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as Identifying Restrictions**

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# Robust Inference in First-Price Auctions: Experimental Findings as Identifying Restrictions

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## Abstract

In laboratory experiments bidding in first-price auctions is more aggressive than predicted by the risk-neutral Bayesian Nash Equilibrium (RNBNE) - a finding known as the overbidding puzzle. Several models have been proposed to explain the overbidding puzzle, but no canonical alternative to RNBNE has emerged, and RNBNE remains the basis of the structural auction literature. Instead of estimating a particular model of overbidding, we use the overbidding restriction itself for identification, which allows us to bound the valuation distribution, the seller's payoff function, and the optimal reserve price. These bounds are consistent with RNBNE and all models of overbidding and remain valid if different bidders employ different bidding strategies. We propose simple estimators and evaluate the validity of the bounds numerically and in experimental data.

JEL Codes: C57, D44, C14

Keywords: First-Price Auction, Robust Inference, Experimental Findings, Structural Estimation, Partial Identification

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# 1 Introduction

Identification and estimation of first-price auctions typically assumes that bidders play the risk neutral Bayesian Nash Equilibrium (RNBNE) (Guerre, Perrigne, and Vuong (2000)). However, a long series of laboratory experiments with independent private valuations have consistently found that bidders tend to bid more aggressively than predicted by the RNBNE - a finding known as the “overbidding puzzle”.

Several models have been proposed that can explain the overbidding puzzle. Thus one way to account for overbidding in structural estimation is to study the identification of these models. Indeed, under additional restrictions, identification results have been established for some overbidding models, including models with risk aversion (Perrigne and Vuong (2007), Lu and Perrigne (2008), Guerre, Perrigne, and Vuong (2009), Campo, Guerre, Perrigne, and Vuong (2011), Campo (2012), Gentry, Li, and Lu (2015), Kong (2017)), ambiguity aversion (Aryal, Grundl, Kim, and Zhu (2018)), and the level-k model (An (2017), Gillen (2009)).

This paper proposes a different approach. Instead of assuming a particular model of overbidding, we use the overbidding restriction itself for identification. As a result our approach is consistent with RNBNE and all models of overbidding. Such a robust approach is needed because no canonical alternative to RNBNE has emerged in the experimental literature, and it is unclear which overbidding model should be used in empirical work.

We assume that bidders (weakly) overbid compared to the RNBNE and bid no more than their valuations, which allows us to partially identify the valuation distribution. The bounds can be tightened by exploiting variation in the number of bidders under assumptions that are common in the structural auction literature. The bounds for the valuation distribution can be translated into bounds for the seller’s payoff as a function of the reserve price under an additional assumption on the bidders’ counterfactual bidding behavior.<sup>1</sup> This in turn allows us to bound the optimal reserve price.

The identifying restriction of overbidding compared to the RNBNE is motivated by findings of laboratory experiments. We also consider an alternative notion of overbidding that is motivated by its direct connection to the structural auction literature. This restriction is that bidders overbid compared to the risk-neutral best

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<sup>1</sup>This additional assumption does not help to tighten the bounds for the valuation distribution.

response to the bid distribution (RNBR).<sup>2</sup> We derive analogous identification results under RNBR overbidding for each of the identification results described above. An attractive feature of these bounds is that the estimates obtained under the standard approach (Guerre, Perrigne, and Vuong (2000)) can be reinterpreted as a sharp upper bound of the bidders' valuations and the seller's profit.

While we know from the experimental literature that overbidding compared to RNBNE is common, we do not know whether bidders also overbid compared to RNBR. To answer this question we analyze experimental data by Dyer, Kagel, and Levin (1989) and find that overbidding compared to RNBR is equally prevalent as overbidding compared to RNBNE.<sup>3</sup> Moreover, we show in Grundl and Zhu (2019) that RNBNE and RNBR overbidding go hand in hand in most models of overbidding.

Most of our identification results remain valid even if different bidders use different bidding strategies. Intuitively, our approach only requires every bidder to bid somewhere between RNBNE or RNBR and their valuations, but does not require all bidders to use the same bid function. Robustness to such unobservable bidder heterogeneity is important because laboratory experiments show not only that bidders tend to overbid, but also that there is a high level of heterogeneity in bids for a given valuation. This is an important advantage compared to point-identification results in the literature that typically do not allow for unobservable heterogeneity in bid functions.

We propose simple estimators for the bounds under both overbidding restrictions, and evaluate the performance of the bounds in numerical examples and in an empirical application to the data from Dyer, Kagel, and Levin (1989). We find that the bounds are valid and that the assumption of RNBR overbidding yields tighter bounds than RNBNE overbidding. Overall, we favor the RNBR restriction over the RNBNE restriction, because it is directly related to the standard approach (Guerre, Perrigne, and Vuong (2000)), tends to yield tighter bounds, and is found to be equally prevalent in the experimental data.

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<sup>2</sup>The standard approach to identification in first-price auctions (Guerre, Perrigne, and Vuong (2000)) relies on the fact that the bidders best respond to the bid distribution if they play the RNBNE. We show that if the bidders use the same bid function then overbidding compared to RNBR implies overbidding compared to RNBNE, but not vice versa. In general however, neither overbidding restriction implies the other.

<sup>3</sup>In the data, not all bidders use the same bid function. Therefore, the result discussed in footnote 2 does not apply and overbidding compared to RNBR is not a stronger restriction than overbidding compared to RNBNE.

This paper is motivated by a large experimental literature on the overbidding puzzle. The overbidding puzzle goes back at least to [Coppinger, Smith, and Titus \(1980\)](#), followed by a series of papers by Cox et al. ([Cox, Roberson, and Smith \(1982\)](#), [Cox, Smith, and Walker \(1983a,b, 1985, 1988\)](#)), who proposed a model of heterogeneous bidders with constant relative risk aversion (CRRA) as the explanation of the puzzle. [Harrison \(1989\)](#) and [Harrison \(1990\)](#) criticized the risk aversion explanation, which sparked a controversial discussion about the correct interpretation of the overbidding puzzle in the December 1992 issue of the *American Economic Review* ([Friedman \(1992\)](#), [Kagel and Roth \(1992\)](#), [Cox, Smith, and Walker \(1992\)](#), [Merlo and Schotter \(1992\)](#) and [Harrison \(1992\)](#)). Since then, other models of overbidding have been proposed and experimental studies have aimed not only at finding out whether there is overbidding, but also at distinguishing different potential explanations.<sup>4</sup> The literature has not settled on a particular model as the correct explanation of the overbidding puzzle, and likely different factors that are captured by different models all play a role.

Detecting overbidding in field data is more difficult than in the laboratory.<sup>5</sup> Nevertheless, there is some evidence that is consistent with overbidding in the field. [Burkart \(1995\)](#) and [de Bodt, Cousin, and Roll \(2018\)](#) find overbidding in takeover contests. There are several papers that find risk aversion in first-price auctions, which leads to overbidding ([Lu and Perrigne \(2008\)](#), [Campo, Guerre, Perrigne, and Vuong \(2011\)](#), [Campo \(2012\)](#), [Kong \(2017\)](#)). [Aryal, Grundl, Kim, and Zhu \(2018\)](#) find evidence of ambiguity aversion, which can also lead to overbidding. In light of the evidence from lab experiments and the suggestive evidence from the field, it is desirable to develop inference for first-price auctions that is robust to overbidding.

This paper is close in spirit to [Haile and Tamer \(2003\)](#), who show how to bound the valuation distribution in English auctions if the assumptions of the point-identified button model are replaced with two weak behavioral restrictions. The papers are

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<sup>4</sup>An incomplete list of references includes [Kagel and Levin \(1993\)](#), [Selten, Buchta, et al. \(1994\)](#), [Chen and Plott \(1998\)](#), [Isaac and James \(2000\)](#), [Dufwenberg and Gneezy \(2002\)](#), [Goeree, Holt, and Palfrey \(2002\)](#), [Dorsey and Razzolini \(2003\)](#), [Ockenfels and Selten \(2005\)](#), [Neugebauer and Selten \(2006\)](#), [Engelbrecht-Wiggans and Katok \(2007\)](#), [Neugebauer and Perote \(2008\)](#), [Engelbrecht-Wiggans and Katok \(2009\)](#), [Kirchkamp and Reiß \(2011\)](#), [Shachat and Wei \(2012\)](#), [Astor, Adam, Jähnig, and Seifert \(2013\)](#), [Georganas, Levin, and McGee \(2015\)](#), [Ratan \(2015\)](#) and [Füllbrunn, Janssen, and Weitzel \(2018\)](#). See also the surveys of [Kagel \(1995\)](#) and [Kagel and Levin \(2010\)](#), and the references therein.

<sup>5</sup>Valuations are not observed in the field and the RNBNE model is just identified, so some additional restrictions are required to detect overbidding.

similar in that they do not rely on a particular model of bidding and the estimates are therefore robust to model misspecification, but they differ regarding the source of identifying restrictions. [Haile and Tamer \(2003\)](#) relax the assumptions of the canonical model of English auctions in a very intuitive way and still obtain informative bounds. It is more difficult to follow this approach in first-price auctions. For example, [Aradillas-Lopez and Tamer \(2008\)](#) find that a similar approach in first-price auctions (level-k rationalizability) bounds the valuation distribution only from above. In this paper, we therefore rely on restrictions from experiments to bound the valuation distribution from both sides.

More broadly our paper proposes a new connection between experimental and structural work. Usually experiments are conducted to compare their findings to theoretical predictions. Thus experiments affect structural work indirectly through their impact on theoretical models. In this paper we point out that experimental work can also be helpful for structural work directly, because it can provide identifying restrictions. Interestingly, this might require different experiments or a different look at existing experimental data. For example, in our context, checking whether bids are above RNBR is relevant, whereas the experimental literature has focused on comparing bids to RNBNE.<sup>6</sup>

The remainder of this paper is organized as follows. Section 2 documents the extent of overbidding compared to RNBNE and RNBR in the experimental data by [Dyer, Kagel, and Levin \(1989\)](#). Section 3 presents the identification results under both overbidding restrictions. Section 4 illustrates the identification results through numerical examples. We discuss estimation in Section 5. Section 6 evaluates the approach using experimental data, and Section 7 concludes. Some proofs are presented in the Appendix.

## 2 Overbidding in the Data

Before presenting the identification results we briefly motivate both of our identifying restrictions using the experimental bid data from [Dyer, Kagel, and Levin \(1989\)](#),

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<sup>6</sup>This is not the first paper to connect experimental and structural work. [Bajari and Hortacsu \(2005\)](#) use experiments to evaluate whether structural auction estimates are reasonable and to compare the fit of competing models. [Hickman, List, Price, and Cotton \(2016\)](#) use a structural model to guide the design of a field experiment. To the best of our knowledge however, this is the first paper to use experimental findings as identifying restrictions.

which was also studied by [Bajari and Hortacsu \(2005\)](#). The subjects in their experiments were MBA students at the University of Houston. As in other experimental work on auctions, the bidders were assigned valuations randomly and then asked to bid given the valuation they drew. The valuations were drawn from a uniform distribution from \$0 to \$30. The payoff for the winning bidder is the difference between bid and valuation. The data set contains bids for  $n = 3$  and  $n = 6$  bidders for the same underlying valuation from the same bidder. After discarding bids from training rounds there are 414 bids for  $n = 3$  and 414 bids for  $n = 6$ .

**Overbidding Compared to RNBNE** Our first identifying restriction is that bidders overbid compared to RNBNE. The RNBNE bidding strategy is  $\sigma_n(t) = \frac{n-1}{n}t$ , where  $t$  is the valuation of a bidder for the auctioned good. Figure 1 shows scatter plots of the bids (blue circles), the RNBNE strategy (red line), and “truthful bidding”, i.e., bidding one’s valuation (black line). Figure 1(a) shows  $n = 3$  and Figure 1(b)  $n = 6$ . For  $n = 3$ , 91.3 percent of the bidders overbid compared to the RNBNE. For  $n = 6$ , 74.4 percent overbid. Overbidding tends to be more prevalent in auctions with fewer opponents. This is because in auctions with many opponents, the RNBNE prediction is already close to bidding one’s valuation, so there is less scope for overbidding while still bidding less than one’s valuation.

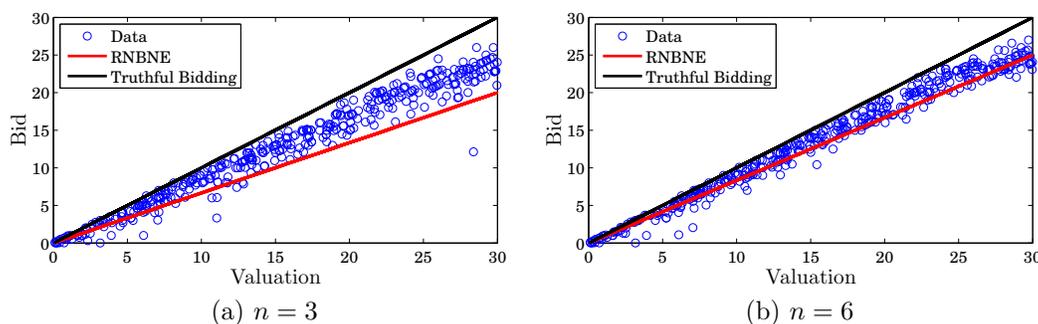


Figure 1: **Overbidding Compared to RNBNE:** Experimental bid data from [Dyer, Kagel, and Levin \(1989\)](#). Our first identifying restriction is that the bids (blue circles) are between the RNBNE (red line) and their valuations (black line).

The scatter plots also show that most bids below the RNBNE prediction come from bidders with low valuations, especially for  $n = 3$ . This pattern of overbidding is common in the experimental literature, as described by [Kagel and Levin \(2010\)](#).

In Section 6, we apply our identification approach to the data from [Dyer, Kagel, and Levin \(1989\)](#), which allows us to see whether such violations of the overbidding restriction by some bidders invalidate the bounds we propose.

Notice also that no bidders bid above their valuation. We will use this restriction to bound the valuations from the other side.

**Overbidding Compared to RNBR** Our second overbidding restriction is that bidders overbid compared to the RNBR. While overbidding compared to RNBNE has been established in a long series of experimental studies, much less is known about overbidding compared to RNBR. Experimental studies have naturally focused on comparing observed bidding to theoretical predictions such as RNBNE. Overbidding compared to RNBR, however, is relevant for this study because RNBR is the basis of [Guerre, Perrigne, and Vuong \(2000\)](#), which is the main approach to identification in first-price auctions.

Figure 2(a) shows the same bid data as Figure 1(a), but here the lower bound is given by RNBR rather than by RNBNE. In three bidder auctions, 91.8 percent of the bidders overbid compared to RNBR. In six bidder auctions, 71.8 percent overbid. Thus overbidding compared to RNBR is approximately just as prevalent as overbidding compared to RNBNE.

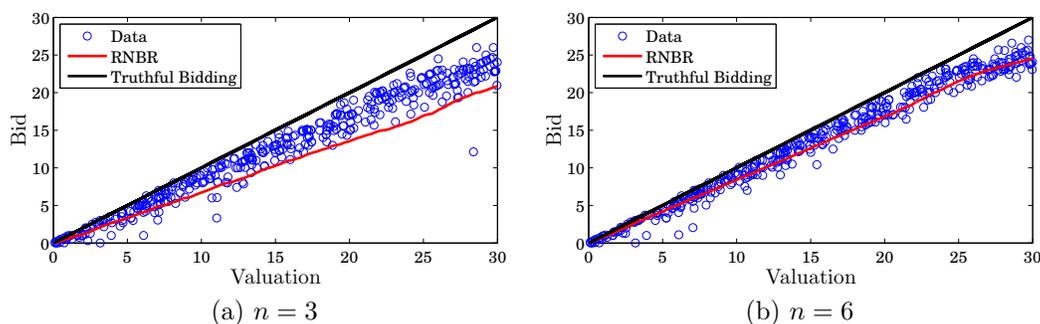


Figure 2: **Overbidding Compared to RNBR:** Experimental bid data from [Dyer, Kagel, and Levin \(1989\)](#). Our second identifying restriction is that the bids (blue circles) are between the RNBR (red line) and their valuations (black line).

In this data set, RNBNE and RNBR are very similar. If all bidders use the same bidding strategy and play RNBR, they must play RNBNE. Moreover, as we show later, if they overbid compared to RNBR, they must also overbid compared to

RNBNE.<sup>7</sup> However, these results do not apply here, because bidding behavior differs considerably across bidders. With heterogeneous bidders, generally neither form of overbidding implies the other.

### 3 Identification

We consider a symmetric independent private-value environment. There are  $n \geq 2$  bidders participating in a first-price auction for an indivisible good. Each of them simultaneously submits a sealed bid. The bidder with the highest bid wins the good and pays the bid. Bidders' valuations for the good are independently drawn from the distribution  $F_n$  with density  $f_n$ . The bid distribution is denoted by  $G_n$  with density  $g_n$ . Our goal is to identify the valuation distribution  $F_n$  from the bid distribution  $G_n$ . We assume that both  $f_n$  and  $g_n$  are supported on bounded intervals, and bounded away from 0 on their supports. Without loss of generality, we assume that the support of  $f_n$  has a lower bound of 0.

In the rest of the paper, instead of working directly with the valuation and bid distributions, we focus on their quantile functions. Define  $v_n(\alpha) = F_n^{-1}(\alpha)$  for  $\alpha \in [0, 1]$ . Because  $f_n$  has a support with a lower bound of 0,  $v_n(0) = 0$ . Let  $b_n(\alpha) = G_n^{-1}(\alpha)$  be the bid quantile function. Denote the bidding strategy of a bidder with valuation  $t$  as  $s_n(t)$ . We assume that all bidders adopt the same bidding strategy for now, but later we discuss how our results generalize to heterogeneous bidding strategies.

We will maintain the following regularity assumption on  $s_n$  and  $G_n$  throughout the paper.

**Assumption 1** (Regularity).

- (1) *The bidders' strategy  $s_n(t)$  is strictly increasing and continuously differentiable with  $s_n(0) = 0$ .*
- (2)  *$b + G_n(b) / [g_n(b)(n - 1)]$  is strictly increasing in  $b$  on the support of  $g_n$ .*

Assumption 1(1) implies that the bid function is smooth and invertible.<sup>8</sup> Assumption 1(2) ensures that the RBNR strategy is strictly increasing in the valuation and

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<sup>7</sup>Interestingly, if RNBNE and RBNR coincide this does not mean that the bidders bid RNBNE.

<sup>8</sup>This assumption is not crucial for our identification results but it simplifies the proofs. One can show that our bounds stay valid even if bidders use non-monotone strategies.

is fully characterized by its first-order condition.

### 3.1 RNBNE Overbidding

Structural analysis of first-price auctions typically assumes that the bidders play the symmetric RNBNE. Under this assumption a bidder with valuation  $t$  bids  $\sigma_n(t)$ , which satisfies the first-order condition,

$$\sigma'_n(t) = (n-1) [t - \sigma_n(t)] \frac{f_n(t)}{F_n(t)}, \quad (1)$$

and the boundary condition,  $\sigma_n(0) = 0$ . This differential equation has the following closed form solution:

$$\sigma_n(t) = t - \int_0^t \left[ \frac{F_n(x)}{F_n(t)} \right]^{n-1} dx.$$

**Definition 1** (RNBNE Overbidding). Bidders overbid compared to RNBNE if  $s_n(t) \geq \sigma_n(t)$  for all  $t$ .

**Definition 2** (*Rationality*). Bidders are rational if  $s_n(t) \leq t$  for all  $t$ .

Rationality is a weak restriction that is also used in [Haile and Tamer \(2003\)](#). This assumption holds in the experimental data by [Dyer, Kagel, and Levin \(1989\)](#) as shown in the previous section. Next, we derive the bounds for the valuation quantile function and the counterfactual profit function under RNBNE overbidding and rationality.

**Valuation Quantile Function** First notice that rationality directly implies  $v_n(\cdot) \geq b_n(\cdot)$ . Therefore,  $\underline{v}_n(\cdot) = b_n(\cdot)$  is a valid lower bound for  $v_n(\cdot)$ . It is also sharp because it can be attained by a model with constant relative risk aversion (CRRA) as the CRRA coefficient approaches 1.

Now we construct an upper bound for  $v_n(\cdot)$ . The idea is that if we can obtain the set of all quantile functions that are consistent with  $b_n$  and satisfy RNBNE overbidding and rationality, then the upper contour of this set is an upper bound for  $v_n(\cdot)$ . If a quantile function  $q_n(\cdot)$  satisfies rationality,  $q_n(\alpha) \geq b_n(\alpha)$  for all  $\alpha \in [0, 1]$ . To check RNBNE overbidding, the following lemma will be useful.

**Lemma 1.** *A quantile function  $q(\cdot)$  is consistent with  $b_n(\cdot)$  and RNBNE overbidding*

if and only if  $\beta_n(\alpha, q) \leq b_n(\alpha)$  for all  $\alpha \in [0, 1]$ , where

$$\beta_n(\alpha, q) = \begin{cases} \frac{1}{\alpha^{n-1}} \int_0^\alpha q(t) dt^{n-1} & \text{if } \alpha > 0; \\ q(0) & \text{if } \alpha = 0. \end{cases} \quad (2)$$

*Proof.* Notice that  $q(\cdot)$  is consistent with  $b_n(\cdot)$  and RNBNE overbidding if and only if  $s_n(q(\alpha)) = b_n(\alpha)$  for some bidding strategy  $s_n(\cdot)$  that is weakly larger than the RNBNE strategy under  $q$ . If  $\tilde{s}_n$  is the RNBNE bidding strategy under  $q$ ,  $\tilde{s}_n(q(\alpha)) \leq s_n(q(\alpha)) = b_n(\alpha)$ . Lastly, by [Gimenes and Guerre \(2016\)](#),  $\tilde{s}_n(q(\alpha)) = \beta_n(\alpha, q)$ .  $\square$

This lemma says that  $q(\cdot)$  can rationalize  $b_n(\cdot)$  under RNBNE overbidding if and only if the RNBNE bid quantile function under  $q(\cdot)$  lies below  $b_n(\cdot)$ .

In light of Lemma 1, an upper bound for  $v_n(\alpha)$  can be defined as

$$\bar{v}_n(\alpha) = \sup_{q \in \Theta} q(\alpha) \text{ s.t. } \beta_n(t, q) \leq b_n(t) \leq q(t), \forall t \in [0, 1], \quad (3)$$

where  $\Theta$  is the set of non-negative, non-decreasing and continuous functions that are supported on  $[0, 1]$  and differentiable almost everywhere. The inequalities impose RNBNE overbidding and rationality. The supremum is well-defined because at least  $q(\cdot) = b_n(\cdot)$  satisfies all the constraints. By construction,  $\bar{v}_n(\alpha)$  is a sharp upper bound. The following result characterizes the properties of this bound.

**Proposition 1.** *If the bidders are rational and overbid compared to RNBNE, then  $\underline{v}_n(\alpha)$  and  $\bar{v}_n(\alpha)$  are sharp bounds for  $v_n(\alpha)$ . In addition,  $\bar{v}_n(\alpha)$  satisfies*

- (1)  $\bar{v}_n(\alpha) < b_n(1) / (1 - \alpha)$  for any  $\alpha < 1$ ;
- (2)  $\exists C > 0$  such that  $\bar{v}_n(\alpha) > C / (1 - \alpha)$  for  $\alpha$  close to 1;
- (3)  $\bar{v}_n$  is strictly increasing and continuous.

The most surprising part of this proposition is perhaps that  $\bar{v}_n(\alpha)$  is bounded for any  $\alpha < 1$  but it diverges to infinity as  $\alpha \rightarrow 1$ . The easiest way to understand this is to start with a quantile function  $q$  and investigate what happens to its RNBNE bid quantile if we increase  $q(\alpha)$  for some  $\alpha < 1$ . First notice that this increase does not change the RNBNE bidding behavior of the bidders with valuations lower than  $q(\alpha)$ . Therefore, only the bid quantile function evaluated at points larger than  $\alpha$  can

provide information to bound  $q(\alpha)$  from above. Now if  $q(\alpha)$  is increased then the bidder with valuation  $q(\alpha)$  has an incentive to increase his bid to increase his winning probability. In the RNBNE, a bidder with a valuation higher than  $q(\alpha)$  never bids lower than a bidder with valuation  $q(\alpha)$ . Therefore the bidders with valuations higher than  $q(\alpha)$  have to increase their bids as well. These upwards shifts of the RNBNE bid function could drive the RNBNE bid quantile function above the observed bid quantile function. This bounds  $v_n(\alpha)$  from above if  $\alpha < 1$ . However, a bidder with the highest valuation  $v_n(1)$  does not have an incentive to bid higher even if his valuation increases, because he wins with probability 1 even with the current bid. Therefore,  $v_n(1)$  is not bounded from above.

Alternatively, one can recast (3) as a one-dimensional maximization problem, which provides some further intuition for  $\bar{v}_n$ . Notice that if we ignore the continuity requirement, the lowest possible valuation quantile that satisfies rationality and takes value  $x$  at a some  $\alpha \in [0, 1]$  is

$$q_{\alpha,x}(t) = \begin{cases} b_n(t) & \text{if } t < \alpha \\ \max\{x, b_n(t)\} & \text{if } t \geq \alpha \end{cases}.$$

If  $\beta_n(\cdot, q_{x,\alpha})$  does not lie below  $b_n(\cdot)$ , there cannot exist a  $q$  that satisfies rationality and RNBNE overbidding and  $q(\alpha) = x$ . Hence, if there were no continuity requirement on  $q$ ,  $\bar{v}_n(\alpha)$  can be obtained by finding the maximum  $x$  such that  $\beta_n(\cdot, q_{x,\alpha})$  does not exceed  $b_n(\cdot)$ . Next, notice that any function with a jump can be approximated by a continuous function. Therefore, we arrive at the following equivalent formulation.

**Proposition 2.** *The upper bound for the valuation quantile can be expressed as follows:*

$$\bar{v}_n(\alpha) = \bar{v}_n^a(\alpha) = \max_{x \geq b_n(\alpha)} x, \quad \text{s.t. } \beta_n(t, q_{\alpha,x}) \leq b_n(t), \quad \forall t \in [\alpha, 1]. \quad (4)$$

Notice that only  $t \in [\alpha, 1]$  enter the expression for  $\bar{v}_n^a(\alpha)$ , which confirms, as discussed earlier, that only points larger than  $\alpha$  can provide information to bound  $q(\alpha)$  from above.

**Profits** The identification result for the valuation distribution yields bounds that are consistent with many models of overbidding. This raises the question how to do counterfactual analysis without assuming a particular model of overbidding. In this

section we provide bounds on the seller’s profit as a function of the reserve price  $r$  by placing a weak restriction on the counterfactual bidding behavior with a binding reserve price, but without choosing a particular model of overbidding.

Let  $s_n(t, r)$  be the bidding function under reserve price  $r$  and let  $b_n(\alpha, r)$  be the bid quantile function. Suppose the seller has valuation  $c$  for the auctioned good. We first construct a lower bound for the profit function. Recall that we cannot rule out the possibility that bidders are bidding arbitrarily close to their valuations. In this case, bids cannot be increased by setting a reserve price, so  $b_n(\alpha, r) = b_n(\alpha)$  if  $\underline{v}_n(\alpha) \geq r$  and  $b_n(\alpha, r) = 0$  otherwise. The profit function obtained under this scenario therefore is a lower bound:

$$\underline{\pi}_n(r) = \int_{\alpha \geq \underline{v}_n^{-1}(r)} [b_n(\alpha) - c] d\alpha^n + c.$$

A naive upper bound for the profit function can be obtained by assuming that the valuation quantile is  $\bar{v}_n(\cdot)$  and that the bidders bid their valuations under any alternative  $r$ . However, this upper bound is not very informative. This is because  $\bar{v}_n$  is obtained under the assumption that bidders bid much lower than their valuations. Combining it with “truthful bidding” leads to a very conservative upper bound. In order to obtain a more informative upper bound, we impose the following restriction.

**Assumption 2.** *Under any reserve price  $r \geq 0$ , (1) if  $t < r$ , the bidding strategy  $s_n(t, r)$  satisfies  $s_n(t, r) = 0$ ; (2) if  $t \geq r$ ,  $s_n$  solves the following differential equation with initial condition  $s_n(r, r) = r$ ,*

$$\frac{\partial s_n(t, r)}{\partial t} = (n - 1) \lambda_n(t - s_n(t, r)) H_n(t) \frac{f_n(t)}{F_n(t)}$$

where  $\lambda_n(\cdot)$  is a positive, weakly increasing function and  $H_n(\cdot) > 0$ ; (3)  $s_n(t, r) \geq \sigma_n(t, r)$  where  $\sigma_n(t, r)$  is the RNBNE bidding strategy under  $r$ .

Assumption 2 is very general and satisfied by most first-price auction models. All models studied in [Grundl and Zhu \(2019\)](#) satisfy this assumption except for the level-k model. Imposing this assumption does not improve the bounds for the valuation quantile function but can greatly improve the bounds for the profit function. The functions  $\lambda_n$  and  $H_n$  allow for deviations from RNBNE that can lead to overbidding, where  $\lambda_n$  captures deviations of bidder preferences and  $H_n$  captures changes of bidder beliefs.

Assumption 2 and rationality imply that  $s_n(t, r) \geq s_n(t, 0)$  for all  $t \geq r$  and  $r \geq 0$ . Because  $\lambda_n$  is weakly increasing, the slope of the bidding strategy under  $r > 0$  is bounded by that under  $r = 0$ . Therefore, if  $v_n(\alpha) > r$ ,

$$\begin{aligned} b_n(\alpha, r) = s_n(v_n(\alpha), r) &= r + \int_r^{v_n(\alpha)} \frac{\partial s_n(t, r)}{\partial t} dt \leq r + \int_r^{v_n(\alpha)} \frac{\partial s_n(t, 0)}{\partial t} dt \\ &= r + b_n(\alpha) - b_n(v_n^{-1}(r)) \leq r + b_n(\alpha) - b_n(\bar{v}_n^{-1}(r)). \end{aligned} \quad (5)$$

In light of this observation, define

$$\bar{b}_n(\alpha, r) = \begin{cases} r + b_n(\alpha) - b_n(\bar{v}_n^{-1}(r)) & \text{if } \alpha \geq \bar{v}_n^{-1}(r) \\ 0 & \text{otherwise} \end{cases}. \quad (6)$$

Then, an upper bound for the profit function is

$$\bar{\pi}_n(r) = \int_{\alpha \geq \bar{v}_n^{-1}(r)} [\bar{b}_n(\alpha, r) - c] d\alpha^n + c. \quad (7)$$

**Proposition 3.** *Suppose that bidders are rational and overbid compared to RNBNE. If Assumption 2 holds, then  $\underline{\pi}_n(r) \leq \pi_n(r) \leq \bar{\pi}_n(r)$ . The lower bound is sharp.*

*Proof.* We start by showing that  $s_n(t, r) \geq s_n(t, 0)$  for all  $t \geq r$  if  $r \geq 0$  by contradiction. Suppose there exists  $t > r$  such that  $s_n(t, r) < s_n(t, 0)$ . Because the bidders never bid more than their valuations by rationality,  $s_n(r, r) = r \geq s_n(r, 0)$ . There must then exist an interval  $[x, x + \epsilon]$  such that  $s_n(x, r) = s_n(x, 0)$  and  $s_n(y, r) < s_n(y, 0)$  for all  $y \in (x, x + \epsilon]$ . Then, there exists at least one  $\tilde{y} \in [x, x + \epsilon]$  such that  $\partial s_n(\tilde{y}, r) / \partial t < \partial s_n(\tilde{y}, 0) / \partial t$ , because otherwise  $s_n(y, r) \geq s_n(y, 0)$  for all  $y \in (x, x + \epsilon]$ . By Assumption 2, this implies  $s_n(\tilde{y}, r) > s_n(\tilde{y}, 0)$  because  $\lambda_n(\cdot)$  is weakly increasing. Then we reach a contradiction. Therefore,  $s_n(v_n(\alpha), r) \geq b_n(\alpha) \mathbf{1}(b_n(\alpha) \geq r)$  and  $\underline{\pi}_n(r)$  is a valid bound because it is the profit function under  $b_n(\alpha) \mathbf{1}(b_n(\alpha) \geq r)$ . To see that it is a sharp lower bound, consider a model with where bidders have constant relative risk aversion (CRRA). If the CRRA coefficient approaches one, the bid function gets arbitrarily close to truthful bidding. Then, the lower bound is arbitrarily close to the true counterfactual profit function. In addition, because  $s_n(t, r) \geq s_n(t, 0)$  for all  $t \geq r$ ,  $\bar{\pi}_n(r) \geq \pi_n(r)$  follows from equation (5) and

the discussion around it.<sup>9</sup> □

## 3.2 RNBR Overbidding

The seminal contributions by [Laffont and Vuong \(1996\)](#) and [Guerre, Perrigne, and Vuong \(2000\)](#) are based on the observation that if bidders play the RNBNE then they also play the risk-neutral best response to the bid distribution (RNBR). This observation forms the building block of structural analysis of first-price auctions, because it means that the bidding strategy can be directly calculated from the observed bid distribution without solving for the equilibrium.

In light of the importance of RNBR for in the structural literature, we also obtain identification results if the bidders overbid compared to RNBR. Unlike overbidding compared to RNBNE this form of overbidding has not been considered in the experimental literature, because the experimental literature is focused on testing the RNBNE theory. Recall from Section 2 however that in the data by [Dyer, Kagel, and Levin \(1989\)](#) overbidding compared to RNBR is just as prevalent as overbidding compared to RNBNE.

Define RNBR as

$$\rho_n(t) = \arg \max_b (t - b) G_n(b)^{n-1}.$$

Under Assumption 1(2),  $\rho_n(t)$  is fully characterized by the first-order condition. Therefore, one can obtain  $\rho_n^{-1}(b) = b + G_n(b) / [(n - 1) g_n(b)]$ .<sup>10</sup> This expression is sometimes referred to as the ‘‘GPV inverse’’. If bidders indeed play RNBNE,

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<sup>9</sup>Although  $\bar{v}_n(\alpha)$  diverges to infinity when  $\alpha$  approaches 1,  $\bar{\pi}_n(r)$  is bounded uniformly in  $r$ . To see this, notice that by Proposition 1,  $\bar{v}_n(\alpha) \leq b_n(1) / (1 - \alpha)$ . Consequently,  $\bar{v}_n^{-1}(r) \geq \max\{1 - b_n(1)/r, 0\}$ . Therefore,

$$\begin{aligned} \bar{\pi}_n(r) &= \int_{\alpha \geq \bar{v}_n^{-1}(r)} [\bar{b}_n(\alpha) - c] d\alpha^n + c \\ &= \int_{\alpha \geq \bar{v}_n^{-1}(r)} [r + b_n(\alpha) - b_n(\bar{v}_n^{-1}(r)) - c] d\alpha^n + c \\ &\leq [r + b_n(1) - c] \left[ 1 - \left( \max \left\{ 1 - \frac{b_n(1)}{r}, 0 \right\} \right)^n \right] + c, \end{aligned}$$

where the last inequality follows because  $b_n(\alpha) - b_n(\bar{v}_n^{-1}(r)) \leq b_n(1) - b_n(0) = b_n(1)$ . This inequality implies that  $\bar{\pi}_n(r) < \infty$  for every  $r \geq 0$ . Because  $\bar{\pi}_n(r)$  is continuous in  $r$  and  $\lim_{r \rightarrow \infty} \bar{\pi}_n(r) \leq c + nb_n(1) < \infty$ , it is bounded uniformly in  $r$ .

<sup>10</sup>A sufficient condition for this is strict log-concavity of  $(t - b) G_n(b)^{n-1}$ , which is implied by the strict log-concavity of  $G_n(b)$ .

$$\rho_n^{-1}(b) = \sigma_n^{-1}(b).$$

**Definition 3** (RNBR Overbidding). Bidders overbid compared to RNBR if  $s_n(\cdot) \geq \rho_n(\cdot)$ .

The following proposition describes how RNBNE and RNBR overbidding are related.

**Proposition 4.** *Suppose Assumption 1 holds and all bidders adopt the same strategy  $s_n(\cdot)$ .*

(1) *If  $s_n(\cdot) = \rho_n(\cdot)$ , then  $s_n(\cdot) = \rho_n(\cdot) = \sigma_n(\cdot)$ .*

(2) *If  $s_n(\cdot) = \sigma_n(\cdot)$ , then  $s_n(\cdot) = \sigma_n(\cdot) = \rho_n(\cdot)$ .*

(3) *If  $s_n(\cdot) \geq \rho_n(\cdot)$ , then  $s_n(\cdot) \geq \sigma_n(\cdot)$ .*

Notice that  $\rho_n(\cdot) = \sigma_n(\cdot)$  does not imply  $s_n(\cdot) = \rho_n(\cdot) = \sigma_n(\cdot)$ .<sup>11</sup> The third part of Proposition 4 shows that overbidding compared to RNBR implies overbidding compared to RNBNE. Notice however, that  $s_n(\cdot) \geq \sigma_n(\cdot)$  does not imply that  $s_n(\cdot) \geq \rho_n(\cdot)$ .<sup>12</sup> Hence, the assumption of overbidding compared to RNBR is stronger than overbidding compared to RNBNE under symmetry.

**Valuation Quantile Function** As in the RNBNE case, the bid quantile remains a sharp lower bound under rationality and RNBR overbidding. This is because a model with very risk averse bidders is also consistent with RNBR overbidding.

For the upper bound, notice that if bidders are bidding the RNBR, the GPV inverse implies the valuation that corresponds to  $b_n(\alpha)$  for any  $\alpha \in [0, 1]$  is

$$\tilde{v}_n(\alpha) = b_n(\alpha) + \frac{1}{n-1} \frac{G_n(b_n(\alpha))}{g_n(b_n(\alpha))}.$$

Because bidders bid more than the RNBR strategy, the bidder who bids  $b_n(\alpha)$  must have a valuation of at most  $\tilde{v}_n(\alpha)$ . Therefore,  $\tilde{v}_n(\alpha)$  is a valid upper bound for  $v_n(\alpha)$ . In fact, it is also a sharp bound, because we cannot rule out RNBR bidding.

<sup>11</sup>For a counter-example let  $F$  be the uniform distribution on  $[0, 1]$  and  $n = 2$ . Now let  $s_n(t) = ct$  for some constant  $c \in (0.5, 1]$ ; then  $\sigma_n(t) = \frac{t}{2} = \rho_n(t) < s_n(t)$ .

<sup>12</sup>For a counter-example let  $F$  be the uniform distribution on  $[0, 1]$  and  $n = 2$ . Hence  $\sigma_n(t) = \frac{t}{2}$ . Now let  $s_n(t) = 0.7t - 0.2t^2 \geq \sigma_n(t)$ . It can be shown that  $\rho_n(0.75) > s_n(0.75)$ . Intuitively,  $s'_n$  becomes small for large  $t$  and therefore the best response is to reduce the bid shading because a slight increase in the bid increases the winning probability substantially.

**Proposition 5.** *If bidders are rational and overbid compared to RNBR, then  $\underline{v}_n(\alpha)$  and  $\tilde{v}_n(\alpha)$  are sharp bounds for  $v_n(\alpha)$ .*

One nice feature of RNBR overbidding is that applied researchers can continue to use the standard method in the literature that assumes RNBNE. If they want robustness to overbidding, they only need to reinterpret the estimate as the upper bound.

**Profits** The lower bound of the profit function comes from truthful bidding. This leads to a sharp lower bound because we cannot rule out that bidders have arbitrarily high risk aversion. As in the last section, additional restrictions are needed to obtain an informative upper bound. One natural approach is to impose Assumption 2. Then an upper bound for the profit function can be constructed by using (6) and (7) after replacing  $\bar{v}_n(\cdot)$  by  $\tilde{v}_n(\cdot)$ . However, it turns out that models consistent with overbidding compared to RNBR usually imply additional restrictions, which lead to tighter bounds. In this section, we focus on the following strengthened version of Assumption 2.

**Assumption 3.** *Under a reserve price  $r \geq 0$ , (1) if  $t < r$ , the bidding strategy  $s_n(t, r)$  satisfies  $s_n(t, r) = 0$ ; (2) if  $t \geq r$ ,  $s_n$  solves the following differential equation with initial condition  $s_n(r, r) = r$ ,*

$$\frac{\partial s_n(t, r)}{\partial t} = (n-1) \lambda_n(t - s_n(t, r)) H_n(t) \frac{f_n(t)}{F_n(t)}$$

where  $H_n \geq 1$ ,  $\lambda'_n(x) \geq 1$  and  $\lambda_n(0) = 0$ .

Assumption 3 is the same as Assumption 2 except that it requires  $H_n$  and  $\lambda'_n$  to be at least 1. Lemma 1 in Grundl and Zhu (2019) shows that Assumption 3 is a sufficient condition for RNBR overbidding. Grundl and Zhu (2019) also shows that the risk aversion model, the loser regret model, loss aversion, and Choquet Expected Utility model all satisfy this assumption.<sup>13</sup> This assumption implies that for any  $\alpha$  such that  $v_n(\alpha) > r$ ,

$$\frac{\partial}{\partial \alpha} b_n(\alpha, r) = (n-1) \frac{\lambda_n(v_n(\alpha) - b_n(\alpha, r))}{\alpha} \mathcal{H}_n(\alpha), \quad (8)$$

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<sup>13</sup>The MaxMin Expected Utility generally does not necessarily predict overbidding, but under the restriction we consider such that it leads to overbidding it satisfies this assumption as well.

where  $\mathcal{H}_n(\alpha) = H_n(v_n(\alpha)) > 1$ .

Let  $\tilde{\sigma}_n(\alpha, r)$  be the RNBNE bid quantile function under  $\tilde{v}_n(\cdot)$  and  $r$  in  $n$ -bidder auctions. Define

$$\tilde{\pi}_n(r) = \int_{\tilde{v}_n^{-1}(r)}^1 [\tilde{\sigma}_n(\alpha, r) - c] d\alpha^n + c.$$

**Proposition 6.** *Suppose bidders are rational and overbid compared to RNBR. If Assumption 3 holds, then  $\underline{\pi}_n(r) \leq \pi_n(r) \leq \tilde{\pi}_n(r)$ . These bounds are sharp.*

*Proof.* We only show that  $\tilde{\pi}_n(r)$  is a sharp upper bound. Applying Lemma 3 in the appendix with  $\gamma(\alpha) = 1/\alpha$ , we obtain  $\tilde{\sigma}_n(\alpha, r) \geq b_n(\alpha, r)$ . Because  $v_n(\cdot) \leq \tilde{v}_n(\cdot)$

$$\pi_n(r) = \int_{v_n^{-1}(r)}^1 [b_n(\alpha, r) - c] d\alpha^n + c \leq \int_{\tilde{v}_n^{-1}(r)}^1 [\tilde{\sigma}_n(\alpha, r) - c] d\alpha^n + c = \tilde{\pi}_n(r)$$

To see that this bound is sharp, notice that the observed bid quantile function  $b_n(\cdot, 0)$  can be rationalized by  $\tilde{v}_n(\cdot)$  with RNBNE, in which case the upper bound is attained.  $\square$

Now we have shown that if bidders overbid compared to RNBR, the standard approach in the structural auction literature yields not only a sharp upper bound for the valuation quantile function but also a sharp upper bound for the profit function. Intuitively, there are many pairs of a valuation quantile function and a bidding model which are consistent with  $b_n(\cdot)$  under  $r = 0$ . For  $r = 0$  they all yield the same profit because they all generate the observed bid distribution. As  $r$  increases, the RNBNE bidding strategy increases most among all strategies that satisfy Assumption 3. Intuitively, the RNBNE bidding strategy has the largest bid shading and therefore leaves the most room for more aggressive bidding. As a result, the profit function increases most under the RNBNE bidding strategy, implying that the RNBNE profit function is an upper bound.

This result shows that [Guerre, Perrigne, and Vuong \(2000\)](#) is a robust approach for choosing the reserve price in the sense that the resulting reserve price can be interpreted as a max-max reserve price, i.e. a reserve price maximizing the upper profit bound.

### 3.3 Tightening Bounds with Variation in $n$

In many applications, auctions with different numbers of bidders are observed. This variation in  $n$  can be used to tighten the bounds. To exploit this variation, we must restrict how the valuation distribution varies with  $n$ . To this end, we consider the following two assumptions:

**Assumption 4.**  $v_n(\cdot) = v(\cdot)$  for all  $n \in \mathbb{N}$ , where  $\mathbb{N}$  is a finite set of integers.

**Assumption 5.**  $v_n(\alpha)$  is weakly increasing in  $n$  for all  $\alpha \in [0, 1]$ .

Assumption 4 requires that the valuation distribution is independent of  $n$ . This assumption is sometimes called exogenous participation (EP) in the literature. It holds trivially if bidders are randomly assigned to auctions. However, even if the bidders decide which auctions to enter, it holds in many cases.<sup>14</sup> EP is used in many papers to either improve inference and/or to achieve identification.<sup>15</sup> Testing EP empirically in field data is usually difficult due to the presence of unobserved auction level heterogeneity.

Assumption 5 is usually referred to as stochastically increasing valuations (SIV). It allows for some positive selection and thereby relaxes Assumption 4. This assumption is also considered in [Aradillas-López, Gandhi, and Quint \(2011\)](#) and [Grundl and Zhu \(2016\)](#). Notice that we do not restrict how bidding behavior changes with  $n$ . This is important for robustness because different overbidding models have different predictions on how bidding changes with  $n$ .

**RNBNE Overbidding** First, we focus on RNBNE overbidding. If  $v_n$  satisfies Assumption 4, the lower bound can be tightened to  $\underline{v}^{EP}(\alpha) = \max_{n \in \mathbb{N}} \{b_n(\alpha)\}$ . The upper bound can be tightened by incorporating additional restrictions introduced by different  $n$  into (3), i.e.,

$$\bar{v}^{EP}(\alpha) = \sup_{q \in \Theta} q(\alpha), \text{ s.t. } \beta_m(t, q) \leq b_m(t) \leq q(t) \quad \forall m \in \mathbb{N}, \quad \forall t \in [0, 1]. \quad (9)$$

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<sup>14</sup> [Grundl and Zhu \(2016\)](#) show that this assumption holds very generally as long as potential bidders observe independent signals before making their entry decisions. If there is auction heterogeneity, we need to condition on all auction level heterogeneity (observed and unobserved) that is observed by the bidders, including the number of potential bidders.

<sup>15</sup> An incomplete list includes [Haile and Tamer \(2003\)](#), [Haile, Hong, and Shum \(2003\)](#), [Guerre, Perrigne, and Vuong \(2009\)](#), [Aradillas-López, Gandhi, and Quint \(2011\)](#) and [Aryal, Grundl, Kim, and Zhu \(2018\)](#).

The only difference from (3) is that now the inequality constraints are required for all  $m \in \mathbb{N}$ , i.e., rationality and RNBNE overbidding have to hold for all  $m \in \mathbb{N}$ . These additional constraints help to improve the upper bound.

If, instead, Assumption 5 holds, the lower bound is  $\underline{v}_n^{SIV}(\alpha) = \max_{m \in \mathbb{N}: m \leq n} \{b_m(\alpha)\}$ . Notice now we only include  $m$  smaller than  $n$ . The upper bound for  $v_n(\alpha)$  is

$$\begin{aligned} \bar{v}_n^{SIV}(\alpha) &= \sup_{q \in \Theta} q(\alpha) \\ \text{s.t. } \beta_m(t, q) &\leq b_m(t) \quad \forall m \in \mathbb{N}, m \geq n, \forall x \in [0, 1] \\ b_m(t) &\leq q(t) \quad \forall m \in \mathbb{N}, m \leq n, \forall x \in [0, 1]. \end{aligned} \quad (10)$$

The constraints are slightly different compared to (9). The first constraint says that the RNBNE bid quantile under  $q$  is below the observed bid quantile for all auctions with no less than  $n$  bidders. This is because in auctions with at least  $n$  bidders, the valuation quantile functions are at least the same as  $v_n$ . Because the RNBNE bid quantile function is weakly increasing in the valuation quantile function, the observed bid quantile functions in auctions with at least  $n$  bidders should lie above the RNBNE bid quantile functions under  $v_n$ . On the other hand,  $b_m(\cdot) \leq v_n(\cdot)$  only if  $m \leq n$ , because Assumption 5 does not rule out the possibility that  $b_m(\alpha) > v_n(\alpha)$  if  $m$  is larger than  $n$ . Therefore, the second constraint is required only for  $m \leq n$ .

**Proposition 7.** *Suppose bidders are rational and overbid compared to RNBNE.*

(1) *If Assumption 4 holds, the sharp bounds for  $v_n(\alpha)$  are  $\underline{v}^{EP}(\alpha)$  and  $\bar{v}^{EP}(\alpha)$ .*

(2) *If Assumption 5 holds, the sharp bounds for  $v_n(\alpha)$  are  $\underline{v}_n^{SIV}(\alpha)$  and  $\bar{v}_n^{SIV}(\alpha)$ .*

*Proof.* We only need to show that the bounds are sharp. The upper bounds are sharp by construction because they are obtained by taking the supremum of all valid  $q$ . The lower bounds are sharp because our identifying assumption cannot rule out the possibility that bidders are truthful bidding at least for some  $n$ .  $\square$

Variation in  $n$  also helps to tighten the profit bounds. Without variation in  $n$ , the lower profit bound is constructed under the truthful bidding scenario, which cannot be ruled out with a single  $n$ . But under Assumption 4, if the bid quantile function changes with  $n$ , bidders are not bidding their valuation at least for some  $n$ . This information helps to tighten the lower bound. To construct the lower bound,

define  $\underline{\sigma}_n^{EP}(\alpha, r)$  and  $\underline{\sigma}_n^{SIV}(\alpha, r)$  to be the RNBNE bidding function under reserve price  $r$  with valuation quantile function  $\underline{v}^{EP}(\alpha)$  and  $\underline{v}_n^{SIV}(\alpha)$ , respectively.<sup>16</sup> Because bidders overbid compared to RNBNE, the counterfactual bid quantile function is at least  $\max\{\underline{\sigma}_n^{EP}(\alpha, r), b_n(\alpha) \mathbf{1}(\underline{v}^{EP}(\alpha) \geq r)\}$  under Assumption 4, and at least  $\max\{\underline{\sigma}_n^{SIV}(\alpha, r), b_n(\alpha) \mathbf{1}(\underline{v}_n^{SIV}(\alpha) \geq r)\}$  under Assumption 5. Then we can obtain lower bounds for the profit function,

$$\begin{aligned}\underline{\pi}_n^{EP}(r) &= \int_{\underline{v}^{EP}(\alpha) \geq r} [\max\{\underline{\sigma}_n^{EP}(\alpha, r), b_n(\alpha)\} - c] d\alpha^n + c \\ \underline{\pi}_n^{SIV}(r) &= \int_{\underline{v}_n^{SIV}(\alpha) \geq r} [\max\{\underline{\sigma}_n^{SIV}(\alpha, r), b_n(\alpha)\} - c] d\alpha^n + c.\end{aligned}$$

For the upper bound, because we do not impose restrictions on how bidding strategies change with  $n$ , bids from auctions with sizes different from  $n$  only provide information on the valuation quantile function. Under Assumption 4 and rationality, the counterfactual bid quantile function is bounded from above by  $\bar{v}^{EP}(\alpha)$  and

$$\bar{b}_n^{EP}(\alpha, r) = \begin{cases} r + b_n(\alpha) - b_n\left(\left(\bar{v}_n^{EP}\right)^{-1}(r)\right) & \text{if } \bar{v}_n^{EP}(\alpha) \geq r \\ 0 & \text{otherwise} \end{cases}. \quad (11)$$

Similarly, under Assumption 5, the bid quantile function is bounded from above by  $\bar{v}_n^{SIV}(\alpha)$  and

$$\bar{b}_n^{SIV}(\alpha, r) = \begin{cases} r + b_n(\alpha) - b_n\left(\left(\bar{v}_n^{SIV}\right)^{-1}(r)\right) & \text{if } \bar{v}_n^{SIV}(\alpha) \geq r \\ 0 & \text{otherwise} \end{cases}. \quad (12)$$

Consequently, we have the following two upper bounds for the profit functions:

$$\begin{aligned}\bar{\pi}_n^{EP}(r) &= \int_{\bar{v}^{EP}(\alpha) \geq r} [\min\{\bar{b}_n^{EP}(\alpha, r), \bar{v}^{EP}(\alpha)\} - c] d\alpha^n + c \\ \bar{\pi}_n^{SIV}(r) &= \int_{\bar{v}_n^{SIV}(\alpha) \geq r} [\min\{\bar{b}_n^{SIV}(\alpha, r), \bar{v}_n^{SIV}(\alpha)\} - c] d\alpha^n + c.\end{aligned}$$

**Proposition 8.** *Suppose bidders are rational and overbid compared to RNBNE. In addition, Assumption 2 holds.*

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<sup>16</sup>If  $\underline{v}^{EP}(\alpha) < r$ , then  $\underline{\sigma}_n^{EP}(\alpha, r) = 0$ . The same holds for  $\underline{\sigma}_n^{SIV}(\alpha, r)$ .

(1) If Assumption 4 holds, then  $\underline{\pi}_n^{EP}(r) \leq \pi_n(r) \leq \bar{\pi}_n^{EP}(r)$ .

(2) If Assumption 5 holds, then  $\underline{\pi}_n^{SIV}(r) \leq \pi_n(r) \leq \bar{\pi}_n^{SIV}(r)$ .

*Proof.* We only show that the lower bounds are valid. Compared to the case without using variation in the number of bidders, we also use  $\underline{\sigma}_n^{EP}(\alpha, r)$  and  $\underline{\sigma}_n^{SIV}(\alpha, r)$  to bound  $b_n(\alpha, r)$  from below. By Lemma 2,  $\underline{\sigma}_n^{EP}(\alpha, r)$  and  $\underline{\sigma}_n^{SIV}(\alpha, r)$  are smaller than the RNBNE bid quantile under the true valuation quantile, which is a valid lower bound for  $b_n(\alpha, r)$ . As a result,  $\underline{\sigma}_n^{EP}(\alpha, r)$  and  $\underline{\sigma}_n^{SIV}(\alpha, r)$  are also valid lower bounds for  $b_n(\alpha, r)$  under corresponding assumptions.  $\square$

**RNBR Overbidding** Now we move to RNBR overbidding. It is straightforward to exploit variation in  $n$  to tighten the upper bounds of the valuation quantile function. We only need to obtain upper and lower bounds for each  $n$  and take the minimum of the upper bounds and the maximum of the lower bounds over the relevant set of auction sizes. Define  $\tilde{v}^{EP}(\alpha) = \min_{m \in \mathbb{N}} \tilde{v}_m(\alpha)$  and  $\tilde{v}_n^{SIV}(\alpha) = \min_{m \geq n} \tilde{v}_m(\alpha)$ . Notice that  $\tilde{v}^{EP}$  is obtained by first applying [Guerre, Perrigne, and Vuong \(2000\)](#) to auctions with different numbers of bidders and then by forming their lower contour. The lower bound remains the same as under RNBNE overbidding. In the interest of space, we omit the proofs in this section.

**Proposition 9.** *Suppose bidders are rational and overbid compared to RNBR.*

(1) If Assumption 4 holds, then  $\underline{v}^{EP}(\alpha) \leq v_n(\alpha) \leq \tilde{v}^{EP}(\alpha)$ .

(2) If Assumption 5 holds, then  $\underline{v}_n^{SIV}(\alpha) \leq v_n(\alpha) \leq \tilde{v}_n^{SIV}(\alpha)$ .

For profits,  $\underline{\pi}_n^{EP}(r)$  and  $\underline{\pi}_n^{SIV}(r)$  remain valid lower bounds. The upper bound can be obtained by a slight change to the above results. Define

$$\begin{aligned}\tilde{\pi}_n^{EP}(r) &= \int_{\tilde{v}^{EP}(\alpha) \geq r}^1 [\min \{ \tilde{v}^{EP}(\alpha), \tilde{\sigma}_n(\alpha, r) \} - c] d\alpha^n + c \\ \tilde{\pi}_n^{SIV}(r) &= \int_{\tilde{v}_n^{SIV}(\alpha) \geq r}^1 [\min \{ \tilde{v}_n^{SIV}(\alpha), \tilde{\sigma}_n(\alpha, r) \} - c] d\alpha^n + c.\end{aligned}$$

Notice here we simply plug in the tighter bounds for valuation quantiles obtained by using variation in  $n$ .

**Proposition 10.** *Suppose bidders are rational and overbid compared to RNBR. In addition, Assumption 2 holds.*

(1) *If Assumption 4 holds, then  $\underline{\pi}_n^{EP}(r) \leq \pi_n(r) \leq \tilde{\pi}_n^{EP}(r)$ .*

(2) *If Assumption 5 holds, then  $\underline{\pi}_n^{SIV}(r) \leq \pi_n(r) \leq \tilde{\pi}_n^{SIV}(r)$ .*

### 3.4 Heterogeneous Bidding Behavior

So far we have assumed that all bidders use the same bidding strategy  $s_n$ . However, the scatter plots in Section 2 show that there is a lot of variation in bids conditional on the bidder's valuation. In this section, we show that most of the identification results in the previous sections remain valid with heterogeneous bidding strategies, i.e., if bidder  $i$  bids  $s_{n,i}$ . This is an important extension because most existing point-identification results for particular models do not allow for such heterogeneity.<sup>17</sup> In this section, we will consider identification if every  $s_{n,i}$  satisfies overbidding:

**Definition 4** (Overbidding with Heterogeneity).

(1) Bidders overbid compared to RNBNE if  $s_{n,i}(\cdot) \geq \sigma_n(\cdot)$ .

(2) Bidders overbid compared to RNBR if  $s_{n,i}(\cdot) \geq \rho_n(\cdot)$ .

Notice that  $\sigma_n$  and  $\rho_n$  do not have bidder subscripts  $i$ , because  $\sigma_n$  is defined as the symmetric BNE strategy of a risk-neutral bidder and  $\rho_n$  is defined as the best response to the bid distribution. Moreover,  $\sigma_n$  is fully determined by the valuation distribution and  $n$  and does not depend on the composition of bidder types  $i$ . However,  $\rho_n$  depends on the composition of bidder types  $i$  because the composition affects the bid distribution.

With heterogeneous bidding strategies, Proposition 4 no longer holds. In particular, it is no longer the case that overbidding compared to RNBR is necessarily a stronger restriction than overbidding compared to RNBNE. A counterexample is given in Appendix B.

**Proposition 11.** *If bidders use heterogeneous bidding functions  $s_{n,i}$  then*

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<sup>17</sup>One exception is An (2017), who studies identification with heterogeneity in bidding strategies with multiple observations per bidder.

- (1) Propositions 1, 5, 7, and 9 remain valid.
- (2) The lower bounds on the profit function in Propositions 3 and 6 remain valid.
- (3) If, in addition,  $s_{n,i}(\cdot, r) \geq \sigma_n(\cdot, r)$  for all  $r$ , the lower bounds on profit in Proposition 8 and 10 remain valid.

*Proof.* The first claim holds because all these bounds exploit only the fact that bidders bid between their valuations and  $\sigma_n(\cdot)$  or  $\rho_n(\cdot)$ . The second claim follows because the lower bounds in Propositions 3 and 6 are obtained from truthful bidding, which remain valid under rationality. Lastly, notice that the lower bounds in Proposition 8 and 10 are obtained from truthful bidding and RNBNE bidding under the lower bound of the valuation quantile function. Truthful bidding leads to a valid bound under rationality. RNBNE bidding under the lower bound of the valuation quantile leads to a valid lower bound under the additional assumption that  $s_{n,i}(\cdot, r) \geq \sigma_n(\cdot, r)$  for all  $r$ . Therefore, the third claim follows.  $\square$

Unlike in previous sections, Proposition 11 does assume that the bidding strategies have some general form. Only in the third claim, we require that bidders bid at least the RNBNE bidding strategy under any reserve price  $r$ . This assumption is needed to exploit variation in  $n$  to tighten lower bound on profit. We were not able to establish general results for the upper bound of the profit function, but we conjecture that our upper bounds also remain valid if Assumption 2 or 3 holds with  $\lambda_n$  and  $H_n$  varying across bidders. Numeric examples shown in the next section are consistent with this conjecture.

## 4 Illustrating the Identification Results

### 4.1 Computation

Solving (3) is a difficult problem because it requires taking the supremum over a set of functions. We now provide a computationally easy approach to approximate the solution. The idea is to approximate quantile functions with B-splines. Let  $P_K = (p_1, p_2, \dots, p_K)$  be a vector of basis functions supported on  $[0, 1]$ . Then  $q(\alpha) \approx P_K(\alpha)\theta$ , where  $\theta$  is a  $K$ -dimensional column vector. The constraints can be imposed

on a set of points  $\{t_j\}_{j=1}^J \subset [0, 1]$ . Then, an approximate upper bound can be obtained by solving

$$\max_{\theta} P_K(\alpha)\theta, \text{ s.t. } P_K(t_j)\theta \geq b_n(t_j) \geq A_K(t_j)\theta, P_K(0)\theta \geq 0, P'_K(t_j)\theta \geq 0 \forall j, \quad (13)$$

where  $A_K(\alpha) = \frac{1}{\alpha^{n-1}} \int_0^\alpha P_K(t) dt^{n-1}$  if  $\alpha > 0$  and  $A_K(0) = 0$ . This is a linear programming problem that can be solved very quickly even if  $K$  and  $J$  are both very large.

The numerical and empirical results we show in the remainder of this paper are based on (13). Alternatively one can solve (4), which is a one-dimensional nonlinear optimization problem that can be solved even faster than (13). One advantage of using (13) is that it is straightforward to impose additional shape restrictions on the valuation quantile function.

## 4.2 Valuation Quantile Function

In this section we illustrate the identification results graphically. We begin with the six graphs in Figure 3, which illustrate the identification results for the valuation distribution with  $n = 2$ . For the three graphs in the upper half, bidders play the RNBNE so there is no overbidding. For the three graphs in the lower half, bidders have CRRA and therefore overbid. The coefficient of relative risk aversion is 0.8. The valuations are drawn from a mixture of a Beta distribution with parameters  $\alpha = 2$  and  $\beta = 7$  and a uniform distribution on  $[0, 1]$ . The mixture weights are 97.5 percent on the Beta distribution and 2.5 percent on the uniform distribution. The small weight on the uniform distribution has a negligible effect on the shape of the mixture, but ensures that the density is strictly positive on  $[0, 1]$ .

The two graphs on the left of Figure 3 do not use variation in the number of bidders. The two graphs in the middle use Assumption 5 (SIV) to tighten the bounds, and the graphs on the right use the stronger Assumption 4 (EP) to tighten the bounds further.

For the graphs in the middle and on the right, bid data with  $n = 7$  are used in addition to data with  $n = 2$  to bound  $v_2(\alpha)$ . Thus, the graphs only use variation in  $n$  from “one side” to tighten the bounds, i.e., from auctions with more bidders. In some cases this only allows us to tighten one of the bounds. This allows us to better illustrate the workings of the identification results than, for example, using data from

$n = 2$  and  $n = 7$  to tighten the bounds for  $n = 3$ .

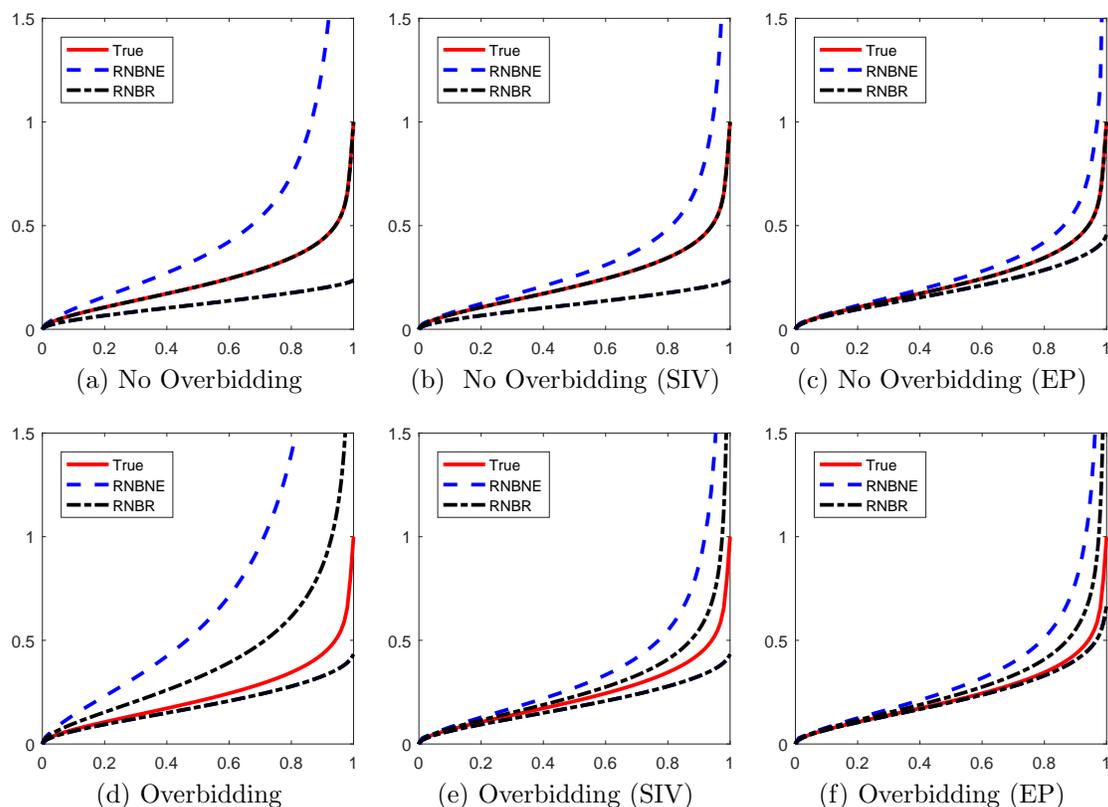


Figure 3: **Bounding the Valuation Distribution:** These graphs show how to bound the valuation distribution for  $n = 2$ . In the upper row the bidders play the RNBNE, so there is no overbidding. In the lower row bidders have constant relative risk aversion, so they overbid. The graphs on the left do not use variation in the number of bidders and only use bid data from  $n = 2$ . The graphs in the middle and the right also use bid data from  $n = 7$  to tighten the bounds. The graphs in the middle impose Assumption 5 (SIV), and on the right Assumption 4 (EP). The coefficient of relative risk aversion is 0.8.

**No Variation in  $n$**  First consider Figure 3(a). The true valuation quantile function  $v_2(\alpha)$  is shown in red. The lower bound stemming from the assumption of rationality ( $v_2(\alpha) = b_2(\alpha)$ ) is shown in black. The upper bound obtained from overbidding compared to RNBNE ( $\bar{v}_2(\alpha)$ ) is shown in blue, and the upper bound from overbidding compared to RNBR ( $\tilde{v}_2(\alpha)$ ) in black. Because the bidders play the RNBNE, they also play the RNBR, and therefore  $\tilde{v}_2(\alpha) = v_2(\alpha)$ .

Next consider Figure 3(d). Here the bidders are risk averse and overbid. Because

the bidders overbid,  $\tilde{v}_2(\alpha)$  no longer coincides with  $v_2(\alpha)$ , but instead  $b_2(\alpha)$  is now close to  $v_2(\alpha)$ . If the bidders would overbid more, then  $b_2(\alpha)$  would get arbitrarily close to  $v_2(\alpha)$ , illustrating that the lower bound is sharp.

**Variation in  $n$  under SIV** Now consider Figure 3(b), which uses bid data from  $n = 7$  and imposes Assumption 5 (SIV) to tighten the bounds in Figure 3(a).

Notice that bid data from  $n = 7$  do not allow us to tighten the lower bound for  $n = 2$ , i.e.,  $\underline{v}_2^{SIV}(\alpha) = b_2(\alpha)$ . To understand this, recall that  $\underline{v}_n^{SIV}(\alpha) = \max_{m \in \mathbb{N}: m \leq n} \{b_m(\alpha)\}$ , so the lower bound can only be tightened by using bid data from auctions with fewer bidders. Intuitively, observing that bidders in auctions with  $n = 7$  bid more aggressively does allow us to tighten the lower bound for  $v_2(\alpha)$ , because  $v_7(\alpha)$  may be higher than  $v_2(\alpha)$  under SIV.

The upper bound from overbidding compared to RNBR cannot be tightened either, i.e.,  $\tilde{v}_2^{SIV}(\alpha) = \tilde{v}_2(\alpha)$ . This is simply because even without using variation in  $n$ , the bound already coincides with the true quantile function ( $\tilde{v}_2(\alpha) = v_2(\alpha)$ ) in Figure 3(a). The upper bound from overbidding compared to RNBNE, however, can be tightened considerably, i.e.  $\bar{v}_n^{SIV}(\alpha) < \bar{v}_n(\alpha)$  for almost all  $\alpha$ .

In the case with overbidding, shown in Figure 3(e), the upper bounds under both overbidding restrictions can be tightened considerably compared to Figure 3(d). In particular, in contrast to the case without overbidding,  $\tilde{v}_2^{SIV}(\alpha) = \min_{m \geq 2} \tilde{v}_m(\alpha) = \tilde{v}_7(\alpha) < \tilde{v}_2(\alpha)$  for almost all  $\alpha$ . Intuitively, the bidders are risk averse and bid close to their valuations even for  $n = 2$ . Therefore, the implied valuation quantile function under the assumption of RNBR is lower for  $n = 7$  than for  $n = 2$ , which allows us to tighten the upper bound for  $v_2(\alpha)$ .

As in the case without overbidding, the lower bound cannot be tightened by using data from auctions with more bidders.

**Variation in  $n$  under EP** Figure 3(c) shows that Assumption 4 (EP) allows us to tighten the lower bound relative to Assumption 5 (SIV) (Figure 3(b)), i.e.,  $\underline{v}^{EP}(\alpha) = b_7(\alpha) > \underline{v}_2^{SIV}(\alpha) = b_2(\alpha)$  for almost all  $\alpha$ . Again, the upper bound from overbidding compared to RNBR cannot be tightened as it already coincides with the true valuation distribution so  $\tilde{v}^{EP}(\alpha) = \tilde{v}_2^{SIV}(\alpha) = \tilde{v}_2(\alpha) = v_2(\alpha)$ . The upper bound from overbidding compared to RNBNE, however, can be tightened further, i.e.,  $\bar{v}_n^{EP}(\alpha) < \bar{v}_n^{SIV}(\alpha)$  for almost all  $\alpha$ .

Lastly, Figure 3(f) shows that, as in the case without overbidding, the lower bound can be tightened. The upper bound under RNBR cannot be tightened further compared to SIV because  $\tilde{v}_n^{EP}(\alpha) = \min_{m \in \mathbb{N}} \tilde{v}_m(\alpha) = \tilde{v}_7(\alpha) = \min_{m \geq n} \tilde{v}_m(\alpha) = \tilde{v}_n^{SIV}(\alpha)$ . The upper bound under RNBNE, however, can be tightened slightly, i.e.,  $\bar{v}_n^{EP}(\alpha) < \bar{v}_n^{SIV}(\alpha)$  for almost all  $\alpha$ .

### 4.3 Profit

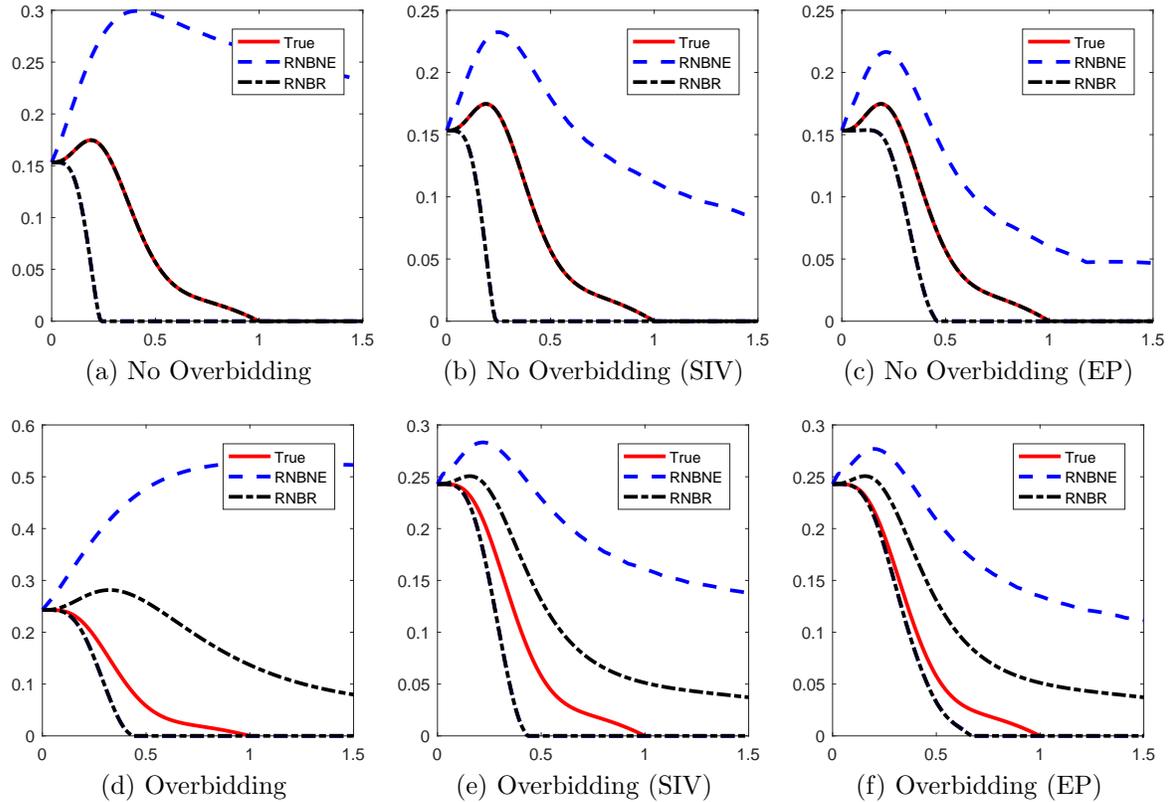


Figure 4: **Bounding the Seller's Payoff:** These graphs show how to bound the seller's payoff for  $n = 2$ . In the upper row the bidders play the RNBNE, so there is no overbidding. In the lower row bidders have CRRA, so they overbid. The graphs on the left do not use variation in the number of bidders and only use bid data from  $n = 2$ . The graphs in the middle and the right also use bid data from  $n = 7$  to tighten the bounds. The graphs in the middle impose Assumption 5 (SIV), and on the right Assumption 4 (EP). The CRRA coefficient is 0.8.

Figure 4 illustrates how the bounds for the valuation quantile function in Figure 3 can be translated into bounds for the seller's profit. We assume that  $c = 0$ , i.e.,

the seller does not value the auctioned good. Without using variation in  $n$ , the bounds from overbidding compared to RNBNE are fairly wide, but the bounds from overbidding compared to RNBR are more informative. Using variation in  $n$  leads to a considerable tightening of the bounds corresponding to the tighter bounds for the valuation distribution.

**Optimal Reserve Price** Figure 5 shows how the profit bounds can be used to bound the optimal reserve price. The two graphs use the bounds from overbidding compared to RNBR and impose Assumption 4 (EP) in Figure 4(c) (left) and 4(f) (right). The pink lines demarcate the range of reserve prices that may be optimal.

How can we translate the profit bounds into the bounds for the reserve price? If the maximum profit under some reserve price  $r_1$  is smaller than the minimum profit under some alternative reserve price  $r_2$ , i.e.,  $\bar{\pi}_2^{EP}(r_1) < \bar{\pi}_2^{EP}(r_2)$ , then  $r_1$  can be ruled out. The set of potentially optimal reserve prices contains all reserve prices that cannot be ruled out in this fashion.

This is shown in Figure 5. The horizontal pink lines in the graphs indicate the maximum of the lower profit bound,  $\bar{\pi}_2^{EP, \max}$ . Any reserve price  $r$  such that  $\bar{\pi}_2^{EP}(r) < \bar{\pi}_2^{EP, \max}$  can therefore be ruled out. The reserve prices between the two vertical pink lines satisfy  $\bar{\pi}_2^{EP}(r) \geq \bar{\pi}_2^{EP, \max}$  and can therefore not be ruled out. In the case without overbidding (Figure 5(a)) the range of potential reserve prices is  $[0.03, 0.29]$  and in the case with overbidding (Figure 5(b)) it is  $[0, 0.23]$ .

The graphs also show the estimated seller profit under the assumption that there is no overbidding, as in [Guerre, Perrigne, and Vuong \(2000\)](#), in green. If this assumption is correct (Figure 5(a)), it identifies the optimal reserve price. If there is overbidding, however, (Figure 5(b)) the optimal reserve price recovered through [Guerre, Perrigne, and Vuong \(2000\)](#) even lies outside the reserve price bounds. This is because the bounds use data from  $n = 7$  to tighten the upper bound for the profit.

Which reserve price inside the bounds should the seller choose? The procedure outlined above only bounds the set of reserve prices, but ultimately the seller has to choose a particular reserve price. One way to choose the reserve price in partially identified models is to maximize the minimum profit, i.e., maximize the profit under a worst case scenario ([Aryal and Kim \(2013\)](#)).

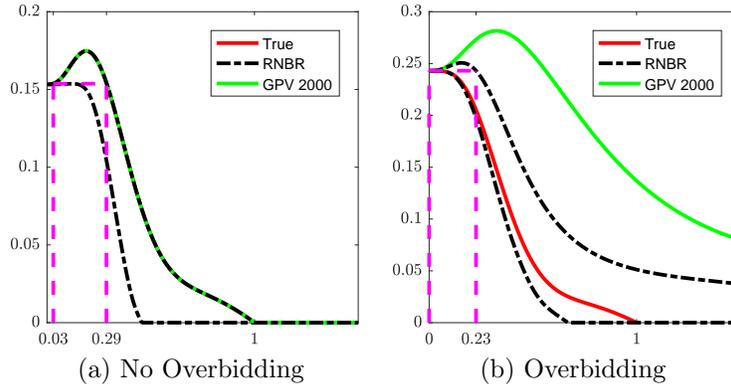


Figure 5: **Bounding the Optimal Reserve Price:** These graphs show how to bound the optimal reserve price for  $n = 2$  using the bounds from overbidding compared to RNBR in Figures 4(c) (left) and 4(f) (right). The pink lines demarcate the range of reserve prices that may be optimal. On the left the bidders play the RNBNE, so there is no overbidding. On the right bidders have CRRA, so they overbid. Both graphs impose Assumption 4 (EP) for  $n = 2$  and  $n = 7$  to tighten the bounds. The coefficient of relative risk aversion is 0.8.

#### 4.4 Heterogeneous Bidding Behavior

Lastly we consider the case of heterogeneous bidding behavior. Figure 6 shows bounds for valuations (top) and profits (bottom) if different bidders use different bidding strategies. There are three bidder groups, each accounts for one-third of the bids. The first group overbids moderately compared to RNBNE. These bidders use the bid function of a BNE where bidders have CRRA with a coefficient of 0.3. The second group overbids more, as their CRRA coefficient is 0.5. The third group overbids the most, as their CRRA coefficient is 0.8.

The graphs on the left row only use data with  $n = 2$ . The graphs in the middle and on the right use data from  $n = 7$  in addition to  $n = 2$  to tighten the bounds. The graphs in the middle use Assumption 5 (SIV) and the graphs at the right use Assumption 4 (EP).

These graphs not only illustrate the identification results in Proposition 11, but are also consistent with the conjecture that the upper bounds for the profit function remain valid even with heterogeneous bidding behavior.

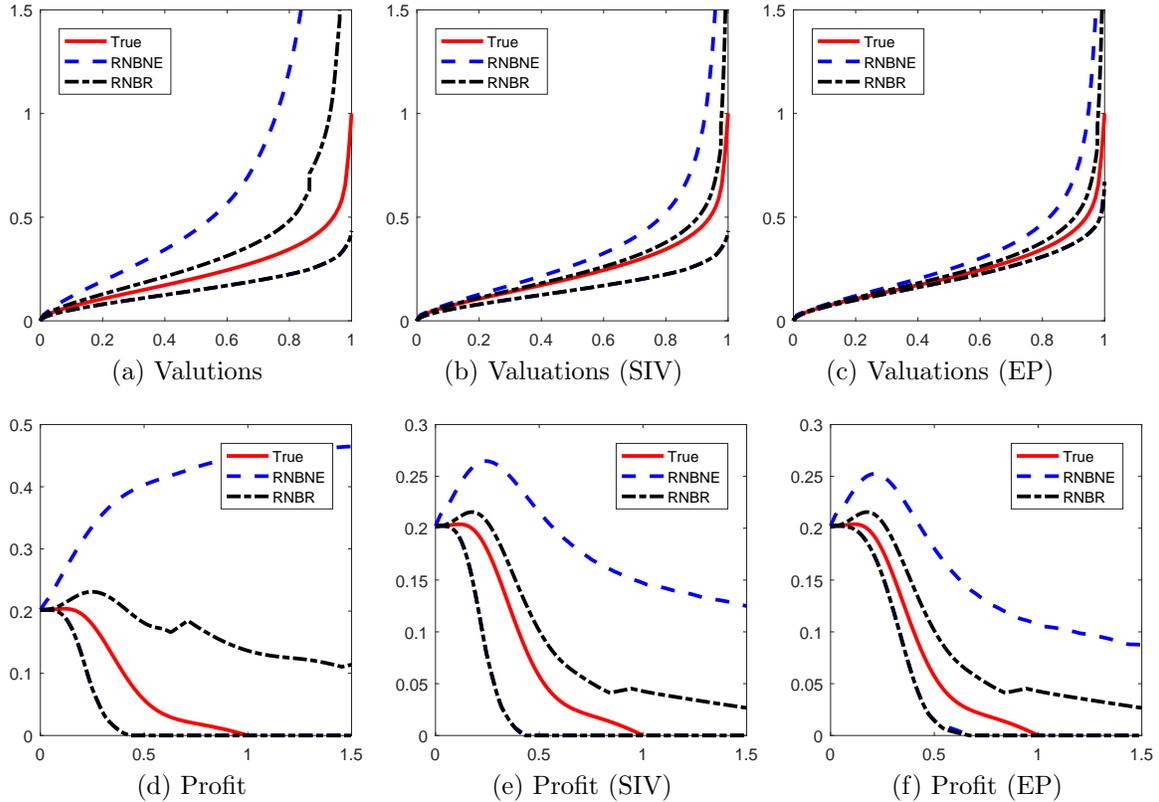


Figure 6: **Heterogeneous Bidding Behavior:** These graphs show bounds for the valuation distribution (top) and profits (bottom) if the bidders use different bidding strategies. Here there are three bidder groups, which each account for one third of the bids. The first group overbids moderately compared to RNBNE. These bidders use the bid function of a BNE where bidders have CRRA with a coefficient of 0.3. The second group overbids more as their CRRA coefficient is 0.5. The third group overbids the most as their CRRA coefficient is 0.8. The graphs on the left only use data with  $n = 2$ . The graphs in the middle and on the right use data from  $n = 7$  in addition to  $n = 2$  to tighten the bounds. The graphs in the middle use Assumption 5 (SIV) and the graphs at the right use Assumption 4 (EP).

## 5 Estimation

Our identification results are constructive and lead to simple estimators for the bounds. Suppose we observe  $N_n$  bids,  $\{y_i\}_{i=1}^{N_n}$ , from  $n$ -bidder auctions. Without loss of generality, assume that  $y_1 \leq y_2 \leq \dots \leq y_{N_n}$ . Let  $\hat{G}_n$  be the empirical distribution of bids from  $n$ -bidder auctions and let  $\hat{g}_n$  be a kernel density estimate.

**Lower Bound** As an estimator for the lower bound of the valuation quantile function we define  $\hat{v}_n(\alpha) = \hat{b}_n(\alpha) = \inf \left\{ b : \hat{G}_n(b) \geq \alpha \right\}$ . An estimator for the lower bound of the profit function can be obtained by plugging  $\hat{b}_n(\alpha)$  into  $\underline{\pi}_n$ .

**Upper Bound under RNBNE Overbidding** Next, consider the upper bound under overbidding compared to RNBNE. In principle, an estimator for the upper bound  $\bar{v}_n(\alpha)$  can be obtained by plugging  $\hat{b}_n(\alpha)$  into (3). We now establish a consistency result of an infinite-dimensional sieve estimator.

$$\Theta_R(b_n) = \{q \in \Theta : \beta_n(t, q) \leq b_n(t) \leq q(t) \forall t \in [0, 1]\}.$$

Therefore,  $\bar{v}_n(\alpha) = \sup_{q \in \Theta_R} q(\alpha)$ . Next, define

$$\tilde{\Theta}_R(b_n) = \left\{ q \in \tilde{\Theta} : \beta_n(t, q) \leq b_n(t) \leq q(t) \forall t \in [0, 1] \right\},$$

where  $\tilde{\Theta}$  is a density subset of  $\Theta$ . Then we can define  $\hat{v}_n(\alpha) = \sup_{q \in \tilde{\Theta}_R(\hat{b}_n^c)} q(\alpha)$  where  $\hat{b}_n^c(\cdot)$  is defined by  $\hat{b}_n^c(0) = y_1$  and  $\hat{b}_n^c\left(\frac{i}{N_n}\right) = y_i$  for all  $i > 1$  and then interpolate between these points.<sup>18</sup> An estimator for the upper bound of the profit function can then be obtained by plugging  $\hat{v}_n(\alpha)$  into the expression for  $\bar{\pi}_n(r)$ .<sup>19</sup>

**Proposition 12.** *Suppose  $\tilde{\Theta}$  contains all the piece-wise linear functions and  $g_n$  is continuous and bounded away from 0 on its support. As  $N_n \rightarrow \infty$ , (1) for any  $\epsilon > 0$ ,  $\sup_{\alpha \in [0, 1-\epsilon]} |\hat{v}_n(\alpha) - \bar{v}_n(\alpha)| \rightarrow 0$  and  $\sup_{\alpha \in [0, 1]} |\hat{v}_n(\alpha) - \underline{v}_n(\alpha)| \rightarrow 0$  almost surely; (2)  $\hat{\pi}_n(r) \rightarrow \bar{\pi}_n(r)$  and  $\hat{\pi}_n(r) \rightarrow \bar{\pi}_n(r)$  and  $\hat{\pi}_n(r) \rightarrow \underline{\pi}_n(r)$  almost surely for every  $r$ .*

Notice that the consistency result holds even though  $\tilde{\Theta}$  is very complex. And unlike the case with large sieve space considered in [Chen and Pouzo \(2012\)](#), we do not need any penalty. This is because that only  $\hat{b}_n^c$  has estimation errors and it is a function defined on  $[0, 1]$ . To implement the estimator, we choose  $\tilde{\Theta}_K$  to be the space spanned by B-splines with  $K$  knots. Then  $\tilde{\Theta} = \cup_{K=1}^{\infty} \tilde{\Theta}_K$  is a dense subset of  $\Theta$  and

<sup>18</sup>We use  $\hat{b}_n^c(\cdot)$  instead of  $\hat{b}_n(\cdot)$  because  $\hat{b}_n(\cdot)$  is a step function and we cannot find a continuous  $q$  that satisfies both overbidding and rationality in a neighborhood of 0. To see this, notice that  $\hat{b}_n(\alpha) = y_1$  for all  $\alpha \in (0, 1/N_n]$ . To satisfy overbidding and rationality, we must have  $q(\alpha) = y_1$  on  $(0, 1/N_n]$ . However,  $\hat{b}_n(\alpha) = y_2 > y_1$  if  $\alpha \in (1/N_n, 2/N_n]$ . As a result,  $q$  has to jump at  $\alpha = 1/N_n$ .

<sup>19</sup>One can construct confidence set following [Hsieh, Shi, and Shum \(2018\)](#). In the interest of space, we do not pursue it in this paper.

contains all piece-wise linear functions. We approximate  $\hat{v}_n(\alpha)$  by taking maximum on  $\tilde{\Theta}_K$  with a very large  $K$ . This leads to a linear programming problem.

**Upper Bound for RNBR** Under overbidding compared to RNBR,

$$\hat{v}_n(\alpha) = \hat{b}_n(\alpha) + \frac{1}{n-1} \frac{\alpha}{\hat{g}_n(\hat{b}_n(\alpha))}.$$

Again this is a step function defined as

$$\hat{v}_n(\alpha) = y_{(i)} + \frac{1}{n-1} \frac{1}{\hat{g}_n(y_{(i)})} \frac{i}{N_n}, \text{ if } \frac{i}{N_n} \leq \alpha < \frac{i+1}{N_n}.$$

An estimator for the upper bound of the profit function can be obtained by plugging in  $\hat{v}_n(\alpha)$  into the the expression for  $\tilde{\pi}_n(r)$ . Consistency results can be established following [Guerre, Perrigne, and Vuong \(2000\)](#) with additional smoothness assumption on  $g_n$ .

**Variation in  $n$**  If there is variation in  $n$ , we can construct similar estimators for the bounds of the valuation quantile function following the identification argument. These estimators can be plugged into the expressions for the profit bounds to obtain estimators for the profit bounds. Similar consistency results can also be established.

## 6 Validity of Bounds in Experimental Data

In this section we check whether the proposed bounds are valid in the experimental data from [Dyer, Kagel, and Levin \(1989\)](#). This is important because the identifying assumptions are violated for some bidders, especially those with low valuations, as discussed in [Section 2](#).

The data set contains contingent bids for  $n = 3$  and  $n = 6$  for the same underlying valuation from the same bidder. After discarding bids from training rounds there are 414 bids for  $n = 3$  and 414 bids for  $n = 6$ . As the valuation distribution does not vary with  $n$ , we can exploit variation in the number of bidders to tighten the bounds. The valuations are drawn from a uniform distribution on  $[0, 30]$ , so  $\sigma_n(t) = \frac{n-1}{n}t$ .

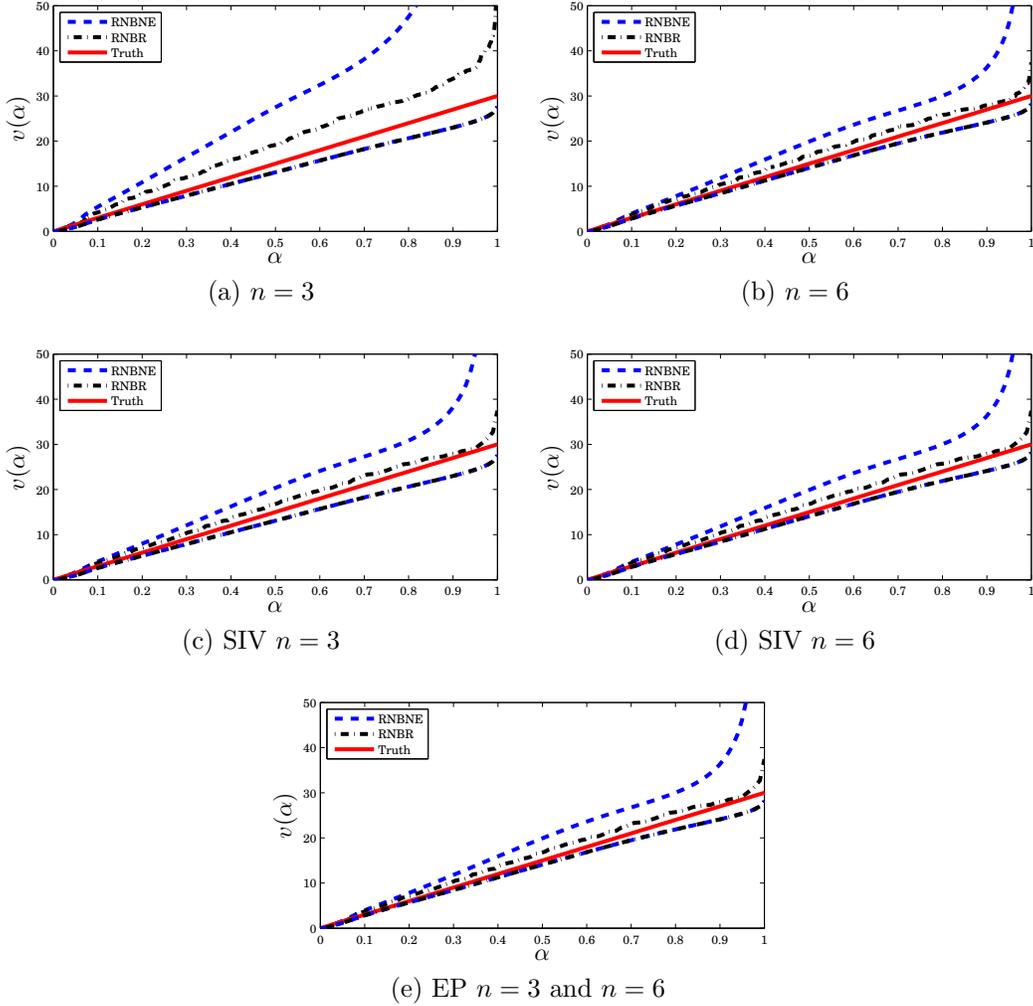


Figure 7: **Bounding the Valuation Distribution:** Bounds on the valuation quantile function for  $n = 3$  (left) and  $n = 6$  (right). The top row does not use variation in  $n$ . The middle row uses variation in  $n$  under the SIV assumption, and the bottom row under the EP assumption.

**Valuation Quantile Function** Figure 7 shows the estimated bounds for the valuation quantile function for  $n = 3$  on the left and  $n = 6$  on the right. The top row does not use variation in  $n$ . The middle row uses variation in  $n$  under the SIV assumption, and the bottom row under the EP assumption. The true valuation distribution is shown in red, the RNBNE bounds in blue, and the RNBR bounds in black.

The graphs show that the bounds are valid even though the identifying assumptions are violated for some bidders. The RNBR bounds are substantially tighter than

the RNBNE bounds and the bounds for  $n = 6$  are tighter than for  $n = 3$ . Using variation in  $n$  helps to tighten the bounds considerably.

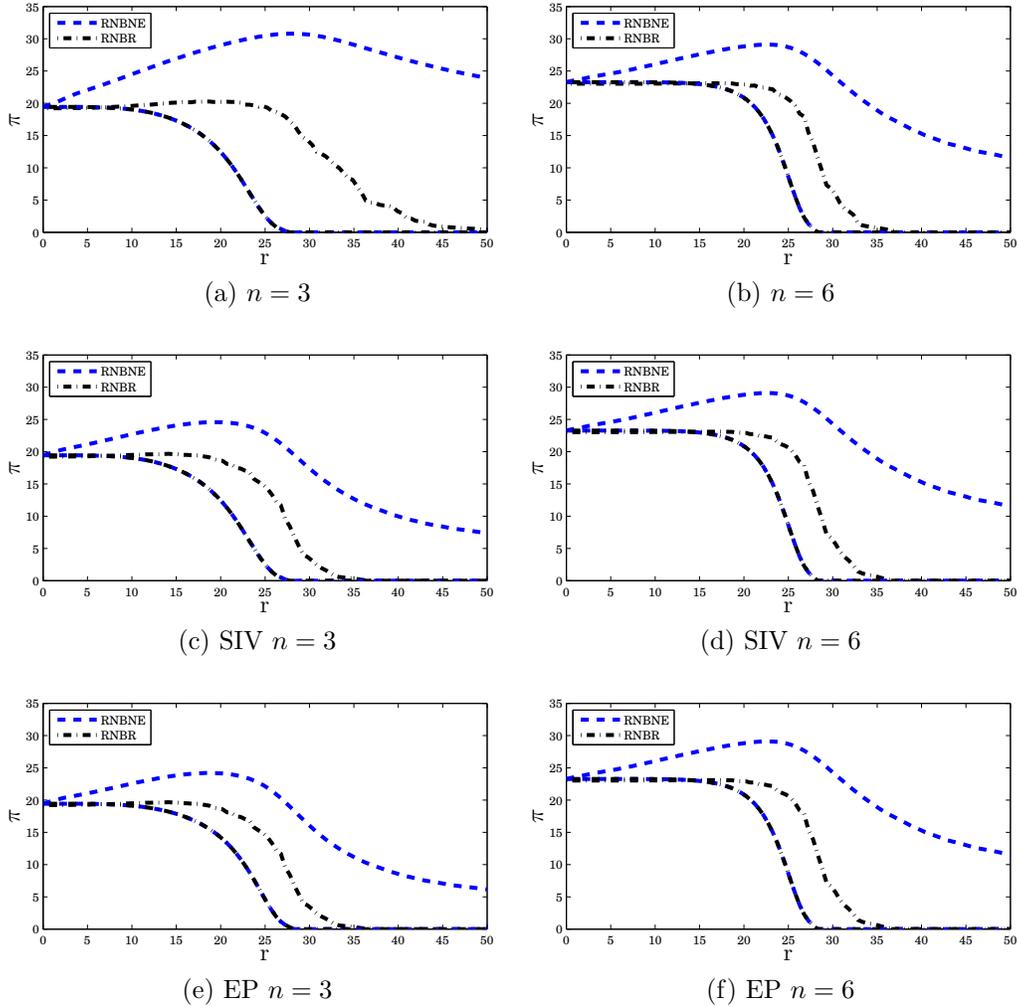


Figure 8: **Bounding the Seller's Payoff:** Bounds on the profit function for  $n = 3$  (left) and  $n = 6$  (right). The top row does not use variation in  $n$ . The middle row uses variation in  $n$  under the SIV assumption, and the bottom row under the EP assumption.

**Seller Payoff** Figure 8 shows the estimated bounds for the seller's payoff that correspond to valuation bounds in Figure 7. Naturally, tighter bounds for valuations are translated into tighter bounds for the seller's payoff. Therefore the RNBR bounds are substantially tighter than the RNBNE bounds, and the bounds for  $n = 6$  are tighter than for  $n = 3$ . The bounds for low reserve prices are tight and widen for

higher reserve prices. Intuitively, this is because the payoff without a reserve price is point identified from the data, so the bounds are tighter for lower reserve prices.

**Reserve Price** Figure 9 illustrates how the bounds for the seller’s profit function can be used to bound the reserve price. Here we show the RNBR bounds under the assumption of EP.

For  $n = 3$ , shown in panel (a), reserve prices between 0 and 16.6 may be optimal as indicated by the two pink vertical lines. Reserve prices above 16.6 can be ruled out because the maximum payoff under these reserve prices is lower than the minimum payoff under some other reserve price, as indicated by the pink horizontal line.

To choose among reserve prices between 0 and 16.6 requires some assumption about the preferences of the seller. For example, a maxmin seller would choose a reserve price of 0 because this yields the highest minimum payoff whereas a maxmax seller would choose a reserve price of 14.1 because this yields the highest maximum payoff.<sup>20</sup>

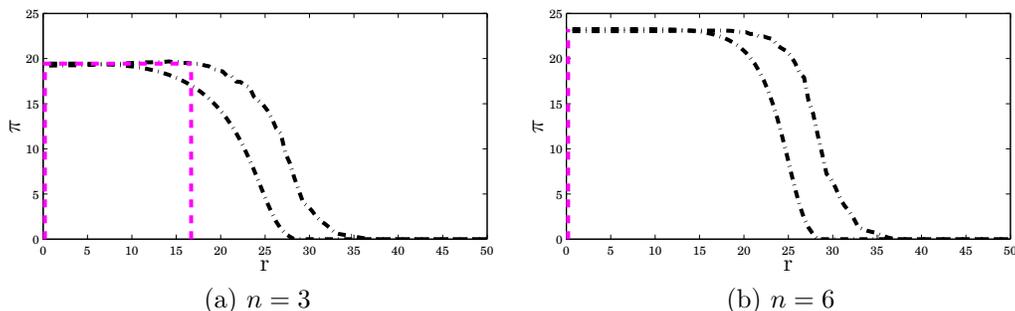


Figure 9: **Bounding the Reserve Price:** These graphs show the bounds for the optimal reserve price. The bounds are shown for overbidding compared to RNBR and under the assumption of EP. For  $n = 3$  reserve prices between 0 and 16.6 may be optimal as indicated by the two vertical pink lines. Reserve prices above 16.6 can be ruled out because the maximum payoff under these reserve prices is lower than the minimum payoff under some other reserve price, as indicated by the pink horizontal line. For  $n = 6$  a reserve price of 0 is optimal. All reserve prices above 0 can be ruled out because they yield a maximum payoff below the known payoff at 0.

For  $n = 6$ , shown in panel (b), all reserve prices above 0 can be ruled out, because their maximum seller payoff is below the payoff at 0. Hence, this is a case where the

<sup>20</sup>See Aryal and Kim (2013) for choosing reserve prices with partially identified models.

optimal reserve price can be determined exactly even though the identifying restrictions are weak and only yield partial identification of the valuation distribution and the payoff function. This is an instance where weakening the identifying restrictions does not come at the expense of having to content oneself with less precise policy recommendations.

## 7 Conclusion

This paper suggests a new link between experimental economics and structural estimation. Usually experiments are conducted to compare their findings to theoretical predictions. Hence, experimental work affects structural work only indirectly if it sparks the development of new models. However, in this paper we demonstrate that experimental findings can be used directly in structural estimation without imposing a particular model. This approach is particularly useful in cases where there is no agreement which model is best supported by the experimental findings, as is the case for first-price auctions.

In the context of first-price auctions, imposing the restrictions from experiments allows us to relax the assumptions of the canonical model, which appears to be inconsistent with the experimental findings, and still obtain informative bounds on primitives. In other contexts it could be the case that while the canonical model is not rejected by experimental findings, it is also not identified from field data. In this case, imposing the restrictions from experiments in addition to the restrictions from the canonical model could help in identification.

We see two broad avenues for future work. First, structural researchers can ask whether using existing experimental findings to inform structural estimation is helpful in other contexts, for example, for identification and estimation of bargaining models. Second, experimental researchers can ask whether it is useful to conduct experiments specifically to inform structural estimation rather than to test theory.

## A Proofs

### A.1 Proof of Results in Section 3.1

We first prove Proposition 2 because it will be handy in proving Proposition 1.

*Proof of Proposition 2.* Notice that  $\bar{v}_n^a(0) = \bar{v}_n(0) = b_n(0)$ . Therefore, the statement holds for  $\alpha = 0$ .

Now consider  $\alpha > 0$ . First we show that  $\bar{v}_n(\alpha) \leq \bar{v}_n^a(\alpha)$ . If this is not the case there exists some  $q$  that satisfies overbidding and rationality such that  $q(\alpha) > \bar{v}_n^a(\alpha)$ . Then  $q_{\alpha, q(\alpha)}(\cdot)$  satisfies rationality by construction and because  $b_n(\cdot) \geq \beta_n(\cdot, q) \geq \beta_n(\cdot, q_{\alpha, q(\alpha)})$  it also satisfies overbidding. By the definition of  $\bar{v}_n^a$ ,  $\bar{v}_n^a(\alpha) \geq q_{\alpha, q(\alpha)}(\alpha) = q(\alpha)$ , which contradicts  $\bar{v}_n^a(\alpha) < q(\alpha)$ .

Next, we show that  $\bar{v}_n(\alpha) \geq \bar{v}_n^a(\alpha)$ . Define

$$q_{\alpha, \epsilon, \delta}(t) = \begin{cases} b_n(t) & \text{if } t < \alpha - \epsilon \\ \frac{\bar{v}_n^a(\alpha) - \delta - b_n(\alpha - \epsilon)}{\epsilon} (t - \alpha + \epsilon) + b_n(\alpha - \epsilon) & \text{if } t \in [\alpha - \epsilon, \alpha], \\ \max\{x - \delta, b_n(t)\} & \text{if } t \geq \alpha \end{cases} \quad (14)$$

where  $\epsilon$  and  $\delta$  are two positive numbers. Notice that  $q_{\alpha, \epsilon, \delta}$  is continuous, weakly increasing and satisfies rationality. If we can show that for every sufficiently small  $\delta$ , there exists  $\epsilon > 0$  such that  $\beta(\cdot, q_{\alpha, \epsilon, \delta}) \leq b_n(\cdot)$ , then  $\tilde{q}_{\alpha, \epsilon, \delta} = (1 - \kappa)q_{\alpha, \epsilon, \delta} + \kappa b_n \in \Theta$  satisfies overbidding and rationality for all  $\kappa \in (0, 1)$ . This would imply that  $\bar{v}_n(\alpha) \geq \lim_{\kappa \rightarrow 0} (1 - \kappa)q_{\alpha, \epsilon, \delta}(\alpha) = \bar{v}_n^a(\alpha) - \delta$ . Because  $\delta$  can be arbitrarily small,  $\bar{v}_n(\alpha) \geq \bar{v}_n^a(\alpha)$ .

Now we show that for every  $\delta$  sufficiently small, there exists  $\epsilon > 0$  such that  $\beta(\cdot, q_{\alpha, \epsilon, \delta}) \leq b_n(\cdot)$ . First, notice that  $\beta_n(t, b_n) < b_n(t)$  for all  $t > 0$  because  $b_n$  is strictly increasing. Therefore,

$$\max_{t \geq \alpha} [\beta_n(t, b_n) - b_n(t)] > 0.$$

In addition,  $\beta_n(\cdot, q)$  is uniformly continuous in  $q$ . This means that

$$q(t) = b_n(t) 1(t < \alpha) + 1(t \geq \alpha) \max\{b_n(\alpha) + 2\delta, b_n(t)\}$$

satisfies RNBNE overbidding if  $\delta$  is sufficiently small. Because  $\bar{v}_n^a(\alpha) \geq q(\alpha) = b_n(\alpha) + 2\delta$ ,  $\bar{v}_n^a(\alpha) - \delta > b_n(\alpha)$  if  $\delta$  is sufficiently small. By continuity of  $b_n$ , there exists  $\eta_1 > 0$  such that  $\bar{v}_n^a(\alpha) - \delta > b_n(t)$  for all  $t < \alpha + \eta_1$ .

Second, notice that  $\beta_n(\alpha, q_{\alpha, \bar{v}_n(\alpha)}) - b_n(\alpha) = \beta(\alpha, b_n) - b_n(\alpha) < 0$  and both  $b_n(\cdot)$  and  $\beta(t, q_{\alpha, \bar{v}_n(\alpha)})$  are continuous in  $t$ . There exists  $\eta_2 > 0$  such that  $\beta(t, q_{\alpha, \bar{v}_n(\alpha)}) < b_n(t) - \eta_2$  for  $t \in (\alpha - \eta_2, \alpha + \eta_2)$ . Define  $\eta = \min(\eta_1, \eta_2)$ . In addition, notice that

$\beta(\cdot, q_{\alpha, \epsilon, \delta})$  converges uniformly to  $\beta(\cdot, q_{\alpha, \bar{v}_n^a(\alpha) - \delta})$  uniformly if  $\epsilon \rightarrow 0$ . This suggests that if  $t \in (\alpha - \eta, \alpha + \eta)$ , for all sufficiently small  $\epsilon > 0$

$$\beta(t, q_{\alpha, \epsilon, \delta}) \leq \beta(t, q_{\alpha, \bar{v}_n(\alpha) - \delta}) \leq \beta(t, q_{\alpha, \bar{v}_n(\alpha)}) < b_n(t).$$

If  $t \geq \alpha + \eta$  and  $\epsilon$  is sufficiently small,

$$\begin{aligned} t^{n-1} \beta(t, q_{\alpha, \epsilon, \delta}) - t^{n-1} \beta_n(t, q_{\alpha, \bar{v}_n(\alpha)}) &\leq t^{n-1} \beta(t, q_{\alpha - \epsilon, \bar{v}_n(\alpha) - \delta}) - \int_0^t q_{\alpha, \bar{v}_n(\alpha)}(s) ds^{n-1} \\ &= \int_0^t [q_{\alpha - \epsilon, \bar{v}_n(\alpha) - \delta}(s) - q_{\alpha, \bar{v}_n(\alpha)}(s)] ds^{n-1} \leq \int_{\alpha - \epsilon}^{\alpha + \eta} [q_{\alpha - \epsilon, \bar{v}_n(\alpha) - \delta}(s) - q_{\alpha, \bar{v}_n(\alpha)}(s)] ds^{n-1} \\ &\leq [\alpha^{n-1} - (\alpha - \epsilon)^{n-1}] (\bar{v}_n(\alpha) - \delta) - \delta [(\alpha + \eta)^{n-1} - \alpha^{n-1}] < 0, \end{aligned}$$

where the second inequality holds because  $q_{\alpha - \epsilon, \bar{v}_n(\alpha) - \delta}(t) - q_{\alpha, \bar{v}_n(\alpha)}(t) = -\delta < 0$  for all  $t > \alpha + \eta$ . This means that  $\beta(t, q_{\alpha, \epsilon, \delta}) \leq \beta_n(t, q_{\alpha, \bar{v}_n(\alpha)}) \leq b_n(t)$  for  $\epsilon$  sufficiently small for all  $t > \alpha + \eta$ . In addition, if  $t \leq \alpha - \epsilon$ ,  $\beta_n(t, q_{\alpha, \epsilon, \delta}) = \beta_n(t, b_n) \leq b_n(t)$ . Hence,  $q_{\alpha, \epsilon, \delta}$  satisfies RNBNE overbidding and rationality for all  $\epsilon$  that are sufficiently small and  $q_{\alpha, \epsilon, \delta}(\alpha) = \bar{v}_n^a(\alpha) - \delta$ . Because  $\delta > 0$  can be arbitrarily small, we can conclude that  $\bar{v}_n(\alpha) \geq \bar{v}_n^a(\alpha)$ .  $\square$

*Proof of Proposition 1.* By definition,  $\bar{v}_n(\alpha)$  is an upper bound for the true valuation quantile. Now we show that it is a sharp bound. Given  $\alpha$ , by the definition of  $\bar{v}_n$ , for any  $\eta > 0$ , there exists a  $\tilde{q}$  that satisfies all the constraints and  $\tilde{q}(\alpha) > \bar{v}_n(\alpha) - \eta/2$ . Define  $\tilde{\tilde{q}} = (1 - \delta)\tilde{q} + \delta b_n$  for some  $\delta \in (0, 1)$ . By assumption,  $\tilde{\tilde{q}} \geq b_n$  and it is strictly increasing. Because  $\tilde{\tilde{q}} < \tilde{q}$ , by Proposition 2 in [Grundl and Zhu \(2016\)](#), the RNBNE bid quantile function of  $\tilde{\tilde{q}}$  is weakly lower than that of  $\tilde{q}$  and hence weakly lower than  $b_n$ . This means that  $\tilde{\tilde{q}}$  is a valid candidate quantile function for any  $\delta \in (0, 1)$ . Then pick a sufficiently small  $\delta$  such that  $\tilde{\tilde{q}}(\alpha) > \tilde{q}(\alpha) - \eta/2 > \bar{v}_n(\alpha) - \eta$ . Therefore,  $\bar{v}_n(\alpha) - \eta$  cannot be a valid upper bound for any  $\eta$ . This shows that  $\bar{v}_n(\alpha)$  is sharp. The lower bound is sharp because we cannot rule out the case where bidders have arbitrarily high risk aversion. This would induce them to bid arbitrarily close to their valuations.

**Proposition 1(1)** Next we show that  $\bar{v}_n(\alpha) < b_n(1)/(1-\alpha)$  for any  $\alpha < 1$ . Notice that by the constraints,  $\int_0^1 q(t) dt^{n-1} \leq b_n(1)$ . As  $q(t) \geq 0$

$$\int_{\alpha}^1 q(t) dt^{n-1} \leq \int_0^1 q(t) dt^{n-1} \leq b_n(1).$$

Because  $q(t)$  is non-decreasing,  $q(\alpha)(1-\alpha^{n-1}) \leq \int_{\alpha}^1 q(t) dt^{n-1}$ . Therefore,

$$q(\alpha) \leq \frac{b_n(1)}{1-\alpha^{n-1}} < \frac{b_n(1)}{1-\alpha}.$$

The last inequality follows because  $\alpha < 1$ .

**Proposition 1(2)** Now we show that  $\exists C > 0$  such that  $\bar{v}_n(\alpha) > C/(1-\alpha)$  for  $\alpha$  close to 1. Define  $2y = b_n(1) - \beta_n(1, b_n)$  where  $y > 0$  because the RNBNE bid is strictly lower than the valuation for  $\alpha > 0$ . We will show that Proposition 1(2) is satisfied for  $C = y/4$ .

By continuity, there exists some  $\gamma > 0$  such that  $b_n(\alpha) - \beta_n(\alpha, b_n) > y$  if  $\alpha \in [1-\gamma, 1]$ . Next, define

$$\tilde{q}(\alpha) = \begin{cases} b_n(\alpha) & \text{if } \alpha < 1-\eta \\ [2M - b_n(1-\eta)] \frac{\alpha-(1-\eta)}{\eta} + b_n(1-\eta) & \text{if } \alpha \geq 1-\eta \end{cases},$$

where  $M = \frac{y}{2} \left( \frac{1}{\eta} - 1 \right)$ . We will show that for any sufficiently small  $\eta$ ,  $\tilde{q}$  is a valid quantile function. Once we establish this,  $\bar{v}_n(1-\eta) \geq \tilde{q}(\alpha) = \frac{y}{4\eta}$  for all sufficiently small  $\eta$ . Then Proposition 1(2) holds with  $C = y/4$ .

Because the bid density is bounded away from 0,  $z = \sup_{\alpha \in [0,1]} b'_n(\alpha)$  is bounded. If  $\eta$  is sufficiently small such that  $M - b_n(1) > z$ , it is guaranteed that  $\tilde{q}(\cdot) \geq b_n(\cdot)$ . For  $\alpha < 1-\eta$ ,  $\beta_n(\alpha, b_n) = \beta_n(\alpha, \tilde{q})$ . Notice that

$$\beta_n(1, \tilde{q}) - \beta_n(1-\eta, \tilde{q}) \leq \frac{1}{(1-\eta)^{n-1}} \int_{1-\eta}^1 \tilde{q}(t) dt^{n-1} \leq \frac{2M}{(1-\eta)^{n-1}} [1 - (1-\eta)^{n-1}]. \quad (15)$$

For all  $\eta < \gamma$ , by the definition of  $M$

$$\frac{2M}{(1-\eta)^{n-1}} [1 - (1-\eta)^{n-1}] \leq \frac{2M}{1-\eta} - 2M = y. \quad (16)$$

As a result, if  $\eta < \gamma$ , for all  $\alpha \in [1 - \eta, 1]$ , by (15), (16) and the fact that  $b_n(1 - \eta) - \beta_n(1 - \eta, b_n) > y$ , we obtain that

$$\begin{aligned}\beta_n(\alpha, \tilde{q}) &< \frac{2M}{(1 - \eta)^{n-1}} [1 - (1 - \eta)^{n-1}] + \beta_n(1 - \eta, \tilde{q}) \\ &< y + \beta_n(1 - \eta, \tilde{q}) < b_n(1 - \eta) \leq b_n(\alpha),\end{aligned}$$

where the third inequality hold because  $\eta < \gamma$ . Therefore,  $\tilde{q}(\cdot)$  is a valid candidate valuation quantile function if  $\eta < \gamma$  and  $M = \frac{y}{2} \left( \frac{1}{\eta} - 1 \right) > b_n(1) + z$ , which are satisfied for all  $\eta$  that are sufficiently small. This concludes the proof for Proposition 1(2).

**Proposition 1(3)** To see that  $\bar{v}_n$  is continuous, notice that for any  $\epsilon > 0$ , we can find a  $q$  that satisfies all the constraints and  $q(\alpha) > \bar{v}_n(\alpha) - \epsilon/2$ . Because  $q$  is continuous, we can find a  $\delta > 0$  such that  $|q(x) - q(\alpha)| < \epsilon/2$  for all  $x \in (\alpha - \delta, \alpha]$ . This means that  $\bar{v}_n(x) - \bar{v}_n(\alpha) > q(x) - q(\alpha) - \epsilon/2 > -\epsilon$ , which in turn implies that  $|\bar{v}_n(x) - \bar{v}_n(\alpha)| < \epsilon$  for all  $x \in (\alpha - \delta, \alpha]$  because  $\bar{v}_n(x)$  is weakly increasing. This shows that  $\bar{v}_n(\cdot)$  is left continuous.

Now we prove  $\bar{v}_n(\cdot)$  is right continuous at any  $\alpha \in (0, 1)$ . Let  $\{\alpha_k\}_{k=1}^\infty$  be a decreasing sequence that converges to  $\alpha$ . Define  $\lim_{k \rightarrow \infty} \bar{v}_n(\alpha_k) = x$  for some  $\epsilon > 0$ . Notice that  $q_{\alpha, x}$  satisfies rationality by construction. Now we show that  $q_{\alpha, x}$  satisfies RNBNE overbidding. First notice that as  $k \rightarrow \infty$

$$\sup_{t \in [\alpha, 1]} |\beta_n(t, q_{\alpha, x}) - \beta_n(t, q_{\alpha_k, \bar{v}_n(\alpha_k)})| \leq \frac{1}{\alpha^{n-1}} \int_0^1 |q_{\alpha, x}(t) - q_{\alpha_k, \bar{v}_n(\alpha_k)}(t)| dt^{n-1} \rightarrow 0$$

As a result,  $\beta_n(t, q_{\alpha, x}) \leq b_n(t)$  for all  $t \in [\alpha, 1]$  because by proposition 2,  $q_{\alpha_k, \bar{v}_n(\alpha_k)}(t) \leq b_n(t)$  for all  $t \in [0, 1]$ . And by construction,  $q_{\alpha, x}$  satisfies RNBNE overbidding on  $[0, \alpha]$ . Therefore, for any  $\alpha \in (0, 1)$

$$\bar{v}_n(\alpha) = q_{\alpha, x}(\alpha) = x = \lim_{k \rightarrow \infty} \bar{v}_n(\alpha_k).$$

Now consider the case where  $\alpha = 0$ . Suppose that  $\lim_{k \rightarrow \infty} \bar{v}_n(\alpha_k) = x > b_n(0)$ . Then for any  $\epsilon > 0$

$$b_n(\epsilon) \geq \lim_{k \rightarrow \infty} \beta(\epsilon, q_{\alpha_k, \bar{v}_n(\alpha_k)}) = x > b_n(0),$$

which contradicts the continuity of  $b_n$ . In addition,  $\lim_{k \rightarrow \infty} \bar{v}_n(\alpha_k) \geq b_n(0)$  by rationality. As a result,  $\lim_{k \rightarrow \infty} \bar{v}_n(\alpha_k) = b_n(0) = \bar{v}_n(0)$ .

Lastly, we prove that  $\bar{v}_n(\cdot)$  is strictly increasing. Notice that it is weakly increasing because it is the supremum of a set of increasing functions. Now suppose towards contradiction that  $\bar{v}_n(\cdot)$  is not strictly increasing. There must exist  $0 < \alpha_1 < \alpha_2 < 1$  such that  $\bar{v}_n(\alpha_1) = \bar{v}_n(\alpha_2) = x$ . By Proposition 2, both  $q_{\alpha_1, x}$  and  $q_{\alpha_2, x}$  satisfies RNBNE overbidding and rationality. Notice that  $b_n(t) \geq \beta(t, q_{\alpha_1, x}) > \beta(t, q_{\alpha_2, x})$  if  $t \in [\alpha_2, x]$ . Because both  $b_n$  and  $\beta(\cdot, q_{\alpha_2, x})$  are continuous, there exists  $\epsilon > 0$  such that  $b_n(t) > \beta(t, q_{\alpha_2, x}) + \epsilon$  if  $t \in [\alpha_2, x]$ . Then  $\beta(t, q_{\alpha_2, x+\epsilon}) \leq \beta(t, q_{\alpha_2, x}) + \epsilon < b_n(t)$  if  $t \in [\alpha_2, x]$ . This implies that  $\bar{v}_n(\alpha_2) > x$ . Therefore,  $\bar{v}_n(\cdot)$  is strictly increasing.  $\square$

## A.2 Proof of Results in Section 3.2

*Proof of Proposition 4.* Notice that  $G_n(b) = F_n(s_n^{-1}(b))$ . Therefore,  $s_n(\cdot) = \rho_n(\cdot)$  implies that  $s_n(t) = \arg \max_b (t - b) F_n(s_n^{-1}(b))^{n-1}$ . Since the unique solution to this functional equation is  $\sigma_n(\cdot)$ ,  $s_n(\cdot) = \sigma_n(\cdot) = \rho_n(\cdot)$ . If  $s_n(\cdot) = \sigma_n(\cdot)$ , then it is the best response to  $F_n(\sigma_n^{-1}(b)) = G_n(b)$ , which implies  $s_n(\cdot) = \sigma_n(\cdot) = \rho_n(\cdot)$ . To see the second part, suppose that  $F_n$  is uniform on  $[0, 1]$  and  $s_n(t) = ct$  for some  $1 > c > (n-1)/n$ . One can show that  $\rho_n(t) = \sigma_n(t) = (n-1)t/n \neq ct$ .

If  $s_n(\cdot) \geq \rho_n(\cdot)$ , then  $\forall t, t \leq \rho_n^{-1}(s_n(t))$ , so  $t \leq s_n(t) + G_n(s_n(t)) / [(n-1)g_n(s_n(t))]$ . Notice that  $G_n(s_n(t)) = F_n(t)$  and  $g_n(s_n(t)) = f_n(t)/s'_n(t)$ . After some algebra, one can show that  $s_n(\cdot) \geq \rho_n(\cdot)$  is equivalent to

$$s'_n(t) \geq (n-1)(t - s_n(t)) \frac{f_n(t)}{F_n(t)}.$$

Because  $s_n(0) = \sigma_n(0)$ , this inequality implies that  $s_n(\cdot) \geq \sigma_n(\cdot)$ . One can see this through contradiction. Suppose there exists some  $t_1$  such that  $s_n(t_1) < \sigma_n(t_1)$ . One can always find  $t_2 < t_1$  such that  $s_n(t_2) = \sigma_n(t_2)$  and  $s_n(t) < \sigma_n(t)$  on  $(t_2, t_1]$ . This is because  $s_n(\cdot)$  is bounded and has a uniformly bounded first-order derivative by Assumption 1. By the first-order conditions,  $s'_n(t) > \sigma'_n(t)$  on  $(t_2, t_1]$ . This implies that  $s_n(t_2) < \sigma_n(t_2)$ , because  $s_n(t_1) < \sigma_n(t_1)$ . This is a contradiction.  $\square$

### A.3 Proof of Results in Section 5

*Proof.* We start with Proposition 12(1). First,  $\sup_b \left| \hat{G}_n(b) - G_n(b) \right| \rightarrow 0$  almost surely as  $N_n \rightarrow \infty$ . Because  $g_n(\cdot)$  is bounded away from 0 on its support, then almost surely  $\sup_{t \in [0,1]} \left| \hat{b}_n(\alpha) - b_n(\alpha) \right| \rightarrow 0$ , which is the second part of Proposition 12(1). In addition, because  $\hat{b}_n$  is increasing and  $\hat{b}_n^c(\alpha) = \hat{b}_n(\alpha)$  for  $\alpha = 0$  and  $i/N_n$  for all  $i > 2$ ,  $\sup_{t \in [0,1]} \left| \hat{b}_n^c(\alpha) - b_n(\alpha) \right| \rightarrow 0$  almost surely. Now notice that  $\tilde{\Theta}$  contains all piece-wise linear functions and  $\hat{b}_n^c(\cdot)$  is piece-wise linear. Then for any  $\alpha \in [0, 1)$ ,  $\hat{v}_n(\alpha) = \sup_{q \in \Theta_R(\hat{b}_n^c)} q(\alpha)$ , i.e. the supreme on  $\tilde{\Theta}$  agrees with the supreme on  $\Theta$ . To see this, just notice that in the proof of Proposition 2, we establish that the supremum on  $\Theta$  can be attained by the supremum of a sequence of functions that has the form (14). This sequence of functions are piece-wiselinear because  $\hat{b}_n^c$  is piece-wise linear. For every  $\alpha \in [0, 1)$ ,  $\sup_{q \in \Theta_R(b_n)} q(\alpha)$  is continuous in  $b_n$  under the sup norm. This implies that  $\hat{v}_n(\alpha) \rightarrow \bar{v}_n(\alpha)$  almost surely for every  $\alpha \in [0, 1)$ . Next, notice that  $\bar{v}_n(\cdot)$  is continuous and increasing on  $[0, 1 - \epsilon]$ . Then it is uniformly continuous on  $[0, 1 - \epsilon]$ . For any  $\delta > 0$ , we can find  $J$  points  $0 = t_1 < t_2 < \dots < t_J = 1 - \epsilon$  such that

$$\sup_{1 < j \leq J} |\bar{v}_n(t_j) - \bar{v}_n(t_{j-1})| < \delta.$$

In addition, notice that by the previous argument, almost surely

$$\lim_{N_n \rightarrow \infty} \sup_{1 < j \leq J} |\hat{v}_n(t_j) - \bar{v}_n(t_j)| = 0.$$

In addition,  $\hat{v}_n(t_j)$  is non-decreasing almost surely because for almost all realization, it is a supreme over a set of non-decreasing functions. Consequently,

$$\sup_{t \in [0, 1 - \epsilon]} |\hat{v}_n(t) - \bar{v}_n(t)| \leq 2 \sup_{1 < j \leq J} |\hat{v}_n(t_j) - \bar{v}_n(t_j)| + \sup_{1 < j \leq J} |\bar{v}_n(t_j) - \bar{v}_n(t_{j-1})|.$$

As a result,  $\lim_{N_n \rightarrow \infty} \sup_{t \in [0, 1 - \epsilon]} |\hat{v}_n(t) - \bar{v}_n(t)| < \delta$ . Because  $\delta$  can be arbitrarily small,

$$\lim_{N_n \rightarrow \infty} \sup_{t \in [0, 1 - \epsilon]} |\hat{v}_n(t) - \bar{v}_n(t)| = 0.$$

Then notice that

$$\hat{\pi}_n(r) = \int_{\alpha \geq \hat{v}_n^{-1}(r)} \left[ \hat{b}_n(\alpha, r) - c \right] d\alpha^n + c.$$

And  $\hat{b}_n(\alpha, r)$  converges to  $\bar{b}_n(\alpha, r)$  uniformly in  $\alpha$  almost surely and  $\hat{v}_n^{-1}(r)$  converges to  $\bar{v}_n^{-1}(r)$  almost surely because  $\bar{v}_n$  is strictly increasing. Therefore,  $\hat{\pi}_n(r) \rightarrow \bar{\pi}_n(r)$  almost surely. Similarly,  $\hat{\pi}_n(r) \rightarrow \bar{\pi}_n(r)$  almost surely for every  $r$ .  $\square$

## A.4 Additional Lemmas

**Lemma 2.** *Let  $\sigma_n^l(\cdot, r)$  be the RNBNE bid quantile function under valuation quantile  $v^l$  and reserve price  $r$  in  $n$  bidder auctions. If  $v^1(\cdot) \geq v^2(\cdot)$ , then  $\sigma_n^1(\cdot, r) \geq \sigma_n^2(\cdot, r)$ .*

*Proof.* Under RNBNE, the bid quantile function satisfies

$$\frac{\partial \sigma_n^l(\alpha, r)}{\partial \alpha} = (n-1) \frac{v^l(\alpha) - \sigma_n^l(\alpha, r)}{\alpha},$$

with the initial condition  $\sigma_n^l(\alpha, r) = r$  if  $\sigma^l(\alpha) = r$ . And if  $v^l(\alpha) < r$ ,  $\sigma_n^l(\alpha) = 0$ . Let  $v^1(\alpha_1) = r$  and  $v^2(\alpha_2) = r$ . Because  $v^1(\cdot) \geq v^2(\cdot)$ ,  $\alpha_1 \leq \alpha_2$ . In addition,  $\sigma_n^1(\alpha, r) \geq r \forall \alpha \geq \alpha_1$ . Therefore,  $\sigma_n^1(\cdot, r) \geq \sigma_n^2(\cdot, r) \forall \alpha \in [0, \alpha_2]$ . Next, we conclude the proof by showing the statement holds  $\forall \alpha \in [\alpha_2, 1]$  using a contradiction argument. Suppose there is some  $\alpha_3 \in [\alpha_2, 1]$  such that  $\sigma_n^1(\alpha_3, r) < \sigma_n^2(\alpha_3, r)$ . Because  $\sigma_n^1(\alpha_2, r) \geq r = \sigma_n^2(\alpha_2, r)$ , by the continuity of the bid function, there must be some  $\alpha_3 > \alpha_4 \geq \alpha_2$  such that  $\sigma_n^1(\alpha_4, r) = \sigma_n^2(\alpha_4, r)$  and  $\sigma_n^1(\alpha, r) < \sigma_n^2(\alpha, r)$  for all  $\alpha \in (\alpha_4, \alpha_3]$ . Then by the first order condition,  $\frac{\partial}{\partial \alpha} \sigma_n^1(\alpha, r) > \frac{\partial}{\partial \alpha} \sigma_n^2(\alpha, r)$ . This implies  $\sigma_n^1(\alpha_3, r) > \sigma_n^2(\alpha_3, r)$ , which is a contradiction.  $\square$

Next, we prove a general result on bid quantile dominance. Let  $v_n(\alpha)$  and  $v_{n,G}(\alpha)$  be two valuation quantile functions with  $v_n(0) = v_{n,G}(0) = 0$ . Let  $b_n(\alpha, r)$  and  $\beta_{n,G}(\alpha, r)$  be bid quantile functions, which are defined as follows. We set  $b_n(\alpha, r) = 0$  if  $v_n(\alpha) < r$  and elsewhere it solves the following differential equation with the initial condition  $b_n(v_n^{-1}(\alpha), r) = r$ :

$$\frac{\partial}{\partial \alpha} b_n(\alpha, r) = (n-1) \lambda_n(v_n(\alpha) - b_n(\alpha, r)) \gamma(\alpha) \mathcal{H}_n(\alpha)$$

where  $\gamma$  is some positive function and  $\lambda_n(0) = 0$ ,  $\lambda'_n \geq 1$  and  $\mathcal{H}_n \geq 1$ . The quantile function  $\beta_{n,G}$  is defined in the same way based on  $v_{n,G}$  and the differential equation,

$$\frac{\partial}{\partial \alpha} \beta_{n,G}(\alpha, r) = (n-1) (v_{n,G}(\alpha) - \beta_{n,G}(\alpha, r)) \gamma(\alpha).$$

**Lemma 3.** *If  $b_n(\alpha, 0) = \beta_{n,G}(\alpha, 0)$  for all  $\alpha \in [0, 1]$ , then  $b_n(\alpha, r) \leq \beta_{n,G}(\alpha, r)$  for all  $\alpha \in [0, 1]$  and  $r > 0$ .*

*Proof.* By assumption,

$$(n-1)\lambda_n(v_n(\alpha) - b_n(\alpha, 0))\mathcal{H}_n(\alpha)\gamma(\alpha) = (n-1)(v_{n,G}(\alpha) - b_n(\alpha, 0))\gamma(\alpha).$$

Because  $\mathcal{H}_n \geq 1$ ,  $\lambda'_n(x) \geq 1$ , we must have  $v_n(\alpha) \leq v_{n,G}(\alpha)$  for all  $\alpha$ . Now define  $\alpha_r$  and  $\bar{\alpha}_r$  through  $v_n(\alpha_r) = r$  and  $v_{n,G}(\bar{\alpha}_r) = r$ . Obviously,  $\bar{\alpha}_r \leq \alpha_r$ . Because  $b_n(\alpha, r) = 0$  if  $\alpha < \alpha_r$  and  $\beta_{n,G}(\alpha_r, r) \geq \beta_{n,G}(\bar{\alpha}_r, r) = r = b_n(\alpha_r, r)$ ,  $\beta_{n,G}(\alpha, r) \geq b_n(\alpha, r)$  for all  $\alpha \leq \alpha_r$ . We next show by contradiction that

$$\beta_{n,G}(\alpha, r) \geq b_n(\alpha, r), \quad \forall \alpha \in [\alpha_r, 1].$$

Suppose there is an  $\alpha_1 \in [\alpha_r, 1]$  such that  $\beta_{n,G}(\alpha_1, r) < b_n(\alpha_1, r)$ . Define

$$\bar{\alpha} = \sup \{ \alpha \in [\alpha_r, \alpha_1] : \beta_{n,G}(\alpha, r) \geq b_n(\alpha, r) \}.$$

By continuity of the bid function and the fact that  $\beta_{n,G}(\alpha_r, r) \geq b_n(\alpha_r, r)$ , we must have  $\beta_{n,G}(\bar{\alpha}, r) = b_n(\bar{\alpha}, r)$  and there exists a  $\Delta > 0$  such that for any  $\alpha \in (\bar{\alpha}, \bar{\alpha} + \Delta)$ ,  $\beta_{n,G}(\alpha, r) < b_n(\alpha, r)$ . This means that there exists at least some  $\alpha \in [\bar{\alpha}, \bar{\alpha} + \Delta]$  such that  $\partial\beta_{n,G}(\alpha, r)/\partial\alpha < \partial b_n(\alpha, r)/\partial\alpha$  or equivalently,

$$\lambda(v_n(\alpha) - b_n(\alpha, r))\mathcal{H}_n(\alpha) > v_{n,G}(\alpha) - \beta_{n,G}(\alpha, r).$$

Now notice that  $b_n(\alpha, r) - b_n(\alpha, 0) \geq 0$ ,  $\lambda'_n \geq 1$  and  $\mathcal{H}_n \geq 1$ ,

$$\begin{aligned} \lambda_n(v_n(\alpha) - b_n(\alpha, 0))\mathcal{H}_n(\alpha) &\geq \lambda_n(v_n(\alpha) - b_n(\alpha, r))\mathcal{H}_n(\alpha) + b_n(\alpha, r) - b_n(\alpha, 0) \\ &> v_{n,G}(\alpha) - \beta_{n,G}(\alpha, r) + b_n(\alpha, r) - b_n(\alpha, 0) \\ &> v_{n,G}(\alpha) - b_n(\alpha, 0). \end{aligned}$$

which violates the fact that

$$(n-1)\lambda_n(v_n(\alpha) - b_n(\alpha))\gamma(\alpha)\mathcal{H}_n(\alpha) = (n-1)[v_{n,G}(\alpha) - b_n(\alpha)]\gamma(\alpha)$$

for all  $\alpha \in (0, 1]$ . Thus,  $\beta_{n,G}(\alpha, r) \geq b_n(\alpha, r)$ ,  $\forall \alpha \in [\alpha_r, 1]$ .  $\square$

## B Counterexample for Proposition 4 with Heterogeneous Bidders

With heterogeneous bidding strategies, Proposition 4 no longer holds. In particular, it is no longer the case that overbidding compared to RNBR is necessarily a stronger restriction than overbidding compared to RNBNE.

To see this consider the following counter-example. Suppose that  $n = 2$  and the valuation distribution  $F_n$  is the standard uniform distribution. Hence,  $\sigma_n^{-1}(b) = 2b$ . There are two bidder types, distinguished by their bidding strategies  $s_{n,1}$  and  $s_{n,2}$ . The first type represents a share of  $1 - \epsilon$  of all bidders and the second type represents the remaining share  $\epsilon$ . First suppose that  $\epsilon = 0$  so there is no heterogeneity among bidders and that  $s_{n,1}^{-1}(b) = 2b - \frac{1}{3}b^3$ . Hence,  $G(b) = s_{n,1}^{-1}(b) = 2b - \frac{1}{3}b^3$  and  $g(b) = 2 - b^2$ . This implies that  $\rho_n^{-1}(b) = b + \frac{G(b)}{g(b)} = b + b\frac{2 - \frac{1}{3}b^2}{2 - b^2}$ . Because  $\frac{2 - \frac{1}{3}b^2}{2 - b^2} > 1$ ,  $\rho_n^{-1}(b) > 2b = \sigma_n^{-1}(b)$  for all  $b > 0$  or  $\rho_n(v) < \sigma_n(v)$  for all  $v \in (0, 1]$ . Therefore, RNBNE is more aggressive than RNBR in this case. Moreover,  $s_{n,1}^{-1}(b) = 2b - \frac{1}{3}b^3 < 2b = \sigma_n^{-1}(b)$  for  $b > 0$ , so  $\rho_n(v) < \sigma_n(v) < s_{n,1}(v)$  for all  $v \in (0, 1]$ .

Now we introduce a small amount of bidder heterogeneity into this example with  $\epsilon > 0$ . Let the new RNBR and RNBNE be  $\rho_n^\epsilon$  and  $\sigma_n^\epsilon$ . The introduction of the second bidder type has no effect on the RNBNE, i.e.  $\sigma_n^\epsilon = \sigma_n$ . If  $\epsilon$  is sufficiently small then  $\rho_n^\epsilon$  is almost entirely determined by bidder type 1 and therefore very close to  $\rho_n$ . Now let  $s_{n,2}(v) = \frac{\rho_n(v) + \sigma_n(v)}{2}$ . Hence, bidder type 2 bids more aggressively than RNBR, but less aggressively than RNBNE. If  $\epsilon$  is sufficiently small then  $\rho_n^\epsilon(v) < s_{n,2}(v) < \sigma_n^\epsilon(v) < s_{n,1}(v)$  for all  $v \in (0, 1]$ . Hence, in this example all bidder types overbid compared to RNBR, but not compared to RNBNE.

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