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**Managing Counterparty Risk in OTC Markets**

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# Managing Counterparty Risk in Over-the-Counter Markets\*

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## Abstract

We study how banks manage their default risk to optimally negotiate quantities and prices of contracts in over-the-counter markets. We show that costly actions exerted by banks to reduce their default probabilities are inefficient. Negative externalities due to counterparty concentration may lead banks to reduce their default probabilities even below the social optimum. The model provides new implications which are supported by empirical evidence: (i) intermediation is done by low-risk banks with medium initial exposure; (ii) the risk-sharing capacity of the market is impaired, even when the trade size limit is not binding; and (iii) intermediaries play the fundamental role of diversifying the idiosyncratic risk in CDS contracts, besides increasing the risk-sharing capacity of the market.

*Keywords:* over-the-counter markets, counterparty risk, negative externalities, counterparty concentration

*JEL Classification:* G11, G12, G21

## 1 Introduction

Counterparty risk is one of the most prominent sources of risk faced by market participants and financial institutions in over-the-counter (OTC) markets. During the global financial

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crisis of 2007–2008, roughly two thirds of credit related losses were attributed to the market price of counterparty credit risk, and only about one third to actual default events; see Bank for International Settlements (2011). We develop a framework to investigate the implications of risk management on trading decisions, characteristics of the intermediaries, and structure of OTC markets.

We show that an inefficiency arises when each bank manages its default risk through costly actions to maximize its private certainty equivalent. Because the system-wide benefits of risk reduction are only partially reflected in bilaterally negotiated prices, banks' decisions on their default probabilities may deviate from the social optimum. Surprisingly, when banks are restricted by a trade size limit and subject to low risk management costs, they may act conservatively (e.g., implement stricter risk management strategies), and reduce their default risk below the socially optimal level. These actions entail opportunity costs for forsaken business activities deemed too risky. Thus, through the choice of default probabilities, banks face a tradeoff between conservative strategies entailing these opportunity costs and high-risk behavior decreasing earnings from trading in the OTC market.

Our theoretical findings highlight the critical role played by counterparty risk in shaping the structure of OTC markets. We show that low-risk banks with medium initial exposure endogenously emerge as intermediaries, profiting from price dispersion, and providing intermediation services to banks with high or low initial exposures. These intermediaries have the least need to trade for their own risk management as their pre-trade exposure is close to the equilibrium post-trade exposure, and they also command a small counterparty risk. Using a proprietary data set of bilateral exposures from the market of credit default swaps (CDSs), we highlight the prominent role played by five banks acting as the main intermediaries (see the network graph in Figure 2).

In our model, all banks engage in risk sharing through OTC contracts. However, high-risk banks with low initial exposure impair the risk-sharing capacity of the market. Because they are not guaranteed to fulfil the obligations towards their counterparties, they do not sell as much insurance as they would if they were riskless. Intermediaries increase the participation rate of the high-risk banks by diversifying counterparty risk. As a consequence of partial risk sharing, post-trade exposures are closer together than initial exposures. In particular, safe banks have the same post-trade exposure if the trade size limit is big enough, whereas risky banks maintain diverse post-trade exposures. Safer banks maintain a higher post-trade exposure in riskier markets because riskier banks prefer to buy protection from them to avoid excessive exposure to counterparty risk.

Our paper contributes to the post-financial-crisis discussion on the role played by counterparty risk in the network of OTC derivative transactions. Derivatives account for more than two thirds of the banks' most prominent USD asset classes.<sup>1</sup> The OTC market for credit derivatives has been identified as the one that has contributed the most to the onset and transmission of systemic risk during the global financial crisis.<sup>2</sup> Stulz (2010) highlights

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<sup>1</sup>A breakdown of the amounts allocated by banks to different asset classes is presented by the Market Participants Group on Reforming Interest Rate Benchmarks. In its final report released on March 17, 2014, Figure 1 of the USD currency section highlights that the largest interbanking exposure on the US market is through OTC derivative trading (69 percent), while bilateral corporate loans and syndicated loans account for only 2 percent of the key asset classes that reference USD-LIBOR and T-bill rates.

<sup>2</sup>The most prominent solution proposed for reducing counterparty risk is the central clearing of OTC

that counterparty credit risk is the highest in CDS markets. This is because the event of a joint default of the underlying reference credit and protection seller, unique to the class of OTC credit derivatives, cannot be anticipated. Hence, collateralizing the contract at all times to cover the full claim would require posting large amount of excess collateral, which would be too costly and economically unfeasible (see also Giglio (2014) for further details).

Our framework is as follows. Before engaging into trading, each bank may choose to reduce its default probability from a target regulatory level. Such an action is costly, depends on the decisions made by other banks in equilibrium, and takes the subsequent trading decisions into consideration. Once the default risk profile has been determined in equilibrium, all banks are granted access to the same technology to trade contracts resembling CDSs. As in Atkeson et al. (2015), each bank is a coalition of many risk-averse agents, called traders; banks have heterogeneous initial exposures to a nontradable risky loan portfolio, which creates heterogeneous exposures to an aggregate risk factor and determines the profitability of the trade. The trading process consists of two stages. First, banks' traders are paired uniformly, and each pair negotiates over the terms of the contract subject to a uniform trade size limit. The resulting prices and quantities are endogenous and depend on the risk profile of market participants, the heterogeneity in their initial exposures, and the dispersion in their marginal valuations. When a trader of a bank purchases a contract from a trader of another bank, it pays a bilaterally agreed-upon fee upfront and receives the contractually agreed-upon payment if the credit event occurs, provided the bank of its trading counterparty does not default. In case of the counterparty's default, the received payment is reduced by an exogenously specified loss rate. Second, each bank consolidates the swaps signed by its traders and executes the contracts. Because banks are risk averse, they value the risk of not receiving the full payment from a defaulted counterparty more than the potential gain obtained when they are protection sellers and default.

Our normative analysis identifies an inefficiency in the banks' risk management decisions. Because traders rely on their banks' profitability, the value of a trader's credit protection decreases as other traders from the same bank purchase protection from the same counterparty. This is because the purchasing bank increases the concentration of its exposure to the selling bank. The decreased value of credit protection caused by counterparty concentration is analogous to the snob effect in the consumption of luxury goods, which lose value because of the reduced prestige when more people own them. Analogously to the snob effect, the demand curve becomes less elastic. A decrease in the selling bank's default probability reduces the size of the negative effect caused by counterparty concentration, and in turn, makes the demand curve more elastic. However, a more elastic demand curve means a smaller buyers' surplus, which corresponds to the trade benefit for the protection buyers. The buyers' surplus is accounted for by the social planner, but not in the individual optimizations of the protection sellers. Thus, the protection selling banks neglect the decrease in buyers' surplus, and as a result may find it optimal to lower their default probabilities below the socially optimal level. Hence, an externality arises because sellers are only partially compensated for their contribution to the system-wide risk reduction.

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derivatives. The Dodd-Frank Wall Street Reform and Consumer Protection Act in the United States and the European Market Infrastructure in Europe have mandated central clearing for standardized OTC derivatives, including CDSs. The centrally cleared CDS market currently captures only approximately 55 percent of the entire U.S. CDS market, as measured by gross notional.

Our findings have direct policy implications: alignment of private incentives with social efficiency can be achieved by collecting a tax from CDS buyers and using it to compensate CDS sellers for their contribution in reducing the system-wide risk exposure. The degree of overinvestment in risk management is higher if the banks’ bargaining power is large when they sell insurance. A policy maker can remedy this inefficiency by reducing the bargaining power of the seller relative to that of the buyer, for example, by reducing concentration in the provision of credit insurance. This is especially important for the CDS market, in which the sell side is twice as much concentrated as the buy side; see Siriwardane (2018).<sup>3</sup> Our findings suggest that merging safe banks with low initial exposure is not welfare-enhancing. Because these banks are likely to be protection sellers, there would be a higher misalignment between individually and socially optimal trading decisions.

The rest of paper is organized as follows. We review related literature in Section 2. We develop the model in Section 3. We study the equilibrium trading decisions of banks in Section 4. We study the normative implications of our model in Section 5. Section 6 concludes. Proofs of all results are delegated to the Appendix.

## 2 Literature Review

Our main contribution to the literature is the development of a tractable framework to study counterparty risk in OTC markets, along with its implications on banks’ risk management and bilateral trading decisions.

In our model, CDSs are used by banks to hedge against the default risk of a nontradable loan portfolios. Oehmke and Zawadowski (2017) provide empirical evidence consistent with this view. They show that banks with a high notional of outstanding bonds also have large outstanding notional of CDS contracts. They also examine trading volumes in the bond and CDS markets and observe a similar pattern—that is, hedging motives are associated with comparable amounts of trading volume in the bond and CDS markets.

Our model implication that high-risk banks engage in imperfect risk sharing is supported by empirical evidence provided by Du et al. (2016). They develop a statistical multinomial logit model for the counterparty choice of buyers in the CDS market. They find that market participants are more likely to trade with counterparties whose credit quality is high. Our model predictions are also consistent with the empirical results of Arora et al. (2012), who find a significant negative relation between the credit risk of the dealer and the prices at which the dealer sells credit protection.

Our framework is based on that developed by Atkeson et al. (2015). The main deviation from their setting is that the sellers of insurance might default without delivering the full contracted amount. While Atkeson et al. (2015) focus on the effect of entry and exit decisions of a continuum of banks on the structure of OTC markets, we consider a fixed finite number of banks and allow them to decide on the level of their credit riskiness. The second, albeit minor, difference is our specification of the aggregate risk factor. Unlike Atkeson et al. (2015), who consider it to be a generic random variable, we model the aggregate risk factor

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<sup>3</sup>The top five sellers of the CDS market account for nearly half of all net selling, and 50% of net selling is in the hands of less than 0.1 percent of the total number of CDS traders net selling is handled by less than 0.1% of the total number of CDS traders.

using a binary random variable. Through this specification, we can parsimoniously capture the joint default risk of the protection seller and the underlying reference credit. Relevant for the valuation of the CDS contract is the seller's default risk conditioned on the default of the underlying reference credit. Our model accounts directly for this conditional default risk, which has been shown by Giglio (2014) to be significantly different from the marginal default risk. As in Atkeson et al. (2015), we also find that banks with medium initial exposure endogenously emerge as intermediaries. However, the counterparty risk friction in our model has important ramifications with regard to the market structure: First, bilateral trading positions are unique in equilibrium because the counterparty risk of the sellers makes the traded contracts imperfect substitutes. Second, the risk-sharing capacity of the market is impaired, even when the trade size limit is not binding. Finally, intermediaries play the additional role of diversifying the idiosyncratic risk in CDS contracts sold by banks, besides increasing the risk-sharing capacity of the market.

The classical setup used to study OTC markets is the search-and-bargaining framework proposed by Duffie et al. (2005), which models the trading friction characteristics typical of these markets. Their model was generalized along several dimensions, including the relaxation of the constraint of zero-one units of assets holdings (see Lagos and Rocheteau (2009)), the entry of dealers (see Lagos and Rocheteau (2007)), and investors' valuations drawn from an arbitrary distribution as opposed to being binary (see Hugonnier et al. (2018)). All these studies do not allow for the inclusion of counterparty risk, mainly because the framework cannot keep track of the identities of the counterparties for the continuum of traders.

The interactions between counterparty risk and derivatives activities are also studied by Thompson (2010) and Biais et al. (2016). Thompson (2010) shows that a moral hazard problem for the protection seller, whose type is exogenously given, causes the protection buyer to be exposed to excessive counterparty risk. In turn, this mitigates the classical adverse selection problem because the protection buyer is incentivized to reveal superior information that it may have relative to the seller. In Biais et al. (2016), risk-averse protection buyers insure against a common exposure to risk by contacting protection sellers. Differently from our model, the protection buyers are risk neutral and avoid costly risk-prevention effort by choosing weaker internal risk controls. It is precisely the failure of protection sellers to exert risk-prevention effort that creates counterparty risk for protection buyers in our model.

Our paper is also related to the emerging literature on endogenous OTC networks. Wang (2018) shows that the trading network which emerges endogenously in OTC markets is of the core-periphery type. In his model, intermediaries exploit their central position to balance inventory risk, while in our model they help diversify counterparty risk in the network. Gofman (2014) provides a network model to study the intermediation friction in OTC markets. As in our model, trading decisions and bilateral prices are jointly determined in equilibrium. Traders can only transact if they have a trading relationship and extract a surplus which depends both on the private value of the buyer and on the resale opportunities of the asset. While the focus of his study is on welfare losses due to intermediation frictions, we study the negative externality originating from counterparty concentration. Babus and Hu (2017) consider an infinite-horizon model of endogenous intermediation and analyze two important frictions of OTC markets. The first is the limited commitment of market participants who can renege on due payments, and the second is the opaqueness of OTC markets in which participants have incomplete information on the past behavior of others.

A related branch of literature has studied the incentives behind the formation of interbank loan networks.<sup>4</sup> Farboodi (2017) proposes a model of financial intermediation where profit-maximizing institutions strategically decide on borrowing and lending activities. Her model predicts that banks which make risky investments voluntarily expose themselves to excessive counterparty risk, while banks that mainly provide funding establish connections with a small number of counterparties in the network. A related study by Acemoglu et al. (2014-b) analyzes the endogenous formation of interbanking loan networks. In their model, banks borrow to finance risky investments, charging an interest rate that is increasing in the risk-taking behavior of the borrower. They find that banks may overlend in equilibrium and do not spread their lending among a sufficiently large number of potential borrowers, thus creating insufficiently connected financial networks prone to defaults. Different from their settings, our framework captures stylized features of derivatives trading in OTC markets, as opposed to markets for interbanking loans. Meetings between traders are random, and the equilibrium trading patterns are the outcome of bilateral bargaining that accounts for counterparty risk.

### 3 The Model

There is a unit continuum of *traders*, that are risk-averse agents. They have constant absolute risk aversion with parameter  $\eta$ . The traders are organized into  $M$  *banks*, which are coalitions of traders. All banks are granted access to the same technology to trade swaps.

The banks are heterogeneous in two dimensions: their initial exposures and their sizes. They are exposed to an aggregate risk factor  $D$ , taking binary values 0 (no default) and 1 (default), with  $P[D = 1] = q$ . We denote by  $s_i > 0$  the size of bank  $i$  and by  $\omega_i$  the initial exposure per trader of bank  $i$  to the aggregate risk factor. The traders are paired uniformly across the different banks. Therefore, the frequency at which a trader of bank  $j \neq i$  is paired with a trader of bank  $i$  is proportional to  $s_i$ , and we normalize it to be equal to  $s_i$ . Both the initial exposure  $\omega_i$  and the size  $s_i$  of bank  $i$  are exogenously specified and observable to the traders. Because the size does not play a crucial role in our main results, we restrict the main body of the paper to the case  $s_i = 1$ . For completeness, we present the results and their proofs in the Appendix for any  $s_i$ .

Before trading begins, each bank manages its default risk at an exogenously specified cost. We assume that given a realization  $D = 1$  of the aggregate risk factor, bank  $i$ 's default probability has a maximal value of  $\bar{p}_i$ . This value can be thought of as the default probability of bank  $i$  if it meets the imposed regulatory standards, and does not engage in risk management or hedging procedures to further reduce its default risk. To become a more attractive trading counterparty in the OTC market, bank  $i$  can decrease its probability to  $p_i \in [0, \bar{p}_i]$  at a cost  $C(p_i)$ . Therefore, depending on its initial exposure, each bank  $i$  needs to decide before trading starts how much it is willing to pay (cost  $C(p_i)$ ) in order to reduce its default probability to  $p_i$ . These decisions also take into consideration the subsequent

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<sup>4</sup>Another branch of literature has studied counterparty risk in an *exogenously* specified network of financial liabilities. The focus of these studies is on how the topology of the network affects the amplification of an initial shock through the network. Relevant contributions in this direction include Eisenberg and Noe (2011), Elliott et al. (2013), and Acemoglu et al. (2014-a).

trading transactions that banks will establish and that are uniquely specified in terms of bilateral prices and quantities (see Theorem 4.3 for details). For a bank  $i$ , we denote by  $A_i$  the event that the bank defaults with  $P[A_i|D = 1] = p_i$ . Because banks will trade contracts of CDS type on the aggregate risk factor  $D$ , only the conditional default event  $A_i|D = 1$  of bank  $i$  and not the unconditional default event  $A_i$  matters to the trading counterparties of bank  $i$ . Therefore, it is precisely the conditional default probability  $p_i$  to determine the attractiveness of bank  $i$  on the OTC market. We assume that the conditional events  $A_i|D = 1$  are independent but do not impose that the banks' defaults themselves are independent. In particular, each bank can have different default probabilities depending on the realization of the aggregate risk factor. This setting allows for a dependence structure among the banks' defaults. A special role will be taken by banks that choose  $p_i = 0$ . We call such banks *safe*, while banks with  $p_i > 0$  are referred to as *risky*.

When a trader from bank  $i$  meets a trader from bank  $n$ , they bargain a contract similar to a CDS. They agree that the trader of bank  $i$  sells  $\gamma_{i,n}$  contracts to the trader of bank  $n$ . If  $\gamma_{i,n} > 0$ , bank  $n$  makes an immediate payment of  $\gamma_{i,n}R_{i,n}$ , and at the end of the period, bank  $i$  makes a payment of  $\gamma_{i,n}D$  to bank  $n$  if bank  $i$  has not defaulted by then; if it has defaulted, the payment is reduced to  $r\gamma_{i,n}R_{i,n}$ . In summary, the payment at the end of the period is  $\gamma_{i,n}D(\mathbb{1}_{A_i^c} + r\mathbb{1}_{A_i})$  from bank  $i$  to bank  $n$  if  $\gamma_{i,n} > 0$ . For the case  $\gamma_{i,n} < 0$ , the roles of  $i$  and  $n$  are interchanged. Therefore, the bilateral constraint  $\gamma_{i,n} = -\gamma_{n,i}$  holds. We further assume that there is a trade size constraint per trader so that  $-k \leq \gamma_{i,n} \leq k$  for some constant  $k > 0$ . We call a set of contracts  $(\gamma_{i,n})_{i,n=1,\dots,M}$  *feasible* if both the bilateral constraint  $\gamma_{i,n} = -\gamma_{n,i}$  and the trade size constraint  $-k \leq \gamma_{i,n} \leq k$  hold for all  $i, n = 1, \dots, M$ . For notational convenience, we will use the abbreviation  $\gamma_i := (\gamma_{i,1}, \dots, \gamma_{i,M})$  to denote the collection of contracts that bank  $i$  has with the other banks.

At the end of the trading period, traders of every bank come together and consolidate all their long and short positions. The consolidated per-capita wealth of bank  $i$  with contracts  $\gamma_{i,1}, \dots, \gamma_{i,M}$  is

$$X_i = \omega_i(1 - D) + \sum_{n \neq i} \gamma_{i,n}(R_{i,n} - D(\mathbb{1}_{A_n^c} + r\mathbb{1}_{A_n})\mathbb{1}_{\gamma_{i,n} < 0} - D(\mathbb{1}_{A_i^c} + r\mathbb{1}_{A_i})\mathbb{1}_{\gamma_{i,n} > 0}),$$

where

- $\omega_i(1 - D)$  is the per-capita payout associated with the initial exposure.
- $\sum_{n \neq i} \gamma_{i,n}R_{i,n}$  is the aggregate net payment received (if positive) or made (if negative) during trading, corresponding to the CDS protection fees.
- $-D\gamma_{i,n}(\mathbb{1}_{A_n^c} + r\mathbb{1}_{A_n})\mathbb{1}_{\gamma_{i,n} < 0}$  is the per-capita payment that bank  $i$  will receive from bank  $n$ . This payment will be executed only if the realization of the aggregate risk factor is  $D = 1$  and bank  $i$  net bought protection from bank  $n$  ( $\gamma_{i,n} < 0$ ). In this case, bank  $i$  will receive  $-\gamma_{i,n}$  if bank  $n$  does not default (event  $A_n^c$ , namely, the complement of the default event  $A_n$ ) or  $-r\gamma_{i,n}$  if bank  $n$  defaults (event  $A_n$ ).
- $D\gamma_{i,n}(\mathbb{1}_{A_i^c} + r\mathbb{1}_{A_i})\mathbb{1}_{\gamma_{i,n} > 0}$  is the per-capita payment that bank  $i$  will make to bank  $n$ . This payment will be executed only if the realization of the aggregate risk factor is

$D = 1$  and bank  $i$  net sold protection to bank  $n$  ( $\gamma_{i,n} > 0$ ). In this case, bank  $i$  will pay  $\gamma_{i,n}$  if it does not default (event  $A_i^c$ ) or  $r\gamma_{i,n}$  if it defaults (event  $A_i$ )

We calculate the certainty equivalent  $x_i$  of  $X_i$  by solving  $U(x_i) = E[U(X_i)]$ , which yields

$$x_i = \omega_i + \sum_{n \neq i} \gamma_{i,n} R_{i,n} - \Gamma^i(\gamma_{i,1}, \dots, \gamma_{i,M}), \quad (1)$$

where

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log E \left[ \exp \left( \eta D \left( \omega_i + \sum_{n \neq i} y_n \left( (\mathbb{1}_{A_i^c} + r \mathbb{1}_{A_i}) \mathbb{1}_{y_n > 0} + (\mathbb{1}_{A_n^c} + r \mathbb{1}_{A_n}) \mathbb{1}_{y_n < 0} \right) \right) \right) \right].$$

The following result gives an explicit formula for  $\Gamma^i$ .

**Lemma 3.1.** *We have*

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log \left( 1 - q + q e^{\eta \omega_i + \eta f(\sum_{n: y_n \geq 0} y_n, p_i) + \eta \sum_{n: y_n < 0} f(y_n, p_n)} \right),$$

where

$$f(y, p) = \frac{1}{\eta} \log \left( (1 - p) e^{\eta y} + p e^{\eta r y} \right). \quad (2)$$

For  $p > 0$ , the functions

$$y \mapsto \Xi(y) := \frac{1}{\eta} \log(1 - q + q e^{\eta y}) \quad \text{and} \quad y \mapsto f(y, p)$$

are strictly increasing and strictly convex so that the function  $\Gamma^i(y_1, \dots, y_M)$  is strictly increasing and convex. If  $p_n > 0$ , then the function  $\Gamma^i$ , viewed as a function of  $y_n$ , is strictly convex on  $(-\infty, 0)$ . Moreover, the function  $f$  satisfies

$$f(y_1, p_1) + f(y_2, p_2) > f(y_1 + y_3, p_1) + f(y_2 - y_3, p_2) \quad (3)$$

for all  $y_1 < y_2$ ,  $y_3 \in (0, \frac{y_2 - y_1}{2}]$  and  $p_1 \geq p_2$ .

The value  $f(y, p)$  quantifies how the exposure of bank  $i$  to the aggregate risk factor  $D$  changes when it sells  $y$  (or buys  $y$  if  $y < 0$ ) contracts to (from) bank  $n$ , where  $p$  is the default probability of the bank selling the contracts. If the bank that sells the contracts is safe ( $p = 0$ ), then  $f(y, p) = y$  as the increase in exposure corresponds to the number of traded contracts in this case. However, if the bank that is selling the contracts is risky ( $p > 0$ ), the increase in exposure is smaller given that

$$\frac{1}{\eta} \log \left( (1 - p) e^{\eta y} + p e^{\eta r y} \right) \begin{cases} < \frac{1}{\eta} \log \left( (1 - p) e^{\eta y} + p e^{\eta y} \right) = y & \text{if } y > 0 \\ > \frac{1}{\eta} \log \left( (1 - p) e^{\eta y} + p e^{\eta y} \right) = y & \text{if } y < 0. \end{cases} \quad (4)$$

The inequality (3) has a very intuitive interpretation. Suppose a bank buys CDS protection from banks 1 and 2 with default probabilities  $p_1 > p_2$ . If the bank were to buy additional protection from bank 1, its certainty equivalent would be lower with respect to the case in which it makes balanced purchases from the two banks.

**Remark 3.2.** *By analyzing the structure of (4), we observe that counterparty risk is accounted for asymmetrically in the contract valuation. Assume that bank  $i$  is selling contracts to bank  $n$ . Bank  $n$  takes the default risk of bank  $i$  into account by applying a credit valuation adjustment (CVA) to the contract. The CVA is defined as the adjustment to the default-free contract valuation made by bank  $n$  in the amount of the difference between (i) the value of the contract if its counterparty bank  $i$  were safe and (ii) the actual contract value which accounts for bank  $i$ 's default risk. Because of the CVA, the actual exposure of bank  $n$  to the aggregate risk factor is reduced by less than the total volume purchased from bank  $i$ . The reason for this smaller reduction is that bank  $i$  is risky and may not deliver the promised payment to bank  $n$  if it defaults. Analogously to the CVA, the debit valuation adjustment (DVA) is the adjustment made to the default-free contract valuation by bank  $i$  to account for its own default risk. Because bank  $i$  may default and not deliver the payment promised to bank  $n$ , its actual exposure is increases less than the nominal amount.*

*CVA and DVA are generally accepted principles for fair-value accounting (see also Duffie and Huang (1996) and Capponi (2013)). Because of risk aversion, DVA and CVA are not symmetric, i.e.,  $DVA \neq -CVA$ . The loss incurred by the buyer for not receiving the payment at default of the seller is higher than the gain of the seller for not making the promised payment to the protection buyer. In the limiting case of risk-neutral investors, we recover symmetry with  $DVA = -CVA$ .*

The extent to which the change in exposure is reduced by trading CDS contracts depends on the default probability  $p$  of the protection seller and its recovery rate  $r$ . Because of the bank's risk aversion, the change in exposure after trading is always smaller for a buyer (and higher for a seller) compared with the case where investors are risk neutral. This asymmetry means that risk-averse investors value their counterparty risk benefit (DVA) less than risk-neutral investors when they are selling and value their counterparty risk cost (CVA) more than risk-neutral investors when they are buying protection. Mathematically,

$$\frac{1}{\eta} \log((1-p)e^{\eta y} + pe^{\eta r y}) > \frac{1}{\eta} \log(e^{(1-p)\eta y + p\eta r y}) = y - (1-r)py,$$

using the strict convexity of the exponential function. For a buyer ( $y < 0$ ), this means that the negative quantity  $\frac{1}{\eta} \log((1-p)e^{\eta y} + pe^{\eta r y})$  is smaller in absolute value than  $y - (1-r)py$ .

## 4 Market Equilibrium Conditional on Banks' Default Risk

This section studies the market equilibrium, fixing the default risk profile of the banks in the system. In Section 4.1, we establish the existence of such an equilibrium. In Section 4.2, we study the implications of counterparty risk on the banks' post-trade exposures. In Section 4.3, we analyze which banks emerge as intermediaries and how this depends on their relative initial exposures and default risks.

## 4.1 Market Equilibrium Existence and Properties

Suppose that the default probability of bank  $i$  is  $p_i$ . Because traders are assumed to be small relative to their banks, they only have a marginal effect. When bank  $i$  sells protection to bank  $n$ , the cost of risk bearing increases by  $\gamma_{i,n}\Gamma_{y_n}^i(\gamma_i)$  for bank  $i$  and decreases by  $\gamma_{i,n}\Gamma_{y_i}^n(\gamma_n)$  for bank  $n$ , where  $\Gamma_{y_i}^n(\gamma_n)$  denotes the partial derivative of  $\Gamma^n(\gamma_n)$  with respect to the  $i$ -th component. Therefore, when traders of banks  $i$  and  $n$  bargain, their trading surplus is given by

$$\gamma_{i,n}(\Gamma_{y_i}^n(\gamma_n) - \Gamma_{y_n}^i(\gamma_i)).$$

This trading surplus is maximized by

$$\gamma_{i,n} \begin{cases} = k & \text{if } \Gamma_{y_n}^i(\gamma_i) < \Gamma_{y_i}^n(\gamma_n), \\ \in [-k, k] & \text{if } \Gamma_{y_n}^i(\gamma_i) = \Gamma_{y_i}^n(\gamma_n),^5 \\ = -k & \text{if } \Gamma_{y_n}^i(\gamma_i) > \Gamma_{y_i}^n(\gamma_n), \end{cases} \quad (5)$$

which is the traded quantity when two traders of banks  $i$  and  $n$  meet. The unit price  $R_{i,n}$  of a CDS is decided via bargaining between a protection seller with bargaining power  $\nu \in [0, 1]$  and a protection buyer with bargaining power  $1 - \nu$ . Hence,

$$R_{i,n} = \nu \max \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \} + (1 - \nu) \min \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \}. \quad (6)$$

If bank  $i$  sells contracts to bank  $n$ , it receives a fraction  $\nu$  of the trading surplus. Indeed, bank  $i$ 's cost of risk bearing increases by  $\gamma_{i,n}\Gamma_{y_n}^i(\gamma_i)$ , but it receives a payment  $\gamma_{i,n}R_{i,n}$  so that the net effect on bank  $i$  is

$$\begin{aligned} & -\gamma_{i,n}\Gamma_{y_n}^i(\gamma_i) + \gamma_{i,n}R_{i,n} \\ &= \gamma_{i,n} \left( \underbrace{\nu \max \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \}}_{= \Gamma_{y_i}^n(\gamma_n) \text{ by (5)}} + (1 - \nu) \underbrace{\min \{ \Gamma_{y_n}^i(\gamma_i), \Gamma_{y_i}^n(\gamma_n) \}}_{= \Gamma_{y_n}^i(\gamma_i) \text{ by (5)}} \right) - \Gamma_{y_n}^i(\gamma_i) \\ &= \nu \underbrace{\gamma_{i,n}(\Gamma_{y_i}^n(\gamma_n) - \Gamma_{y_n}^i(\gamma_i))}_{\text{trading surplus}}. \end{aligned}$$

Because of the translation invariance property of the exponential utility, the relative bargaining power between buyers and sellers does not affect how traded quantities are chosen in equilibrium. However, it has an effect on how banks choose their default probabilities before trading starts, as we will see in Section 5.

**Definition 4.1.** *Feasible contracts  $(\gamma_{i,n})_{i,n=1,\dots,M}$  build a market equilibrium if they are optimal in the sense that they satisfy (5).*

The following result shows that finding a market equilibrium is equivalent to solving a planning problem.

---

<sup>5</sup>For  $\gamma_{i,n} = 0$  where  $\Gamma^i$  is not differentiable with respect to  $y_n$ , both one-sided partial derivatives must match, i.e.,  $\lim_{\gamma_{i,n} \searrow 0} \Gamma_{y_n}^i(\gamma_i) = \lim_{\gamma_{n,i} \nearrow 0} \Gamma_{y_i}^n(\gamma_n)$  and  $\lim_{\gamma_{i,n} \nearrow 0} \Gamma_{y_n}^i(\gamma_i) = \lim_{\gamma_{n,i} \searrow 0} \Gamma_{y_i}^n(\gamma_n)$ , as  $\gamma_{i,n} = 0$  needs to be optimal with respect to both positive and negative changes.

**Theorem 4.2.** *Feasible contracts  $(\gamma_{i,n})_{i,n=1,\dots,M}$  are a market equilibrium if and only if they solve the optimization problem*

$$\text{minimize } \sum_{i=1}^M \Gamma^i(\gamma_i) \quad \text{over } \gamma \text{ subject to } \gamma_{i,n} = -\gamma_{n,i} \text{ and } -k \leq \gamma_{i,n} \leq k. \quad (7)$$

This result follows from the fact that certainty equivalents are quasi-linear so that feasible contracts are a solution to the planning problem if and only if they are Pareto optimal for the banks. Based on the quasi-linearity of certainty equivalents, Atkeson et al. (2015) find that, conditional on entry decisions, the pairwise traded contracts are socially optimal.

In our model, a market equilibrium on the level of the individual traders is thus equivalent to a Pareto optimal allocation for the banks. However, this holds only for given banks' default probabilities, and Pareto optimality for banks is only a statement about quantities and does not characterize prices. In our model, prices are determined in each meeting between two traders, as is standard in OTC market models.

**Theorem 4.3.** *There exists a market equilibrium  $(\gamma_{i,n})_{i,n=1,\dots,M}$ . The  $\gamma_{i,n}$ 's are unique for  $p_n > 0$  and  $\gamma_{i,n} < 0$ , or  $p_i > 0$  and  $\gamma_{i,n} > 0$ . For every  $i$ , the value of  $\sum \gamma_{i,n}$  is unique in equilibrium, where the sum is over  $n$  such that  $p_n = 0$  and  $\gamma_{i,n} < 0$ , or  $p_i = 0$  and  $\gamma_{i,n} > 0$ . In particular, the values of  $\Gamma(\gamma_n)$ 's are uniquely determined for a market equilibrium  $(\gamma_{i,n})_{i,n=1,\dots,M}$ .*

Theorem 4.3 establishes the existence of a market equilibrium, and states that quantities bilaterally traded with risky protection sellers are unique in equilibrium. This uniqueness result contrasts with Theorem 1 of Atkeson et al. (2015), where bilaterally traded volumes are not unique. As soon as counterparty risk is involved in a trade, bilaterally traded volumes are uniquely pinned down in equilibrium. The reason is that counterparty risk makes CDS contracts purchased from traders of different banks imperfect substitutes. Even if banks have the same default probability, CDSs purchased from them cannot be perfectly substituted due to counterparty concentration: because of risk aversion, if a trader buys two CDS contracts, he/she prefers to choose the two trading counterparties from different banks, rather than purchasing both contracts from traders of the same bank. However, if the seller of protection is a safe bank, there is an indifference to increasing or decreasing the trading volume as long as it can be balanced by other trades not involving counterparty risk. For example, trades between three safe banks A, B and C could be increased without changing the planning problem (7) if A buys  $n$  additional CDS contracts from B, B buys  $n$  additional CDS contracts from C, and C buys  $n$  additional CDS contracts from A.

## 4.2 Post-trade Exposures

Suppose that each bank has decided on its default risk  $p_i$  at cost  $C(p_i)$ , and denote by  $(\gamma_{i,n})_{i,n=1,\dots,M}$  a market equilibrium from Theorem 4.3. We define the per-capita post-trade exposure of bank  $i$  by

$$\Omega_i := \omega_i + f\left(\sum_{n:\gamma_{i,n} \geq 0} \gamma_{i,n}, p_i\right) + \sum_{n:\gamma_{i,n} < 0} f(\gamma_{i,n}, p_n), \quad (8)$$

where  $f$  is defined in Lemma 3.1. Note that  $\Omega_i$  accounts for counterparty risk: if bank  $i$  and all its counterparties are safe, then  $\Omega_i$  simplifies to  $\omega_i + \sum_{n \neq i} \gamma_{i,n}$ , as in Atkeson et al. (2015). Observe also that  $\Omega_i$  is uniquely determined by Theorem 4.3. If bank  $i$  buys  $-\gamma_{i,n}$  contracts on average from each trader of a risky bank  $n$ , the exposure of bank  $i$  is effectively reduced by less than  $\gamma_{i,n}$  — namely by  $f(\gamma_{i,n}, p_n)$  — to adjust for counterparty risk, taking the bank’s risk aversion into consideration. Similarly, if bank  $i$  is risky and sells  $\gamma_{i,n}$  contracts to each trader of bank  $n$ , then its effective increase in exposure is less than  $\gamma_{i,n}$  due to its own default risk (DVA), as discussed in Remark 3.2.

The next result says that the post-trade exposures are increasing and closer together than initial exposures. This result generalizes the first part of Proposition 1 of Atkeson et al. (2015) to our counterparty risk setting, while we will see in our Proposition 4.5 below that the second part of Proposition 1 of Atkeson et al. (2015) takes a quite different form in our model.

**Proposition 4.4.** *Assume that*

$$p_i = p_j \text{ or } p_i \leq 1/2 \text{ or } p_j \leq 1/2.^6 \tag{9}$$

*We then have the following relations between initial and post-trade exposures:*

1. *If  $\omega_i \geq \omega_j$  and  $p_i \leq p_j$ , then  $\Omega_i \geq \Omega_j$ .*
2. *If  $\omega_i > \omega_j$  and  $p_i \geq p_j$ , then  $\omega_i - \omega_j > \Omega_i - \Omega_j$ .*

Under condition (9), Proposition 4.4 states that

1. The banks’ order in post-trade exposures is the same as that in the initial exposures, provided that their default probabilities are ordered in the opposite direction.
2. Post-trade exposures are closer together than initial exposures if the bank with larger initial exposure is at least as risky as the bank with smaller initial exposure.

To see why conditions on the default risks of the banks need to be imposed, consider two banks  $i$  and  $j$  whose initial exposures  $\omega_i > \omega_j$  are smaller than the average initial exposure. Because both banks have initial exposures below the average, they are interested in selling protection and earning the CDS protection fee. These trading motives imply that their post-trade exposures  $\Omega_i$  and  $\Omega_j$  are bigger than  $\omega_i$  and  $\omega_j$ , respectively. However, if bank  $i$  is safer than bank  $j$ , it is likely that the other banks will buy a higher amount of protection from bank  $i$  so that  $\Omega_i - \omega_i > \Omega_j - \omega_j$ . This inequality stands in contrast with that in the second statement of Proposition 4.4, noting that  $p_i \geq p_j$  does not hold, either. Yet if bank  $j$  is safer than bank  $i$ , it is likely that the other banks will buy a larger amount of protection from bank  $j$ , leading to  $\Omega_j > \Omega_i$  even though the initial exposures had the reverse order. We will graphically demonstrate later in Figure 1 that both of these cases can indeed happen so that conditions on the default probabilities in Proposition 4.4 are needed. Building on Proposition 4.4, we next analyze which banks engage in full risk sharing if the trade size limit is big enough.

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<sup>6</sup>The condition  $p \leq 1/2$  is essentially used to avoid that the probability-weighted CDS protection payment in the seller’s default event,  $rp$ , is higher than  $1 - p$ , which is the probability-weighted payment if the protection seller does not default.

**Proposition 4.5.** *Assume that the trade size limit is not binding and that there are at least two safe banks.<sup>7</sup> Then*

1. *All safe banks have the same post-trade exposure, say,  $\bar{\Omega}$ .*
2. *Risky banks with initial exposure above some threshold  $\alpha$  also have the same post-trade exposure  $\bar{\Omega}$ . The threshold  $\alpha$  depends only on the distribution of initial exposures and not on the banks' default probabilities.*
3. *Risky banks with initial exposure below  $\alpha$  will have post-trade exposures strictly smaller than  $\bar{\Omega}$ .*

The first part of the proposition is consistent with Proposition 3 in Atkeson et al. (2015): if the trade size limit is big enough, safe banks perfectly share their risk to the aggregate risk factor so that they all end up with the same post-trade exposure. Risky banks with large initial exposures are active as buyers on the OTC market. Hence, their default risk does not matter to traders of other banks and they have the same post-trade exposure as the safe banks, as stated by the second part of the proposition. By contrast, risky banks with small initial exposures would like to sell credit protection, but they are not very attractive as trading counterparties because they bear high default risk. Consequently, they will have a lower post-trade exposure than the safe banks, as stated in the third part of the proposition.

To highlight these findings, we construct a parametric example of 30 banks and show the resulting post-trade exposures in Figure 1. Note that the dashed and dotted curves hit the blue line at the same point, which means that the initial exposure needed to guarantee that risky banks have the same post-trade exposure does not depend on their default probabilities. This observation is a consequence of Proposition 4.5, and follows from the fact that the threshold  $\alpha$  in Proposition 4.5 does not depend on the banks' default probabilities. This is because when the bank's initial exposure becomes sufficiently high, the bank will trade in only one direction, buying (and not selling) protection against the aggregate risk factor. As the protection fee is paid upfront, the default risk of the bank does not matter to the seller. Proposition 4.5 implies that the post-trade exposure of safe banks is higher than their average initial exposure, and this is visually confirmed in Figure 1.

An immediate consequence of Proposition 4.5 is the sensitivity of the post-trade exposures to the banks' default probabilities.

**Corollary 4.6.** *If the trade size limit is big enough, the post-trade exposures of banks with sufficiently high initial exposure are not sensitive to their default probabilities, while the post-trade exposures of banks with small initial exposures are sensitive to their default probabilities.*

The statement in Corollary 4.6 is intuitive. If the risk bearing capacity of the market is not impaired by the presence of trade size limits, banks with sufficiently large initial exposures are protection buyers and thus their own default probabilities do not matter. However, banks with low initial exposures are protection sellers, so their default probabilities matter when other banks decide to trade with them.

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<sup>7</sup>For risk sharing, we need at least two banks. Because perfect risk sharing is done by safe banks, we consider in the proposition a market environment with at least two safe banks.

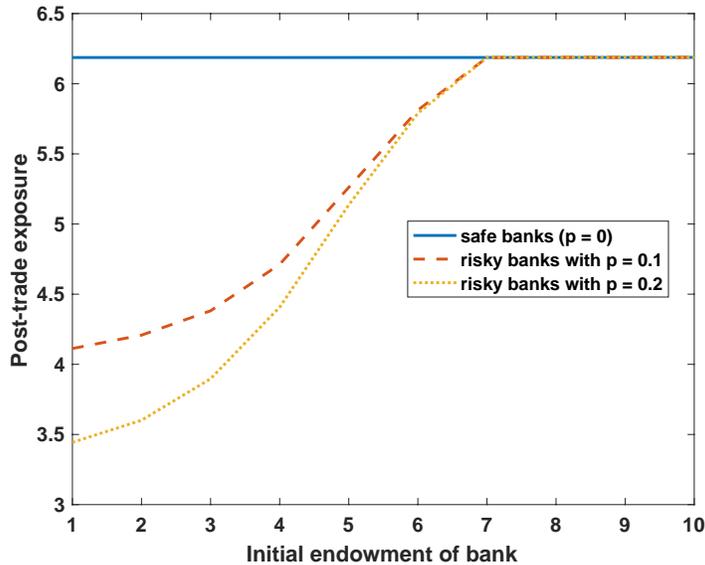


Figure 1: A market model consisting of 30 banks: for each initial exposure  $1, 2, \dots, 10$ , we consider three banks, respectively with default probabilities  $p = 0$ ,  $p = 0.1$  and  $p = 0.2$ . For large enough  $k$ , all safe banks and all risky banks with big initial exposures have the same post-trade exposure. The corresponding value 6.19 is higher than the average initial exposure of the safe banks,  $5.5 (= (1 + 2 + \dots + 10)/10)$ . Risky banks with small initial exposures have a smaller post-trade exposure than safe banks. Risky banks with  $p = 0.2$  (dotted curve) have a smaller post-trade exposure than risky banks with  $p = 0.1$  (dashed curve). We set the risk aversion parameter  $\eta = 1$ , the recovery rate  $r = 0$ , the default probability of the aggregate risk factor  $q = 0.1$ , and the trade size limit  $k = 2$ .

### 4.3 Intermediation Volume

We study which banks endogenously emerge as intermediaries. These banks participate on both sides of the CDS market, as opposed to taking large net positions, either long or short. We consider per-capita gross numbers of sold or purchased contracts, accounting for counterparty risk similarly to the post-trade exposure in (8). For a trader of bank  $i$ , these quantities are given by

$$G_i^+ = f\left(\sum_{n:\gamma_{i,n} \geq 0} \gamma_{i,n}, p_i\right) \quad \text{and} \quad G_i^- = - \sum_{n:\gamma_{i,n} < 0} f(\gamma_{i,n}, p_n).$$

If bank  $i$  is safe ( $p_i = 0$ ), then  $G_i^+ = \sum_{n \neq i} \max\{\gamma_{i,n}, 0\}$ . Similarly, we have  $G_i^- = \sum_{n \neq i} \max\{-\gamma_{i,n}, 0\}$  if all its counterparties  $n$  are safe. In general, however, the actual exposure is obtained by adjusting for DVA and CVA so that  $G_i^+ \leq \sum_{n \neq i} \max\{\gamma_{i,n}, 0\}$  and  $G_i^- \leq \sum_{n \neq i} \max\{-\gamma_{i,n}, 0\}$ ; see Remark 3.2. The per-capita intermediation volume of bank  $i$  is defined as  $I_i = \min\{G_i^+, G_i^-\}$ . By Theorem 4.3,  $G_i^+$  and  $G_i^-$  and thus the intermediation volume  $I_i$  are uniquely determined if all  $p_n$ 's are strictly positive. Hence, we work under this assumption in this section. We analyze separately the effects of banks' initial exposures and default probabilities on their intermediation volume.

**Proposition 4.7.** *If the trade size limit  $k$  is small enough and there are at least three banks with different initial exposures  $\omega_i$ 's, then the intermediation volume  $I_i$  as a function of  $\omega_i$  is a hump-shaped curve, taking its maximum at or next to the median initial exposure.*

*Additionally, if (9) holds and two banks  $i$  and  $j$  have the same initial exposure, then  $I_i \leq I_j$  for  $p_i \geq p_j$ .*

The most prominent implication of Proposition 4.7 is that banks with intermediate exposures and small default probabilities are the main intermediaries. This prediction is consistent with empirical data from the bilateral CDS market, as shown in Figure 2. We describe the data set and the procedure followed to generate the intermediation plots in Appendix B. As it appears from the figure, most of the traded CDS volume is either between two intermediaries or between an intermediary and a non-intermediary. The volume of traded CDS contracts between intermediaries is high, but each intermediary has large trading positions with only a few, and not all, other intermediaries. There is high heterogeneity in the volume of traded CDS contracts for the banks that are not intermediaries: some banks (mainly those with very small initial exposures) trade a very small volume of CDSs, while others are either large buyers or large sellers of CDSs. Figure 3 further highlights that the main intermediaries are the banks with medium initial exposures and low default probabilities, relative to all banks in the data set.

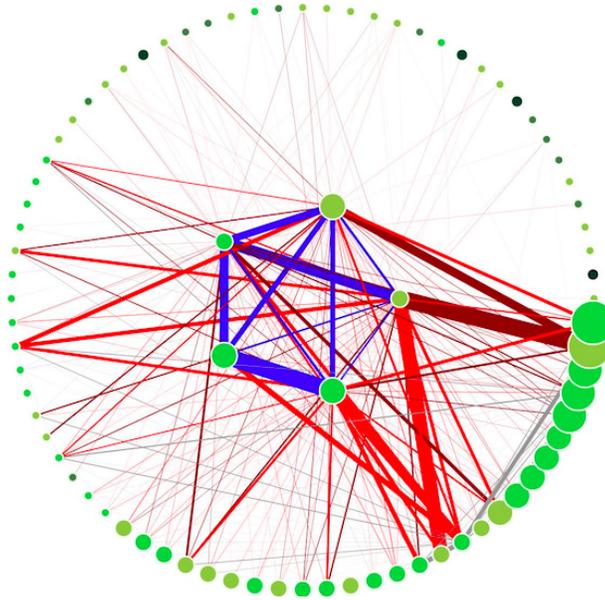


Figure 2: Network of banks' bilateral CDS exposures. Each node corresponds to a bank. The inner nodes are the 5 intermediaries, while the remaining 76 banks are arranged as nodes on an outside circle. Both in the inner area and the outside circle, the nodes are ordered by their initial exposures, which correspond to the sizes of the nodes. The darker a node is, the higher is the default probability of the corresponding bank. The widths of the edges are proportional to the banks' bilateral net CDS volume. We use blue for the CDS volume between two intermediaries; gray for CDS volume between two non-intermediaries; light red for CDS protection sold by an intermediary to a non-intermediary; dark red for CDS protection sold by a non-intermediary to an intermediary.

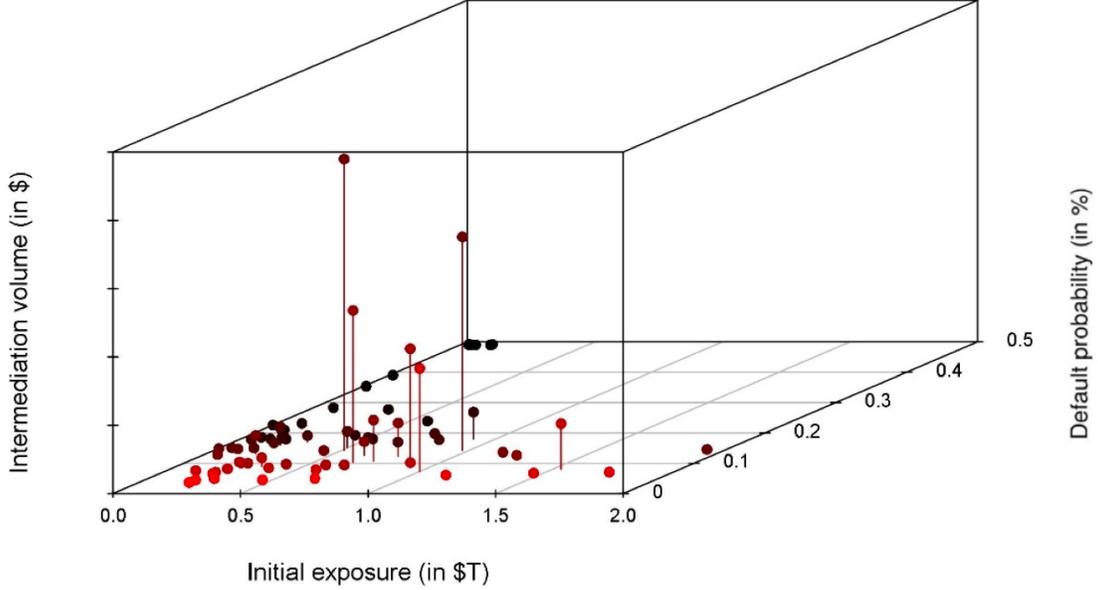


Figure 3: Intermediation volume as a function of initial exposures, measured in trillion USD (\$ T), and default probabilities: each point denotes a bank. The higher the default probability of a bank, the darker the color of the corresponding point.

## 5 Private versus Socially Optimal Default Risk Levels

In this section, we compare the banks' decisions on their default probabilities with the socially optimal levels. Recall that each bank  $i$  manages its risk by choosing the conditional default probability  $p_i \in [0, \bar{p}_i]$ , where  $\bar{p}_i$  is the given maximal value. Bank  $i$  can lower its conditional default probability to  $p_i$  at a cost  $C(p_i)$ . We assume that  $C : [0, \bar{p}_i] \rightarrow [0, \infty)$  is a decreasing, convex, and continuous function. Let  $p_i \in [0, \bar{p}_i]$  be the decision of bank  $i$ . Theorem 4.3 yields that for given  $p_1, \dots, p_M$ , there exists a market equilibrium  $(\gamma_{i,n})_{i,n=1,\dots,M}$ . As we focus in this section on the choice of  $p_1, \dots, p_M$ , we write

$$x_i(p_1, \dots, p_M) = \omega_i + \sum_{n \neq i} \gamma_{i,n} R_{i,n} - \Gamma^i(\gamma_i) \quad (10)$$

to denote bank  $i$ 's per-capita certainty equivalent (1) in a market equilibrium.

**Lemma 5.1.** *The value of  $x_i(p_1, \dots, p_M)$  is uniquely determined.*

Because each bank chooses individually its default risk, we are looking for a Nash equilibrium.

**Definition 5.2.** *A choice of  $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$  is an equilibrium if*

$$x_i(p_1, \dots, p_M) - C(p_i) \geq x_i(p_1, \dots, p_{i-1}, \tilde{p}_i, p_{i+1}, \dots, p_M) - C(\tilde{p}_i)$$

for all  $i$  and  $\tilde{p}_i \in [0, \bar{p}_i]$ .

**Proposition 5.3.** *If the cost function  $C$  is such that*

$$\arg \max_{p_i \in [0, \bar{p}_i]} (x_i(p_1, \dots, p_M) - C(p_i)) \quad (11)$$

*is a convex set for each  $i$ , then there exists an equilibrium  $p_1, \dots, p_M$ .*

The assumption that (11) is a convex set means that if  $\hat{p}_i$  and  $p_i^*$  are maximizers of  $x_i(p_1, \dots, p_M) - C(p_i)$ , then so is any convex combination of  $\hat{p}_i$  and  $p_i^*$ . Note that, in particular, this assumption is satisfied if there is a unique maximizer.

We consider a social planner who chooses the banks' default probabilities  $p_1, \dots, p_M$  and the quantities of traded contracts  $(\gamma_{i,n})_{i,n=1,\dots,M}$  so to maximize the banks' aggregate certainty equivalent minus the risk management costs. The planner maximizes the objective function

$$\sum_{i=1}^M x_i(p_1, \dots, p_M) - \sum_{i=1}^M C(p_i) \quad (12)$$

over  $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$  and  $(\gamma_{i,n})_{i,n=1,\dots,M}$  subject to  $\gamma_{i,n} = -\gamma_{n,i}$  and  $-k \leq \gamma_{i,n} \leq k$ , where

- $\sum_{i=1}^M x_i(p_1, \dots, p_M)$  is the aggregate certainty equivalent of the banks with default probabilities  $p_1, \dots, p_M$ .
- $\sum_{i=1}^M C(p_i)$  is the sum of the costs incurred to reduce the default risk probabilities to the levels  $p_1, \dots, p_M$ .

It follows from  $\gamma_{i,n} = -\gamma_{n,i}$  and  $R_{i,n} = R_{n,i}$  that  $\sum_{i=1}^M x_i(p_1, \dots, p_M) = \sum_{i=1}^M \omega_i - \sum_{i=1}^M \Gamma^i(\gamma_i)$ . Therefore, the social planner's optimization problem (12) is equivalent to minimize

$$\sum_{i=1}^M \Gamma^i(\gamma_i) + \sum_{i=1}^M C(p_i)$$

over the same optimization variables  $p_1 \in [0, \bar{p}_1], \dots, p_M \in [0, \bar{p}_M]$  and  $(\gamma_{i,n})_{i,n=1,\dots,M}$ . Hence, the social planner minimizes the aggregate cost of pre-trade default reduction and the level of post-trade risk.

**Proposition 5.4.** *The social planner's optimization problem has a solution.*

We will next analyze when and how a solution chosen by the individual banks differs from the social optimum. The difference between individual and social optimization problems comes from the buyers' surplus, which corresponds to the trade benefit of the banks purchasing CDS protection. This surplus is reflected in the social welfare, but is not taken into account in the individual optimization problems of the CDS selling banks. As we will demonstrate, this surplus depends on the banks' default probabilities, and thus leads to different choices of individually and socially optimal default probabilities, giving rise to an externality. The buyers' surplus is equal to the portion of the trading benefit that the CDS buyer receives. When the trade size limit is binding, this portion depends on the seller bargaining power  $\nu$ . To highlight the primary economic forces, we first consider the case

where sellers have full bargaining power (i.e.,  $\nu = 1$ ), and we discuss later how the size of this externality depends on  $\nu$ . When the trade size limit is binding, the buyers' surplus changes only because the elasticity of the demand curve varies, while traded quantities remain constant at the trade size limit. This is also illustrated in Figure 4, where it can be seen that the buyers' surplus decreases when the default probability decreases. This can be explained by the fact that, below the trade size limit, the demand curve is more elastic when the default probability of the protection seller decreases. This is because a higher credit quality of protection sellers mitigates the risk due to counterparty concentration. We recall here that counterparty concentration reduces the elasticity of the demand curve because each additionally purchased CDS contract carries less value for the protection buyer. These intuitions are formalized in the following lemma.

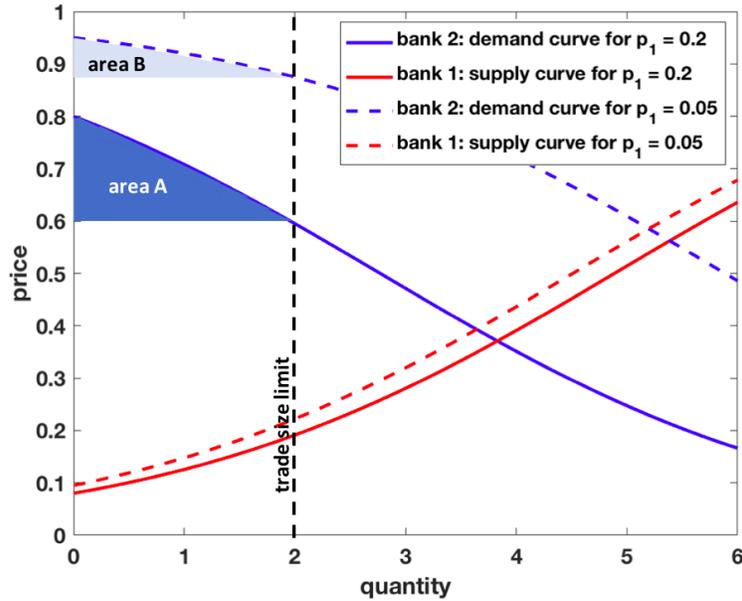


Figure 4: The dependence of the buyers' surplus on default probabilities. The protection seller has full bargaining power, and a trade size limit  $k = 2$  is imposed. Interestingly, the buyers' surplus for bank 2 is higher when bank 1 has default probability 0.2 (area A) than when it has default probability 0.05 (area B). We choose the risk aversion parameter  $\eta = 1$ , the recovery rate  $r = 0$ , the default probability of the aggregate risk factor  $q = 0.1$ , and the initial exposures of banks 1 and 2 to  $\omega_1 = 0$ , and  $\omega_2 = 100$ , respectively.

**Lemma 5.5.** *For large enough  $q$ , the demand curve for CDSs by traders of the same bank becomes more elastic on the short end, if the default probability of the protection seller decreases.*

A more elastic demand curve, combined with a binding trade size limit, means that the buyers' surplus decreases, as shown in Figure 4. Because of the decreasing buyers' surplus (reflected in the social welfare), the socially optimal default probabilities may be higher than those individually chosen by each bank. The following result shows that a policy maker may mitigate the inefficiency in the banks' risk management decisions and enhance welfare through a suitably designed tax and subsidy system.

**Theorem 5.6.** *A solution to the social planner's optimization satisfies the first-order conditions of an equilibrium if bank  $i$  receives a subsidy equal to  $S = S_1 + k(1 - \nu)S_2$  with*

$$S_1 := - \sum_{n \neq i} (\gamma_{i,n} \Gamma_{y_i}^n(\gamma_n, p) + \Gamma^n(\gamma_n, p)), \quad S_2 := \sum_{n \neq i} (\Gamma_{y_i}^n(\gamma_n, p) - \Gamma_{y_n}^i(\gamma_i, p)),$$

where we highlighted the dependence on  $p = (p_1, \dots, p_M)$  in the above expressions.

Assuming a small enough trade size limit, we have  $\frac{\partial S_1}{\partial p_i} > 0$  and  $\frac{\partial S_2}{\partial p_i} < 0$  for small enough  $p_i$  and large enough  $q$ . In this case, the privately chosen  $p_i$ 's are lower than the socially optimal ones if sellers have full bargaining power. The difference between the socially and individually optimal choices of  $p_i$  increases as a function of the seller bargaining power.

To induce optimal risk management, a policy maker needs to give a subsidy to some banks and collect a tax from other banks in the amount of the difference between marginal social and marginal private value. The tax would be collected from CDS buyers maximal to the amount of the buyers' surplus and given as a subsidy to CDS sellers to compensate them for their social contribution in reducing exposure to the aggregate risk factor. Such a tax would depend on the volume of traded CDS contracts and on the default probabilities of the CDS sellers, which in turn depend on their initial exposures.

The first part of Theorem 5.6 states that the subsidy can be decomposed into a part ( $S_1$ ) that is independent of the bargaining power and a part ( $k(1 - \nu)S_2$ ) that depends linearly on the bargaining power. The second part of Theorem 5.6 relates the banks' optimal choice of default probabilities to the socially optimal choice. As the seller bargaining power increases, a phase transition may occur: the banks' choice of default probabilities may switch from being above the socially optimal level to falling below it. This phenomenon can be explained as follows. When the buyers have high bargaining power, their trade benefit may increase if the sellers' default probabilities are reduced. By contrast, if the buyers' bargaining power is low, the benefits of bargaining are more than offset by the reduced surplus resulting from the increased elasticity of the demand curve.

The threshold on the seller bargaining power at which a phase transition occurs depends crucially on the banks' initial exposures. This dependence is graphically illustrated in the left panel of Figure 5 for a market consisting of three banks. Bank 1 has zero initial exposure,  $\omega_1 = 0$ , and acts as a CDS seller. It chooses a smaller default probability than bank 2, which has medium initial exposure,  $\omega_2 = 2$ , and intermediates between banks 1 and 3. Bank 3 has the highest initial exposure,  $\omega_3 = 10$ , and thus chooses to purchase CDS protection. Being a protection buyer, bank 3 does not reduce its default probability from the maximal level  $\bar{p}_3 = 0.2$ . It can be seen from Figure 5 that a higher level of bargaining power is necessary for bank 1 ( $\nu \geq 0.30$ ) to reduce its default probability below the social optimum, as compared with bank 2 ( $\nu \geq 0.22$ ). The phase transitions occur at different levels because the individually chosen default probabilities of banks 1 and 2 are closer together than their socially optimal ones; see the left panel of Figure 5. Targeting the aggregated certainty equivalents, the social planner chooses a lower default probability for bank 1 than for bank 2, because a default of bank 1 affects the two other banks (banks 2 and 3), while a default of bank 2 affects only bank 3. On an individual level, bank 1 will also reduce its default probability more than bank 2. However, the difference in default probabilities resulting from the banks' equilibrium choices is smaller than what would be socially optimal. The right

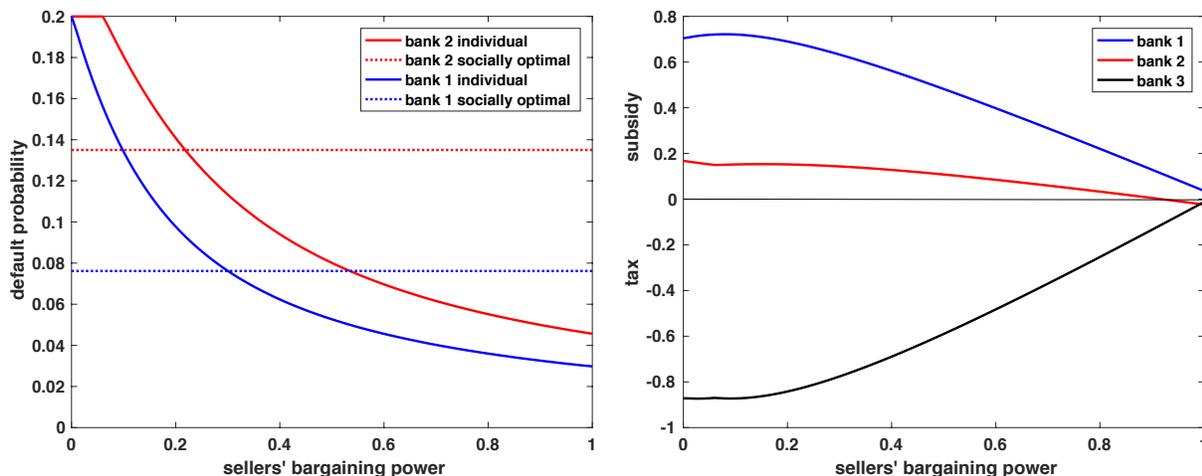


Figure 5: Left panel: individually and socially optimal choices of default probabilities. Right panel: subsidy and tax required to make individual choices efficient. The bank with the lowest initial exposure (Bank 1), acting as a CDS seller, chooses the lowest default probability and receives the highest subsidy. The bank with the highest initial exposure (Bank 3), acting as a CDS buyer, is the primary tax payer. The bank with the medium initial exposure (Bank 2), which intermediates between banks 1 and 3, receives a subsidy for acting as CDS seller to bank 3 and pays a tax for being a CDS buyer from bank 1, resulting in the net subsidy displayed on the right panel. We set the risk aversion parameter  $\eta = 1$ , the recovery rate  $r = 0$ , the default probability of the risk factor  $q = 0.1$ , the trade size limit  $k = 0.5$ , the cost function  $C(p) = 1/p^{0.05}$ , and the maximal values of default probabilities  $\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = 0.2$ . The banks' initial exposures are  $\omega_1 = 0$ ,  $\omega_2 = 2$  and  $\omega_3 = 10$ .

panel of Figure 5 shows that a subsidy-tax policy that achieves efficiency would subsidize bank 1 with the highest amount, as it acts on the sell side for both banks 2 and 3, and impose the highest tax on bank 3. Bank 2, acting as an intermediary, would benefit from the net effect of subsidies received for selling CDS contracts to bank 2 and tax paid for buying CDS contracts from bank 1. Bank 2's subsidy is much smaller than that of bank 1, but its individually chosen default probability deviates more from the social optimum, relative to that of bank 1. This is seemingly puzzling observation can be understood as follows: as an intermediary, the choice of bank 2's default probability is more sensitive to subsidies and taxes than that of the protection selling bank 2.

## 6 Conclusion

How do participants of OTC markets account for counterparty risk when they negotiate prices and quantities of traded contracts? Do they manage their own default risk to be more attractive trading counterparties? Do market participants diversify counterparty risk in OTC markets? If so, how can they achieve this? Answering these questions is of critical importance to understand the structure of OTC markets, the risk management decisions of their participants, and the social implications of their trading patterns.

In this paper, we study the incentives behind the choices of banks' default probabilities,

along with the role played by counterparty risk in influencing trading decisions and the resulting structure of OTC markets. We show that banks' trading and risk management decisions arising in equilibrium are inefficient. Negative externalities arise because protection selling banks are not exactly compensated for their contribution in reducing the system-wide risk exposure. Our results show that banks may reduce their default probabilities below what is socially optimal to benefit from higher fees. These decisions depend on the banks' initial exposures to an aggregate risk factor and on their bargaining power as sellers. Intermediaries contribute to social welfare by reducing the frictions caused by the trade size limit, and more importantly, counterparty risk. Our model predicts that the main intermediaries have medium initial exposure and low default risk, and that banks engage in less risk-sharing in a market with higher counterparty risk.

Our framework can be extended along several directions. A first extension is to construct a model that can capture the dynamic formation of interbank trading relations, taking counterparty risk into consideration. Secondly, it can be generalized to include a role for the real economy. In such a model extension, banks might have obligations to the private sector and, additionally, fees charged as a result of a CDS trade affect the lending activities of the banks to the real economy. A third extension is to compare trading decisions when market participants have the choice between bilateral OTC trading, which exposes them to counterparty risk, and centralized trading. In the latter case, the clearinghouse insulates banks from counterparty risk, but they would be required to additionally pay clearing fees. A recent work by Dugast et al. (2018) studies the welfare implications of central clearing, building on the framework of Atkeson et al. (2015). Their main focus is on the trading capacity and costs of joining the centralized clearing platform. Our proposed work would complement theirs by accounting for the netting and counterparty risk reduction benefits of a clearinghouse.

## A Results and their Proofs

This section contains the proofs of our results done for arbitrary sizes  $s_i$  of the banks. When the formulation of the statement is different from that in the case  $s_i = 1$ , we restate the result.

### A.1 Proof of Lemma 3.1

Using  $P[D = 1] = q$ , we compute

$$\begin{aligned}
\Gamma^i(y_1, \dots, y_M) &= \frac{1}{\eta} \log E \left[ \exp \left( \eta D \left( \omega_i + \sum_{n \neq i} y_n \left( (\mathbb{1}_{A_i^c} + r \mathbb{1}_{A_i}) \mathbb{1}_{y_n > 0} + (\mathbb{1}_{A_n^c} + r \mathbb{1}_{A_n}) \mathbb{1}_{y_n < 0} \right) \right) \right) \right] \\
&= \frac{1}{\eta} \log \left( 1 - q + q E \left[ \exp \left( \eta \omega_i + \eta \sum_{n \neq i} y_n \left( (\mathbb{1}_{A_i^c} + r \mathbb{1}_{A_i}) \mathbb{1}_{y_n > 0} + (\mathbb{1}_{A_n^c} + r \mathbb{1}_{A_n}) \mathbb{1}_{y_n < 0} \right) \right) \right] \right) \\
&= \frac{1}{\eta} \log \left( 1 - q + q e^{\eta \omega_i} E \left[ \exp \left( \eta (\mathbb{1}_{A_i^c} + r \mathbb{1}_{A_i}) \sum_{n \neq i: y_n > 0} y_n \right) \right] \prod_{n \neq i: y_n < 0} E \left[ \exp \left( \eta (\mathbb{1}_{A_n^c} + r \mathbb{1}_{A_n}) y_n \right) \right] \right).
\end{aligned}$$

Using that  $p_i = P[A_i]$ , we obtain

$$\Gamma^i(y_1, \dots, y_M) = \frac{1}{\eta} \log \left( 1 - q + qe^{\eta\omega_i} \left( (1 - p_i)e^{\eta \sum_{n: y_n > 0} y_n} + p_i e^{\eta r \sum_{n: y_n > 0} y_n} \right) \right. \\ \left. \times \prod_{n \neq i, y_n < 0} \left( (1 - p_n)e^{\eta y_n} + p_n e^{\eta r y_n} \right) \right),$$

which can be brought into the form  $\Gamma^i(y_1, \dots, y_M)$  written in the statement of Lemma 3.1.

To show the additional properties of  $\Gamma^i(y_1, \dots, y_M)$ , we first note that the function  $\Xi$  given by

$$\Xi(y) = \frac{1}{\eta} \log (1 - q + qe^{\eta y}) \quad (13)$$

is strictly increasing and strictly convex. Indeed, we can calculate

$$\Xi'(y) = \frac{qe^{\eta y}}{1 - q + qe^{\eta y}} > 0, \quad \Xi''(y) = \frac{(1 - q)q\eta e^{\eta y}}{(1 - q + qe^{\eta y})^2} > 0.$$

Next, we consider

$$f(y, p) = \frac{1}{\eta} \log ((1 - p)e^{\eta y} + pe^{\eta r y})$$

for  $p > 0$  and calculate

$$f_y(y, p) = \frac{(1 - p)e^{\eta y} + rpe^{\eta r y}}{(1 - p)e^{\eta y} + pe^{\eta r y}} > 0, \quad (14)$$

$$f_{yy}(y, p) = \eta \frac{((1 - p)e^{\eta y} + pe^{\eta r y})((1 - p)e^{\eta y} + r^2 pe^{\eta r y}) - ((1 - p)e^{\eta y} + rpe^{\eta r y})^2}{((1 - p)e^{\eta y} + pe^{\eta r y})^2} \\ = \eta \frac{p(1 - p)(1 - r)^2 e^{\eta(1+r)y}}{((1 - p)e^{\eta y} + pe^{\eta r y})^2} > 0. \quad (15)$$

These inequalities show that the function  $y \mapsto f(y, p)$  is strictly increasing and strictly convex for  $p > 0$ . Because  $f(y, p)$  either equals  $y$  (if  $p = 0$ ) or is strictly increasing and strictly convex (if  $p > 0$ ), we see that  $\Gamma^i(y_1, \dots, y_M)$  is strictly increasing, and the statements on convexity of  $\Gamma^i(y_1, \dots, y_M)$  now follow from the fact that convexity is maintained under sums and compositions with a convex, nondecreasing function.

Finally, to prove (3), let  $y_1 < y_2$ ,  $y_3 \in (0, \frac{y_2 - y_1}{2}]$  and  $p_1 \geq p_2$ . We first note that (3) is equivalent to

$$\left( (1 - p_1)e^{\eta y_1} + p_1 e^{\eta r y_1} \right) \left( (1 - p_2)e^{\eta y_2} + p_2 e^{\eta r y_2} \right) \\ > \left( (1 - p_1)e^{\eta(y_1 + y_3)} + p_1 e^{\eta r(y_1 + y_3)} \right) \left( (1 - p_2)e^{\eta(y_2 - y_3)} + p_2 e^{\eta r(y_2 - y_3)} \right),$$

which can be further simplified to

$$(1 - p_1)p_2 e^{\eta(y_1 + r y_2)} + (1 - p_2)p_1 e^{\eta(y_2 + r y_1)} > (1 - p_1)p_2 e^{\eta(y_1 + r y_2 + y_3(1 - r))} + (1 - p_2)p_1 e^{\eta(y_2 + r y_1 - y_3(1 - r))}.$$

This inequality follows from

$$ae^{x_1} + be^{x_2} > ae^{x_1 + x_3} + be^{x_2 - x_3} \quad (16)$$

for all  $a \leq b$ ,  $x_1 < x_2$  and  $x_3 \in (0, \frac{x_2 - x_1}{2}]$  by choosing

$$a = (1 - p_1)p_2, \quad b = (1 - p_2)p_1, \quad x_1 = \eta(y_1 + ry_2), \quad x_2 = \eta(y_2 + ry_1), \quad x_3 = \eta y_3(1 - r),$$

where we note that  $p_1 \geq p_2$ ,  $y_1 < y_2$ , and  $y_3 \in (0, \frac{y_2 - y_1}{2}]$  imply  $a \leq b$ ,  $x_1 < x_2$ , and  $x_3 \in (0, \frac{x_2 - x_1}{2}]$ . The inequality (16) can be seen from the convexity of the exponential function or checked directly by calculating the partial derivative

$$\frac{\partial}{\partial z}(ae^{x_1+z} + be^{x_2-z}) = ae^{x_1+z} - be^{x_2-z} \leq be^{x_1+z} - be^{x_2-z} < 0$$

for all  $z \in [0, \frac{x_2 - x_1}{2})$ . □

## A.2 Results of Section 4.1 and their Proofs

**Theorem A.1** (Theorem 4.2). *Feasible contracts  $(\gamma_{i,n})_{i,n=1,\dots,M}$  are a market equilibrium if and only if they solve the optimization problem*

$$\text{minimize } \sum_{i=1}^M s_i \Gamma^i(\gamma_i s) \quad \text{over } \gamma \text{ subject to } \gamma_{i,n} = -\gamma_{n,i} \text{ and } -k \leq \gamma_{i,n} \leq k, \quad (17)$$

where  $\gamma_i s := (\gamma_{i,1}s_1, \dots, \gamma_{i,M}s_M)$ .

*Proof.* The Lagrangian function corresponding to (17) is

$$\sum_{i=1}^M s_i \Gamma^i(\gamma_i s) - \sum_{i,n=1}^M s_i s_n \alpha_{i,n} (\gamma_{i,n} + \gamma_{n,i}) - \sum_{i,n=1}^M s_i s_n \underline{\beta}_{i,n} (k - \gamma_{i,n}) - \sum_{i,n=1}^M s_i s_n \bar{\beta}_{i,n} (k + \gamma_{i,n}).$$

The optimality conditions are

$$\begin{aligned} \Gamma_{y_n}^i(\gamma_i s) &= \alpha_{i,n} + \alpha_{n,i} - \underline{\beta}_{i,n} + \bar{\beta}_{i,n}, & \underline{\beta}_{i,n} &\geq 0, & \bar{\beta}_{i,n} &\geq 0, \\ \underline{\beta}_{i,n}(k - \gamma_{i,n}) &= 0, & \bar{\beta}_{i,n}(k + \gamma_{i,n}) &= 0. \end{aligned} \quad (18)$$

All of them are satisfied for

$$\underline{\beta}_{n,i} = \bar{\beta}_{i,n} = \frac{1}{2} \max \{ \Gamma_{y_n}^i(\gamma_i s) - \Gamma_{y_i}^n(\gamma_n s), 0 \}, \quad \alpha_{i,n} + \alpha_{n,i} = \frac{1}{2} (\Gamma_{y_n}^i(\gamma_i s) + \Gamma_{y_i}^n(\gamma_n s))$$

if  $\gamma$  satisfies (5) and  $\gamma_{i,n} = -\gamma_{n,i}$ . This means that if  $\gamma$  is a market equilibrium, it is a solution to (17). Conversely, if  $\gamma$  is a solution to (17), then (18) implies

$$\Gamma_{y_n}^i(\gamma_i s)(k^2 - \gamma_{i,n}^2) = (\alpha_{i,n} + \alpha_{n,i})(k^2 - \gamma_{i,n}^2) = (\alpha_{n,i} + \alpha_{i,n})(k^2 - \gamma_{n,i}^2) = \Gamma_{y_i}^n(\gamma_n s)(k^2 - \gamma_{n,i}^2).$$

This equation shows that if  $\gamma_{i,n} \neq \pm k$ , we need  $\Gamma_{y_n}^i(\gamma_i s) = \Gamma_{y_i}^n(\gamma_n s)$ . In turn,  $\Gamma_{y_n}^i(\gamma_i s) \neq \Gamma_{y_i}^n(\gamma_n s)$  implies  $\gamma_{i,n} = \pm k$ . Consider the case  $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$  and assume  $\gamma_{i,n} = -k$ , then  $\gamma_{n,i} = k$ ; it follows from (18) that  $\underline{\beta}_{i,n} = 0$ ,  $\bar{\beta}_{n,i} = 0$  and

$$\Gamma_{y_n}^i(\gamma_i s) = \alpha_{i,n} + \alpha_{n,i} + \bar{\beta}_{i,n} \geq \alpha_{i,n} + \alpha_{n,i} \geq \alpha_{n,i} + \alpha_{i,n} - \underline{\beta}_{i,n} = \Gamma_{y_i}^n(\gamma_n s),$$

which is a contradiction to  $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$ . Therefore,  $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$  implies  $\gamma_{i,n} = k$ . By symmetry,  $\Gamma_{y_n}^i(\gamma_i s) > \Gamma_{y_i}^n(\gamma_n s)$  implies  $\gamma_{i,n} = -k$ . This shows that a solution to (17) satisfies (5) and thus is a market equilibrium. □

**Theorem A.2** (Theorem 4.3). *There exists a market equilibrium  $(\gamma_{i,n})_{i,n=1,\dots,M}$ . The  $\gamma_{i,n}$  are unique for  $p_n > 0$  and  $\gamma_{i,n} < 0$ , or  $p_i > 0$  and  $\gamma_{i,n} > 0$ . For every  $i$ , the value is the same for  $\sum \gamma_{i,n} s_n$  where the sum is over  $n$  such that  $p_n = 0$  and  $\gamma_{i,n} < 0$ , or  $p_i = 0$  and  $\gamma_{i,n} > 0$ . In particular,  $\Gamma(\gamma_n)$  are uniquely determined for a market equilibrium  $(\gamma_{i,n})_{i,n=1,\dots,M}$ .*

*Proof.* We prove first the existence of a market equilibrium. To this end, we will apply Kakutani's fixed-point theorem (see, for example, Corollary 15.3 in Border (1985)). Fix  $k$ , set  $S = [-k, k]^{M(M-1)/2}$ , and define a mapping  $\Phi : S \rightarrow 2^S$  as follows, where  $2^S$  denotes the power set of  $S$ , i.e., the set of all subsets of  $S$ . Each element in  $S$  corresponds to the lower triangular matrix of  $(\gamma_{i,n})_{i,n=1,\dots,M}$ , where we set the diagonal elements  $\gamma_{ii}$  equal to zero and the upper diagonal elements are defined by  $\gamma_{i,n} = -\gamma_{n,i}$ . Let  $\Phi(\gamma)$  consist of all  $(\tilde{\gamma}_{i,n})_{i,n=1,\dots,M}$  that satisfy  $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$ ,  $-k \leq \tilde{\gamma}_{i,n} \leq k$ , and

$$\tilde{\gamma}_{i,n} \begin{cases} = k & \text{if } \Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s), \\ \in [-k, k] & \text{if } \Gamma_{y_n}^i(\gamma_i s) = \Gamma_{y_i}^n(\gamma_n s), \\ = -k & \text{if } \Gamma_{y_n}^i(\gamma_i s) > \Gamma_{y_i}^n(\gamma_n s). \end{cases}$$

Note that these “if” conditions depend on  $\gamma$  and not on  $\tilde{\gamma}$ . We can see that  $\Phi(\gamma)$  is nonempty, compact and convex. To show that  $\Phi$  has a closed graph, consider a sequence  $(\gamma^{(m)}, \tilde{\gamma}^{(m)})$  converging to  $(\gamma, \tilde{\gamma})$  with  $\tilde{\gamma}^{(m)} \in \Phi(\gamma^{(m)})$  for all  $m$ . Because  $\tilde{\gamma}^{(m)} \rightarrow \tilde{\gamma}$  and  $\tilde{\gamma}^{(m)} \in \Phi(\gamma^{(m)})$ , we have  $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$  and  $-k \leq \tilde{\gamma}_{i,n} \leq k$ . Moreover, if  $\Gamma_{y_n}^i(\gamma_i s) < \Gamma_{y_i}^n(\gamma_n s)$ , we have  $\Gamma_{y_n}^i(\gamma_i^{(m)} s) < \Gamma_{y_i}^n(\gamma_n^{(m)} s)$  for all  $m$  big enough, as  $\gamma^{(m)} \rightarrow \gamma$ . This yields  $\tilde{\gamma}_{i,n}^{(m)} = k$  for all  $m$  big enough; hence,  $\tilde{\gamma}_{i,n} = k$ . Similarly,  $\Gamma_{y_n}^i(\gamma_i s) > \Gamma_{y_i}^n(\gamma_n s)$  implies  $\tilde{\gamma}_{i,n} = -k$ . The condition is also satisfied for the last case  $\Gamma_{y_n}^i(\gamma_i s) = \Gamma_{y_i}^n(\gamma_n s)$ , as we have already shown  $-k \leq \tilde{\gamma}_{i,n} \leq k$ . Therefore, there exists  $\gamma$  with  $\Phi(\gamma) = \gamma$  by Kakutani's fixed-point theorem; hence, there is a market equilibrium.

To prove uniqueness, we first apply Theorem 4.2, which says that finding a market equilibrium is equivalent to solving (17). We then write the objective function in (17) as

$$\sum_{i=1}^M s_i \Gamma^i(\gamma_i s) = \sum_{i=1}^M s_i \Xi \left( \omega_i + f \left( \sum_{n: \gamma_{i,n} s_n \geq 0} \gamma_{i,n} s_n, p_i \right) + \sum_{n: \gamma_{i,n} s_n < 0} f(\gamma_{i,n} s_n, p_n) \right),$$

where the function  $\Xi$  is given in Lemma 3.1. The uniqueness statements now follow from the statements on convexity in Lemma 3.1.  $\square$

### A.3 Results of Section 4.2 and their Proofs

**Proposition A.3** (Proposition 4.4). *Assume that at least one of the following conditions holds:*

- (a)  $p_i = p_j$ , or
- (b)  $p_i \leq 1/2$  and  $\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell \geq s_i \max_\ell \gamma_{i,\ell}$ , or
- (c)  $p_j \leq 1/2$  and  $\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell \geq s_j \max_\ell \gamma_{j,\ell}$ .

We then have the following relations between initial and post-trade exposures:

1. If  $\omega_i \geq \omega_j$ ,  $p_i \leq p_j$ , and  $s_i \leq s_j$ , then  $\Omega_i \geq \Omega_j$ .
2. If  $\omega_i > \omega_j$ ,  $p_i \geq p_j$ , and  $s_i \geq s_j$ , then  $\omega_i - \omega_j > \Omega_i - \Omega_j$ .

*Proof.* We first note that, for general sizes, the post-trade exposure is given by

$$\Omega_i := \omega_i + f\left(\sum_{n:\gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i\right) + \sum_{n:\gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n).$$

We split the proof in several steps, starting with some preparation.

*Claim 1a.* For two banks  $i$  and  $j$ , we have

$$\Omega_j > \Omega_i \implies \gamma_{j,i} < 0. \quad (\text{C1a})$$

*Proof of Claim 1a.* From Lemma 3.1, it follows that

$$\Gamma_{y_i}^j(\gamma_j s) = \begin{cases} \Xi'(\Omega_j) \eta f_y(\sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j) & \text{if } \gamma_{j,i} > 0, \\ \Xi'(\Omega_j) \eta f_y(\gamma_{j,i} s_i, p_i) & \text{if } \gamma_{j,i} < 0, \end{cases} \quad (19)$$

with an analogous expression for  $\Gamma_{y_j}^i(\gamma_i s)$ . If  $\gamma_{j,i} > 0$  (and thus  $\gamma_{i,j} < 0$ ), we obtain

$$\begin{aligned} \Gamma_{y_i}^j(\gamma_j s) &= \Xi'(\Omega_j) \eta f_y\left(\sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j\right) \\ &> \Xi'(\Omega_i) \eta f_y\left(\sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j\right) \\ &\geq \Xi'(\Omega_i) \eta f_y(\gamma_{i,j} s_j, p_j) \\ &= \Gamma_{y_j}^i(\gamma_i s) \end{aligned}$$

by strict convexity of  $\Xi$  and convexity of  $f(\cdot, p_j)$  from Lemma 3.1. However, this implies  $\gamma_{j,i} = -k$  by (5) in contradiction to the assumption  $\gamma_{j,i} > 0$ . Similarly, we obtain a contradiction for  $\gamma_{j,i} = 0$ , using Footnote 5, which concludes the proof of (C1a).

*Claim 1b.* For two banks  $i$  and  $j$ , we have

$$\Omega_j > \Omega_i \implies \gamma_{j,n} < \gamma_{i,n} \text{ or } \gamma_{j,n} = -k \text{ for all } n \text{ with } \Omega_n < \Omega_j. \quad (\text{C1b})$$

*Proof of Claim 1b.* We distinguish the following three cases:

- If  $\Omega_n \in (\Omega_i, \Omega_j)$ , we have  $\gamma_{j,n} < 0$  and  $\gamma_{i,n} > 0$  by (C1a) so that  $\gamma_{j,n} < \gamma_{i,n}$  holds.
- If  $\Omega_n < \Omega_i$ , we have  $\gamma_{j,n} < 0$  and  $\gamma_{i,n} < 0$  by (C1a); thus,

$$\Gamma_{y_n}^j(\gamma_j s) = \Xi'(\Omega_j) \eta f_y(\gamma_{j,n} s_n, p_n), \quad (20)$$

$$\Gamma_{y_n}^i(\gamma_i s) = \Xi'(\Omega_i) \eta f_y(\gamma_{i,n} s_n, p_n), \quad (21)$$

$$\Gamma_{y_j}^n(\gamma_n s) = \Xi'(\Omega_n) \eta f_y \left( \sum_{\ell: \gamma_{n,\ell} \geq 0} \gamma_{n,\ell} s_\ell, p_n \right) = \Gamma_{y_i}^n(\gamma_n s). \quad (22)$$

Assume that  $\gamma_{j,n} \neq -k$ , which implies

$$\Gamma_{y_n}^j(\gamma_j s) = \Gamma_{y_j}^n(\gamma_n s) = \Gamma_{y_i}^n(\gamma_n s) \leq \Gamma_{y_n}^i(\gamma_i s)$$

by (5) and (22); thus,

$$1 < \frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \leq \frac{f_y(\gamma_{i,n} s_n, p_n)}{f_y(\gamma_{j,n} s_n, p_n)}$$

by (20) and (21). This is only possible if  $\gamma_{j,n} < \gamma_{i,n}$ .

- If  $\Omega_n = \Omega_i$ , we argue as in the first item if  $\gamma_{i,n} \geq 0$ , or as in the second item if  $\gamma_{i,n} < 0$ .

Note that (C1b) holds regardless of the default risks of banks  $i$  and  $j$ . This is because we are considering banks  $n$  with smaller post-trade exposures; thus, banks that are seller of protection by (C1a) so that the same counterparty risk  $p_n$  applies to trades with  $i$  and  $j$ .

*Claim 1c.* For two banks  $i$  and  $j$ , we have

$$\Omega_j > \Omega_i \implies \frac{f_y \left( \sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j \right)}{f_y \left( \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i \right)} < \frac{f_y(\gamma_{n,j} s_j, p_j)}{f_y(\gamma_{n,i} s_i, p_i)} \text{ or } \gamma_{n,i} = -k \text{ for all } n \text{ with } \Omega_n > \Omega_j. \quad (C1c)$$

*Proof of Claim 1c.*  $\Omega_n > \Omega_j$  implies  $\gamma_{j,n} > 0$  by (C1a), and thus  $\Gamma_{y_n}^j(\gamma_j s) \leq \Gamma_{y_j}^n(\gamma_n s)$ . If  $\gamma_{n,i} \neq -k$ , it follows that  $\Gamma_{y_n}^i(\gamma_i s) \geq \Gamma_{y_i}^n(\gamma_n s)$ ; hence,

$$\begin{aligned} \Gamma_{y_j}^n(\gamma_n s) &\geq \Gamma_{y_n}^j(\gamma_j s) = \Xi'(\Omega_j) \eta f_y \left( \sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j \right) > \Xi'(\Omega_i) \eta f_y \left( \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i \right) \\ &= \Gamma_{y_n}^i(\gamma_i s) \frac{f_y \left( \sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j \right)}{f_y \left( \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i \right)} \geq \Gamma_{y_i}^n(\gamma_n s) \frac{f_y \left( \sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j \right)}{f_y \left( \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i \right)}, \end{aligned}$$

which shows (C1c), as  $\Gamma_{y_i}^n(\gamma_n s) = \Xi'(\Omega_n) \eta f_y(\gamma_{n,i} s_i, p_i)$  and  $\Gamma_{y_j}^n(\gamma_n s) = \Xi'(\Omega_n) \eta f_y(\gamma_{n,j} s_j, p_j)$ .

*Claim 1d.* For three banks  $i$ ,  $j$ , and  $n$ , we have

$$\Omega_i < \Omega_j = \Omega_n \implies \gamma_{j,n} \leq \gamma_{i,n} \text{ or (C1c) holds.} \quad (C1d)$$

*Proof of Claim 1d.* If  $\gamma_{j,n} \leq 0$ , we obtain  $\gamma_{j,n} \leq \gamma_{i,n}$ , as  $\gamma_{i,n} > 0$  by (C1a). If  $\gamma_{j,n} > 0$ , we can argue as (C1c).

We can summarize (C1a)–(C1d) as

$$\Omega_j > \Omega_i \implies \begin{cases} \gamma_{j,n} \leq \gamma_{i,n} & \text{for all } \gamma_{j,n} \leq 0, \\ \text{(C1c) holds} & \text{for all } \gamma_{j,n} > 0. \end{cases} \quad (C1)$$

*Claim 2.* For two banks  $i$  and  $j$ , we have

$$\omega_i \geq \omega_j, p_j \geq p_i, s_j \geq s_i, \text{ and (a), (b) or (c) of the proposition holds} \implies \Omega_i \geq \Omega_j. \quad (C2)$$

*Proof of Claim 2.* We prove the claim by contradiction and assume that  $\Omega_i < \Omega_j$ . This implies  $\gamma_{j,n} \leq \gamma_{i,n}$  for all  $\gamma_{j,n} \leq 0$  by (C1); hence,

$$\begin{aligned} f\left(\sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}S_\ell,p_j\right) &= \Omega_j - \omega_j - \sum_{n:\gamma_{j,n}<0}f(\gamma_{j,n}S_n,p_n) \\ &> \Omega_i - \omega_i - \sum_{n:\gamma_{i,n}<0}f(\gamma_{i,n}S_n,p_n) \\ &= f\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}S_\ell,p_i\right) \\ &\geq f\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}S_\ell,p_j\right), \end{aligned}$$

using (8),  $p_j \geq p_i$ , and that  $f(y, p)$  is decreasing in  $p$  for  $y \geq 0$  because, using the definition (2),

$$f_p(y, p) = \frac{\partial}{\partial p} \frac{1}{\eta} \log((1-p)e^{\eta y} + pe^{\eta r y}) = \frac{-e^{\eta y} + e^{\eta r y}}{\eta((1-p)e^{\eta y} + pe^{\eta r y})} < 0 \quad \text{for } y \geq 0. \quad (23)$$

This yields  $\sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}S_\ell > \sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}S_\ell$ , as  $y \mapsto f(y, p_j)$  is strictly increasing by Lemma 3.1. This implies that there exists  $n$  with  $\gamma_{j,n} > \gamma_{i,n} \geq 0$ ; thus,

$$\gamma_{n,j} < \gamma_{n,i} \leq 0 \quad \text{and} \quad \gamma_{n,j}S_j < \gamma_{n,i}S_i \quad (24)$$

because  $S_j \geq S_i$  by assumption. Moreover,  $\gamma_{j,n} > 0$  implies  $\Omega_n \geq \Omega_j$  by (C1a). On the other hand,  $\Omega_i < \Omega_j$  implies by (C1c) and (C1d) that  $\gamma_{n,i} = -k$  (which stands in contradiction to (24) because  $\gamma_{n,j} \geq -k$ ) or  $\gamma_{j,n} \leq \gamma_{i,n}$  (also a contradiction to (24)) or

$$\frac{f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}S_\ell,p_i\right)}{f_y(\gamma_{n,i}S_i,p_i)} > \frac{f_y\left(\sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}S_\ell,p_j\right)}{f_y(\gamma_{n,j}S_j,p_j)}. \quad (25)$$

We will show that (25) contradicts

$$p_j \geq p_i, \quad \sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}S_\ell > \sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}S_\ell \quad \text{and} \quad \gamma_{n,j}S_j < \gamma_{n,i}S_i \quad (26)$$

if one of the conditions (a)–(c) of the proposition holds.

As an auxiliary step, we next analyze the function  $p \mapsto \frac{f_y(y_1,p)}{f_y(y_2,p)}$  and show that

$$\frac{\partial}{\partial p} \frac{f_y(y_1,p)}{f_y(y_2,p)} \geq 0 \quad \text{for all } p \in [0, 1/2] \text{ and } y_1 \geq -y_2 \geq 0. \quad (27)$$

Indeed, we use (14) and

$$f_{yp}(y, p) = \frac{\partial}{\partial p} \frac{(1-p)e^{\eta y} + rpe^{\eta r y}}{(1-p)e^{\eta y} + pe^{\eta r y}}$$

$$\begin{aligned}
&= \frac{((1-p)e^{\eta y} + pe^{\eta r y})(-e^{\eta y} + re^{\eta r y}) - ((1-p)e^{\eta y} + rpe^{\eta r y})(-e^{\eta y} + e^{\eta r y})}{((1-p)e^{\eta y} + pe^{\eta r y})^2} \\
&= \frac{(r-1)e^{\eta(1+r)y}}{((1-p)e^{\eta y} + pe^{\eta r y})^2}
\end{aligned}$$

to deduce that

$$\begin{aligned}
\frac{\partial f_y(y_1, p)}{\partial p f_y(y_2, p)} &= \frac{f_y(y_2, p)f_{yp}(y_1, p) - f_{yp}(y_2, p)f_y(y_1, p)}{(f_y(y_2, p))^2} \\
&= \frac{\frac{(1-p)e^{\eta y_2} + rpe^{\eta r y_2}}{(1-p)e^{\eta y_2} + pe^{\eta r y_2}} \frac{(r-1)e^{\eta(1+r)y_1}}{((1-p)e^{\eta y_1} + pe^{\eta r y_1})^2} - \frac{(r-1)e^{\eta(1+r)y_2}}{((1-p)e^{\eta y_2} + pe^{\eta r y_2})^2} \frac{(1-p)e^{\eta y_1} + rpe^{\eta r y_1}}{(1-p)e^{\eta y_1} + pe^{\eta r y_1}}}{(f_y(y_2, p))^2} \\
&= \frac{(r-1)e^{\eta(1+r)y_1} \left( (1-p)e^{\eta y_2} + rpe^{\eta r y_2} \right) \left( (1-p)e^{\eta y_2} + pe^{\eta r y_2} \right)}{\left( (1-p)e^{\eta y_1} + pe^{\eta r y_1} \right)^2 \left( (1-p)e^{\eta y_2} + pe^{\eta r y_2} \right)^2 (f_y(y_2, p))^2} \\
&\quad - \frac{(r-1)e^{\eta(1+r)y_2} \left( (1-p)e^{\eta y_1} + rpe^{\eta r y_1} \right) \left( (1-p)e^{\eta y_1} + pe^{\eta r y_1} \right)}{\left( (1-p)e^{\eta y_1} + pe^{\eta r y_1} \right)^2 \left( (1-p)e^{\eta y_2} + pe^{\eta r y_2} \right)^2 (f_y(y_2, p))^2} \\
&= \frac{(1-r)e^{\eta(1+r)(y_1+y_2)}}{\left( (1-p)e^{\eta y_1} + pe^{\eta r y_1} \right)^2 \left( (1-p)e^{\eta y_2} + pe^{\eta r y_2} \right)^2 (f_y(y_2, p))^2} \\
&\quad \times \left( \left( (1-p)e^{\eta(1-r)y_1} + rp \right) \left( 1-p + pe^{-\eta(1-r)y_1} \right) \right. \\
&\quad \left. - \left( (1-p)e^{\eta(1-r)y_2} + rp \right) \left( 1-p + pe^{-\eta(1-r)y_2} \right) \right).
\end{aligned}$$

From this, we obtain  $\frac{\partial f_y(y_1, p)}{\partial p f_y(y_2, p)} \geq 0$  because

$$\begin{aligned}
&\left( (1-p)e^{\eta(1-r)y_1} + rp \right) \left( 1-p + pe^{-\eta(1-r)y_1} \right) - \left( (1-p)e^{\eta(1-r)y_2} + rp \right) \left( 1-p + pe^{-\eta(1-r)y_2} \right) \\
&= (1-p)^2 e^{\eta(1-r)y_1} + rp^2 e^{-\eta(1-r)y_1} - (1-p)^2 e^{\eta(1-r)y_2} - rp^2 e^{-\eta(1-r)y_2} \\
&\geq rp^2 \left( e^{\eta(1-r)y_1} + e^{-\eta(1-r)y_1} - e^{\eta(1-r)y_2} - e^{-\eta(1-r)y_2} \right) \\
&\geq 0,
\end{aligned}$$

using  $(1-p)^2 \geq rp^2$  for  $p \leq 1/2$  and  $y_1 \geq y_2$  for the second last inequality, and  $e^{\eta(1-r)y_1} + e^{-\eta(1-r)y_1} \geq e^{\eta(1-r)y_2} + e^{-\eta(1-r)y_2}$  for  $|y_1| \geq |y_2|$  for the last inequality. This concludes the proof of (27).

We now consider each of the three conditions (a)–(c) of the proposition.

*Condition (a).* From (26), we deduce

$$\frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,j} s_j, p_j)} \geq \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,i} s_i, p_j)} = \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,i} s_i, p_i)},$$

using the convexity of  $y \mapsto f(y, p_j)$  by Lemma 3.1 and  $p_i = p_j$  in condition (a).

*Condition (b).* We apply (27) choosing  $p = p_i$ ,  $y_1 = \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell$ , and  $y_2 = \gamma_{n,i} s_i$ . This implies

$$\frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,i} s_i, p_i)} \leq \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,i} s_i, p_j)} \leq \frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,j} s_j, p_j)},$$

where we use (26) and the convexity of  $y \mapsto f(y, p_j)$  for the second inequality.

*Condition (c).* This time, we apply (27) choosing  $p = p_j$ ,  $y_1 = \sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell$ , and  $y_2 = \gamma_{n,j} s_j$ . We obtain

$$\frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right)}{f_y(\gamma_{n,j} s_j, p_j)} \geq \frac{f_y\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,j} s_j, p_i)} \geq \frac{f_y\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right)}{f_y(\gamma_{n,i} s_i, p_i)},$$

where we again use (26) and the convexity of  $y \mapsto f(y, p_i)$  for the second inequality.

Under each of the three conditions (a)–(c), we obtain a contradiction to (25). Hence,  $\Omega_i < \Omega_j$  cannot hold, which concludes the proof of (C2).

*Claim 3.* For two banks  $i$  and  $j$ , we have

$$\omega_i > \omega_j, p_j \leq p_i, s_j \leq s_i \implies \Omega_i - \omega_i < \Omega_j - \omega_j.$$

*Proof of Claim 3.* We proceed similarly to the proof of (C2). We prove the claim by contradiction and assume that  $\Omega_i - \omega_i \geq \Omega_j - \omega_j$ . This implies  $\Omega_i > \Omega_j$ ; hence,  $\gamma_{i,n} \leq \gamma_{j,n}$  for all  $\gamma_{i,n} \leq 0$  by (C1) and  $\gamma_{i,j} < 0 < \gamma_{j,i}$  by (C1a), and thus

$$\begin{aligned} f\left(\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right) &= \Omega_j - \omega_j - \sum_{n: \gamma_{j,n} < 0} f(\gamma_{j,n} s_n, p_n) \\ &< \Omega_i - \omega_i - \sum_{n: \gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n) \\ &= f\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right) \\ &\leq f\left(\sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_j\right) \end{aligned}$$

using  $p_j \leq p_i$  and (23), which yields  $\sum_{\ell: \gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell < \sum_{\ell: \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell$  because  $y \mapsto f(y, p_j)$  is strictly increasing by Lemma 3.1. We conclude the proof in the same way as the proof of (C2) after (24), with  $i$  and  $j$  interchanged.  $\square$

**Proposition A.4** (Proposition 4.5). *Assume that the trade size limit is not binding and that there are at least two safe banks. Then*

1. *All safe banks have the same post-trade exposure, say,  $\bar{\Omega}$ .*
2. *Risky banks with initial exposure above some level  $\alpha$  also have the same post-trade exposure  $\bar{\Omega}$ . The level  $\alpha$  depends only on the distribution of initial exposures and sizes, but not on the banks' default probabilities.*
3. *Risky banks with initial exposure below  $\alpha$  will have post-trade exposures strictly smaller than  $\bar{\Omega}$ .*

*Proof.* We define  $\bar{k}_1$  by

$$\bar{k}_1 = \inf \{k > 0 : \Omega_i = \Omega_j \text{ for all } i, j \text{ with } p_i = p_j = 0\}. \quad (28)$$

We can prove that  $0 < \bar{k}_1 < \infty$  and that the infimum in (28) is attained along the same lines as on page 2273 of Atkeson et al. (2015), restricting their arguments to the safe banks. We choose  $\bar{k}$  as the smallest number  $k \geq \bar{k}_1$  such that

$$\Omega_i \leq \Omega_j \quad (29)$$

for all  $i, j$  with  $p_i > 0$  and  $p_j = 0$ . We next show that  $\bar{k}$  is well defined. If (29) holds for  $k = \bar{k}_1$ , we set  $\bar{k} = \bar{k}_1$ . Moreover, (29) always holds for  $k$  big enough. Indeed, let  $i$  be with  $p_i > 0$  and, working towards a contradiction, assume that

$$\Omega_i > \Omega_j \quad (30)$$

for some  $j$  with  $p_j = 0$ . From (C1a) and (C1) in the proof of Proposition 4.4 with  $p_j = 0$ , it follows that  $\gamma_{i,j} < 0$  and  $\gamma_{i,n} \leq \gamma_{j,n}$  for all  $n$ ; hence,

$$\begin{aligned} \Gamma_{y_i}^j(\gamma_j s) &= \Xi'(\Omega_j) \eta f_y \left( \sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j \right) = \Xi'(\Omega_j) \eta \\ &< \Xi'(\Omega_i) \eta = \Xi'(\Omega_i) \eta f_y(\gamma_{i,j} s_j, p_j) = \Gamma_{y_j}^i(\gamma_i s) \end{aligned}$$

using that  $f_y(y, p_j) = 1$  because  $p_j = 0$ ,  $\Xi$  is strictly increasing and  $\Omega_i > \Omega_j$ . Then  $\gamma_{i,j} = -k$  follows from  $\Gamma_{y_i}^j(\gamma_j s) < \Gamma_{y_j}^i(\gamma_i s)$  by (5), and thus

$$\begin{aligned} \Omega_j &= \omega_j + f \left( \sum_{n:\gamma_{j,n} \geq 0} \gamma_{j,n} s_n, p_j \right) + \sum_{n:\gamma_{j,n} < 0} f(\gamma_{j,n} s_n, p_n) \\ &\geq k s_i + \omega_j + f \left( \sum_{n:\gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i \right) + \sum_{n:\gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n) \\ &= k s_i + \omega_j - \omega_i + \Omega_i. \end{aligned}$$

However, for  $k \geq (\omega_i - \omega_j)/s_i$ , this gives  $\Omega_j \geq \Omega_i$  in contradiction to (30). Hence, we have (29) for  $k$  big enough. By a compactness argument similar to page 2273 of Atkeson et al. (2015), we deduce that (29) holds for  $k = \bar{k}$ . By definition of  $\bar{k}$ , for  $k < \bar{k}$ , there exist  $i$  and  $j$  with  $p_j = 0$  such that  $\Omega_i > \Omega_j$ .

We now consider  $k \geq \bar{k}$  and

$$\begin{aligned} \beta(p, s) &= \max_{i:p_i=p, s_i=s} \Omega_i, \quad \bar{i}(p, s) = \begin{cases} \arg \max_{i:p_i=p, s_i=s} \Omega_i & \text{if } \beta(p, s) = \Omega_j \text{ for } j \text{ with } p_j = 0, \\ \emptyset & \text{otherwise.} \end{cases} \\ \bar{\delta}(p, s) &= \min_{i \in \bar{i}(p, s)} \omega_i, \quad \underline{\delta}(p, s) = \max_{\{i:p_i=p, s_i=s\} \setminus \bar{i}(p, s)} \omega_i \end{aligned}$$

for  $p \in \{p_1, \dots, p_M\}$  and  $s \in \{s_1, \dots, s_M\}$  where the minimum (and maximum) over an empty set equals  $+\infty$  and  $-\infty$  by the usual convention. Several  $p_j$  and  $s_j$  for different  $j$  can take the same values, and thus  $\bar{i}(p, s)$  can be a set with several entries because the maximum does not need to be attained at a unique  $i$ . We can choose a function  $\bar{\alpha} : (0, 1] \times [0, 1] \rightarrow [0, \infty)$  for all  $s$  such that  $\underline{\delta}(p, s) < \bar{\alpha}(p, s) \leq \bar{\delta}(p, s)$  for all  $p \in \{p_1, \dots, p_M\}$  and  $s \in \{s_1, \dots, s_M\}$ .

Note that  $\bar{\alpha}(p, s)$  may depend here on both arguments  $p$  and  $s$ , but in the next paragraph, we will show that  $\bar{\alpha}$  can be chosen independently of  $p$ . From  $\bar{\alpha}(p, s) \leq \bar{\delta}(p, s)$ , it follows that  $A(\bar{\alpha})$  defined by

$$A(\bar{\alpha}) = \{i : \omega_i \geq \bar{\alpha}(s_i, p_i) \text{ or } p_i = 0\}$$

contains all  $i$  with  $\Omega_i = \Omega_j$  for  $j$  with  $p_j = 0$ . To show that  $A(\bar{\alpha})$  contains only such  $i$ , assume that there exists  $i \in A(\bar{\alpha})$  with  $\Omega_i < \Omega_j$  for  $j$  with  $p_j = 0$ . This implies

$$\omega_i \geq \bar{\alpha}(p_i, s_i) > \underline{\delta}(p_i, s_i);$$

hence,  $\omega_i > \omega_\ell$  for all  $\omega_\ell$  with  $\Omega_\ell < \Omega_j$ , which contradicts  $\Omega_i < \Omega_j$ . Therefore, all banks  $i \in A(\bar{\alpha})$  have the same post-trade exposure  $\Omega_i$  while banks  $i \notin A(\bar{\alpha})$  have a strictly smaller post-trade exposure. Thus, we can set  $\bar{\Omega} = \Omega_i$  for some  $i \in A(\bar{\alpha})$ .

Finally, we show that  $\bar{\alpha}$  can be chosen independently of  $p$ , consider  $k \geq \bar{k}$  and  $i$  with  $p_i > 0$  and  $\Omega_i = \Omega_j$  for  $j$  with  $p_j = 0$ . Because of  $k \geq \bar{k}$ , we have  $\Omega_i \geq \Omega_\ell$  for all  $\ell$ , using (29). In the case  $\Omega_i > \Omega_\ell$ , we obtain  $\gamma_{i,\ell} < 0$  by (C1a). In the case  $\Omega_i = \Omega_\ell$ , we argue similarly to the proof of (C1a) to show  $\gamma_{i,\ell} \leq 0$ . Indeed, to derive a contradiction, we assume that  $\gamma_{i,\ell} > 0$  and  $\Omega_i = \Omega_\ell$ , which implies

$$\begin{aligned} \Gamma_{y_\ell}^i(\gamma_i s) &= \Xi'(\Omega_i) \eta f_y \left( \sum_{n: \gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i \right) \\ &= \Xi'(\Omega_\ell) \eta f_y \left( \sum_{n: \gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i \right) \\ &> \Xi'(\Omega_\ell) \eta f_y(\gamma_{\ell,i} s_i, p_i) \\ &= \Gamma_{y_i}^\ell(\gamma_\ell s) \end{aligned}$$

by strict convexity of  $f(\cdot, p_i)$  from Lemma 3.1, using that  $p_i > 0$ . However, this implies  $\gamma_{i,\ell} = -k$  by (5) in contradiction to the assumption  $\gamma_{i,\ell} > 0$ . Hence, we have  $\gamma_{i,\ell} \leq 0$ , and  $p_i$  does not matter for the trading of bank  $i$ . Indeed, Lemma 3.1 shows then that, for all  $\ell$ ,  $\Gamma^\ell(\gamma_\ell s)$  does not depend on  $p_i$  if  $\gamma_{i,\ell} \leq 0$ , and thus the objective function  $\sum_{\ell=1}^M s_\ell \Gamma^\ell(\gamma_\ell s)$  in (17) does not depend on  $p_i$  in the optimum. Therefore,  $\bar{\alpha}$  can be chosen independently of  $p$ .  $\square$

## A.4 Proposition 4.7 and its Proof

For general sizes  $s_i$ , the per-capita gross numbers of sold or purchased contracts are given by

$$G_i^+ = f \left( \sum_{n: \gamma_{i,n} \geq 0} \gamma_{i,n} s_n, p_i \right) \text{ and } G_i^- = - \sum_{n: \gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n).$$

**Proposition A.5** (Proposition 4.7). *1. If the trade size limit  $k$  is small enough and there are at least three banks with different initial exposures  $\omega_i$ , then the intermediation volume  $I_i$  as a function of  $\omega_i$  is a hump-shaped curve, taking its maximum at or next to the median initial exposure.*

*2. Assume that at least one of the conditions (a)–(c) of Proposition A.3 holds. If two banks  $i$  and  $j$  have the same initial exposure, then  $I_i \leq I_j$  for  $p_i \geq p_j$  and  $s_i \geq s_j$ .*

*Proof.* 1. If  $k$  is small enough, then  $\omega_i < \omega_j$  implies  $\Omega_i < \Omega_j$  and  $\gamma_{i,n}$  equals  $\pm k$  for all  $i$  and  $n$  with  $\omega_i \neq \omega_n$ ; hence,

$$\gamma_{i,n} \begin{cases} = k & \omega_i < \omega_n \\ = -k & \omega_i > \omega_n \\ \in [k, -k] & \omega_i = \omega_n \end{cases}$$

by (C1a) and (5). This yields

$$G_i^+ = f\left(k \sum_{n:\omega_n > \omega_i} s_n + \sum_{\substack{n:\omega_n = \omega_i, \\ \gamma_{i,n} > 0}} s_n \gamma_{i,n}, p_i\right),$$

$$G_i^- = - \sum_{n:\omega_n < \omega_i} f(-k s_n, p_n) - \sum_{\substack{n:\omega_n = \omega_i, \\ \gamma_{i,n} < 0}} f(s_n \gamma_{i,n}, p_n).$$

As a function of  $\omega_i$ ,  $G_i^+$  is decreasing with zero at the largest value of  $\omega_i$  and  $G_i^-$  is increasing with zero at the smallest value of  $\omega_i$  so that  $I_i = \min\{G_i^+, G_i^-\}$  is a hump-shaped curve. If we order the banks by their initial exposures such that  $\omega_1 \leq \omega_2 \leq \dots \leq \omega_M$  and use the values of  $G_i^+$  to determine the order when  $\omega_i = \omega_j$ , then the intermediation volume  $I_i$  takes its maximum at  $\omega_{i^*}$  or the next bigger  $\omega_i$  where  $G_{i^*}^+ \geq G_{i^*}^-$ .

2. If two banks  $i$  and  $j$  have the same initial exposure, then  $p_i \geq p_j$  and  $s_i \geq s_j$  imply  $\Omega_i \leq \Omega_j$  by Proposition 4.4. Working towards a contradiction, we assume that

$$f\left(\sum_{\ell:\gamma_{j,\ell} \geq 0} \gamma_{j,\ell} s_\ell, p_j\right) < f\left(\sum_{\ell:\gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\right). \quad (31)$$

Together with  $\Omega_i \leq \Omega_j$  and  $\omega_i = \omega_j$ , this implies

$$\sum_{n:\gamma_{i,n} < 0} f(\gamma_{i,n} s_n, p_n) < \sum_{n:\gamma_{j,n} < 0} f(\gamma_{j,n} s_n, p_n)$$

by (8) so that there exists  $n$  with  $\gamma_{i,n} < \gamma_{j,n} \leq 0$  because  $f_y > 0$  by (14); in particular,  $\gamma_{j,n} > -k$ . From (5) and (19), we thus obtain

$$\Xi'(\Omega_j) \eta f_y(\gamma_{j,n} s_n, p_n) = \Xi'(\Omega_n) \eta f_y\left(\sum_{\ell:\gamma_{n,\ell} \geq 0} \gamma_{n,\ell} s_\ell, p_n\right),$$

$$\Xi'(\Omega_i) \eta f_y(\gamma_{i,n} s_n, p_n) \geq \Xi'(\Omega_n) \eta f_y\left(\sum_{\ell:\gamma_{n,\ell} \geq 0} \gamma_{n,\ell} s_\ell, p_n\right),$$

and hence

$$\frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \leq \frac{f_y(\gamma_{i,n} s_n, p_n)}{f_y(\gamma_{j,n} s_n, p_n)}.$$

However, this leads to a contradiction because  $\Omega_i \leq \Omega_j$  implies  $\frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \geq 1$  by (13) and  $\gamma_{i,n} < \gamma_{j,n}$  gives  $\frac{f_y(\gamma_{i,n} s_n, p_n)}{f_y(\gamma_{j,n} s_n, p_n)} < 1$  because  $f_{yy} > 0$  by (15). Therefore, (31) does not hold, which implies  $G_j^+ \geq G_i^+$ . Next, we assume directly that there exists  $n$  with  $\gamma_{i,n} < \gamma_{j,n} \leq 0$ , which leads to a contradiction by the above arguments. Therefore, we deduce  $G_j^- \geq G_i^-$  and thus  $I_j \geq I_i$ .  $\square$

## A.5 Results of Section 5 and their Proofs

**Lemma A.6** (Lemma 5.1). *For given  $s_1, \dots, s_M$ , the value of  $x_i(p_1, \dots, p_M)$  is uniquely determined.*

*Proof.* For general  $s_i$ , (10) becomes

$$x_i(p_1, \dots, p_M) = \omega_i + \sum_{n \neq i} \gamma_{i,n} s_n R_{i,n} - \Gamma^i(\gamma_i s).$$

Using the definition (6) of  $R_{i,n}$  and (5), we can write

$$\begin{aligned} x_i(p_1, \dots, p_M) &= \omega_i - \Gamma^i(\gamma_i s) + \sum_{n: \gamma_{i,n} > 0} \gamma_{i,n} s_n (\nu \Gamma_{y_i}^n(\gamma_n s) + (1 - \nu) \Gamma_{y_n}^i(\gamma_i s)) \\ &\quad + \sum_{n: \gamma_{i,n} < 0} \gamma_{i,n} s_n (\nu \Gamma_{y_n}^i(\gamma_i s) + (1 - \nu) \Gamma_{y_i}^n(\gamma_n s)) \\ &= \omega_i - \Gamma^i(\gamma_i s) + \nu \sum_{n: \gamma_{i,n} > 0} \gamma_{i,n} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) \\ &\quad + (1 - \nu) \sum_{n: \gamma_{i,n} < 0} \gamma_{i,n} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) + \sum_{n \neq i} \gamma_{i,n} s_n \Gamma_{y_n}^i(\gamma_i s) \\ &= \omega_i - \Gamma^i(\gamma_i s) + \nu k \sum_{n: \gamma_{i,n} > 0} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) \\ &\quad - (1 - \nu) k \sum_{n: \gamma_{i,n} < 0} s_n (\Gamma_{y_i}^n(\gamma_n s) - \Gamma_{y_n}^i(\gamma_i s)) + \sum_{\substack{p_n > 0, \gamma_{i,n} < 0, \text{ or} \\ p_i > 0, \gamma_{i,n} > 0}} \gamma_{i,n} s_n \Gamma_{y_n}^i(\gamma_i s) \\ &\quad + \bar{\Gamma}^i(\gamma_i s) \sum_{\substack{p_n = 0, \gamma_{i,n} < 0, \text{ or} \\ p_i = 0, \gamma_{i,n} > 0}} \gamma_{i,n} s_n, \end{aligned}$$

where

$$\bar{\Gamma}^i(y) := \Gamma_{y_n}^i(y) = \frac{q e^{\eta \omega_i + \eta f(\sum_{n: y_n \geq 0} y_n, p_i) + \eta \sum_{n: y_n < 0} f(y_n, p_n)}}{1 - q + q e^{\eta \omega_i + \eta f(\sum_{n: y_n \geq 0} y_n, p_i) + \eta \sum_{n: y_n < 0} f(y_n, p_n)}}$$

does not depend on the specific  $n$  for all  $n$  with  $p_n = 0$  and  $\gamma_{i,n} < 0$ , or  $p_i = 0$  and  $\gamma_{i,n} > 0$ . This means that  $\Gamma_{y_n}^i$  is the same for all banks  $n$  that are (I) default-free protection sellers to  $i$ , or (II) protection buyers from  $i$ , and  $i$  is default-free. All these pairwise transactions do not bear any counterparty risk. Uniqueness of  $x_i(p_1, \dots, p_M)$  now follows from Theorem A.2.  $\square$

*Proof of Proposition 5.3.* We first note that the mapping  $p_i \mapsto x_i(p_1, \dots, p_M)$  is continuous. This follows from the Envelope theorem using that  $\Gamma^i$  and its partial derivatives are differentiable. For  $p_{-i} = (p_j)_{j \neq i}$ , we define set-valued functions

$$r_i(p_{-i}) = \arg \max_{p_i \in [0, \bar{p}_i]} (x_i(p_1, \dots, p_M) - C(p_i)), \quad r(p) = (r_1(p_{-1}), \dots, r_M(p_{-M}))$$

so that  $r$  is a mapping from  $[0, \bar{p}_1] \times \dots \times [0, \bar{p}_m]$  onto its power set. It has the following properties:

- $[0, \bar{p}_1] \times \cdots \times [0, \bar{p}_m]$  is compact, convex, and nonempty.
- For each  $p$ ,  $r(p)$  is nonempty because a continuous function over a compact set has always a maximizer.
- $r(p)$  is convex by assumption.
- It follows from Berge's maximum theorem that  $r(p)$  has a closed graph.

Thanks to these properties, Kakutani's fixed point theorem implies that there exists a fixed point of the mapping  $r$ , which means that there exists an equilibrium.  $\square$

*Proof of Proposition 5.4.* Because the function

$$\sum_{i=1}^M s_i \Gamma^i(\gamma_i s, p) + \sum_{i=1}^M s_i C(p_i) \quad (32)$$

is continuous over the compact set  $[0, \bar{p}_1] \times \cdots \times [0, \bar{p}_M]$ , it has a maximum, which shows the statement of the proposition, using that the social planner's optimization problem over  $(\gamma_{i,n})_{i,n=1,\dots,M}$  conditional on the choice of the default probabilities has a solution by Theorems 4.2 and 4.3.  $\square$

*Proof of Lemma 5.5.* The demand of bank  $n$  on CDS contracts to bank  $i$  is given by the curve  $-y_i \mapsto \Gamma_{y_i}^n(y, p)$  for  $y_i < 0$  because the marginal cost of risk bearing decreases by  $\Gamma_{y_i}^n(y, p)$  per unit purchased CDS. The slope of the demand curve is obtained by taking the negative  $y_i$ -partial derivative of the demand curve, resulting in  $-\Gamma_{y_i, y_i}^n(y, p)$ . Hence, the demand curve becomes flatter for decreased  $p_i$  if  $-\Gamma_{y_i, y_i, p_i}^n(y, p) < 0$ , or equivalently,  $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$ . We show  $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$  for small enough  $p_i$  and big enough  $q$ .

By Lemma 3.1, we have

$$\begin{aligned} \Gamma^n(y, p) &= \frac{1}{\eta} \log \left( 1 - q + qe^{\eta\omega_n + \eta f(\sum_{\ell: y_\ell \geq 0} y_\ell, p_n) + \eta \sum_{\ell: y_\ell < 0} f(y_\ell, p_\ell)} \right) \\ &= \frac{1}{\eta} \log \left( 1 - q + a(1 - p_i)e^{\eta y_i} + ap_i e^{\eta r y_i} \right) \end{aligned}$$

for  $y_i < 0$ , where we use the abbreviation  $a = qe^{\eta\omega_n + \eta f(\sum_{\ell: y_\ell \geq 0} y_\ell, p_n) + \eta \sum_{\ell \neq i: y_\ell < 0} f(y_\ell, p_\ell)}$  in this proof. Because  $\Gamma^n(y, p)$  is a smooth function, the order of taking partial derivatives does not matter. For simplicity, we start with the  $p_i$ -partial derivative, which equals

$$\Gamma_{p_i}^n(y, p) = \frac{1}{\eta} \frac{-ae^{\eta y_i} + ae^{\eta r y_i}}{1 - q + a(1 - p_i)e^{\eta y_i} + ap_i e^{\eta r y_i}} = \frac{1}{\eta} \frac{e^{\eta r y_i} - e^{\eta y_i}}{b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i}}, \quad (33)$$

using the abbreviation  $b = (1 - q)/a$ . Next, we compute

$$\begin{aligned} \Gamma_{y_i, p_i}^n(y, p) &= \frac{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})(re^{\eta r y_i} - e^{\eta y_i}) - (e^{\eta r y_i} - e^{\eta y_i})((1 - p_i)e^{\eta y_i} + rp_i e^{\eta r y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \\ &= \frac{b(re^{\eta r y_i} - e^{\eta y_i}) + (r - rp_i - p_i)e^{\eta(1+r)y_i} - (1 - p_i)e^{2\eta y_i} + rp_i e^{2\eta r y_i}}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \end{aligned}$$

$$\begin{aligned}
& - \frac{(1 - rp_i - p_i)e^{\eta(1+r)y_i} - (1 - p_i)e^{2\eta y_i} + rp_i e^{2\eta r y_i}}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \\
& = \frac{b(re^{\eta r y_i} - e^{\eta y_i}) + (r - 1)e^{\eta(1+r)y_i}}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2}. \tag{34}
\end{aligned}$$

Finally, we determine the third-order partial derivative

$$\begin{aligned}
\Gamma_{y_i, y_i, p_i}^n(y, p) &= \eta \frac{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2 (b(r^2 e^{\eta r y_i} - e^{\eta y_i}) + (r^2 - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^4} \\
&\quad - \eta \frac{2((1 - p_i)e^{\eta y_i} + rp_i e^{\eta r y_i})(b(re^{\eta r y_i} - e^{\eta y_i}) + (r - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^3} \\
&= \eta \frac{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})(b(r^2 e^{\eta r y_i} - e^{\eta y_i}) + (r^2 - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^3} \\
&\quad - \eta \frac{2((1 - p_i)e^{\eta y_i} + rp_i e^{\eta r y_i})(b(re^{\eta r y_i} - e^{\eta y_i}) + (r - 1)e^{\eta(1+r)y_i})}{(b + (1 - p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^3}.
\end{aligned}$$

For  $p_i = 0$  and  $q = 1$ , we have

$$\Gamma_{y_i, y_i, p_i}^n(y, p) = \eta(r - 1)^2 e^{\eta(-1+r)y_i} > 0,$$

and hence,  $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$  for small enough  $p_i$  and big enough  $q$  by continuity of  $\Gamma_{y_i, y_i, p_i}^n$ .  $\square$

**Theorem A.7** (Theorem 5.6). *The social planner's optimization satisfies the first-order conditions of an equilibrium if bank  $i$  receives a per-trader subsidy equal to  $S = S_1 + k(1 - \nu)S_2$  with*

$$S_1 := - \sum_{n \neq i} s_n \left( \gamma_{i,n} \Gamma_{y_i}^n(\gamma_n s, p) + \frac{1}{s_i} \Gamma^n(\gamma_n s, p) \right), \quad S_2 := \sum_{n \neq i} s_n (\Gamma_{y_i}^n(\gamma_n, p) - \Gamma_{y_n}^i(\gamma_i, p)). \tag{35}$$

Assuming a small enough trade size limit, we have  $\frac{\partial S_1}{\partial p_i} > 0$  and  $\frac{\partial S_2}{\partial p_i} < 0$  for small enough  $p_i$  and large enough  $q$ . In this case, privately chosen  $p_i$ 's are lower than the socially optimal level if sellers have full bargaining power. The difference between the socially and individually optimal choices of  $p_i$  increases as a function of the seller bargaining power.

*Proof.* For the first part of the theorem, we compare the marginal social value  $MSV_i$ , defined as the partial derivative of the social planner's objective function with respect to the default probability  $p_i$  of bank  $i$  assuming that banks trade optimally, with the corresponding marginal private value  $MPV_i$ . To do so, we highlight the dependence on the banks' default probabilities  $p = (p_i)_{i=1, \dots, M}$  by using notations such as  $\Gamma^i(\gamma_i s, p)$  and  $R_{i,n}(\gamma_i s, p)$ . For arbitrary bank sizes, (12) becomes

$$\sum_{i=1}^M s_i x_i(p_1, \dots, p_M) - \sum_{i=1}^M s_i C(p_i) \tag{36}$$

so that the  $MSV_i$ , given as its  $p_i$ -partial derivative, equals

$$MSV_i = \sum_{n=1}^M s_n \frac{\partial x_n}{\partial p_i}(p_1, \dots, p_M) - s_i C'(p_i) = - \sum_{n=1}^M s_n \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) - s_i C'(p_i),$$

where we have used the equivalence between (32) and (36). The marginal private value  $MPV_i$  is the partial derivative of the bank  $i$ 's certainty equivalent (10) minus its risk-management costs, with respect to its default probability  $p_i$  — namely,

$$MPV_i = -s_i C'(p_i) - s_i \frac{\partial \Gamma^i}{\partial p_i}(\gamma_i s, p) + s_i \sum_{n \neq i} \gamma_{i,n} s_n \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p).$$

The difference between marginal private and social value for bank  $i$  is

$$MPV_i - MSV_i = \sum_{n \neq i} s_n \left( s_i \gamma_{i,n} \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right)$$

If  $\gamma_{i,n} \leq 0$ , then we obtain from (6) that

$$R_{i,n}(\gamma_i s, \gamma_n s, p) = \nu \Gamma_{y_n}^i(\gamma_i s, p) + (1 - \nu) \Gamma_{y_i}^n(\gamma_n s, p).$$

We then have that  $\frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) = 0$  and  $\frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) = 0$  because  $\Gamma^n(\gamma_n s, p)$ ,  $\Gamma_{y_n}^i(\gamma_i s, p)$ , and  $R_{i,n}(\gamma_i s, \gamma_n s, p)$  do not depend on  $p_i$  for  $\gamma_{i,n} \leq 0$ ; if traders of bank  $i$  are buying CDSs from bank  $n$ , the default probability of bank  $i$  does not affect the terms of trade between traders of banks  $i$  and  $n$ . For  $\gamma_{i,n} > 0$ , we find

$$R_{i,n}(\gamma_i s, \gamma_n s, p) = \nu \Gamma_{y_i}^n(\gamma_n s, p) + (1 - \nu) \Gamma_{y_n}^i(\gamma_i s, p)$$

by (5) and (6) so that

$$\begin{aligned} MPV_i - MSV_i &= \sum_{n: \gamma_{i,n} > 0} s_n \left( s_i \gamma_{i,n} \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right) \\ &= \sum_{n: \gamma_{i,n} > 0} s_n \left( s_i \gamma_{i,n} \frac{\partial \Gamma_{y_i}^n}{\partial p_i}(\gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right) \\ &\quad + \sum_{n: \gamma_{i,n} > 0} s_n s_i \gamma_{i,n} (1 - \nu) \left( \frac{\partial \Gamma_{y_n}^i}{\partial p_i}(\gamma_i s, p) - \frac{\partial \Gamma_{y_i}^n}{\partial p_i}(\gamma_n s, p) \right) \\ &= \frac{\partial}{\partial p_i} \sum_{n \neq i} s_n (s_i \gamma_{i,n} \Gamma_{y_i}^n(\gamma_n s, p) + \Gamma^n(\gamma_n s, p)) \\ &\quad + \frac{\partial}{\partial p_i} \sum_{n \neq i} s_n s_i k (1 - \nu) (\Gamma_{y_n}^i(\gamma_i s, p) - \Gamma_{y_i}^n(\gamma_n s, p)), \end{aligned}$$

using for the last equality that  $\frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) = 0$ ,  $\frac{\partial \Gamma_{y_i}^n}{\partial p_i}(\gamma_n s, p) = 0$  and  $\frac{\partial \Gamma_{y_n}^i}{\partial p_i}(\gamma_i s, p) = 0$  for  $\gamma_{i,n} \leq 0$  and  $\Gamma_{y_i}^i(\gamma_i s, p) = \Gamma_{y_i}^n(\gamma_n s, p)$  for  $\gamma_{i,n} \in (-k, k)$ .

We next prove the second part of Theorem A.7. By Lemma 5.5, we have  $\Gamma_{y_i, y_i, p_i}^n(y, p) > 0$  for small enough  $p_i$  and big enough  $q$ . This implies

$$\Gamma_{y_i, p_i}^n(\gamma_n s, p) < \Gamma_{y_i, p_i}^n(((\gamma_{\ell, n} s_\ell)_{\ell \neq i}, y_i), p) \quad (37)$$

for all  $y_i \in (\gamma_{n,i} s_i, 0)$  where  $((\gamma_{\ell,n} s_\ell)_{\ell \neq i}, y_i) := (\gamma_{1,n} s_1, \dots, \gamma_{i-1,n} s_{i-1}, y_i, \gamma_{i+1,n} s_{i+1}, \dots, \gamma_{M,n} s_M)$ . From  $\Gamma_{p_i}^n(((\gamma_{\ell,n} s_\ell)_{\ell \neq i}, 0), p) = 0$  by (33), we deduce

$$\begin{aligned} \frac{\partial}{\partial p_i} (s_i \gamma_{i,n} \Gamma_{y_i}^n(\gamma_n s, p) + \Gamma^n(\gamma_n s, p)) &= s_i \gamma_{i,n} \Gamma_{y_i, p_i}^n(\gamma_n s, p) - \int_{\gamma_{n,i} s_n}^0 \Gamma_{y_i, p_i}^n(((\gamma_{\ell,n} s_\ell)_{\ell \neq i}, y_i), p) dy_i \\ &= \int_{\gamma_{n,i} s_n}^0 (\Gamma_{y_i, p_i}^n(\gamma_n s, p) - \Gamma_{y_i, p_i}^n(((\gamma_{\ell,n} s_\ell)_{\ell \neq i}, y_i), p)) dy_i \\ &< 0, \end{aligned}$$

using (37). Hence, we obtain  $\frac{\partial S_1}{\partial p_i} > 0$  by the definition (35) of  $S_1$ . To show  $\frac{\partial S_2}{\partial p_i} < 0$  for small enough  $p_i$  and big enough  $q$ , we compare  $\Gamma_{y_i, p_i}^n(\gamma_n, p)$  and  $\Gamma_{y_n, p_i}^i(\gamma_i, p)$ . We first note that  $\Gamma_{y_i, p_i}^n(\gamma_n, p) = 0$  and  $\Gamma_{y_n, p_i}^i(\gamma_i, p) = 0$  for  $\gamma_{n,i} = -\gamma_{i,n} \geq 0$ . For  $p_i = 0$  and  $q = 1$ , we obtain from (34) that

$$\Gamma_{y_i, p_i}^n(y, p) \Big|_{p_i=0, q=1} = \frac{b(re^{\eta r y_i} - e^{\eta y_i}) + (r-1)e^{\eta(1+r)y_i}}{(b + (1-p_i)e^{\eta y_i} + p_i e^{\eta r y_i})^2} \Big|_{p_i=0, q=1} = (r-1)e^{\eta(r-1)y_i}$$

for  $y_i < 0$ . A calculation similar to (34) gives

$$\begin{aligned} \Gamma_{y_n, p_i}^i(y, p) \Big|_{p_i=0, q=1} &= \frac{\tilde{b}(re^{\eta r \sum_{\ell: y_\ell \geq 0} y_\ell} - e^{\eta \sum_{\ell: y_\ell \geq 0} y_\ell}) + (r-1)e^{\eta(1+r) \sum_{\ell: y_\ell \geq 0} y_\ell}}{(\tilde{b} + (1-p_i)e^{\eta \sum_{\ell: y_\ell \geq 0} y_\ell} + p_i e^{\eta r \sum_{\ell: y_\ell \geq 0} y_\ell})^2} \Big|_{p_i=0, q=1} \\ &= (r-1)e^{\eta(r-1) \sum_{\ell: y_\ell \geq 0} y_\ell} \end{aligned}$$

for  $y_n > 0$ , where  $\tilde{b} = (1-q)/(qe^{\eta \omega_i + \eta \sum_{\ell: y_\ell < 0} f(y_\ell, p_\ell)})$ . Therefore, for  $\gamma_{n,i} = -\gamma_{i,n} < 0$ , we obtain

$$\Gamma_{y_i, p_i}^n(\gamma_n s, p) \Big|_{p_i=0, q=1} < \Gamma_{y_n, p_i}^i(\gamma_i s, p) \Big|_{p_i=0, q=1}$$

and thus  $\frac{\partial S_2}{\partial p_i} < 0$  in this case. Using that the cost function  $C$  is convex, it follows from  $\frac{\partial S}{\partial p_i} = MSV_i - MPV_i$  that the individual choice of  $p_i$  is lower than socially optimal if  $\frac{\partial S}{\partial p_i} > 0$ .  $\square$

## B Description of Data and Plot Generation Procedure

In this section, we test the empirical predictions of our model conditional on banks' default risk choices. We use different data sources for the bilateral exposures in the CDS market, the initial exposures of banks, and their default probabilities.

**CDS volume.** CDS data come from the confidential Trade Information Warehouse of the DTCC. We use position data from December 31, 2011. This data set allows for a post-crisis analysis in which a large part of CDS trades were not yet centrally cleared.<sup>8</sup> We eliminate from our data set the following transactions:

<sup>8</sup>Distortion on the CDS market due to the ‘‘London Whale’’ (large unauthorized trading activities in JPMorgan’s Chief Investment Office) occurred only after December 31, 2011 and, thus, does not affect our analysis.

- All swaps with governments, states, or sovereigns as reference entities. We eliminate these transactions because we expect the default risk profile of corporate reference entities to have stronger dependence on the risk stemming from banks’ exposures than on that of sovereign entities.
- All swaps with reference entities that are considered systemically important financial institutions. By doing so, we avoid problems related to specific wrong-way risk, where the seller of the transaction also happens to be the reference entity.
- All transactions done by nonbanking institutions. For nonbanking institutions, there is no consistent way to measure initial exposures, which are needed in our analysis. While we consider only banks, we adjust their initial exposures by including CDS trades done with nonbanks. This procedure is consistent with our model and means that initial exposures of banks are determined after they have traded with nonbanks.
- The transactions done by two small private banks for which there were no data available on their initial exposures. Because these two banks are small players, the conclusions of our analysis are not affected by their exclusion.

Other than these four restrictions, we do not make any further adjustments. In particular, our data set also includes settlement locations outside of the United States, which allows for a more complete coverage of CDS trades and, importantly, guarantees symmetry in the inclusion of CDS trades (the transactions of both buyers and sellers are accounted for). The resulting set consists of CDS data for 81 banks.

**Initial exposure.** For each of these 81 banks, we compute its initial exposure by using 2011 data from the Federal Financial Institutions Examination Council (FFIEC) form 031 (“call report”), as in Begenau et al. (2015). We compute the initial exposure of each bank as the discounted valuation of its securities and loan portfolio, including CDSs traded with nonbanks as explained above. For large banks that book their assets mainly in holding companies, we use securities and loan portfolios at the bank holding company level. We group the securities and loans into three categories and use a specific discount factor for each group: less than one year (using the six-month U.S. Treasury rate to discount), one to five years (using the two-year U.S. Treasury rate to discount), and more than five years (using the seven-year U.S. Treasury rate to discount). Given the low interest rate environment in 2011, the precise choice of the discounting date and rate does not have a significant effect on our results. For foreign banks that do not report to the FFIEC, we analyze individual annual reports from 2011 to find the maturity profile of their securities and loans. Most of these annual reports are dated December 31, 2011, making them consistent with the domestic bank data. Some of them were released in March, June, or October of 2011, in line with the respective country’s regulatory guidelines.

**Default probabilities.** The banks’ default probabilities are calculated using CDS spread data from IHS Markit Ltd. (2018) via Wharton Research Data Services (WRDS). Because the default probabilities that are relevant for the analysis are those around the time of the transaction, we fix January 3, 2011 as the proxy date for CDS transactions and use the spread on this date to infer the default probability. We use the average five-year spread for Senior Unsecured Debt (Corporate/Financial) and Foreign Currency Sovereign Debt (Government) (SNRFOR). We compute the default probabilities from the CDS spreads applying standard techniques (credit triangle relation), assuming a recovery rate of 40 percent. For

19 among the 81 banks, CDS spread data were not available. For each of these banks, we instead use Moody’s credit rating as of January 2011 for its Senior Unsecured Debt, and relate the ratings to default probabilities by using corporate default rates over the 1982–2010 period from Moody’s.

**Intermediation volume.** We compute the intermediation volume without taking the counterparty risk into account. Table 1 shows that applying (4.3) with different values for the risk-aversion parameter only has a minor effect on the results.

	$\eta = 0.1$	$\eta = 1$	$\eta = 10$
total change	0.08 %	0.07 %	0.04 %
biggest change across banks	0.29 %	0.27 %	0.16 %

Table 1: Variation in intermediation volume for different levels of risk aversion.

**Intermediaries.** We define an intermediary to be a bank that provides at least 5 percent of the total intermediation volume. Using this definition, we obtain 5 intermediaries among the 81 banks. The selection of these 5 banks is not sensitive to the chosen threshold level of intermediation volume. Indeed, it is evident in Figure 3 that 5 banks account for the majority of the intermediation volume while the contributions of all other banks are small.

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