

**Finance and Economics Discussion Series
Divisions of Research & Statistics and Monetary Affairs
Federal Reserve Board, Washington, D.C.**

Managing Counterparty Risk in OTC Markets

Christoph Frei, Agostino Capponi, and Celso Brunetti

2017-083

Please cite this paper as:

Frei, Christoph, Agostino Capponi, and Celso Brunetti (2017). "Managing Counterparty Risk in OTC Markets," Finance and Economics Discussion Series 2017-083. Washington: Board of Governors of the Federal Reserve System, <https://doi.org/10.17016/FEDS.2017.083>.

NOTE: Staff working papers in the Finance and Economics Discussion Series (FEDS) are preliminary materials circulated to stimulate discussion and critical comment. The analysis and conclusions set forth are those of the authors and do not indicate concurrence by other members of the research staff or the Board of Governors. References in publications to the Finance and Economics Discussion Series (other than acknowledgement) should be cleared with the author(s) to protect the tentative character of these papers.

Managing Counterparty Risk in OTC Markets*

Christoph Frei[†] Agostino Capponi[‡] Celso Brunetti[§]

July 19, 2017

Abstract

We study how banks manage their default risk before bilaterally negotiating the quantities and prices of over-the-counter (OTC) contracts resembling credit default swaps (CDSs). We show that the costly actions exerted by banks to reduce their default probabilities are not socially optimal. Depending on the imposed trade size limits, risk-management costs and sellers' bargaining power, banks may switch from choosing default risk levels above the social optimum to reducing them even below the social optimum. We use a unique and comprehensive data set of bilateral exposures from the CDS market to test the main model implications on the OTC market structure: (i) intermediation is done by low-risk banks with medium credit exposure; (ii) all banks with high credit exposures are net buyers of CDSs, and low-risk banks with low credit exposures are the main net sellers; and (iii) heterogeneity in post-trade credit exposures is higher for riskier banks and smaller for safer banks.

1 Introduction

Counterparty risk is a key consideration in over-the-counter (OTC) markets because of its importance to market participants, financial institutions and the real economy. Analyzing data of the credit default swap (CDS) market, we identify counterparty risk and banks' credit exposures as key drivers of the market structure, regarding both bilateral trading decisions and the resulting emergence of intermediaries. Accounting for heterogeneity in banks' credit

*We are grateful for constructive comments by Jack Bao, Garth Baughman, Michael Gordy, Julien Hugonnier, Pete Kyle, Ricardo Lagos, Semyon Malamud, Artem Neklyudov, Alireza Tahbaz-Salehi, Pierre-Olivier Weill, and seminar participants at the Board of Governors of the Federal Reserve System and the Swiss Finance Institute at EPFL. We would like to thank Alice Moore for her help in the data work. Part of this project was completed when Christoph Frei was visiting the Board of Governors of the Federal Reserve System, whose hospitality is gratefully acknowledged. Christoph Frei thanks the Natural Sciences and Engineering Council of Canada for support under grant RGPIN/402585-2011. Agostino Capponi acknowledges a generous grant from the Global Risk Institute. Any views are of the authors alone and do not represent the views of the Board of Governors of the Federal Reserve System.

[†]cfrei@ualberta.ca, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton AB T6G 2G1, Canada

[‡]ac3827@columbia.edu, Department of Industrial Engineering and Operations Research, Columbia University, New York, 10027, NY, USA

[§]celso.brunetti@frb.gov, Board of Governors of the Federal Reserve System, Washington D.C., 20551, USA

exposures, we introduce a framework to investigate, first, how default risk is managed and, second, the extent to which it affects the formation of bilateral trading exposures in OTC markets.

We compare the banks' risk management decisions with the optimal choice of a social planner, who maximizes the banks' aggregate certainty equivalent. Because fees earned from selling CDSs do not fully reflect the social benefits from risk management activities, banks' decisions on their default probabilities may deviate from the social optimum. Perhaps surprisingly, when banks are restricted by a trade size limit and subject to relatively low risk management costs, they may act conservatively and reduce their default risk below the socially optimal level. Examples of these actions are the implementation of stricter risk management policies or the reduction of the debt-to-assets ratio. These actions entail opportunity costs for forsaken business activities deemed too risky. Thus, through the choice of default probabilities, banks have a tradeoff between conservative policies entailing these opportunity costs and high-risk behavior decreasing earnings from selling CDSs.

Our model highlights the critical role played by counterparty risk in shaping the structure of OTC markets. It predicts that low-risk banks with medium credit exposure endogenously emerge as dealers, profiting from price dispersion, and providing intermediation services to banks with higher or lower credit exposures.¹ Banks with high credit exposures are net buyers of CDSs, and banks with low credit exposures are the main net sellers provided that their default probabilities are sufficiently low. Consequently, post-trade credit exposures are closer together than pre-trade credit exposures. In particular, safe banks have the same post-trade credit exposure if the trade size limit is big enough, whereas risky banks maintain diverse post-trade credit exposures. The post-trade credit exposure of safe banks is higher in riskier markets. This implication follows from the fact that riskier banks prefer to buy protection from safer banks to reduce their risk of counterparty's default. As a result, safe banks become the prime CDS protection sellers in markets with heightened default risk.

This paper contributes to the post-financial-crisis discussion on the role played by counterparty risk in the network of OTC derivative transactions. The OTC market for credit derivatives has been identified as the one that has contributed the most to the onset and transmission of systemic risk during the Global Financial Crisis.² Stulz (2010) highlights that counterparty credit risk is the highest in CDS markets. Because of the joint default risk of reference entity and protection seller, unique to the class of OTC credit derivatives, counterparty credit risk in general cannot be fully mitigated by collateral agreements.³

¹Our definition of dealers is also consistent with Stulz (2010), who defines dealers as large institutions maintaining a matched book, i.e., their net exposures on each given reference entity are very low compared with their total gross exposures.

²The most prominent solution proposed for reducing counterparty risk is the central clearing of OTC derivatives. The Dodd-Frank Wall Street Reform and Consumer Protection Act in the United States and the European Market Infrastructure in Europe have mandated central clearing for standardized OTC derivatives, including CDSs. The centrally cleared CDS market currently captures only approximately 30 percent of the entire U.S. CDS market, as measured by gross notional.

³The extent to which counterparty risk can be alleviated by collateral requirements depends on the jump properties of default events. Defaults of financial intermediaries are often difficult to anticipate and occur over short time periods, so that the protection buyer is usually unlikely to hold enough collateral to cover all potential losses.

Our framework is as follows. Before engaging into trading, each bank manages its risk so as to reduce its default probability.⁴ Such an action is costly, depends on the decisions made by other banks in equilibrium, and takes the subsequent trading decisions into consideration. Once the equilibrium default risk profile has been determined, all banks are granted access to the same technology to trade contracts resembling CDSs. As in Atkeson et al. (2015), each bank is a coalition of many risk-averse agents, called traders; banks have heterogeneous initial exposures to a nontradable risky loan portfolio, which creates heterogeneous exposures to an aggregate risk factor and determines the profitability of the trade. The trading process consists of two stages. First, banks' traders are paired uniformly, and each pair negotiates over the terms of the contract subject to a uniform trade size limit. The resulting prices and quantities are endogenous and depend on the risk profile of market participants, the heterogeneity in their initial exposures, and the dispersion in their marginal valuations. When a trader of a bank purchases a contract from the trader of another bank, it pays a bilaterally agreed-upon fee upfront and receives the contractually agreed-upon payment if the credit event occurs, provided the bank of its trading counterparty does not default. In case of the counterparty's default, the received payment is reduced by an exogenously specified loss rate. Second, each bank consolidates the swaps signed by its traders and executes the contracts. Because banks are risk averse, they value the risk of not receiving the full payment from a defaulted counterparty more than the potential gain obtained when they are protection sellers and default.

We test the main implications of the model using an extensive data set of bilateral exposures from the CDS market. This data set includes over 50 of the most active banks in the CDS market, with a wide coverage of global settlement locations. Our sample, obtained from the Depository Trust & Clearing Corporation (DTCC), covers all CDS bilateral exposures on corporate reference entities, containing 3,174 single names and 384 indexes. The network graph (see Figure 6) of the bilateral exposures of OTC market participants highlights the prominent role played by five banks acting as the main intermediaries. Our analysis confirms statistically that intermediation is done by banks with medium initial exposure and low default risk relative to all banks in the market. Post-trade credit exposures maintain the same order as initial exposures and are closer together provided that the bank selling CDS protection is not too risky.

Our normative analysis identifies inefficiencies in the banks' risk management decisions. To better understand the sources of these inefficiencies, consider a bilateral trade between a protection seller and a protection buyer. Such a trade affects the certainty equivalents in three different ways: (I) the seller increases its exposure to the aggregate risk factor, (II) the buyer decreases its exposure to the aggregate risk factor, and (III) fees are paid by the buyer to the seller. The social planner captures (I) and (II) in the aggregate certainty equivalent, while (III) has no net effect on it because fees are just wealth transfers from buyers to sellers. In contrast, the CDS protection seller's certainty equivalent depends on (I) and (III). Therefore, the difference between the social planner's and CDS protection seller's optimization problems is given by the benefit (II) net of the fees paid by the buyer (III). This

⁴Stulz (2014) argues that the primary reason for a bank to manage its default probability is to prevent the occurrence of adverse outcomes that can lead to distress, which is costly. When this happens, the bank loses the ability to implement its business strategy, and has more difficulty in finding trading counterparties.

net benefit is precisely the trade benefit of the CDS protection buyer. Under a trade size limit, the trade benefit depends crucially on the slope of the CDS protection buyer’s demand curve. Two factors determine the slope: increased concentration risk from purchasing additional CDS contracts from the same seller and, as usual, the law of diminishing marginal utility. Concentration risk is attenuated if the seller decreases its default probability. Consequently, the demand curve may become flatter, reducing the buyer’s trade benefit as the seller’s default probability decreases. Recalling that the trade benefits equals the difference between the certainty equivalents of the social planner and the CDS protection seller, the latter would choose a lower default probability than what is socially optimal under these circumstances. This effect is more pronounced if the seller has a higher bargaining power because it then receives a higher compensation for reducing its default probability.⁵

The rest of paper is organized as follows. We review related literature in Section 2. We develop the model in Section 3. We study the equilibrium trading decisions of banks conditional on their choices of default probabilities in Section 4. We compare private versus socially optimal default risk incentives in Section 5. Section 6 tests the empirical implications of our model. Section 7 concludes. Proofs of all results are delegated to the Appendix.

2 Literature Review

Our main contribution to the literature is the development of a tractable model to explain the endogenous formation of OTC derivatives trading networks, along with the risk-management decisions of its market participants. The predictions of our model have direct policy implications in regard to the banks’ initial risk exposures, trade size limits, and banks’ risk-management costs.

In our model, CDSs are used by banks to hedge against the default of corporate bonds or loans. Oehmke and Zawadowski (2017) provide empirical evidence consistent with this view. They show that banks with a large notional of outstanding bonds also have larger net notional outstanding for CDS contracts. They find 7.77 cents of net CDS positions per dollar of outstanding bonds, suggesting that this hedging role is economically significant. They also examine trading volumes in the bond and CDS markets and observe a similar pattern — that is, hedging motives are associated with comparable amounts of trading volume in the bond and the CDS market.

Our findings are consistent with Du et al. (2016), who develop a statistical multinomial logit model for the counterparty choice of buyers in the CDS market. They find that market participants are more likely to trade with safer counterparties and tend to avoid trading with counterparties whose default risk is highly correlated with that of the reference entity of the CDS contract. Our model predictions are also supported by the empirical analysis of Arora

⁵The existence of a relation between allocation inefficiency and bargaining power is also observed by Lagos and Rocheteau (2009), albeit in a different context. They consider a dynamic OTC market model in which a continuum of banks with a finite set of preference types decide on both traded asset quantities and intermediation fees. They find that the competitive allocation of quantities is efficient provided that the bargaining power of the dealer is zero.

et al. (2012), who find a significant negative relation between the credit risk of the dealer and the prices at which the dealer can sell credit protection.

To the extent that counterparty risk is ignored, our predictions are in line with those obtained by Atkeson et al. (2015). They consider a model similar to ours, but focus on the effect of entry and exit decisions. As in their model, we also find that banks with medium initial exposure endogenously emerge as dealers. Differently from Atkeson et al. (2015), in our model banks account for the default risk of their counterparties when entering into a trade. These counterparty-risk considerations have profound implications both for the size of the traded contracts and for the bilaterally negotiated price.

The classical setup used to study OTC markets is the search-and-bargaining framework proposed by Duffie et al. (2005), which models the trading friction characteristics typical of these markets. This model was generalized along several dimensions, including relaxation of the constraint of zero-one unit of assets holdings (see Lagos and Rocheteau (2009)), the entry of dealers (see Lagos and Rocheteau (2007)), and investors' valuations drawn from an arbitrary distribution as opposed to being binary (see Hugonnier et al. (2016)). All these studies do not allow for the inclusion of counterparty risk, mainly because the framework cannot keep track of the identities of the counterparties for the continuum of traders.

The interactions between counterparty risk and derivatives activities are also studied by Thompson (2010) and Biais et al. (2016). Thompson (2010) shows that a moral hazard problem for the protection seller, whose type is exogenously given, causes the protection buyer to be exposed to excessive counterparty risk. In turn, this mitigates the classical adverse selection problem because the protection buyer is incentivized to reveal superior information that it may have relative to the seller. In Biais et al. (2016), risk-averse protection buyers insure against a common exposure to risk by contacting protection sellers. Differently from our model, the protection buyers are risk neutral and avoid costly risk-prevention effort by choosing weaker internal risk controls. It is precisely the failure of protection sellers to exert risk-prevention effort that creates counterparty risk for protection buyers in our model.

Our paper is also related to the emerging, yet scarce, literature on endogenous network formation in interbank lending markets. Farboodi (2014) proposes a model of financial intermediation where profit-maximizing institutions strategically decide on borrowing and lending activities. Her model predicts that banks that make risky investments voluntarily expose themselves to excessive counterparty risk, while banks that mainly provide funding establish connections with a small number of counterparties in the network. Acemoglu et al. (2014-b) also study the endogenous formation of interbanking loan networks. In their model, banks borrow to finance risky investments, charging an interest rate that is increasing in the risk-taking behavior of the borrower. Their finding suggests that banks may overlend in equilibrium and do not spread their lending among a sufficiently large number of potential borrowers, thus creating insufficiently connected financial networks prone to defaults. While the above referenced works focus on the network of interbanking loans, our model targets derivatives trading in OTC markets. These instruments account for more than two thirds of the banks' most prominent USD asset classes, in contrast to bilateral corporate and syndicated loans which only account for about 2 percent of these assets.⁶ Most recently,

⁶A breakdown of the amounts allocated by banks to different asset classes is presented by the Market

Klimenko et al. (2016) develop a dynamic general equilibrium model to analyze the role of banking capital as a loss absorbing buffer in an economy populated by the real and the banking sectors. As compared with the solution of a social planner, they find that banks lend too much, excessively exposing themselves to risk when the equity is low, and banks lend too little when the equity is high. In contrast to all these studies, in which banks accrue a counterparty risk exposure above the socially desirable target, in our model banks may decide to reduce their default risk below the socially optimal level.⁷

3 The Model

There is a unit continuum of *traders*, that are risk-averse agents. They have constant absolute risk aversion with parameter η . The traders are organized into *banks*, which are coalitions of traders. We only consider banks that have already entered the OTC market and do not impose any entry costs. The effect of entry costs on the interbank dealer structure has been studied in Atkeson et al. (2015). In our model, all banks are granted access to the same technology to trade swaps. The banks are heterogeneous in two dimensions: their initial exposures and their sizes.

We consider M banks that are exposed to an aggregate risk factor D , taking binary values 0 (no default) and 1 (default), with $P[D = 1] = q$. We denote by ω_i the initial exposure per trader of bank i to the aggregate risk factor. The traders are paired uniformly across the different banks. We denote by $s_i > 0$ the size of bank i . Therefore, the frequency at which a trader of bank $j \neq i$ is paired with a trader of bank i is s_i . Both the initial exposure ω_i and the size s_i of bank i are exogenously specified and observable to the traders. Because the size does not play a crucial role in our main results, we restrict the main body of the paper to the case $s_i = 1$, but for completeness, we present the results and their proofs in the Appendix for any s_i .

Before trading begins, each bank i manages its default risk at an exogenously specified cost. We assume that given a realization $D = 1$ of the aggregate risk factor, bank i has maximal default probability \bar{p}_i . This value can be thought of as the default probability of bank i if it meets the imposed regulatory standards and does not engage in risk management or hedging procedures to further reduce its default risk. To become a more attractive trading counterparty in the OTC market, bank i can decrease its probability to $p_i \in [0, \bar{p}_i]$ at a cost $C(p_i)$. Therefore, depending on its initial exposure, each bank i needs to decide before trading starts how much it is willing to pay (cost $C(p_i)$) in order to reduce its default probability to p_i . These decisions also take into consideration the subsequent trading transactions that

Participants Group on Reforming Interest Rate Benchmarks. In its final report released on March 17, 2014, Figure 1 of the USD currency section highlights that the largest interbanking exposure on the US market is through OTC derivative trading (69 percent), while bilateral corporate loans and syndicated loans account for only 2 percent of the key asset classes that reference USD-LIBOR and T-bill rates.

⁷A branch of literature has studied counterparty risk in an exogenously specified network of financial liabilities. The focus of these studies is to investigate how the topology of the network affects the amplification of an initial shock hitting banks using fixed point methods. Relevant contributes include Acemoglu et al. (2014-a), Elliott et al. (2013), and Eisenberg and Noe (2011).

banks will establish and that are uniquely specified in terms of bilateral prices and quantities (see Theorem 4.3 for details). For a bank i , we denote by A_i the event that the bank defaults with $P[A_i|D = 1] = p_i$. Because banks will trade contracts of CDS type on the aggregate risk factor D , only the conditional default $A_i|D = 1$ of bank i and not the unconditional default A_i matters to the trading counterparties of bank i . Therefore, it is precisely the conditional default probability p_i to determine the attractiveness of bank i on the OTC market. We assume that the conditional events $A_i|D = 1$ are independent but do not impose that the banks' defaults themselves are independent. In particular, each bank can have different default probabilities depending on the realization of the aggregate risk factor. This setting allows for a dependence structure among the banks' defaults. A special role will be taken by banks that choose $p_i = 0$. We call such banks *safe*, while banks with $p_i > 0$ are referred to as *risky*.

When a trader from bank i meets a trader from bank n , they bargain a contract similar to a CDS. They agree that the trader of bank i sells $\gamma_{i,n}$ contracts to the trader of bank n . If $\gamma_{i,n} > 0$, bank n makes an immediate payment of $\gamma_{i,n}R_{i,n}$, and at the end of the period, bank i makes a payment of $\gamma_{i,n}D$ to bank n if bank i has not defaulted by then; if it has defaulted, the payment is reduced to $r\gamma_{i,n}R_{i,n}$. In summary, the payment at the end of the period is $\gamma_{i,n}D(\mathbb{1}_{A_i^c} + r\mathbb{1}_{A_i})$ from bank i to bank n if $\gamma_{i,n} > 0$. For the case $\gamma_{i,n} < 0$, the roles of i and n are interchanged. Therefore, the bilateral constraint $\gamma_{i,n} = -\gamma_{n,i}$ holds. We further assume that there is a trade size constraint per trader so that $-k \leq \gamma_{i,n} \leq k$ for some constant $k > 0$. We call a set of contracts $(\gamma_{i,n})_{i,n=1,\dots,M}$ *feasible* if both the bilateral constraint $\gamma_{i,n} = -\gamma_{n,i}$ and the trade size constraint $-k \leq \gamma_{i,n} \leq k$ hold for all $i, n = 1, \dots, M$. For notational convenience, we will use the abbreviation $\gamma_i := (\gamma_{i,1}, \dots, \gamma_{i,M})$ for the collection of contracts that bank i has with the other banks.

At the end of the trading period, traders of every bank come together and consolidate all their long and short positions. The consolidated per-capita wealth of bank i with contracts $\gamma_{i,1}, \dots, \gamma_{i,M}$ is

$$X_i = w_i(1 - D) + \sum_{n \neq i} \gamma_{i,n} (R_{i,n} - D(\mathbb{1}_{A_n^c} + r\mathbb{1}_{A_n})\mathbb{1}_{\gamma_{i,n} < 0} - D(\mathbb{1}_{A_i^c} + r\mathbb{1}_{A_i})\mathbb{1}_{\gamma_{i,n} > 0}),$$

where

- $w_i(1 - D)$ is the contingent payoff related to the initial exposure to the aggregate risk factor D
- $\sum_{n \neq i} \gamma_{i,n}R_{i,n}$ is the aggregate net payment received (if positive) or made (if negative) during trading, corresponding to the CDS protection fees
- $-D\gamma_{i,n}(\mathbb{1}_{A_n^c} + r\mathbb{1}_{A_n})\mathbb{1}_{\gamma_{i,n} < 0}$ is the per-capita payment that bank i will receive from bank n . This payment will be executed only if the realization of the aggregate risk factor is $D = 1$ and bank i net bought protection from bank n ($\gamma_{i,n} < 0$). In this case, bank i will receive $-\gamma_{i,n}$ if bank n does not default (event A_n^c) or $-r\gamma_{i,n}$ if bank n defaults (event A_n)

- $D\gamma_{i,n}(\mathbb{1}_{A_i^c} + r\mathbb{1}_{A_i})\mathbb{1}_{\gamma_{i,n}>0}$ is the per-capita payment that bank i will make to bank n . This payment will be executed only if the realization of the aggregate risk factor is $D = 1$ and bank i net sold protection to bank n ($\gamma_{i,n} > 0$). In this case, bank i will pay $\gamma_{i,n}$ if it does not default (event A_i^c) or $r\gamma_{i,n}$ if it defaults (event A_i)

We calculate the certainty equivalent x_i of X_i by solving $U(x_i) = E[U(X_i)]$, which yields

$$x_i = \omega_i + \sum_{n \neq i} \gamma_{i,n} R_{i,n} - \Gamma^i(\gamma_{i,1}, \dots, \gamma_{i,M}), \quad (1)$$

where

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log E \left[\exp \left(\eta D \left(w_i + \sum_{n \neq i} y_n \left((\mathbb{1}_{A_i^c} + r\mathbb{1}_{A_i}) \mathbb{1}_{y_n > 0} + (\mathbb{1}_{A_n^c} + r\mathbb{1}_{A_n}) \mathbb{1}_{y_n < 0} \right) \right) \right) \right].$$

The following result gives an explicit formula for Γ^i .

Lemma 3.1. *We have*

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log \left(1 - q + qe^{\eta w_i + \eta f(\sum_{n: y_n \geq 0} y_n, p_i) + \eta \sum_{n: y_n < 0} f(y_n, p_n)} \right),$$

where

$$f(y, p) = \frac{1}{\eta} \log \left((1 - p)e^{\eta y} + pe^{\eta r y} \right). \quad (2)$$

For $p > 0$, the functions

$$y \mapsto \Xi(y) := \frac{1}{\eta} \log(1 - q + qe^{\eta y}) \quad \text{and} \quad y \mapsto f(y, p)$$

are strictly increasing and strictly convex so that the function $\Gamma^i(y_1, \dots, y_M)$ is strictly increasing and convex. If $p_n > 0$, then the function Γ^i , viewed as a function of y_n , is strictly convex on $(-\infty, 0)$. Moreover, the function f satisfies

$$f(y_1, p_1) + f(y_2, p_2) > f(y_1 + y_3, p_1) + f(y_2 - y_3, p_2) \quad (3)$$

for all $y_1 < y_2$, $y_3 \in (0, \frac{y_2 - y_1}{2}]$ and $p_1 \geq p_2$.

The value $f(y, p)$ quantifies how the exposure of bank i to the aggregate risk factor D changes when it sells y (or buys y if $y < 0$) contracts to (from) bank n , where p is the default probability of the bank selling the contracts. If the bank that sells the contracts is safe ($p = 0$), then $f(y, p) = y$ as the increase in exposure corresponds to the number of traded contracts in this case. However, if the bank that is selling the contracts is risky ($p > 0$), the increase in exposure is smaller given that

$$\frac{1}{\eta} \log \left((1 - p)e^{\eta y} + pe^{\eta r y} \right) \begin{cases} < \frac{1}{\eta} \log \left((1 - p)e^{\eta y} + pe^{\eta y} \right) = y & \text{if } y > 0 \\ > \frac{1}{\eta} \log \left((1 - p)e^{\eta y} + pe^{\eta y} \right) = y & \text{if } y < 0. \end{cases} \quad (4)$$

The inequality (3) has a very intuitive interpretation. Suppose a bank buys CDS protection from banks 1 and 2 with default probabilities $p_1 > p_2$. If the bank were to buy additional protection from bank 1, its certainty equivalent would be lower with respect to the case in which it makes balanced purchases from the two banks.

Remark 3.2. *Risk aversion leads to an asymmetric role of credit valuation adjustment (CVA) and debit valuation adjustment (DVA). A CVA is deducted from the bank's assets to account for the default risk of its counterparty, while a DVA is deducted from the value of the bank's liabilities to account for the default risk of the bank itself. These adjustments are generally accepted principles for fair-value accounting (see also Duffie and Huang (1996) and Capponi (2013)). They can be understood by analyzing the structure of (4). If bank i is selling a total of α contracts to all its counterparties, then its exposure increases by less than α . The reason for this smaller increase is that bank i is risky, and would not deliver the promised payment to its buyer counterparties if it defaults prematurely. The reduction in risk exposure of bank i , given by $\alpha - \frac{1}{\eta} \log((1-p)e^{\eta\alpha} + pe^{\eta r\alpha})$, corresponds to the DVA. Consider now the situation that bank i buys α contracts from bank n . In that case, its exposure to the aggregate risk factor is reduced by less than α . The reason for this smaller reduction is that bank n is risky and may not deliver the promised payment to bank i if it defaults prematurely. The increase in risk exposure of bank i , attributed to the default risk of its counterparty n , is given by $\frac{1}{\eta} \log((1-p)e^{-\eta\alpha} + pe^{-\eta r\alpha}) + \alpha$ and is referred to as CVA. Because of risk aversion, DVA and CVA are not symmetric, i.e., $DVA \neq -CVA$. Indeed, the loss incurred by the buyer for not receiving the payment at default of the seller is higher than the gain of the protection seller for not making the promised payment to the protection buyer. In the limiting case that investors are risk neutral, we recover symmetry and $DVA = \alpha(1 - (1-r)p) = -CVA$.*

The extent to which the change in exposure is reduced by trading CDS contracts depends on the default probability p of the protection seller and its recovery rate r . Because of the bank's risk aversion, the change in exposure after trading is always smaller for a buyer (and higher for a seller) compared with the case where investors are risk neutral. This asymmetry means that risk-averse investors value their counterparty risk benefit (DVA) less than risk-neutral investors when they are selling and value their counterparty risk cost (CVA) more than risk-neutral investors when they are buying protection. Mathematically,

$$\frac{1}{\eta} \log((1-p)e^{\eta y} + pe^{\eta r y}) > \frac{1}{\eta} \log(e^{(1-p)\eta y + p\eta r y}) = y - (1-r)py,$$

using the strict convexity of the exponential function. For a buyer ($y < 0$), this means that the negative quantity $\frac{1}{\eta} \log((1-p)e^{\eta y} + pe^{\eta r y})$ is smaller in absolute value than $y - (1-r)py$.

4 Market Equilibrium Conditional on Banks' Default Risk

This section studies the market equilibrium under a given default risk profile of the banks in the system. In Section 4.1, we establish the existence of such an equilibrium. Section 4.2 studies the interplay of counterparty risk and the banks' post-trade exposures. In Section 4.3, we analyze which banks emerge as intermediaries and how this depends on their relative initial exposures and default risks.

4.1 Market Equilibrium Existence and Properties

Suppose that the default probability of bank i is p_i . Because traders are assumed to be small relative to their banks, they only have a marginal effect. When bank i sells protection to bank n , the cost of risk bearing increases by $\gamma_{i,n}\Gamma_{y_n}^i(\gamma_i)$ for bank i and decreases by $\gamma_{i,n}\Gamma_{y_i}^n(\gamma_n)$ for bank n , where $\Gamma_{y_i}^n(\gamma_n)$ denotes the partial derivative of $\Gamma^n(\gamma_n)$ with respect to the i -th component. Therefore, when traders of banks i and n bargain, their trading surplus is given by

$$\gamma_{i,n}(\Gamma_{y_i}^n(\gamma_n) - \Gamma_{y_n}^i(\gamma_i)).$$

This trading surplus is maximized by

$$\gamma_{i,n} \begin{cases} = k & \text{if } \Gamma_{y_n}^i(\gamma_i) < \Gamma_{y_i}^n(\gamma_n), \\ \in [-k, k] & \text{if } \Gamma_{y_n}^i(\gamma_i) = \Gamma_{y_i}^n(\gamma_n),^8 \\ = -k & \text{if } \Gamma_{y_n}^i(\gamma_i) > \Gamma_{y_i}^n(\gamma_n), \end{cases} \quad (5)$$

which is the traded quantity when two traders of banks i and n meet. The unit price $R_{i,n}$ of a CDS is decided via bargaining between a protection seller with bargaining power $\nu \in [0, 1]$ and a protection buyer with bargaining power $1 - \nu$. Hence,

$$R_{i,n} = \nu \max \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \} + (1 - \nu) \min \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \}. \quad (6)$$

If bank i sells contracts to bank n , it receives a fraction ν of the trading surplus. Indeed, bank i 's cost of risk bearing increases by $\gamma_{i,n}\Gamma_{y_n}^i(\gamma_i)$, but it receives a payment $\gamma_{i,n}R_{i,n}$ so that the net effect on bank i is

$$\begin{aligned} & -\gamma_{i,n}\Gamma_{y_n}^i(\gamma_i) + \gamma_{i,n}R_{i,n} \\ &= \gamma_{i,n} \left(\underbrace{\nu \max \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \}}_{= \Gamma_{y_i}^n(\gamma_n) \text{ by (5)}} + (1 - \nu) \underbrace{\min \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \}}_{= \Gamma_{y_n}^i(\gamma_i) \text{ by (5)}} - \Gamma_{y_n}^i(\gamma_i) \right) \\ &= \underbrace{\nu \gamma_{i,n} (\Gamma_{y_i}^n(\gamma_n) - \Gamma_{y_n}^i(\gamma_i))}_{\text{trading surplus}}. \end{aligned}$$

Because of the translation invariance property of the exponential utility, the relative bargaining power between buyers and sellers does not affect how traded quantities are chosen in equilibrium. However, it has an effect on how banks choose their default probabilities before trading starts, as we will see in Section 5.

Definition 4.1. *Feasible contracts $(\gamma_{i,n})_{i,n=1,\dots,M}$ build a market equilibrium if they are optimal in the sense that they satisfy (5).*

The following result shows that finding a market equilibrium is equivalent to solving a planning problem.

⁸For $\gamma_{i,n} = 0$ where Γ^i is not differentiable with respect to y_n , both one-sided partial derivatives must match, i.e., $\lim_{\gamma_{i,n} \searrow 0} \Gamma_{y_n}^i(\gamma_i) = \lim_{\gamma_{i,n} \nearrow 0} \Gamma_{y_i}^n(\gamma_n)$ and $\lim_{\gamma_{i,n} \nearrow 0} \Gamma_{y_n}^i(\gamma_i) = \lim_{\gamma_{i,n} \searrow 0} \Gamma_{y_i}^n(\gamma_n)$, as $\gamma_{i,n} = 0$ needs to be optimal with respect to both positive and negative changes.

Theorem 4.2. *Feasible contracts $(\gamma_{i,n})_{i,n=1,\dots,M}$ are a market equilibrium if and only if they solve the optimization problem*

$$\text{minimize } \sum_{i=1}^M \Gamma^i(\gamma_i) \quad \text{over } \gamma \text{ subject to } \gamma_{i,n} = -\gamma_{n,i} \text{ and } -k \leq \gamma_{i,n} \leq k. \quad (7)$$

This result follows from the fact that certainty equivalents are quasi-linear so that feasible contracts are a solution to the planning problem if and only if they are Pareto optimal for the banks. Based on the quasi-linearity of certainty equivalents, Atkeson et al. (2015) find that, conditional on entry decisions, the pairwise traded contracts are socially optimal.⁹

In our model, a market equilibrium on the level of the individual traders is thus equivalent to a Pareto optimal allocation for the banks. However, this holds only for given banks' default probabilities, and Pareto optimality for banks is only a statement about quantities and does not characterize prices. In our model, prices are determined in each meeting between two traders, as is standard in OTC market models.

Theorem 4.3. *There exists a market equilibrium $(\gamma_{i,n})_{i,n=1,\dots,M}$. The $\gamma_{i,n}$'s are unique for $p_n > 0$ and $\gamma_{i,n} < 0$, or $p_i > 0$ and $\gamma_{i,n} > 0$. For every i , the value of $\sum \gamma_{i,n}$ is unique in equilibrium, where the sum is over n such that $p_n = 0$ and $\gamma_{i,n} < 0$, or $p_i = 0$ and $\gamma_{i,n} > 0$. In particular, the values of $\Gamma(\gamma_n)$'s are uniquely determined for a market equilibrium $(\gamma_{i,n})_{i,n=1,\dots,M}$.*

Theorem 4.3 establishes the existence of a market equilibrium and states that volumes bilaterally traded with risky protection sellers are unique in equilibrium. This uniqueness result contrasts with Theorem 1 of Atkeson et al. (2015), where bilaterally traded volumes are not unique in their setting without counterparty risk. As soon as counterparty risk is involved in a trade, bilaterally traded volumes are unique in equilibrium. The reason is that counterparty risk makes CDS contracts purchased from traders of different banks imperfect substitutes. Even if banks have the same default probability, CDSs purchased from them are imperfect substitutes because of counterparty risk concentration: because of risk aversion, if a trader buys two CDS contracts, he/she prefers to choose the two trading counterparties from different banks, rather than purchasing both contracts from traders of the same bank. However, if the seller of protection is a safe bank, there is an indifference to increasing or decreasing the trading volume as long as it can be balanced by other trades not involving counterparty risk. For example, trades between three safe banks A, B and C could be increased without changing the planning problem (7) if A buys n additional CDS contracts from B, B buys n additional CDS contracts from C, and A buys n additional CDS contracts from C.

4.2 Post-trade Exposures

Suppose that each bank has decided on its default risk p_i at cost $C(p_i)$ and we denote by $(\gamma_{i,n})_{i,n=1,\dots,M}$ a market equilibrium from Theorem 4.3. We define the per-capita post-trade

⁹In a dynamic OTC market model for federal funds, Afonso and Lagos (2015) find that equilibrium loan sizes pairwise negotiated between banks are privately efficient.

exposure of bank i by

$$\Omega_i := \omega_i + f\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n}, p_i\right) + \sum_{n:\gamma_{i,n}< 0} f(\gamma_{i,n}, p_n), \quad (8)$$

where f is defined in Lemma 3.1. Note that Ω_i accounts for counterparty risk: if bank i and all its counterparties are safe, then Ω_i simplifies to $\omega_i + \sum_{n\neq i} \gamma_{i,n}$, as in Atkeson et al. (2015). Observe also that Ω_i is uniquely determined by Theorem 4.3. If bank i buys $-\gamma_{i,n}$ contracts on average from each trader of a risky bank n , the exposure of bank i is effectively reduced by less than $\gamma_{i,n}$ — namely by $f(\gamma_{i,n}, p_n)$ — which makes an adjustment for the counterparty risk, taking the bank’s risk aversion into consideration. Similarly, if bank i is risky and sells $\gamma_{i,n}$ contracts to each trader of bank n , then its effective increase in exposure is less than $\gamma_{i,n}$ due to its own default risk (DVA), as discussed in Remark 3.2.

The next result says that the post-trade exposures are increasing and closer together than pre-trade exposures. This result generalizes the first part of Proposition 1 of Atkeson et al. (2015) to our setting, while we will see in our Proposition 4.5 below that the second part of Proposition 1 of Atkeson et al. (2015) takes a quite different form in our model.

Proposition 4.4. *Assume that*

$$p_i = p_j \text{ or } p_i \leq 1/2 \text{ or } p_j \leq 1/2.^{10} \quad (9)$$

We then have the following:

1. *If $\omega_i \geq \omega_j$ and $p_i \leq p_j$, then $\Omega_i \geq \Omega_j$.*
2. *If $\omega_i > \omega_j$ and $p_i \geq p_j$, then $\omega_i - \omega_j > \Omega_i - \Omega_j$.*

Under condition (9), Proposition 4.4 states that

1. The banks’ order in post-trade exposures is the same as that in the initial exposures, provided that their default probabilities are ordered in the opposite direction.
2. Post-trade exposures are closer together than initial exposures if the bank with larger initial exposure is at least as risky as the bank with smaller initial exposure.

To see why conditions on the default risks of the banks need to be imposed, consider two banks i and j whose initial exposures $\omega_i > \omega_j$ are smaller than the average initial exposure. Because both banks have initial exposures below the average, they are interested in selling protection and earning the CDS protection fee. These trading motives imply that their post-trade exposures Ω_i and Ω_j are bigger than ω_i and ω_j , respectively. However, if bank i is safer than bank j , it is likely that the other banks will buy a higher amount of protection from bank i so that $\Omega_i - \omega_i > \Omega_j - \omega_j$. This inequality stands in contrast with that in the second statement of Proposition 4.4, noting that $p_i \geq p_j$ does not hold, either. Yet if bank j

¹⁰The condition $p \leq 1/2$ is essentially used to avoid that the probability-weighted CDS protection payment in the seller’s default event, rp , is higher than $1 - p$, which is the payment if the protection seller does not default.

is safer than bank i , it is likely that the other banks will buy a larger amount of protection from bank j , leading to $\Omega_j > \Omega_i$ even though the initial exposures had the reverse order. We will graphically demonstrate later in Figure 1 that both of these cases can indeed happen so that conditions on the default probabilities in Proposition 4.4 are needed.

We next study the conditions under which full risk sharing is possible for a subset of the banks in the system. If the trade size limit is big enough, then all safe banks are expected to perfectly share their risk. This is also consistent with the findings of Atkeson et al. (2015), see Proposition 3 therein. However, there are different degrees of achieved risk sharing when counterparty risk is taken into account. Regardless of the trade size limits we expect that, among the risky banks, only those with sufficiently large initial exposures will have the same post-trade exposure. The reason is that these banks primarily act as buyers of protection, and hence their default risk does not matter to the selling counterparties. We formalize these statements in the following proposition.

Proposition 4.5. *Assume that there are at least two safe banks¹¹ and set*

$$A(\alpha) = \{i : \omega_i \geq \alpha \text{ or } p_i = 0\}.$$

There exist $C \in \left[\frac{1}{M} \sum_{j=1}^M \omega_j, \frac{1}{\#\{i:p_i=0\}} \sum_{j=1}^M \omega_j \right]$, $\bar{k} > 0$, and $\bar{\alpha} \geq 0$ such that

- *for all $k \geq \bar{k}$, $\Omega_i = C > \Omega_\ell$ for all $i \in A(\bar{\alpha})$ and $\ell \notin A(\bar{\alpha})$*
- *for all $k < \bar{k}$, there exist i and j with $p_j = 0$ such that $\Omega_i > \Omega_j$*

Risky banks with small initial exposures would like to sell protection. However, other banks account for the default risk of their trading counterparties and, hence, they will trade with them only to a limited extent and there is no perfect risk sharing. This phenomenon is illustrated in Figure 1 through a numerical example. Note that the dashed and dotted curves hit the blue line at the same point, which means that the initial exposure needed to guarantee that risky banks have the same post-trade exposure does not depend on their default probabilities. This observation is a consequence of Proposition 4.5, and follows from the fact that $\bar{\alpha}$ does not depend on the banks' default probabilities. The reason behind this is that when the bank's initial exposure becomes sufficiently high, the bank will trade in only one direction, buying (and not selling) protection against the aggregate risk factor. Under these circumstances, the default risk of the bank does not matter to the seller.

The first statement of Proposition 4.5 implies that if the trade size limit is big enough, safe banks will have a higher post-trade exposure than they would have in absence of the risky banks. Figure 1 provides visual support for this implication and confirms that the post-trade exposure of safe banks is higher than their average initial exposure. In contrast, if there were only safe banks, their post-trade exposure would equal their average initial exposure for big enough trade size limit. More generally, it follows from Proposition 4.5 that all safe banks and risky banks with sufficiently large initial exposures have the same

¹¹For risk sharing, we need at least two banks. Because perfect risk sharing is done by safe banks, we consider in the proposition a market environment with at least two safe banks.

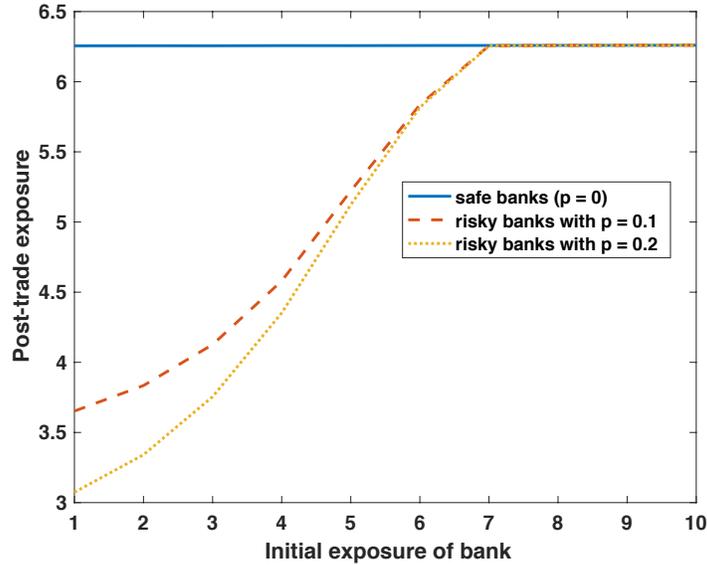


Figure 1: A market model consisting of 30 banks: for each initial exposure $1, 2, \dots, 10$, we consider three banks, respectively with default probabilities $p = 0$, $p = 0.1$ and $p = 0.2$. For large enough k , all safe banks and all risky banks with big initial exposures have the same post-trade exposure. The corresponding value 6.26 is higher than the average initial exposure of the safe banks, $5.5 (= (1 + 2 + \dots + 10)/10)$. Risky banks with small initial exposures have a smaller post-trade exposure than safe banks. Risky banks with $p = 0.2$ (dotted curve) have a smaller post-trade exposure than risky banks with $p = 0.1$ (dashed curve). The parameters chosen in this example are $\eta = 1$, $r = 0.4$, $q = 0.3$, $k = 1$.

post-trade exposure. The reason is that risky banks with large initial exposures also want to buy protection leading to higher demand for protection; however, while risky banks with small initial exposures would like to sell protection, they are not very attractive as trading counterparties because they bear high default risk. If the trade size limit is not big enough, the second statement of Proposition 4.5 implies that only partial risk sharing is done even in the absence of counterparty risk.

An immediate and interesting consequence of Proposition 4.5 is the sensitivity of the post-trade exposures to the banks' default probabilities.

Corollary 4.6. *If the trade size limit is big enough, the post-trade exposures of banks with sufficiently high initial exposure (banks in the set $A(\bar{\alpha})$ of Proposition 4.5) are not sensitive to their default probabilities, while the post-trade exposures of banks with small initial exposures are sensitive to their default probabilities.*

The statement in Corollary 4.6 is intuitive. Banks with sufficiently large initial exposures are protection buyers and thus their own default probabilities do not matter. However, banks with low initial exposures are protection sellers, so their default probabilities matter when other banks decide to trade with them.

4.3 Intermediation Volume

We study which banks endogenously emerge as intermediaries. These banks participate on both sides of the CDS market, as opposed to taking large net positions, either long or short. We consider per-capita gross numbers of sold or purchased contracts, accounting for the counterparty risk similarly to the post-trade exposure in (8). For a trader of bank i , these quantities are given by

$$G_i^+ = f\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n}, p_i\right) \quad \text{and} \quad G_i^- = -\sum_{n:\gamma_{i,n}< 0} f(\gamma_{i,n}, p_n). \quad (10)$$

If bank i is safe ($p_i = 0$), then $G_i^+ = \sum_{n\neq i} \max\{\gamma_{i,n}, 0\}$. Similarly, we have $G_i^- = \sum_{n\neq i} \max\{-\gamma_{i,n}, 0\}$ if all its counterparties n are safe. In general, however, the effect of the sold and purchased volumes are adjusted for DVA and CVA so that $G_i^+ \leq \sum_{n\neq i} \max\{\gamma_{i,n}, 0\}$ and $G_i^- \leq \sum_{n\neq i} \max\{-\gamma_{i,n}, 0\}$; see Remark 3.2. The per-capita intermediation volume of bank i is defined as $I_i = \min\{G_i^+, G_i^-\}$. By Theorem 4.3, G_i^+ and G_i^- and thus the intermediation volume I_i are uniquely determined if all p_n 's are strictly positive. Hence, we work under this assumption in this section. We analyze separately the effects of banks' initial exposures and default probabilities on their intermediation volume.

Proposition 4.7. *1. If the trade size limit k is small enough and there are at least three banks with different initial exposures ω_i 's, then the intermediation volume I_i as a function of ω_i is a hump-shaped curve, taking its maximum at or next to the median initial exposure weighted by counterparty risk.*

2. Assume that (9) holds. If two banks i and j have the same initial exposure, then $I_i \leq I_j$ for $p_i \geq p_j$.

The most prominent implication of Proposition 4.7 is that banks with intermediate exposures and small default probabilities are the main intermediaries. This implication is also illustrated via a numerical example in Figure 2. Note that banks in the example have a trade size limit high enough to not restrict their trading activities. In our model, intermediation activities are beneficial even for a high trade size limit. This result stands in contrast with Atkeson et al. (2015), where the intermediation activity vanishes when the trade size limit is sufficiently large. The reason why this is not the case in our framework is the presence of counterparty risk and can be understood from the second part of Proposition 4.7. Banks avoid buying protection from risky counterparties. By buying contracts through intermediaries, part of the counterparty risk is transferred to the intermediaries, which in turn benefit from the received CDS protection fees. Because of these transactions, counterparty risk is split between the customer bank and the intermediary, and the concentration of counterparty risk in the OTC market is reduced. Therefore, in our model, the intermediaries have two functions: they help diversify the aggregate level of counterparty risk and, as in Atkeson et al. (2015), they facilitate partial sharing of risk toward the aggregate risk factor.

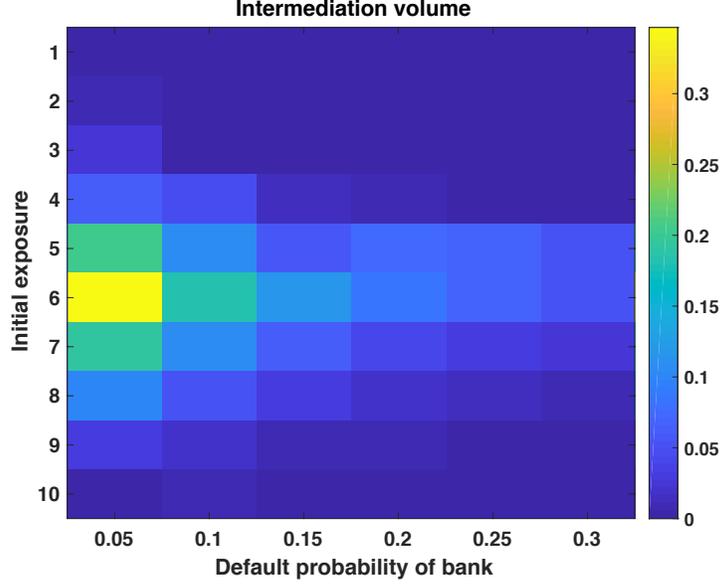


Figure 2: This figure presents a heat map of the volume of intermediation activity as a function of the default probability and initial exposure of the banks. It shows that the highest intermediation is done by banks with low default probability and medium initial exposure. The biggest intermediation is offered by banks with initial exposure around 6 and low default risk. The intermediation volume is small for banks with high default risk, in particular, for those with extreme initial exposures. The parameters chosen in this example are $\eta = 1$, $r = 0.4$, $q = 0.3$, $k = 10$ (trade size limit is large enough so that trading is unrestricted), default probabilities are between 0.05 and 0.3, and the initial exposures are between 1 and 10.

5 Private versus Socially Optimal Default Risk Levels

In this section, we compare the banks' decisions on their default probabilities with the socially optimal levels. Recall that each bank i manages its risk by choosing the conditional default probability $p_i \in [0, \bar{p}_i]$, where \bar{p}_i is the given maximal value. Bank i can lower its conditional default probability to p_i at a cost $C(p_i)$. We assume that $C : [0, \bar{p}_i] \rightarrow [0, \infty)$ is a decreasing, convex, and continuous function. Let $p_i \in [0, \bar{p}_i]$ be the decision of bank i . Theorem 4.3 yields that for given p_1, \dots, p_M , there exists a market equilibrium $(\gamma_{i,n})_{i,n=1,\dots,M}$. As we focus in this section on the choice of p_1, \dots, p_M , we write

$$x_i(p_1, \dots, p_M) = \omega_i + \sum_{n \neq i} \gamma_{i,n} R_{i,n} - \Gamma^i(\gamma_i) \quad (11)$$

for bank i 's per-capita certainty equivalent (1) in a market equilibrium.

Lemma 5.1. *The value of $x_i(p_1, \dots, p_M)$ is uniquely determined.*

Because each bank chooses individually its default risk, we are looking for a Nash equilibrium.

Definition 5.2. A choice of $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$ is an equilibrium if

$$x_i(p_1, \dots, p_M) - C(p_i) \geq x_i(p_1, \dots, p_{i-1}, \tilde{p}_i, p_{i+1}, \dots, p_M) - C(\tilde{p}_i)$$

for all i and $\tilde{p}_i \in [0, \bar{p}_i]$.

Proposition 5.3. If the cost function C is such that

$$\arg \max_{p_i \in [0, \bar{p}_i]} (x_i(p_1, \dots, p_M) - C(p_i)) \quad (12)$$

is a convex set for each i , then there exists an equilibrium p_1, \dots, p_M .

The assumption that (12) is a convex set means that if \hat{p}_i and p_i^* are maximizers of $x_i(p_1, \dots, p_M) - C(p_i)$, then so is any convex combination of \hat{p}_i and p_i^* . Note that, in particular, this assumption is satisfied if there is a unique maximizer.

We consider a social planner who decides over the banks' default probabilities p_1, \dots, p_M as well as the size of the traded contracts $(\gamma_{i,n})_{i,n=1,\dots,M}$ so as to maximize the banks' aggregate certainty equivalent minus the costs for managing their default probabilities. In other words, the social planner maximizes the objective function

$$\sum_{i=1}^M x_i(p_1, \dots, p_M) - \sum_{i=1}^M C(p_i) \quad (13)$$

over $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$ and $(\gamma_{i,n})_{i,n=1,\dots,M}$ subject to $\gamma_{i,n} = -\gamma_{n,i}$ and $-k \leq \gamma_{i,n} \leq k$, where

- $\sum_{i=1}^M x_i(p_1, \dots, p_M)$ is the aggregate certainty equivalent of the banks with risk types t_1, \dots, t_M
- $\sum_{i=1}^M C(p_i)$ is the sum of the costs incurred to reduce the default risk probabilities to the levels p_1, \dots, p_M

It follows from $\gamma_{i,n} = -\gamma_{n,i}$ and $R_{i,n} = R_{n,i}$ that $\sum_{i=1}^M x_i(p_1, \dots, p_M) = \sum_{i=1}^M \omega_i - \sum_{i=1}^M \Gamma^i(\gamma_i)$. Therefore, the social planner's optimization problem (13) is equivalent to minimize

$$\sum_{i=1}^M \Gamma^i(\gamma_i) + \sum_{i=1}^M C(p_i)$$

over the same optimization variables $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$ and $(\gamma_{i,n})_{i,n=1,\dots,M}$.

Proposition 5.4. The social planner's optimization problem has a solution.

We will next compare the optimization problem of the social planner with that of the individual banks. First, notice that the default probability p_n of bank n is relevant only to bank n itself and protection buyers from bank n . If p_n decreases, the DVA of the protection selling bank n decreases, and this is reflected in the same way in the social planner's and

bank n 's optimizations. Simultaneously, the CVAs of protection buyers from bank n decrease. This decrease in CVA is accounted for directly in the social planner's optimization, but only indirectly in bank n 's optimization problem through a higher fee revenue generated from the larger number of sold contracts and/or higher charged prices. Therefore, the different choices of default probabilities made by the social planner and the individual banks depend crucially on (I) how the fees earned from selling CDS protection change as a function of bank n 's default risk, and (II) the change in the protection buyers' certainty equivalent.

Before analyzing how changes in default probabilities affect prices and traded quantities, we recall the price building mechanism. When traders of bank 1 sell protection to traders of bank 2, their unit marginal cost of risk bearing is $\Gamma_{y_2}^1(\gamma_1)$, while the unit marginal benefit of risk reduction for the CDS buyers is $\Gamma_{y_1}^2(\gamma_2)$. Viewing $\Gamma_{y_2}^1(\gamma_1)$ and $\Gamma_{y_1}^2(\gamma_2)$ as functions of the quantity $\gamma_{1,2} = -\gamma_{2,1}$, they determine supply and demand curves, respectively. The traded quantity in equilibrium is determined by either the point where $\Gamma_{y_1}^2(\gamma_2)$ equals $\Gamma_{y_2}^1(\gamma_1)$ or the trade size limit; see Figure 3 for an illustration. If the trade size limit is binding, the negotiated price will be between $\Gamma_{y_1}^2(\gamma_2)$ and $\Gamma_{y_2}^1(\gamma_1)$, depending on the bargaining power. The highlighted area in Figure 3 marks the trade benefit of bank 2. The trade benefit equals the infinitesimal change in certainty equivalent after trading occurs.

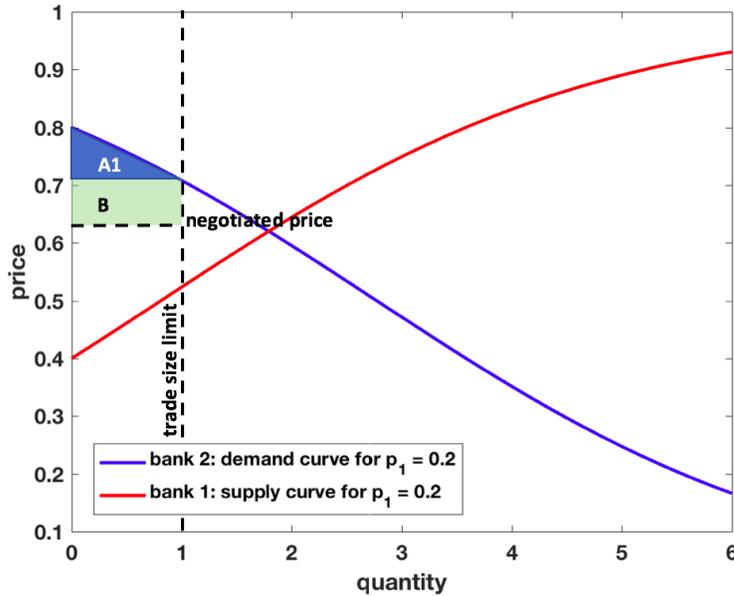


Figure 3: The supply and demand curves in a market consisting of two banks. The highlighted area shows the split of the trade benefit into guaranteed trade benefit (area A1) and negotiated trade benefit (area B) for trade size limit $k = 1$. The chosen parameters are $\eta = 1$, $r = 0$, $q = 0.5$, $\omega_1 = 0$, $\omega_2 = 100$; in this example, only the bank 1's default probability $p_1 = 0.2$ matters.

The trade benefit can be split into two parts:

$$\text{trade benefit} = \underbrace{\Gamma^2(\gamma_2) - \gamma_{1,2}\Gamma_{y_1}^2(\gamma_2)}_{\text{guaranteed (Area A1)}} + \underbrace{k(1 - \nu)(\Gamma_{y_1}^2(\gamma_2) - \Gamma_{y_2}^1(\gamma_1))}_{\text{negotiated (Area B)}} \quad (14)$$

1. Area A1 is defined as the part of the buyer's trade benefit that is independent of the bargaining power. We call it the *guaranteed trade benefit*. It is given by $\Gamma^2(\gamma_2) - \gamma_{1,2}\Gamma_{y_1}^2(\gamma_2)$.¹²
2. Area B is the part of the buyer's trade benefit that is determined by the buyer's bargaining power. We call it the *negotiated trade benefit*. It equals $k(1 - \nu)(\Gamma_{y_1}^2(\gamma_2) - \Gamma_{y_2}^1(\gamma_1))$.¹³ For big enough trade size limit k , it is zero because then $\Gamma_{y_1}^2(\gamma_2) = \Gamma_{y_2}^1(\gamma_1)$ by (5), while for binding trade size limit, it is linearly decreasing in ν .

This split of the trade benefit is important, as it provides a microfoundation for the difference between individually and socially optimal default probabilities.

The trade benefit of bank 2 (the highlighted area in Figure 3) corresponds to the contribution of bank 2 to the social planner's optimization problem, after fees have been paid to bank 1. While the whole certainty equivalent of bank 2 matters for the social planner, only the fees received from bank 2 contribute to the certainty equivalent of bank 1. Hence, the dependence of the trade benefit of bank 2 on the default probability p_1 explains why the social planner and bank 1 may choose different default probabilities.

Assume that p_1 decreases. On the one hand, if the trade benefit of bank 2 increases, the incremental fees paid by bank 2 are smaller than bank 2's benefit from counterparty risk reduction. Hence, bank 1 has individually smaller incentives to reduce its default probability compared with the social planner. On the other hand, if the trade benefit of bank 2 decreases when p_1 decreases, the additional fees that bank 2 needs to pay outweigh the benefit coming from the reduced default risk of its counterparty. In this case, the additional fees earned by bank 1 as a result of a default risk reduction are higher than its social contribution coming from lower counterparty risk in the market. Hence, bank 1 will reduce its default probability below the socially optimal level. While one would expect that the first case holds (bank 1 chooses a higher default probability than socially optimal), interestingly, the second case (bank 1 chooses a lower than socially optimal default probability) can also occur. To explain the reason behind this behavior, we provide a graphical example in Figure 4: assuming the seller (bank 1) has full bargaining power, the trade benefit of bank 2 equals its guaranteed trade benefit. Figure 4 shows that this is smaller for $p_1 = 0.05$ (area A2) than for $p_1 = 0.2$ (area A1).

For determining the optimal choices of default probabilities, the dependence of the infinitesimal changes in the trade benefit on the default probability plays a crucial role. When the trade size limit is binding, the change in the trade benefit is given by the change of the slope of the demand curve; see Figure 4. The following result formalizes the observation that

¹²This formula follows from computing area A1 as the difference between the area below the demand curve $\Gamma_{y_1}^2(\gamma_2)$, given by its antiderivative $\Gamma^2(\gamma_2)$, and the rectangle below area B, with size $\gamma_{1,2}\Gamma_{y_1}^2(\gamma_2)$.

¹³Area B in Figure 3 is a rectangle with width k and height $(1 - \nu)(\Gamma_{y_1}^2(\gamma_2) - \Gamma_{y_2}^1(\gamma_1))$.

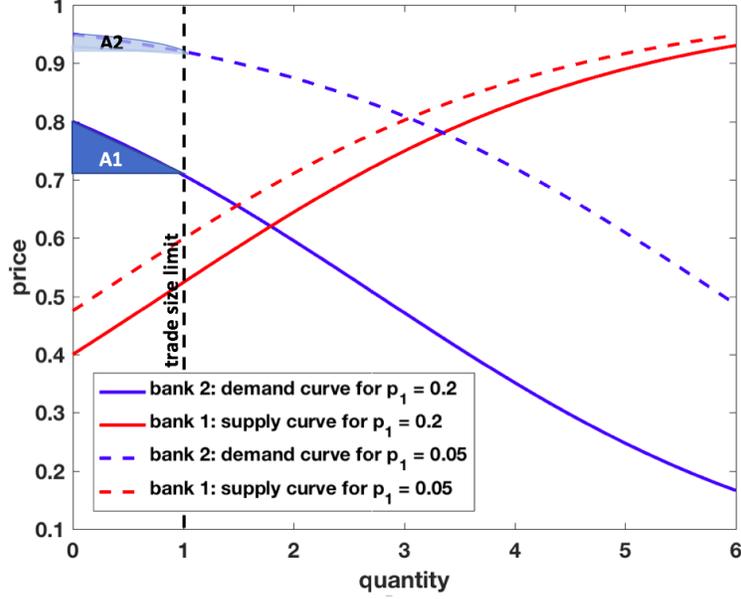


Figure 4: The dependence of the trade benefit on default probabilities. The protection seller has full bargaining power, and a trade size limit $k = 1$ is imposed. Interestingly, the trade benefit for bank 2 is higher when bank 1 has default probability 0.2 (area A1) than when it has default probability 0.05 (area A2). The parameters are the same as in Figure 3.

the demand curves can become flatter (and thus the trade benefit of bank i decreases) when the default risk of bank n decreases.

Lemma 5.5. *For large enough q , the demand curve for CDS protection on the short end becomes flatter if the default probability of the protection seller decreases.*

A flatter demand curve results in a decrease in the guaranteed trade benefit, which can lead to an individually chosen default probability below the socially optimal level, as discussed above. The following result shows that the decomposition into guaranteed and negotiated trade benefits for individual traders can be translated to a subsidy on the banking level in order to remedy inefficiencies in the banks' risk management decisions.

Theorem 5.6. *A solution to the social planner's optimization satisfies the first-order conditions of an equilibrium if bank i receives a subsidy equal to $S = S_1 + k(1 - \nu)S_2$ with*

$$S_1 := - \sum_{n \neq i} (\gamma_{i,n} \Gamma_{y_i}^n(\gamma_n, p) + \Gamma^n(\gamma_n, p)), \quad S_2 := \sum_{n \neq i} (\Gamma_{y_i}^n(\gamma_n, p) - \Gamma_{y_n}^i(\gamma_i, p)),$$

where we highlighted the dependence on $p = (p_1, \dots, p_M)$ in $\Gamma_{y_i}^n(\gamma_n, p)$, etc.

Assuming a small enough trade size limit, we have $\frac{\partial S_1}{\partial p_i} > 0$ and $\frac{\partial S_2}{\partial p_i} < 0$ for small enough p_i and large enough q . In this case, the privately chosen p_i 's are lower than the socially optimal level if sellers have full bargaining power. The difference between the socially optimal and individual choices of p_i increases as a function of the sellers' bargaining power.

To induce optimal risk management, a policymaker needs to give a subsidy to some banks and collect a tax from other banks in the amount of the difference between marginal social and marginal private value. The tax would be collected from CDS buyers maximal to the amount of their trade benefit (areas A1 and B in Figure 3) and given as a subsidy to CDS sellers to compensate them for their social contribution in reducing exposure to the aggregate risk factor. Such a tax would depend on the traded CDS volume and the default probabilities of the CDS sellers, which in turn depend on their initial exposures.

The first part of Theorem 5.6 states that the subsidy can be decomposed analogously to the trade benefit in (14):

1. S_1 is independent of the bargaining power and equals the sum of the guaranteed trade benefits of all counterparties of bank i .
2. $k(1 - \nu)S_2$ depends linearly on the bargaining power and corresponds to the sum of the negotiated trade benefits of all counterparties of bank i .

The second part of Theorem 5.6 relates the banks' optimal choice of default probabilities to the corresponding choice of a social planner. We position ourselves in a scenario in which marginal costs of risk management are sufficiently low so that the default probabilities privately chosen by the banks are low. As the sellers' bargaining power increases, a phase transition may occur: the banks' choice of default probabilities may switch from being above the socially optimal level to falling below it. This phenomenon can be explained as follows. When the sellers have high bargaining power, the term S_1 of the guaranteed trade benefits has a higher impact than S_2 on the banks' choice of default probabilities. A decrease in the sellers' default probability means that the additional protection fees earned are higher than the additional social benefit from the counterparty risk reduction, as a consequence of Lemma 5.5. Consequently, the protection seller would choose a lower default probability than what is socially optimal. However, if the sellers' bargaining power is low, the term S_2 of the negotiated trade benefits has a higher impact than S_1 on the banks' choice of default probabilities. The protection fees received by the sellers reflect only a fraction of their social contribution to the counterparty risk reduction. Hence, they have fewer incentives to reduce their default probabilities. In general, the difference between private and socially optimal choices of default probabilities will depend on the convexity of the cost function; higher convexity leads to smaller differences because the marginal cost from a reduction in default probability is higher in that case.

The level of sellers' bargaining power at which a phase transition occurs depends crucially on the initial exposures of the banks. This dependence is graphically illustrated in the left panel of Figure 5 for a market consisting of three banks. Bank 1 has zero initial exposure, $\omega_1 = 0$, and acts as a CDS seller. It chooses a smaller default probability than bank 2, which has medium initial exposure, $\omega_2 = 5$, and intermediates between banks 1 and 3. Bank 3 has the highest initial exposure, $\omega_3 = 10$, and thus chooses to purchase CDS protection. Being a protection buyer, bank 3 does not reduce its default probability from the maximal level $\bar{p}_3 = 0.2$. It can be seen from Figure 5 that a higher level of bargaining power is necessary for bank 1 ($\nu \geq 0.30$) to reduce its default probability below the social optimum,

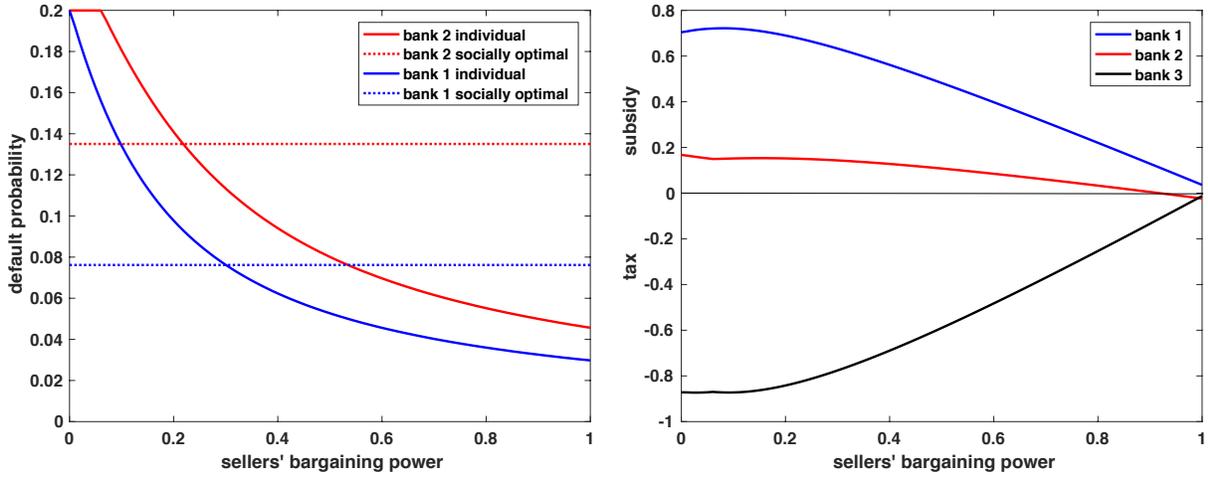


Figure 5: Left panel: Individual and socially optimal choices of default probabilities. Right panel: subsidy and tax required to make individual choices efficient. The bank with the lowest initial exposure (Bank 1), acting as a CDS seller, chooses the lowest default probability and would receive the highest subsidy. The bank with the highest initial exposure (Bank 3), acting as a CDS buyer, would be the primary tax payer. The bank with the medium initial exposure (Bank 2), which intermediates between banks 1 and 3, would receive a subsidy for acting as CDS seller to bank 3 and pay a tax as CDS buyer from bank 1, resulting in the net subsidy displayed in the right panel. We use the following benchmark parameters: $\eta = 1$, $r = 0$, $q = 0.1$, $k = 0.5$, $C(p) = 1/p^{0.05}$ and $\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = 0.2$. The banks' initial exposures are $\omega_1 = 0$, $\omega_2 = 5$ and $\omega_3 = 10$.

as compared with bank 2 ($\nu \geq 0.22$). The phase transitions occur at different levels because the individually chosen default probabilities of banks 1 and 2 are closer together than their socially optimal levels; see the left panel of Figure 5. Targeting the aggregated certainty equivalents, the social planner chooses a lower default probability for bank 1 than for bank 2, because a default of bank 1 affects the two other banks (banks 2 and 3), while a default of bank 2 affects only bank 3. On an individual level, bank 1 will also reduce its default probability more than bank 2. However, the difference in default probabilities resulting from the banks' equilibrium choices is smaller than what would be socially optimal. The right panel of Figure 5 shows that when implementing a subsidy-tax policy to achieve efficiency, bank 1 would receive the highest amount, as it acts on the sell side for both banks 2 and 3, while bank 3 would be the primary tax payer. Bank 2, acting as an intermediary, would benefit from the net effect of subsidies received for selling CDS contracts to bank 2 and tax paid for buying CDS contracts from bank 1. Bank 2's subsidy is much smaller than that of bank 1, but its individually chosen default probability deviates more from the socially optimal level, relative to that of bank 1. This is not a contradiction. Indeed, as an intermediary, the choice of bank 2's default probability is more sensitive to subsidies and taxes than that of the CDSs selling bank 2.

6 Empirical Analysis

In this section, we test the empirical predictions of our model conditional on banks' default risk choices. We describe the data set in Section 6.1, provide descriptive statistics of the data in Section 6.2, and test the empirical predictions related to our model in Section 6.3.

6.1 Data Set

We use different data sources for the bilateral exposures in the CDS market, the initial exposures of banks, and their default probabilities.

CDS volume. CDS data come from the confidential Trade Information Warehouse of the DTCC. We use position data from December 31, 2011. This data set allows for a post-crisis analysis in which a large part of CDS trades were not yet centrally cleared.¹⁴ We eliminate from our data set the following transactions:

- All swaps with governments, states, or sovereigns as reference entities. We eliminate these transactions because we expect the default risk profile of corporate reference entities to have stronger dependence on the risk stemming from banks' exposures than on that of sovereign entities.
- All swaps with reference entities that are considered systemically important financial institutions. By doing so, we avoid problems related to specific wrong-way risk, where the seller of the transaction also happens to be the reference entity.
- All transactions done by nonbanking institutions. For nonbanking institutions, there is no consistent way to measure initial exposures, which are needed in our analysis. While we consider only banks, we adjust their initial exposures by including CDS trades done with nonbanks. This procedure is consistent with our model and means that initial exposures of banks are determined after they have traded with nonbanks.
- The transactions done by two small private banks for which there were no data available on their initial exposures. Because these two banks are small players, the conclusions of our analysis are not affected by their exclusion.

Other than these four restrictions, we do not make any further adjustments. In particular, our data set also includes settlement locations outside of the United States, which allows for a more complete coverage of CDS trades and, importantly, guarantees symmetry in the inclusion of CDS trades (the transactions of both buyers and sellers are accounted for). The resulting set consists of CDS data for 56 banks.

Initial exposure. For each of these 56 banks, we compute its initial exposure by using 2011 data from the Federal Financial Institutions Examination Council (FFIEC) form 031 ("call report"), as in Begenau et al. (2015). We compute the initial exposure of each bank as the discounted valuation of its securities and loan portfolio, including CDSs traded with

¹⁴Distortion on the CDS market due to the "London Whale" (large unauthorized trading activities in JPMorgan's Chief Investment Office) occurred only after December 31, 2011 and, thus, does not affect our analysis.

nonbanks as explained above. For large banks that book their assets mainly in holding companies, we use securities and loan portfolios at the bank holding company level. We group the securities and loans into three categories and use a specific discount factor for each group: less than one year (using the six-month U.S. Treasury rate to discount), one to five years (using the two-year U.S. Treasury rate to discount), and more than five years (using the seven-year U.S. Treasury rate to discount). Given the low interest rate environment in 2011, the precise choice of the discounting date and rate does not have a significant effect on our results. For foreign banks that do not report to the FFIEC, we analyze individual annual reports from 2011 to find the maturity profile of their securities and loans. Most of these annual reports are dated December 31, 2011, making them consistent with the domestic bank data. Some of them were released in March, June, or October of 2011, in line with the respective country’s regulatory guidelines.

Default probabilities. The banks’ default probabilities are calculated using CDS spread data from IHS Markit Ltd. (2017) via Wharton Research Data Services (WRDS). Because the default probabilities that are relevant for the analysis are those around the time of the transaction, we fix January 3, 2011 as the proxy date for CDS transactions and use the spread on this date to infer the default probability. We use the average five-year spread for Senior Unsecured Debt (Corporate/Financial) and Foreign Currency Sovereign Debt (Government) (SNRFOR). We compute the default probabilities from the CDS spreads applying standard techniques (credit triangle relation), assuming a recovery rate of 40 percent. For 11 among the 56 banks, CDS spread data were not available. For each of these banks, we instead use Moody’s credit rating as of January 2011 for its Senior Unsecured Debt, and relate the ratings to default probabilities by using corporate default rates over the 1982–2010 period from Moody’s.

Post-trade exposure and intermediation volume. We set the post-trade exposure equal to the initial exposure plus the net effect from CDS trading. Similarly, we compute the intermediation volume without taking the counterparty risk into account. Table 1 shows that applying (10) with different values for the risk-aversion parameter only has a minor effect on the results.

	$\eta = 0.1$	$\eta = 1$	$\eta = 10$
total change	0.08 %	0.07 %	0.04 %
biggest change across banks	0.29 %	0.27 %	0.16 %

Table 1: Variation in intermediation volume for different levels of risk aversion.

Intermediaries. We define an intermediary to be a bank that provides at least 5 percent of the total intermediation volume. Using this definition, we obtain 5 intermediaries among the 56 banks. The selection of these 5 banks is not sensitive to the chosen threshold level of intermediation volume. Indeed, it will be evident below in Figure 7 that 5 banks account for the majority of the intermediation volume while the contributions of all other banks are small.

6.2 Summary and Overview

This section gives the descriptive statistics of the data set and provides a network visualization of the CDS exposures. Table 2 summarizes the CDS exposures data set, splitting it by activities by intermediaries (5 banks, as defined above) and by all other banks (51 banks). The table also distinguishes CDSs on single names (3,174 names) and indexes (384 indexes). Single name CDSs account for 41 percent of the gross notional volume. Table 3 provides the descriptive statistics of the banks' balance sheet characteristics relevant to our study — namely, their initial exposures as well as loans and securities — and of their default probabilities.

We visualize the network of bilateral CDS exposures in Figure 6. The inner nodes are the 5 intermediaries, while the remaining 51 banks are arranged as nodes on an outside circle. Both in the inner area and the outside circle, the nodes are ordered by their initial exposures, which correspond to the sizes of the nodes. The darker a node is, the higher is the default probability of the corresponding bank. The widths of the edges denote the notional net CDS

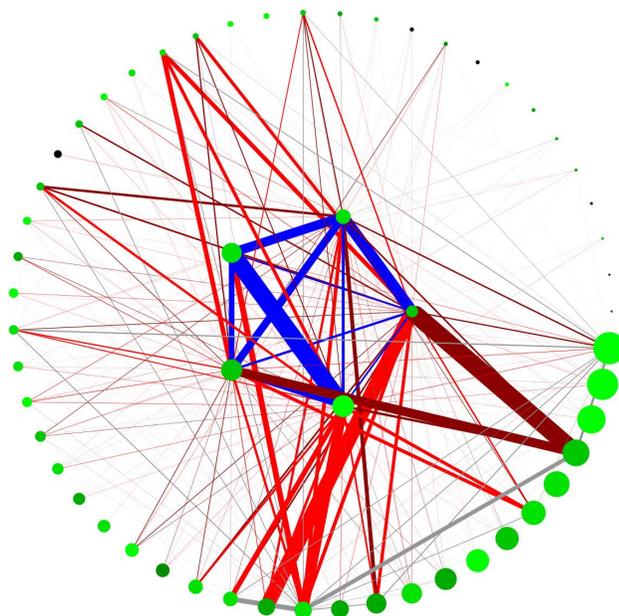


Figure 6: Network of banks' bilateral CDS exposures. Each node corresponds to a bank. The sizes of the nodes are proportional to the banks' initial exposures. The darker a node is, the higher is the default probability of the corresponding bank. The widths of the edges are proportional to the banks' bilateral notional net CDS volume. The five banks in the inner area are the main intermediaries. We use the blue color for CDS volume between two intermediaries; gray for CDS volume between two non-intermediaries; light red for CDS protection sold by an intermediary to a non-intermediary; dark red for CDS protection sold by a non-intermediary to an intermediary.

Bank type CDS type	Intermediaries			Other banks			All banks		
	Single	Indexes	Total	Single	Indices	Total	Single	Indices	Total
Gross notional	1,759	2,293	4,051	1,412	1,429	2,841	3,171	3,722	6,893
Net notional	-32	-45	-77	32	45	77	0	0	0
Notional total	895	1,169	2,064	690	692	1,382	1,585	1,861	3,446
purchased min	73	144	216	0	0	0	0	0	0
CDS med	185	227	407	1	1	3	2	2	3
mean	179	234	413	14	14	27	28	33	62
max	247	297	527	120	145	264	247	297	527
Notional total	863	1,124	1,987	722	737	1,459	1,585	1,861	3,446
sold CDS min	122	104	227	0	0	0	0	0	0
med	161	206	422	2	1	4	2	2	4
mean	173	225	397	14	14	29	28	33	62
max	223	327	528	164	166	330	223	327	528

Table 2: Summary statistics of the CDS exposures data set (3,174 single names and 384 indices). This is split into two categories: intermediaries (5 banks) and other banks (51 banks). All numbers are in billion USD.

		Intermediaries	Other banks	All banks
Initial exposure (billion USD)	total	4,031	25,551	29,582
	minimum	507	10	10
	median	906	387	410
	mean	806	501	528
	mean weighted by notional gross CDS volume	784	788	786
	maximum	999	1,512	1,512
Loans and securities (billion USD)	maturities ≤ 1 year	945	10,505	11,450
	maturities > 1 and ≤ 5 years	2,379	7,096	9,475
	maturities > 5 years	781	8,860	9,641
Default probability (in %)	minimum	0.072	0.036	0.036
	median	0.103	0.134	0.133
	mean	0.111	0.178	0.172
	maximum	0.143	1.190	1.190

Table 3: Summary statistics on banks' initial exposures, loans and securities as well as default probabilities. These are split into two categories: intermediaries (5 banks) and other banks (51 banks).

volume. The figure highlights that the intermediaries are banks with medium initial exposures and low default probabilities. We will provide statistical support for this statement in the next section. Most of the traded CDS volume is either between two intermediaries or between an intermediary and a non-intermediary. The volume of traded CDS contracts between intermediaries is high, but each intermediary has large trading positions with only a few, and not all, other intermediaries. There is high heterogeneity in the volume of traded CDS contracts for the banks that are not intermediaries: some banks (mainly those with very small initial exposures) trade a very small volume of CDSs, while others are either big buyers or big sellers of CDSs.

6.3 Testable Implications

We examine the following empirical implications (I1–I4) related to our study:

- I1 The main intermediaries have medium initial exposure (see item 1 of Proposition 4.7).
- I2 The banks that provide higher intermediation have lower default probabilities (see item 2 of Proposition 4.7).
- I3 The banks’ order in post-trade exposures is the same as that in their initial exposures, provided that their default probabilities are ordered in the opposite direction (see item 1 of Proposition 4.4).
- I4 Post-trade exposures are closer together than initial exposures if the bank with larger initial exposure is at least as risky as the bank with smaller initial exposure (see item 2 of Proposition 4.4).

Implications I1 and I2

Before testing I1 and I2, we provide in Figure 7 a visual overview of the relation between initial exposures, default probabilities and intermediation volumes. It is evident from Figure 7 that the main intermediaries have medium initial exposures and low default probabilities, relative to all banks in the data set.

To provide statistical support for implication I1, we test if intermediaries have medium initial exposures. This test requires us to define a medium value of initial exposures and a measure for the deviation from such a medium value. We set the medium value of initial exposures equal to the mean, over all banks, of their initial exposure weighted by their notional gross CDS volume. The weighting accounts for the relative participation of the banks in the CDS trading activity. We measure the deviations from this weighted mean by taking squared differences. The null hypothesis is that these squared differences of initial exposures for the banks in the two groups (main intermediaries and other banks) come from two populations with equal means. We perform a Welch’s t-test. The results in Table 4 show clear rejection of the null hypothesis. Hence, this confirms statistically that the intermediaries are banks with medium initial exposures.

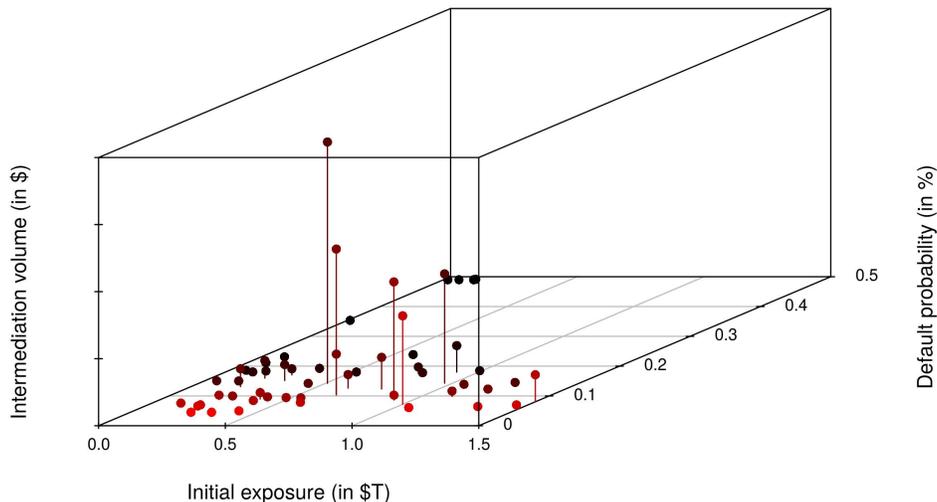


Figure 7: Intermediation volume as a function of initial exposures, measured in trillion USD (\$ T), and of default probabilities: each point denotes a bank. To increase visibility, banks with higher default probability correspond to darker points.

	deg. of freedom	test statistic	p-value	95 %-conf. int.
abs. initial exp. (in B\$)	48.548	-7.428	7.6×10^{-10}	$(-\infty, -162)$
rank-based initial exp.	52.793	-6.210	4.3×10^{-8}	$(-\infty, -290)$

Table 4: Results of a one-sided Welch’s t-test used to test the null hypothesis that the initial exposures of the intermediaries are not more centered than those of the other banks.

We next test implication I2 that the default probabilities are lower for banks with high intermediation volume. We perform a Welch’s t-test to determine if the difference in default probabilities between the two groups is statistically significant. With p-values below 0.05, the results for one-sided tests shown in Table 5 reject on the 5 percent level the null hypothesis that the samples of the two groups come from two populations with equal means.

	degrees of freedom	test statistic	p-value	95 %-conf. int.
all data	40.286	-2.305	0.0132	$[-100, -0.018)$
excl. one outlier*	18.869	-2.147	0.0225	$[-100, -0.001)$

Table 5: Results of a one-sided Welch’s t-test, used to test the null hypothesis that the mean of the intermediaries’ default probabilities is equal to that of the other banks in the market. The confidence intervals describe percentage changes in default probabilities. The result is significant at the 5%-level even when an outlier (bank with much higher default probability) that lowers the p-value is excluded from the analysis.

*There is one bank with much higher default probability, which is not displayed in Figure 7.

Implications I3 and I4

We first test implication I3 about the banks' order in post-trade exposures. We consider all pairs (i, j) of banks such that bank i has an equal or bigger initial exposure and an equal or lower default risk than bank j . The total number of such pairs is 988. We then assign 1 to the pair if bank i has an equal or bigger post-trade exposure than j , and 0 otherwise. This procedure yields a sequence of zeros and ones. Using our data set we find that the total value (sum of all entries) is 977, which gives strong statistical support for implication I3. Indeed, the relation holds in 99 percent of all pairs.

We proceed similarly to test implication I4. We consider all pairs (i, j) of banks such that bank i has an equal or bigger initial exposure and an equal or bigger default risk than bank j . The total number of such pairs is 573. We again get a sequence of zeros and ones by assigning 1 if the difference between the initial exposures of banks i and j is greater than the difference in their post-trade exposures. Under the null hypothesis that the relation between initial exposures and default probabilities of the two banks has no effect on the relation between the differences in initial and post-trade exposures, we expect to observe on average an equal number of zero and one entries—hence, a value of 286.5. Using our data set we find a value of 372, which gives strong statistical evidence for implication I4. For instance, assuming that the 0-1 observations are independent, the test statistics would follow a binomial distribution and the p-value would be 2.2×10^{-13} . We expect that this large value of 372 is unlikely to be observed under the null hypothesis even if we impose a dependence structure.

7 Conclusion

How do participants of OTC markets account for counterparty risk when they negotiate prices and quantities of traded contracts? Do they manage their own default risk to be more attractive trading counterparties? Do market participants diversify counterparty risk in OTC markets? If so, how can they achieve this? Answering these questions is of critical importance for the development of policies aiming for financial stability.

In this paper, we study the incentives behind the choices of banks' default probabilities and the role played by counterparty risk in shaping the structure of OTC markets. Our results show that banks may choose to reduce their default probabilities below what is socially optimal in order to benefit from higher fees when selling CDS protection. These decisions depend on the banks' initial exposures to an aggregate risk factor and on their bargaining power as sellers, relative to the buyer counterparty. Heterogeneity in trading decisions arises endogenously from banks' incentives to share their common risk exposure, taking counterparty risk into consideration. Intermediaries contribute to social welfare by diversifying counterparty risk and circumventing trade size limits. Our model produces several testable implications: the main intermediaries have medium initial exposure and low default risk. Banks share their exposures to the aggregate risk factor to a greater extent if they have low default risk. These empirical predictions produced by our model are strongly supported by CDS data.

Our framework can be extended along several directions. A first extension is to construct a model that can capture the dynamic formation of interbank trading relations, taking counterparty risk into consideration. Secondly, it would be desirable to include a role for the real economy. In such a model extensions banks might have obligations to the private sector and, additionally, fees paid and received as a result of a CDS trade affect the lending activities of the banks to the real economy. A third interesting extension is to compare trading decisions when market participants have the choice between bilateral OTC trading, exposing them to counterparty risk, and centralized trading. In the latter case, the clearinghouse insulates them from counterparty risk, but they are required to pay clearing fees.

A Results and their Proofs

This section contains the proofs of our results done for arbitrary sizes s_i of the banks. When the result formulation is different from that in the case $s_i = 1$, we restate the result.

A.1 Proof of Lemma 3.1

Using $P[D = 1] = q$, we compute

$$\begin{aligned} \Gamma^i(y_1, \dots, y_M) &= \frac{1}{\eta} \log E \left[\exp \left(\eta D \left(w_i + \sum_{n \neq i} y_n \left((\mathbb{1}_{A_i^c} + r \mathbb{1}_{A_i}) \mathbb{1}_{y_n > 0} + (\mathbb{1}_{A_n^c} + r \mathbb{1}_{A_n}) \mathbb{1}_{y_n < 0} \right) \right) \right) \right] \\ &= \frac{1}{\eta} \log \left(1 - q + q E \left[\exp \left(\eta w_i + \eta \sum_{n \neq i} y_n \left((\mathbb{1}_{A_i^c} + r \mathbb{1}_{A_i}) \mathbb{1}_{y_n > 0} + (\mathbb{1}_{A_n^c} + r \mathbb{1}_{A_n}) \mathbb{1}_{y_n < 0} \right) \right) \right] \right) \\ &= \frac{1}{\eta} \log \left(1 - q + q e^{\eta w_i} E \left[\exp \left(\eta (\mathbb{1}_{A_i^c} + r \mathbb{1}_{A_i}) \sum_{n \neq i: y_n > 0} y_n \right) \right] \prod_{n \neq i: y_n < 0} E \left[\exp \left(\eta (\mathbb{1}_{A_n^c} + r \mathbb{1}_{A_n}) y_n \right) \right] \right). \end{aligned}$$

Using that $p_i = P[A_i]$, we obtain

$$\begin{aligned} \Gamma^i(y_1, \dots, y_M) &= \frac{1}{\eta} \log \left(1 - q + q e^{\eta w_i} \left((1 - p_i) e^{\eta \sum_{n: y_n > 0} y_n} + p_i e^{\eta r \sum_{n: y_n > 0} y_n} \right) \right. \\ &\quad \left. \times \prod_{n \neq i: y_n < 0} \left((1 - p_n) e^{\eta y_n} + p_n e^{\eta r y_n} \right) \right), \end{aligned}$$

which can be brought into the form $\Gamma^i(y_1, \dots, y_M)$ written in the statement of Lemma 3.1.

To show the additional properties of $\Gamma^i(y_1, \dots, y_M)$, we first note that the function Ξ given by

$$\Xi(y) = \frac{1}{\eta} \log (1 - q + q e^{\eta y}) \tag{15}$$

is strictly increasing and strictly convex. Indeed, we can calculate

$$\Xi'(y) = \frac{q e^{\eta y}}{1 - q + q e^{\eta y}} > 0, \quad \Xi''(y) = \frac{(1 - q) q \eta e^{\eta y}}{(1 - q + q e^{\eta y})^2} > 0.$$

Next, we consider

$$f(y, p) = \frac{1}{\eta} \log((1-p)e^{\eta y} + pe^{\eta r y})$$

for $p > 0$ and calculate

$$f_y(y, p) = \frac{(1-p)e^{\eta y} + rpe^{\eta r y}}{(1-p)e^{\eta y} + pe^{\eta r y}} > 0, \quad (16)$$

$$\begin{aligned} f_{yy}(y, p) &= \eta \frac{((1-p)e^{\eta y} + pe^{\eta r y})((1-p)e^{\eta y} + r^2pe^{\eta r y}) - ((1-p)e^{\eta y} + rpe^{\eta r y})^2}{((1-p)e^{\eta y} + pe^{\eta r y})^2} \\ &= \eta \frac{p(1-p)(1-r)^2e^{\eta(1+r)y}}{((1-p)e^{\eta y} + pe^{\eta r y})^2} > 0. \end{aligned} \quad (17)$$

These inequalities show that the function $y \mapsto f(y, p)$ is strictly increasing and strictly convex for $p > 0$. Because $f(y, p)$ either equals y (if $p = 0$) or is strictly increasing and strictly convex (if $p > 0$), we see that $\Gamma^i(y_1, \dots, y_M)$ is strictly increasing, and the statements on convexity of $\Gamma^i(y_1, \dots, y_M)$ now follow from the fact that convexity is maintained under sums and compositions with a convex, nondecreasing function.

Finally, to prove (3), let $y_1 < y_2$, $y_3 \in (0, \frac{y_2 - y_1}{2}]$ and $p_1 \geq p_2$. We first note that (3) is equivalent to

$$\begin{aligned} &((1-p_1)e^{\eta y_1} + p_1e^{\eta r y_1})((1-p_2)e^{\eta y_2} + p_2e^{\eta r y_2}) \\ &> ((1-p_1)e^{\eta(y_1+y_3)} + p_1e^{\eta r(y_1+y_3)})((1-p_2)e^{\eta(y_2-y_3)} + p_2e^{\eta r(y_2-y_3)}), \end{aligned}$$

which can be further simplified to

$$(1-p_1)p_2e^{\eta(y_1+ry_2)} + (1-p_2)p_1e^{\eta(y_2+ry_1)} > (1-p_1)p_2e^{\eta(y_1+ry_2+y_3(1-r))} + (1-p_2)p_1e^{\eta(y_2+ry_1-y_3(1-r))}.$$

This inequality follows from

$$ae^{x_1} + be^{x_2} > ae^{x_1+x_3} + be^{x_2-x_3} \quad (18)$$

for all $a \leq b$, $x_1 < x_2$ and $x_3 \in (0, \frac{x_2-x_1}{2}]$ by choosing

$$a = (1-p_1)p_2, \quad b = (1-p_2)p_1, \quad x_1 = \eta(y_1 + ry_2), \quad x_2 = \eta(y_2 + ry_1), \quad x_3 = \eta y_3(1-r),$$

where we note that $p_1 \geq p_2$, $y_1 < y_2$, and $y_3 \in (0, \frac{y_2-y_1}{2}]$ imply $a \leq b$, $x_1 < x_2$, and $x_3 \in (0, \frac{x_2-x_1}{2}]$. The inequality (18) can be seen from the convexity of the exponential function or checked directly by calculating the partial derivative

$$\frac{\partial}{\partial z}(ae^{x_1+z} + be^{x_2-z}) = ae^{x_1+z} - be^{x_2-z} \leq be^{x_1+z} - be^{x_2-z} < 0$$

for all $z \in [0, \frac{x_2-x_1}{2}]$. □

A.2 Results of Section 4.1 and their Proofs

Theorem A.1 (Theorem 4.2). *Feasible contracts $(\gamma_{i,n})_{i,n=1,\dots,M}$ are a market equilibrium if and only if they solve the optimization problem*

$$\text{minimize } \sum_{i=1}^M s_i \Gamma^i(\gamma_i s) \quad \text{over } \gamma \text{ subject to } \gamma_{i,n} = -\gamma_{n,i} \text{ and } -k \leq \gamma_{i,n} \leq k, \quad (19)$$

where $\gamma_i s := (\gamma_{i,1} s_1, \dots, \gamma_{i,M} s_M)$.

Proof. The Lagrangian function corresponding to (19) is

$$\sum_{i=1}^M s_i \Gamma^i(\gamma_i s) - \sum_{i,n=1}^M s_i s_n \alpha_{i,n} (\gamma_{i,n} + \gamma_{n,i}) - \sum_{i,n=1}^M s_i s_n \underline{\beta}_{i,n} (k - \gamma_{i,n}) - \sum_{i,n=1}^M s_i s_n \bar{\beta}_{i,n} (k + \gamma_{i,n}).$$

The optimality conditions are

$$\begin{aligned} \Gamma_{y_n}^i(\gamma_i s) &= \alpha_{i,n} + \alpha_{n,i} - \underline{\beta}_{i,n} + \bar{\beta}_{i,n}, & \underline{\beta}_{i,n} &\geq 0, & \bar{\beta}_{i,n} &\geq 0, \\ \underline{\beta}_{i,n}(k - \gamma_{i,n}) &= 0, & \bar{\beta}_{i,n}(k + \gamma_{i,n}) &= 0. \end{aligned} \quad (20)$$

All of them are satisfied for

$$\underline{\beta}_{n,i} = \bar{\beta}_{i,n} = \frac{1}{2} \max \{ \Gamma_{y_n}^i(\gamma_i s) - \Gamma_{y_i}^n(\gamma_n s), 0 \}, \quad \alpha_{i,n} + \alpha_{n,i} = \frac{1}{2} (\Gamma_{y_n}^i(\gamma_i s) + \Gamma_{y_i}^n(\gamma_n s))$$

if γ satisfies (5) and $\gamma_{i,n} = -\gamma_{n,i}$. This means that if γ is a market equilibrium, it is a solution to (19). Conversely, if γ is a solution to (19), then (20) implies

$$\Gamma_{y_n}^i(\gamma_i s)(k^2 - \gamma_{i,n}^2) = (\alpha_{i,n} + \alpha_{n,i})(k^2 - \gamma_{i,n}^2) = (\alpha_{n,i} + \alpha_{i,n})(k^2 - \gamma_{n,i}^2) = \Gamma_{y_i}^n(\gamma_n s)(k^2 - \gamma_{n,i}^2).$$

This equation shows that if $\gamma_{i,n} \neq \pm k$, we need $\Gamma_{y_n}^i(\gamma_i s) = \Gamma_{y_i}^n(\gamma_n s)$. In turn, $\Gamma_{y_n}^i(\gamma_i s) \neq \Gamma_{y_i}^n(\gamma_n s)$ implies $\gamma_{i,n} = \pm k$. Consider the case $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$ and assume $\gamma_{i,n} = -k$, then $\gamma_{n,i} = k$; it follows from (20) that $\underline{\beta}_{i,n} = 0$, $\bar{\beta}_{n,i} = 0$ and

$$\Gamma_{y_n}^i(\gamma_i s) = \alpha_{i,n} + \alpha_{n,i} + \bar{\beta}_{i,n} \geq \alpha_{i,n} + \alpha_{n,i} \geq \alpha_{n,i} + \alpha_{i,n} - \underline{\beta}_{i,n} = \Gamma_{y_i}^n(\gamma_n s),$$

which is a contradiction to $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$. Therefore, $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$ implies $\gamma_{i,n} = k$. By symmetry, $\Gamma_{y_n}^i(\gamma_i s) > \Gamma_{y_i}^n(\gamma_n s)$ implies $\gamma_{i,n} = -k$. This shows that a solution to (19) satisfies (5) and thus is a market equilibrium. \square

Theorem A.2 (Theorem 4.3). *There exists a market equilibrium $(\gamma_{i,n})_{i,n=1,\dots,M}$. The $\gamma_{i,n}$ are unique for $p_n > 0$ and $\gamma_{i,n} < 0$, or $p_i > 0$ and $\gamma_{i,n} > 0$. For every i , the value is the same for $\sum \gamma_{i,n} s_n$ where the sum is over n such that $p_n = 0$ and $\gamma_{i,n} < 0$, or $p_i = 0$ and $\gamma_{i,n} > 0$. In particular, $\Gamma(\gamma_n)$ are uniquely determined for a market equilibrium $(\gamma_{i,n})_{i,n=1,\dots,M}$.*

Proof. We prove first the existence of a market equilibrium. To this end, we will apply Kakutani's fixed-point theorem (see, for example, Corollary 15.3 in Border (1985)). Fix k , set $S = [-k, k]^{M(M-1)/2}$, and define a mapping $\Phi : S \rightarrow 2^S$ as follows, where 2^S denotes the power set of S , i.e., the set of all subsets of S . Each element in S corresponds to the lower triangular matrix of $(\gamma_{i,n})_{i,n=1,\dots,M}$, where we set the diagonal elements γ_{ii} equal to zero and the upper diagonal elements are defined by $\gamma_{i,n} = -\gamma_{n,i}$. Let $\Phi(\gamma)$ consist of all $(\tilde{\gamma}_{i,n})_{i,n=1,\dots,M}$ that satisfy $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$, $-k \leq \tilde{\gamma}_{i,n} \leq k$, and

$$\tilde{\gamma}_{i,n} \begin{cases} = k & \text{if } \Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s), \\ \in [-k, k] & \text{if } \Gamma_{y_n}^i(\gamma_i s) = \Gamma_{y_i}^n(\gamma_n s), \\ = -k & \text{if } \Gamma_{y_n}^i(\gamma_i s) > \Gamma_{y_i}^n(\gamma_n s). \end{cases}$$

Note that these “if” conditions depend on γ and not on $\tilde{\gamma}$. We can see that $\Phi(\gamma)$ is nonempty, compact and convex. To show that Φ has a closed graph, consider a sequence $(\gamma^{(m)}, \tilde{\gamma}^{(m)})$ converging to $(\gamma, \tilde{\gamma})$ with $\tilde{\gamma}^{(m)} \in \Phi(\gamma^{(m)})$ for all m . Because $\tilde{\gamma}^{(m)} \rightarrow \tilde{\gamma}$ and $\tilde{\gamma}^{(m)} \in \Phi(\gamma^{(m)})$, we have $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$ and $-k \leq \tilde{\gamma}_{i,n} \leq k$. Moreover, if $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$, we have $\Gamma_{y_n}^i(\gamma_i^{(m)} s) < \Gamma_{y_i}^n(\gamma_n^{(m)} s)$ for all m big enough, as $\gamma^{(m)} \rightarrow \gamma$. This yields $\tilde{\gamma}_{i,n}^{(m)} = k$ for all m big enough; hence, $\tilde{\gamma}_{i,n} = k$. Similarly, $\Gamma_{y_n}^i(\gamma_i s) > \Gamma_{y_i}^n(\gamma_n s)$ implies $\tilde{\gamma}_{i,n} = -k$. The condition is also satisfied for the last case $\Gamma_{y_n}^i(\gamma_i s) = \Gamma_{y_i}^n(\gamma_n s)$, as we have already shown $-k \leq \tilde{\gamma}_{i,n} \leq k$. Therefore, there exists γ with $\Phi(\gamma) = \gamma$ by Kakutani's fixed-point theorem; hence, there is a market equilibrium.

To prove uniqueness, we first apply Theorem 4.2, which says that finding a market equilibrium is equivalent to solving (19). We then write the objective function in (19) as

$$\sum_{i=1}^M s_i \Gamma^i(\gamma_i s) = \sum_{i=1}^M s_i \Xi \left(w_i + f \left(\sum_{n: \gamma_{i,n} s_n \geq 0} \gamma_{i,n} s_n, p_i \right) + \sum_{n: \gamma_{i,n} s_n < 0} f(\gamma_{i,n} s_n, p_n) \right),$$

where the function Ξ is given in Lemma 3.1. The uniqueness statements now follow from the statements on convexity in Lemma 3.1. \square

A.3 Results of Section 4.2 and their Proofs

Proposition A.3 (Proposition 4.4). *Assume that at least one of the following conditions holds:*

- (a) $p_i = p_j$, or
- (b) $p_i \leq 1/2$ and $\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell \geq s_i \max_{\ell} \gamma_{i,\ell}$, or
- (c) $p_j \leq 1/2$ and $\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell \geq s_j \max_{\ell} \gamma_{j,\ell}$.

We then have the following:

1. If $\omega_i \geq \omega_j$, $p_i \leq p_j$, and $s_i \leq s_j$, then $\Omega_i \geq \Omega_j$.
2. If $\omega_i > \omega_j$, $p_i \geq p_j$, and $s_i \geq s_j$, then $\omega_i - \omega_j > \Omega_i - \Omega_j$.

Proof. We first note that, for general sizes, the post-trade exposure is given by

$$\Omega_i := \omega_i + f\left(\sum_{n:\gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i\right) + \sum_{n:\gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n).$$

We split the proof in several steps, starting with some preparation.

Claim 1a. For two banks i and j , we have

$$\Omega_j > \Omega_i \implies \gamma_{j,i} < 0. \quad (\text{C1a})$$

Proof of Claim 1a. From Lemma 3.1, it follows that

$$\Gamma_{y_i}^j(\gamma_j s) = \begin{cases} \Xi'(\Omega_j) \eta f_y(\sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j) & \text{if } \gamma_{j,i} > 0, \\ \Xi'(\Omega_j) \eta f_y(\gamma_{j,i} s_i, p_i) & \text{if } \gamma_{j,i} < 0, \end{cases} \quad (21)$$

with an analogous expression for $\Gamma_{y_j}^i(\gamma_i s)$. If $\gamma_{j,i} > 0$ (and thus $\gamma_{i,j} < 0$), we obtain

$$\begin{aligned} \Gamma_{y_i}^j(\gamma_j s) &= \Xi'(\Omega_j) \eta f_y\left(\sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j\right) \\ &> \Xi'(\Omega_i) \eta f_y\left(\sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j\right) \\ &\geq \Xi'(\Omega_i) \eta f_y(\gamma_{i,j} s_j, p_j) \\ &= \Gamma_{y_j}^i(\gamma_i s) \end{aligned}$$

by strict convexity of Ξ and convexity of $f(\cdot, p_j)$ from Lemma 3.1. However, this implies $\gamma_{j,i} = -k$ by (5) in contradiction to the assumption $\gamma_{j,i} > 0$. Similarly, we obtain a contradiction for $\gamma_{j,i} = 0$, using Footnote 8, which concludes the proof of (C1a).

Claim 1b. For two banks i and j , we have

$$\Omega_j > \Omega_i \implies \gamma_{j,n} < \gamma_{i,n} \text{ or } \gamma_{j,n} = -k \text{ for all } n \text{ with } \Omega_n < \Omega_j. \quad (\text{C1b})$$

Proof of Claim 1b. We distinguish the following three cases:

- If $\Omega_n \in (\Omega_i, \Omega_j)$, we have $\gamma_{j,n} < 0$ and $\gamma_{i,n} > 0$ by (C1a) so that $\gamma_{j,n} < \gamma_{i,n}$ holds.
- If $\Omega_n < \Omega_i$, we have $\gamma_{j,n} < 0$ and $\gamma_{i,n} < 0$ by (C1a); thus,

$$\Gamma_{y_n}^j(\gamma_j s) = \Xi'(\Omega_j) \eta f_y(\gamma_{j,n} s_n, p_n), \quad (22)$$

$$\Gamma_{y_n}^i(\gamma_i s) = \Xi'(\Omega_i) \eta f_y(\gamma_{i,n} s_n, p_n), \quad (23)$$

$$\Gamma_{y_j}^n(\gamma_n s) = \Xi'(\Omega_n) \eta f_y\left(\sum_{\ell:\gamma_{n,\ell} \geq 0} \gamma_{n,\ell} s_\ell, p_n\right) = \Gamma_{y_i}^n(\gamma_n s). \quad (24)$$

Assume that $\gamma_{j,n} \neq -k$, which implies

$$\Gamma_{y_n}^j(\gamma_j s) = \Gamma_{y_j}^n(\gamma_n s) = \Gamma_{y_i}^n(\gamma_n s) \leq \Gamma_{y_n}^i(\gamma_i s)$$

by (5) and (24); thus,

$$1 < \frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \leq \frac{f_y(\gamma_{i,n} s_n, p_n)}{f_y(\gamma_{j,n} s_n, p_n)}$$

by (22) and (23). This is only possible if $\gamma_{j,n} < \gamma_{i,n}$.

- If $\Omega_n = \Omega_i$, we argue as in the first item if $\gamma_{i,n} \geq 0$, or as in the second item if $\gamma_{i,n} < 0$.

Note that (C1b) holds regardless of the default risks of banks i and j . This is because we are considering banks n with smaller post-trade exposures; thus, banks that are seller of protection by (C1a) so that the same counterparty risk p_n applies to trades with i and j .

Claim 1c. For two banks i and j , we have

$$\Omega_j > \Omega_i \implies \frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)} < \frac{f_y(\gamma_{n,j} s_j, p_j)}{f_y(\gamma_{n,i} s_i, p_i)} \text{ or } \gamma_{n,i} = -k \text{ for all } n \text{ with } \Omega_n > \Omega_j. \quad (\text{C1c})$$

Proof of Claim 1c. $\Omega_n > \Omega_j$ implies $\gamma_{j,n} > 0$ by (C1a), and thus $\Gamma_{y_n}^j(\gamma_j s) \leq \Gamma_{y_j}^n(\gamma_n s)$. If $\gamma_{n,i} \neq -k$, it follows that $\Gamma_{y_n}^i(\gamma_i s) \geq \Gamma_{y_i}^n(\gamma_n s)$; hence,

$$\begin{aligned} \Gamma_{y_j}^n(\gamma_n s) &\geq \Gamma_{y_n}^j(\gamma_j s) = \Xi'(\Omega_j) \eta f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right) > \Xi'(\Omega_i) \eta f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right) \\ &= \Gamma_{y_n}^i(\gamma_i s) \frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)} \geq \Gamma_{y_i}^n(\gamma_n s) \frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}, \end{aligned}$$

which shows (C1c), as $\Gamma_{y_i}^n(\gamma_n s) = \Xi'(\Omega_n) \eta f_y(\gamma_{n,i} s_i, p_i)$ and $\Gamma_{y_j}^n(\gamma_n s) = \Xi'(\Omega_n) \eta f_y(\gamma_{n,j} s_j, p_j)$.

Claim 1d. For three banks i , j , and n , we have

$$\Omega_i < \Omega_j = \Omega_n \implies \gamma_{j,n} \leq \gamma_{i,n} \text{ or (C1c) holds.} \quad (\text{C1d})$$

Proof of Claim 1d. If $\gamma_{j,n} \leq 0$, we obtain $\gamma_{j,n} \leq \gamma_{i,n}$, as $\gamma_{i,n} > 0$ by (C1a). If $\gamma_{j,n} > 0$, we can argue as (C1c).

We can summarize (C1a)–(C1d) as

$$\Omega_j > \Omega_i \implies \begin{cases} \gamma_{j,n} \leq \gamma_{i,n} & \text{for all } \gamma_{j,n} \leq 0, \\ (\text{C1c) holds} & \text{for all } \gamma_{j,n} > 0. \end{cases} \quad (\text{C1})$$

Claim 2. For two banks i and j , we have

$$\omega_i \geq \omega_j, p_j \geq p_i, s_j \geq s_i, \text{ and (a), (b) or (c) of the proposition holds} \implies \Omega_i \geq \Omega_j. \quad (\text{C2})$$

Proof of Claim 2. We prove the claim by contradiction and assume that $\Omega_i < \Omega_j$. This implies $\gamma_{j,n} \leq \gamma_{i,n}$ for all $\gamma_{j,n} \leq 0$ by (C1); hence,

$$f\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right) = \Omega_j - \omega_j - \sum_{n: \gamma_{j,n} < 0} f(\gamma_{j,n} s_n, p_n)$$

$$\begin{aligned}
&> \Omega_i - \omega_i - \sum_{n:\gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n) \\
&= f\left(\sum_{\ell:\gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right) \\
&\geq f\left(\sum_{\ell:\gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_j\right),
\end{aligned}$$

using (8), $p_j \geq p_i$, and that $f(y, p)$ is decreasing in p for $y \geq 0$ because, using the definition (2),

$$f_p(y, p) = \frac{\partial}{\partial p} \frac{1}{\eta} \log((1-p)e^{\eta y} + pe^{\eta r y}) = \frac{-e^{\eta y} + e^{\eta r y}}{\eta((1-p)e^{\eta y} + pe^{\eta r y})} < 0 \text{ for } y \geq 0. \quad (25)$$

This yields $\sum_{\ell:\gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell > \sum_{\ell:\gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell$, as $y \mapsto f(y, p_j)$ is strictly increasing by Lemma 3.1. This implies that there exists n with $\gamma_{j,n} > \gamma_{i,n} \geq 0$; thus,

$$\gamma_{n,j} < \gamma_{n,i} \leq 0 \text{ and } \gamma_{n,j} s_j < \gamma_{n,i} s_i \quad (26)$$

because $s_j \geq s_i$ by assumption. Moreover, $\gamma_{j,n} > 0$ implies $\Omega_n \geq \Omega_j$ by (C1a). On the other hand, $\Omega_i < \Omega_j$ implies by (C1c) and (C1d) that $\gamma_{n,i} = -k$ (which stands in contradiction to (26) because $\gamma_{n,j} \geq -k$) or $\gamma_{j,n} \leq \gamma_{i,n}$ (also a contradiction to (26)) or

$$\frac{f_y\left(\sum_{\ell:\gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,i} s_i, p_i)} > \frac{f_y\left(\sum_{\ell:\gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,j} s_j, p_j)}. \quad (27)$$

We will show that (27) contradicts

$$p_j \geq p_i, \quad \sum_{\ell:\gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell > \sum_{\ell:\gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell \text{ and } \gamma_{n,j} s_j < \gamma_{n,i} s_i \quad (28)$$

if one of the conditions (a)–(c) of the proposition holds.

As an auxiliary step, we next analyze the function $p \mapsto \frac{f_y(y_1, p)}{f_y(y_2, p)}$ and show that

$$\frac{\partial}{\partial p} \frac{f_y(y_1, p)}{f_y(y_2, p)} \geq 0 \text{ for all } p \in [0, 1/2] \text{ and } y_1 \geq -y_2 \geq 0. \quad (29)$$

Indeed, we use (16) and

$$\begin{aligned}
f_{yp}(y, p) &= \frac{\partial}{\partial p} \frac{(1-p)e^{\eta y} + rpe^{\eta r y}}{(1-p)e^{\eta y} + pe^{\eta r y}} \\
&= \frac{((1-p)e^{\eta y} + pe^{\eta r y})(-e^{\eta y} + re^{\eta r y}) - ((1-p)e^{\eta y} + rpe^{\eta r y})(-e^{\eta y} + e^{\eta r y})}{((1-p)e^{\eta y} + pe^{\eta r y})^2} \\
&= \frac{(r-1)e^{\eta(1+r)y}}{((1-p)e^{\eta y} + pe^{\eta r y})^2}
\end{aligned}$$

to deduce that

$$\begin{aligned}
\frac{\partial f_y(y_1, p)}{\partial p f_y(y_2, p)} &= \frac{f_y(y_2, p)f_{yp}(y_1, p) - f_{yp}(y_2, p)f_y(y_1, p)}{(f_y(y_2, p))^2} \\
&= \frac{\frac{(1-p)e^{\eta y_2} + rpe^{\eta y_2}}{(1-p)e^{\eta y_2} + pe^{\eta y_2}} \frac{(r-1)e^{\eta(1+r)y_1}}{((1-p)e^{\eta y_1} + pe^{\eta y_1})^2} - \frac{(r-1)e^{\eta(1+r)y_2}}{((1-p)e^{\eta y_2} + pe^{\eta y_2})^2} \frac{(1-p)e^{\eta y_1} + rpe^{\eta y_1}}{(1-p)e^{\eta y_1} + pe^{\eta y_1}}}{(f_y(y_2, p))^2} \\
&= \frac{(r-1)e^{\eta(1+r)y_1} \left((1-p)e^{\eta y_2} + rpe^{\eta y_2} \right) \left((1-p)e^{\eta y_2} + pe^{\eta y_2} \right)}{\left((1-p)e^{\eta y_1} + pe^{\eta y_1} \right)^2 \left((1-p)e^{\eta y_2} + pe^{\eta y_2} \right)^2 (f_y(y_2, p))^2} \\
&\quad - \frac{(r-1)e^{\eta(1+r)y_2} \left((1-p)e^{\eta y_1} + rpe^{\eta y_1} \right) \left((1-p)e^{\eta y_1} + pe^{\eta y_1} \right)}{\left((1-p)e^{\eta y_1} + pe^{\eta y_1} \right)^2 \left((1-p)e^{\eta y_2} + pe^{\eta y_2} \right)^2 (f_y(y_2, p))^2} \\
&= \frac{(1-r)e^{\eta(1+r)(y_1+y_2)}}{\left((1-p)e^{\eta y_1} + pe^{\eta y_1} \right)^2 \left((1-p)e^{\eta y_2} + pe^{\eta y_2} \right)^2 (f_y(y_2, p))^2} \\
&\quad \times \left(\left((1-p)e^{\eta(1-r)y_1} + rp \right) \left(1-p + pe^{-\eta(1-r)y_1} \right) \right. \\
&\quad \left. - \left((1-p)e^{\eta(1-r)y_2} + rp \right) \left(1-p + pe^{-\eta(1-r)y_2} \right) \right).
\end{aligned}$$

From this, we obtain $\frac{\partial f_y(y_1, p)}{\partial p f_y(y_2, p)} \geq 0$ because

$$\begin{aligned}
&\left((1-p)e^{\eta(1-r)y_1} + rp \right) \left(1-p + pe^{-\eta(1-r)y_1} \right) - \left((1-p)e^{\eta(1-r)y_2} + rp \right) \left(1-p + pe^{-\eta(1-r)y_2} \right) \\
&= (1-p)^2 e^{\eta(1-r)y_1} + rp^2 e^{-\eta(1-r)y_1} - (1-p)^2 e^{\eta(1-r)y_2} - rp^2 e^{-\eta(1-r)y_2} \\
&\geq rp^2 \left(e^{\eta(1-r)y_1} + e^{-\eta(1-r)y_1} - e^{\eta(1-r)y_2} - e^{-\eta(1-r)y_2} \right) \\
&\geq 0,
\end{aligned}$$

using $(1-p)^2 \geq rp^2$ for $p \leq 1/2$ and $y_1 \geq y_2$ for the second last inequality, and $e^{\eta(1-r)y_1} + e^{-\eta(1-r)y_1} \geq e^{\eta(1-r)y_2} + e^{-\eta(1-r)y_2}$ for $|y_1| \geq |y_2|$ for the last inequality. This concludes the proof of (29).

We now consider each of the three conditions (a)–(c) of the proposition.

Condition (a). From (28), we deduce

$$\frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,j} s_j, p_j)} \geq \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,i} s_i, p_j)} = \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,i} s_i, p_i)},$$

using the convexity of $y \mapsto f(y, p_j)$ by Lemma 3.1 and $p_i = p_j$ in condition (a).

Condition (b). We apply (29) choosing $p = p_i$, $y_1 = \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell$, and $y_2 = \gamma_{n,i} s_i$. This implies

$$\frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,i} s_i, p_i)} \leq \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,i} s_i, p_j)} \leq \frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,j} s_j, p_j)},$$

where we use (28) and the convexity of $y \mapsto f(y, p_j)$ for the second inequality.

Condition (c). This time, we apply (29) choosing $p = p_j$, $y_1 = \sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell$, and $y_2 = \gamma_{n,j} s_j$. We obtain

$$\frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,j} s_j, p_j)} \geq \frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,j} s_j, p_i)} \geq \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,i} s_i, p_i)},$$

where we again use (28) and the convexity of $y \mapsto f(y, p_i)$ for the second inequality.

Under each of the three conditions (a)–(c), we obtain a contradiction to (27). Hence, $\Omega_i < \Omega_j$ cannot hold, which concludes the proof of (C2).

Claim 3. For two banks i and j , we have

$$\omega_i > \omega_j, p_j \leq p_i, s_j \leq s_i \implies \Omega_i - \omega_i < \Omega_j - \omega_j.$$

Proof of Claim 3. We proceed similarly to the proof of (C2). We prove the claim by contradiction and assume that $\Omega_i - \omega_i \geq \Omega_j - \omega_j$. This implies $\Omega_i > \Omega_j$; hence, $\gamma_{i,n} \leq \gamma_{j,n}$ for all $\gamma_{i,n} \leq 0$ by (C1) and $\gamma_{i,j} < 0 < \gamma_{j,i}$ by (C1a), and thus

$$\begin{aligned} f\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right) &= \Omega_j - \omega_j - \sum_{n: \gamma_{j,n} < 0} f(\gamma_{j,n} s_n, p_n) \\ &< \Omega_i - \omega_i - \sum_{n: \gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n) \\ &= f\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right) \\ &\leq f\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_j\right) \end{aligned} \tag{30}$$

using $p_j \leq p_i$ and (25), which yields $\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell < \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell$ because $y \mapsto f(y, p_j)$ is strictly increasing by Lemma 3.1. We conclude the proof in the same way as the proof of (C2) after (26), with i and j interchanged. \square

Proposition A.4 (Proposition 4.5). *Assume that there are at least two safe banks and set*

$$A(\alpha) = \{i : \omega_i \geq \alpha(s_i) \text{ or } p_i = 0\}$$

for a function $\alpha : [0, 1] \rightarrow [0, \infty)$. There exist $C \in \left[\frac{1}{M} \sum_{j=1}^M \omega_j, \frac{1}{\#\{i: p_i=0\}} \sum_{j=1}^M \omega_j\right]$, $\bar{k} > 0$, and a function $\bar{\alpha} : [0, 1] \rightarrow [0, \infty)$ such that

- for all $k \geq \bar{k}$, $\Omega_i = C > \Omega_\ell$ for all $i \in A(\bar{\alpha})$ and $\ell \notin A(\bar{\alpha})$;
- for all $k < \bar{k}$, there exist i and j with $p_j = 0$ such that $\Omega_i > \Omega_j$.

Proof. We define \bar{k}_1 by

$$\bar{k}_1 = \inf \{k > 0 : \Omega_i = \Omega_j \text{ for all } i, j \text{ with } p_i = p_j = 0\}. \tag{31}$$

We can prove that $0 < \bar{k}_1 < \infty$ and that the infimum in (31) is attained along the same lines as on page 2273 of Atkeson et al. (2015), restricting their arguments to the safe banks. We choose \bar{k} as the smallest number $k \geq \bar{k}_1$ such that

$$\Omega_i \leq \Omega_j \quad (32)$$

for all i, j with $p_i > 0$ and $p_j = 0$. We next show that \bar{k} is well defined. If (32) holds for $k = \bar{k}_1$, we set $\bar{k} = \bar{k}_1$. Moreover, (32) always holds for k big enough. Indeed, let i be with $p_i > 0$ and, working towards a contradiction, assume that

$$\Omega_i > \Omega_j \quad (33)$$

for some j with $p_j = 0$. From (C1a) and (C1) in the proof of Proposition 4.4 with $p_j = 0$, it follows that $\gamma_{i,j} < 0$ and $\gamma_{i,n} \leq \gamma_{j,n}$ for all n ; hence,

$$\begin{aligned} \Gamma_{y_i}^j(\gamma_j s) &= \Xi'(\Omega_j) \eta f_y \left(\sum_{n: \gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j \right) = \Xi'(\Omega_j) \eta \\ &< \Xi'(\Omega_i) \eta = \Xi'(\Omega_i) \eta f_y(\gamma_{i,j} s_j, p_j) = \Gamma_{y_j}^i(\gamma_i s) \end{aligned}$$

using that $f_y(y, p_j) = 1$ because $p_j = 0$, Ξ is strictly increasing and $\Omega_i > \Omega_j$. Then $\gamma_{i,j} = -k$ follows from $\Gamma_{y_i}^j(\gamma_j s) < \Gamma_{y_j}^i(\gamma_i s)$ by (5), and thus

$$\begin{aligned} \Omega_j &= \omega_j + f \left(\sum_{n: \gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j \right) + \sum_{n: \gamma_{j,n} < 0} f(\gamma_{j,n} s_n, p_n) \\ &\geq k s_i + \omega_j + f \left(\sum_{n: \gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i \right) + \sum_{n: \gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n) \\ &= k s_i + \omega_j - \omega_i + \Omega_i. \end{aligned}$$

However, for $k \geq (\omega_i - \omega_j)/s_i$, this gives $\Omega_j \geq \Omega_i$ in contradiction to (33). Hence, we have (32) for k big enough. By a compactness argument similar to page 2273 of Atkeson et al. (2015), we deduce that (32) holds for $k = \bar{k}$. By definition of \bar{k} , for $k < \bar{k}$, there exist i and j with $p_j = 0$ such that $\Omega_i > \Omega_j$, which shows the second item of the proposition.

To finish the proof of the first item of the proposition, we consider $k \geq \bar{k}$ and

$$\begin{aligned} \beta(p, s) &= \max_{i: p_i = p, s_i = s} \Omega_i, \quad \bar{i}(p, s) = \begin{cases} \arg \max_{i: p_i = p, s_i = s} \Omega_i & \text{if } \beta(p, s) = \Omega_j \text{ for } j \text{ with } p_j = 0, \\ \emptyset & \text{otherwise.} \end{cases} \\ \bar{\delta}(p, s) &= \min_{i \in \bar{i}(p, s)} w_i, \quad \underline{\delta}(p, s) = \max_{\{i: p_i = p, s_i = s\} \setminus \bar{i}(p, s)} w_i \end{aligned}$$

for $p \in \{p_1, \dots, p_M\}$ and $s \in \{s_1, \dots, s_M\}$ where the minimum (and maximum) over an empty set equals $+\infty$ and $-\infty$ by the usual convention. Several p_j and s_j for different j can take the same values, and thus $\bar{i}(p, s)$ can be a set with several entries because the maximum does not need to be attained at a unique i . We can choose a function $\bar{\alpha} : (0, 1] \times [0, 1] \rightarrow [0, \infty)$ for all s such that $\underline{\delta}(p, s) < \bar{\alpha}(p, s) \leq \bar{\delta}(p, s)$ for all $p \in \{p_1, \dots, p_M\}$ and $s \in \{s_1, \dots, s_M\}$.

Note that $\bar{\alpha}(p, s)$ may depend here on both arguments p and s , but in the next paragraph, we will show that $\bar{\alpha}$ can be chosen independently of p . From $\bar{\alpha}(p, s) \leq \bar{\delta}(p, s)$, it follows that $A(\bar{\alpha})$ defined by

$$A(\bar{\alpha}) = \{i : \omega_i \geq \bar{\alpha}(s_i, p_i) \text{ or } p_i = 0\}$$

contains all i with $\Omega_i = \Omega_j$ for j with $p_j = 0$. To show that $A(\bar{\alpha})$ contains only such i , assume that there exists $i \in A(\bar{\alpha})$ with $\Omega_i < \Omega_j$ for j with $p_j = 0$. This implies

$$\omega_i \geq \bar{\alpha}(p_i, s_i) > \underline{\delta}(p_i, s_i);$$

hence, $\omega_i > \omega_\ell$ for all ω_ℓ with $\Omega_\ell < \Omega_j$, which contradicts $\Omega_i < \Omega_j$.

To show that $\bar{\alpha}$ can be chosen independently of p , consider $k \geq \bar{k}$ and i with $p_i > 0$ and $\Omega_i = \Omega_j$ for j with $p_j = 0$. Because of $k \geq \bar{k}$, we have $\Omega_i \geq \Omega_\ell$ for all ℓ , using (32). In the case $\Omega_i > \Omega_\ell$, we obtain $\gamma_{i,\ell} < 0$ by (C1a). In the case $\Omega_i = \Omega_\ell$, we argue similarly to the proof of (C1a) to show $\gamma_{i,\ell} \leq 0$. Indeed, to derive a contradiction, we assume that $\gamma_{i,\ell} > 0$ and $\Omega_i = \Omega_\ell$, which implies

$$\begin{aligned} \Gamma_{y_\ell}^i(\gamma_i s) &= \Xi'(\Omega_i) \eta f_y \left(\sum_{n: \gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i \right) \\ &= \Xi'(\Omega_\ell) \eta f_y \left(\sum_{n: \gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i \right) \\ &> \Xi'(\Omega_\ell) \eta f_y(\gamma_{\ell,i} s_i, p_i) \\ &= \Gamma_{y_i}^\ell(\gamma_\ell s) \end{aligned}$$

by strict convexity of $f(\cdot, p_i)$ from Lemma 3.1, using that $p_i > 0$. However, this implies $\gamma_{i,\ell} = -k$ by (5) in contradiction to the assumption $\gamma_{i,\ell} > 0$. Hence, we have $\gamma_{i,\ell} \leq 0$, and p_i does not matter for the trading of bank i . Indeed, Lemma 3.1 shows then that, for all ℓ , $\Gamma^\ell(\gamma_\ell s)$ does not depend on p_i if $\gamma_{i,\ell} \leq 0$, and thus the objective function $\sum_{\ell=1}^M s_\ell \Gamma^\ell(\gamma_\ell s)$ in (19) does not depend on p_i in the optimum. Therefore, $\bar{\alpha}$ can be chosen independently of p .

The constant C in Proposition 4.5 takes the value $C = \Omega_j$ for any $j \in A(\bar{\alpha})$; hence, $C = \frac{1}{\#A(\bar{\alpha})} \sum_{j \in A(\bar{\alpha})} \Omega_j$. Because post-trade exposures are nonnegative and the safe banks belong to $A(\bar{\alpha})$, we obtain

$$C \leq \frac{1}{\#A(\bar{\alpha})} \sum_{j=1}^M \Omega_j = \frac{1}{\#A(\bar{\alpha})} \sum_{j=1}^M \omega_j \leq \frac{1}{\#\{i : p_i = 0\}} \sum_{j=1}^M \omega_j,$$

where the equality is obtained from the market-clearing condition. Moreover, we have

$$C \geq \frac{1}{M} \sum_{j=1}^M \Omega_j = \frac{1}{M} \sum_{j=1}^M \omega_j$$

because the banks that are not in $A(\bar{\alpha})$ have a lower post-trade exposure because no bank has a higher post-trade exposure than the banks in $A(\bar{\alpha})$ by construction of $A(\bar{\alpha})$. \square

A.4 Proposition 4.7 and its Proof

For general sizes s_i , the per-capita gross numbers of sold or purchased contracts are given by

$$G_i^+ = f\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n}s_n, p_i\right) \text{ and } G_i^- = -\sum_{n:\gamma_{i,n}< 0} f(\gamma_{i,n}s_n, p_n).$$

Proposition A.5 (Proposition 4.7). *1. If the trade size limit k is small enough and there are at least three banks with different initial exposures ω_i , then the intermediation volume I_i as a function of ω_i is a hump-shaped curve, taking its maximum at or next to the median initial exposure weighted by size and counterparty risk.*

2. Assume that at least one of the conditions (a)–(c) of Proposition A.3 holds. If two banks i and j have the same initial exposure, then $I_i \leq I_j$ for $p_i \geq p_j$ and $s_i \geq s_j$.

Proof. 1. If k is small enough, then $\omega_i < \omega_j$ implies $\Omega_i < \Omega_j$ and $\gamma_{i,n}$ equals $\pm k$ for all i and n with $\omega_i \neq \omega_n$; hence,

$$\gamma_{i,n} \begin{cases} = k & \omega_i < \omega_n \\ = -k & \omega_i > \omega_n \\ \in [k, -k] & \omega_i = \omega_n \end{cases}$$

by (C1a) and (5). This yields

$$G_i^+ = f\left(k \sum_{n:\omega_n > \omega_i} s_n + \sum_{\substack{n:\omega_n = \omega_i, \\ \gamma_{i,n} > 0}} s_n \gamma_{i,n}, p_i\right),$$

$$G_i^- = -\sum_{n:\omega_n < \omega_i} f(-k s_n, p_n) - \sum_{\substack{n:\omega_n = \omega_i, \\ \gamma_{i,n} < 0}} f(s_n \gamma_{i,n}, p_n).$$

As a function of ω_i , G_i^+ is decreasing with zero at the largest value of ω_i and G_i^- is increasing with zero at the smallest value of ω_i so that $I_i = \min\{G_i^+, G_i^-\}$ is a hump-shaped curve. If we order the banks by their initial exposures such that $\omega_1 \leq \omega_2 \leq \dots \leq \omega_M$ and use the values of G_i^+ to determine the order when $\omega_i = \omega_j$, then the intermediation volume I_i takes its maximum at ω_{i^*} or the next bigger ω_i where $G_{i^*}^+ \geq G_{i^*}^-$.

2. If two banks i and j have the same initial exposure, then $p_i \geq p_j$ and $s_i \geq s_j$ imply $\Omega_i \leq \Omega_j$ by Proposition 4.4. Working toward a contradiction, we assume that

$$f\left(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell}s_\ell, p_j\right) < f\left(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell}s_\ell, p_i\right). \quad (34)$$

Together with $\Omega_i \leq \Omega_j$ and $\omega_i = \omega_j$, this implies

$$\sum_{n:\gamma_{i,n}< 0} f(\gamma_{i,n}s_n, p_n) < \sum_{n:\gamma_{j,n}< 0} f(\gamma_{j,n}s_n, p_n)$$

by (8) so that there exists n with $\gamma_{i,n} < \gamma_{j,n} \leq 0$ because $f_y > 0$ by (16); in particular, $\gamma_{j,n} > -k$. From (5) and (21), we thus obtain

$$\begin{aligned}\Xi'(\Omega_j)\eta f_y(\gamma_{j,n}s_n, p_n) &= \Xi'(\Omega_n)\eta f_y\left(\sum_{\ell:\gamma_{n,\ell}\geq 0}\gamma_{n,\ell}s_\ell, p_n\right), \\ \Xi'(\Omega_i)\eta f_y(\gamma_{i,n}s_n, p_n) &\geq \Xi'(\Omega_n)\eta f_y\left(\sum_{\ell:\gamma_{n,\ell}\geq 0}\gamma_{n,\ell}s_\ell, p_n\right),\end{aligned}$$

and hence

$$\frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \leq \frac{f_y(\gamma_{i,n}s_n, p_n)}{f_y(\gamma_{j,n}s_n, p_n)}.$$

However, this leads to a contradiction because $\Omega_i \leq \Omega_j$ implies $\frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \geq 1$ by (15) and $\gamma_{i,n} < \gamma_{j,n}$ gives $\frac{f_y(\gamma_{i,n}s_n, p_n)}{f_y(\gamma_{j,n}s_n, p_n)} < 1$ because $f_{yy} > 0$ by (17). Therefore, (34) does not hold, which implies $G_j^+ \geq G_i^+$. Next, we assume directly that there exists n with $\gamma_{i,n} < \gamma_{j,n} \leq 0$, which leads to a contradiction by the above arguments. Therefore, we deduce $G_j^- \geq G_i^-$ and thus $I_j \geq I_i$. \square

A.5 Results of Section 5 and their Proofs

Lemma A.6 (Lemma 5.1). *For given s_1, \dots, s_M , the value of $x_i(p_1, \dots, p_M)$ is uniquely determined.*

Proof. For general s_i , (11) becomes

$$x_i(p_1, \dots, p_M) = \omega_i + \sum_{n \neq i} \gamma_{i,n} s_n R_{i,n} - \Gamma^i(\gamma_i s).$$

Using the definition (6) of $R_{i,n}$ and (5), we can write

$$\begin{aligned}x_i(p_1, \dots, p_M) &= \omega_i - \Gamma^i(\gamma_i s) + \sum_{n:\gamma_{i,n}>0} \gamma_{i,n} s_n (\nu \Gamma_{y_i}^n(\gamma_n s) + (1-\nu) \Gamma_{y_n}^i(\gamma_i s)) \\ &\quad + \sum_{n:\gamma_{i,n}<0} \gamma_{i,n} s_n (\nu \Gamma_{y_n}^i(\gamma_i s) + (1-\nu) \Gamma_{y_i}^n(\gamma_n s)) \\ &= \omega_i - \Gamma^i(\gamma_i s) + \nu \sum_{n:\gamma_{i,n}>0} \gamma_{i,n} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) \\ &\quad + (1-\nu) \sum_{n:\gamma_{i,n}<0} \gamma_{i,n} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) + \sum_{n \neq i} \gamma_{i,n} s_n \Gamma_{y_n}^i(\gamma_i s) \\ &= \omega_i - \Gamma^i(\gamma_i s) + \nu k \sum_{n:\gamma_{i,n}>0} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) \\ &\quad - (1-\nu) k \sum_{n:\gamma_{i,n}<0} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) + \sum_{\substack{p_n > 0, \gamma_{i,n} < 0, \text{ or} \\ p_i > 0, \gamma_{i,n} > 0}} \gamma_{i,n} s_n \Gamma_{y_n}^i(\gamma_i s)\end{aligned}$$

$$+ \bar{\Gamma}^i(\gamma_i s) \sum_{\substack{p_n = 0, \gamma_{i,n} < 0, \text{ or} \\ p_i = 0, \gamma_{i,n} > 0}} \gamma_{i,n} s_n,$$

where

$$\bar{\Gamma}^i(y) := \Gamma_{y_n}^i(y) = \frac{q e^{\eta w_i + \eta f(\sum_{n: y_n \geq 0} y_n, p_i) + \eta \sum_{n: y_n < 0} f(y_n, p_n)}}{1 - q + q e^{\eta w_i + \eta f(\sum_{n: y_n \geq 0} y_n, p_i) + \eta \sum_{n: y_n < 0} f(y_n, p_n)}}$$

does not depend on the specific n for all n with $p_n = 0$ and $\gamma_{i,n} < 0$, or $p_i = 0$ and $\gamma_{i,n} > 0$. This means that $\Gamma_{y_n}^i$ is the same for all banks n that are (I) default-free protection sellers to i , or (II) protection buyers from i , and i is default-free. All these pairwise transactions do not bear any counterparty risk. Uniqueness of $x_i(p_1, \dots, p_M)$ now follows from Theorem A.2. \square

Proof of Proposition 5.3. We first note that the mapping $p_i \mapsto x_i(p_1, \dots, p_M)$ is continuous. This follows from the Envelope theorem using that Γ^i and its partial derivatives are differentiable. For $p_{-i} = (p_j)_{j \neq i}$, we define set-valued functions

$$r_i(p_{-i}) = \arg \max_{p_i \in [0, \bar{p}_i]} (x_i(p_1, \dots, p_M) - C(p_i)), \quad r(p) = (r_1(p_{-1}), \dots, r_M(p_{-M}))$$

so that r is a mapping from $[0, \bar{p}_1] \times \dots \times [0, \bar{p}_m]$ onto its power set. It has the following properties:

- $[0, \bar{p}_1] \times \dots \times [0, \bar{p}_m]$ is compact, convex, and nonempty.
- For each p , $r(p)$ is nonempty because a continuous function over a compact set has always a maximizer.
- $r(p)$ is convex by assumption.
- It follows from Berge's maximum theorem that $r(p)$ has a closed graph.

Thanks to these properties, Kakutani's fixed point theorem implies that there exists a fixed point of the mapping r , which means that there exists an equilibrium. \square

Proof of Proposition 5.4. Because the function

$$\sum_{i=1}^M s_i \Gamma^i(\gamma_i s, p) + \sum_{i=1}^M s_i C(p_i) \tag{35}$$

is continuous over the compact set $[0, \bar{p}_1] \times \dots \times [0, \bar{p}_M]$, it has a maximum, which shows the statement of the proposition, using that the social planner's optimization problem over $(\gamma_{i,n})_{i,n=1, \dots, M}$ conditional on the choice of the default probabilities has a solution by Theorems 4.2 and 4.3. \square

Proof of Lemma 5.5. The demand of bank n on CDS contracts to bank i is given by the curve $-y_i \mapsto \Gamma_{y_i}^n(y, p)$ for $y_i < 0$ because the marginal cost of risk bearing decreases by $\Gamma_{y_i}^n(y, p)$ per unit purchased CDS. The slope of the demand curve is obtained by taking the negative

y_i -partial derivative of the demand curve, resulting in $-\Gamma_{y_i, y_i}^n(y, p)$. Hence, the demand curve becomes flatter for decreased p_i if $-\Gamma_{y_i, y_i, p_i}^n(y, p) < 0$, or equivalently, $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$. We show $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$ for small enough p_i and big enough q .

By Lemma 3.1, we have

$$\begin{aligned}\Gamma^n(y, p) &= \frac{1}{\eta} \log \left(1 - q + qe^{\eta w_n + \eta f(\sum_{\ell: y_\ell \geq 0} y_\ell, p_n) + \eta \sum_{\ell: y_\ell < 0} f(y_\ell, p_\ell)} \right) \\ &= \frac{1}{\eta} \log \left(1 - q + a(1 - p_i)e^{\eta y_i} + ap_i e^{\eta r y_i} \right)\end{aligned}$$

for $y_i < 0$, where we use the abbreviation $a = qe^{\eta w_n + \eta f(\sum_{\ell: y_\ell \geq 0} y_\ell, p_n) + \eta \sum_{\ell \neq i: y_\ell < 0} f(y_\ell, p_\ell)}$ in this proof. Because $\Gamma^n(y, p)$ is a smooth function, the order of taking partial derivatives does not matter. For simplicity, we start with the p_i -partial derivative, which equals

$$\Gamma_{p_i}^n(y, p) = \frac{1}{\eta} \frac{-ae^{\eta y_i} + ae^{\eta r y_i}}{1 - q + a(1 - p_i)e^{\eta y_i} + ap_i e^{\eta r y_i}} = \frac{1}{\eta} \frac{e^{\eta r y_i} - e^{\eta y_i}}{b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i}}, \quad (36)$$

using the abbreviation $b = (1 - q)/a$. Next, we compute

$$\begin{aligned}\Gamma_{y_i, p_i}^n(y, p) &= \frac{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})(re^{\eta r y_i} - e^{\eta y_i}) - (e^{\eta r y_i} - e^{\eta y_i})((1 - p_i)e^{\eta y_i} + rp_i e^{\eta r y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \\ &= \frac{b(re^{\eta r y_i} - e^{\eta y_i}) + (r - rp_i - p_i)e^{\eta(1+r)y_i} - (1 - p_i)e^{2\eta y_i} + rp_i e^{2\eta r y_i}}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \\ &\quad - \frac{(1 - rp_i - p_i)e^{\eta(1+r)y_i} - (1 - p_i)e^{2\eta y_i} + rp_i e^{2\eta r y_i}}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \\ &= \frac{b(re^{\eta r y_i} - e^{\eta y_i}) + (r - 1)e^{\eta(1+r)y_i}}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2}.\end{aligned} \quad (37)$$

Finally, we determine the third-order partial derivative

$$\begin{aligned}\Gamma_{y_i, y_i, p_i}^n(y, p) &= \eta \frac{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2 (b(r^2 e^{\eta r y_i} - e^{\eta y_i}) + (r^2 - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^4} \\ &\quad - \eta \frac{2((1 - p_i)e^{\eta y_i} + rp_i e^{\eta r y_i})(b(re^{\eta r y_i} - e^{\eta y_i}) + (r - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^3} \\ &= \eta \frac{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})(b(r^2 e^{\eta r y_i} - e^{\eta y_i}) + (r^2 - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^3} \\ &\quad - \eta \frac{2((1 - p_i)e^{\eta y_i} + rp_i e^{\eta r y_i})(b(re^{\eta r y_i} - e^{\eta y_i}) + (r - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^3}.\end{aligned}$$

For $p_i = 0$ and $q = 1$, we have

$$\Gamma_{y_i, y_i, p_i}^n(y, p) = \eta(r - 1)^2 e^{\eta(-1+r)y_i} > 0,$$

and hence, $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$ for small enough p_i and big enough q by continuity of Γ_{y_i, y_i, p_i}^n . \square

Theorem A.7 (Theorem 5.6). *The social planner's optimization satisfies the first-order conditions of an equilibrium if bank i receives a per-trader subsidy equal to $S = S_1 + k(1 - \nu)S_2$ with*

$$S_1 := - \sum_{n \neq i} s_n \left(\gamma_{i,n} \Gamma_{y_i}^n(\gamma_n s, p) + \frac{1}{s_i} \Gamma^n(\gamma_n s, p) \right), \quad S_2 := \sum_{n \neq i} s_n (\Gamma_{y_i}^n(\gamma_n, p) - \Gamma_{y_n}^i(\gamma_i, p)). \quad (38)$$

Assuming a small enough trade size limit, we have $\frac{\partial S_1}{\partial p_i} > 0$ and $\frac{\partial S_2}{\partial p_i} < 0$ for small enough p_i and large enough q . In this case, privately chosen p_i 's are lower than the socially optimal level if sellers have full bargaining power. The difference between the individual and socially optimal choices of p_i increases as a function of the sellers' bargaining power.

Proof. For the first part of the theorem, we compare the marginal social value MSV_i , defined as the partial derivative of the social planner's objective function with respect to the default probability p_i of bank i assuming that banks trade optimally, with the corresponding marginal private value MPV_i . To do so, we highlight the dependence on the banks' default probabilities $p = (p_i)_{i=1, \dots, M}$ by using notations such as $\Gamma^i(\gamma_i s, p)$ and $R_{i,n}(\gamma_i s, p)$. For arbitrary bank sizes, (13) becomes

$$\sum_{i=1}^M s_i x_i(p_1, \dots, p_M) - \sum_{i=1}^M s_i C(p_i) \quad (39)$$

so that the MSV_i , given as its p_i -partial derivative, equals

$$MSV_i = \sum_{n=1}^M s_n \frac{\partial x_n}{\partial p_i}(p_1, \dots, p_M) - s_i C'(p_i) = - \sum_{n=1}^M s_n \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) - s_i C'(p_i),$$

where we have used the equivalence between (35) and (39). The marginal private value MPV_i is the partial derivative of the bank i 's certainty equivalent (11) minus its risk-management costs, with respect to its default probability p_i — namely,

$$MPV_i = -s_i C'(p_i) - s_i \frac{\partial \Gamma^i}{\partial p_i}(\gamma_i s, p) + s_i \sum_{n \neq i} \gamma_{i,n} s_n \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p).$$

The difference between marginal private and social value for bank i is

$$MPV_i - MSV_i = \sum_{n \neq i} s_n \left(s_i \gamma_{i,n} \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right)$$

If $\gamma_{i,n} \leq 0$, then we obtain from (6) that

$$R_{i,n}(\gamma_i s, \gamma_n s, p) = \nu \Gamma_{y_n}^i(\gamma_i s, p) + (1 - \nu) \Gamma_{y_i}^n(\gamma_n s, p).$$

We then have that $\frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) = 0$ and $\frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) = 0$ because $\Gamma^n(\gamma_n s, p)$, $\Gamma_{y_n}^i(\gamma_i s, p)$, and $R_{i,n}(\gamma_i s, \gamma_n s, p)$ do not depend on p_i for $\gamma_{i,n} \leq 0$; if traders of bank i are buying CDSs

from bank n , the default probability of bank i does not affect the terms of trade between traders of banks i and n . For $\gamma_{i,n} > 0$, we find

$$R_{i,n}(\gamma_i s, \gamma_n s, p) = \nu \Gamma_{y_i}^n(\gamma_n s, p) + (1 - \nu) \Gamma_{y_n}^i(\gamma_i s, p)$$

by (5) and (6) so that

$$\begin{aligned} MPV_i - MSV_i &= \sum_{n:\gamma_{i,n}>0} s_n \left(s_i \gamma_{i,n} \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right) \\ &= \sum_{n:\gamma_{i,n}>0} s_n \left(s_i \gamma_{i,n} \frac{\partial \Gamma_{y_i}^n}{\partial p_i}(\gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right) \\ &\quad + \sum_{n:\gamma_{i,n}>0} s_n s_i \gamma_{i,n} (1 - \nu) \left(\frac{\partial \Gamma_{y_n}^i}{\partial p_i}(\gamma_i s, p) - \frac{\partial \Gamma_{y_i}^n}{\partial p_i}(\gamma_n s, p) \right) \\ &= \frac{\partial}{\partial p_i} \sum_{n \neq i} s_n (s_i \gamma_{i,n} \Gamma_{y_i}^n(\gamma_n s, p) + \Gamma^n(\gamma_n s, p)) \\ &\quad + \frac{\partial}{\partial p_i} \sum_{n \neq i} s_n s_i k (1 - \nu) (\Gamma_{y_n}^i(\gamma_i s, p) - \Gamma_{y_i}^n(\gamma_n s, p)), \end{aligned}$$

using for the last equality that $\frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) = 0$, $\frac{\partial \Gamma_{y_i}^n}{\partial p_i}(\gamma_n s, p) = 0$ and $\frac{\partial \Gamma_{y_n}^i}{\partial p_i}(\gamma_i s, p) = 0$ for $\gamma_{i,n} \leq 0$ and $\Gamma_{y_n}^i(\gamma_i s, p) = \Gamma_{y_i}^n(\gamma_n s, p)$ for $\gamma_{i,n} \in (-k, k)$.

We next prove the second part of Theorem A.7. By Lemma 5.5, we have $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$ for small enough p_i and big enough q . This implies

$$\Gamma_{y_i, p_i}^n(\gamma_n s, p) < \Gamma_{y_i, p_i}^n(((\gamma_{\ell, n} s_{\ell})_{\ell \neq i}, y_i), p) \quad (40)$$

for all $y_i \in (\gamma_{n, i} s_i, 0)$ where $((\gamma_{\ell, n} s_{\ell})_{\ell \neq i}, y_i) := (\gamma_{1, n} s_1, \dots, \gamma_{i-1, n} s_{i-1}, y_i, \gamma_{i+1, n} s_{i+1}, \dots, \gamma_{M, n} s_M)$. From $\Gamma_{p_i}^n(((\gamma_{\ell, n} s_{\ell})_{\ell \neq i}, 0), p) = 0$ by (36), we deduce

$$\begin{aligned} \frac{\partial}{\partial p_i} (s_i \gamma_{i, n} \Gamma_{y_i}^n(\gamma_n s, p) + \Gamma^n(\gamma_n s, p)) &= s_i \gamma_{i, n} \Gamma_{y_i, p_i}^n(\gamma_n s, p) - \int_{\gamma_{n, i} s_n}^0 \Gamma_{y_i, p_i}^n(((\gamma_{\ell, n} s_{\ell})_{\ell \neq i}, y_i), p) dy_i \\ &= \int_{\gamma_{n, i} s_n}^0 (\Gamma_{y_i, p_i}^n(\gamma_n s, p) - \Gamma_{y_i, p_i}^n(((\gamma_{\ell, n} s_{\ell})_{\ell \neq i}, y_i), p)) dy_i \\ &< 0, \end{aligned}$$

using (40). Hence, we obtain $\frac{\partial S_1}{\partial p_i} > 0$ by the definition (38) of S_1 . To show $\frac{\partial S_2}{\partial p_i} < 0$ for small enough p_i and big enough q , we compare $\Gamma_{y_i, p_i}^n(\gamma_n, p)$ and $\Gamma_{y_n, p_i}^i(\gamma_i, p)$. We first note that $\Gamma_{y_i, p_i}^n(\gamma_n, p) = 0$ and $\Gamma_{y_n, p_i}^i(\gamma_i, p) = 0$ for $\gamma_{n, i} = -\gamma_{i, n} \geq 0$. For $p_i = 0$ and $q = 1$, we obtain from (37) that

$$\Gamma_{y_i, p_i}^n(y, p) \Big|_{p_i=0, q=1} = \frac{b(re^{\eta r y_i} - e^{\eta y_i}) + (r-1)e^{\eta(1+r)y_i}}{(b + (1-p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \Big|_{p_i=0, q=1} = (r-1)e^{\eta(r-1)y_i}$$

for $y_i < 0$. A calculation similar to (37) gives

$$\begin{aligned}\Gamma_{y_n, p_i}^i(y, p)\Big|_{p_i=0, q=1} &= \frac{\tilde{b}(re^{\eta r \sum_{\ell: y_\ell \geq 0} y_\ell} - e^{\eta \sum_{\ell: y_\ell \geq 0} y_\ell}) + (r-1)e^{\eta(1+r) \sum_{\ell: y_\ell \geq 0} y_\ell}}{(\tilde{b} + (1-p_i)e^{\eta \sum_{\ell: y_\ell \geq 0} y_\ell} + p_i e^{\eta r \sum_{\ell: y_\ell \geq 0} y_\ell})^2} \Big|_{p_i=0, q=1} \\ &= (r-1)e^{\eta(r-1) \sum_{\ell: y_\ell \geq 0} y_\ell}\end{aligned}$$

for $y_n > 0$, where $\tilde{b} = (1-q)/(qe^{\eta w_i + \eta \sum_{\ell: y_\ell < 0} f(y_\ell, p_\ell)})$. Therefore, for $\gamma_{n,i} = -\gamma_{i,n} < 0$, we obtain

$$\Gamma_{y_i, p_i}^n(\gamma_n S, p)\Big|_{p_i=0, q=1} < \Gamma_{y_n, p_i}^i(\gamma_i S, p)\Big|_{p_i=0, q=1}$$

and thus $\frac{\partial S_2}{\partial p_i} < 0$ in this case. Using that the cost function C is convex, it follows from $\frac{\partial S}{\partial p_i} = MSV_i - MPV_i$ that the individual choice of p_i is lower than socially optimal if $\frac{\partial S}{\partial p_i} > 0$. \square

References

- D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi (2014): Systemic Risk and Stability in Financial Networks. *American Economic Review* 105, 564–608.
- D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi (2014): Systemic Risk in Endogenous Financial Networks. Working paper, Columbia Business School.
- G. Afonso and R. Lagos (2015): Trade Dynamics in the Market for Federal Funds. *Econometrica* 83, 263–313.
- N. Arora, P. Gandhi, and F.A. Longstaff (2012): Counterparty Credit Risk and the Credit Default Swap Market. *Journal of Financial Economics* 103, 280–293.
- A. Atkeson, A. Eisfeldt, and P.-O. Weill (2015): Entry and Exit in OTC Derivative Markets. *Econometrica* 83, 2231–2292.
- J. Begenau, M. Piazzesi, and M. Schneider (2015): Banks’ Risk Exposures. NBER Working Paper No. 21334.
- B. Biais, F. Heider, and M. Horova (2016): Risk-Sharing or Risk-Taking? Counterparty Risk, Incentives, and Margins. *Journal of Finance* 71, 1669–1698.
- K. Border (1985): Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University Press.
- A. Capponi (2013): Pricing and Mitigation of Counterparty Credit Exposure. In: *Handbook of Systemic Risk*, edited by J.-P. Fouque and J. Langsam. Cambridge University Press.
- W. Du, S. Gadgil, M. Gordy, and C. Vega (2016): Counterparty Risk and Counterparty Choice in the Credit Default Swap Market. Finance and Economics Discussion Series 2016-087, Board of Governors of the Federal Reserve System.

- D. Duffie, N. Gârleanu and L.H. Pedersen (2005): Over-the-Counter Markets, *Econometrica* 73, 1815–1847.
- D. Duffie and M. Huang (1996): Swap Rates and Credit Quality. *Journal of Finance* 51, 921–949.
- D. Duffie, A. Li, and T. Lubke (2010): Policy Perspectives on OTC Derivatives Market Infrastructure. Federal Reserve Bank of New York Staff Report No. 424.
- L. Eisenberg and T. Noe (2011): Systemic Risk in Financial Systems. *Management Science* 47, 236–249.
- M. Elliott, B. Golub, and M. Jackson (2013): Financial Networks and Contagion. *American Economic Review* 104, 3115–3153.
- M. Farboodi (2014): Intermediation and Voluntary Exposure to Counterparty Risk. Working paper, The Chicago Booth School of Business.
- J. Hugonnier, B. Lester, and O. Weill (2016): Heterogeneity in Decentralized Asset Markets. NBER Working Paper No. 20746.
- IHS Markit Ltd. Markit Credit Default Swap. Wharton Research Data Services (WRDS). <https://wrds-web.wharton.upenn.edu/wrds/about/>
- N. Klimenko, S. Pfeil, J.-C. Rochet, and G. De Nicolo (2016): Aggregate Bank Capital and Credit Dynamics. Swiss Finance Institute Research Paper No. 16-42.
- R. Lagos, and G. Rocheteau (2007): Search in Asset Markets: Market Structure, Liquidity, and Welfare. *American Economic Review (Papers & Proceedings)* 97, 198–202.
- R. Lagos, and G. Rocheteau (2009): Liquidity in Asset Markets with Search Frictions. *Econometrica* 72, 403–426.
- M. Oehmke, and A. Zawadowski (2017): The Anatomy of the CDS Market. *Review of Financial Studies* 30, 80–119.
- R. Stulz (2010): Credit Default Swaps and the Credit Crisis. *Journal of Economic Perspectives* 24, 73–92.
- R. Stulz (2014): Governance, Risk Management, and Risk-Taking in Banks. European Corporate Governance Institute (ECGI), Finance Working Paper No. 427.
- J.R. Thompson (2010): Counterparty Risk in Financial Contracts: Should the Insured Worry about the Insurer? *Quarterly Journal of Financial Economics* 125, 1195–1252.