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Garth Baughman and Stanislav Rabinovich

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Self-confirming Price Dispersion in Monetary Economies

Garth Baughman
Federal Reserve Board

Stanislav Rabinovich*
University of North Carolina
Chapel Hill

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Abstract

In a monetary economy, we show that price dispersion arises as an equilibrium outcome without the need for costly simultaneous search or any heterogeneity in preferences, production costs, or search technologies. A distribution of money holdings among buyers makes sellers indifferent across a set of posted prices, leading to a non-degenerate price distribution. This price distribution, in turn, makes buyers indifferent across a range of money balances, rationalizing the non-degenerate distribution of money holdings. We completely characterize the distribution of posted prices and money holdings in any equilibrium. Equilibria with price dispersion admit higher maximum prices than observed in any single-price equilibrium. Also, price dispersion reduces welfare by creating mismatch between posted prices and money balances. Inflation exacerbates this welfare loss by shifting the distribution towards higher prices.

Keywords: Search, Price Dispersion, Money, Inflation.
JEL codes: D43; E31; E40.

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1 Introduction

Under what conditions is there price dispersion for identical goods? A well-known result going back to Diamond (1971) is that simple search frictions do not alone suffice to generate price dispersion. In this paper, we identify a natural and intuitive source of price dispersion: monetary trade. In a frictional product market where buyers must carry money in order to make purchases, we show that price dispersion arises as an equilibrium outcome without recourse to costly simultaneous search, or any heterogeneity in production technology, preferences, or search opportunities. In this sense, price dispersion can be a purely monetary phenomenon.

We consider a random matching monetary model in which sellers post prices, and buyers simultaneously decide on their quantity of money balances. When choosing prices, a seller trades off revenue per sale against the chance of selling a discrete good. Similarly, when choosing money balances, a buyer trades off the cost of holding money against the probability of being able to afford a posted price. A non-degenerate distribution of money balances among buyers makes sellers indifferent across a range of prices; a non-degenerate distribution of prices, in turn, makes buyers indifferent across a range of money balances. Thus, as in any mixed strategy equilibrium, the distribution of money balances rationalizes the distribution of prices and vice versa.

Due to the complementarity between price posting and money holdings, the model has many equilibria. We fully characterize the distribution of posted prices and money holdings in any equilibrium. Our main result is a necessary and sufficient condition for a set to be the support of an equilibrium price distribution. We derive this in three steps. First, we show that mass points in the distribution of money holdings imply gaps in the distribution of prices, and vice versa. Second, we provide closed-form expressions for the distributions on any interval. Third, these two results enables us to characterize the distributions on any closed set. We illustrate our results by describing two special cases: equilibria with a discrete support, i.e. exactly N prices, and equilibria whose support is an interval. We also derive expressions for average prices, real balances, and welfare.

There are three main lessons from our analysis. First, the set of equilibria is large. Consider the interval of prices that can obtain in a single-price equilibrium and yield sellers positive profits.[1] Our results imply that any closed subset of this interval can be the support of the price distribution in some dispersed-price equilibrium. Importantly, this does not exhaust the possibilities for equilibria. Equilibria with price dispersion can feature prices higher than those observed in any single-price equilibrium. This is because, in a dispersed-price equilibrium, a chance of sometimes paying a low price compensates buyers for carrying high money balances. In other words, price dispersion allows for higher maximum prices, a result quite different from the predictions of other models of price dispersion, as we discuss below.

[1] The fact that there is an interval of such prices is a very special case of our results and is straightforward to show; see. e.g. Jean *et al.* (2010).

Second, price dispersion reduces welfare, which depends in our model on the fraction of meetings that result in trade. In an equilibrium with price dispersion, there are meetings where the seller's posted price exceeds the buyer's money balances, and trade does not occur despite being mutually beneficial. Price dispersion reduces welfare by creating mismatch between posted prices and money balances.

Third, inflation reduces welfare by exacerbating the mismatch caused by price dispersion. This welfare cost of inflation is quite different from the standard inflation tax channel, whereby inflation lowers the demand for real balances. Instead, in our model, inflation exacerbates the welfare loss from mismatch by shifting the weight of the equilibrium price distribution onto higher prices. Importantly, this effect of inflation only exists in equilibria with price dispersion, and differs substantially from menu cost driven effects. Our analysis provides a new link between price dispersion and monetary theory.

1.1 Relationship to the literature

Our work complements the existing literature on price dispersion in non-monetary models. In order to generate price dispersion, such models often need to assume some form of heterogeneity such as in outside options (as in Albrecht and Axell (1984)), in search opportunities (as in Lester (2011), Menzio and Trachter (2018), or Kaplan *et al.* (2016)), or in seller size (as in Menzio and Trachter (2015)). In contrast, all the buyers and sellers in our model are identical in every respect, and price dispersion is entirely self-confirming. Another strand of the literature relies on nominal rigidities, such as menu costs, to generate price dispersion, as in Benabou (1988) and, more recently, Burdett and Menzio (2017). We have no such rigidities, so sellers are endogenously indifferent over a range of prices. Two previous papers deliver dispersion through purely strategic effects without any underlying heterogeneity: the classic model of Burdett and Judd (1983), which assumes costly simultaneous search, and the monetary model of Galenianos and Kircher (2008), which relies on auctions rather than posted prices. Both models, however, depend on implicit competition among multiple agents – on the seller's side for Burdett and Judd (1983) and on the buyer's side for Galenianos and Kircher (2008) – and so do not directly address the paradoxical finding of Diamond (1971) that simple sequential search leads to monopoly pricing. Here, we show that there is no paradox if consumers must pay with money, i.e. if there are both ex ante and ex post costs of paying higher prices.

Our model not only differs in the mechanism leading to price dispersion, but also implies results qualitatively different from the above literature. In all of the above models, the mechanism delivering price dispersion can not raise prices above those which obtain without dispersion. In contrast, in our environment, dispersed-price equilibria allow for prices *higher* than could be observed in any single-price equilibrium. Next, the result that price dispersion results in some mutually beneficial trades being turned down, and therefore a welfare loss, is present in some, but not all, models in the literature. Like Albrecht and Axell (1984) or Menzio and Trachter (2015), and unlike, e.g. Lester (2011), our model has the property that some prices get rejected on the equilibrium path. In Albrecht and Axell (1984), e.g., exogenous heterogeneity in buyers' willingness to pay causes them to reject prices. This makes seller's indifference over a range of prices. In our model, buyers would be willing to pay, but some cannot because buyers *endogenously* differ in their *ability* to pay as a result of choices of money holdings.

In addition to generating a novel source of price dispersion, our results contribute to the literature on monetary search models. This literature has recognized, starting with Green and Zhou (1998), that monetary models with price posting often possess a multiplicity of steady-state *single price* equilibria; our environment is closest to Jean *et al.* (2010), who show the same in the Lagos and Wright (2005) framework. Multiple single-price equilibria arise due to a coordination problem: if sellers post high prices, buyers have an incentive to carry large money holdings, and vice versa.[2] However, the literature emanating from Green and Zhou (1998), including Jean *et al.* (2010), has focused exclusively on single-price equilibria.[3] The existence and multiplicity of dispersed-price equilibria has been an open question, which we answer here.

The predictions concerning welfare and the effects of inflation are also markedly different from those obtained in a single-price equilibrium of a monetary model. Because of indivisible goods, all the single-price equilibria in our model have the same level of welfare, and, moreover, this level of welfare is independent of inflation. By contrast, not only does price dispersion lead to a welfare loss, but also inflation is detrimental for welfare when it affects the distribution of prices. This effect of inflation is also distinct from the well-known effects of inflation in other monetary models. For example, it is different from the standard inflation-tax effect on the intensive margin in e.g. Lagos and Wright (2005) or cash-in-advance models, which is shut down here because of indivisibility. Instead, it is the probability of trade conditional on a meeting that is affected here, due to the effect on the distribution of posted prices.

Section 2 below lays out the model environment, and Section 3 defines a steady-state equilibrium. Our main results concerning the characterization of the set of equilibria are in Section 4. Section 5 then uses these insights to describe two specific classes of equilibria: those with a discrete (finite or countably infinite) support of the price distribution, and those with a connected support. We compute welfare and examine the effects of inflation in Section 6. Appendix A discusses an extension of the model that allows for credit in some

[2]The same multiplicity of equilibria arises if money is replaced by an asset with an intrinsic value, as in Zhou (2003) or Rabinovich (2017). This multiplicity differs from classic results on indeterminacy in monetary models such as for exchange rates (Kareken and Wallace, 1981), or the price level (Sargent and Wallace, 1975), because our model features a multiplicity in real prices and trade volumes instead of nominal multiplicities.

[3]There exists a growing literature on monetary models with price dispersion, but these models resort to one of the other aforementioned channels to generate price dispersion - it is not due to monetary trade per se. For example, Head *et al.* (2012) rely on non-sequential search, as in Burdett and Judd (1983), and Bethune *et al.* (2018) assume heterogeneity in search opportunities, as in Lester (2011). Our mechanism is completely distinct from these.

meetings. Proofs are in Appendix B.

2 Environment

Time is discrete and the time horizon is infinite. There is a $[0, 1]$ continuum of ex ante identical agents with discount factor $\beta \in (0, 1)$. Each period is divided into two subperiods. In the second subperiod all agents consume a general good Q and supply labor H in a Walrasian centralized market, CM. Labor and the general good are perfectly divisible. As is common in the literature, and without loss of generality, it is assumed that H produces Q one-for-one. Let $U(Q) - H$ be the utility of consuming Q and working H in the CM, where $U(Q)$ satisfies standard assumptions.

In the first subperiod agents produce and consume specialized goods in a decentralized market, DM, with random bilateral matching. In a random match between two agents A and B, the probability of a single-coincidence meeting (B wants to consume the specialized good A can produce but not vice-versa) is $\sigma \in \left(0, \frac{1}{2}\right]$, and the probability of a double-coincidence meeting (B wants to consume a good A can produce and vice-versa) is 0. When B wants to consume what A produces, the former is called a buyer and the latter a seller. The specialized good is indivisible. Let $c > 0$ the cost of producing one unit, and $u > c$ be the utility from consuming one unit, of the indivisible good, conditional on it being one that the buyer consumes and the seller produces.

Trade in the DM requires a medium of exchange. We assume that this role is served by fiat money, whose supply is augmented via lump-sum transfers and grows at the gross rate γ, assumed to satisfy $\gamma > \beta$. Define the Fisherian nominal rate as $\iota = \gamma/\beta - 1$.

Search in the DM is random, and the terms of trade in the DM (i.e. the price of the indivisible good) are determined by price posting. Agents post selling prices y in units of the CM general good.

2.1 Discussion of the environment

The mechanism developed below depends on the specification of the environment – random matching, posted prices, indivisible goods, and money as a means of payment. We discuss each of these assumptions in turn.

We assume that matching is random and that firms post prices. We find both assumptions eminently realistic. Firms post prices in most markets. In some instances, these posted prices are merely suggestions – as, for example, with automobiles – that set the stage for a bargaining game. Most of the time, however, employees at the point of sale are not empowered to bargain, and the transaction price is the posted price. In the same vein as bargaining frictions, information is costly. Buyers usually do not know the exact price that a given seller will quote before contacting that seller. We show that this most natural technology of random matching with price posting can deliver price dispersion if one acknowledges the endogeneity of consumers' budget constraints.

We assume that goods are indivisible. This is again in line with almost all of the literature on price dispersion, reviewed above. In practice, many goods are discrete, and even those goods that are in principle divisible usually are not treated as such in practice – the grocer will not typically carve half an orange and sell it at half the price. Of course, indivisibility removes an important margin for adjustment which, in many models, drives the welfare effects of inflation. We show that, with dispersed prices, inflation can have real effects despite the lack of an intensive margin. This effect of inflation does not exist in equilibria without price dispersion. Unless so high as to shut down all trade, inflation has no allocative effects in any single-price equilibrium of our model, yet reduces welfare in any dispersed-price equilibrium. This link between price dispersion and the effects of inflation is novel and made transparent by the indivisible goods environment.

The crucial assumption is that trade requires money. In practice, while credit cards or other forms of credit may be important in some cases, most sellers will not extend credit at the point of sale, only 57 percent of American adults owned a general purpose credit card as of 2015, and credit payments amount to only a fifth of transactions.[4] We also show in the Appendix A that the results reported below – in particular, the existence of self-confirming price dispersion – are robust to the introduction of credit, as long as it is not ubiquitous, i.e. at least some transactions require cash. One implication of our results is that the availability of credit affects the nature and shape of price dispersion – a point that, to our knowledge, has not been explored in the literature.[5]

With regard to the specific choice of monetary model, we adopt the quasi-linear preferences environment following a long literature beginning with Lagos and Wright (2005). The quasi-linearity assumption is not critical. Essential is the fact that buyers face a cost of earning money, not that this cost is linear. Assuming quasilinear preferences, in addition to making the model tractable, also makes transparent the mechanism: it shuts down all other distributional concerns, allowing us to focus on coordination as the source of equilibrium price dispersion.[6]

[4]General purpose cards are distinguished from private-label cards in that the former can be used at a wide variety instead of only a single firm. See Consumer Financial Protection Bureau (2015) for discussion and data on ownership rates. No administrative data exist for the proportion of transactions involving credit cards in the United States, the survey data reported here derive from (Greene *et al.*, 2017, 30) and are from 2015.

[5]Liu *et al.* (2017) provide a monetary model with Burdett and Judd (1983) pricing and costly credit. Price dispersion is the result of simultaneous search as in Burdett and Judd (1983). That study provides further information on the data, but credit does not affect the *shape* of the price distribution in that model, rather the level of prices.

[6]It is well known that uninsured idiosyncratic risk can generate a distribution of money holdings, which, in a frictional product market, can lead to a distribution of prices; see e.g. Molico (2006) or Menzio *et al.* (2013). We abstract from these considerations here, so as to isolate our novel mechanism.

3 Equilibrium

We focus throughout on steady states. Let $V(z,y)$ be the expected payoff of an agent entering the DM with z units of real money balances (measured in units of the general good) and a posted price of y; and $W(z)$ the expected payoff of an agent entering the CM with z. Note that inflation – caused by lump-sum transfers T – implies that, in order to possess z' real balances in the next DM, one must commit $\gamma z'$ of the general good today. Therefore, we have

$$\begin{aligned} W(z) &= \max_{Q,H,z',y'} \{U(Q) - H + \beta V(z',y')\} \quad (1) \\ \text{s.t.} \quad & Q + \gamma z' = H + T + z \quad (2) \end{aligned}$$

After substituting out H using the budget constraint, one observes (as is standard in models with quasilinear preferences) that $W'(z) = 1$, which will be used below.

Moving on to the DM, let F and G denote the cumulative distributions of posted prices money balances, respectively. It is easy to see that a buyer with money holdings $\tilde{z}$ is willing to buy at price $\tilde{y}$ if and only if $\tilde{y} \leq \tilde{z}$ and $u + W(\tilde{z} - \tilde{y}) \geq W(\tilde{z})$. Since $W'(z) = 1$, this implies

$$\begin{aligned} V(z,y) &= W(z) + \sigma \int_{(-\infty,z]} \max\{0, u - \tilde{y}\} \, dF(\tilde{y}) \\ &\quad + \sigma \int_{[y,\infty)} \mathbf{1}_{y \leq u} (y - c) \, dG(\tilde{z}) \quad (3) \end{aligned}$$

Substituting back into (1), we obtain $W(z) = z + W(0)$, with

$$(1-\beta) W(0) = T + \max_Q \{U(Q) - Q\} + \beta \max_{z'} \nu(z') + \beta \max_{y'} \pi(y'), \quad (4)$$

where

$$\nu(z) = -\iota z + \sigma \int_{(-\infty,z]} \max\{0, u - \tilde{y}\} \, dF(\tilde{y}) \quad (5)$$

is the utility gain from carrying z units of real balances into the DM, and

$$\pi(y) = \sigma \int_{[y,\infty)} \mathbf{1}_{y \leq u} (y - c) \, dG(\tilde{z}) \quad (6)$$

is the profit from posting a selling price of y. Optimizing behavior requires that $\nu(z)$ is maximized at all z in the support of G, henceforth denoted by $\mathcal{G}$, and $\pi(y)$ is maximized at all y in the support of F, denoted by $\mathcal{F}$.[7]

Definition 1 *A* steady-state equilibrium *consists of distributions G and F, with supports $\mathcal{G}$ and $\mathcal{F}$, respectively, and numbers $\bar{\nu} \geq 0$ and $\bar{\pi} \geq 0$ such that:*

[7] The support of a distribution is the smallest closed set with probability 1.

1. $\bar{\nu} \geq \nu(z)$ *for all* z*, with equality if* $z \in \mathcal{G}$*,*

2. $\bar{\pi} \geq \pi(y)$ *for all* y*, with equality if* $y \in \mathcal{F}$*,*

where $\nu(z)$ *and* $\pi(y)$ *are given by (5) and (6).*

4 Analysis

We are interested in characterizing the set of possible equilibrium distributions, F and G. Beginning with some preliminary observations, we go on to consider mass points and then continuous points of the distributions. These allow us to formulate our main result providing necessary and sufficient conditions for equilibrium. First, we define additional notation.

4.1 Notation

For any function $h(x)$, write $h(x-) = \lim_{\epsilon \downarrow 0} h(x-\epsilon)$ for the left limit of h at x; similarly, write $h(x+)$ for the right limit. Then, we define $\delta_F(x) = F(x) - F(x-)$ for the mass on x under F; and similarly, define $\delta_G(x) = G(x) - G(x-)$ for the mass on x under G. Finally, let $\bar{G}(x) = \int_{[x,\infty)} dG(\tilde{x}) = 1 - G(x) + \delta_G(x)$ be the left truncated probability.

4.2 Preliminaries

First, we establish conditions necessary for a monetary equilibrium to exist, i.e. for trade to occur in the DM in an equilibrium. Supposing there is trade, write $\underline{x} = \min(\mathcal{F} \cap \mathcal{G})$ for the lowest transaction price.[8] Individual rationality requires that both buyers and sellers make weakly positive surplus. For sellers, this requires $\underline{x} \geq c$. For buyers, we need

$$0 \leq \nu(\underline{x}) = -\iota\underline{x} + \sigma\delta_F(\underline{x})(u - \underline{x}), \tag{7}$$

in other words, buyers are at least as well off *ex ante* carrying $\underline{x}$ as they would be if carrying zero. Rearranging,

$$\underline{x} \leq u\frac{\sigma\delta_F(\underline{x})}{\iota + \sigma\delta_F(\underline{x})} \leq u\frac{\sigma}{\iota+\sigma} \quad \text{since} \quad \delta_F(\underline{x}) \leq 1. \tag{8}$$

Combining these, a monetary equilibrium exists only if

$$c \leq \frac{\sigma}{\iota+\sigma}u \tag{9}$$

Where this holds with equality, (8) implies that $\delta_F(\underline{x}) = 1$. In this case the only transaction price is $c = u\sigma/(\iota+\sigma)$ and neither buyers nor sellers derive any surplus. Indeed, Jean *et al.* (2010) show that there exist equilibria with $\mathcal{F} = \mathcal{G} = \{x\}$ for any $x \in [c, u\sigma/(\iota+\sigma)]$.

[8] We refer to those posted prices which may lead to a sale, $\mathcal{F} \cap \mathcal{G}$, as transaction prices.

Lemma 1 *Equilibria with DM trade exist if and only if parameters satisfy (9). If (9) holds with equality, all equilibria with DM trade features* $\mathcal{F} \cap \mathcal{G} = \{c\}$.

We now examine the relationship between the support sets for the distributions of prices and money holdings. Except for prices too high for anyone to buy, or zero money holdings, these two sets will be equal. Furthermore, these exceptions obtain only if either some sellers never sell, or some buyers never buy.

Lemma 2 *If* $\bar{\pi} > 0$ *and* $\bar{\nu} > 0$*, then* $\mathcal{F} = \mathcal{G}$*. If* $\mathcal{F} \neq \mathcal{G}$ *and* $\bar{\nu} = 0$ *then* $\mathcal{G} \setminus \mathcal{F} \subseteq \{0\}$*. If* $\mathcal{F} \neq \mathcal{G}$ *and* $\bar{\pi} = 0$ *then* $\mathcal{F} \subseteq \{c\} \cup (\max(\mathcal{G}), \infty)$ *and* $\mathcal{G} \cap \mathcal{F} \subseteq \{c\}$.

Proof. See Appendix B. ■

In equilibrium, no one can have a strict incentive to deviate. For sellers, this means that marginally raising prices must reduce sales, so some buyers must hold money at or arbitrarily closely above any posted price. Similarly, for every quantity of money, buyers must not want to deviate to slightly lower, less costly, money balances. This can only occur if they would loose buying opportunities from the lower balances, so if some seller has posted a price at or arbitrarily close below a given balance. Equilibrium tends to drive buyers and sellers towards each other.

Indeed, buyers and sellers can only differ from one another if they do not lose in doing so. Hence, in the cases where $\mathcal{F} \neq \mathcal{G}$, either buyers, sellers, or both make no surplus. In this case, they are indifferent to also playing no-trade strategies. If $\bar{\pi} = 0$, there can be no dispersion in transaction prices, and if there is trade it occurs only at a price c. In this case, there are always a continuum of equilibria which differ in the proportion of sellers who post c versus some high price that never transacts. If $\bar{\nu} = 0$, there can still exist price dispersion, as we show in several examples in Section 5. In this case, given a set of traded prices, there are always a continuum of equilibria which differ in the proportion of buyers who choose to carry no money. We will ignore these multiplicities, and assume that, when indifferent, buyers carry positive balances and sellers post a transactable price. Henceforth, explicitly assume that $\mathcal{F} = \mathcal{G}$, denote this set by $\mathcal{X}$, and refer to it as *the equilibrium support.* Given this discussion, we can combine Lemmas 1 and 2 and further state

Lemma 3 *If* $\mathcal{X}$ *contains more than one point, then* $\underline{x} = \min \mathcal{X}$ *satisfies*

$$c < \underline{x} < u\frac{\sigma}{\iota + \sigma}.$$

Proof. $c < \underline{x}$ follows from the last part of Lemma (2). If $\underline{x} = u\frac{\sigma}{\iota+\sigma}$ then (8) implies $\delta_F(\underline{x}) = 1$ so that $\mathcal{X} = \{\underline{x}\}$. ■

Henceforth, we will assume that (9) holds with strict inequality, so that dispersed price equilibria are possible.

4.3 Gaps and Mass Points

Mass points in the distribution of money holdings induce gaps in the equilibrium support of the price distribution, and vice-versa.

Theorem 1 *In any equilibrium, if there is a mass point in G at $x > c$, then the interval $(x, x + k_F(x))$ does not intersect the equilibrium support $\mathcal{X}$, where*

$$k_F(x) = (x - c)\frac{\delta_G(x)}{1 - G(x)}. \tag{10}$$

Similarly, if there is a mass point in F at $x < u$ in the support of money holdings, then the interval $(x - k_G(x), x)$ does not intersect $\mathcal{X}$, where

$$\iota k_G(x) = \sigma\delta_F(x)(u - x). \tag{11}$$

Conversely, if $a < b \in \mathcal{X}$ and $(a, b) \cap \mathcal{X} = \emptyset$ - a gap in the distribution - then

$$\delta_F(b) = \frac{\iota}{\sigma}\frac{b - a}{u - b} > 0, \quad \textit{and} \quad \delta_G(a) = (1 - G(a))\frac{b - a}{a - c} > 0 \tag{12}$$

Proof. See Appendix B. ■

Mass points induce gaps because they cause discontinuities in payoffs. A mass in the money holding distribution induces a gap below because sellers want to shade up their prices to the mass-point. A mass in the price distribution induces a gap above because buyers want to shade down their money holdings to the mass-point. A point with positive mass in both distributions must, therefore, be isolated. The converse statement holds for much the same reason. A gap (a, b) in the equilibrium support can only be supported if the end points are sufficiently remunerative. To compensate buyers for holding b rather than a, there must be a mass of sellers at b. The mass $\delta_F(b)$ is pinned down by the buyers' indifference condition, which equalizes the benefits of extra transactions to the extra cost of carrying higher balances.

$$\sigma\delta_F(b)(u - b) = \iota(b - a), \tag{13}$$

which, rearranged, gives the expression in Theorem 1. Similarly, to make sellers indifferent to a discrete price cut from b to a, they must make discretely more sales because of a mass of buyers at a. By lowering the price from b to a, the seller gains $\delta_G(a)$ buyers at a but loses the extra revenue $(1 - G(a))(b - a)$ from the buyers above a, so the mass $\delta_G(a)$ must satisfy

$$\delta_G(a)(a - c) = (1 - G(a))(b - a) \tag{14}$$

Hence, any gap in the equilibrium support must feature a mass in F at the top and a mass in G at the bottom.

4.4 Intervals and Continuity

Theorem 1 has characterized behavior around mass points, one conclusion being that no interval in the equilibrium support can contain such. Now we characterize behavior on open intervals in the equilibrium support, where the distribution must be continuous. For any interval in the equilibrium support, sellers must be indifferent across all the prices on the interval, and buyers must be indifferent across all the money holdings on the interval. These indifference conditions pin down F and G. For sellers to be indifferent over all prices x in an interval, it must be that $\pi(x) = \bar{\pi}$ everywhere in that interval. Using the profit equation $\pi(x) = (x - c)(1 - G(x-))$ gives

$$G(x-) = 1 - \frac{\bar{\pi}}{x - c} \tag{15}$$

Moreover, on an open interval contained in the support, there can be no mass points, so $G(x-)$ can be replaced by $G(x)$ in the above formula. Next, for buyers to mix over money holdings in an interval, it must be that $\nu(x) = \bar{\nu}$ in that region. This indifference condition implies that F is differentiable (as formally shown in the Appendix) and therefore

$$0 = \frac{d\nu}{dx} = -\iota + \sigma(u - x) f(x), \tag{16}$$

where f is the density of F. The last two equations yield the following:

Theorem 2 *On any open interval contained in the equilibrium support,*

$$G(x) = 1 - \frac{\bar{\pi}}{x - c}, \tag{17}$$

and F is differentiable with density given by

$$f(x) = \frac{\iota}{\sigma(u - x)}. \tag{18}$$

Proof. See Appendix B. ■

4.5 Necessary and Sufficient Condition for Equilibrium

Our main result provides necessary and sufficient conditions for a set X to constitute the equilibrium support in a dispersed-price equilibrium. Recall that every closed set can be represented uniquely as the complement of a countable collection of disjoint open intervals. Hence, write $X = [\underline{x}, \overline{x}] \setminus \cup_{i=1}^{\infty} I_n$ for some countable, disjoint collection of intervals $I_n = (a_n, b_n) \subset [\underline{x}, \overline{x}]$ with $\underline{x} = \min X$ and $\overline{x} = \max X$. Using Theorems 1 and 2, we can fully characterize the distribution on X, thereby obtaining a necessary condition for an equilibrium with support X to exist. Because our proof is constructive, the necessary condition we derive is also sufficient and the equilibrium on any support X, whenever one exists, is unique.

Consider the finite approximations of X, letting $X_N = [\underline{x}, \overline{x}] \setminus \cup_{n=1}^N I_n$. Theorems 1 and 2 allow us to construct equilibria on each X_N. These converge uniformly to limiting distributions which give an equilibrium on the set X. For the distribution of real balances G, we can write

$$G_N(x) = \begin{cases} 0 & \text{if } x \leq \underline{x}, \\ 1 - \frac{\underline{x}-c}{x-c} & \text{if } \overline{x} > x \in X \setminus \cup_{i=1}^N [a_n, b_n), \\ 1 - \frac{\underline{x}-c}{b_n-c} & \text{if } x \in [a_n, b_n) \text{ for } n \leq N, \\ 1 & \text{if } x \geq \overline{x}. \end{cases} \tag{19}$$

Transparently, $G(x) = \lim_{N\to\infty} G_N(x)$ is a proper probability distribution and satisfies equal profit on X.

To construct the distribution of posted prices, F, we first recognize that, by Theorem 1, there must be mass points at the top of each interval, b_n:

$$\delta_{F_N}(b_n) = \frac{\iota}{\sigma} \frac{b_n - a_n}{u - b_n} \tag{20}$$

Next, the distribution must be constant on each I_n and have the density f given in Theorem 2 elsewhere. This can be expressed as

$$\begin{aligned} F_N(x) &= \delta_{F_N}(\underline{x}) + \sum_{n=1}^N \delta_{F_N}(b_n) \mathbf{1}_{\{b_n \leq x\}} + \int_{\underline{x}}^x f(y) \left[1 - \sum_{n=1}^N \mathbf{1}_{y \in I_n}\right] dy \\ &= \delta_{F_N}(\underline{x}) + \frac{\iota}{\sigma} \left[\sum_{n=1}^N \mathbf{1}_{\{b_n \leq x\}} \frac{b_n - a_n}{u - b_n} + \ln\left(\frac{u - \underline{x}}{u - x}\right) - \sum_{n=1}^N \ln\left(\frac{u - \min\{a_n, x\}}{u - \min\{b_n, x\}}\right)\right]. \end{aligned} \tag{21}$$

The mass point $\delta_{F_N}(\underline{x})$ is pinned down by the restriction that the distribution sum up to 1:

$$\delta_{F_N} = 1 - \frac{\iota}{\sigma} \left[\sum_{n=1}^N \frac{b_n - a_n}{u - b_n} + \ln\left(\frac{u - \underline{x}}{u - \overline{x}}\right) - \sum_{n=1}^N \ln\left(\frac{u - a_n}{u - b_n}\right)\right]. \tag{22}$$

Note that $(u - a_n)/(u - b_n) = 1 + (b_n - a_n)/(u - b_n)$ and $\ln(1 + w) \leq w$ for any $w > -1$, so

$$\frac{b_n - a_n}{u - b_n} - \ln\left(\frac{u - a_n}{u - b_n}\right) \geq 0. \tag{23}$$

Hence, $\delta_{F_N}(\underline{x})$ defined by (22) is a decreasing sequence in N. Individual rationality for sellers is guaranteed by $\underline{x} > c$. Individual rationality for buyers requires

$$\nu_N(\underline{x}) = -\iota\underline{x} + \sigma\delta_{F_N}(\underline{x})(u - \underline{x}) \geq 0 \tag{24}$$

in other words, a buyer ex ante prefers carrying $\underline{x}$ real balances to carrying zero. Because $\delta_{F_N}(\underline{x})$ is decreasing in N, we only need to check the limit as $N \to \infty$. Substituting (22) into (24) and taking the limit, we conclude that the buyer's individual rationality condition is satisfied if and only if

$$\frac{\sigma}{\iota} \geq \frac{\underline{x}}{u-\underline{x}} + \ln\left(\frac{u-\underline{x}}{u-\overline{x}}\right) + \sum_{n=1}^{\infty}\left[\frac{b_n-a_n}{u-b_n} - \ln\left(\frac{u-a_n}{u-b_n}\right)\right] \tag{25}$$

This is our necessary and sufficient condition that fully characterizes the set of equilibria.

Theorem 3 *Given a closed set $X = [\underline{x}, \overline{x}] \setminus \cup_{n=1}^{\infty} I_n$ with $I_n = (a_n, b_n)$ disjoint and $c < \underline{x} < \frac{\sigma}{\iota+\sigma}u$, there exists a unique equilibrium with support equal to X if and only if (25) holds. G and F are then given by the limits as $N \to \infty$ of (19)-(22).*

Notice that, in a single-price equilibrium, buyer individual rationality (24) amounts to $z \leq \frac{\sigma}{\iota+\sigma}u$, where z is the unique price posted in equilibrium. This means that any price $z \in \left[c, \frac{\sigma}{\iota+\sigma}u\right]$ can constitute a single-price equilibrium. Interestingly, any combination of these prices, excluding c, can be the support of a dispersed-price equilibrium.

Proposition 1 *Let X be a closed subset of $\left(c, \frac{\sigma}{\iota+\sigma}u\right]$. There exists an equilibrium with support X.*

Proof. See Appendix B. The proof amounts to showing that $\overline{x} \leq \frac{\sigma}{\iota+\sigma}u$ is a sufficient condition for (25). The lower bound c is excluded as a result of Lemma 3: if sellers get the same profits from posting two different prices in X, these profits cannot be zero. ■

The message of Proposition 1 is two-fold. First, it shows that the set of equilibria is large. Any closed subset of the prices that can be observed in a single-price equilibrium, while yielding sellers positive profits, can serve as the equilibrium support. Second, while $\overline{x} \leq \frac{\sigma}{\iota+\sigma}u$ is sufficient, it is not necessary since it is strictly stronger than (25): there exist dispersed-price equilibria in which the upper bound on prices is even higher than $\frac{\sigma}{\iota+\sigma}u$, which would be impossible in a single-price equilibrium. Price dispersion allows for higher maximum prices.

4.6 Upper Bound on Equilibrium Prices

The above discussion raises the question of what is the maximum price $\overline{x}$ that can be observed in any equilibrium. We characterize this maximum price by inspecting (25). First, the right-hand side of (25) is increasing in $\overline{x}$, so the maximum is attained when (25) holds with equality. Second, the infinite summation term is non-negative (as observed in (23)), so that the right-hand side of (25) is greater than or equal to $\underline{x}/\left(u-\underline{x}\right)+\ln\left(\left(u-\underline{x}\right)/\left(u-\overline{x}\right)\right)$. This means that if $X = [\underline{x}, \overline{x}] \setminus \cup_{n=1}^{\infty} I_n$ satisfies (25), then so does $X' = [\underline{x}, \overline{x}]$. In other words, the maximum possible price is attained when the support is an interval - a special case that we return to in section 5.1. Third, (25) holding with equality defines $\overline{x}$ as a decreasing function of $\underline{x}$, so that the upper bound $\overline{x}$ is attained when $\underline{x} \to c$. Summing up, we conclude that the upper bound on $\overline{x}$ satisfies

$$\frac{c}{u-c} + \ln\left(\frac{u-c}{u-\overline{x}}\right) = \frac{\sigma}{\iota} \tag{26}$$

Solving for $\overline{x}$, we obtain

Proposition 2 *The least upper bound on the set of prices that can obtain in any equilibrium is*

$$\sup \overline{x} = u - (u-c)\exp\{c/(u-c) - \sigma/\iota\} \tag{27}$$

This upper bound is tight since $\underline{x}$ can be arbitrarily close, but not equal, to c. It is straightforward to verify that $\sup \overline{x}$ is strictly higher than $\frac{\sigma}{\iota+\sigma}u$. However, $\sup \overline{x}$ is strictly lower than u, since

$$u - (u-c)\exp\{c/(u-c) - \sigma/\iota\} \leq u(1 - \exp\{-\sigma/\iota\}) < u, \tag{28}$$

and equal to $u(1 - \exp\{-\sigma/\iota\})$ in the limiting case $c = 0$. In other words, the upper bound on equilibrium prices is higher than the maximum price attainable in any *single-price* equilibrium, but still lower than the buyer's ex post valuation.

We now provide some intuition for this finding. For any equilibrium support $\mathcal{X}$, let us denote by $y^* = \int_{\mathcal{X}} y dF(y)$ the average posted price. Then the utility of a buyer carrying the maximum real balances $\overline{x}$ is $\nu(\overline{x}) = -\iota\overline{x} + \sigma(u - y^*)$, since such a buyer trades at the posted price whenever meeting a seller. The maximum possible $\overline{x}$ is obtained when $\nu(\overline{x}) = 0$, which gives, after re-arranging,

$$\overline{x} = \frac{\sigma}{\iota+\sigma}u + \frac{\sigma}{\iota+\sigma}(\overline{x} - y^*) \tag{29}$$

This is greater than or equal to $\frac{\sigma}{\iota+\sigma}u$, with equality only if $\overline{x} = y^*$, that is, if there is no price dispersion. Buyers get non-negative utility despite carrying high real balances, because of the chance of sometimes paying a lower price. This also clarifies why $\overline{x}$ cannot be arbitrarily close to u. Suppose that a high posted price, say b, coexists in equilibrium with some lower posted price, a. The fraction of sellers posting b must be sufficiently low to ensure that buyers carrying real balances b get non-negative utility. On the other hand, the fraction of sellers posting b must be sufficiently high to induce some buyers to carry real balances b rather than a. The tradeoff between these two opposing forces keeps the maximum price bounded away from the buyer's valuation.

5 Special cases

In this section, we apply the Theorem 3 to characterize two special classes of equilibria – equilibria with a connected support, and those with a discrete support.

5.1 Connected Support Equilibria

We first construct equilibria in which the equilibrium support is an interval. This is a special case of Theorem 3 with $X = [\underline{x}, \bar{x}]$. By Theorem 2, $\bar{\pi} = \underline{x} - c$ implies

$$G(x) = 1 - \frac{\underline{x} - c}{x - c} \quad \text{for} \quad x \in [\underline{x}, \bar{x}) \tag{30}$$

And, since $G(\bar{x}) = 1$ by definition, there is a mass point at the top of the money holding distribution, with mass equal to

$$\delta_G(\bar{x}) = \frac{\underline{x} - c}{\bar{x} - c} \tag{31}$$

Turning now to F, for the support to be connected, there can be no mass points in $(\underline{x}, \bar{x}]$ by Theorem 1, and $f(x)$ is given by (18) on the interior by Theorem 2. Since the only possible mass point is at $\underline{x}$, we can write the distribution function for all $x \in [\underline{x}, \bar{x}]$ as

$$F(x) = \delta_F(\underline{x}) + \int_{(\underline{x}, x]} f(x)\, dx = \delta_F(\underline{x}) + \frac{\iota}{\sigma} \ln\left(\frac{u - \underline{x}}{u - x}\right) \tag{32}$$

The restriction $F(\bar{x}) = 1$ then implies

$$\delta_F(\underline{x}) = 1 - \frac{\iota}{\sigma} \ln\left(\frac{u - \underline{x}}{u - \bar{x}}\right) \tag{33}$$

Buyer utility is

$$\bar{\nu} = -\iota \underline{x} + \sigma \delta_F(\underline{x})(u - \underline{x}) \tag{34}$$

To ensure $\bar{\nu} \geq 0$, we need

$$\delta_F(\bar{x}) \geq \frac{\iota}{\sigma} \frac{\underline{x}}{u - \underline{x}}, \tag{35}$$

which, by (33), requires

$$\frac{\underline{x}}{u - \underline{x}} + \ln\left(\frac{u - \underline{x}}{u - \bar{x}}\right) \leq \frac{\sigma}{\iota} \tag{36}$$

Condition (36) is a special case of of condition (25) for a connected support. We have thus established the following:

Corollary 1 *There exists a unique equilibrium with connected support equal to $\mathcal{X} = [\underline{x}, \bar{x}]$ if and only if $c < \underline{x}$ and $\bar{x}$ satisfies (36). The equilibrium distribution of real balances G is given by (30) and (31), and the equilibrium distribution of prices is given by (32) and (33). The equilibrium payoffs are given by $\bar{\pi} = \underline{x} - c$ and $\bar{\nu} = -\iota \underline{x} + \sigma \delta_F(\underline{x})(u - \underline{x})$.*

Figure 1 illustrates the cumulative distributions F and G in a connected-support equilibrium. In the example shown here, we assume $u = 10$, $c = 1$, $\sigma = 0.2$, and $\iota = 0.05$, and look at the equilibrium with the support $[2, 7.8526]$.

We now discuss the maximum attainable equilibrium price. The left-hand side of (36) is increasing in both $\underline{x}$ and $\overline{x}$. Therefore, for every $\underline{x}$, the maximum attainable $\overline{x}$ is given by (36)

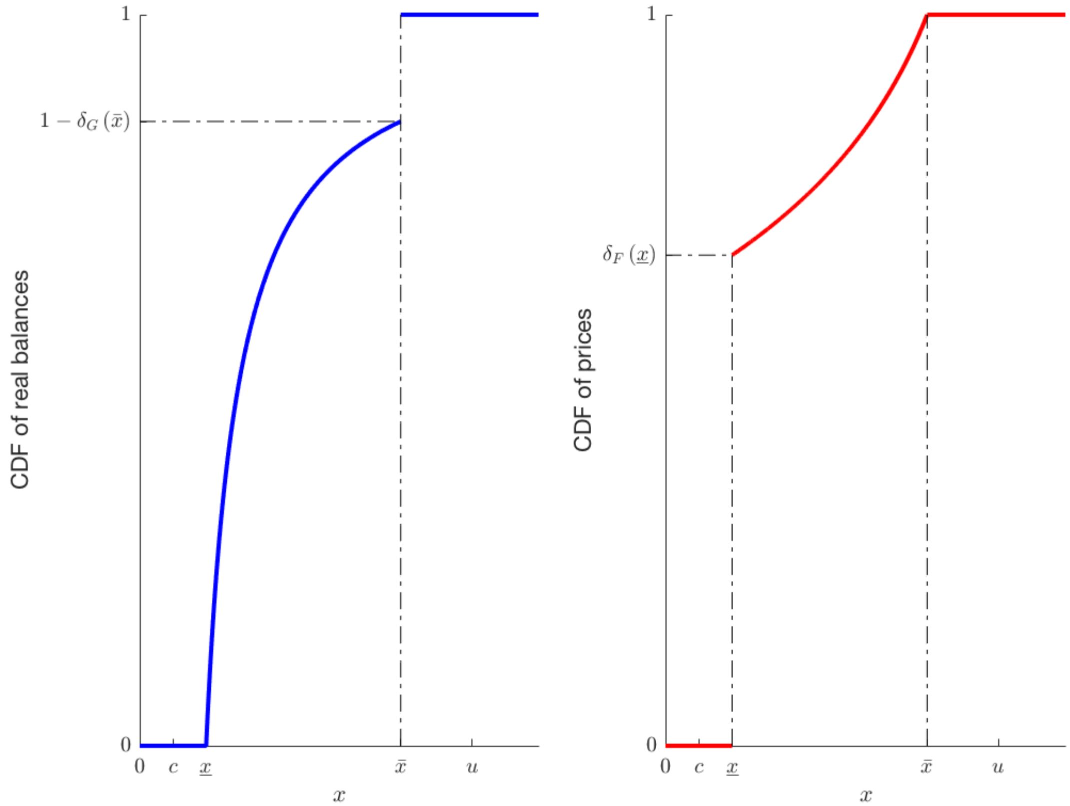

Figure 1: Distributions of prices and real balances for $u = 10$, $c = 1$, $\sigma = 0.2$, $\iota = 0.05$ and $\mathcal{X} = [2, 7.8526]$

holding with equality, wherefore the maximum attainable $\overline{x}$ is decreasing in $\underline{x}$. Furthermore, for $\underline{x}$ close to c, the highest attainable $\overline{x}$ exceeds $\frac{\sigma}{\iota+\sigma}u$, since

$$\frac{c}{u-c} + \ln(u-c) < \frac{\sigma}{\iota} + \ln\left(u - \frac{\sigma}{\iota+\sigma}u\right) \tag{37}$$

The finding that the maximum attainable $\overline{x}$ is decreasing in $\underline{x}$ reflects the same intuition as for the general case: in order to compensate buyers for carrying high real balances, there must be a chance of paying a sufficiently low price. Finally, the upper bound on $\overline{x}$ is achieved with (36) holding at equality at $\underline{x} \to c$, and is given by Proposition 2.

5.2 Discrete Point Equilibria

We now construct equilibria whose support has exactly N points, where N can be finite or infinite. Note that this is trivial for $N = 1$: for any $x \in \left[c, \frac{\sigma}{\iota+\sigma}u\right]$ there is an equilibrium with $\mathcal{X} = \{x\}$. We focus on constructing dispersed-price equilibria, i.e. those with $N > 1$.

Consider a candidate equilibrium with $\mathcal{X} = \{x_1, x_2, ..., x_N\}$, where $x_i < x_{i+1}$. This is a special case of Theorem 3 with $\underline{x} = x_1$, $\overline{x} = x_N$, and $I_n = (x_n, x_{n+1})$.

First, by Lemma 3, we must have

$$c < x_1 \leq \frac{\sigma}{\iota + \sigma} u, \tag{38}$$

since if sellers get equal profits from posting two different prices, these profits cannot be zero. Second, we characterize G. The x_i must all yield equal profit to sellers, so

$$\bar{\pi} = x_1 - c = \sum_{j=i+1}^{N} \delta_G(x_j)(x_{i+1} - c) \tag{39}$$

for all $i \geq 1$. This immediately implies

$$\delta_G(x_i) = \frac{x_1 - c}{x_i - c} - \frac{x_1 - c}{x_{i+1} - c} \tag{40}$$

for every i with $1 \leq i < N$. Note that this is simply an application of Theorem 3, with the substitution $G_N(x_i) = \sum_{j=1}^{i} \delta_G(x_j)$. We next verify that (40) results in a probability distribution. Clearly, $\delta_G(x_i) > 0$ for $i < N$ follows trivially from $x_{i+1} > x_i$. For $N < \infty$, we set

$$\delta_G(x_N) = 1 - \sum_{i=1}^{N-1} \delta_G(x_i), \tag{41}$$

Note that (40) leads to telescoping sums, i.e.

$$\sum_{i=1}^{N-1} \delta_G(x_i) = 1 - \frac{x_1 - c}{x_2 - c} + \frac{x_1 - c}{x_2 - c} - \frac{x_1 - c}{x_3 - c} + \ldots = 1 - \frac{x_1 - c}{x_N - c}, \tag{42}$$

which implies $\delta_G(x_N) > 0$. For $N = \infty$, we observe that the sequence x_i is strictly increasing and bounded, so must have a limit $x_\infty = \lim_{n \to \infty} x_n$. This implies

$$\lim_{n \to \infty} \sum_{i=1}^{n} \delta_G(x_i) = \lim_{n \to \infty} \left(1 - \frac{x_1 - c}{x_{n+1} - c}\right) = 1 - \frac{x_1 - c}{x_\infty - c} \tag{43}$$

We therefore set

$$\delta_G(x_\infty) = 1 - \lim_{n \to \infty} \sum_{i=1}^{n} \delta_G(x_i) = \frac{x_1 - c}{x_\infty - c} \tag{44}$$

and observe that $\delta_G(x_\infty) > 0$. This shows that G, as defined by (40) and either (41) or (44), adds to one.

Third, we characterize F. Applying Theorem 3 again, we have

$$F(x) = \delta_F(x_1) + \frac{\iota}{\sigma} \sum_{i=1}^{N} \mathbf{1}_{\{x_{i+1} \leq x\}} \frac{x_{i+1} - x_i}{u - x_{i+1}} \tag{45}$$

In other words, there is a mass point at each x_{i+1} equal to

$$\delta_F(x_{i+1}) = \frac{\iota}{\sigma} \frac{x_{i+1} - x_i}{u - x_{i+1}} \tag{46}$$

for every $i \geq 1$. As above, this states that a buyer is indifferent between carrying x_i real balances and carrying x_{i+1}, whereby the latter raises his probability of trade by $\delta_F(x_{i+1})$. The mass $\delta_F(x_1)$ at x_1 is pinned down by

$$\delta_F(x_1) = 1 - \sum_{i=1}^{N} \delta_F(x_{i+1}) \tag{47}$$

Buyer utility is then given by

$$\bar{\nu} = -\iota x_1 + \sigma \delta_F(x_1)(u - x_1) \tag{48}$$

Equilibrium requires $\bar{\nu} \geq 0$, implying

$$\frac{x_1}{u - x_1} + \sum_{i=1}^{N} \frac{x_{i+1} - x_i}{u - x_{i+1}} \leq \frac{\sigma}{\iota} \tag{49}$$

The restriction (49) is a special case of (25) for a discrete point support. The above analysis can be summarized as

Corollary 2 *Let $\mathcal{X}$ be any discrete, but not necessarily finite, increasing sequence $\{x_i\}$. There exists an equilibrium with support equal to the closure of $\mathcal{X}$ if and only if $\mathcal{X}$ satisfies (38) and (49). The distribution of real balances G is given by (40) and either (41) if $\mathcal{X}$ is finite, or (44) if $\mathcal{X}$ is infinite. The distribution of prices F is given by (46) and (47). The equilibrium payoffs $\bar{\pi}$ and $\bar{\nu}$ are given by (39) and (48), respectively.*

Condition (49) puts an upper bound on prices. This upper bound, however, depends on the number of prices. For a single-price equilibrium ($N = 1$) this condition reduces to $x_1 \leq \frac{\sigma}{\iota+\sigma} u$. However, in dispersed-price equilibria, we can obtain prices above $\frac{\sigma}{\iota+\sigma} u$, as shown in Proposition 2. We now derive an explicit formula for the upper bound on prices for any N-point support.

For any given x_1, consider the problem of choosing $x_2, ..., x_N$ to maximize x_N, subject to the constraint (49). The necessary first-order conditions for this problem (derived in detail in the Appendix) imply

$$\frac{u - x_{i-1}}{u - x_i} = \frac{u - x_i}{u - x_{i+1}} \tag{50}$$

for every $i = 2, ..., N-1$, which, by (46), indicates that the prices x_2 through x_N need to be equally likely, i.e. $\delta_F(x_2) = ... = \delta_F(x_N)$. At the maximum x_N, the restriction (49) clearly holds with equality; substituting (50) into (49) can then be shown to yield

$$x_N = u - (u - x_1)\left(\frac{N-1}{N-1+\sigma/\iota - x_1/(u-x_1)}\right)^{(N-1)} \tag{51}$$

Equation (51) defines the maximum x_N as a function of x_1. Thus defined, x_N is hump-shaped in x_1 for any N and increasing in N for any x_1. As an illustration, Figure 2 plots x_N defined by (51) as a function of x_1 for $N = 2, 3, 4, 5$.

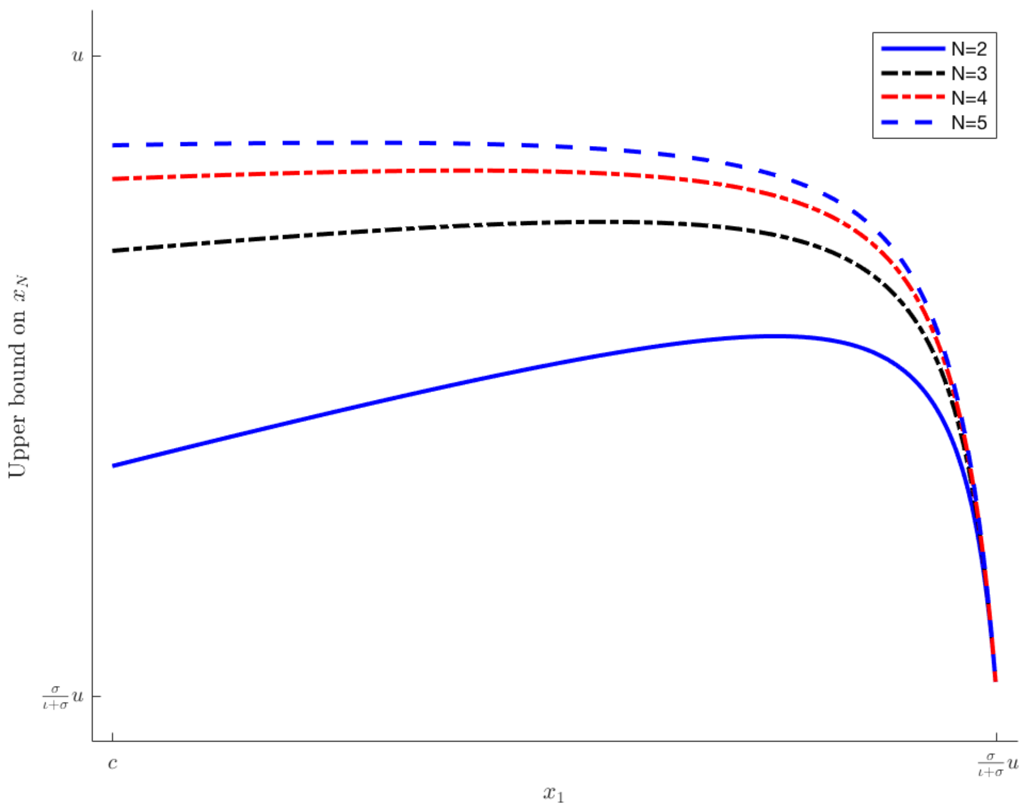

Figure 2: Upper bound on x_N for a given x_1, when $u = 10$, $c = 4.2857$, $\sigma = 0.3$, $\iota = 0.05$.

For an equilibrium with price dispersion to exist, the lowest price must strictly exceed c. Hence, determining the highest possible x_N amounts to maximizing (51) with respect to x_1, subject to the restriction $x_1 \geq c$.[9] When the constraint $x_1 \geq c$ does not bind, the necessary

[9] As explained earlier, x_1 must be *strictly* greater than c in a dispersed-price equilibrium. Therefore, if the

first-order condition for this problem implies $x_1 = u\sigma / (\iota N + \sigma)$ and

$$x_N = u\left(1 - \left(\frac{\iota N}{\iota N + \sigma}\right)^N\right) \tag{52}$$

Incidentally, in this case all the prices x_i are equally likely, i.e. $\delta_F(x_1) = ... = \delta_F(x_N)$. On the other hand, if the restriction $x_1 \geq c$ binds, then the maximum x_N is obtained in the limit when $x_1 \to c$, and $\delta_F(x_1)$ is pinned down by (49) holding with equality at $x_1 \to c$. In this case, the upper bound on prices satisfies

$$x_N = u - (u - c)\left(\frac{N-1}{N - 1 + \sigma/\iota - c/(u-c)}\right)^{(N-1)} \tag{53}$$

The constraint $x_1 \geq c$ binds when $c > u\sigma / (\iota N + \sigma)$ and does not bind otherwise. Since $c < u\sigma / (\iota + \sigma)$, the constraint does not bind for small enough N, and will bind for a large enough N unless $c = 0$. This then implies the following characterization of the upper bound:

Corollary 3 *Let $\hat{x}_N$ be the upper bound on the price x_N that can obtain in an equilibrium with N-price support $X = \{x_1, ..., x_N\}$. If $c < u\sigma / (\iota N + \sigma)$, the upper bound is attained at $x_1 = u\sigma / (\iota N + \sigma)$ and is given by (52). If $c \geq u\sigma / (\iota N + \sigma)$, the upper bound is attained in the limit as $x_1 \to c$ and is given by (53). Moreover, $\hat{x}_N$ is strictly increasing in N and*

$$\lim_{N\to\infty} \hat{x}_N = u - (u - c)\exp\{c/(u-c) - \sigma/\iota\} \tag{54}$$

Proof. See Appendix B for full proof. ■

Figure 3 illustrates the upper bound, $\hat{x}_N$, as a function of N. Note that at $N = 1$, the upper bound is $u\sigma / (\iota + \sigma)$, which equals 8 in this numerical example. In the limit as $N \to \infty$, x_N is given by (53) rather than (52); taking the limit of (53) give (54). Note that the expression in (54) is the same as the limit for the general case in Proposition 2.

The result that the upper bound on prices is increasing in N has a very intuitive interpretation. Note that (49) is simply the restriction that buyer utility must be non-negative. Suppose that buyers receive non-negative utility in some equilibrium with N prices. Now, suppose that some of the sellers posting x_N instead post a price between x_{N-1} and x_N; the non-negativity constraint on buyer utility now holds with slack. Thus, equilibria with more prices allow for higher prices.

6 Aggregate implications

In this section, we examine the implications of price dispersion for aggregate real balances, profits, and welfare, and the effects of inflation on these aggregates. The model endogenously

constraint $x_1 \geq c$ in fact binds in the price maximization problem, then the upper bound on x_N is attained only in the limit as $x_1 \to c$.

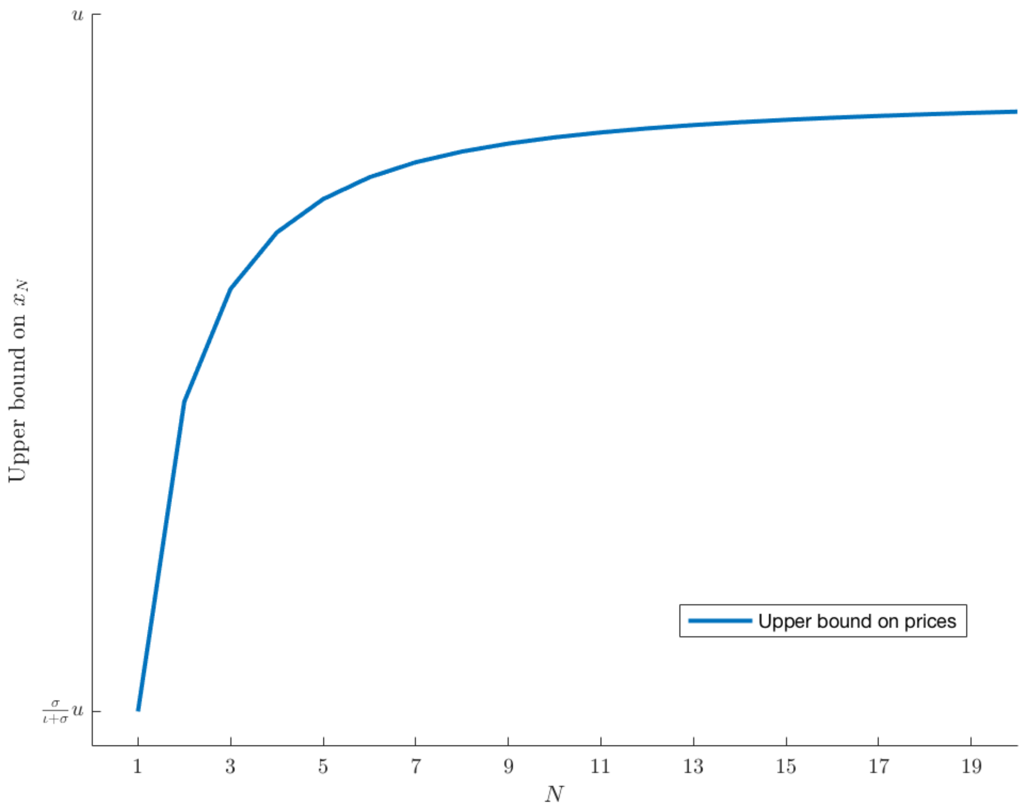

Figure 3: Upper bound on x_N as a function of N, when $u = 10$, $c = 4$, $\sigma = 0.2$, $\iota = 0.05$.

generates inequality in income and spending. Some agents work more in the CM and carry more money balances into the DM than others. Interestingly, agents are ex ante indifferent among all the levels of money balances: higher money balances incur a higher inflation cost but yield a higher probability of trade. However, this does not imply that equilibria with such inequality are welfare-equivalent to equilibria without it. As a result of price dispersion, some meetings do not result in trade because the posted price exceeds the buyer's money balances with positive probability. Thus, price dispersion generates mismatch between posted prices and real balances, which is detrimental for overall welfare. We show this formally below. We then show that inflation exacerbates this welfare loss from mismatch by shifting the distribution toward higher prices.

6.1 Real balances, price, and welfare

Given equilibrium distributions F and G on equilibrium support $\mathcal{X}$, it is straightforward to calculate aggregate real balances,

$$z^* \equiv \int_{\mathcal{X}} z dG(z), \tag{55}$$

and the average posted price in the DM,

$$y^* \equiv \int_{\mathcal{X}} y dF(y). \tag{56}$$

We next derive an expression for steady-state welfare. Abstractly, this is simply

$$\mathcal{W} = \int_{\mathcal{X}} \int_{\mathcal{X}} V(z, y)\, dG(z)\, dF(y). \tag{57}$$

Recall from (3) and (4) that $V(z, y)$ is given by

$$\begin{aligned} V(z, y) &= W(z) + \iota z + \nu(z) + \pi(y) \\ &= (1 + \iota) z + W(0) + \nu(z) + \pi(y) \end{aligned} \tag{58}$$

where

$$W(0) = \frac{1}{1-\beta}\left[T + \max_Q (U(Q) - Q) + \beta(\bar{\nu} + \bar{\pi})\right]. \tag{59}$$

Market clearing requires the growth of the money supply to satisfy $T = (\gamma - 1) z^*$. Substituting back into (59) and (58), integrating over z and y, and noting that $\gamma - 1 = \beta(1+\iota) - 1$, one obtains

$$\begin{aligned} \mathcal{W} &= \frac{1}{1-\beta}\left[\max_Q (U(Q) - Q) + \bar{\pi} + \bar{\nu} + \iota z^*\right] \\ &= \frac{1}{1-\beta}\left[\max_Q (U(Q) - Q) + \mathcal{W}_{DM}\right], \end{aligned} \tag{60}$$

where $\mathcal{W}_{DM} = \bar{\pi} + \bar{\nu} + \iota z^*$ is the per-period social welfare in the DM. Equation (60) illustrates that, because of quasi-linear preferences, the social surplus in the DM is all we need to know in order to welfare-rank equilibrium allocations. The expression for $\mathcal{W}_{DM}$, in turn, simply says that this social surplus equals the seller surplus $\bar{\pi}$ plus the (gross) ex ante buyer surplus

$\bar{\nu} + \iota z^*$. To better understand this social surplus, note that we can re-write it as

$$\begin{aligned}
\mathcal{W}_{DM} &= \bar{\pi} + \bar{\nu} + \iota z^* \\
&= \int_{\mathcal{X}} \pi(y)\, dF(y) + \int_{\mathcal{X}} (\nu(z) + \iota z)\, dG(z) \\
&= \int_{\mathcal{X}} \int_{\mathcal{X}} \sigma \mathbf{1}_{y \le z} (y - c)\, dG(z)\, dF(y) + \int_{\mathcal{X}} \int_{\mathcal{X}} \sigma \mathbf{1}_{y \le z} (u - y)\, dF(y)\, dG(z) \\
&= \int_{\mathcal{X}} \int_{\mathcal{X}} \mathbf{1}_{y \le z} [\sigma(y - c) + \sigma(u - y)]\, dG(z)\, dF(y) \\
&= \sigma(u - c) \int_{\mathcal{X}} \int_{\mathcal{X}} \mathbf{1}_{y \le z} dG(z)\, dF(y) \qquad (61)
\end{aligned}$$

In other words, because of indivisible goods in the DM, the social surplus in the DM depends only on the probability that a trade takes place, since the surplus from each trade is $u - c$. This surplus is maximized, moreover, when all meetings result in trade, in which case the social surplus is simply $\sigma(u - c)$.

Motivated by this observation, we examine how price dispersion impacts $\mathcal{W}_{DM}$. Notice that $\bar{\pi} = \pi(\underline{x}) = \sigma(\underline{x} - c)$ and $\bar{\nu} = \nu(\bar{x}) = -\iota\bar{x} + \sigma(u - y^*)$, where $\underline{x} = \min \mathcal{X}$ and $\bar{x} = \max \mathcal{X}$. This means that

$$\begin{aligned}
\mathcal{W}_{DM} &= \sigma(\underline{x} - c) + \sigma(u - y^*) - \iota\bar{x} + \iota z^* \\
&= \sigma(u - c) - \sigma(y^* - \underline{x}) - \iota(\bar{x} - z^*) \qquad (62)
\end{aligned}$$

The first term of this expression, $\sigma(u - c)$ is the maximum social surplus obtainable in the DM, since this is the surplus that would obtain if every meeting resulted in trade. So, (62) states that welfare equals the maximum possible welfare $\sigma(u - c)$ minus a wedge that depends on price dispersion. Intuitively, to induce trade with probability 1, a fictitious social planner would need to subsidize each seller posting y by $y - \underline{x}$, and subsidize each buyer with x real balances by $\iota(\bar{x} - x)$ ex ante, leading to an aggregate welfare loss of $\sigma(y^* - \underline{x}) + \iota(\bar{x} - z^*)$.

This intuitive expression immediately implies that price dispersion is detrimental for welfare. Formally, let $E = (\bar{\pi}, \bar{\nu}, F, G)$ be an equilibrium with support $\mathcal{X}$ and let $E' = (\bar{\pi}', \bar{\nu}', F', G')$ be an equilibrium with support $\mathcal{X}'$. We will say that E' is a mean-preserving spread of E if

$$\int_{\mathcal{X}} y dF(y) = \int_{\mathcal{X}'} y dF'(y), \quad \text{and} \quad \int_{\mathcal{X}} x dG(x) = \int_{\mathcal{X}'} x dG'(x),$$

and either $\min \mathcal{X}' \le \min \mathcal{X}$ or $\max \mathcal{X}' \ge \max \mathcal{X}$, with at least one inequality strict. Then we can state

Proposition 3 *If an equilibrium E' is a mean-preserving spread of another equilibrium E, then welfare is lower under E' than under E.*

The proof follows directly from inspecting (62). Fixing y^* and z^*, welfare is lower the lower is $\underline{x}$, and the higher is $\bar{x}$. In fact, it is straightforward to see that both buyers and sellers are worse off as a result of higher price dispersion: fixing y^* and z^*, a buyer wishing to buy with maximum probability needs to carry a higher level $\overline{x}$ of real balances, and a seller wishing to sell with maximum probability needs to post a lower price $\underline{x}$.

Price dispersion leads to mismatch. Note that in every single-price equilibrium, we have $\underline{x} = y^* = z^* = \bar{x}$, so that welfare equals the first-best level of welfare, $\sigma(u-c)$, because trade takes place with probability one. Price dispersion, on the other hand, allows a positive probability of meetings in which the seller's posted price exceeds the buyer's money balances. The proportion of such meetings determines the welfare loss.

6.2 Effects of inflation

We next examine the effects of inflation, captured by ι, on real balances, prices and welfare. Proposition 3 concerned the comparison between two equilibria under a given inflation rate. In this section, we fix an equilibrium support $\mathcal{X}$ and consider how a change in ι impacts the equilibrium distributions F and G on that support.

Take F and G in turn. For G, (19) shows that a change in inflation has no effect on G and therefore on z^*. This is because G is pinned down by the seller's indifference condition, which remains unaffected when ι changes. Since G is unaffected, so is z^*. F, however, is determined by the buyer's ex ante indifference condition and will therefore depend on the cost of carrying money balances. Combining (21) with (22), we obtain

$$\begin{aligned} F(x) &= 1+\frac{\iota}{\sigma}\left[\ln\left(\frac{u-\overline{x}}{u-x}\right)+\sum_{n=1}^{\infty}\mathbf{1}_{\{b_n\leq x\}}\frac{b_n-a_n}{u-b_n}-\sum_{n=1}^{\infty}\ln\left(\frac{u-\min\{a_n,x\}}{u-\min\{b_n,x\}}\right)\right.\\ &\qquad\left.-\sum_{n=1}^{\infty}\frac{b_n-a_n}{u-b_n}+\sum_{n=1}^{\infty}\ln\left(\frac{u-a_n}{u-b_n}\right)\right]\\ &= 1+\frac{\iota}{\sigma}\left[\ln\left(\frac{u-\overline{x}}{u-x}\right)-\sum_{n=1}^{\infty}\mathbf{1}_{\{x<b_n\}}\left(\frac{b_n-a_n}{u-b_n}-\ln\left(\frac{u-a_n}{u-b_n}\right)\right)\right.\\ &\qquad\left.-\sum_{n=1}^{\infty}\mathbf{1}_{\{a_n\leq x<b_n\}}\ln\left(\frac{u-a_n}{u-x}\right)\right] \end{aligned} \tag{63}$$

The expression in brackets in (63) is negative, so an increase in ι lowers $F(x)$ for any x, and therefore shifts the cumulative distribution of prices upward in the sense of first-order stochastic dominance. An upward shift in the distribution raises its mean, so an increase in ι raises y^*. Inflation raises the average *relative* price of the good whose consumption requires cash. For example, for the connected support example introduced in Section 5.1, it is straightforward to derive

$$y^* = \underline{x} - \frac{\iota}{\sigma}\left(\overline{x} - \underline{x}\right) + \frac{\iota}{\sigma}\left(u - \underline{x}\right)\ln\left(\frac{u - \underline{x}}{u - \overline{x}}\right) \tag{64}$$

The mechanism at work here is mixed-strategy equilibrium reasoning. In equilibrium, buyers are indifferent among all the levels of real balances in the support. An increase in inflation, all else equal, makes it more costly to carry high, rather than low, real balances. To keep buyers indifferent in spite of the higher inflation, it must be that the benefit of carrying high real balances also rises, which is only possible if high prices are now posted with a higher probability.

Once we know the effects of inflation on z^* and y^*, it is straightforward to determine its effects on $\bar{\pi}$ and $\bar{\nu}$. Since $\bar{\pi} = \sigma\left(\underline{x} - c\right)$, seller profit does not depend on inflation. On the other hand, buyer utility equals the utility of a buyer who takes out the maximum real balances and pays the average price: $\bar{\nu} = -\iota\overline{x} + \sigma\left(u - y^*\right)$. Thus, inflation unambiguously reduces $\bar{\nu}$, for two reasons. First, it directly raises the cost $\iota\overline{x}$ of carrying real balances. Second, it has the equilibrium effect of shifting the price distribution toward higher prices, thus raising the average price the buyer pays.

Finally, we deduce the effect on welfare through similar reasoning. Welfare, given by (62), falls when ι rises, both through the direct effect of ι, and the equilibrium effect on y^*. We can summarize our findings as follows.

Proposition 4 *Fix an equilibrium support $\mathcal{X}$. An increase in inflation leaves the distribution of real balances and seller profit unaffected, but raises the average DM price, and lowers buyer utility and social welfare.*

Figure 4 illustrates the effect of inflation on the average price y^*, buyer utility $\bar{\nu}$ and overall welfare $\mathcal{W}_{DM}$ for the connected support example introduced in Section 5.1. As explained above, an increase in inflation shifts the price distribution so as to put more weight on higher posted prices. Since the distribution of real balances remains unchanged, this exacerbates the mismatch between posted prices and real balances, leading to lower welfare. This detrimental effect of inflation is distinct from the standard inflation tax channel present in most monetary models, including monetary search models such as Lagos and Wright (2005), as well as their predecessors featuring either cash-in-advance constraints or money in the utility function. There, inflation leads to a fall in demand for real balances, which in turn lowers the quantity purchased in the decentralized market. This channel is shut down here since the goods purchased in the decentralized market are indivisible, and hence there is no intensive-margin quantity distortion. In fact, in our model inflation leaves the distribution of real balances unaffected, unlike the aforementioned models where it operates entirely through real balances. What changes instead is the distribution of posted prices, which in turn leads to a welfare loss because it distorts the extensive-margin probability of trade through mismatch.

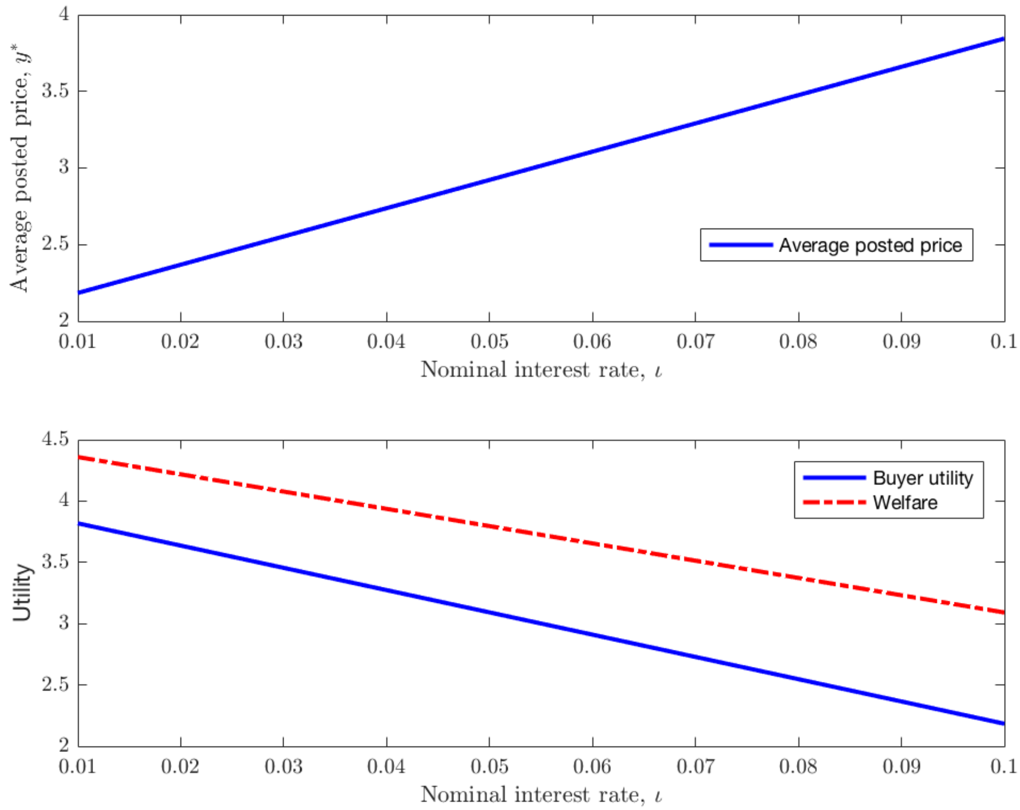

Figure 4: Average posted price y^*, buyer utility $\bar{\nu}$ and welfare $\mathcal{W}_{DM}$ as functions of ι, for $u = 10$, $c = 1$, $\sigma = 0.5$, and $\mathcal{X} = [2, 8.939]$.

7 Conclusion

The main lesson from our findings is that price dispersion can exist as an equilibrium outcome of monetary trade combined with search frictions, in the absence of simultaneous search or ex ante heterogeneity. This price dispersion is a purely self-confirming phenomenon: a non-degenerate distribution of money holdings rationalizes a non-degenerate distribution of prices, and vice versa. The fact that a distribution of money holdings rationalizes price dispersion is a consequence of buyers being constrained, at least sometimes, by their money holdings at the point of sale. Our analysis thus points to a new link between cash constraints and price dispersion and a new channel through which inflation affects prices and allocations.

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A Economy with Credit

This section considers an altered version of the model where some agents have access to credit. One might suppose that, if agents have access to credit so are not bound by their money holdings, the mechanism at the heart of the paper would fail. The main result of this section is that dispersed-price equilibria survive the introduction of credit as long as it is not ubiquitous. The most striking effect of credit is to fix the profit of sellers, so this is no longer indeterminate. Multiplicity still persists, but credit places tighter bounds on the set of sustainable prices in addition to affecting the shape of money and price distributions. As the analysis largely parallels the text, it is somewhat abbreviated.

Suppose that a proportion λ of buyer-seller meetings have access to costless credit. That is, in a proportion λ of buyer-seller meetings, instead of requiring cash as a medium of exchange, there exists some mechanism whereby the buyer can work and repay the seller in the next centralized market. One might imagine credit cards, although these are costly in reality. Alternately, one might imagine that some meetings are not anonymous in that the buyer and seller know one another and can commit to meet again in the next subperiod. In either event, we maintain the assumption of posted prices, so sellers must choose the same price in both cash and credit meetings.

Linear production in the centralized market implies that buyers can commit to repay any amount, so credit removes the budget constraint. Hence, in credit meetings, the buyer can and will pay any amount up to and including u. In cash-only meetings, individual rationality for money holdings, equation 9, still applies, so a seller posting a price $y = u$ only sells in credit meetings. Hence,

$$\pi(u) = \sigma\lambda(u - c),$$

and

$$\pi(y) = \sigma\left[(1-\lambda)\bar{G}(y) + \lambda\right](y - c). \tag{65}$$

Setting $\pi(\underline{x}) = \pi(u)$ fixes the bottom of the equilibrium support at

$$\underline{x} = \lambda u + (1-\lambda)c. \tag{66}$$

Writing y^* as the expected price, then buyer's value becomes

$$\nu(z) = -\iota z + \sigma\left[(1-\lambda)\int^{z}(u-x)dF(x) + \lambda(u - y^*)\right]. \tag{67}$$

If one considers gaps in the distribution (a, b), setting $\nu(a) = \nu(b)$ and $\pi(a) = \pi(b)$ gives an analog of Theorem 1:

$$\delta_F(b) = \frac{\iota(b-a)}{\sigma(1-\lambda)(u-b)} \quad \text{and} \quad \delta_G(a) = \frac{\left[(1-\lambda)(1-G(a)) + \lambda\right](b-a)}{(1-\lambda)(a-c)}. \tag{68}$$

Similarly, one can use Equations 65 and 67 to get an analog of Theorem 2 which characterizes distributions on the interior of the joint support:

$$G(x) = 1 - \frac{\lambda(u-y)}{(1-\lambda)(y-c)}, \tag{69}$$

and

$$f(x) = \frac{\iota}{\sigma(1-\lambda)(u-z)}. \tag{70}$$

Towards reconstructing our full characterization, let X be some closed set with $\min X = \underline{x} = \lambda u + (1-\lambda)c$, $\max X = \overline{x} < u$, and $X = [\underline{x}, \overline{x}] \setminus (\cup_{n=1}^{\infty}(a_n, b_n))$ for some collection of disjoint open intervals (a_n, b_n). We will seek to construct an equilibrium with money holdings in X and prices in $X \cup \{u\}$. The distribution of money holdings is similar to Equation 19:

$$G(x) = \begin{cases} 0 & \text{if } x \leq \underline{x}, \\ 1 - \frac{\lambda(u-x)}{(1-\lambda)(x-c)} & \text{if } \overline{x} > x \in X \setminus \left(\cup_{i=1}^{\infty}[a_n, b_n)\right), \\ 1 - \frac{\lambda(u-b_n)}{(1-\lambda)(b_n-c)} & \text{if } x \in [a_n, b_n) \text{ for } n \leq N, \\ 1 & \text{if } x \geq \overline{x}. \end{cases} \tag{71}$$

For the distribution of prices, there are two changes. First, the appearance of $1-\lambda$ terms according to Equations 68 and 70. Second, there is not full mass on X, so one must also calculate $\delta_F(u)$. For $x \in X$,

$$\begin{aligned} F(x) &= \delta_F(\underline{x}) + \sum_{n=1}^{\infty} \delta_F(b_n) \mathbf{1}_{\{b_n \leq x\}} + \int_{\underline{x}}^{x} f(y) \left[1 - \sum_{n=1}^{\infty} \mathbf{1}_{y \in I_n}\right] dy \\ &= \delta_F(\underline{x}) + \frac{\ln\left(\frac{u-\underline{x}}{u-x}\right) + \sum_{n=1}^{\infty}\left[\mathbf{1}_{\{b_n \leq x\}} \frac{b_n - a_n}{u-b_n} - \ln\left(\frac{u-\min\{a_n,x\}}{u-\min\{b_n,x\}}\right)\right]}{\sigma(1-\lambda)/\iota} \end{aligned} \tag{72}$$

Evaluating this at $\overline{x}$ we get

$$1 - \delta_F(u) = \delta_F(\underline{x}) + \frac{\iota}{\sigma(1-\lambda)}\left\{\ln\left(\frac{u-\underline{x}}{u-\overline{x}}\right) + \sum_{n=1}^{\infty}\left[\frac{b_n-a_n}{u-b_n} - \ln\left(\frac{u-a_n}{u-b_n}\right)\right]\right\}$$

Using this, and noting that

$$\int_a^b x f(x) dx = \frac{\iota}{\sigma(1-\lambda)} \int_a^b \frac{x}{u-x} dx = u \ln\left(\frac{u-a}{u-b}\right) - (b-a),$$

one can evaluate

$$y^* = \int_{\underline{x}}^{u} x dF(x) \tag{73}$$

$$= \delta_F(\underline{x})\underline{x} + \delta_F(u)u + \sum_{n=1}^{\infty} \left(\delta_F(b_n)b_n - \int_{a_n}^{b_n} xf(x)dx \right) + \int_{\underline{x}}^{\overline{x}} xf(x)dx \tag{74}$$

$$= \delta_F(\underline{x})\underline{x} + \delta_F(u)u + \frac{\iota}{\sigma(1-\lambda)} \left\{ \sum_{n=1}^{\infty} \left[\left(\frac{b_n - a_n}{u - b_n} \right) b_n - u \ln \left(\frac{u - a_n}{u - b_n} \right) + (b_n - a_n) \right] + u \ln \left(\frac{u - \underline{x}}{u - \overline{x}} \right) - (\overline{x} - \underline{x}) \right\}. \tag{75}$$

Individual rationality for buyers requires weakly higher value from carrying positive balances over nothing, $\nu(x) \geq \nu(0)$ for all $x \in \mathcal{X}$. Here, however, $\nu(0) = \sigma\lambda(u - y^*)$. Hence, some manipulation of 67 gives us a condition for there to exist an equilibrium which satisfies individual rationality:

$$\nu(\overline{x}) - \nu(0) = -\iota\overline{x} + \sigma(1-\lambda)(u - y^*) \geq 0.$$

Note, in the credit economy, the support set of money holding does not uniquely determine equilibrium as a range of values for $\delta_F(u)$ and $\delta_F(\underline{x})$ can satisfy adding up and individual rationality. For existence, however, one need only check at $\delta_F(u) = 0$. Solving $1 = F(\overline{x})$ for $\delta_F(\underline{x})$ with $\delta_F(u) = 0$ gives

$$\delta_F(\underline{x}) = 1 - \frac{\iota}{\sigma(1-\lambda)} \left\{ \ln \left(\frac{u - \underline{x}}{u - \overline{x}} \right) + \sum_{n=1}^{\infty} \left[\frac{b_n - a_n}{u - b_n} - \ln \left(\frac{u - a_n}{u - b_n} \right) \right] \right\}.$$

This makes

$$y^* = \underline{x} + \frac{\iota}{\sigma(1-\lambda)} \left\{ \sum_{n=1}^{\infty} \left[\left(\frac{b_n - a_n}{u - b_n} \right) (b_n - \underline{x}) - (u - \underline{x}) \ln \left(\frac{u - a_n}{u - b_n} \right) + (b_n - a_n) \right] + (u - \underline{x}) \ln \left(\frac{u - \underline{x}}{u - \overline{x}} \right) - (\overline{x} - \underline{x}) \right\}.$$

Finally, substituting this and y^* into $\nu(\overline{x}) - \nu(0)$ gives a condition analogous to (25):

$$0 \leq -\iota\overline{x} + \sigma(1-\lambda)(u - \underline{x}) - \iota \left\{ \sum_{n=1}^{\infty} \left[\left(\frac{b_n - a_n}{u - b_n} \right) (b_n - \underline{x}) - (u - \underline{x}) \ln \left(\frac{u - a_n}{u - b_n} \right) + (b_n - a_n) \right] + (u - \underline{x}) \ln \left(\frac{u - \underline{x}}{u - \overline{x}} \right) - (\overline{x} - \underline{x}) \right\}. \tag{76}$$

If one sends $\overline{x} \to \underline{x}$ in (76), one obtains a condition on $\underline{x}$ in an equilibrium with $\mathcal{X} = \{\underline{x}\}$

reminiscent of (8).

$$\underline{x} \leq u \frac{\sigma(1-\lambda)}{\iota + \sigma(1-\lambda)}.$$

Since we know $\underline{x} = \lambda u + (1-\lambda)c$, this can be expresed directly as a condition of underlying parameters, similar to (9).

$$c \leq u \frac{\sigma(1-\lambda)^2 - \iota\lambda}{(1-\lambda)\iota + \sigma(1-\lambda)^2}.$$

This is the condition for the existence of equilibria with prices other than u. Some manipulation shows that $c \geq 0$ implies a bound on λ which is necessary.

$$\frac{(1-\lambda)^2}{\lambda} \geq \frac{\iota}{\sigma}.$$

If one returns to (76), as was true in the case without credit, the right hand side is maximized in the case with connected support (without the gaps (a_n, b_n)). Assuming this and rearranging gives an upper bound for $\overline{x}$.

$$\overline{x} \leq u - (u - \underline{x}) \exp\left(\frac{\underline{x}}{u - \underline{x}} - \frac{\sigma(1-\lambda)}{\iota}\right).$$

Substituting the value of $\underline{x}$ gives an explicit bound similar to (27).

$$\overline{x} \leq u - (1-\lambda)(u-c) \exp\left(\frac{\lambda u + (1-\lambda)c}{(1-\lambda)(u-c)} - \frac{\sigma(1-\lambda)}{\iota}\right).$$

While we have focused on dispersed-price equilibria, there exist single price equilibria as well. First, there is always a pure credit equilibrium where no buyer brings cash because they expect all sellers to post u and vice versa. Second, there exist pure cash equilibria so long as the single price x satisfies (9) and $\pi(x) \geq \pi(u)$. Working this out, one derives a necessary condition for the existence of pure cash equilibria.

$$\lambda \leq \frac{u\frac{\sigma}{\iota+\sigma} - c}{u - c}.$$

We conclude with the remark that, while different in some interesting ways, limited credit does not alter the basic results we derive in the body of the paper.

B Omitted Proofs

Proof of Lemma 2. We show that, if $\bar{\pi} > 0$ and $\bar{\nu} > 0$, then $\mathcal{G}$ and $\mathcal{F}$ are mutually dense; because they are also closed, they must be equal. First, if $\bar{\pi} > 0$, then $\mathcal{G}$ is dense in $\mathcal{F}$. Suppose not: then there exist $x \in \mathcal{F}$ and $\eta > 0$ such that $(x-\eta, x+\eta) \cap \mathcal{G} = \emptyset$. But then, for $\eta > \epsilon > 0$, $\bar{G}(x + \eta - \epsilon) = \bar{G}(x)$, where we defined $\bar{G}(w) \equiv 1 - G(w) + \delta_G(w)$ for any w. Hence, either $G(x) = 0$, which is ruled out if $\bar{\pi} > 0$, or $\pi(x + \delta - \epsilon) > \pi(x)$, contradicting profit maximization. On the other hand, if $\bar{\pi} = 0$, then, since $\pi(x) = \bar{G}(x)(x-c)$, we must have either $x = c$ or $\bar{G}(x) = 0$ for all $x \in \mathcal{F}$.

Next, if $\bar{\nu} > 0$, then $\mathcal{F}$ is dense in $\mathcal{G}$. Suppose not: then there exist $x \in \mathcal{G}$ and $\eta > 0$ such that $(x - \eta, x + \eta) \cap \mathcal{F} = \emptyset$. Then $\int \mathbf{1}\{x - \eta < y \leq x\}\, dF(y) = 0$. Hence, $\nu(x - \eta) - \nu(x) = \iota\eta > 0$. But this contradicts utility maximization unless $x - \eta$ is infeasible; the latter implies $x = 0$, which would imply $\bar{\nu} = 0$. This also proves that $\mathcal{G} \setminus \mathcal{F} \subseteq \{0\}$ if $\bar{\nu} = 0$, so $\mathcal{G} \setminus \mathcal{F} = \{0\}$ if $\mathcal{G}$ is non-empty. ■

Proof of Theorem 1. As above, define $\bar{G}(w) \equiv 1 - G(w) + \delta_G(w)$ for any w. For $\epsilon > 0$, we can calculate

$$\begin{aligned} \pi(x) - \pi(x + \epsilon) &= (x - c)\, \bar{G}(x) - (x + \epsilon - c)\, \bar{G}(x + \epsilon) \\ &= (x - c)\left[\bar{G}(x) - \bar{G}(x + \epsilon)\right] - \epsilon\left[\bar{G}(x + \epsilon)\right] \end{aligned} \tag{77}$$

Taking the limit, we obtain

$$\lim_{\epsilon \downarrow 0} \pi(x) - \pi(x + \epsilon) = (x - c)\, \delta_G(x) > 0 \tag{78}$$

Hence, for $\epsilon > 0$ small, pricing at $x + \epsilon$ is dominated by pricing at x, so there must exist some $k > 0$ such that the interval $(x, x + k)$ does not intersect $\mathcal{X}$. Hence, $1 - G(x) = \bar{G}(x + k)$. So we can write

$$\pi(x) - \pi(x + k) = \{(x - c)\, \delta_G(x) - k[1 - G(x)]\} \tag{79}$$

Setting this equal to zero – as would be required for constant profit – we derive the formula for $k_F(x)$ in (10).

Similarly, if there is a mass point in F at z, we can calculate, for $\epsilon > 0$ small,

$$\begin{aligned} \nu(z) - \nu(z - \epsilon) = -\iota z + \sigma \int_{[0,z]} (u - x)\, dF(x) \\ - \left\{-\iota(z - \epsilon) + \sigma \int_{[0,z-\epsilon]} (u - x)\, dF(x)\right\} \\ = -\iota\epsilon + \sigma \int_{(z-\epsilon,z]} (u - x)\, dF(x) \end{aligned} \tag{80}$$

Taking the limit, we obtain

$$\lim_{\epsilon \downarrow 0} \nu(z) - \nu(z - \epsilon) = \sigma\delta_F(z)\,(u - z) > 0 \tag{81}$$

Hence, for ϵ small, carrying $z - \epsilon$ is dominated by carrying z, so there must exist some $k > 0$ such that the interval $(z - k, z)$ does not intersect $\mathcal{X}$. This implies that $F(z - k) = F(z) - \delta_F(z)$. Given this, we have

$$\nu(z) - \nu(z - k) = -\iota k + \sigma\delta_F(z)\,(u - z) \tag{82}$$

Setting this equal to zero and solving gives our formula for $k_G(z)$ in (11).

That there must be mass points on opposite ends whenever there is a gap follows from the same calculations in reverse. Suppose that $a, b \in \mathcal{X}$ but (a, b) does not intersect $\mathcal{X}$. This implies $G(a) = G(b) - \delta_G(b)$. Now, we know that

$$\pi(a) = (1 - G(a) + \delta_G(a))(a - c) \tag{83}$$

and

$$\begin{aligned}\pi(b) &= (1 - G(b) + \delta_G(b))(b - c) \\ &= (1 - G(a))(b - c)\end{aligned} \tag{84}$$

Setting $\pi(b) = \pi(a)$ then immediately yields the expression for $\delta_G(a)$ in (12). Similarly, the same calculations that yielded (80) imply $\nu(b) - \nu(a) = -\iota(b - a) + \sigma\delta_F(b)(u - b)$; setting this to zero gives the expression for $\delta_F(b)$ in (12). ■

Proof of Theorem 2. The derivation of expression (17) for G is in the text. We next derive the expression for F. For agents to be indifferent over money holdings in a region, it must be that $\nu(x) = \bar{\nu}$. Because F is continuous, one can apply integration by parts to the integral defining ν in (5), and rearranging to obtain

$$F(z) = \frac{1}{\sigma(u - z)}\left[\bar{\nu} + \iota z - \sigma \int_0^z F(x)\, dx.\right] \tag{85}$$

From this we can see that, by the fundamental theorem of calculus, on any interval where F is continuous, it must also be differentiable. Differentiating (85) gives (18). ■

Proof of Proposition 1. We want to show that $\overline{x} \leq \frac{\sigma}{\iota+\sigma}u$ implies (25). Without loss of generality, let us label $b_0 = \underline{x}$, and $a_{N+1} = \overline{x}$. Then, observe that

$$\begin{aligned}
\ln\left(\frac{u-\underline{x}}{u-\overline{x}}\right) - \sum_{n=1}^{N} \ln\left(\frac{u-a_n}{u-b_n}\right) &= \ln(u-b_0) - \ln(u-a_{N+1}) - \sum_{n=1}^{N}\left(\ln(u-a_n) - \ln(u-b_n)\right) \\
&= \sum_{n=1}^{N+1} \ln\left(\frac{u-b_{n-1}}{u-a_n}\right) \\
&= \sum_{n=1}^{N+1} \ln\left(\frac{a_n-b_{n-1}}{u-a_n} - 1\right) \\
&< \sum_{n=1}^{N+1} \frac{a_n-b_{n-1}}{u-a_n} \qquad (86)
\end{aligned}$$

Adding $\frac{\underline{x}}{u-\underline{x}} + \sum_{n=1}^{N} \frac{b_n-a_n}{u-b_n}$ to both sides, we get

$$\begin{aligned}
&\frac{\underline{x}}{u-\underline{x}} + \ln\left(\frac{u-\underline{x}}{u-\overline{x}}\right) + \sum_{n=1}^{N} \frac{b_n-a_n}{u-b_n} - \sum_{n=1}^{N} \ln\left(\frac{u-a_n}{u-b_n}\right) \\
<\; & \frac{\underline{x}}{u-\underline{x}} + \sum_{n=1}^{N} \frac{b_n-a_n}{u-b_n} + \sum_{n=1}^{N+1} \frac{a_n-b_{n-1}}{u-a_n} \\
<\; & \frac{\underline{x}}{u-\overline{x}} + \sum_{n=1}^{N} \frac{b_n-a_n}{u-\overline{x}} + \sum_{n=1}^{N+1} \frac{a_n-b_{n-1}}{u-\overline{x}} \\
=\; & \frac{\underline{x}}{u-\overline{x}} + \sum_{n=1}^{N} \frac{b_n-b_{n-1}}{u-\overline{x}} + \frac{\overline{x}-b_N}{u-\overline{x}} \qquad (87) \\
=\; & \frac{\underline{x}}{u-\overline{x}} + \frac{\overline{x}-\underline{x}}{u-\overline{x}} \\
=\; & \frac{\overline{x}}{u-\overline{x}} \\
\leq\; & \frac{\sigma}{\iota}
\end{aligned}$$

where the last line is assured by $\overline{x} \leq \frac{\sigma}{\iota+\sigma}u$. ■

Proof of Corollary 3. We consider the problem of maximizing x_N subject to (25) and subject to $x_1 \geq c$. First, observe that (25) is equivalent to

$$\frac{u}{u-x_1} + \sum_{i=1}^{N-1} \frac{u-x_i}{u-x_{i+1}} \leq N + \frac{\sigma}{\iota} \tag{88}$$

Let η be the Lagrange multiplier on the constraint (88) and let μ be the Lagrange multiplier on the constraint $x_1 \geq c$. Note that (88) clearly binds, so $\eta > 0$. For every $i = 2, ..., N-1$, the first-order necessary condition for x_i is

$$\frac{u-x_{i-1}}{(u-x_i)^2} = \frac{1}{u-x_{i+1}} \tag{89}$$

Rearranging, we get

$$\frac{u-x_{i-1}}{u-x_i} = \frac{u-x_i}{u-x_{i+1}} \tag{90}$$

for every $i = 2, ..., N-1$. Substituting back into the binding constraint (88), we get

$$\begin{aligned} \frac{u-x_1}{u-x_2} &= \frac{N + \sigma/\iota - u/(u-x_1)}{N-1} \\ &= \frac{N-1+\sigma/\iota - x_1/(u-x_1)}{N-1} \end{aligned} \tag{91}$$

But (90) also implies

$$\frac{u-x_1}{u-x_N} = \left(\frac{u-x_1}{u-x_2}\right)^{N-1} \tag{92}$$

Using (91), we then obtain

$$x_N = u - (u-x_1)\left(\frac{N-1}{N-1+\sigma/\iota - x_1/(u-x_1)}\right)^{N-1} \tag{93}$$

Next, the first-order condition for x_1 is

$$\frac{1}{u-x_2} - \frac{u}{(u-x_1)^2} = \mu \tag{94}$$

If $x_1 \geq c$ binds, then we have $x_1 = c$, and then (93) evaluated at $x_1 = c$ implies (53). If $x_1 \geq c$ does not bind, then $\mu = 0$ and hence (94) simplifies to

$$\frac{u}{u-x_1} = \frac{u-x_1}{u-x_2}, \tag{95}$$

which, combined with (90) and (92), gives us

$$\frac{u}{u-x_1} = \frac{N+\sigma/\iota}{N} \tag{96}$$

and

$$\frac{u - x_1}{u - x_N} = \left(\frac{u}{u - x_1} \right)^N \tag{97}$$

Combining the last two equations implies that x_1 satisfies $x_1 = u\sigma / (\iota N + \sigma)$ and x_N satisfies (52). This also means that $x_1 \geq c$ binds if and only if $c < u\sigma / (\iota N + \sigma)$. If $c > 0$, this means that for large enough N, x_N is determined by (53). Finally, if $c = 0$, both (52) and (53) converge to $u (1 - \exp \{-\sigma/\iota\})$. ■