Finance and Economics Discussion Series
Divisions of Research & Statistics and Monetary Affairs
Federal Reserve Board, Washington, D.C.

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2019-033

Please cite this paper as:
Bianchi, Francesco, and Giovanni Nicolò (2019). "A Generalized Approach to Indeterminacy in Linear Rational Expectations Models," Finance and Economics Discussion Series 2019-033. Washington: Board of Governors of the Federal Reserve System, https://doi.org/10.17016/FEDS.2019.033.

A Generalized Approach to Indeterminacy in Linear Rational Expectations Models*

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29th April 2019

Abstract

We propose a novel approach to deal with the problem of indeterminacy in Linear Rational Expectations models. The method consists of augmenting the original state space with a set of auxiliary exogenous equations to provide the adequate number of explosive roots in presence of indeterminacy. The solution in this expanded state space, if it exists, is always determinate, and is identical to the indeterminate solution of the original model. The proposed approach accommodates determinacy and any degree of indeterminacy, and it can be implemented even when the boundaries of the determinacy region are unknown. Thus, the researcher can estimate the model using standard packages without restricting the estimates to the determinacy region. We apply our method to estimate the New-Keynesian model with rational bubbles by Galí (2017) over the period 1982:Q4 until 2007:Q3. We find that the data support the presence of two degrees of indeterminacy, implying that the central bank was not reacting strongly enough to the bubble component.

JEL classification: C19, C51, C62, C63.

Keywords: Indeterminacy, General Equilibrium, Solution method, Bayesian methods.

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*The views expressed in this paper are those of the authors and do not reflect the views of the Federal Reserve Board or the Federal Reserve System. We thank Jonas Arias, Jess Benhabib, Roger Farmer, François Geerolf, Frank Schorfheide and all participants at UCLA seminars, NBER Multiple Equilibria and Financial Crises Conference, CEPR-IMFS New Methods for Macroeconomic Modeling, Model Comparison and Policy Analysis Conference, Federal Reserve Bank of St. Louis, Society of Economic Dynamics, 12th Dynare Conference, 2017 NBER-NSF conference on Bayesian Inference in Econometrics and Statistics. Correspondence: Francesco Bianchi, Duke University, Department of Economics. Email: francesco.bianchi@duke.edu. Giovanni Nicolò, Board of Governors of the Federal Reserve System. Email: giovanni.nicolo@frb.gov.

1 Introduction

Sunspot shocks and multiple equilibria have been at the center of economic thinking at least since the seminal work of Cass and Shell (1983), Farmer and Guo (1994) and Farmer and Guo (1995). The zero lower bound has brought renovated interest to the problem of indeterminacy (Aruoba et al. (2018)). Furthermore, in many of the Linear Rational Expectation (LRE) models used to study the properties of the macroeconomy the possibility of multiple equilibria arises for some parameter values, but not for others. This paper proposes a novel approach to solve LRE models that easily accommodates both the case of determinacy and indeterminacy. As a result, the proposed methodology can be used to easily estimate a LRE model that could potentially be characterized by multiplicity of equilibria. Our approach is implementable even when the analytic conditions for determinacy or the degrees of indeterminacy are unknown. Importantly, the proposed method can be easily implemented to study indeterminacy in standard software packages, such as Dynare and Sims' (2001) code Gensys.

To understand how our approach works, it is useful to recall the conditions for determinacy as stated by Blanchard and Kahn (1980). Indeterminacy arises when the parameter values are such that the number of explosive roots is smaller than the number of non-predetermined variables. The key idea behind our methodology consists of augmenting the original model by appending additional autoregressive processes that can be used to provide the missing explosive roots. The innovations of these exogenous processes are assumed to be linear combinations of a subset of the forecast errors associated with the expectational variables of the model and a newly defined vector of sunspot shocks. When the Blanchard-Kahn condition for determinacy is satisfied, all the roots of the auxiliary autoregressive processes are assumed to be within the unit circle and the auxiliary process is irrelevant for the dynamics of the model. The law of motion for the endogenous variables is in this case equivalent to the solution obtained using standard solution algorithms (King and Watson (1998), Klein (2000), Sims (2001)). When the model is indeterminate, the appropriate number of appended autoregressive processes is assumed to be explosive. For example, if there are two degrees of indeterminacy, two of the auxiliary processes are assumed to be explosive. The solution that we obtain for the endogenous variables is equivalent to the one obtained with the methodology of Lubik and Schorfheide (2003) or, equivalently, Farmer et al. (2015).

Our methodology can be used with standard estimation packages such as Dynare. The solution or estimation under indeterminacy is not generally implementable in standard packages. Our method solves this problem by expanding the state space and making sure that in this expanded state space the conditions for determinacy always hold. Thus, our approach allows the researcher to solve and estimate a model under indeterminacy using standard software packages.

Our methodology also simplifies the common approach used to deal with indeterminacy. The common procedure requires the researcher to solve the model differently depending on the area of the parameter space that is being studied. Under indeterminacy, existing methods require to construct the solution ex-post following the seminal contribution of Lubik and Schorfheide (2003) or to rewrite the model based on the existing degree of indeterminacy (Farmer et al. (2015)). In itself, this is not an insurmountable task, but it implies that the researcher interested in a structural estimation of the model would need to write the estimation codes and not just the solution codes. Our proposed method only requires the researcher to augment the original system of equations to reflect the maximum degree of indeterminacy and can therefore be used with no modification of the solution approach. Finally, we show that our approach can facilitate the transition between the determinacy and indeterminacy regions of the parameter space. This method works because our auxiliary processes can be used to make more likely a draw that crosses the threshold of determinacy and to keep track of the distance from such threshold. This idea is particularly easy to implement when the threshold of the determinacy region is known.

Our work is related to the vast literature that studies the role of indeterminacy in explaining the evolution of the macroeconomy. Prominent examples in the monetary policy literature include the work of Clarida et al. (2000) and Kerr and King (1996), that study the possibility of multiple equilibria as a result of violations of the Taylor Principle in New-Keynesian (NK) models. Applying the methods developed in Lubik and Schorfheide (2003) to the canonical NK model, Lubik and Schorfheide (2004) test for indeterminacy in U.S. monetary policy. Using a calibrated small-scale model, Coibon and Gorodnichenko (2011) find that the reduction of the target inflation rate in the United States also played a key role in explaining the Great Moderation, and Arias et al. (2017) support this finding in the context of a medium-scale model *à la* Christiano et al. (2005). In a similar spirit, Arias (2013) studies the dynamic properties of medium-sized NK models with trend inflation. More recently, Aruoba et al. (2018) study inflation dynamics at the Zero Lower Bound (ZLB) and during an exit from the ZLB.

The paper closest to our is Farmer et al. (2015). As explained above, the main difference between the two approaches is that our method accommodates the case of both determinacy and indeterminacy while considering the same augmented system of equations. Instead, the method proposed by Farmer et al. (2015) requires us to rewrite the model based on the existing degree of indeterminacy. With respect to Lubik and Schorfheide (2003), the main novelty of our approach is to provide a unified approach to study determinacy and indeterminacy of different degrees.[1] Finally, we deliberately use Dynare in all the examples presented in this paper to show that our method can be combined with standard packages. However, our solution method can be

[1] Ascari et al. (forthcoming) allow for temporarily unstable paths, while we require all solutions to be stationary, in line with previous contributions in the literature.

combined with more sophisticated estimation techniques such as the ones developed in Herbst and Schorfheide (2015).

To show how to use our methodology in practice, we estimate the small-scale NK model of Galí (2017) using Bayesian techniques using U.S. data over the period 1982:Q4 until 2007:Q3. Galí's model extends a conventional NK model to allow for the existence of rational bubbles. An interesting aspect of the model is that it displays up to two degrees of indeterminacy for realistic parameter values. We find that the data support the version of the model with two degrees of indeterminacy, implying that the central bank was not reacting strongly enough to the bubble component.

The remainder of the paper is organized as follows. Section 2 builds the intuition by using a univariate example in the spirit of Lubik and Schorfheide (2004). Section 3 describes the methodology and shows that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). In Section 4, we provide guidance on how to properly implement our methodology, and suggestions on how it could be used to improve the efficiency of existing estimation algorithms. In Section 5, we apply our theoretical results to estimate NK model with rational bubbles of Galí (2017) using Bayesian techniques. We present our conclusions in Section 6.

2 Building the intuition

Before presenting the theoretical results of the paper, this section builds the intuition behind our approach by considering a univariate example similar to the one proposed in Lubik and Schorfheide (2004). While Section 2.1 explains our approach from an analytical perspective, Section 2.2 addresses questions which could arise at the time of its practical implementation.

2.1 A useful example

Consider a classical monetary model characterized by the Fisher equation

$$i_t = E_t(\pi_{t+1}) + r_t, \tag{1}$$

and the simple Taylor rule

$$i_t = \phi_\pi \pi_t, \tag{2}$$

where i_t denotes the nominal interest rate, π_t represents the inflation rate, and $\phi_\pi > 0$ is a parameter controlling the response of the nominal interest to inflation. We assume that the real interest rate r_t is given and described by a mean-zero Gaussian i.i.d. shock.[2] To properly specify the model, we also define the one-step ahead forecast error associated with the expectational variable, π_t, as

$$\eta_t \equiv \pi_t - E_{t-1}(\pi_t). \tag{3}$$

Combining (1) and (2), we obtain the univariate model

$$E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t. \tag{4}$$

First, we consider the case $\phi_\pi > 1$. Rewriting equation (4), it is clear that this case is associated with the determinate solution,

$$\begin{aligned} \pi_t &= \frac{1}{\phi_\pi} E_t(\pi_{t+1}) + \frac{1}{\phi_\pi} r_t \\ &= \frac{1}{\phi_\pi} r_t. \end{aligned} \tag{5}$$

where the last equality is obtained by solving the equation forward and recalling the assumptions on r_t. The strong response of the monetary authority to changes in inflation ($\phi_\pi > 1$) guarantees that inflation is pinned down as a function of the exogenous real interest r_t. From a technical perspective, when $\phi_\pi > 1$ the Blanchard-Kahn condition for uniqueness of a solution is satisfied: The number of explosive roots matches the number of expectational variables, that in this univariate case is one.

The second case corresponds to $\phi_\pi \leq 1$. The solution is obtained by combining (4) with (3), and it corresponds to *any* process that takes the following form

$$\pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \eta_t. \tag{6}$$

The previous equation also holds under determinacy, but in that case the central bank's behavior induces restrictions on the expectation error η_t. Instead, when the monetary authority does not respond aggressively enough to changes in inflation ($\phi_\pi \leq 1$), there are multiple solutions for the inflation rate, π_t, each indexed by the expectations that the representative agent holds about future inflation, η_t. Equivalently, the solution to the univariate model is indeterminate: The

[2] In the classical monetary model, the real interest rate results from the equilibrium in labor and goods market, and it depends on the technology shocks. We are considering an exogenous process for the technology shocks, and therefore we take the process for the real interest rate as given.

Blanchard-Kahn solution is not satisfied as there is no explosive root to match the number of expectational variables.

The simple model considered here can be solved pencil and paper. However, when considering richer models with multiple endogenous variables, indeterminacy represents a challenge from a methodological and computational perspective. Standard software packages such as Dynare do not allow for indeterminacy. Of course, a researcher could in principle code an estimation algorithm herself, following the methods outlined in Lubik and Schorfheide (2004). However, this approach requires a substantial amount of time and technical skills. The researcher would need to write a code that not only finds the solution, but also implements the estimation algorithm. Hence, the result is that in practice most of the papers simply rule out the possibility of indeterminacy, even if the model at hand could in principle allow for such a feature.

The problem that a researcher faces when solving a LRE model under indeterminacy using standard solution algorithms can be easily understood based on the example provided above. Under determinacy, the model already has a sufficient number of unstable roots to match the number of expectational variables. However, under indeterminacy, the model is missing one explosive root. Thus, we propose to augment the original state space of the model by appending an independent process which could be either stable or unstable.

The key insight consists of choosing this auxiliary processes in a way to deliver the correct solution. When the original model is determinate, the auxiliary process must be stationary so that also the augmented representation satisfies the Blanchard-Kahn condition. In this case, the auxiliary process represents a separate block that does not affect the law of motion of the model variables. When the model is indeterminate, the additional process should however be explosive so that the Blanchard-Kahn condition is satisfied for the augmented system, even if not for the original model. By choosing the auxiliary process in the appropriate way, the solution under determinacy in this expanded state space corresponds to the solution under indeterminacy under the original state space. In what follows, we apply this intuition to the example considered above. In Section 3, we show that the approach can be easily extended to richer models to accommodate any degree of indeterminacy.

Our methodology proposes to solve an augmented system of equations which can be dealt with by using standard solution algorithms such as Sims (2001) under both determinacy and indeterminacy. Consider the following augmented system

$$\begin{cases} E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t, \\ \omega_t = \left(\frac{1}{\alpha}\right) \omega_{t-1} - \nu_t + \eta_t, \end{cases} \tag{7}$$

where ω_t is an independent autoregressive process, $\alpha \in [0, 2]$ and ν_t is a newly defined mean-zero

Blanchard-Kahn condition in the augmented representation			
	Unstable Roots	B-K condition in augmented model (7)	Solution
Determinacy $\phi_\pi > 1$ in original model (4)			
$\frac{1}{\alpha} < 1$	1	Satisfied	$\left\{\pi_t = \frac{1}{\phi_\pi} r_t,\ \omega_t = \alpha\omega_{t-1} - \nu_t + \varepsilon_t\right\}$
$\frac{1}{\alpha} > 1$	2	Not satisfied	-
Indeterminacy $\phi_\pi \leq 1$ in original model (4)			
$\frac{1}{\alpha} < 1$	0	Not satisfied	-
$\frac{1}{\alpha} > 1$	1	Satisfied	$\{\pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \eta_t, \omega_t = 0\}$

Table 1: The table reports the regions of the parameter space for which the Blanchard-Kahn condition in the augmented representation is satisfied, even when the original model is indeterminate.

sunspot shock with standard deviation σ_ν.

Table 1 summarizes the intuition behind our approach. When the original LRE model in (4) is determinate, $\phi_\pi > 1$, the Blanchard-Kahn condition for the augmented representation in (7) is satisfied when $1/\alpha < 1$. Indeed, for $\phi_\pi > 1$ the original model has the same number of unstable roots as the number of expectational variables. Our methodology thus suggests to append a stable autoregressive process such that $1/\alpha < 1$. In this case, the method of Sims (2001) delivers the same solution for the endogenous variable π_t as in equation (5) and for the autoregressive process ω_t. Importantly, ω_t is an independent autoregressive process, and its dynamics do not impact the endogenous variable π_t.

Considering the case of indeterminacy (i.e. $\phi_\pi \leq 1$), the original model has one expectational variable, but no unstable root, thus violating the Blanchard-Kahn condition. By appending an explosive autoregressive process, the augmented representation that we propose satisfies the Blanchard-Kahn condition and delivers the same solution as the one resulting from the methodology of Lubik and Schorfheide (2003) or Farmer et al. (2015) described by equation (6). Moreover, stability imposes conditions such that ω_t is always equal to zero at any time t, and even in this case, the solution for the endogenous variable does not depend on the appended autoregressive process.

Summarizing, the choice of the coefficient $\frac{1}{\alpha}$ should be made as follows. For values of ϕ_π greater than 1, the Blanchard-Kahn condition for the augmented representation is satisfied for values of α greater than 1. Conversely, under indeterminacy (i.e. $\phi_\pi \leq 1$) the condition is satisfied when α is smaller than 1. The choice of parametrizing the auxiliary process with $1/\alpha$ instead of α induces a positive correlation between ϕ_π and α that facilitates the implementation of our method when estimating a model.

Finally, note that under both determinacy and indeterminacy, the exact value of $1/\alpha$ is irrelevant for the law of motion of π_t. Under determinacy, the auxiliary process ω_t is stationary, but its evolution does not affect the law of motion of the model variables. Under indeterminacy, ω_t is always equal to zero. Thus, the introduction of the auxiliary process does not affect the properties of the solution in the two cases. However, this process serves two important purposes: It provides the correct number of explosive roots under indeterminacy and creates a mapping between the sunspot shock and the expectation errors. As we will see in Section 3, this result can be generalized and applies to more complicated models with potentially multiple degrees of indeterminacy.

2.2 Choosing α

Before presenting detailed suggestions for the practical implementation of our method in Section 4, it is useful to provide the intuition for the choice of the parameter α in the context of the simple model presented above. First of all, from the discussion above, it should be clear that what matters is only if this parameter is smaller or larger than 1. Its exact value does not affect the solution for π_t. Thus, if a researcher wants to solve the model only under indeterminacy (determinacy), it can simply fix the parameter to a value smaller (larger) than 1. In this way, standard solution algorithms proceed to solve the model in the augmented state space only when the model under the original state space is characterized by indeterminacy (determinacy).

However, a researcher might want to allow for both determinacy and indeterminacy when solving the model. We consider the following two cases: (1) The analytic condition defining the region of determinacy are known; (2) The analytic condition defining the region of determinacy are unknown. We consider the two cases separately.

We first consider the case in which the researcher is able to analytically derive the condition which defines when the model is determinate or indeterminate. For the example considered in this section, this case corresponds to knowing that when $\phi_\pi \leq 1$ the model in (4) is indeterminate. We thus suggest to write the parameter α as a function of the parameter ϕ_π so that the augmented representation in (7) always satisfies the Blanchard-Kahn condition. In this example, we set $\alpha \equiv \phi_\pi$. When the original model is determinate ($\phi_\pi > 1$), the appended autoregressive process is stationary because $1/\alpha < 1$. If the original model is indeterminate ($\phi_\pi \leq 1$), the coefficient $1/\alpha$ is greater than 1 and the appended process is therefore explosive. Hence, when the region of determinacy is known, the researcher can easily choose α such that the augmented representation always delivers a solution under both determinacy and indeterminacy. Note that in this case α is a transformation of ϕ_π and effectively no auxiliary extra parameters are introduced.

There are however instances in which the researcher does not know the exact properties of the determinacy region. In this case, the researcher can start with an arbitrary value of α for a given sets of parameters θ. Suppose that the researcher starts with a value less than 1 and finds that the model is indeterminate for the given set of parameters θ. Then, the researcher can just change α to a value larger than 1, for example $\alpha' = 1/\alpha$. A similar logic applies to the case with multiple degrees of indeterminacy that we discuss below: If the solution algorithm returns a solution with m degrees of indeterminacy, m explosive auxiliary processes are necessary.

3 Methodology

We now present the main contribution of the paper generalizing the intuition provided above to a multivariate model with potentially multiple degrees of indeterminacy. Given the general class of LRE models described in Sims (2001), this paper proposes an augmented representation which embeds the solution for the model under both determinacy and indeterminacy. In particular, the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) or equivalently Farmer et al. (2015). In the following, we generalize the intuition built in the previous section. Consider the following LRE model

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{8}$$

where $X_t \in R^k$ is a vector of endogenous variables, $\varepsilon_t \in R^\ell$ is a vector of exogenous shocks, $\eta_t \in R^p$ collects the p one-step ahead forecast errors for the expectational variables of the system and $\theta \in \Theta$ is a vector of parameters. The matrices Γ_0 and Γ_1 are of dimension $k \times k$, possibly singular, and the matrices Ψ and Π are of dimension $k \times \ell$ and $k \times p$, respectively. Also, we assume

$$E_{t-1}(\varepsilon_t) = 0, \quad \text{and} \quad E_{t-1}(\eta_t) = 0.$$

We also define the $\ell \times \ell$ matrix $\Omega_{\varepsilon\varepsilon}$,

$$\Omega_{\varepsilon\varepsilon} \equiv E_{t-1}(\varepsilon_t\varepsilon_t'),$$

which represents the covariance matrix of the exogenous shocks.

Consider a model whose maximum degree of indeterminacy is denoted by m.[3] The proposed methodology appends to the original LRE model in (8) the following system of m equations

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \frac{1}{\alpha_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{1}{\alpha_m} \end{bmatrix} \tag{9}$$

where the vector $\eta_{f,t}$ is a subset of the endogenous shocks and the vectors $\{\omega_t, \nu_t, \eta_{f,t}\}$ are of dimension $m \times 1$. The equations in (9) are autoregressive processes whose innovations are linear combinations of a vector of newly defined sunspot shocks, ν_t, and a subset of forecast errors, $\eta_{f,t}$, where $E_{t-1}(\nu_t) = E_{t-1}(\eta_{f,t}) = 0$. As we will show below, the choice of which expectational errors to include in (9) does not affect the solution.

The intuition behind the proposed methodology works as in the example considered in the previous section. Let $m^*(\theta)$ denote the actual degree of indeterminacy associated with the parameter vector θ. Under indeterminacy the Blanchard-Kahn condition for the original LRE model in (8) is not satisfied. Given that the system is characterized by $m^*(\theta)$ degrees of indeterminacy, it is necessary to introduce $m^*(\theta)$ explosive roots to solve the model using standard solution algorithms. In this case, $m^*(\theta)$ of the diagonal elements of the matrix Φ are assumed to be outside the unit circle (in absolute value), and the augmented representation is therefore determinate because the Blanchard-Kahn condition is now satisfied. On the other hand, under determinacy the (absolute value of the) diagonal elements of the matrix Φ are assumed to be all inside the unit circle, as the number of explosive roots of the original LRE model in (8) already equals the number of expectational variables in the model ($m^*(\theta) = 0$). Also, in this case the augmented representation is determinate due to the stability of the appended auxiliary processes. Importantly, as shown for the univariate example in Section 2, the block structure of the proposed methodology guarantees that the autoregressive process, ω_t, never affects the solution for the endogenous variables, X_t.

Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)'$ and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the system in (8) and (9) can be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{10}$$

where

[3] Denoting by n the minimum number of unstable roots of a LRE model and p the number of one-step ahead forecast errors, the maximum degrees of indeterminacy are defined as $m \equiv p - n$. When the minimum number of unstable roots of a model is unknown, then m coincides with number of expectational variables p. This represents the maximum degree of indeterminacy in any model with p expectational variables.

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1(\theta) & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n(\theta) & \Pi_f(\theta) \\ 0 & -\mathbf{I} \end{bmatrix},$$

and without loss of generality the matrix Π in (8) is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$, where the matrices Π_n and Π_f are respectively of dimension $k \times (p-m)$ and $k \times m$.[4]

Section 3.1 and 3.2 show that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). In order to simplify the exposition, when analyzing the case of indeterminacy we assume, without loss of generality, $m^*(\theta) = m$. As it will become clear, the case of $m^*(\theta) < m$ is a special case of what we present below.

3.1 Equivalence under determinacy

This section considers the case in which the original LRE is determinate, and shows the equivalence of the solution obtained using the proposed augmented representation with the one from the standard solution method described in Sims (2001).

3.1.1 Canonical solution

Consider the LRE model in (8) and reported in the following equation

$$\underset{k\times k}{\Gamma_0}\underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\underset{k\times 1}{X_{t-1}} + \underset{k\times l}{\Psi}\underset{l\times 1}{\varepsilon_t} + \underset{k\times p}{\Pi}\underset{p\times 1}{\eta_t}. \tag{11}$$

The method described in Sims (2001) delivers a solution, if it exists, by following four steps. First, Sims (2001) shows how to write the model in the form

$$SZ'X_t = TZ'X_{t-1} + Q\Psi\varepsilon_t + Q\Pi\eta_t, \tag{12}$$

where $\Gamma_0 = Q'SZ'$ and $\Gamma_1 = Q'TZ'$ result from the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$, and the $k \times k$ matrices Q and Z are orthonormal, upper triangular and possibly complex. Also, the diagonal

[4] Suppose that $\underset{k\times 3}{\Pi} \equiv \begin{bmatrix} \underset{k\times 1}{\Pi_1} & \underset{k\times 1}{\Pi_2} & \underset{k\times 1}{\Pi_3} \end{bmatrix}$. The proposed augmented representation would therefore allow for the following three possible alternatives, $\hat{\Pi}_1 \equiv \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ 0 & 0 & -1 \end{bmatrix}$, $\hat{\Pi}_2 \equiv \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ 0 & -1 & 0 \end{bmatrix}$ and $\hat{\Pi}_3 \equiv \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ -1 & 0 & 0 \end{bmatrix}$. In Appendix B, we show with an analytic example that the alternative representations have a unique mapping that ensures the equivalence among each of them.

elements of S and T contain the generalized eigenvalues of $\{\Gamma_0, \Gamma_1\}$.

Second, given that the QZ decomposition is not unique, Sims' algorithm chooses a decomposition that orders the equations so that the absolute values of the ratios of the generalized eigenvalues are placed in an increasing order, that is

$$|t_{jj}| \,/\, |s_{jj}| \geq |t_{ii}| \,/\, |s_{ii}| \quad \text{for } j > i.$$

The algorithm then partitions the matrices S, T, Q and Z as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \qquad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad Z' = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where the first block corresponds to the system of equations for which $|t_{jj}| \,/\, |s_{jj}| \leq 1$ and the second block groups the equations which are characterized by explosive roots, $|t_{jj}| \,/\, |s_{jj}| > 1$.

The third step imposes conditions on the second, explosive block to guarantee the existence of at least one bounded solution. Defining the transformed variables

$$\xi_t \equiv Z' X_t = \begin{bmatrix} \underset{(k-n)\times 1}{\xi_{1,t}} \\ \underset{n\times 1}{\xi_{2,t}} \end{bmatrix},$$

where n is the number of explosive roots, and the transformed parameters

$$\widetilde{\Psi} \equiv Q'\Psi, \quad \text{and} \quad \widetilde{\Pi} \equiv Q'\Pi,$$

the second block can be written as

$$\xi_{2,t} = S_{22}^{-1} T_{22} \xi_{2,t-1} + S_{22}^{-1} (\widetilde{\Psi}_2 \varepsilon_t + \widetilde{\Pi}_2 \eta_t).$$

As this system of equations contains the explosive roots of the original system, then a bounded solution, if it exists, will set

$$\underset{n\times 1}{\xi_{2,0}} = 0 \tag{13}$$

$$\underset{n\times\ell}{\widetilde{\Psi}_2} \underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{\widetilde{\Pi}_2} \underset{p\times 1}{\eta_t} = 0, \tag{14}$$

where n also denotes the number of equations in (14). A necessary condition for the existence of a solution requires that the number of unstable roots (n) equals the number of expectational variables (p). In this section, we are considering the solution under determinacy, and this guarantees that there are no degrees of indeterminacy $m^*(\theta) = 0$. The sufficient condition then requires that the columns of the matrix $\widetilde{\Pi}_2$ are linearly independent so that there is at least one bounded solution. In that case, the matrix $\widetilde{\Pi}_2$ is a square, non-singular matrix and equation (14) imposes linear restrictions on the forecast errors, η_t, as a function of the fundamental shocks, ε_t,

$$\eta_t = -\widetilde{\Pi}_2^{-1}\widetilde{\Psi}_2\varepsilon_t. \tag{15}$$

The fourth and last step finds the solution for the endogenous variables, X_t, by combining the restrictions in (13) and (15) with the system of stable equations in the first block,

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t) \\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\left(\widetilde{\Psi}_1 - \widetilde{\Pi}_1\widetilde{\Pi}_2^{-1}\widetilde{\Psi}_2\right)\varepsilon_t \end{aligned} \tag{16}$$

Using the algorithm by Sims (2001), we can describe the solution under determinacy of the LRE model in (11) with equations (13), (15), and (16).

3.1.2 Augmented representation

We now consider the methodology proposed in this paper, and we augment the LRE model in (11) with the following system of m equations

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \frac{1}{\alpha_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{1}{\alpha_m} \end{bmatrix}$$

where Φ is a $m \times m$ diagonal matrix. As the original model in (11) is determinate, then we assume that all the diagonal elements α_i belong to the interval $[1, 2]$. Therefore, we are appending a system of stable equations, and we show that the solution for the endogenous variables, X_t, is equivalent to the one found in Section 3.1.1. Defining the augmented vector of endogenous variables, $\hat{X}_t \equiv (X_t, \omega_t)'$ and the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the representation that we propose takes the form

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{17}$$

where

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix},$$

and without loss of generality the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$, where the matrices Π_n and Π_f are respectively of dimension $k \times (p-m)$ and $k \times m$.

We can find a solution to the augmented representation in (17) by using Sims' algorithm. Similarly to the previous section, we follow the four steps which describe the algorithm. First, the solution algorithm performs the QZ decomposition of the matrices $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and the augmented representation takes the form

$$\hat{S}\hat{Z}'\hat{X}_t = \hat{T}\hat{Z}'\hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{18}$$

where $\hat{\Gamma}_0 = \hat{Q}'\hat{S}\hat{Z}'$ and $\hat{\Gamma}_1 = \hat{Q}'\hat{T}\hat{Z}'$ result from the QZ decomposition of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$, and

$$\hat{S} = \begin{bmatrix} S_{11} & \mathbf{0} & S_{12} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_{22} \end{bmatrix}, \qquad \hat{T} = \begin{bmatrix} T_{11} & \mathbf{0} & T_{12} \\ \mathbf{0} & \Phi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T_{22} \end{bmatrix}, \qquad \hat{Z}^T = \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Z_2 & \mathbf{0} \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Q_2 & \mathbf{0} \end{bmatrix}.$$

Importantly, note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z, Q\}$ found in the previous section using the canonical solution method.

Second, the algorithm chooses a QZ decomposition which groups the equations in a stable and an explosive block. Because this section assumes that the original model is determinate and that the diagonal elements of the matrix Φ are within the unit circle, the explosive block corresponds to the third system of equations in (18) which is characterized by explosive roots. Recalling the definition of the matrices $\hat{\Psi}$ and $\hat{\Pi}$, the system of equations in the third block is

$$\xi_{2,t} = S_{22}^{-1}T_{22}\xi_{2,t-1} + S_{22}^{-1}(\widetilde{\Psi}_2\varepsilon_t + \widetilde{\Pi}_2\eta_t). \tag{19}$$

The third step imposes conditions to guarantee the existence of a bounded solution. However, the explosive block in (19) is identical to the system of equations found in the previous section. Therefore, the algorithm imposes the same restrictions to guarantee the existence of a bounded solution, that is

$$\xi_{2,0} = 0 \tag{20}$$

and as found earlier

$$\eta_t = -\widetilde{\Pi}_2^{-1}\widetilde{\Psi}_2\varepsilon_t. \tag{21}$$

Finally, the last step combines these restrictions with the system of equations in the stable block which corresponds to the first and second systems of equations in (18),

$$\xi_{1,t} = S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\left(\widetilde{\Psi}_1 - \widetilde{\Pi}_1\widetilde{\Pi}_2^{-1}\widetilde{\Psi}_2\right)\varepsilon_t, \tag{22}$$

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}. \tag{23}$$

Recalling that $\xi_t \equiv Z'X_t$, the solution in (20)∼(23) obtained for the augmented representation of the LRE model delivers the same solution for the endogenous variables of interest, X_t, found in the previous section and defined in equations (13), (15), and (16).

Two remarks should be made when comparing the two solutions. First, as shown in (21), the forecast errors are only a function of the exogenous shocks ε_t, and *not* of the newly defined sunspot shocks, ν_t. It is therefore clear that the endogenous variables, X_t, of the original LRE model do not respond to sunspot shocks either, as expected under determinacy. Second, (22) and (23) indicate that under determinacy the appended system of equations constitutes a separate block, which does not affect the dynamics of the endogenous variables, X_t. Thus, the likelihood associated with a vector of observables Z_t that represents a linear transformation of the variables in X_t is invariant with respect to the method used to compute the solution. This statement holds because the latent processes do not affect Z_t.

3.2 Equivalence under indeterminacy

This section shows the equivalence of the solutions obtained for a LRE model under indeterminacy using the proposed augmented representation and the methodology of Lubik and Schorfheide (2003, 2004).

3.2.1 Lubik and Schorfheide (2003)

As in Section 3.1, we consider the LRE model in (11) and reported below as (24)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\varepsilon_t + \Pi\eta_t. \tag{24}$$

In this section we assume that the model is indeterminate, and we present the method used by Lubik and Schorfheide (2003). The authors implement the first two steps of the algorithm by Sims (2001) and described in Section 3.1.1.[5] They proceed by first applying the QZ decomposition to the LRE model in (24) and then ordering the resulting system of equations in a stable and

[5] It is relevant to mention that in this section the matrices obtained from the QZ decomposition and the ordering of the equations into a stable and an explosive block differ from those in Section 3.1 both in terms of their dimensionality and the elements constituting them. However, we opted to use the same notation for simplicity.

an explosive block as defined in equation (12). However, their approach differs in the third step when the algorithm imposes restrictions to guarantee the existence of a bounded solution. In particular, the restrictions in (13) and (14) reported below as (25) and (26) require that

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{25}$$

$$\underset{n\times\ell}{\widetilde{\Psi}_2}\,\underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{\widetilde{\Pi}_2}\,\underset{p\times 1}{\eta_t} = 0. \tag{26}$$

Nevertheless, it is clear that the system of equation in (26) is indeterminate as the number of forecast errors exceeds the number of explosive roots ($p > n$). Equivalently, there are less equations (n) than the number of variables to solve for (p). To characterize the full set of solutions to equation (26), Lubik and Schorfheide (2003) decompose the matrix $\widetilde{\Pi}_2$ using the following singular value decomposition

$$\underset{n\times p}{\widetilde{\Pi}_2} \equiv \underset{n\times n}{U}\left[\begin{array}{cc} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{array}\right]\underset{p\times p}{V'},$$

where m represents the degrees of indeterminacy. Given the partition $\underset{p\times p}{V} \equiv \left[\begin{array}{cc} \underset{p\times n}{V_1} & \underset{p\times m}{V_2} \end{array}\right]$, equation (26) can be written as

$$\underset{n\times n}{D_{11}^{-1}}\,\underset{n\times n}{U'}\,\underset{n\times\ell}{\widetilde{\Psi}_2}\,\underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{V_1'}\,\underset{p\times 1}{\eta_t} = 0. \tag{27}$$

Given that the system is indeterminate, Lubik and Schorfheide (2003) append additional m equations,

$$\underset{m\times\ell}{\widetilde{M}}\,\underset{\ell\times 1}{\varepsilon_t} + \underset{m\times m}{M_\zeta}\,\underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2'}\,\underset{p\times 1}{\eta_t}. \tag{28}$$

The $m \times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero, covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamental shocks, ε_t, that is

$$E\left[\zeta_t\right] = 0, \quad E\left[\zeta_t \varepsilon_t'\right] = 0, \quad E\left[\zeta_t \zeta_t'\right] = \Omega_{\zeta\zeta}.$$

The matrix $\widetilde{M}$ captures the correlation of the forecast errors, η_t, with fundamentals, ε_t, and Lubik and Schorfheide (2003) choose the normalization $M_\zeta = I_m$. The reason to append the system of equations in (28) to the equations in (27) is to exploit the properties of the orthonormal matrix V. Premultiplying the system by the matrix V and recalling that $V * V' = I$, the expectational errors can be written as a function of the fundamental shocks, ε_t, and the sunspot shocks, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(-\underset{p\times n}{V_1} \underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1'} \underset{n\times \ell}{\widetilde{\Psi}_2} + \underset{p\times m}{V_2} \underset{m\times \ell}{\widetilde{M}} \right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}.$$

More compactly,

$$\underset{p\times 1}{\eta_t} = \left(\underset{p\times n}{V_1} \underset{n\times \ell}{N} + \underset{p\times m}{V_2} \underset{m\times \ell}{\widetilde{M}} \right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}, \tag{29}$$

where

$$\underset{n\times \ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1'} \underset{n\times \ell}{\widetilde{\Psi}_2}.$$

is a function of the parameters of the model. Given the restriction in (25) and (29), the fourth step in the algorithm combines these equations with the system of stable equations in the first block as in Section 3.1.1,

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} (\widetilde{\Psi}_1 \varepsilon_t + \widetilde{\Pi}_1 \eta_t) \\ &= S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} \left(\widetilde{\Psi}_1 + \widetilde{\Pi}_1 V_1 N + \widetilde{\Pi}_1 V_2 \widetilde{M} \right) \varepsilon_t + S_{11}^{-1} \left(\widetilde{\Pi}_1 V_2 \right) \zeta_t. \end{aligned} \tag{30}$$

Using the method in Lubik and Schorfheide (2003), we can describe the solution for the original LRE model under indeterminacy with equations (25), (29) and (30).

3.2.2 Augmented representation

We now consider the augmented representation as in (17) and reported below as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t, \tag{31}$$

where $\hat{X}_t \equiv (X_t, \omega_t)'$, $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$ and

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix}.$$

where the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$ without loss of generality.

The novelty of our approach is that, given our representation, we can easily obtain the solution by using Sims' algorithm even when the original LRE is assumed to be indeterminate. It is enough to assume that the auxiliary processes ω_t are characterized by explosive roots, or equivalently that the diagonal elements of the matrix Φ are outside the unit circle. This approach guarantees that the Blanchard-Kahn condition for the augmented representation is satisfied and, given the analytic form that we propose for the auxiliary processes, we show that the solution for the endogenous variables of interest, X_t, is equivalent to the method of Lubik and Schorfheide (2003).

To show this result, we simply apply the four steps of the algorithm described in Sims (2001) to the proposed augmented representation. First, the QZ decomposition of (31) takes the form

$$\hat{S}\hat{Z}'\hat{X}_t = \hat{T}\hat{Z}'\hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{32}$$

where $\hat{\Gamma}_0 = \hat{Q}'\hat{S}\hat{Z}'$ and $\hat{\Gamma}_1 = \hat{Q}'\hat{T}\hat{Z}'$ result from the QZ decomposition[6] of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and

$$\hat{S} = \begin{bmatrix} S_{11} & S_{12} & \mathbf{0} \\ \mathbf{0} & S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{T} = \begin{bmatrix} T_{11} & T_{12} & \mathbf{0} \\ \mathbf{0} & T_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{Z}^T = \begin{bmatrix} Z_1 & \mathbf{0} \\ Z_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{33}$$

Note that in the expression above the auxiliary matrix Φ is in the lower (explosive) block because of our simplifying assumption that $m^*(\theta) = m$. When $m^*(\theta) < m$, part of the matrix Φ would belong in the stable block. As mentioned above, we made this simplifying assumption without loss of generality and only to simplify the exposition.

Second, the QZ decomposition chosen by the algorithm groups the explosive dynamics of the model in the second and third system of equations in (32), which are reported below as (34)

$$\begin{bmatrix} S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \xi_2 \\ \omega_t \end{bmatrix} = \begin{bmatrix} T_{22} & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix} \begin{bmatrix} \xi_{2,t-1} \\ \omega_{t-1} \end{bmatrix} + \begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t \right). \tag{34}$$

In the third step, the following restrictions are imposed,

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{35}$$

$$\underset{m\times 1}{\omega_0} = 0, \tag{36}$$

$$\begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t \right) = 0. \tag{37}$$

[6] Note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}^T, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z^T, Q\}$ found from the QZ decomposition using the methodology of Lubik and Schorfheide (2003).

Recalling the definition of $\hat{\Psi}$ and $\hat{\Pi}$ in (31), then equation (37) can be written as

$$\underbrace{\begin{bmatrix} \widetilde{\Psi}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{p\times(\ell+m)} \underset{(\ell+m)\times 1}{\hat{\varepsilon}_t} + \underbrace{\begin{bmatrix} \widetilde{\Pi}_{n,2} & \widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}}_{p\times p} \underset{p\times 1}{\eta_t} = 0, \tag{38}$$

where $\widetilde{\Psi} \equiv Q'\Psi$ and $\widetilde{\Pi} \equiv Q'\Pi$. Equation (38) shows transparently how the explosive auxiliary process that we append in our augmented representation helps to solve the original LRE model under indeterminacy. The system of equations in (38) is determinate, as the number of equations defined by the explosive roots of the system equals the number of expectational errors of the model. Thus, the necessary condition for the existence of a bounded solution for the augmented representation is satisfied. Assuming that the columns of the matrix associated with the vector of non-fundamental shocks, η_t, are linearly independent, we can impose linear restrictions on the forecast errors as a function of the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$,

$$\eta_t = -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 & \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \hat{\varepsilon}_t.$$

More compactly,

$$\eta_t = C_1\varepsilon_t + C_2\nu_t, \tag{39}$$

where $C_1 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 \\ \mathbf{0} \end{bmatrix}$ and $C_2 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ -\mathbf{I} \end{bmatrix}$ are a function of the structural parameters of the model.

The last step of Sims' algorithm combines the restrictions in (35), (36) and (39) with the stationary block derived from the QZ decomposition in (32),

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t) \\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\left(\widetilde{\Psi}_1 + \widetilde{\Pi}_1 C_1\right)\varepsilon_t + S_{11}^{-1}\left(\widetilde{\Pi}_1 C_2\right)\nu_t. \end{aligned} \tag{40}$$

3.2.3 Indeterminate equilibria and equivalent characterizations

The indeterminate equilibria found using the methodology of Lubik and Schorfheide (2003) are parametrized by two sets of parameters. The first set is defined by $\theta_1 \in \Theta_1$, where $\theta_1 \equiv vec(\Gamma_0, \Gamma_1, \Psi, \Omega_{\varepsilon\varepsilon})'$ is a vector of structural parameters of the model as well as the covariance matrix of the exogenous shocks. The second set corresponds to $\theta_2 \in \Theta_2$, where $\theta_2 \equiv vec\left(\Omega_{\zeta\zeta}, \widetilde{M}\right)'$ is a parameter vector related to the additional equations introduced in

(28) and reported below as (41),

$$\underset{m\times\ell}{\widetilde{M}}\,\underset{\ell\times 1}{\varepsilon_t} + \underset{m\times m}{M_\zeta}\,\underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2'}\,\underset{p\times 1}{\eta_t}\,. \tag{41}$$

Given the normalization $M_\zeta = I$ chosen by Lubik and Schorfheide (2004), equation (41) introduces $m \times (m+1)/2$ parameters associated with the covariance matrix of the sunspot shocks, $\Omega_{\zeta\zeta}$, and additional $m \times \ell$ parameters of the matrix $\widetilde{M}$ that is related to the covariances between η_t and ε_t. In Appendix A, we show how the normalization chosen by Lubik and Schorfheide (2004) maps one-to-one into a specific covariance matrix for the exogenous shocks under the methodology proposed in this paper.

The characterization of a Lubik-Schorfheide equilibrium is a vector $\theta^{LS} \in \Theta^{LS}$, where Θ^{LS} is defined as

$$\Theta^{LS} \equiv \{\Theta_1, \Theta_2\}\,.$$

Similarly, the full characterization of the solutions under indeterminacy using the proposed augmented representation is parametrized by the set of parameters $\theta_1 \in \Theta_1$ common between the two methodologies, and the set of additional parameters $\theta_3 \in \Theta_3$, where $\theta_3 \equiv vec(\Omega_{\nu\nu}, \Omega_{\nu\varepsilon})'$. Using our approach, we also introduce $m \times (m+1)/2$ parameters associated with the covariance matrix of the sunspot shocks, $\Omega_{\nu\nu}$, and $m \times \ell$ parameters of the covariances, $\Omega_{\nu\varepsilon}$, between the sunspot shock ν_t and the exogenous shocks ε_t. A Bianchi-Nicolò equilibrium is characterized by a parameter vector $\theta^{BN} \in \Theta^{BN}$, where Θ^{BN} is defined as

$$\Theta^{BN} \equiv \{\Theta_1, \Theta_3\}\,.$$

The following theorem establishes the equivalence between the characterizations of indeterminate equilibria obtained by using the methodology in Lubik and Schorfheide (2003) and the proposed augmented representation.

Theorem 1 *Let θ^{LS} and θ^{BN} be two alternative parametrizations of an indeterminate equilibrium of the model*

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\varepsilon_t + \Pi\eta_t.$$

For every BN equilibrium, parametrized by θ^{BN}, there exists a unique matrix $\widetilde{M}$ and a unique matrix $\Omega_{\zeta\zeta}$ such that $\theta_2 = vec(\Omega_{\zeta\zeta}, \widetilde{M})'$, and $\{\theta_1, \theta_2\} \in \Theta^{LS}$ defines an equivalent LS equilibrium. Conversely, for every LS equilibrium, parametrized by θ^{LS}, there is a unique matrix $\Omega_{\nu\nu}$ and a unique covariance matrix $\Omega_{\nu\varepsilon}$ such that $\theta_3 = vec(\Omega_{\nu\nu}, \Omega_{\nu\varepsilon})'$, and $\{\theta_1, \theta_3\} \in \Theta^{BN}$ defines an equivalent BN equilibrium.

Proof. See Appendix A. ■

In the paper Farmer et al. (2015), the authors also show that their characterization of indeterminate equilibria is equivalent to Lubik and Schorfheide (2003). Therefore, the following corollary holds.

Corollary 2 *Given a parametrization θ^{BN} of a Bianchi-Nicolò indeterminate equilibrium, there exists a unique mapping into the parametrization of an indeterminate equilibrium for Farmer et al. (2015), and vice-versa.*

Moreover, the following two considerations support Corollary 3 below, which describes a relevant result on the likelihood function of the augmented representation. First, as emphasized in this section, the solution of the model in the augmented state space has a block structure which ensures that the evolution of the endogenous variables in X_t is not a function of the autoregressive processes, ω_t. Second, note that the appended autoregressive processes in ω_t only serves the purpose of providing the necessary explosive roots under indeterminacy and creating a mapping from the sunspot shocks to the expectational errors. These auxiliary processes are not mapped into the observable variables through the measurement equation. These two considerations imply that the parameters of the matrix Φ introduced with the augmented representation are not identified within certain parameter region. The algorithm only requires them to be inside or outside the unit circle. Corollary 3 then follows.[7]

Corollary 3 *Conditional on the existence of a solution, the likelihood function associated with the newly defined vector of endogenous variables, $\hat{X}_t$, does not depend on the additional parameters included in the augmented representation, Φ, and is equivalent to the likelihood function associated with the endogenous variables, X_t.*

While Section 3.1 shows that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, Theorem 1 proves that under indeterminacy the solutions of our methodology are equivalent to those obtained using Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). This theoretical result is crucial for the application of our methodology to the New-Keynesian (NK) model with rational bubbles of Galí (2017) in Section 5.

[7] Notice that Corollary 3 holds when the augmented representation has a unique solution. This happens in two cases. First, values of the structural parameters θ which guarantee determinacy in the original LRE model should be combined with values for α_i in the matrix Φ whose absolute value lies within the unit circle. Second, values of the structural parameters θ for which the original model is indeterminate should be combined with (absolute) values of α_i outside the unit circle.

4 Estimation

In this section, we present some suggestions for the practical implementation of our method with an emphasis on the use of standard software packages such as Dynare. We consider both the model and the data that Lubik and Schorfheide (2004) use to test for indeterminacy in U.S. monetary policy, as it is possible to derive analytically the boundary of the determinacy region and to estimate the model under determinacy and indeterminacy. The model is described by equations (42)∼(47) below and consists of a dynamic IS curve,

$$y_t = E_t\left(y_{t+1}\right) - \tau\left(i_t - E_t\left(\pi_{t+1}\right)\right) + g_t, \tag{42}$$

a NK Phillips curve,

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa\left(y_t - z_t\right), \tag{43}$$

and a Taylor rule,

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(\psi_1 \pi_t + \psi_2\left(y_t - z_t\right)\right) + \varepsilon_{i,t}. \tag{44}$$

The demand shock, g_t, and the supply shock, z_t, follow univariate AR(1) processes

$$\begin{aligned} g_t &= \rho_g g_{t-1} + \varepsilon_{g,t}, & (45)\\ z_t &= \rho_z z_{t-1} + \varepsilon_{z,t}, & (46)\end{aligned}$$

where the standard deviations of the fundamental shocks $\varepsilon_{g,t}$, $\varepsilon_{z,t}$ and $\varepsilon_{i,t}$ are denoted by σ_g, σ_z and σ_i, respectively. As in Lubik and Schorfheide (2004), we allow for the correlation between demand and supply shocks, ρ_{gz}, to be nonzero. The rational expectation forecast errors are defined as

$$\eta_{y,t} \equiv y_t - E_{t-1}\left[y_t\right], \qquad \eta_{\pi,t} \equiv \pi_t - E_{t-1}\left[\pi_t\right]. \tag{47}$$

We define the vector of endogenous variables as $X_t \equiv \left(y_t, \pi_t, i_t, E_t\left(y_{t+1}\right), E_t\left(\pi_{t+1}\right), g_t, z_t\right)'$, the vectors of fundamental shocks and expectation errors,

$$\varepsilon_t = \left(\varepsilon_{i,t}, \varepsilon_{g,t}, \varepsilon_{z,t}\right)', \qquad \eta_t = \left(\eta_{y,t}, \eta_{\pi,t}\right)'$$

and the vector of parameters $\theta = \left(\psi_1, \psi_2, \rho_i, \beta, \kappa, \tau, \rho_g, \rho_z, \sigma_g, \sigma_z, \sigma_i, \rho_{gz}\right)'$. We can therefore represent the model as in the following equation,

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t. \tag{48}$$

The LRE model in (48) is determinate when the following analytic condition is satisfied,

$$|\psi^*| \equiv \left|\psi_1 + \frac{(1-\beta)}{\kappa}\psi_2\right| > 1. \tag{49}$$

When the model is indeterminate, $0 < |\psi^*| \leq 1$, the system is characterized by one degree of indeterminacy ($m = 1$) because there are two expectational variables ($E_t\left(y_{t+1}\right)$ and $E_t\left(\pi_{t+1}\right)$) and at most one root outside the unit circle. Thus, to implement our methodology we need to augment the original state space of the model in (48) with the autoregressive process

$$\omega_t = \left(\frac{1}{\alpha}\right)\omega_{t-1} + \nu_t - \eta_{\pi,t}. \tag{50}$$

where, without loss of generality, we have parameterized the auxiliary process with respect to $\eta_{\pi,t}$.[8] We then define a new vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)'$ and a newly defined vector of exogenous shocks as $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)' = (\varepsilon_{i,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \nu_t)'$. The system in (48) and (50) can now be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{51}$$

As in Lubik and Schorfheide (2004), we estimate the model using Bayesian methods using three series as observables: The percentage deviations of (log) real GDP per capita from an HP-trend ($y_{obs,t}$), the annualized percentage change in the Consumer Price Index for all Urban Consumers ($\pi_{obs,t}$), and the annualized Federal Funds Rate ($i_{obs,t}$). We focus on the data for the pre-Volcker period (1960Q1 - 1979Q2) as Lubik and Schorfheide (2004) show that during this period the monetary authority did not respond aggressively enough to changes in inflation, thus not suppressing self-fulfilling inflation expectations. We repeat the estimation of the model of Lubik and Schorfheide (2004) by adopting the same prior distributions for the structural parameters. The Bayesian estimation is conducted using conventional Metropolis-Hastings algorithm in Dynare.

Priors for the auxiliary parameters. As a first step, we discuss how we choose the prior distribution for the additional parameters introduced under our methodology. Our augmented representation introduces the vector of parameters α and parametrizes the continuum of equilibria under indeterminacy by introducing the standard deviation of a sunspot shock and its correlations with the exogenous shocks.

Regarding the vector of parameters α, we can distinguish three cases. Case 1: When the determinacy threshold is known, then α can be expressed as a function of the other parameters.

[8] In Appendix B, we show an analytic example of the unique mapping that exists between the alternative representations that can be considered using our augmented representation. We therefore ensure that the representations are equivalent up to a transformation of the correlations between the exogenous shocks and the forecast error included in the auxiliary process.

In this case, there is no need to specify a prior on α and the prior probability of determinacy is given by the prior on the parameter vector θ. Case 2: When the threshold is unknown and the researcher writes her own code, she can start with all the roots inside the unit circle for α at each draw of θ and then flip the appropriate number of elements in the vector α. Thus, even in this case, there is no need to specify a prior on α and the prior probability of indeterminacy depends on the prior on the parameter vector θ. Case 3: The researcher is using Dynare and the region of the parameter state is unknown. In this case, we suggest to choose priors that are symmetric between the two regions, i.e. that attach 50% probability to determinacy, and orthogonal with respect to the priors on the other parameters.

In what follows, we focus on Case 3 and discuss how to proceed under the assumption that the researcher does not know the region of determinacy and might be interested in using an estimation package such as Dynare. For a given draw of the structural parameters, the researcher would like to make draws of α smaller or greater than 1 with equal probabilities. In this case, the researcher could use a uniform distribution over the interval $[0, 2]$ or any symmetric interval around 1 as a prior distribution.[9] Note that when the determinacy region is not known, the efficiency of the algorithm can be improved when the researcher writes her own estimation/solution algorithm. We describe how to improve the efficiency of traditional MCMC algorithms in the subsection "Efficiency" below.

The correlations of the sunspot shocks with the exogenous disturbances are crucial parameters that affect the fit of the model. In line with the theoretical results, a given set of correlations under the representation that includes the forecast error for the inflation rate has a unique mapping to (different values of) the correlations in the representation with the forecast errors for the output gap, and vice versa. Therefore, in order for the two representations to deliver the same fit to the data, a researcher has to leave the correlations unrestricted. One simple option is to set a uniform prior distributions over the interval $(-1, 1)$ for the correlations of the sunspot shocks. This approach guarantees that the fit of the model does not depend on which forecast error is included in the auxiliary process.[10] This is the approach that we follow in the estimation of the model of Galí (2017) in Section 5.

Lubik and Schorfheide (2004) center the prior distributions for the additional parameters introduced in their representation on values that minimize the distance between the impulse responses of the model under indeterminacy and determinacy evaluated at the boundary of the region of

[9] Note that if the researcher writes her own code, the prior distribution does not necessarily have to be continuous. A discrete probability distribution that allows to make draws of α to be either equal 0.5 or 1.5 could also be specified as a prior.

[10] Similarly, imposing a zero correlation between the sunspot shock and the exogenous shocks under a given representation does not map into an assumption of zero correlation under the alternative representations. In this case, the fit of the model differs across the different specifications.

determinacy. Thus, they estimate parameters controlling deviations from this centering point. We showed the mapping between the two representations in Section 3 and we illustrate this equivalence with an analytical example in Appendix B. Given this equivalence, the priors for the correlations between sunspot shocks and structural shocks could also be specified in a way to replicate the approach of Lubik and Schorfheide (2004). Specifically, we could specify a prior on the auxiliary matrices used in Lubik and Schorfheide (2004) and then map the matrices into the correlations used in our approach. However, to easily implement our approach using standard packages, we suggest choosing a flat prior. Thus, we do not center our priors on the impulse response at the boundary of the region of determinacy, but still cover this case as equally likely with respect to the others.

Convergence. We are interested in showing that the methodology allows a standard estimation algorithm such as the one implemented in Dynare to travel to the correct region of the parameter space. At the same time, we also want to emphasize the importance of conducting standard convergence diagnostics. To achieve these goals, we set the initial parametrization in the "wrong" region of the parameter space and consider 1,000,000 draws to show that the methodology accommodates the case of determinacy and indeterminacy, as well as to highlight the importance of checking convergence before interpreting the estimation results. Figure 1 reports the posterior distribution for the parameter ψ_1 and α obtained for an initial parametrization such that the Taylor Principle holds (i.e. we set $\psi_1 = 2$). At first glance, the posterior distribution of the parameter ψ_1 would appear to be bimodal. This finding is consistent with the fact that the proposed augmented representation allows the Metropolis-Hastings algorithm to visit both regions of the parameter space. At the same time, the posterior distribution for the parameter α is very similar to the prior distribution, which is specified as a uniform distribution over the interval $[0, 2]$. Such result conveys the same evidence derived from the posterior for ψ_1 because the algorithm explores both regions by considering draws of α which are within as well as outside the unit circle.

A researcher should then verify the occurrence of either of the following two circumstances. This bimodal distribution could arise because the log-likelihood is highly discontinuous between the two regions. In this case, the algorithm could have jumped towards the region where the peak of the posterior lies, without having spent a significant time there. In other words, convergence has not occurred yet. Alternatively, if the log-likelihood function varies smoothly between the two regions of the parameter space, the posterior distribution plotted in Figure 1 could be the result of the algorithm traveling across the two regions multiple times.

We therefore recommend the researcher to analyze the draws of the parameter α which have been accepted during the MCMC algorithm. By inspecting the behavior of the auxiliary parameter α, a researcher can detect if the algorithm reached convergence or not. We report the draws

Posterior distribution of parameter ψ_1 and α

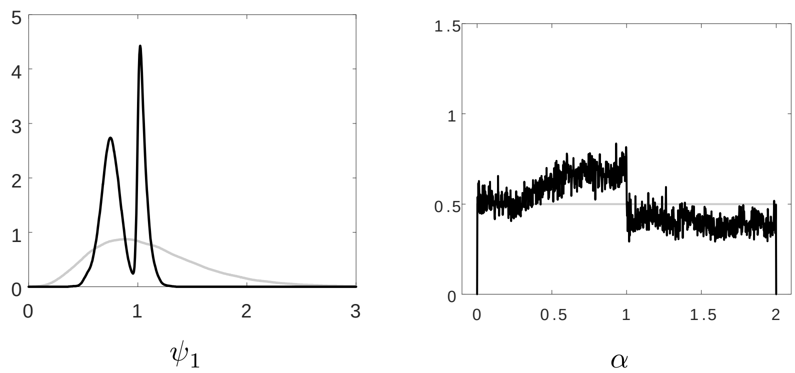

Figure 1: Initial parametrization $\psi_1 = 2$. The grey line represents the prior distribution and the black line is the posterior distribution.

that we obtained during our exercise in Figure 2. After approximately 400,000 draws of α in the region of determinacy (i.e. outside the unit circle), the algorithm jumps to the indeterminate region and never visits the determinacy region again.

Draws of the parameter α

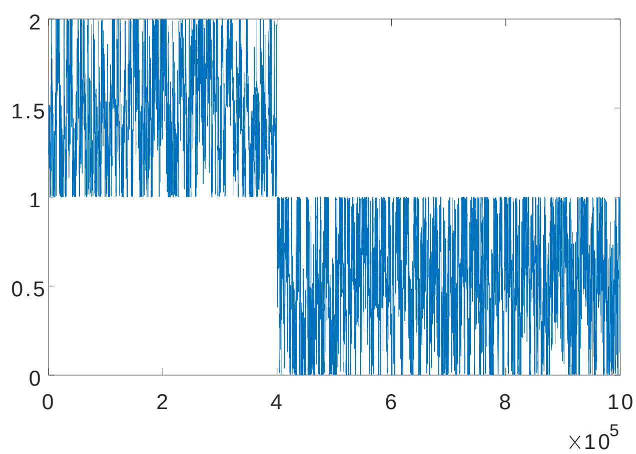

Figure 2: Sequence of draws for α given an initial parametrization $\psi_1 = 2$.

Thus, Figure 1 and 2 suggest that the algorithm is in fact able to jump toward the correct region of the parameter space, but also that convergence has not occurred yet. Therefore, the researcher should repeat the estimation exercise, increase the number of draws, and make sure that the parameter α stabilizes on one region of the parameter space. Under different circumstances, the researcher could face the second scenario, for which the log-likelihood function transitions smoothly between the two regions. In this case, the parameter α would repeatedly transition between the two areas of the parameter space and could be used to infer the probability attached to determinacy. Below we discuss how our methodology can be used to efficiently facilitate such transition.

Only (in)determinacy. In some cases, a researcher might want to estimate the model exclusively under determinacy or exclusively under indeterminacy. Our approach easily accommodates this need. If the researcher is only interested in the solution under determinacy, the parameter vector of α should be chosen in a way to guarantee stationarity of the auxiliary process (for example, fixing all values of the alphas to 2). Furthermore, all parameters that are relevant only under indeterminacy should be fixed to zero or any other constant, given that they do not affect the fit of the model under determinacy. If instead the researcher is only interested in estimating the model under indeterminacy, the parameters of the auxiliary process can be chosen in a way to guarantee that the correct number of explosive roots are provided. In this case, the parameters describing the properties of the sunspot disturbances should also be estimated.

Model comparison. A researcher might also be interested in comparing the fit of the model under determinacy and under indeterminacy. Model comparison can be conducted by using standard techniques, such as the harmonic mean estimator proposed by Geweke (1999). If the researcher is interested in comparing the same model under determinacy and under indeterminacy, we recommend the following procedure that adapts the approach used by Lubik and Schorfheide (2004):

1. Estimate the model under determinacy by fixing the parameter(s) alpha to a value larger than one in a way that the model is solved only under determinacy. Note that in this case all parameters that pertain to the solution under indeterminacy, such as the volatility of the sunspot shocks and its correlations with the exogenous shocks, *should* be restricted to zero (or any other constant). This restriction avoids penalizing the model for extra parameters that do not affect its fit under determinacy.

2. Estimate the model under indeterminacy by fixing the parameter(s) alpha to a value smaller than one in a way that the model is solved only under indeterminacy. Note that in this case all parameters that pertain to the solution under indeterminacy, such as the volatility of the sunspot shocks and its correlations, should be estimated.

3. Use standard methods to compare the fit of the model under determinacy with the fit of the same model under indeterminacy.

Efficiency. As mentioned above, our method can also be used to make traditional MCMC algorithms more efficient. We have left the discussion of this point last because, unlike the procedures described above, it generally requires the researcher to write its own code. The key idea is that in many cases the auxiliary parameter α can be used to summarize the distance of the current parameter vector from the threshold separating determinacy and indeterminacy regions. This approach is substantially easier when the partition of the parameter space is known, as in Lubik and Schorfheide (2004). However, the idea can be used even when the region of the parameter space is not known. For the sake of presenting the general idea, we focus on the former case.

As illustrated above, our method prescribes to set the parameter α to a value smaller than 1 when the original model presents indeterminacy. Therefore, the value of this auxiliary parameter can be considered an indicator variable for the presence of indeterminacy in the original model. When determinacy or indeterminacy depends on a large number of parameters, access to this indicator variable can facilitate transition between the two regions of the parameter space. To see why, suppose that indeterminacy depends on k parameters and that the threshold for indeterminacy is known. Then, we can easily obtain a draw for α and $k-1$ of the parameters that control determinacy, check whether the drawn α implies determinacy or indeterminacy, and, finally, solve for the $k-th$ parameter. Therefore, the probability of jumping between the two regions is controlled by a single parameter. The proposal distribution for this parameter can then be chosen to ensure that once it approaches the threshold, the proposal distribution is such that a jump is more likely. Instead, in the standard approach the k parameters are drawn without consideration of how far the current parameter vector is from the threshold separating the two areas of the parameter space.

There are of course many possible ways to choose the proposal distribution for α in a way that jumps between the two regions is more likely. One simple way consists of choosing a mixture of normals and then using a standard Metropolis-Hastings algorithm that corrects for the asymmetry in the proposal distribution. In what follows, we present this approach in the context of the model of Lubik and Schorfheide (2004).

Let's choose α in a way that every draw implies existence and uniqueness of a solution in the augmented parameter space:

$$\alpha \equiv \psi_1 + \frac{1-\beta}{\kappa}\psi_2$$

In this case, determinacy depends on four parameters, $\{\psi_1, \psi_2, \kappa, \beta\}$. However, we know that whenever $\alpha > 1$, these four parameters are such that determinacy holds, while whenever $\alpha < 1$, the model is in the indeterminacy region. Thus, we could implement an MCMC algorithm in which the proposal distribution draws $\{\alpha, \psi_2, \kappa, \beta\}$ and then derives ψ_1:

$$\psi_1 = \alpha - \frac{1-\beta}{\kappa}\psi_2.$$

Let $d \equiv \alpha - \overline{\alpha} = \alpha - 1$ be the distance between the current value for α and the boundary of the determinacy region for this auxiliary parameter, $\overline{\alpha} \equiv 1$. Note that when the distance is negative, α is below the threshold and the model is under indeterminacy. Suppose that we specify the proposal distribution to be a mixture of normals: One centered on the current parameter value and one just beyond the threshold of the determinacy region. Specifically, we assume the following proposal distribution for the proposed draw $\widetilde{\alpha}$, given the current value, α_n:

$$\begin{aligned}
\widetilde{\alpha} &\sim \begin{cases} N\left(1 - sign\,(d)\,\mu_c, \sigma_c^2\right) & \text{with probability } w \\ N\left(\alpha_n, \sigma_\alpha^2\right) & \text{with probability } 1-w \end{cases} \\
w &= K_0 \exp\left(-\left|d\right| K_1\right), \; w_m = \exp\left(-\left|d\right| K_2\right) \\
\mu_c &= (1-w_m)\mu_b + w_m * .5 \\
\sigma^2 &= (1-w_m)\sigma_b + w_m * .1
\end{aligned}$$

where K_0 is a parameter between 0 and 1 that controls the maximum weight on the auxiliary normal and $K_1 > 0$ is a parameter controlling the speed with which the weight on the auxiliary normal goes to zero as the MCMC algorithm gets further from the threshold region. The parameters $\mu_b > 0$ and $\sigma_b > 0$ control the position and shape of the auxiliary normal. These, in turn, depend on the parameter $K_2 > 0$ that makes sure that as the current alpha approaches the threshold of the determinacy region, the location and shape of the auxiliary distribution are adjusted accordingly. The parameter σ_α controls the variance for the typical normal proposal distribution centered on the current value of the parameter. Note that the typical Metropolis-Hastings algorithm would have $w = 0$ implying that the proposal distribution does not vary in response to the distance from the boundary of the determinacy region.

Figure 3 presents the proposal distribution for different values of the current α_n. To facilitate the interpretation of the graphs, the top panel plots the proposal distribution for a series of values implying determinacy, while the lower panel considers a series of values implying indeterminacy.[11] When α_n is far from the threshold separating the two regions (dotted black line), the proposal distribution is symmetric. As the current α_n becomes closer to 1, the weight on the auxiliary normal increases and more and more mass is assigned to drawing a value of α that implies a jump between the two regions. Furthermore, as the the current α_n gets closer to 1, the mean of

[11] We use these hyperparameters $K_0 = .5$, $K_1 = 2$, $K_2 = 10$, $\mu_b = .01$, $\sigma_b = .01$.

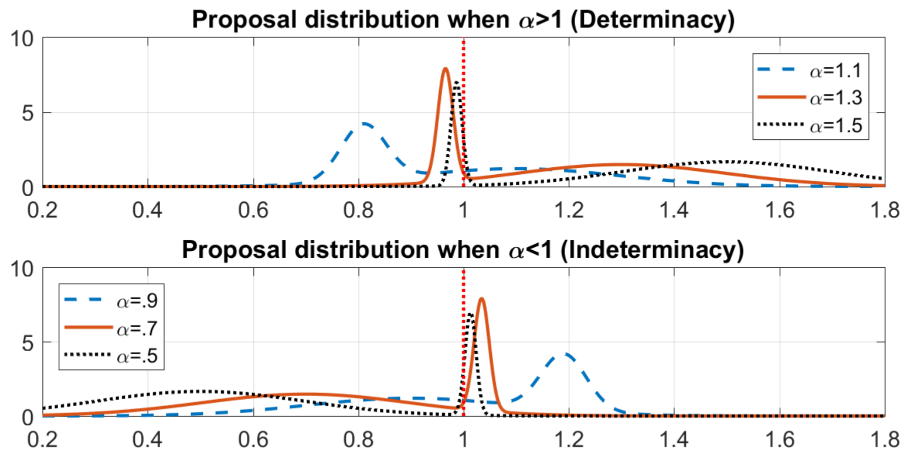

Figure 3: Proposal distribution for different values of α. The proposal distribution is chosen to facilitate crossing the determinacy threshold and is obtained with a mixture of normals. The upper (lower) panel assumes that α is currently above (below) the threshold of the determinacy region.

the auxiliary normal distribution moves further from the determinacy threshold.

To understand how the algorithm helps in crossing the determinacy threshold, we estimate the model of Lubik and Schorfheide (2004) for the post-1982 period using the modified Metropolis algorithm involving the parameter α and the traditional algorithm that only involves the model parameters and a symmetric proposal distribution. We start the two algorithms 1,000 times by making a draw from the posterior mode. For each iteration, we count the number of draws necessary for the parameters to cross the determinacy threshold for the first time. We stop when the algorithm has reached 100,000 iterations.

Figure 4 reports the distribution for the number of draws necessary to cross the determinacy threshold for the first time in the two cases. The blue/dark colored bars correspond to the algorithm implemented by drawing values for the auxiliary parameter α and then using the value of α to obtain the corresponding value of ψ_1. Instead, the yellow/light colored bars correspond to the traditional algorithm that makes draws for the original parameter space. The distribution is truncated at 100,000 draws. From the graph, it is clear that the modified algorithm greatly facilitates crossing the determinacy region. The median value for the number of draws necessary to cross the determinacy region is only $16,555$ for the modified algorithm. Instead, for the traditional algorithm in $74,2\%$ of the cases the parameters have *not* crossed the determinacy threshold after $100,000$ iterations.

Finally, we also verify that the modified MCMC algorithm is able to repeatedly jump back and

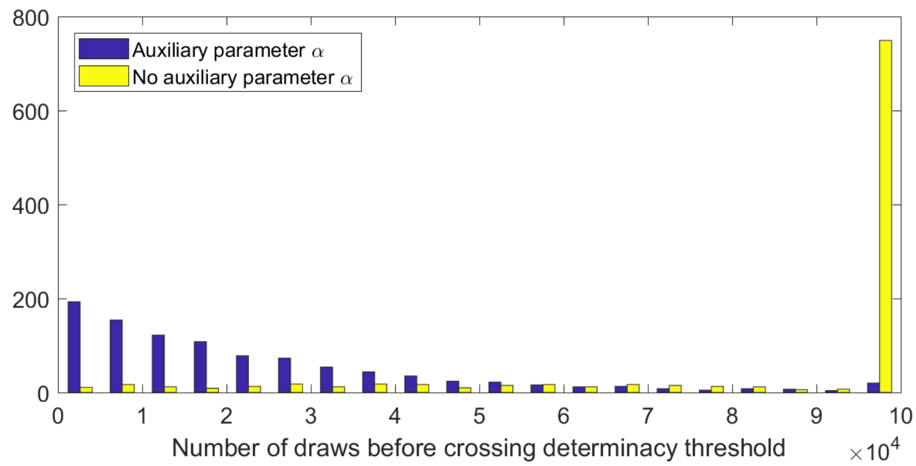

Figure 4: The figure reports the distribution for the number of draws necessary to cross the determinacy threshold for the first time when using a Metropolis-Hastings algorithm to estimate the model of Lubik and Schorfheide (2004). Two cases are considered. In the first case (blue/dark colored bars), the algorithm is implemented by drawing values for the auxiliary parameter α. The value of α is then used to obtain the corresponding value of ψ_π. In the second case (yellow/light colored bars), the algorithm is implemented by drawing directly the parameters of the model. The distribution is truncated at 100,000 draws.

forth between the two regions of the parameter space. We then make $2,100,000$ draws from the posterior using the modified algorithm. We find that the algorithm transitions a total of 34 times between the two regions. The posterior probability of being under determinacy, computed as the fraction of draws for which $\alpha > 1$, is 98.9%. Therefore, the algorithm is able to explore the entire area of the parameter space despite the fact that the determinacy region is overwhelmingly favored by the data.

When conducting the same exercise for the pre-1979 period, we found that the algorithm was able to quickly move to the indeterminacy region independently of the starting point. However, once the algorithm had reached such region, it was not able to leave it because of a large discontinuity in the likelihood. This result is important to highlight that while our approach can facilitate the transition across regions, it cannot overcome the fact that for some models and some data samples the boundary of the determinacy regions might imply a large discontinuity in the posterior. In this case, jumping between the two regions becomes extremely unlikely, even when a clever proposal distribution is used. In these cases, more recent methods, such as the ones described in Herbst and Schorfheide (2015), can be used to make sure that the entire parameter space is explored. However, for the example considered in this paper, it is worth emphasizing that the conclusions of the analysis are unlikely to change because the lack of jumps between the two regions reflects the fact that the data strongly favor indeterminacy.

5 Monetary Policy and Asset Bubbles

In this section, we implement the proposed methodology to estimate the small-scale NK model of Galí (2017) using Bayesian techniques. The model extends a conventional NK model to allow for the existence of rational expectations equilibria with asset price bubbles. Interestingly, the model displays up to two degrees of indeterminacy for realistic parameter values.

We estimate the model using U.S. data over the period 1982:Q4 until 2007:Q3, and we consider the case that the U.S. monetary policy aimed at stabilizing the inflation rate and leaning against the bubble. We find that the strength of such responses was not enough to guarantee a stabilization of the U.S. economy and to avoid that unexpected changes in expectations could drive U.S. business cycles. In particular, we show that the model specification that provides the best fit to the data is characterized by two degrees of indeterminacy.[12]

5.1 The Model

The model of Galí (2017) is described by the following equations. First, equation (52) represents a dynamic IS curve

$$y_t = \Phi E_t\left(y_{t+1}\right) - \Psi\left(i_t - E_t\left(\pi_{t+1}\right)\right) + \Theta q_t, \tag{52}$$

where the variables are expressed in deviations from a balanced growth path (henceforth BGP), and the parameters $\{\Phi, \Psi, \Theta\}$ are function of the structural parameters of the model.[13] The term q_t denotes the size of an aggregate bubble in the economy (normalized by trend output) relative to its value along the BGP.

The aggregate bubble plays the role of demand shifter and is defined as

$$q_t = b_t + u_t^q,$$

where b_t denotes the aggregate value in period t of bubble assets that were already available for

[12] In line with our theoretical results, we show that, given the degree of indeterminacy, the estimation delivers the same marginal data densities regardless of which forecast error we include in our representation since there exists a unique mapping among them.

[13] In particular, $\Phi \equiv \frac{\Lambda\Gamma v}{\beta} \in (0,1]$, $\Psi \equiv \Upsilon\Phi\left(1 + \frac{v\gamma(1-\Phi)}{\Phi(1-\beta\gamma)}\right)$, $\Upsilon \equiv \frac{1-\beta\gamma}{1-\Lambda\Gamma v\gamma} \in (0,1]$ and $\Theta \equiv \frac{(1-\beta\gamma)(1-v\gamma)}{\beta\gamma}$. The parameters are function of the following structural parameters of the model: i) γ, the constant probability of each individual in the OLG model to survive to the next period; ii) v, the probability of each individual to be employed in the next period; iii) β, the discount factor of each individual; iv) $\Lambda \equiv 1/(1+r)$, the steady state stochastic discount factor for one-period ahead payoffs derived from a portfolio of securites; v) $\Gamma \equiv (1+g)$, the gross rate of productivity growth.

trade in period $t-1$, and u_t^q is the value of a new bubble at time t. We assume that u_t^q follows an exogenous autoregressive process of the form

$$u_t^q = \rho_q u_{t-1}^q + \varepsilon_t^q, \qquad \varepsilon_t^q \overset{iid}{\sim} N(0, \sigma_q^2).$$

Equation (53) defines the evolution of the value of the asset bubble q_t as

$$q_t = \Lambda\Gamma E_t\left(b_{t+1}\right) - q\left(i_t - E_t\left(\pi_{t+1}\right)\right), \tag{53}$$

where $q \equiv \frac{\gamma(\beta-\Lambda\Gamma v)}{(1-\beta\gamma)(1-\Lambda\Gamma v\gamma)}$ represents the steady state bubble-to-output ratio, $\Lambda \equiv 1/(1+r)$ is the steady state stochastic discount factor for one-period ahead payoffs derived from a portfolio of securities and $\Gamma \equiv (1+g)$ is the gross rate of productivity growth. To guarantee that newly created bubbles along the BGP are non-negative, the model requires that $\Lambda\Gamma = \frac{1+g}{1+r} \geq 1$. Equivalently, it must hold that $r \leq g$ on a BGP characterized by the creation of (non-negative) new asset bubbles. Equation (53) shows how "optimistic" expectations about the future value of the bubble lead to a higher price for those assets today.

The model is then closed by the following NK Phillips curve

$$\pi_t = \Lambda\Gamma v\gamma E_t\left(\pi_{t+1}\right) + \kappa y_t + u_t^s, \tag{54}$$

where $u_t^s = \rho_s u_{t-1}^s + \varepsilon_t^s$ and $\varepsilon_t^s \overset{iid}{\sim} N(0, \sigma_s^2)$.[14] Finally, the conduct of monetary policy is described by an interest rate rule of the form,

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left(\phi_\pi \pi_t + \phi_q q_t\right) + \varepsilon_t^i, \tag{55}$$

according to which monetary policy that displays a certain degree of interest rate inertia, and aims not only at stabilizing inflation, but also at leaning against the bubble.[15] The rational expectation forecast errors are defined as

$$\eta_{y,t} \equiv y_t - E_{t-1}\left[y_t\right], \quad \eta_{\pi,t} \equiv \pi_t - E_{t-1}\left[\pi_t\right], \quad \eta_{b,t} \equiv b_t - E_{t-1}\left[b_t\right]. \tag{56}$$

Equations (52)∼(56) describe the equilibrium dynamics of the model economy around a given BGP. We define the vector of endogenous variables as $X_t \equiv (y_t, \pi_t, b_t, i_t, q_t, E_t\left(y_{t+1}\right), E_t\left(\pi_{t+1}\right),$

[14] In particular, $k \equiv \frac{(1-\theta)(1-\Lambda\Gamma v\gamma\theta)}{\theta}\rho$, where θ represents the Calvo probability that a firm keeps its price unchanged in any given period and ρ is the elasticity of hours worked.

[15] We also assume that $\varepsilon_t^i \overset{iid}{\sim} N(0, \sigma_i^2)$.

$E_t\left(b_{t+1}\right), u_t^q, u_t^s)'$, and the vectors of fundamental shocks, ε_t, and non-fundamental errors, η_t, as

$$\varepsilon_t \equiv \left(\varepsilon_t^q, \varepsilon_t^s, \varepsilon_t^i\right)', \qquad \eta_t \equiv \left(\eta_{y,t}, \eta_{\pi,t}, \eta_{b,t}\right)'.$$

The model can therefore be represented as

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{57}$$

where θ represents the vector of structural parameters of the model. Galí (2017) shows that for realistic parameter values, the model is characterized by up to two degrees of indeterminacy. Therefore, the proposed methodology augments the representation of the model in (57) with two autoregressive processes

$$\omega_{1,t} = \left(\frac{1}{\alpha_1}\right)\omega_{1,t-1} + \nu_{1,t} - \eta_{1,t}, \tag{58}$$

$$\omega_{2,t} = \left(\frac{1}{\alpha_2}\right)\omega_{2,t-1} + \nu_{2,t} - \eta_{2,t}, \tag{59}$$

where $\{\eta_{1,t}, \eta_{2,t}\}$ could be any combination consisting of two of the three forecast errors defined by the vector $\eta_t \equiv \left(\eta_{y,t}, \eta_{\pi,t}, \eta_{b,t}\right)'$. Hence, we define a new vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_{1,t}, \omega_{2,t})'$ and a newly defined vector of exogenous shocks as $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_{1,t}, \nu_{2,t})'$. The system in (57), (58) and (59) can then be written as

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t.$$

5.2 Estimation

We estimate the model to match U.S. data over the period 1982:Q4 until 2007:Q3. We consider a subset of three macroeconomic quarterly time series used in Smets and Wouters (2007) to match the number of exogenous shocks in the model. In particular, we use the growth rate in real GDP, the inflation rate measured by the GDP deflator and the Federal Funds rate. We implement Bayesian techniques, and the measurement equations that relate the macroeconomic data to the endogenous variables of the model are defined as

$$\begin{bmatrix} dlGDP_t \\ dlP_t \\ FFR_t \end{bmatrix} = \begin{bmatrix} g \\ \pi^* \\ i^* \end{bmatrix} + \begin{bmatrix} y_t - y_{t-1} \\ \pi_t \\ i_t \end{bmatrix},$$

where dl denotes the percentage change measured as log difference.

We follow Galí (2017) and set the discount factor of each individual, β, to 0.998. We estimate the remaining structural parameters of the model using Bayesian techniques. We report the prior distributions for the parameters in Table 2. As mentioned when studying equation (53) for the evolution of the value of the asset bubble q_t, the model requires that the real interest rate, r, and the growth rate of output, g, satisfy $r \leq g$ to ensure that newly created bubbles along the BGP are non-negative. To guarantee that this inequality holds for each draw of the Metropolis-Hastings algorithm, we express the real interest rate, r, as $r = \lambda g$, where $\lambda \in (0, 1)$. We then set the prior for the quarterly growth rate of output, g, as a gamma distribution centered at 0.45, and the prior for λ as a beta distribution with mean 0.8. These priors imply that the annualized growth rate of output is 1.6% and the annualized real interest rate is approximately 1.3% over the considered period.

We center the prior for the employment ratio, α, to 0.6. Following the calibration in Galí (2017), the prior distribution for the probability that an individual survives to the next period, γ, is centered at 0.996 . The prior for the slope of the New Keynesian Phillips Curve, κ , is set at 0.04, a value chosen for the calibration in Galí (2017) and consistent with an average duration of individual prices of 4 quarters. The parameter describing the response of the monetary authority to changes in inflation, ϕ_π, follows a gamma distribution with mean 1 and standard error 0.4. The response to deviations of the bubble relative to its value along the BGP follows a gamma distribution with mean 0.3 and standard error 0.15. The parameter which governs the degree of interest rate inertia, ρ_i, follows a beta distribution centered at 0.7.

The priors on the stochastic processes that define the fundamental shocks are inverse gamma distributions centered at 0.3 with a standard deviation of 0.15. Finally, when we estimate the model under indeterminacy, we specify uniform prior distributions for both the standard deviations of the non-fundamental shocks $\{\sigma_{\nu_l}\}$ where $l = \{\pi, y, b\}$, and their correlations with the exogenous shocks of the model $\{\varphi_{\nu_{l,j}}\}$ where $j = \{i, q, s\}$.

We estimate the model in each region of the parameter space: Determinacy, one degree of indeterminacy and two degrees of indeterminacy. When the model is indeterminate, we run the estimation for the different combinations consisting of one or two of the forecast errors defined by the vector $\eta_t \equiv \left(\eta_{y,t}, \eta_{\pi,t}, \eta_{b,t}\right)'$ depending on the degree of indeterminacy. In line with our theoretical results, we show that, given the degree of indeterminacy, the estimation delivers the same marginal data densities regardless of which forecast error(s) we include in our representation. In Appendix C, we also show that the posterior distributions for the model parameters are equivalent up to a transformation of the correlations between the exogenous shocks and the sunspot disturbances considered in each specification.[16]

[16] In Appendix B, we show the analytical equivalence among the different representations applying our methodology to a bivariate model.

Prior distribution for model parameters			
Name	Density	Mean	Std. Dev.
g	$Gamma$	0.45	0.04
λ	$Beta$	0.80	0.10
α	$Beta$	0.60	0.10
$100(\gamma^{-1} - 1)$	$Gamma$	0.4	0.10
κ	$Gamma$	0.04	0.005
π^*	$Gamma$	0.9	0.30
i^*	$Gamma$	1.2	0.30
ϕ_π	$Gamma$	1	0.40
ϕ_q	$Gamma$	0.3	0.10
ρ_i	$Beta$	0.70	0.10
σ_q	$Inv.\ Gamma$	0.30	0.15
σ_s	$Inv.\ Gamma$	0.30	0.15
σ_i	$Inv.\ Gamma$	0.30	0.15
ρ_q	$Beta$	0.70	0.10
ρ_s	$Beta$	0.70	0.10
σ_{ν_i}	$Uniform[0, 10]$	5	2.89
$\varphi_{\nu_i,j}$	$Uniform[-1, 1]$	0	0.57

Table 2: The table reports the prior distribution for the model parameters.

Model comparison	
Specification	*Marginal data densities*
Indet-2	-72.3
Indet-1	-83.0
Determinacy	-158.3

Table 3: The table reports the (log) marginal data densities for each model specification.

Table 3 reports the (log) marginal data density for each of the model specification. We find that the data favor the specification of the model with two degrees of indeterminacy. We attribute this result to the stylized nature of the model, and the observation that indeterminate models are consistent with a richer dynamic and stochastic structure. In future work, it would be valuable to study whether the findings would carry over in the context of a more realistic, medium-scale model that could explain the persistence and volatility in the data without recurring to indeterminate dynamics.

Table 4 reports the mean and 90% probability interval of the posterior distribution of the estimated structural parameters. The probability of surviving to the next period, γ, is estimated to be approximately 99%. The posterior of the slope of the NK Phillips curve is 0.039, which in this model is consistent with a probability of 24.1% that a firm keeps its price unchanged in any given period. The steady-state inflation rate and nominal interest rate are about 0.7% and 1.4% on an quarterly basis. We also find that the strength of the responses of U.S. monetary policy to stabilize the inflation rate and lean against the bubble was not enough to guarantee a stabilization of the U.S. economy and to avoid that unexpected changes in expectations could drive U.S. business cycles.

The mean of the standard error of the bubble component is 0.28, and larger than the standard deviation of the supply and monetary policy shocks that are estimated to be 0.11 and 0.12, respectively. The data also provide evidence that the bubble shock is less persistent than the supply shock.

Finally, we report the standard deviation of the sunspot shocks for the representation that includes the forecast error for the output gap and the inflation rate. The posterior estimates show that the standard error related to forecast errors for the output gap is approximately twice as large as the standard deviation of the sunspot shock associated with the inflation rate. The data also appear to be informative on the correlations of both sunspot shocks with the exogenous shocks of the model. A monetary policy shock is negatively correlated with both sunspot shocks, implying a contemporaneous impact on both inflation and output. A shock due to a new bubble can be interpreted as a demand shifter, and it has no significant correlation with unexpected changes in expectations about future inflation and economic activity. A supply shock has a positive correlation with the sunspot shock associated with inflation, as well as a negative correlation with the sunspot shock for output. These correlations are crucial to interpret the impact that each shock has on the model economy as described next.

Figure 5 plots the impulse response of output, inflation and nominal interest rate. We orthogonalize the fundamental shocks using a Cholesky decomposition with the same order as in the plots $\{\varepsilon_q, \varepsilon_s, \varepsilon_i\}$. The last two panels report the impulse response functions in which each sunspot shock is the most exogenous shock in the Cholesky decomposition. We plot the impulse responses to a one-standard-deviation shock. The solid lines represent the posterior means, while the dashed line correspond to the 90% probability intervals.

Considering the estimated correlations reported in Table 4, we observe that a shock due to the creation of a new bubble generates no significant effect on the economy in line with the estimated correlations. A positive supply shock has both inflationary and contractionary effects on impact. The persistence of the shock on output is then associated to deflationary effects to which the monetary authority responds by decreasing the nominal interest rate. A monetary policy tightening generates contractionary and deflationary pressures. The persistence of these effects on the inflation rate then requires the monetary authority to adopt an accommodative stance to stabilize the economy.

Posterior distribution for model parameters		
	Mean	90% prob. int.
g	0.46	[0.41,0.51]
λ	0.79	[0.64,0.94]
α	0.59	[0.51,0.68]
$100(\gamma^{-1}-1)$	0.44	[0.30,0.58]
κ	0.039	[0.032,0.047]
π^*	0.69	[0.38,1.01]
i^*	1.41	[1.07,1.71]
ϕ_π	0.35	[0.16,0.53]
ϕ_q	0.13	[0.06,0.19]
ρ_i	0.68	[0.54,0.84]
σ_q	0.28	[0.13,0.43]
σ_s	0.11	[0.09,0.13]
σ_i	0.12	[0.09,0.14]
ρ_q	0.70	[0.54,0.86]
ρ_s	0.89	[0.84,0.95]
σ_{ν_π}	0.28	[0.24,0.32]
σ_{ν_y}	0.69	[0.60,0.78]
$\varphi_{\nu_\pi,i}$	-0.42	[-0.67,-0.16]
$\varphi_{\nu_\pi,q}$	0.07	[-0.43,0.59]
$\varphi_{\nu_\pi,s}$	0.61	[0.48,0.73]
$\varphi_{\nu_y,i}$	-0.14	[-0.40,0.13]
$\varphi_{\nu_y,q}$	-0.01	[-0.52,0.55]
$\varphi_{\nu_y,s}$	-0.68	[-0.77,-0.59]

Table 4: The table reports the posterior distribution of the model parameters under two degrees of indeterminacy $\{\nu_\pi, \nu_y\}$.

The last two panels show the impulse response to the sunspot shocks that we assume to be uncorrelated. In this economy, a positive shock to inflation expectations generates self-fulfilling effects on inflation. Given that in this panel the sunspot shock, ε_{ν_π}, is assumed to be the most exogenous, economic activity does not respond on impact, while the increase in the nominal interest rate triggers a contractionary effect in the medium term. Finally, a positive sunspot shock to the expectation about future deviations of output from its trend leads to a rise in economic activity due to its self-fulfilling nature. Given that in the last panel we assume that the sunspot shock, ε_{ν_y}, is the most exogenous, the inflation rate does not respond on impact, while it is characterized by a mild deflationary effect in the medium term that leads to a decrease in the nominal interest rate.

Table 5 reports the variance decompositions for output, inflation and interest rate. The means and the 90% probability intervals are calculated from the output of the Metropolis-Hastings algorithm. Because the estimated correlations of the two sunspot shocks with the exogenous shocks are nonzero, the reported variance decomposition results from the orthogonalization of the shocks using a Cholesky factorization in which the order of the shocks follows the list in Table 5. The results are in line with those in the literature and in particular with Lubik and Schorfheide (2004). The deviations of output from its trend are mostly explained by supply shocks. In addition to supply-side shocks, fluctuations in inflations are also accounted for by unexpected changes in monetary policy. Similar conclusions can be drawn for the decomposition of the nominal interest rate. Interestingly, both sunspot shocks play only a marginal role in explaining business cycle fluctuations for each of the three endogenous variables.

Variance Decomposition

	Output dev. from trend		Inflation		Interest rate	
	Mean	90% prob. int.	Mean	90% prob. int.	Mean	90% prob. int.
ε_s	0.68	[0.37,0.93]	0.27	[0.01,0.57]	0.27	[0.01,0.56]
ε_i	0.22	[0.02,0.45]	0.61	[0.37,0.85]	0.60	[0.36,0.85]
ε_q	0.02	[0.01,0.03]	0.02	[0.01,0.04]	0.02	[0.01,0.04]
ε_{ν_y}	0.07	[0.01,0.14]	0.08	[0.01,0.21]	0.09	[0.01,0.21]
ε_{ν_π}	0.01	[0.001,0.02]	0.02	[0.01,0.03]	0.02	[0.01,0.03]

Table 5: The table reports the means and 90-percent probability intervals for the unconditional variance decomposition. Since the estimated correlations with the two sunspot shocks are nonzero, the decomposition of the orthogonalized shocks via Cholesky decomposition follows the order of the shocks as listed in the Table.

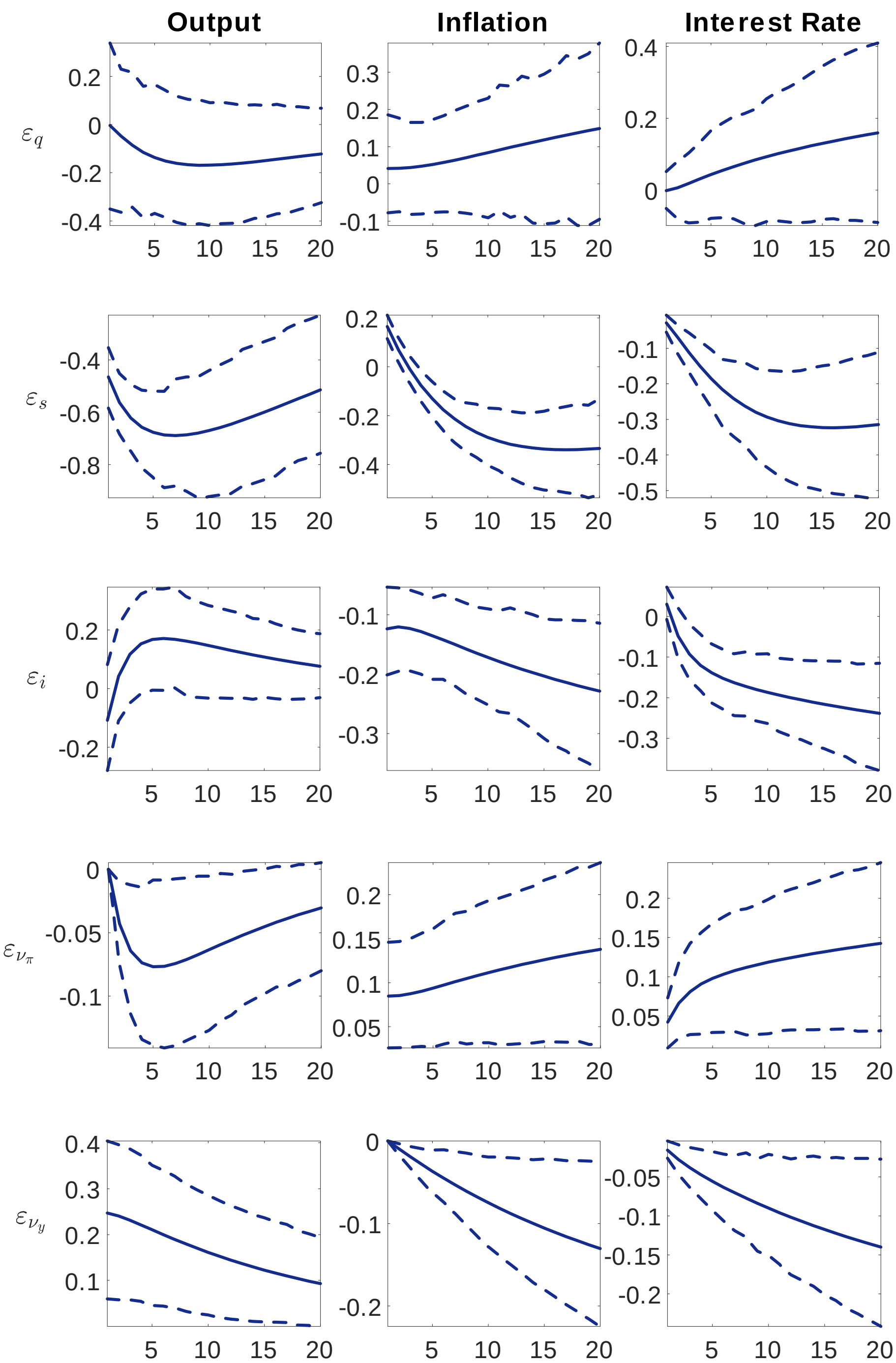

Figure 5: The figures plot the posterior means (solid lines) and 90-percent probability intervals (dashed lines) for the impulse responses of output, inflation and nominal interest rate to a shock of one standard deviation for each orthogonalized disturbance using a Cholesky decomposition with the same order as in the plots.

6 Conclusions

In this paper, we propose a generalized approach to solve and estimate LRE models over the entire parameter space. Our approach accommodates both cases of determinacy and indeterminacy and it does not require the researcher to know the analytic conditions describing the region of determinacy or the degrees of indeterminacy.

When a LRE model is characterized by m degrees of indeterminacy, our approach augments it by appending m autoregressive processes whose innovations are linear combinations of a subset of endogenous shocks and a vector of newly defined sunspot shocks. We show that the solution for the resulting augmented representation embeds both the solution which is obtained under determinacy using standard solution methods and that delivered by solving the model under indeterminacy using the approach of Lubik and Schorfheide (2003) and equivalently Farmer et al. (2015).

We apply our methodology to estimate the small-scale NK model of Galí (2017) using Bayesian techniques. Galí's model extends a conventional NK model to allow for the existence of rational bubbles. An interesting aspect of the model is that it displays up to two degrees of indeterminacy for realistic parameter values. We estimate the model using U.S. data over the period 1982:Q4 until 2007:Q3. Using Bayesian model comparison we find that the data support the version of the model with two degrees of indeterminacy, implying that the central bank was not reacting strongly enough to the bubble component. One caveat to note, however, is that the model of Galí (2017) is quite stylized, but the results are intriguing and merit further exploration in future research.

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7 Appendix

7.1 Appendix A

We prove the equivalence between the parametrization of the Lubik-Schorfheide indeterminate equilibrium $\theta^{LS} \in \Theta^{LS}$ and the Bianchi-Nicolò equilibrium parametrized by $\theta^{BN} \in \Theta^{BN}$. In particular, we show that there is a unique mapping between the linear restrictions imposed in each of the two methodologies on the forecast errors to guarantee the existence of at least a bounded solution. As shown in Section 3.2.1, the method by Lubik and Schorfheide (2003) imposes the following restrictions on the non-fundamental shocks, η_t, as a function of the exogenous shocks, ε_t, and the sunspot shocks introduced in their specification, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(\underset{p\times n}{V_1} \underset{n\times \ell}{N} + \underset{p\times m}{V_2} \underset{\substack{m\times\ell \\ m\times\ell}}{\widetilde{M}} \right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}. \tag{60}$$

Using the methodology proposed in this paper, Section 3.2.2 shows that the restrictions on the non-fundamental shocks, η_t, as a function of the exogenous shocks, ε_t, and the sunspot shocks, v_t, are

$$\underset{p\times 1}{\eta_t} = \underset{p\times \ell}{C_1} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{C_2} \underset{m\times 1}{\nu_t}, \tag{61}$$

where

$$C_1 \equiv - \begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1} \widetilde{\Psi}_2 \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad C_2 \equiv - \begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1} \widetilde{\Pi}_{f,2} \\ -\mathbf{I} \end{bmatrix}.$$

Post-multiplying equation (60) and (61) by ε_t' and taking expectations on both sides,

$$\begin{aligned} \underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times n}{V_1} \underset{n\times \ell}{N} \underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{V_2} \underset{m\times \ell}{\widetilde{M}} \underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}}, \\ \underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times \ell}{C_1} \underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{C_2} \underset{m\times l}{\Omega_{\nu\varepsilon}} \end{aligned}$$

Pre-multiplying by V_2' and equating the equations,

$$\underset{m\times \ell}{\widetilde{M}} \underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} = \left(\underset{m\times p}{V_2'} \underset{p\times \ell}{C_1} - \underset{m\times p}{V_2'} \underset{p\times n}{V_1} \underset{n\times \ell}{N} \right) \underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{m\times p}{V_2'} \underset{p\times m}{C_2} \underset{m\times l}{\Omega_{\nu\varepsilon}}. \tag{62}$$

Using the properties of the vec operator, the following result holds

$$\underset{(m\times\ell)\times 1}{vec(\widetilde{M})} = \underset{(m\times\ell)\times(m\times\ell)}{(\Omega_{\varepsilon\varepsilon} \otimes I_m)^{-1}} \left[\underset{(m\times\ell)\times\ell^2}{\left[I_l \otimes \left(V_2' C_1 - V_2' V_1 N \right) \right]} \underset{\ell^2\times 1}{vec\left(\Omega_{\varepsilon\varepsilon}\right)} + \underset{(m\times\ell)\times(m\times\ell)}{\left(I_l \otimes V_2' C_2 \right)} \underset{(m\times\ell)\times 1}{vec\left(\Omega_{\nu\varepsilon}\right)} \right]. \tag{63}$$

Equation (63) is the first relevant equation to show the mapping between the representation in Lubik and Schorfheide (2003) and our representation. For a given variance-covariance matrix of the exogenous shocks, $\Omega_{\varepsilon\varepsilon}$, that is common between the two representations, equation (63) tells us that the covariance structure, $\Omega_{\nu\varepsilon}$, of the sunspot shock in our representation with the exogenous shocks has a unique mapping to the matrix, $\widetilde{M}$, in Lubik and Schorfheide (2003). Clearly, equation (62) can also be used to derive the mapping from their representation to our method.

We now show how to derive the mapping between the variance-covariance matrix, $\Omega_{\nu\nu}$, of the sunspot shocks in our representation to the variance-covariance matrix, $\Omega_{\zeta\zeta}$, of the sunspot shocks in Lubik and Schorfheide (2003). Considering again equation (60) and (61), we post-multiply by ζ_t' and take expectations on both sides,

$$\begin{aligned} \underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{V_2}\,\underset{m\times m}{\Omega_{\zeta\zeta}}\,, \\ \underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{C_2}\,\underset{m\times m}{\Omega_{\nu\zeta}} \end{aligned}$$

Pre-multiplying both equations by V_2' and equating them,

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \underset{m\times m}{\Omega_{\zeta\nu}}\,\underset{m\times m}{\left(V_2'C_2\right)'}. \tag{64}$$

Finally, to obtain an expression for $\Omega_{\zeta\nu}$, we post-multiply equation (60) and (61) by ν_t' and taking expectations

$$\begin{aligned} \underset{p\times m}{\Omega_{\eta\nu}} &= \left(\underset{p\times n}{V_1}\,\underset{n\times\ell}{N} + \underset{p\times m}{V_2}\,\underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{V_2}\,\underset{m\times m}{\Omega_{\zeta\nu}}\,, \\ \underset{p\times m}{\Omega_{\eta\nu}} &= \underset{p\times\ell}{C_1}\,\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{C_2}\,\underset{m\times m}{\Omega_{\nu\nu}} \end{aligned}$$

Pre-multiplying both equations by V_2' and solving for $\Omega_{\zeta\nu}$,

$$\underset{m\times m}{\Omega_{\zeta\nu}} = \left(\underset{m\times p}{V_2'}\,\underset{p\times\ell}{C_1} - \underset{m\times p}{V_2'}\,\underset{p\times n}{V_1}\,\underset{n\times\ell}{N} - \underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{m\times m}{\left(V_2'C_2\right)}\,\underset{m\times m}{\Omega_{\nu\nu}}\,. \tag{65}$$

Post-multiplying (65) by $\underset{m\times m}{(V_2'C_2)'}$ and using (64), then

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \left(\underset{m\times p}{V_2'}\,\underset{p\times\ell}{C_1} - \underset{m\times p}{V_2'}\,\underset{p\times n}{V_1}\,\underset{n\times\ell}{N} - \underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}}\underset{m\times m}{\left(V_2'C_2\right)'} + \underset{m\times m}{\left(V_2'C_2\right)}\,\underset{m\times m}{\Omega_{\nu\nu}}\,\underset{m\times m}{\left(V_2'C_2\right)'}. \tag{66}$$

Therefore, equation (66) defines the mapping between the variance-covariance matrix, $\Omega_{\nu\nu}$, of the sunspot shocks in our representation to the variance-covariance matrix, $\Omega_{\zeta\zeta}$, of the sunspot shocks in Lubik and Schorfheide (2003). Together with equation (63), we show that this equation defines the one-to-one mapping between the parametrization in Lubik and Schorfheide $\{\Theta, \Theta^{LS}\}$ and the parametrization in Bianchi-Nicolò $\{\Theta, \Theta^{BN}\}$.

7.2 Appendix B

In this Appendix, we provide an analytical example to show the equivalence between the solutions for an indeterminate LRE model using two alternative methodologies: Lubik and Schorfheide (2003) and our proposed method. In particular, we consider the following simple model

$$y_t = \frac{1}{\theta_y} E_t(y_{t+1}) + \frac{1}{\theta_y} E_t(x_{t+1}) + \varepsilon_t \tag{67}$$

$$x_t = \frac{1}{\theta_x} E_t(x_{t+1}) \tag{68}$$

where $\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$ and the corresponding forecast errors are denoted as

$$\eta_{y,t} \equiv y_t - E_{t-1}(y_t) \tag{69}$$

$$\eta_{x,t} \equiv x_t - E_{t-1}(x_t) \tag{70}$$

7.2.1 Lubik and Schorfheide (2003)

The LRE model in (67)$\sim$ (70) can be written in the following matrix form

$$\Gamma_0 S_t = \Gamma_1 S_{t-1} + \Psi\varepsilon_t + \Pi\eta_t, \tag{71}$$

where $S_t \equiv (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}))'$ and $\eta_t \equiv (\eta_{y,t}, \eta_{x,t})'$.
As the matrix Γ_0 is non-singular, the LRE model in (71) can be written as

$$S_t = \Gamma_1^* S_{t-1} + \Psi^*\varepsilon_t + \Pi^*\eta_t, \tag{72}$$

where

$$\Gamma_1^* \equiv \Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} \mathbf{0_{4\times 2}} & \mathbf{A_{4\times 2}} \end{bmatrix}, \qquad \Pi^* \equiv \Gamma_0^{-1}\Pi = \mathbf{A_{4\times 2}}$$

$$\Psi^* \equiv \Gamma_0^{-1}\Psi = \begin{bmatrix} 0 \\ 0 \\ -\theta_x \\ 0 \end{bmatrix}, \qquad \mathbf{A_{4\times 2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \theta_y & -\theta_x \\ 0 & \theta_x \end{bmatrix}$$

Note that equation (72) corresponds to equation (20) in Lubik and Schorfheide (2004). We now show how to solve the model and obtain equation (26) in Lubik and Schorfheide (2004).

Applying the Jordan decomposition, the matrix Γ_1^* can be decomposed as $\Gamma_1^* \equiv J\Lambda J^{-1}$, where the elements of the diagonal matrix Λ denote the roots of the system

$$\Lambda \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_x & 0 \\ 0 & 0 & 0 & \theta_y \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \mathbf{0} \\ \mathbf{0} & \theta_y \end{bmatrix}.$$

Assuming without loss of generality that $|\theta_x| \leq 1$ and $|\theta_y| > 1$, the system in (72) is indeterminate because the number of expectational variables, $\{E_t(y_{t+1}), E_t(x_{t+1})\}$, exceeds the number of explosive roots, θ_y. Defining the vector $w_t \equiv J^{-1}S_t$, the model can be represented as

$$w_t \equiv \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \mathbf{0} \\ \mathbf{0} & \theta_y \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} \varepsilon_t + \begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix} \eta_t, \tag{73}$$

where the first block denotes the stationary block of the system and the second block is unstable.

The adoption of Sims' (2002) code, Gensys, to solve this model is not appropriate as it deals with determinate models. After having obtained the representation in (73), Gensys would construct a matrix Φ such that premultiplying the system by a matrix $[I \ \ -\Phi]$ would eliminate the effect of non-fundamental shocks. Equivalently, the matrix has to satisfy the condition

$$[I \ \ -\Phi] \begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix} = \tilde{\Pi}_1 - \Phi\tilde{\Pi}_2 = 0. \tag{74}$$

Under determinacy, the matrix $\tilde{\Pi}_2$ is square and, assuming that it is also non-singular[17], it is possible to solve for $\Phi = \tilde{\Pi}_1 \left(\tilde{\Pi}_2\right)^{-1}$.

The approach in Lubik and Schorfheide (2003) modifies this intuition to account for the indeterminacy that characterizes the model in (73). Under indeterminacy, the matrix $\tilde{\Pi}_2$ is a vector

[17] Note that Gensys obtains the matrix Φ even when the matrix $\tilde{\Pi}_2$ is singular by applying a singular value decomposition.

with more columns than rows, implying that it is not possible to obtain a matrix Φ that satisfies the above condition in (74). Nevertheless, Lubik and Schorfheide (2003) apply a singular value decomposition (SVD) to the matrix $\tilde{\Pi}_2$ to obtain

$$\tilde{\Pi}_2 \equiv UDV' = \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'_{.1} \\ V'_{.2} \end{bmatrix} = U_{.1} D_{11} V'_{.1}, \tag{75}$$

where D_{11} is a diagonal matrix and U and V are orthonormal matrices. In this particular example, the matrix to decompose is $\tilde{\Pi}_2 = \begin{bmatrix} a & b \end{bmatrix}$, where $a \equiv -\theta_y$ and $b \equiv -\theta_x \theta_y / (\theta_x - \theta_y)$, and the resulting SVD is

$$\tilde{\Pi}_2 \equiv UDV' = 1 \begin{bmatrix} d & 0 \end{bmatrix} \begin{bmatrix} \frac{a}{d} & \frac{b}{d} \\ \frac{b}{d} & -\frac{a}{d} \end{bmatrix}, \tag{76}$$

where $d \equiv \sqrt{a^2 + b^2}$. Lubik and Schorfheide (2003) then proceed by defining the matrix Φ as

$$\Phi = \tilde{\Pi}_1 \left(V_{.1} d^{-1} U'_{.1} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \theta_x \end{bmatrix} \begin{bmatrix} \frac{a}{d} \\ \frac{b}{d} \end{bmatrix} \frac{1}{d} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \theta_x \frac{b}{d^2} \end{bmatrix},$$

and premultiply the system in (73) by the following matrices

$$\begin{aligned} \begin{bmatrix} I & -\Phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} &= \begin{bmatrix} I & -\Phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \theta_y \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \\ &+ \begin{bmatrix} I & -\Phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} \varepsilon_t + \underbrace{\begin{bmatrix} I & -\Phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix}}_{\neq 0} \eta_t, \end{aligned} \tag{77}$$

where the second block represents the constraint that guarantees the boundedness of the solution,

$$w_{2,t} = 0 \Longleftrightarrow E_t(y_{t+1}) = -\frac{b}{a} E_t(x_{t+1}). \tag{78}$$

Importantly, given that the model is indeterminate, the last term in equation (77) differs from zero and therefore non-fundamental disturbances affect the model dynamics. Solving (77) for the endogenous variables, S_t, the system takes the form

$$S_t = \tilde{\Gamma}_1^* S_{t-1} + \tilde{\Psi}^* \varepsilon_t + \tilde{\Pi}^* \eta_t, \tag{79}$$

where

$$\tilde{\Gamma}_1^* \equiv \begin{bmatrix} \mathbf{0_{4\times 2}} & \mathbf{B_{4\times 2}} \end{bmatrix}, \qquad \tilde{\Psi}^* \equiv \left(\frac{a}{d}\right)^2 \begin{bmatrix} 1 \\ b/a \\ -\theta_x (b/a)^2 \\ \theta_x b/a \end{bmatrix},$$

$$\tilde{\Pi}^* \equiv \mathbf{B_{4\times 2}} = \begin{bmatrix} \left(b^2/d^2\right) & -\frac{b}{a}(1-b^2/d^2) \\ -ab/d^2 & (1-b^2/d^2) \\ \theta_x \left(b^2/d^2\right) & -\theta_x \frac{b}{a}(1-b^2/d^2) \\ -\theta_x ab/d^2 & \theta_x(1-b^2/d^2) \end{bmatrix}.$$

The last step that Lubik and Schorfheide (2003) implement is to express the forecast errors as a function of the fundamental shock, ε_t, and a sunspot shock, ζ_t, as

$$\eta_t = -V_{.1} D_{11}^{-1} U_{.1}' \tilde{\Psi}_2 \varepsilon_t + V_{.2}\left(\breve{M}\varepsilon_t + M_\zeta \zeta_t\right), \tag{80}$$

where $V_{.2}' = \begin{bmatrix} \frac{b}{d} & -\frac{a}{d} \end{bmatrix}$. Combining (79) with (80) and normalizing $M_\zeta = 1$, the solution to the LRE model is[18]

$$S_t = \tilde{\Gamma}_1^* S_{t-1} + \tilde{\Psi}^* \varepsilon_t + \tilde{\Pi}^* V_{.2}\left(\breve{M}\varepsilon_t + \zeta_t\right). \tag{81}$$

This solution can be *equivalently* written in a form that explicitly includes the boundedness condition in (78) for which $w_{2,t} = 0$ and therefore $E_t(y_{t+1}) = -\frac{b}{a}E_t(x_{t+1})$. Recalling that $S_t = (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}))'$, the dynamics of the solution in (81) are now expressed as a function of only one state variable,

$$\begin{aligned} S_t &= \begin{bmatrix} -b/a \\ 1 \\ -\theta_x b/a \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \tilde{\Psi}^* \varepsilon_t + \tilde{\Pi}^* V_{.2}\left(\breve{M}\varepsilon_t + \zeta_t\right) \\ &= \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \frac{\theta_y^2}{d^2} \begin{bmatrix} 1 \\ \frac{\theta_x}{(\theta_x-\theta_y)} \\ -\frac{\theta_x^3}{(\theta_x-\theta_y)^2} \\ \frac{\theta_x^2}{(\theta_x-\theta_y)} \end{bmatrix} \varepsilon_t + \frac{\theta_y}{d} \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} \left(\breve{M}\varepsilon_t + \zeta_t\right), \end{aligned} \tag{82}$$

[18] Note that the term $-\tilde{\Pi}^*\left(V_{.1} D_{11}^{-1} U_{.1}' \breve{\Psi}_2\right)\varepsilon_t$ always equals to zero since $\left(\tilde{\Pi}^* V_{.1}\right) = 0$ by the properties of the orthonormal matrix V.

where $d = \sqrt{\theta_y^2 + (\theta_x\theta_y)^2 / (\theta_x - \theta_y)^2}$.

7.2.2 Our proposed methodology

We now provide the derivation of the solution for the LRE model in (71) and reported below in equation (83) using the methodology proposed in this paper

$$\Gamma_0 S_t = \Gamma_1 S_{t-1} + \Psi\varepsilon_t + \Pi\eta_t. \tag{83}$$

The methodology consists of appending the following equation to the original LRE model

$$\omega_t = \frac{1}{\alpha}\omega_{t-1} + \nu_{x,t} - \eta_{x,t},$$

where v_t denotes a newly defined sunspot shock and without loss of generality $\alpha \equiv |\theta_x|$. Denoting the newly defined vector of endogenous variables $\hat{S}_t \equiv (S_t, \omega_t)' = (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}), \omega_t)'$, and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t^x \equiv (\varepsilon_t, \nu_{x,t})'$, the augmented representation of the LRE model is

$$\hat{\Gamma}_0 \hat{S}_t = \hat{\Gamma}_1 \hat{S}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t^x + \hat{\Pi}\eta_t. \tag{84}$$

Pre-multiplying the system in (84) by $\hat{\Gamma}_0^{-1}$, we obtain

$$\hat{S}_t = \hat{\Gamma}_1^* \hat{S}_{t-1} + \hat{\Psi}^*\hat{\varepsilon}_t^x + \hat{\Pi}^*\eta_t, \tag{85}$$

where

$$\hat{\Gamma}_1^* \equiv \begin{bmatrix} \Gamma_1^* & \mathbf{0_{4\times 1}} \\ \mathbf{0_{1\times 4}} & \frac{1}{\alpha} \end{bmatrix}, \quad \hat{\Psi}^* \equiv \begin{bmatrix} \Psi^* & \mathbf{0_{4\times 1}} \\ 0 & -1 \end{bmatrix}, \quad \hat{\Pi}^* \equiv \begin{bmatrix} & \Pi_{4\times 2}^* \\ 0 & 1 \end{bmatrix}.$$

and the matrices $\{\Gamma_1^*, \Psi^*, \Pi^*\}$ are the same as those found in (72). Applying the Jordan decomposition, the matrix $\hat{\Gamma}_1^*$ can be decomposed as $\hat{\Gamma}_1^* \equiv \hat{J}\hat{\Lambda}\hat{J}^{-1}$, where the elements of the diagonal matrix $\hat{\Lambda}$ denote the roots of the system

$$\hat{\Lambda} \equiv \begin{bmatrix} \Lambda & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_x & 0 & 0 \\ 0 & 0 & 0 & \theta_y & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}.$$

Assuming as in the previous section that $|\theta_x| \leq 1$ and $|\theta_y| > 1$, then $1/\alpha = 1/|\theta_x| > 1$ and the diagonal elements of the matrix $\Lambda_{22} = \begin{bmatrix} \theta_y & 0 \\ 0 & 1/\alpha \end{bmatrix}$ correspond to the explosive roots of the system. While the original system in (83) is indeterminate, the augmented representation in (84) is determinate as the number of expectational variables, $\{E_t(y_{t+1}), E_t(x_{t+1})\}$, equals the number of explosive roots, $\{\theta_y, 1/\alpha\}$. Defining the vector $\hat{w}_t \equiv \hat{J}^{-1}\hat{S}_t$, the model can be represented as

$$\hat{w}_t \equiv \begin{bmatrix} \hat{w}_{1,t} \\ \hat{w}_{2,t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \hat{w}_{1,t-1} \\ \hat{w}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \hat{\Psi}_1^{**} \\ \hat{\Psi}_2^{**} \end{bmatrix} \hat{\varepsilon}_t^x + \begin{bmatrix} \hat{\Pi}_1^{**} \\ \hat{\Pi}_{2,x}^{**} \end{bmatrix} \eta_t, \tag{86}$$

where the first block is stationary. Given that the second block is unstable, the following two conditions have to be imposed to guarantee the boundedness of the solution. First, the linear combination of the endogenous variables, $\hat{w}_{2,t}$, is set to zero,

$$\hat{w}_{2,t} = 0 \iff \begin{cases} E_t(y_{t+1}) = -\frac{b}{a} E_t(x_{t+1}) \\ \omega_t = 0 \end{cases} \tag{87}$$

Second, the linear combination of fundamental and non-fundamental shocks also has to equal zero. Therefore, the non-fundamental shocks, η_t, become a function of the augmented vector of exogenous shocks, $\hat{\varepsilon}_t^x$,

$$\eta_t = -\left(\hat{\Pi}_{2,x}^{**}\right)^{-1} \hat{\Psi}_2^{**} \hat{\varepsilon}_t^x \iff \eta_t = \begin{bmatrix} 1 & -\frac{\theta_x}{\theta_x - \theta_y} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \nu_{x,t} \end{bmatrix} \tag{88}$$

Considering equation (86), it is relevant to point out that the matrix $\hat{\Pi}_{2,x}^{**}$ differs from the corresponding matrix for the representation in which we incorporate the forecast error, $\eta_{y,t}$, defined as $\hat{\Pi}_{2,y}^{**}$,

$$\hat{\Pi}_{2,x}^{**} \equiv \begin{bmatrix} \theta_y & \frac{\theta_x \theta_y}{\theta_x - \theta_y} \\ 0 & -1 \end{bmatrix} \qquad \hat{\Pi}_{2,y}^{**} \equiv \begin{bmatrix} \theta_y & \frac{\theta_x \theta_y}{\theta_x - \theta_y} \\ -1 & 0 \end{bmatrix}.$$

Therefore, when the auxiliary process is written as a function of the non-fundamental shock, $\eta_{y,t}$, the restriction imposed on η_t to guarantee the boundedness of the solution also differs from the one found in (88)

$$\eta_t = -\left(\hat{\Pi}_{2,y}^{**}\right)^{-1} \hat{\Psi}_2^{**} \hat{\varepsilon}_t^y \iff \eta_t = \begin{bmatrix} 0 & 1 \\ \frac{\theta_x - \theta_y}{\theta_x} & -\frac{\theta_x - \theta_y}{\theta_x} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \nu_{y,t} \end{bmatrix} \tag{89}$$

Importantly, from equations (88) and (89) it is possible to establish a relationship that links the two non-fundamental disturbances $\{\nu_{x,t}, \nu_{y,t}\}$ and the exogenous shock ε_t,

$$\nu_{x,t} = \frac{\theta_x - \theta_y}{\theta_x}\varepsilon_t - \frac{\theta_x - \theta_y}{\theta_x}\nu_{y,t}. \tag{90}$$

We show below that equations (87) and (90) are crucial for the equivalence between the augmented representations that include different non-fundamental shocks in the auxiliary processes that our methodology proposes.

The augmented model in (86) is determinate as the second block has two explosive roots to match the two expectational variables of the model. It is therefore possible to apply the approach in Sims'(2002) to construct a matrix $\hat{\Phi}_x$ such that premultiplying the system by a matrix $[I \;\; -\hat{\Phi}_x]$ would eliminate the effect of non-fundamental shocks. Equivalently, the matrix has to satisfy the condition

$$[I \;\; -\hat{\Phi}_x]\begin{bmatrix}\hat{\Pi}_1^{**}\\ \hat{\Pi}_{2,x}^{**}\end{bmatrix} = \hat{\Pi}_1^{**} - \hat{\Phi}_x\hat{\Pi}_{2,x}^{**} = 0. \tag{91}$$

Importantly, the matrix $\hat{\Pi}_{2,x}^{**}$ is square under determinacy and, assuming that it is also non-singular[19], it is possible to solve for $\hat{\Phi}_x = \hat{\Pi}_1^{**}\left(\hat{\Pi}_{2,x}^{**}\right)^{-1}$.

To solve the model, the system in (86) is then premultiplied by the following matrices

$$\begin{aligned}\begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & I\end{bmatrix}\begin{bmatrix}\hat{w}_{1,t}\\ \hat{w}_{2,t}\end{bmatrix} &= \begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & 0\end{bmatrix}\begin{bmatrix}\Lambda_{11} & 0\\ 0 & \Lambda_{22}\end{bmatrix}\begin{bmatrix}\hat{w}_{1,t-1}\\ \hat{w}_{2,t-1}\end{bmatrix} + \\ &+ \begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\Psi}_1^{**}\\ \hat{\Psi}_2^{**}\end{bmatrix}\hat{\varepsilon}_t^x + \underbrace{\begin{bmatrix} I & -\hat{\Phi}_x \\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\Pi}_1^{**}\\ \hat{\Pi}_{2,x}^{**}\end{bmatrix}}_{=0}\eta_t, \end{aligned} \tag{92}$$

where the second block represents the constraint that guarantees the boundedness of the solution, $\hat{w}_{2,t} = 0$. Importantly, the augmented representation is determinate, and the last term of the system in (92) equals zero. Nevertheless, the non-fundamental disturbance, $\nu_{x,t}$, affects the dynamics of the original model through vector of exogenous shocks, $\hat{\varepsilon}_t^x \equiv (\varepsilon_t, \nu_{x,t})'$. Solving (91) for the endogenous variables, $\hat{S}_t \equiv (S_t, \omega_t)' = (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}), \omega_t)'$, the system takes

[19] Note that Gensys obtains the matrix $\hat{\Phi}$ even when the matrix $\hat{\Pi}_2^{**}$ is singular by applying a singular value decomposition.

the form

$$\hat{S}_t = \hat{\Gamma}_1^{**}\hat{S}_{t-1} + \hat{\Psi}_S^{**}\hat{\varepsilon}_t^x$$
$$= \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \\ 0 \end{bmatrix} E_{t-1}(x_t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_t + \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \\ 0 \end{bmatrix} \nu_{x,t}. \tag{93}$$

Finally, to rewrite the reduced-form solution for the augmented representation that includes the non-fundamental shock, $\eta_{y,t}$, in the auxiliary process, we recall equations (87) and (90) that we report below in equations (94) and (95)

$$\hat{w}_{2,t} = 0 \iff \begin{cases} E_t(y_{t+1}) = -\frac{\theta_x}{\theta_x-\theta_y}E_t(x_{t+1}) \\ \omega_t = 0 \end{cases} \tag{94}$$

$$\nu_{x,t} = \frac{\theta_x-\theta_y}{\theta_x}\varepsilon_t - \frac{\theta_x-\theta_y}{\theta_x}\nu_{y,t} \tag{95}$$

Using the above equations, we can rewrite the system in (93) as

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\theta_y-\theta_x}{\theta_x} \\ \theta_x \\ \theta_y-\theta_x \end{bmatrix} E_{t-1}(y_t) + \begin{bmatrix} 0 \\ \frac{\theta_x-\theta_y}{\theta_x} \\ -\theta_x \\ \theta_x-\theta_y \end{bmatrix} \varepsilon_t + \begin{bmatrix} 1 \\ -\frac{\theta_x-\theta_y}{\theta_x} \\ \theta_x \\ -(\theta_x-\theta_y) \end{bmatrix} \nu_{y,t}. \tag{96}$$

7.2.3 Equivalence of methodologies

In this section, we show the equivalence of the representations obtained using the two methodologies. In equation (97) below, we report the solution for the endogenous variables, $S_t = (y_t, x_t, E_t(y_{t+1}), E_t(x_{t+1}))'$, using the methodology of Lubik and Schorfheide (2003),

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \frac{\theta_y^2}{d^2}\begin{bmatrix} 1 \\ \frac{\theta_x}{(\theta_x-\theta_y)} \\ -\frac{\theta_x^3}{(\theta_x-\theta_y)^2} \\ \frac{\theta_x^2}{(\theta_x-\theta_y)} \end{bmatrix} \varepsilon_t + \frac{\theta_y}{d}\begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} \left(\breve{M}\varepsilon_t + \zeta_t\right), \tag{97}$$

where $d = \sqrt{\theta_y^2 + (\theta_x\theta_y)^2/(\theta_x - \theta_y)^2}$. We now report in equation (98) below the solution using our methodology when we include the forecast error, $\eta_{x,t}$, in the auxiliary process[20]

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_t + \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} \nu_{x,t}. \tag{98}$$

To show the equivalence between the two representations, we need to recall the restrictions that each methodology imposed on the forecast errors, η_t, as a function of the exogenous shock, ε_t , and the additional sunspot shock. Following Lubik and Schorfheide (2003), we derived that

$$\eta_t = -V_{.1}D_{11}^{-1}U_{.1}'\tilde{\Psi}_2\varepsilon_t + V_{.2}\left(\breve{M}\varepsilon_t + M_\zeta\zeta_t\right),$$

where we know that $V' = \begin{bmatrix} V_{.1}' \\ V_{.2}' \end{bmatrix} = \begin{bmatrix} \frac{a}{d} & \frac{b}{d} \\ \frac{b}{d} & -\frac{a}{d} \end{bmatrix}$, $D_{11} = d = \sqrt{a^2+b^2}$, $U_1 = 1$, $\tilde{\Psi}_2 = -a = \theta_y$ and $b = -\theta_x\theta_y/(\theta_x - \theta_y)$. Therefore, normalizing $M_\zeta = 1$, we obtain

$$\begin{aligned} \eta_t &= \begin{bmatrix} \frac{a}{d} \\ \frac{b}{d} \end{bmatrix} \frac{a}{d}\varepsilon_t + \begin{bmatrix} \frac{b}{d} \\ -\frac{a}{d} \end{bmatrix} \left(\breve{M}\varepsilon_t + \zeta_t\right) \\ &= \left\{ \frac{\theta_y^2}{d^2} \begin{bmatrix} 1 \\ \frac{\theta_x}{(\theta_x-\theta_y)} \end{bmatrix} + \frac{\theta_y}{d} \begin{bmatrix} -\frac{\theta_x}{(\theta_x-\theta_y)} \\ 1 \end{bmatrix} \breve{M} \right\} \varepsilon_t + \frac{\theta_y}{d} \begin{bmatrix} -\frac{\theta_x}{(\theta_x-\theta_y)} \\ 1 \end{bmatrix} \zeta_t. \end{aligned} \tag{99}$$

Similarly, from the derivation using our methodology, we know that

$$\eta_t = -\left(\hat{\Pi}_{2,x}^{**}\right)^{-1} \hat{\Psi}_2^{**}\hat{\varepsilon}_t^x \iff \eta_t = \begin{bmatrix} 1 & -\frac{\theta_x}{\theta_x-\theta_y} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \nu_{x,t} \end{bmatrix} \tag{100}$$

Comparing equations (99) and (100), we also point out that the sunspot shock introduced in our representation, $\nu_{x,t}$, has a clear interpretation: It is always equivalent to the forecast error that is included in the auxiliary process. On the contrary, the sunspot shock, ζ_t, in Lubik and Schorfheide (2003) has a more complex interpretation and the authors provide a formal argument to consider it as a trigger of belief shocks that lead to a revision of the forecasts.

[20] Notice that the last equation of the solution in (93) shows that the auxiliary variable is such that $\omega_t = 0$ and therefore we report here the solution for the endogeonus variables of interest S_t. Also, we showed earlier the equivalence with the representations that includes the forecast error, $\eta_{y,t}$, obtained using the methodology of Bianchi and Nicolò (2018).

We then combine equations (99) and (100) to establish the following relationship

$$\nu_{x,t} = \left[\frac{\theta_y^2}{d^2} \frac{\theta_x}{(\theta_x - \theta_y)} + \frac{\theta_y}{d} \breve{M} \right] \varepsilon_t + \frac{\theta_y}{d} \zeta_t. \tag{101}$$

Plugging this relationship in the solution in equation (98) obtained using our methodology, we derive the solution in (97) derived using the methodology of Lubik and Schorfheide (2004). This result shows that any parametrization in Lubik and Schorfheide (2004) has a unique mapping to our representation. In particular, we now consider the parametrization $\breve{M} = M^*(\theta) + M$, where M is centered at 0 and $M^*(\theta)$ is found by minimizing the distance between the impulse response functions under determinacy and indeterminacy at the boundary of the determinacy region. We can therefore write equation (101) as

$$\nu_{x,t} = \gamma_\varepsilon(M^*(\theta))\varepsilon_t + \gamma_\zeta \zeta_t, \tag{102}$$

where $\gamma_\varepsilon(M^*(\theta)) \equiv \left[\frac{\theta_y^2}{d^2} \frac{\theta_x}{(\theta_x - \theta_y)} + \frac{\theta_y}{d} M^*(\theta) \right]$ and $\gamma_\zeta \equiv \frac{\theta_y}{d}$. Given a parametrization $\{M^*(\theta), \sigma_\zeta\}$ and the normalization $E[\varepsilon_t \zeta_t] = 0$ in Lubik and Schorfheide (2004), we derive the corresponding variance and covariance terms of the non-fundamental shock, $\nu_{x,t}$, introduced in our approach as

$$\sigma_{\nu_x}^2(M^*(\theta)) = \gamma_\varepsilon^2(M^*(\theta))\sigma_\varepsilon^2 + \gamma_\zeta^2 \sigma_\zeta^2 \tag{103}$$

$$\sigma_{\varepsilon,\nu_x}(M^*(\theta)) = \gamma_\varepsilon(M^*(\theta))\sigma_\varepsilon^2 \tag{104}$$

The variance-covariance matrix of the shocks $\hat{\varepsilon}_t^x = \{\varepsilon_t, \nu_{x,t}\}'$ can be written as

$$\Omega_{\hat{\varepsilon}^x}(M^*(\theta)) \equiv \begin{bmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon,\nu_x}(M^*(\theta)) \\ \sigma_{\varepsilon,\nu_x}(M^*(\theta)) & \sigma_{\nu_x}^2(M^*(\theta)) \end{bmatrix}. \tag{105}$$

Implementing a Cholesky decomposition, the shocks $\hat{\varepsilon}_t^x = \{\varepsilon_t, v_t^x\}'$ can be written as

$$\hat{\varepsilon}_t^x = \begin{bmatrix} \varepsilon_t \\ \nu_{x,t} \end{bmatrix} = L(M^*(\theta))u_t \equiv \begin{bmatrix} \sigma_\varepsilon & 0 \\ \frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon} & \sqrt{\sigma_{\nu_x}^2(M^*(\theta)) - \left(\frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon} \right)^2} \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}, \tag{106}$$

where $Var(u_t) = I$ and $E(u_t) = 0$. Finally, the parametrization in Lubik and Schorfheide (2004) can be mapped to the solution we obtained in equation (98) as

$$\begin{bmatrix} y_t \\ x_t \\ E_t(y_{t+1}) \\ E_t(x_{t+1}) \end{bmatrix} = \begin{bmatrix} \frac{\theta_x}{(\theta_y-\theta_x)} \\ 1 \\ \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ \theta_x \end{bmatrix} E_{t-1}(x_t) + \begin{bmatrix} 1 & \frac{\theta_x}{(\theta_y-\theta_x)} \\ 0 & 1 \\ 0 & \frac{\theta_x^2}{(\theta_y-\theta_x)} \\ 0 & \theta_x \end{bmatrix} \begin{bmatrix} \sigma_\varepsilon & 0 \\ \frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon} & \sqrt{\sigma^2_{\nu_x}(M^*(\theta)) - \left(\frac{\sigma_{\varepsilon,\nu_x}(M^*(\theta))}{\sigma_\varepsilon}\right)^2} \end{bmatrix} \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}. \tag{107}$$

7.3 Appendix C

Posterior distribution for model parameters (2-degrees of indeterminacy)						
	$\{\nu_1{=}\nu_\pi, \nu_2 = \nu_y\}$		$\{\nu_1{=}\nu_\pi, \nu_2 = \nu_b\}$		$\{\nu_1{=}\nu_y, \nu_2 = \nu_b\}$	
	Mean	90% prob. int.	Mean	90% prob. int.	Mean	90% prob. int.
g	0.49	[0.45,0.54]	0.47	[0.42,0.52]	0.46	[0.42,0.51]
λ	0.96	[0.93,0.99]	0.80	[0.65,0.94]	0.80	[0.67,0.94]
α	0.66	[0.60,0.72]	0.60	[0.52,0.69]	0.60	[0.52,0.68]
$100(\gamma^{-1}-1)$	0.50	[0.34,0.65]	0.46	[0.31,0.61]	0.45	[0.31,0.59]
κ	0.040	[0.032,0.048]	0.042	[0.034,0.051]	0.039	[0.032,0.047]
π^*	0.68	[0.36,1.00]	0.70	[0.38,1.01]	0.72	[0.37,1.01]
i^*	1.43	[1.09,1.75]	1.41	[1.09,1.74]	1.42	[1.08,1.74]
ϕ_π	0.31	[0.14,0.48]	0.33	[0.17,0.50]	0.30	[0.14,0.46]
ϕ_q	0.08	[0.04,0.12]	0.10	[0.05,0.15]	0.16	[0.08,0.23]
ρ_i	0.75	[0.61,0.88]	0.67	[0.51,0.82]	0.62	[0.45,0.79]
σ_q	0.29	[0.15,0.44]	0.26	[0.13,0.38]	0.51	[0.19,0.85]
σ_s	0.11	[0.09,0.13]	0.11	[0.09,0.12]	0.11	[0.09,0.12]
σ_i	0.10	[0.08,0.12]	0.10	[0.09,0.12]	0.11	[0.09,0.13]
ρ_q	0.94	[0.91,0.97]	0.68	[0.51,0.84]	0.68	[0.54,0.81]
ρ_s	0.89	[0.83,0.94]	0.90	[0.85,0.94]	0.87	[0.81,0.92]
σ_{ν_1}	0.28	[0.24,0.32]	0.27	[0.23,0.31]	0.70	[0.61,0.78]
σ_{ν_2}	0.69	[0.60,0.78]	2.59	[1.12,4.14]	1.74	[0.73,2.76]
$\varphi_{\nu_1,i}$	-0.42	[-0.67,-0.16]	-0.27	[-0.55,0.01]	0.08	[-0.15,0.34]
$\varphi_{\nu_1,q}$	0.07	[-0.43,0.59]	0.11	[-0.45,0.67]	0.60	[0.40,0.81]
$\varphi_{\nu_1,s}$	0.61	[0.48,0.73]	0.61	[0.48,0.73]	-0.58	[-0.71,-0.44]
$\varphi_{\nu_2,i}$	-0.14	[-0.40,0.13]	-0.53	[-0.75,-0.31]	-0.72	[-0.93,-0.51]
$\varphi_{\nu_2,q}$	-0.01	[-0.52,0.55]	-0.11	[-0.51,0.27]	-0.30	[-0.63,0.03]
$\varphi_{\nu_2,s}$	-0.68	[-0.77,-0.59]	-0.67	[-0.80,-0.55]	-0.39	[-0.63,-0.18]
MDD	-72.3		-73.0		-75.1	

Table 6: Posterior distributions of the estimated model with two degrees of indeterminacy.

Posterior distribution for model parameters (1-degree of indeterminacy)						
	$\{\nu_1{=}\nu_b\}$		$\{\nu_2 = \nu_\pi\}$		$\{\nu_1{=}\nu_y\}$	
	Mean	90% prob. int.	Mean	90% prob. int.	Mean	90% prob. int.
g	0.49	[0.45,0.54]	0.50	[0.45,0.54]	0.50	[0.45,0.54]
λ	0.96	[0.93,0.99]	0.97	[0.94,0.99]	0.97	[0.94,0.99]
α	0.66	[0.60,0.72]	0.66	[0.60,0.72]	0.67	[0.61,0.73]
$100(\gamma^{-1}{-}1)$	0.50	[0.34,0.65]	0.47	[0.34,0.60]	0.50	[0.34,0.66]
κ	0.040	[0.032,0.048]	0.043	[0.033,0.049]	0.040	[0.032,0.048]
π^*	0.68	[0.36,1.00]	0.71	[0.39,1.04]	0.70	[0.38,1.00]
i^*	1.43	[1.09,1.75]	1.44	[1.13,1.77]	1.43	[1.11,1.74]
ϕ_π	0.31	[0.14,0.48]	0.33	[0.14,0.50]	0.30	[0.12,0.45]
ϕ_q	0.08	[0.04,0.12]	0.09	[0.04,0.14]	0.10	[0.04,0.16]
ρ_R	0.75	[0.61,0.88]	0.74	[0.61,0.88]	0.79	[0.66,0.92]
σ_q	0.29	[0.15,0.44]	0.30	[0.14,0.46]	0.32	[0.14,0.48]
σ_s	0.11	[0.09,0.13]	0.11	[0.09,0.14]	0.11	[0.09,0.13]
σ_i	0.10	[0.08,0.12]	0.10	[0.09,0.12]	0.10	[0.08,0.12]
ρ_q	0.94	[0.91,0.97]	0.93	[0.90,0.97]	0.92	[0.88,0.96]
ρ_s	0.89	[0.83,0.94]	0.89	[0.84,0.94]	0.89	[0.84,0.94]
σ_{ν_1}	6.11	[3.44,9.38]	0.28	[0.23,0.31]	0.74	[0.63,0.85]
$\varphi_{\nu_1,i}$	-0.41	[-0.65,-0.19]	-0.47	[-0.81,-0.12]	-0.46	[-0.78,-0.10]
$\varphi_{\nu_1,q}$	-0.70	[-0.84,-0.56]	-0.52	[-0.70,-0.33]	0.01	[-0.27,0.29]
$\varphi_{\nu_1,s}$	-0.46	[-0.57,-0.35]	0.46	[0.26,0.64]	-0.64	[-0.76,-0.53]
MDD	-83.2		-84.2		-83.0	

Table 7: Posterior distributions of the estimated model with one degree of indeterminacy.

Posterior distribution for model parameters (determinacy)		
	Mean	90% prob. int.
g	0.40	[0.35,0.44]
λ	0.98	[0.97,0.99]
α	0.69	[0.66,0.71]
$100(\gamma^{-1}-1)$	0.46	[0.41,0.51]
κ	0.050	[0.042,0.058]
π^*	0.60	[0.34,0.85]
i^*	1.38	[1.12,1.65]
ϕ_π	1.58	[1.46,1.74]
ϕ_q	0.03	[0.01,0.04]
ρ_i	0.78	[0.74,0.82]
σ_q	0.23	[0.13,0.33]
σ_s	0.08	[0.07,0.09]
σ_i	0.20	[0.17,0.23]
ρ_q	0.95	[0.94,0.97]
ρ_s	0.94	[0.90,0.97]
MDD	-158.3	

Table 8: Posterior distributions of the estimated model under determinacy.