24–48.
- Coleman II, Wilbur John, Christian Gilles, and Pamela A. Labadie (1996) “A Model of the Federal Funds Market,” *Economic Theory*, Vol. 7, pp. 337–357.
- Furfine, Craig (2002) “The Interbank Market during a Crisis,” *European Economic Review*, Vol. 46, pp. 33–57.
- Furfine, Craig H. (1999) “The Microstructure of the Federal Funds Market,” *Financial Markets, Institutions & Instruments*, Vol. 8, pp. 24–44.
- (2001) “Banks as Monitors of Other Banks: Evidence from the Overnight Federal Funds Market,” *Journal of Business*, Vol. 74, pp. 33–57.
- Gofman, Michael (2013) “Efficiency and Stability of a Financial Architecture with Too-Interconnected-to-Fail Institutions,” *Working Paper*.

- Ho, Thomas S. Y. and Anthony Saunders (1985) “A Micro Model of the Federal Funds Market,” *Journal of Finance*, Vol. 40, pp. 977–988.
- Kamien, Morton I. and Nancy L. Schwartz (1991) *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management, 2nd Ed.*, Amsterdam, The Netherlands: Elsevier Science Publishers B. V.
- Sokolov, Andrey, Rachel Webster, Andrew Melatos, and Tien Kieu (2012) “Loan and Non-loan Flows in the Australian Interbank Network,” *Physica A*, Vol. 391, pp. 2867–2882.
- Tadelis, Steve and Ilya Segal (2005) “Lectures in Contract Theory,” URL: [http://faculty.haas.berkeley.edu/stadelis/econ\\_206\\_notes\\_2006.pdf](http://faculty.haas.berkeley.edu/stadelis/econ_206_notes_2006.pdf).

## 6 Appendix

### 6.1 Proof of Proposition 1

The optimization problem is a variant of the standard price-differentiation problem. The proof broadly follows the steps outlined in Tadelis and Segal (2005).

$x$  is a positive real random variable.

Bank  $L$ 's problem is

$$\max_{r(y)} \int_0^{\infty} [p - r(l(x|r(y)))]l(x|r(y))f(x)dx, \quad (29)$$

subject to the condition that the function  $l(y)$  is an optimal response to the following problem, given the function  $r(y)$ :

$$\max_l \int_0^{x-l} (p - c(y))dy + r(l)l. \quad (30)$$

$l(y) \geq 0$  because bank  $S$  borrowing from bank  $L$  always results in a loss for at least one of the two banks. Also,  $l(0) \leq 0$  because  $l(0) > 0$  is possible only with  $r(l(0)) > p$ , which results in a loss to bank  $L$ . Therefore,  $l(0) = 0$ .

The problem can be transformed into the following equivalent form:

$$\max_{l(y), R(y)} \int_0^{\infty} [pl(x) - R(x)]f(x)dx, \quad (31)$$

subject to  $l(0) = R(0) = 0$  and the truth-telling constraint, which states that  $z = x$  is an optimal response to the following problem for every  $x > 0$ , given the function  $R(y)$ :

$$\max_z \int_0^{x-l(z)} (p - c(y))dy + R(z). \quad (32)$$

This transformation is achieved by replacing  $r(l(y))l(y)$  by  $R(y)$ . The transformation is valid if (i) for any  $y_1$  and  $y_2$  such that  $l(y_1) = l(y_2)$ ,  $R(y_1) = R(y_2)$ ; and (ii) for any  $y$  such that  $l(y) = 0$ ,  $R(y) = 0$ . These conditions make sure that  $R(y)$  found by solving the maximization problem can indeed be written as  $r(y)l(y)$ . It will be shown later that a solution  $(l(y), R(y))$  to this transformed problem satisfies these conditions. Participation constraint is automatically satisfied by  $l(0) = R(0) = 0$ .

In solving the maximization problem,  $l(y)$  and  $R(y)$  are assumed to be continuous and

piecewise continuously differentiable. This assumption is not strictly necessary, but simplifies the proof.<sup>23</sup> Indeed, it is easy to show that  $l(y)$  and  $R(y)$  need to be continuous by showing that any jump in the solution is not optimal.

The truth-telling constraint implies that for any  $x_1 < x_2$ ,  $l(x_1) \leq l(x_2)$ . Suppose that there exists  $x_1 < x_2$  such that  $l(x_1) > l(x_2)$ . Then,  $[\int_0^{x_2-l(x_1)} (p-c(y))dy + R(x_1)] - [\int_0^{x_2-l(x_2)} (p-c(y))dy + R(x_2)] = -\int_{x_2-l(x_1)}^{x_2-l(x_2)} (p-c(y))dy + R(x_1) - R(x_2) > -\int_{x_1-l(x_1)}^{x_1-l(x_2)} (p-c(y))dy + R(x_1) - R(x_2) = [\int_0^{x_1-l(x_1)} (p-c(y))dy + R(x_1)] - [\int_0^{x_1-l(x_2)} (p-c(y))dy + R(x_2)] \geq 0$ . This violates the truth-telling constraint. Therefore,  $l'(y) \geq 0$  except on  $X_0$ , where  $X_0$  is the union of the sets of discontinuities in  $l(y)$  and  $R(y)$ , which is countable. The qualifier ‘except on  $X_0$ ’ will be generally omitted when it is not important.

Wherever  $l(x)$  and  $R(x)$  are differentiable, the following first-order condition for the truth-telling constraint needs to be satisfied:

$$l'(x) \left[ \frac{\partial}{\partial l} \int_0^{x-l} (p-c(y))dy \right]_{l=l(x)} + R'(x) = -(p-c(x-l(x)))l'(x) + R'(x) = 0. \quad (33)$$

Therefore, the truth-telling constraint implies that for every  $x$ ,  $l'(x) \geq 0$  and  $-(p-c(x-l(x)))l'(x) + R'(x) = 0$ .

These two conditions are also sufficient for the truth-telling constraint to hold. For every  $x_1 \neq x_2$ , the type  $x_1$  agent can pretend to be of type  $x_2$  and receive

$$\begin{aligned} \int_0^{x_1-l(x_2)} (p-c(y))dy + R(x_2) &= \int_0^{x_1-l(x_1)} (p-c(y))dy + R(x_1) + \int_{x_1-l(x_1)}^{x_1-l(x_2)} (p-c(y))dy \\ &\quad + \int_{x_1}^{x_2} R'(x)dx \\ &= \int_0^{x_1-l(x_1)} (p-c(y))dy + R(x_1) \\ &\quad + \int_{x_1}^{x_2} [-(p-c(x_1-l(x)))l'(x) + R'(x)]dx. \end{aligned} \quad (34)$$

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<sup>23</sup>See Tadelis and Segal (2005) for a discussion.

$l'(x) \geq 0$  and increasing  $c$  imply that the expression inside the second integral,  $-(p - c(x_1 - l(x)))l'(x) + R'(x)$ , is weakly smaller than  $-(p - c(x - l(x)))l'(x) + R'(x) = 0$  if  $x_1 < x$ . Similarly, the expression is weakly greater than 0 if  $x_1 > x$ . Therefore, the last integral is always nonpositive, and thus,

$$\begin{aligned} \int_0^{x_1-l(x_2)} (p - c(y))dy + R(x_2) &= \int_0^{x_1-l(x_1)} (p - c(y))dy + R(x_1) + \int_{x_1}^{x_2} [-(p - c(x_1 - l(x)))l'(x) \\ &\quad + R'(x)]dx \leq \int_0^{x_1-l(x_1)} (p - c(y))dy + R(x_1). \end{aligned} \quad (35)$$

This proves sufficiency. Therefore, the truth-telling constraint can be replaced by the two conditions,  $l'(x) \geq 0$  and  $-(p - c(x - l(x)))l'(x) + R'(x) = 0$ .

Also, these two conditions imply that for any  $x_1 < x_2$  such that  $l(x_1) = l(x_2)$ ,  $l'(x) = 0$  between  $x_1$  and  $x_2$ . Therefore,  $R'(x) = 0$  between  $x_1$  and  $x_2$  and thus,  $R(x_1) = R(x_2)$ . Also,  $l(x) = 0$  implies that  $l'(y) = 0$  for any  $y$  in  $(0, x)$ , and thus  $R'(y) = 0$ , which implies that  $R(x) = 0$ . This shows that a solution to the optimization problem satisfies the necessary conditions for the initial transformation of the problem to be valid.

The maximand of bank  $L$ 's maximization problem is  $\int_0^\infty [pl(x) - R(x)]f(x)dx$ . It is convenient to remove  $R(x)$  from the expression by using the differential equation  $-(p - c(x - l(x)))l'(x) + R'(x) = 0$ :

$$\begin{aligned} R(x) &= \int_0^x (p - c(y - l(y)))l'(y)dy \\ &= pl(x) + \int_0^x c(y - l(y))(1 - l'(y))dy - \int_0^x c(y - l(y))dy \\ &= pl(x) + \int_0^{x-l(x)} c(y)dy - \int_0^x c(y - l(y))dy. \end{aligned} \quad (36)$$

Therefore, the maximand is

$$\int_0^\infty (pl(x) - R(x))f(x)dx = \int_0^\infty \left[ - \int_0^{x-l(x)} c(y)dy + \int_0^x c(y - l(y))dy \right] f(x)dx. \quad (37)$$

Applying integration by parts to  $\int_0^\infty \int_0^x c(y - l(y)) dy f(x) dx$  with respect to  $x$  yields:

$$\begin{aligned}
\int_0^\infty \int_0^x c(y - l(y)) dy f(x) dx &= \left[ \int_0^x c(y - l(y)) dy F(x) \right]_{x=0}^{x=\infty} - \int_0^\infty c(x - l(x)) F(x) dx \\
&= \int_0^\infty c(x - l(x)) dx - \int_0^\infty c(x - l(x)) F(x) dx \\
&= \int_0^\infty c(x - l(x)) \frac{1 - F(x)}{f(x)} f(x) dx. \tag{38}
\end{aligned}$$

Therefore, the original optimization problem can be written as follows:

$$\max_{l(y), R(y)} \left[ \int_0^\infty \left[ - \int_0^{x-l(x)} c(y) dy + c(x - l(x)) \frac{1 - F(x)}{f(x)} \right] f(x) dx \right], \tag{39}$$

subject to

$$l'(x) \geq 0, \tag{40}$$

$$-(p - c(x - l(x)))l'(x) + R'(x) = 0, \tag{41}$$

$$l(0) = R(0) = 0. \tag{42}$$

This is an optimization problem with  $R'(x)$  as the control variable and  $l(x)$  as the state variable. Note that  $R(x)$  only appears in the form of  $R'(x)$ , except for the boundary condition  $R(0) = 0$ .

Ignoring the constraint  $l'(x) \geq 0$  and replacing  $R'(x)$  by  $u(x)$  for notational convenience, the Hamiltonian for this optimization problem is:<sup>24</sup>

$$H = \left[ - \int_0^{x-l(x)} c(y) dy + c(x - l(x)) \frac{1 - F(x)}{f(x)} \right] f(x) + \lambda \frac{u(x)}{p - c(x - l(x))}. \tag{43}$$

For optimality, it is necessary that (i)  $H_u = 0$ , (ii)  $\lambda' = -H_l$ , and (iii)  $\lim_{x \rightarrow \infty} \lambda(x) = 0$ . Condition (i) implies  $\lambda = 0$ , so condition (ii) can be simply written as :

$$H_l = f(x)c(x - l(x)) - (1 - F(x))c'(x - l(x)) = 0. \tag{44}$$

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<sup>24</sup>Generally following the formulation in Kamien and Schwartz (1991).

Therefore, with  $l'(x) \geq 0$  not binding, the solution to the optimization problem is characterized by the following two equations:

$$-(p - c(x - l(x)))l'(x) + R'(x) = 0. \quad (45)$$

$$f(x)c(x - l(x)) - (1 - F(x))c'(x - l(x)) = 0. \quad (46)$$

The image of  $l(x)$  is an interval of the form  $[0, a]$  or  $[0, a]$  for some  $a > 0$  or  $a = \infty$ . For  $l$  in the image,  $r(l)$  can be defined by choosing any  $x$  such that  $l = l(x)$  and setting  $r(l) = R(x)/l(x)$ . As described earlier in the proof, this definition is valid. If  $a < \infty$ , for any  $l \geq a$ ,  $r(l)$  can be simply set at a level that bank  $S$  will never choose. For example,  $r(l) = (a/l) \lim_{y \rightarrow a-0} R(y)$  for any  $l \geq a$ . Replacing  $R(x)$  by  $r(l(x))l(x)$  proves the proposition.

If the pair of constraints  $l(0) = 0$  and  $l'(x) \geq 0$  bind at some  $x$ , the equation  $f(x)c(x - l(x)) - (1 - F(x))c'(x - l(x)) = 0$  will not hold for  $x$ , and it is necessary to introduce a shadow cost  $\mu(x)$ .  $\mu(x) = 0$  at any  $x$  for which the constraints do not bind. Additional assumptions in proposition 2 results in a very simple form of  $\mu$ , as will be shown in its proof. Under additional assumptions, it is possible to characterize  $\mu(x)$ , as discussed in Tadelis and Segal (2005).

## 6.2 Proof of Proposition 2

Using the assumption  $c(y) = \alpha y^\theta$ , (FOC) in proposition 1 is

$$f(x)\alpha(x - l(x))^\theta - (1 - F(x))\alpha\theta(x - l(x))^{\theta-1} = 0. \quad (47)$$

$$\therefore l(x) = x - \theta \frac{1 - F(x)}{f(x)}. \quad (48)$$

This function  $l(x)$  cannot be used because it violates the constraints  $l(0) = 0$  and  $l'(x) \geq 0$ . Since  $(1 - F(x))/f(x)$  is weakly decreasing in  $x$ ,  $x - \theta(1 - F(x))/f(x)$  is a strictly increasing function of  $x$ . Also, at  $x = 0$ , the expression has a negative value. Therefore, there exists  $x_0 > 0$  such that  $x - \theta(1 - F(x))/f(x) < 0$  if and only if  $x < x_0$ .

For any  $x < x_0$ ,  $f(x)\alpha(x - l)^\theta - (1 - F(x))\alpha\theta(x - l)^{\theta-1} < 0$  for any  $0 \leq l \leq x$ . Recall from the proof of proposition 1 that this expression is the derivative of the profit of bank  $L$  when bank  $S$  is of type  $x$ . Therefore, to maximize bank  $L$ 's pointwise profit,  $l(x) = 0$  needs to hold. At the same time, increasing  $l(x)$  above zero will make the constraints  $l'(x) \geq 0$  more binding. Therefore, it is optimal to set  $l(x) = 0$  for  $x < x_0$ .

For  $x \geq x_0$ ,  $x - \theta(1 - F(x))/f(x)$  is nonnegative and strictly increasing, so the optimal

$l(x) = x - \theta(1 - F(x))/f(x)$ . Therefore, for all  $x \geq 0$ ,

$$l(x) = [x - \theta \frac{1 - F(x)}{f(x)}]^+ \quad (49)$$

is the optimal solution, where  $[\cdot]^+$  denotes the maximum between the expression inside the brackets and 0.

(IC) constraint is

$$l'(x)[r(l) + r'(l)l - p + c(x - l)] = 0. \quad (50)$$

For  $x \leq x_0$ , (IC) constraint trivially holds because  $l'(x) = 0$ . For  $x > x_0$ ,

$$r(l) + r'(l)l - p + c(x - l) = 0. \quad (51)$$

Since  $r(l) + r'(l)l = \frac{d}{dl}(r(l)l)$ ,

$$\frac{d}{dl}(r(l)l) = p - c(x - l) = p - \alpha(\theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))})^\theta. \quad (52)$$

$l^{-1}$  is well defined inside the region  $x \geq x_0$  because  $l'(x) > 0$ .

Integrating from 0 to  $l$  yields

$$r(l)l = pl - \int_0^l \alpha(\theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))})^\theta dl. \quad (53)$$

$$\therefore r(l) = p - \frac{1}{l} \int_0^l \alpha(\theta \frac{1 - F(l^{-1}(l))}{f(l^{-1}(l))})^\theta dl. \quad (54)$$

### 6.3 Proof of Proposition 3

By assumption,  $(r(l), l(x))$  is an optimal solution under cost function  $c(y)$ . Let  $d(y) = C \cdot c(y)$ .

Under  $d(y)$ ,  $(r_c(l), l(x))$  satisfies bank  $S$ 's optimization problem. Recall that bank  $S$ 's problem is

$$\max_l \int_0^{x-l} (p - d(y))dy + r_c(l)l. \quad (55)$$

Note that the maximand can be rewritten as follows:

$$\int_0^{x-l} (p - d(y))dy + r_c(l)l = (1 - C)x + C \int_0^{x-l} (p - c(y))dy + r(l)l. \quad (56)$$

Therefore,  $(r_c(l), l(x))$  solves bank  $S$ 's optimization problem.

Bank  $L$ 's expected profit is

$$\int_0^{\infty} (p - r_c(l(x)))l(x)f(x)dx = C \int_0^{\infty} (p - r(l(x)))l(x)f(x)dx. \quad (57)$$

Suppose that  $(r_c(l), l(x))$  is not optimal for bank  $L$ . Then, there exists  $(r_2(l), l_2(x))$  that satisfies bank  $S$ 's problem and generates a higher profit for bank  $L$  than  $C \int_0^{\infty} (p - r(l(x)))l(x)f(x)dx$ . Let  $r_{2c}(l) = p - (1/C)(p - r_2(l))$ . Then, using preceding arguments, it can be shown that  $(r_{2c}, l_2(x))$  solves bank  $S$ 's optimization problem under cost function  $c(y)$ . Moreover, bank  $L$ 's profit under  $(r_{2c}, l_2(x))$  and  $c(y)$  is greater than  $(1/C) \cdot C \int_0^{\infty} (p - r(l(x)))l(x)f(x)dx = \int_0^{\infty} (p - r(l(x)))l(x)f(x)dx$ . This contradicts the assumption that  $(r(l), l(x))$  is optimal under  $c(y)$ . Therefore,  $(r_c(l), l(x))$  is optimal under  $d(y) = C \cdot c(y)$ . This proves part (i) of the proposition.

Under the original cost function  $c(y)$ ,  $(r_x(l), Cl(x/C))$  solves bank  $S$ 's optimization problem. The objective function of bank  $S$  can be transformed as follows:

$$\begin{aligned} \int_0^{x-l} (p - c(y))dy + r_x(l)l &= (1 - C^\theta)pl + \int_0^{x-l} (p - c(y))dy + C^\theta r\left(\frac{l}{C}\right)l \\ &= (1 - C^\theta)px + C^\theta \left[ \int_0^{x-l} \left(p - c\left(\frac{y}{C}\right)\right)dy + r\left(\frac{l}{C}\right)l \right] \\ &= (1 - C^\theta)px + C^{\theta+1} \left[ \int_0^{x-l} \left(p - c\left(\frac{y}{C}\right)\right)\frac{dy}{C} + r\left(\frac{l}{C}\right)\frac{l}{C} \right] \\ &= (1 - C^\theta)px + C^{\theta+1} \left[ \int_0^{\frac{x}{C} - \frac{l}{C}} (p - c(y))dy + r\left(\frac{l}{C}\right)\frac{l}{C} \right]. \end{aligned} \quad (58)$$

Therefore,  $l$  is optimal if  $l/C$  is an optimal choice for  $x/C$  in the original problem of choosing  $l$  given  $r(l)$ . Therefore,  $Cl(x/C)$  solves bank  $S$ 's problem.

Note that with  $x$  having cumulative distribution function of  $F(x/C)$ , its probability

density function is  $f(x/C)/C$ . Bank  $L$ 's expected profit under  $(r_x(l), Cl(x/C))$  is

$$\begin{aligned} \int_0^{\infty} (p - r_x(Cl(\frac{x}{C})))Cl(\frac{x}{C})\frac{f(\frac{x}{C})}{C}dx &= C^{\theta+1} \int_0^{\infty} (p - r(l(\frac{x}{C})))l(\frac{x}{C})f(\frac{x}{C})\frac{dx}{C} \\ &= C^{\theta+1} \int_0^{\infty} (p - r(x))l(x)f(x)dx. \end{aligned} \quad (59)$$

This form of the expected profit allows applying the argument used to prove part (i) to show that  $(r_x(l), Cl(x/C))$  is optimal for bank  $L$ . This proves part (ii) of the proposition.

## 6.4 Proof of Proposition 4

First, it is shown that for any given  $(\theta, k)$ , there exists a unique  $(\lambda, C, \alpha, \alpha')$  satisfying the following four moment conditions, for each bank:

$$E[(l_t - E_N[l|l > 0])(1 - D_t)] = 0. \quad (\text{M1})$$

$$E[(l_t - E_C[l|l > 0])D_t] = 0. \quad (\text{M2})$$

$$E[(r_t - r_N(l_t))(1 - D_t)] = 0. \quad (\text{M3})$$

$$E[(r_t - r_C(l_t))D_t] = 0. \quad (\text{M4})$$

Proposition 5 shows that  $E_N[l|l > 0] = \lambda g_1(k, \theta)$ , with  $g_1(k, \theta) \equiv \theta(\theta/k)^{1/k} + (1 - \theta)\exp(\theta/k)\Gamma(1 + (1/k), \theta/k)$ . Therefore, the following  $\lambda$  is the solution to equation (M1):

$$\lambda = \frac{E[l_t(1 - D_t)]}{g_1(k, \theta)E(1 - D_t)}. \quad (60)$$

Similarly, the following  $C$  is the solution to equation (M2):

$$\lambda C = \frac{E[l_t D_t]}{g_1(k, \theta)E D_t}. \quad (61)$$

Given the analytic forms of  $\lambda$  and  $C$ , it is easy to calculate them numerically.

Proposition 5 also shows that  $r_N(l_t) - p_t = -\alpha g_2(l_t, \lambda, k, \theta)$  for some function  $g_2$ . Therefore, the following  $\alpha$  is the solution to equation (M3):

$$\alpha = \frac{E[(r_t - p_t)(1 - D_t)]}{E[-g_2(l_t, \lambda, k, \theta)(1 - D_t)]}. \quad (62)$$

Similarly,  $r_C(l_t) - p_t = -\alpha' g_2(l_t, \lambda C, k, \theta)$ , and the following  $\alpha'$  is the solution to equation

(M4):

$$\alpha' = \frac{E[(r_t - p_t)D_t]}{E[-g_2(l_t, \lambda C, k, \theta)D_t]}. \quad (63)$$

According to proposition 5,  $g_2$  has an analytic form in  $x_t, \lambda, k$  and  $\theta$ . Transforming  $l_t$  to  $x_t$  is easy because  $l_t$  is a strictly increasing and strictly concave function of  $x_t$ , which implies that  $x_t$  can easily be computed from  $l_t$  using a derivative-based method such as Newton's method.

For given  $\theta$ , let  $h(k) \equiv E_N[l^2 | l > 0] / (E_N[l | l > 0])^2$ . This quantity is known to be not smaller than 1 (recall that variance is nonnegative). Recall from the proof of proposition 5:

$$E_N[l | l > 0] = \exp\left(\frac{\theta}{k}\right) \int_{(\theta/k)}^{\infty} [\lambda w^{1/k} - \frac{\lambda\theta}{k} w^{-1+(1/k)}] \exp(-w) dw. \quad (64)$$

Following the steps in the proof of proposition 5, it can be shown that

$$E_N[l^2 | l > 0] = \exp\left(\frac{\theta}{k}\right) \int_{(\theta/k)}^{\infty} [\lambda w^{1/k} - \frac{\lambda\theta}{k} w^{-1+(1/k)}]^2 \exp(-w) dw. \quad (65)$$

Therefore,  $h(k)$  depends only on parameters  $k$  and  $\theta$ :

$$h(k) = \exp\left(-\frac{\theta}{k}\right) \frac{\int_{(\theta/k)}^{\infty} [w^{1/k} - \frac{\theta}{k} w^{-1+(1/k)}]^2 \exp(-w) dw}{\left[ \int_{(\theta/k)}^{\infty} [w^{1/k} - \frac{\theta}{k} w^{-1+(1/k)}] \exp(-w) dw \right]^2}. \quad (66)$$

Since  $h(k)$  does not depend on  $\lambda$ ,  $h(k) = E_N[l^2 | l > 0] / (E_N[l | l > 0])^2 = E_C[l^2 | l > 0] / (E_C[l | l > 0])^2$ .

By direct calculation,  $h(1) = 2$ . Also, replacing  $(1/k)$  by 0 and calculating the resulting formula shows that  $h(k) \rightarrow 1$  as  $k \rightarrow \infty$ .

Moment condition (M5) can be transformed as follows:

$$\begin{aligned} 0 &= E[l_t^2 - E_N[l^2 | l > 0](1 - D_t) - E_C[l^2 | l > 0]D_t] \\ &= E[l_t^2] - h(k)[E_N[l | l > 0]^2 E[1 - D_t] + E_C[l | l > 0]^2 E D_t] \\ &= E[l_t^2] - h(k)[E[l_t | D_t = 0]^2 E[1 - D_t] + E[l_t | D_t = 1]^2 E[D_t]]. \end{aligned} \quad (67)$$

Note that  $E[l_t^2] = E[l_t^2 | D_t = 0]E[1 - D_t] + E[l_t^2 | D_t = 1]E[D_t] \geq E[l_t | D_t = 0]^2 E[1 - D_t] + E[l_t | D_t = 1]^2 E[D_t]$ . Therefore, if  $E[l_t^2] \leq 2[E[l_t | D_t = 0]^2 E[1 - D_t] + E[l_t | D_t = 1]^2 E[D_t]]$ ,

there exists  $k \geq 1$  such that

$$h(k) = \frac{E[l_t^2]}{E[l_t|D_t = 0]^2 E[1 - D_t] + E[l_t|D_t = 1]^2 E[D_t]}. \quad (68)$$

If the ratio  $E[l_t^2]/(E[l_t|D_t = 0]^2 E[1 - D_t] + E[l_t|D_t = 1]^2 E[D_t])$  is greater than 2, the left-hand side of (M5),  $E[l_t^2 - E_N[l^2|l > 0](1 - D_t) - E_C[l^2|l > 0]D_t] = E[l_t^2] - h(k)[E[l_t|D_t = 0]^2 E[1 - D_t] + E[l_t|D_t = 1]^2 E[D_t]]$ , is positive and is minimized if  $h(k)$  takes its maximum value of 2 with  $k = 1$ .

Since it is not clear whether  $h(k)$  is monotonic, it can be difficult to compute  $k$  numerically, in principle. However, with the Mexican data, it seems that  $h(k)$  is monotonic, at least over a large range relevant for parameter calibration.

## 6.5 Proof of Proposition 5

Proposition 5 can be proved by just applying proposition 1. Still, a proof is provided to illustrate the computational steps. Recall that  $x$  has cumulative distribution function  $F(x) = 1 - \exp(-(x/\lambda)^k)$  and probability distribution function  $f(x) = kx^{k-1}\lambda^{-k}\exp(-(x/\lambda)^k)$ . Also, the following definition of the upper incomplete gamma function will be used repeatedly:

$$\Gamma(a, z) = \int_z^\infty w^{a-1} e^{-w} dw. \quad (69)$$

The inverse hazard rate of  $x$  is

$$\frac{1 - F(x)}{f(x)} = \frac{\lambda^k}{kx^{k-1}}. \quad (70)$$

This inverse hazard rate is monotonically weakly decreasing in  $x$ . Therefore, proposition 1 can be used to compute  $l$ :

$$l = [x - \theta \frac{1 - F(x)}{f(x)}]^+ = [x - \frac{\theta \lambda^k}{kx^{k-1}}]^+. \quad (71)$$

$x_0$  is the unique positive solution to  $x_0 - (\theta \lambda^k)/(kx_0^{k-1}) = 0$ , which implies

$$x_0 = \lambda \left(\frac{\theta}{k}\right)^{1/k}. \quad (72)$$

$E[l|l > 0]$  can be calculated as follows:

$$\begin{aligned} E[l|l > 0] &= \frac{1}{1 - F(x_0)} \int_{x_0}^{\infty} lf(x)dx \\ &= \exp\left(\left(\frac{x_0}{\lambda}\right)^k\right) \int_{x_0}^{\infty} \left[x - \frac{\theta\lambda^k}{kx^{k-1}}\right] \frac{kx^{k-1}}{\lambda^k} \exp\left(-\left(\frac{x}{\lambda}\right)^k\right) dx. \end{aligned} \quad (73)$$

Note that  $(x_0/\lambda)^k = \theta/k$ . Changing the variable of integral from  $x$  to  $w \equiv (x/\lambda)^k$ , which implies  $dw = kx^{k-1}\lambda^{-k}dx$  and  $x = \lambda w^{1/k}$ ,

$$E[l|l > 0] = \exp\left(\frac{\theta}{k}\right) \int_{(\theta/k)}^{\infty} \left[\lambda w^{1/k} - \frac{\lambda\theta}{k} w^{-1+(1/k)}\right] \exp(-w) dw. \quad (74)$$

Using integration by parts, the second term in the integrand can be transformed as follows:

$$\begin{aligned} \int_{(\theta/k)}^{\infty} -\frac{\lambda\theta}{k} w^{-1+(1/k)} \exp(-w) &= [-\lambda\theta w^{1/k} \exp(-w)]_{(\theta/k)}^{\infty} + \int_{(\theta/k)}^{\infty} -\lambda\theta w^{1/k} \exp(-w) dw \\ &= \theta x_0 \exp\left(-\frac{\theta}{k}\right) + \int_{(\theta/k)}^{\infty} -\lambda\theta w^{1/k} \exp(-w) dw. \end{aligned} \quad (75)$$

Therefore,

$$\begin{aligned} E[l|l > 0] &= \exp\left(\frac{\theta}{k}\right) \left[ \theta x_0 \exp\left(-\frac{\theta}{k}\right) + \lambda(1 - \theta) \int_{(\theta/k)}^{\infty} w^{1/k} \exp(-w) dw \right] \\ &= \theta x_0 + \lambda(1 - \theta) \exp\left(\frac{\theta}{k}\right) \Gamma\left(1 + \frac{1}{k}, \frac{\theta}{k}\right). \end{aligned} \quad (76)$$

According to proposition 1,

$$r - p = -\frac{1}{l} \int_0^l \alpha\left(\theta \frac{1 - F(l^{-1}(z))}{f(l^{-1}(z))}\right)^\theta dz. \quad (77)$$

Changing the variable of integration from  $z$  to  $y \equiv l^{-1}(z)$  gives

$$r - p = -\frac{1}{l} \int_{x_0}^x \alpha \left( \theta \frac{1 - F(y)}{f(y)} \right)^\theta \frac{dz}{dy} dy. \quad (78)$$

$dz/dy$  is simply  $l'(y)$ . Therefore,

$$\begin{aligned} r - p &= -\frac{\alpha x_0^{k\theta}}{l} \int_{x_0}^x \frac{1}{y^{k\theta - \theta}} \left( 1 - \frac{(1-k)x_0^k}{y^k} \right) dy \\ &= -\frac{\alpha x_0^{k\theta}}{l} \left[ \frac{1}{1 + \theta - \theta k} (x^{1+\theta-\theta k} - x_0^{1+\theta-\theta k}) - \frac{x_0^k}{1 + \theta} (x^{(1-k)(1+\theta)} - x_0^{(1-k)(1+\theta)}) \right]. \end{aligned} \quad (79)$$

This expression is not valid if  $1 + \theta - \theta k = 0$ . In that case, the first two terms should be changed to logarithms:

$$r - p = -\frac{\alpha x_0^{k\theta}}{l} \left[ \log\left(\frac{x}{x_0}\right) - \frac{x_0^k}{1 + \theta} (x^{(1-k)(1+\theta)} - x_0^{(1-k)(1+\theta)}) \right]. \quad (80)$$

## 6.6 Calibration and GMM estimation

The calibration procedure relies mostly on choosing parameters to match values of certain empirical moments with their theoretical counterparts. With 25 banks, parameters are chosen to numerically satisfy the following four moment conditions for each bank:

$$E[(l_t - E_N[l|l > 0])(1 - D_t)] = 0. \quad (81)$$

$$E[(l_t - E_C[l|l > 0])D_t] = 0. \quad (82)$$

$$E[(r_t - r_N(l_t))(1 - D_t)] = 0. \quad (83)$$

$$E[(r_t - r_C(l_t))D_t] = 0. \quad (84)$$

The parameters are chosen also to satisfy the following moment condition, which is possible for 18 banks. For the remaining banks, the parameters are chosen to move the disparity between empirical and theoretical moments as close to zero as possible, while satisfying the preceding four moment conditions:

$$E[l_t^2 - E_N[l^2|l > 0](1 - D_t) - E_C[l^2|l > 0]D_t] = 0. \quad (85)$$

So far, the calibration represents a non-optimal solution to a GMM problem involving  $25 \times 5 = 125$  moment conditions. The final step is to choose  $\theta$  to minimize

$$\sum_{t, l_t > 0} [r_t - r_N(l_t)(1 - D_t) - r_C(l_t)D_t]^2. \quad (86)$$

This is not an explicit moment condition. However, it can be turned into multiple moment conditions, at most 126, by differentiating the objective function to be minimized by each of the 126 parameters:

$$v(\theta, \Theta) \cdot u_i(\theta, \Theta) = 0, \quad (87)$$

where  $v$  is the vector of  $r_t - r_N(l_t)(1 - D_t) - r_C(l_t)D_t$  across all  $t$  and all individual banks, and  $u_i$  is the partial derivative of  $v$  with respect to parameter  $i$ ,  $1 \leq i \leq 126$ . For given observations,  $v$  and  $u$  can be regarded as functions of the parameters. For convenience, let  $\theta$  be the first parameter,  $i = 1$ , and let  $\Theta$  denote the rest of the parameters.

The calibration step to determine  $\theta$  can be expressed as a single moment condition. Given observations and a value for  $\theta$ , the calibration procedure prior to the final step determines the value of 125 parameters, which can be represented by a 125-dimensional function  $h(\theta)$ , with the parameters in the same order as in  $u_i$ , for  $2 \leq i \leq 126$ . Then, the final step of square distance minimization is equivalent to finding the solution  $\theta$  to the following equation:

$$\begin{bmatrix} v(\theta, h(\theta)) \cdot u_1(\theta, h(\theta)) \\ v(\theta, h(\theta)) \cdot u_2(\theta, h(\theta)) \\ \vdots \\ v(\theta, h(\theta)) \cdot u_{126}(\theta, h(\theta)) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{dh}{d\theta}(\theta) \end{bmatrix} = 0. \quad (88)$$

Considering the calibration procedure as a moment-matching problem with this equation added to the previous 125 moment conditions, calibrated parameters satisfy 119 of the 126 moment conditions.

The objective function to the minimization problem, as a function of  $\theta$ , is not guaranteed to be well-behaved. However, at least with the Mexican data, it seems so, as the plot of the objective function versus  $\theta$  in figure 8 shows.

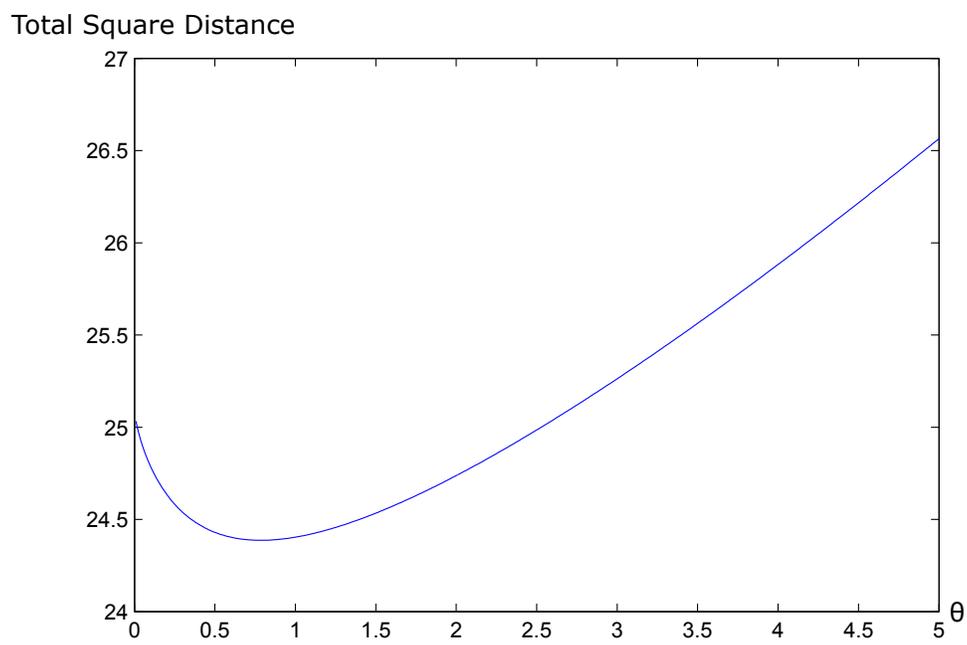


Figure 8: Total Square Distance as a Function of  $\theta$ .