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A Simple Model of Voluntary Reserve Targets with Tolerance Bands

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Abstract

This note presents a simplified version of the model of voluntary reserve targets (VRT) developed in Baughman and Carapella (forthcoming), with a Walrasian inter-bank market. First, the model makes transparent the role of target setting in controlling the market rate. Second, the simplicity of the model allows for an analysis of the interaction between VRT and tolerance bands, which are a common tool for controlling rate variability. We find that the persistent overshooting of interbank rates observed during the Bank of England's experiment with VRT may derive from the interaction between target setting and tolerance bands, a new explanation relative to the literature. We also suggest a simple remedy.

1 Introduction

The Federal Reserve System has been, throughout 2019, conducting a strategic review of its approach to monetary policy (Clarida, 2019). An important part of a central bank's overall monetary policy strategy is the set of short-run tactics employed to implement its

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desired policy rate. Different central banks employ a variety of such tactics including reserve requirements, open market operations, and standing deposit and lending facilities. Taken together, the set of tools employed comprise a central bank’s framework for monetary policy implementation. Building a simplified version of the model in Baughman and Carapella (forthcoming), this note describes a framework based on voluntary reserve targets (VRT). Relative to Baughman and Carapella (forthcoming), this paper provides insight into the interaction between the interbank interest rate and banks’ targets, and considers a technique designed to limit rate variability.¹

Before the 2008 financial crisis, the Bank of England (BoE) experimented with a VRT framework with the goal of reducing rate variability (Clews, 2005). In this framework, banks would set a target for their reserves (targeted reserves), and would then be held to this target up to a small error allowance termed a tolerance band (Bank of England, 2006). The BoE incentivized banks to set targets and manage reserves by remunerating those reserves. Targeted reserves were paid at the BoE’s target rate, balances above the target (excess reserves) were paid the target rate less a spread, while banks with balances below the target (reserve shortages) would need to borrow from the BoE at the target rate plus the same spread; a form of symmetric corridor (Berentsen and Monnet, 2008). Voluntary targets, once set, served a similar function to reserve requirements in other countries, as the BoE lacks the authority to set substantial reserve requirements. Specifically, targets can serve as the basis for two methods to limit rate variability – tolerance bands and reserve averaging – both of which depend on binding requirements.²

Here, complementary to Baughman and Carapella (forthcoming), we consider in detail the effect of target setting on the market rate.³ We provide a simple argument that, given

¹The Federal Reserve Open Market Committee (FOMC) has, as recently as March 2019, committed to continue its current framework (Federal Open Market Committee (2019)), but has previously considered a VRT framework. For citations and discussion of FOMC transcripts, see the introduction of Baughman and Carapella (forthcoming).

²See Clouse and Dow (2002) and Hamilton (1996) on reserve averaging, and Armenter (2016) and Lee (2016) on tolerance bands.

³Baughman and Carapella (forthcoming) considered a decentralized, OTC interbank market, focusing on a variety of complexities of the US system. In the current paper, we offer a streamlined model with a

an appropriate remuneration function, optimal target setting itself helps to guide market rates. A reserve remuneration function which features a certain separability induces banks to set targets which move the expected market rate to equal the rate on targeted reserves.⁴ Hence, simply by setting the rate on targeted reserves, a central bank moves the market rate without the need for open market operations – providing the simplicity and flexibility of a floor system discussed in Keister *et al.* (2008), but still providing an incentive for interbank trade.

A remuneration function without the needed separability may provide perverse incentives, moving market rates away from the rate on targeted reserves. Specifically, with a remuneration function where tolerance bands are based on targeted reserves, the expected market rate exceeds the target rate because of banks’ optimal target setting strategy. This reflects the BoE’s experience. While the VRT system implemented by the BoE was largely successful at controlling rate variability, the average interbank rate consistently exceeded the BoE’s target. One plausible explanation, explored by Lee (2016), places the blame on implicit asymmetries in the corridor due to stigma, collateral costs, or credit risk. Our analysis of target setting instead points to tolerance bands, and rules out the shape of the corridor. We suggest a simple alteration which could resolve the problem, basing tolerance bands on something other than targets.

While not the driving factor of interest rates, we find that symmetry of the corridor is important for quantities such as the level of banks’ targets and individual banks’ demand for reserves. Under symmetry and other sufficient conditions, aggregate targets equal expected aggregate reserves, and individual banks trade to exactly meet their targets when the market rate equals the rate on targeted reserves.

Walrasian market where “the market rate” is a meaningful object that we can clearly connect to banks’ target setting.

⁴In the text, we consider only piecewise linear remuneration functions. In the appendix, we consider a general remuneration function, provide precise definitions for separability and symmetry of the remuneration function, and explore which results depend on which assumptions.

2 Voluntary Reserve Targets with a Walrasian Market

We describe a model of voluntary reserve targets with a perfectly competitive interbank market in the style of Poole (1968), first presenting a simplified version of the analysis in Baughman and Carapella (forthcoming), and then applying the model to study tolerance bands. In the model, banks' choice of targets is endogenous, and so are interest rates in the interbank market. Differently from the over-the-counter, frictional markets of Baughman and Carapella (forthcoming), the interbank market is modeled here as a frictionless Walrasian market. New relative to our previous work, we allow for aggregate shocks and rich heterogeneity.

The economy is populated by a unit mass of banks indexed by i , and lasts for one period, from the evening of one day to the evening of the next day. In the first evening, banks choose a target for their reserve balances for the next day, T_i . The next morning, the aggregate state is drawn, $\omega \in \Omega$. This aggregate state determines the aggregate quantity of reserves, and affects the distribution of reserve balances of different banks. Banks start the day with early balances D_i drawn from some distribution $G_i(D_i|\omega)$. After learning the aggregate state and their individual balances, banks enter the interbank market. Banks observe the market rate, i_F and decide how much to borrow F_i (or lend if $F_i < 0$). After the close of the market, banks receive a shock to their balances, $\epsilon_i \sim H(\epsilon_i)$, which is assumed to have a median of zero ($H(0) = 1/2$).⁵

The central bank pays a high rate on balances up to the target, a low rate on balances over the target, and charges a fee for shortages relative to the target. Specifically, let ϕ be the fee on shortages with respect to the target, i_E be the rate on reserves in excess of the target (excess reserves), and i_T be the rate on reserves up to the target (targeted reserves). Assume $i_E < i_T < i_T + \phi$. For notational convenience we drop the index i , highlighting it again when relevant. The remuneration of end of day reserve holdings D with a target of T can be expressed concisely as

⁵This is without loss of generality: If the median value of ϵ is $\bar{\epsilon}$, simply redefine D to $D + \bar{\epsilon}$ and ϵ to $\epsilon - \bar{\epsilon}$.

$$R(D, T) = \begin{cases} i_T T + i_E(D - T) & \text{if } D \geq T \\ i_T D - \phi(T - D) & \text{if } D < T. \end{cases}$$

Notice, this can be written as $R(D, T) = i_T T + \tilde{R}(D - T)$ where

$$\tilde{R} = \begin{cases} (i_T + \phi)(D - T) & \text{if } D \leq T \\ i_E(D - T) & \text{if } D > T \end{cases}$$

This additively separable form of remuneration into a portion depending on T and a portion depending on $D - T$ drives our core result, Proposition 1, that the expected market rate should equal the rate on targeted reserves, i_T . Before deriving that, however, let us first define value functions and solve for optimal decisions working backwards.

Let $U(D, T) = \mathbb{E}_\epsilon[R(D + \epsilon, T)]$ be the expected value of holding reserve balances D after trading in the interbank market but before learning the final shock. Then

$$U(D, T) = i_T T + \int_{-\infty}^{T-D} (i_T + \phi)(D + \epsilon - T) dH(\epsilon) + \int_{T-D}^{\infty} i_E(D + \epsilon - T) dH(\epsilon). \quad (1)$$

Stepping backward, the expected value entering the interbank market is to choose a quantity of reserves to trade, F (positive if borrowing, negative if lending), at the market rate i_F .

$$W(D, T) = \max_F U(D + F, T) - i_F F.$$

Writing $\hat{D} = D + F$ and using equation (1), the first order condition⁶ for this problem is

$$i_F = (i_T + \phi)H(T - \hat{D}) + i_E[1 - H(T - \hat{D})], \quad (2)$$

⁶The second order condition is satisfied: $\partial^2 U / \partial F^2 = -(i_T + \phi - i_E)H'(T - D) < 0$.

which can be re-written in terms of spreads as

$$i_F - i_T = \phi H(T - \hat{D}) - (i_T - i_E)[1 - H(T - \hat{D})]. \quad (3)$$

The quantities ϕ and $i_T - i_E$ reflect the width of the corridor around the target rate on the top and bottom, respectively. If the corridor is symmetric, $\phi = (i_T - i_E)$, the first order condition (3) reduces to

$$\frac{i_T - i_F}{\phi} = 1 - 2H(T - \hat{D}). \quad (4)$$

When the rate is on target, by which we mean $i_F = i_T$, we have $H(T - \hat{D}) = 1/2$. Given our assumption that the median of H is zero, this implies that $\hat{D} = T$. That is, when the interbank rate equals the rate on targeted reserves and the corridor is symmetric, all banks trade to reach their targets. If the interbank rate is above the rate on targeted reserves, $i_F > i_T$, then banks trade to below their targets, $\hat{D} < T$, and vice versa when rates are under target.

Consider, also, the demand curve of each individual bank, derived from (3):

$$F_i = T_i - D_i + H^{-1} \left(\frac{1}{2} - \frac{i_T - i_F}{2\phi} \right)$$

Integrating over the population and using the market clearing condition $\int F_i di = 0$, yields the result that the relationship between aggregate targets, reserves, and interest rates is the same as for the individual bank.

$$\int_i D_i - \int_i T_i = H^{-1} \left(\frac{1}{2} - \frac{i_T - i_F}{2\phi} \right).$$

Or, rearranging,

$$i_F = i_T + 2\phi \left[H \left(\int_i D_i - \int_i T_i \right) - \frac{1}{2} \right]. \quad (5)$$

Since $H(0) = 1/2$, the market rate will be on target when aggregate reserves equal aggregate

gate targets, over when they are over, and under when they are under – the same as in the individual case. Importantly, we can see that the market rate depends *only* on the difference between aggregate targets and aggregate reserves – an aggregation result familiar from representative-bank models such as Whitesell (2006).

Stepping back to the point when banks choose targets, the night before, the expected value of a given target for bank i , $V_i(T)$, is

$$V_i(T) = \mathbb{E}_{\omega, D_i, i_F}[W(D_i, T)].$$

Taking the first order condition with respect to T and substituting from (2) we get

$$0 = \mathbb{E}_{\omega, D_i, i_F}[i_T - (i_T + \phi)H(T - D_i) - i_E[1 - H(T - D_i)]] = \mathbb{E}[i_T - i_F] \quad (6)$$

Hence, it must be the case that average interbank rates, in equilibrium, equal the rate on targeted reserves. We have proved

Proposition 1 *In equilibrium expected rates equal the target: $\mathbb{E}[i_F] = i_T$.*

Regardless of the symmetry of the corridor or the shape of each banks' distribution of reserves, $G_i(D_i|\omega)$, banks will set targets in a way that results in an average market rate equal to the rate on targeted reserves. Indeed, this holds even if there is correlation across banks, reflected here in the dependence of G_i on the aggregate state, ω . The driving force for this result is the additive separability of the remuneration schedule which makes trading decisions affine functions of targets. If banks think that the interbank rate will be higher than the rate on targeted reserves, they have an incentive to lower targets and lend on the market. If banks think the interbank rate will be lower than the rate on targeted reserves, they have an incentive to raise their target and borrow from the market. The only conjecture for rates consistent with equilibrium is that the expected market rate will equal the rate on targeted reserves.⁷ Hence, target setting controls the average rate.

⁷We restrict attention to the only interesting equilibrium, that is the one with interior choice of targets

What level of targets should banks set? We can derive a simple prediction for aggregate targets when both H and G_i are symmetric about their means.

Proposition 2 *If the aggregate state ω induces a distribution of aggregate reserves, $\bar{D} \equiv \int_i D_i \sim G_{\bar{D}}$, that is symmetric about its mean, $\mathbb{E}[\bar{D}]$, and the corridor is symmetric, $\phi = i_T - i_E$, then aggregate targets equal expected aggregate reserves, $\bar{T} \equiv \int_i T_i = \mathbb{E}[\bar{D}]$.*

Proof. This is a special case of Theorem 7 in appendix section C, but we provide a proof using the current notation in appendix section B. ■

Proposition 1 shows that banks' targets always track expected reserves, in the sense that targets result in expected rates that equal the rate on targeted reserves. Additionally, by Proposition 2, if the corridor and the distribution of aggregate reserves are symmetric, banks set their targets so that aggregate targets equal expected aggregate reserves.

Both of these results are about expected values taken at the point of target setting, before ω or D_i are revealed. If realized aggregate reserves exceed aggregate targets, then (5) shows that the market rate will fall below the rate on targeted reserves, and vice versa. A central bank wishing to control rate volatility could add or subtract reserves according to (5). This must be done carefully, however. If banks anticipate that the central bank will respond to their targets, they may strategically alter targets, creating instability discussed in section 5 of Baughman and Carapella (forthcoming). Hence, in order to preserve the rate control promised by Proposition 1, a central bank should condition its operations not on targets, per se, but instead on information about the aggregate state, ω – e.g. information on so-called autonomous factors, like payments to or from the Treasury.

3 Tolerance Bands and Nonseparability

There are other methods to control variability of market rates besides open market operations that adjust the aggregate quantity of reserves. One technique previously employed by the

and interior interbank rates: $T_i \in (0, \infty), i_F \in (0, \infty)$. To be precise, zero supply (i.e. infinite demand) in the interbank market and zero interbank rates are also an equilibrium.

BoE and others is tolerance bands (see Bank of England, 2006, pp. 14). These are analyzed extensively by Armenter (2016) and Lee (2016), and here we extend their analysis to the case of VRT. With tolerance bands, small errors within a proportion δ of the target, T , are forgiven in the sense that they still earn the rate on targeted reserves. The adjusted remuneration function becomes

$$R(D, T) = \begin{cases} i_T D - \phi((1 - \delta)T - D) & \text{if } D < (1 - \delta)T \\ i_T D & \text{if } (1 - \delta)T \leq D < (1 + \delta)T \\ i_T(1 + \delta)T + i_E(D - (1 + \delta)T) & \text{if } (1 + \delta)T \leq D. \end{cases} \quad (7)$$

With such a remuneration function, the demand equation (3) becomes⁸

$$i_T - i_F = (i_T + \phi)H((1 - \delta)T - \hat{D}) + i_E[1 - H((1 + \delta)T - \hat{D})] \quad (8)$$

and, assuming a symmetric corridor and writing

$$\mathcal{M} = (1 - H((1 + \delta)T - \hat{D})) + H((1 - \delta)T - \hat{D})$$

for the probability of falling outside the tolerance band, the target choice equation, (6), becomes

$$0 = i_T - \mathbb{E}[i_F + \phi\delta\mathcal{M}]. \quad (9)$$

So, unless banks are sure they will fall inside the tolerance band, i.e. $\mathcal{M} = 0$ in all states, the average rate will exceed the rate on targeted reserves. When the width of the tolerance band is increasing in targets, banks have an incentive to shade targets higher than they otherwise might in order to increase the width of their tolerance band, so decrease the cost of falling outside it.

⁸The details are similar to above, so are relegated to Appendix A.

Lee (2016) discusses the fact that the BoE had rates on average higher than their target rate during their operation of VRT, and attributes this to factors that make the corridor implicitly asymmetric to the top: stigma of borrowing from the central bank or collateral costs thereof, and credit risk of lending to private counterparties. While these asymmetries may contribute to the experience of the BoE, equation (9) makes clear that the persistent deviation of market rates from the rate on targeted reserves directly derives from target setting and its interaction with tolerance bands even with a symmetric corridor.

As shown in appendix section A.1, this problematic interaction between tolerance bands and voluntary targets, however, can easily be remedied if tolerance bands are not proportional to targets. For example, one could set bands as a function of a bank’s lagged average balances, or some other balance sheet variable like total deposits. The \mathcal{M} term in the target equation would then disappear, but inverse demands would retain the same shape, and the desirable qualities of a VRT without tolerance bands – that market rates average to the targeted rate and aggregate targets track aggregate reserves – could be retained, while still producing the reduced rate variability which is quantified in Lee (2016).

4 Conclusion

We have provided an analysis of VRT with a Walrasian interbank market, characterizing targets, interbank trades, and rates, all under the assumption of a piecewise linear remuneration function. In section C of the appendix, we re-derive our key results for a general remuneration function. There, we provide precise definitions of separability and symmetry of a reserve remuneration function, and make clear which results depend on what assumptions. Results regarding rates depend only on the separability of the remuneration function into a part depending on the target and a part depending on deviations from targets. Results regarding quantities depend on both separability and symmetry, together.

Regarding rates, our first result is an aggregation result common to representative-bank

versions of the Poole model; the market rate depends only on the difference between aggregate targets and aggregate reserves. The second result is more novel, establishing that VRT automatically controls rates; the expected market rate equals the rate on targeted reserves because of banks' optimal target setting strategy.

Neither result on rates depends on a symmetric corridor. Both results, however, would be affected by a tolerance band scheme similar to that implemented by the BoE, which violates separability. Thus, we offer a new explanation for the rate divergence experienced by the BoE during its operation of VRT. Our explanation links rate divergence to the interaction between target setting and tolerance bands, instead of either stigma, credit risks, or collateral costs as discussed in Lee (2016). Nevertheless, that paper provides a quantification of the effect of tolerance bands on rate variability, and understanding the effects of asymmetries remains important.

Regarding quantities, our first result states that, if the remuneration function and the distribution of final shocks are both symmetric, banks trade to exactly satisfy their targets when the market rate equals the rate on targeted reserves. Second, if one additionally assumes the aggregate distribution of reserves is symmetric, then aggregate targets will equal expected aggregate reserves. Deviations of targets from individual or aggregate reserves are not necessarily cause for concern or intervention, and may simply reflect underlying asymmetries in the distribution of shocks.

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A Derivations for tolerance bands

Given the remuneration function (7), the expected end-of-day value is

$$\begin{aligned}
 U(D, T) = & \int_{-\infty}^{(1-\delta)T-D} -\phi(1-\delta)T + (i_T + \phi)(D + \epsilon) dH \\
 & + \int_{(1-\delta)T-D}^{(1+\delta)T-D} i_T(D + \epsilon) dH + \int_{(1+\delta)T-D}^{\infty} (i_T - i_E)(1 + \delta)T + i_E(D + \epsilon) dH \quad (10)
 \end{aligned}$$

Stepping back to the interbank market, one solves for demand from the first order condition to $W(D, T) = \max_F U(D + F, T) - i_F F$ which gives (8). To derive the target choice equation, take a derivative of $V(T) = \mathbb{E}[U_F(D, T)]$ to get

$$\begin{aligned}
 0 = \mathbb{E}[(i_T + \phi)H((1 - \delta)T - \hat{D}) + i_E[1 - H((1 + \delta)T - \hat{D})] \\
 + \phi\delta(1 - H((1 + \delta)T - \hat{D})) + H((1 - \delta)T - \hat{D})] \quad (11)
 \end{aligned}$$

Which gives (9) after substituting from (7) and the definition of \mathcal{M} .

A.1 Non-proportional tolerance bands

Suppose that the width of the tolerance band is fixed at some constant k instead of being a proportion of the target. The altered remuneration function becomes

$$R(D, T) = \begin{cases} i_T D - \phi(T - k - D) & \text{if } D < T - k \\ i_T D & \text{if } T - k \leq D < T + k \\ i_T(T + k) + i_E(D - T - k) & \text{if } T + k \leq D \end{cases} \quad (12)$$

Notice that this can be re-written as $R(D, T) = i_T T + \tilde{R}(D - T)$ where

$$\tilde{R}(x) = \begin{cases} (i_T + \phi)x - \phi k & \text{if } x < -k \\ i_T x & \text{if } -k \leq x < k \\ (i_T - i_E)k + i_E x & \text{if } k \leq x. \end{cases}$$

When tolerance bands do not depend on T , one restores the additive separability of the remuneration function into a part dependant on T and a part dependent on $D - T$. This, then, removes the \mathcal{M} term from the target setting equation (9), and all of the results from Section 2 go through.

B Proof of Proposition 2

Proof. From (5), we can write

$$H(\bar{D} - \bar{T}) = \frac{1}{2} + \frac{i_T - i_F}{2\phi}.$$

Since $\mathbb{E}[i_F] = i_T$ by Proposition 1, we have $\mathbb{E}[H(\bar{D} - \bar{T})] = 1/2$. So, setting $\bar{T} = \mathbb{E}[\bar{D}]$ yields:

$$\begin{aligned}
\mathbb{E}[H(\bar{T} - \bar{D})] &= \int H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) \\
&= \int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) + \int_{-\infty}^{\mathbb{E}[\bar{D}]} H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) \\
&= \int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) + \int_{-\infty}^{\mathbb{E}[\bar{D}]} [1 - H(\bar{D} - \mathbb{E}[\bar{D}])] dG(\bar{D}) \\
&= \int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) + \frac{1}{2} - \int_{-\infty}^{\mathbb{E}[\bar{D}]} H(\bar{D} - \mathbb{E}[\bar{D}])dG(\bar{D}) \\
&= \int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) - \int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) + \frac{1}{2} \quad (13)
\end{aligned}$$

where we used symmetry of H , that is $H(x) = 1 - H(-x)$, and then symmetry of G about $\mathbb{E}[\bar{D}]$.

Lemma 1 *If G is symmetric about $\mathbb{E}[\bar{D}]$ then*

$$\int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})dG(\bar{D}) = \int_{-\infty}^{\mathbb{E}[\bar{D}]} H(\bar{D} - \mathbb{E}[\bar{D}])dG(\bar{D})$$

Proof. To see this, first rewrite the right hand side of the above equation as

$$\begin{aligned}
\int_{-\infty}^{\mathbb{E}[\bar{D}]} H(\bar{D} - \mathbb{E}[\bar{D}])g(\bar{D}) d(\bar{D}) &= \int_{+\infty}^{-\mathbb{E}[\bar{D}]} H(-\bar{D} - \mathbb{E}[\bar{D}])g(-\bar{D}) (-d(\bar{D})) \\
&= \int_{-\mathbb{E}[\bar{D}]}^{+\infty} H(-\bar{D} - \mathbb{E}[\bar{D}])g(-\bar{D}) d(\bar{D})
\end{aligned}$$

We then want to show that

$$\int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})g(\bar{D}) d(\bar{D}) = \int_{-\mathbb{E}[\bar{D}]}^{+\infty} H(-\bar{D} - \mathbb{E}[\bar{D}])g(-\bar{D}) d(\bar{D}) \quad (14)$$

consider each term and do a change of variable: let $u = \mathbb{E}[\bar{D}] - \bar{D}$, so that

$$\begin{aligned}
\int_{\mathbb{E}[\bar{D}]}^{\infty} H(\mathbb{E}[\bar{D}] - \bar{D})g(\bar{D}) d(\bar{D}) &= \int_0^{-\infty} H(u)g(\mathbb{E}[\bar{D}] - u) d(\mathbb{E}[\bar{D}] - u) \\
&= \int_{-\infty}^0 H(u)g(\mathbb{E}[\bar{D}] - u) du
\end{aligned}$$

then let $u = -\mathbb{E}[\bar{D}] - \bar{D}$, so that

$$\begin{aligned} \int_{-\mathbb{E}[\bar{D}]}^{+\infty} H(-\bar{D} - \mathbb{E}[\bar{D}])g(-\bar{D}) d(\bar{D}) &= \int_0^{-\infty} H(u)g(\mathbb{E}[\bar{D}] + u) d(-\mathbb{E}[\bar{D}] - u) \\ &= \int_{-\infty}^0 H(u)g(\mathbb{E}[\bar{D}] + u) du \end{aligned}$$

then (14) can be rearranged as

$$\int_{-\infty}^0 H(u)g(\mathbb{E}[\bar{D}] - u) du = \int_{-\infty}^0 H(u)g(\mathbb{E}[\bar{D}] + u) du$$

which is always satisfied as $g(\mathbb{E}[\bar{D}] - u) = g(\mathbb{E}[\bar{D}] + u)$ by symmetry of G about $\mathbb{E}[\bar{D}]$. ■

Combining the results from Lemma 1 with (13) yields

$$\mathbb{E}[H(\bar{T} - \bar{D})]_{|\bar{T}=\mathbb{E}[\bar{D}]} = \frac{1}{2}.$$

Hence $\bar{T} = \mathbb{E}[\bar{D}]$. ■

C More General Remuneration Functions

In this section, we reconstruct our results with a more general remuneration function, illustrating which results depend on what properties. First, we give our precise definition of separability.

Definition 1 A remuneration function, $R(D, T)$, is separable if there exists some \tilde{R} such that

$$R(D, T) = i_T T + \tilde{R}(D - T).$$

Assumption 1 Assume the remuneration function is separable, and \tilde{R} is a.e. differentiable, weakly concave and injective. Also assume that H has full support.

Next, let $\gamma_i(x) \equiv \mathbb{E}_{\epsilon_i}[\tilde{R}(x + \epsilon_i)]$, $h(y) = \int_i \gamma_i^{-1}(y) di$ and $\Gamma(x) = h^{-1}(x)$. Note that γ_i^{-1} and so Γ exist if one assumes full support for ϵ_i . Given these preliminaries, we can state

Theorem 2 Under Assumption 1, the market rate depends only on the difference between aggregate targets and aggregate reserves: $i_F = \Gamma(\bar{D} - \bar{T})$.

Proof. When entering the interbank market, a bank solves

$$W(D_i, T_i | i_F) = \max_{F_i} \left\{ \mathbb{E}_{\epsilon_i} [i_T T_i + \tilde{R}(D_i + F_i + \epsilon_i - T_i)] - i_F F_i \right\}$$

with first order condition

$$i_F = \mathbb{E}_{\epsilon_i} [\tilde{R}'(D_i + F_i + \epsilon_i - T_i)] = \gamma_i(D_i + F_i - T_i). \quad (15)$$

Inverting γ_i and integrating over i gives

$$\int_i D_i + F_i - T_i di = \int_i \gamma_i^{-1}(i_F) di = \Gamma^{-1}(i_F). \quad (16)$$

Combining (16) with the market clearing condition, $\int_i F_i di = 0$, gives the result. ■

Theorem 3 *Under Assumption 1, the expected market rate equals the rate on targeted reserves: $i_T = \mathbb{E}[i_F]$.*

Proof. When choosing targets, a bank makes a conjecture about other banks' targets, \bar{T} , and solves

$$\max_{T_i} \mathbb{E}_{\omega, D_i} [W(D_i, T_i | i_F = \Gamma(\bar{D} - \bar{T}))].$$

Substituting in the remuneration function and taking derivatives we arrive at our first order condition

$$i_T = \mathbb{E}_{\omega, D_i} \left[\mathbb{E}_{\epsilon_i} [\tilde{R}'(D_i + F_i + \epsilon_i - T_i)] \right] = \mathbb{E}_{\omega, D_i} [\gamma_i(D_i + F_i - T_i)].$$

We arrive at the result by substituting $\gamma_i(\cdot) = i_F$ from (15). ■

The results regarding the level of targets depend upon additional assumptions. Hence, we introduce some further preliminaries: a definition of symmetry for remuneration functions, and some technical lemmas.

Definition 2 *A remuneration function, $R(D, T)$, is symmetric if it is separable and if \tilde{R} satisfies*

$$\tilde{R}(x) + \tilde{R}'(-x) = 2i_T.$$

Lemma 4 *If $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x) + f(-x) = 2a$ for all x and some constant a , and Y is a random variable, which is symmetrically distributed about 0, then $g(x) \equiv \mathbb{E}_Y[f(x + Y)]$ satisfies $g(x) + g(-x) = 2a$. Further, $f(0) = g(0) = a$.*

Proof.

$$\begin{aligned} g(x) + g(-x) &= \mathbb{E}[f(x + Y)] + \mathbb{E}[f(-x + Y)] \\ &= \mathbb{E}[f(x + Y)] + \mathbb{E}[f(-x - Y)] \\ &= \mathbb{E}[f(x + Y) + f(-(x + Y))] = 2a \end{aligned}$$

where the second line follows from symmetry of Y which implies $\mathbb{E}[h(Y)] = \mathbb{E}[h(-Y)]$ for any function h . To see that $f(0) = g(0) = a$, suppose $f(0) = a + \Delta$ for some Δ . Then $f(-0) = 2a - f(0) = a - \Delta$. Hence $a + \Delta = a - \Delta$, so $\Delta = 0$. This applies equally well to g . ■

Lemma 5 *Suppose $f_i(x) : \mathbb{R} \rightarrow \mathbb{R}$ is a collection of strictly decreasing functions which are integrable over i and satisfy $f_i(x) + f_i(-x) = 2a$ for all i . Define*

$$h(x) \equiv \int_i f_i^{-1}(x).$$

Then $g(x) \equiv h^{-1}$ exists and $g(x) + g(-x) = 2a$.

Proof. First, f_i strictly decreasing implies h is also, so g exists (and is strictly decreasing). Next, if $f_i(x) = a + \Delta$, then $f_i(-x) = 2a - f_i(x) = a - \Delta$. Hence, $f_i^{-1}(a + \Delta) = -f_i^{-1}(a - \Delta)$. This implies that, for each $\Delta \in \mathbb{R}$,

$$h(a + \Delta) = \int_i f_i^{-1}(a + \Delta) = x \implies h(a - \Delta) = \int_i f_i^{-1}(a - \Delta) = -x.$$

Hence, $g(x) = a + \Delta$ and $g(-x) = a - \Delta$, giving $g(x) + g(-x) = 2a$. ■

Assumption 2 Assume that the remuneration function is symmetric and that end-of-day errors, ϵ_i , have a symmetric distribution around zero ($H_i(\epsilon_i) = 1 - H_i(-\epsilon_i)$).

Notice that, under assumption 2, the expected marginal remuneration function, γ_i , satisfies the assumptions of Lemma 4.

Theorem 6 Given Assumptions 1 and 2, whenever rates are on target, $i_F = i_T$, banks trade to their target: $F_i = T_i - D_i$.

Proof. When $i_F = i_T$ the first order condition for F_i , equation (15), implies $\gamma_i(F_i + D_i - T_i) = i_T$. Let $Y = \epsilon_i$ and $a = i_T$, so that $g(F_i + D_i - T_i) = \gamma_i(F_i + D_i - T_i) = i_T$ in Lemma 4. Then Lemma 4 implies that one solution to $g(F_i + D_i - T_i) = i_T$ is $F_i + D_i - T_i = 0$; this is the only solution by monotonicity of γ_i and hence g . ■

Finally, if we add one final assumption, we arrive at the result regarding aggregate targets:

Assumption 3 Assume that the distribution of the aggregate state ω induces a distribution of aggregate reserves, \bar{D} , that is symmetric about its mean.

Theorem 7 Under Assumptions 1, 2, and 3, aggregate targets equal expected aggregate reserves: $\bar{T} = \mathbb{E}[\bar{D}]$.

Proof. From Theorems 2 and 3, we have $i_T = \mathbb{E}_\omega[\Gamma(\bar{D} - \bar{T})]$. Now let $f_i = \gamma_i$ in Lemma 5. As γ_i inherits the properties of \tilde{R} , it, therefore, satisfies the assumptions in Lemma 5. So $\Gamma(x) + \Gamma(-x) = 2a$. As a consequence, $\Gamma(x)$ satisfies the assumptions in Lemma 4, for $f(x) = \Gamma(x)$. By assumption 3 the distribution of aggregate reserves is symmetric, and thus satisfies the assumptions in Lemma 4 by letting $Y = \bar{D} - \mathbb{E}_\omega[\bar{D}]$. Then Lemma 4 implies that $g(0) = f(0) = a$ for $g(x) \equiv \mathbb{E}_\omega[\Gamma(x + Y)]$, $x = \mathbb{E}_\omega[\bar{D}] - \bar{T}$ and $a = i_T$. Hence, for $f(x) = \Gamma(x)$, one solution to $g(x) = i_T$ is $\bar{T} = \mathbb{E}_\omega[\bar{D}]$; this is the only solution because g inherits monotonicity from Γ . ■