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Estimates of r^* Consistent with a Supply-Side Structure and a Monetary Policy Rule for the U.S. Economy

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Abstract

We estimate the natural rate of interest (r^*) using a semi-structural model of the U.S. economy that jointly characterizes the trend and cyclical factors of key macroeconomic variables such as output, the unemployment rate, inflation, and short- and long-term interest rates. We specify a monetary policy rule and an equation that characterizes the 10-year Treasury yield to exploit the information provided by both interest rates to infer r^*. However, the use of a monetary policy rule with a sample that spans the Great Recession and its aftermath poses a challenge because of the effective lower bound. We devise a Bayesian estimation technique that incorporates a Tobit-like specification to deal with the censoring problem. We compare and validate our model specifications using pseudo out-of-sample forecasting exercises and Bayes factors. Our results show that the smoothed value of r^* declined sharply around the Great Recession, eventually falling below zero, and has remained negative since then. Our results also indicate that obviating the censoring would imply higher estimates of r^* than otherwise.

Keywords: natural rate of interest, natural unemployment rate, output gap, shadow interest rate

JEL Classification Numbers: C32, C34, E32

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1 Introduction

The natural rate of interest (or simply referred to as r^*) has become a key concept to understand and characterize monetary policy both in theory and practice. As pointed out by Summers and Rachel (2019), monetary policymakers across the globe have highlighted r^* as a fundamental policy variable to assess the stance of monetary policy. For instance, Chair Jerome Powell cites this factor as one of the benchmarks of the Federal Reserve's monetary policy decisions as follows (see Powell, 2019):

> [W]e set our policy interest rate to achieve our goals of maximum employment and stable prices. In doing so, we often refer to certain benchmarks. One of these is the interest rate that would be neutral—neither restraining the economy nor pushing it upward. We call that rate "r^*" (pronounced "r-star"). A policy rate above r^* would tend to restrain economic activity, while a setting below r^* would tend to speed up the economy. A second benchmark is the natural rate of unemployment, which is the lowest rate of unemployment that would not create upward pressure on inflation. We call that rate "u^*" (pronounced "u-star"). You can think of r^* and u^* as two of the main stars by which we navigate. In an ideal world, policymakers could rely on these stars like mariners before the advent of GPS. But, unlike celestial stars on a clear night, we cannot directly observe these stars, and their values change in ways that are difficult to track in real time.

In this paper, we postulate and estimate a semi-structural model of the U.S. economy that allows us to jointly infer measures for r^* and u^* within a framework in which monetary policy is characterized by an inertial version of the Taylor (1993) rule. In particular, r^* is the time-varying intercept of the monetary policy rule. As such, it is the value of the real interest rate that would prevail in the long run, when the inflation rate is at its target and output is at its potential level (and the unemployment rate is at u^*). Additionally, we take a further step and account for the effective lower bound (ELB) on the federal funds rate, as otherwise the relationship between the policy rule and the observed short-term interest rate breaks down when the latter is at the ELB. It is particularly important to explicitly account for the censoring if one wants, like us, to analyze and estimate a sample that includes the Great Recession and its aftermath.

A comprehensive literature on the estimation of a notion of the natural rate of interest for the U.S. already exists (see Lubik and Matthes, 2015; Kiley, 2015; Cúrdia et al., 2015; Del Negro et al., 2017; Christensen and Rudebusch, 2019; Lewis and Vazquez-Grande, 2019; Johannsen and Mertens, forthcoming, for instance). A seminal and original work on the estimation of the natural rate of interest for the U.S. economy is Laubach and Williams (2003) (LW hereafter), which has been subsequently updated and expanded to other advanced economies in Holston, Laubach and Williams (2017) (HLW hereafter). They exploit the theoretical relationship between the real rate of interest and the growth rate of the economy to estimate r^* based on information from real gross domestic product (GDP), the inflation rate, and the short-term real interest rate. While their estimator is widely popular, several issues have been raised regarding their approach by subsequent work (see Beyer and Wieland, 2019). First, a great deal of uncertainty pertains to the estimate of r^*. Second, there is a significant wedge between the output gap estimate and more conventional ones—such as the Congressional Budget Office's (CBO) starting in the early 2000s and widening from then on (strikingly, the output gap estimate casts the Great Recession as a rather shallow downturn, historically speaking). Third, with the model estimated with maximum likelihood methods, a potential pile-up problem arises in which the estimated variances of some shocks are biased towards zero.

We give considerations to these observations in our model. For instance, we directly inform the estimate of r^* by including short- and long-run interest rates as in Bauer and Rudebusch (2019) to better pin down the evolution of r^*. Like Bauer and Rudebusch, we also introduce an inflation trend and inform it with a measure of survey-based inflation expectations to discriminate movements in yields that come from a long-run trend different from r^*. In addition to real GDP and the inflation rate, we also add information on the unemployment rate and use it to better identify the output gap through an Okun's law, as originally proposed by Clark (1989), as well as to estimate the natural rate of unemployment. Finally, we adopt a more robust estimation approach that relies on Bayesian techniques appropriate for state-space modeling.

As indicated earlier, we exploit the information provided by the federal funds rate by specifying its evolution as a Taylor (1993) rule with inertia. However, the binding of the ELB during the Great Recession and the recovery that followed complicates the use of this policy rule for any data set that extends beyond 2008. We tackle this issue by embedding the model with a Tobit-like specification for the Taylor rule and, hence, a shadow rate. Our approach is similar but not the same as those in Wu and Xia (2016) and Johannsen and Mertens (forthcoming). The failure to account for the ELB can significantly distort the outcomes of the estimation both in terms of parameters and latent states—r^* among them. Our results indicate that it is the case.

We estimate our model with a sample that covers 1962:Q1 to 2020:Q1. Our estimate of r^* gradually declines starting in the mid-1980s and enters negative territory in mid 2009, although the real-time estimate only does so in mid-2011; r^* is estimated to have hovered around -1% since 2016, in line with simple estimates of the short-term real interest rate, before gradually edging down to -1.7% over the last year of our sample. We find that the shadow federal funds rate would have reached -6.7% at the trough of the Great Recession. Regarding the natural unemployment rate, we find that it has been steadily declining since 2010, when it reached 5.7%, to a level of 4.6% in 2020:Q1. Our measure of the (time-varying) trend output growth rate has declined over time and has been around 1.7% per year since 2012. Finally, our estimate of the output gap is similar to those from the CBO and the staff of the Federal Reserve Board. It peaked at slightly above 2% in 2019 and has declined to about 1% since then.

Taking advantage of historical data decomposition techniques, we find that our negative estimates of the natural rate of interest since the Great Recession are based on the information from a small subset of observations. More specifically, the secular decline in (what we refer to as) the medium- to long-run real interest rate and the persistently low realized inflation apply sufficient downward pressures on our estimate of r^* for it to turn negative around the Great Recession and remain below zero thereafter. Also contributing to its dynamics, but more at a business cycle frequency, is the parsing of the federal funds rate in relation to our estimate of the output gap—which is primarily identified by the unemployment rate—through the structure of the monetary policy rule.

We also consider relevant changes to the original specification of our model to either validate its adequacy or assess the robustness of the results. For instance, we estimate the model ignoring the ELB and find that r^* is about 50 basis points higher than in the model that takes into account the censoring of the policy rule. We also try a specification in which the innovations of r^* and the trend output growth rate are correlated, as it is imposed in LW and HLW. The results indicate some degree of correlation between the two trend processes, but the findings are overall unaffected by relaxing the assumption that their innovations are independent. Additionally, we introduce an IS curve specification in which the output gap is affected by the real interest rate gap (see Neiss and Nelson, 2003), which we choose to be with respect to the long-term interest rate, as opposed to LW and HLW, where the short rate gap is chosen. We find that r^* in this case is more precisely estimated, a little more volatile, but about the same at the end of the sample compared with the original specification of the output gap, which does not allow for a direct feedback from the interest rate gap.

Finally, we conduct in-sample and pseudo out-of-sample forecasting exercises to determine which specification performs best. The results indicate that the benchmark specification—that incorporates a shadow rate and in which there is no correlation between r^* and the trend output growth rate innovations or an IS curve—outperforms the other specifications. Additionally, we compute real-time estimates of r^* and notice that its low value around -1% has been remarkably stable over the last five years before recently edging down.

2 Literature review

LW and, subsequently, HLW are seminal works on the estimation of natural rates of interest rates for the U.S. economy and, more recently, other advanced economies. One key element of their identification strategy is the relationship between the growth rate of the economy and the real short-term interest rate implied by standard economic theory. Using information on output, the inflation rate, and the short-term interest rate, they document a downward trending estimate of r^*, which in the case of the U.S. economy eventually falls close to zero. Their estimates have become a staple in the economic and policy discussions of r^*, and updates are regularly made publicly available.[1] Nonetheless, numerous studies have sought to improve the LW methodology and estimates.

Lewis and Vazquez-Grande (2019), Beyer and Wieland (2019), Kiley (2015) and Brand and Mazelis (2019) are fairly recent examples of such work. For instance, all four papers use Bayesian methods rather than a multi-step procedure likelihood-based estimator to address the pile-up problem that often afflicts classical estimation approaches.

Lewis and Vazquez-Grande (2019) also studies the consequences of assuming that the non-growth component of r^* is first-difference stationary (as in LW) rather than persistent but stationary. They argue that a mixture of permanent and transitory processes to characterize the natural rate of interest is preferable to the original specification of LW. Their estimate is more procyclical and displays less of a secular decline than the one shown in LW and HLW.

[1] See the New York Fed webpage "Measuring the Natural Rate of Interest" at https://www.newyorkfed.org/research/policy/rstar.

Beyer and Wieland (2019) argue that a large degree of uncertainty surrounds the estimates of LW and that their methodology and estimation methods are highly sensitive to the choice made by the econometrician. They note the challenge of simultaneously estimating many unobserved variables in a large state-space model. For instance, they find that the precision of the estimates does not increase even after adding more than one decade of data relative to the original set of LW, which ended in 2002.

Kiley (2015) also points out the weak identification of the natural rate of interest in the original LW setup. This observation motivates him to investigate possible ways to improve the identification of r^*. He proposes to add an Okun's law equation to the system and account for the role of additional demand shifters (e.g., asset prices, fiscal policy, and credit conditions) in the IS curve equation. The addition of credit spreads is one factor that significantly helps with improving the identification of r^*. Following these changes, estimates of r^* are more stable over time and do not exhibit the same kind of gradual secular decline as shown by the LW estimates.

Brand and Mazelis (2019) estimate a semi-structural model of the U.S. economy featuring key elements of the LW model but also a Taylor-type policy rule to better identify r^*. Their estimate of the r^* process for the U.S. is far more volatile than that of LW and falls well below zero following the Great Recession. They do not, however, account explicitly for the presence of the ELB and assume instead that the observed short-term interest rate is what would have prevailed under their rule, even in the absence of the ELB.

The studies discussed so far in this section have adopted the definition of r^* from LW and mostly followed or investigated the robustness of the assumptions of their model. However, economists have also come up with other concepts of the short-term interest rate to characterize the stance of monetary policy.

For instance, Christensen and Rudebusch (2019) employ flexible dynamic term structure models and financial data (e.g. inflation-indexed debt) to obtain estimates of the real rate that prevail, on average, between the 5- to 10-year horizon window, once business fluctuations have mostly faded. Their framework allows them to compute an equilibrium rate without having to correctly specify the dynamics of the output gap and inflation. The results show that the natural rate of interest has gradually declined over the past two decades to a level close to zero.

Another paper that computes a longer-run (i.e., 5-year horizon) measure of r^* under a flexible approach is Lubik and Matthes (2015). They estimate a time-varying vector autoregressive (TVP-VAR) model, which imposes much fewer theoretical restrictions than LW. Their measure of r^* is the 5-year conditional forecast of the observed real rate implied by this model. Although using a different approach, Lubik and Matthes estimate a path of r^* that is roughly consistent with that of LW starting around the mid-1980s. Unsurprisingly, with few restrictions and time-varying coefficients, the degree of uncertainty around their estimates is relatively large.

Cúrdia, Ferrero, Ng and Tambalotti (2015) argue that policy rules responding to the efficient real interest rate better characterize the evolution of the federal funds rate since late 1987 than traditional monetary policy rules based on estimates of the output gap.[2]

[2] The efficient real interest rate in a DSGE model is that which would prevail in an economy in which prices are flexible and desired markups are zero.

They refer to the former as a Wicksellian policy rule. It is worth noting that the dynamics of their efficient interest rate—and hence their results—are highly dependent on the model specifications and underlying assumptions.

Del Negro, Giannone, Giannoni and Tambalotti (2017) compare the measure of r^* computed from a low-frequency estimate of the short-term interest rate in a VAR model with common trends to the efficient interest rate in a version of the FRBNY-DSGE model (see Del Negro, Giannoni and Schorfheide, 2015). The two methodologies deliver fairly consistent views regarding the gradual decline in the short-term real interest rate observed over the past few decades.

Finally, the paper that is perhaps most closely related to ours is Johannsen and Mertens (forthcoming). They propose a flexible time series approach that decomposes their data as trends and cycles and explicitly accounts for the presence of the ELB by simulating a shadow rate for the periods when the ELB is binding. They also allow for stochastic volatility in the variance of some of the innovations. However, and in contrast to our methodology, they do not identify and infer the output gap based on the structure of their model and the data. Instead, they take the CBO estimate as observed values. The reliance on a reaction function in which the output gap is a significant determinant of the monetary policy rate entails strong identification linkages between the estimate of r^*, the shadow rate, and the output gap. In our paper, we seek to capture the simultaneous directionality of these influences as well as to take into account the contribution of the uncertainty around the output gap estimate on the uncertainty surrounding the estimate of r^*.

3 The Model

Our model of the U.S. economy includes equations for (the log of) real GDP, denoted as y_t, the unemployment rate, u_t, the core personal consumption expenditures (PCE) price inflation rate, π_t, the federal funds rate, R_t, the 10-year Treasury yield, R_t^{10}, and survey information about long-run inflation expectations, π_t^e.

3.1 Real GDP and the unemployment rate

We adopt a trend-cycle decomposition approach to characterize real GDP and the unemployment rate, similar to that used by Clark (1989), as follows:

$$y_t = \tau_t^y + c_t, \tag{1}$$

$$u_t = \tau_t^u + \theta_1 c_t + \theta_2 c_{t-1} + \upsilon_t, \tag{2}$$

where real GDP is decomposed as the sum of potential output, denoted as τ_t^y, and the output gap, denoted as c_t. In turn, we assume that potential output is a local-linear trend, whereas the output gap is a stationary AR(2) process, as shown below:[3]

$$\tau_t^y = \mu_{t-1} + \tau_{t-1}^y + \eta_t^y, \tag{3}$$

[3]In our benchmark model, the output gap is closed in the long run by construction without the need of a feedback from real interest rates. In a sensitivity analysis, we allow for such a feedback.

$$\mu_t = \mu_{t-1} + \nu_t, \tag{4}$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \varepsilon_t, \tag{5}$$

where $\eta_t^y \sim$ i.i.d. $N(0, \sigma_{\eta^y}^2), \nu_t \sim$ i.i.d. $N(0, \sigma_\nu^2)$, and $\varepsilon_t \sim$ i.i.d. $N(0, \sigma_\varepsilon^2)$. Equation (4) allows potential output to exhibit a (time-varying) trend growth rate, denoted as μ_t. This feature is particularly important given the lower-than-average productivity growth rates observed after the Great Recession.

The unemployment rate in equation (2) is determined by an Okun's law with coefficients θ_1 and θ_2. The natural rate of unemployment is given by τ_t^u, which evolves according to the following random walk process:

$$\tau_t^u = \tau_{t-1}^u + \eta_t^u, \tag{6}$$

where $\eta_t^u \sim$ i.i.d. $N(0, \sigma_{\eta^u}^2)$. The Okun's law error, denoted $\upsilon_t \sim$ i.i.d. $N(0, \sigma_\upsilon^2)$, allows for deviations of the unemployment rate from its trend and cyclical components.[4]

3.2 Inflation

We specify the inflation process as a hybrid New Keynesian Phillips curve in which inflation expectations are a weighted average of trend inflation, denoted π_t^*, and its own lag. Inflation is also a function of the degree of slack (measured by c_t) in the economy. The specification appears below:

$$\pi_t = \beta \pi_t^* + (1 - \beta)\, \pi_{t-1} + \kappa c_t + \eta_t^\pi, \tag{7}$$

with $\eta_t^\pi \sim$ i.i.d. $N(0, \sigma_{\eta^\pi}^2)$ and where κ is the slope of the Phillips curve; we ensure long-run neutrality by assuming that $\beta \in (0, 1]$.

Additionally, we assume that the inflation trend evolves as a random walk process, as follows (see Stock and Watson, 2007; Aruoba and Schorfheide, 2011; Cogley and Sargent, 2015; Mertens, 2016, for example):

$$\pi_t^* = \pi_{t-1}^* + \eta_t^{\pi^*}, \tag{8}$$

with $\eta_t^{\pi^*} \sim$ i.i.d. $N(0, \sigma_{\eta^{\pi^*}}^2)$. We choose a random walk specification also because our sample includes the 1970s, which likely has associated a level of trend inflation much higher than what is implied by the readings of inflation in the last three decades. Furthermore, in a similar fashion to Del Negro, Giannoni and Schorfheide (2015) and Bauer and Rudebusch (2019), we use information on 10-year-ahead inflation expectations, denoted as π_t^e, to pin down the inflation trend by assuming the following:

$$\pi_t^e = \pi_t^* + e_t, \tag{9}$$

[4] All the disturbances in this section are independent of each other. Some authors allow for correlated trend-cycle disturbances in a similar setting (see Morley, Nelson and Zivot, 2003; Basistha and Nelson, 2007, for example). Gonzalez-Astudillo and Roberts (2016) allow for correlated disturbances in a similar setting and find that, even though the correlation coefficient is statistically significant, the results are broadly similar with respect to a model in which there is no correlation.

with $e_t \sim$ i.i.d. $N(0, \sigma_e^2)$. This specification explicitly assumes that survey long-run inflation expectations are an unbiased estimate of the inflation trend.

3.3 Interest rates

So far, the chosen specifications for output, the unemployment rate, and inflation are relatively standard. However, our interest is not only to estimate the supply-side of the economy, but also a monetary policy stance that is consistent with it. For that purpose, we augment our model with equations for the short-term interest rate and the 10-year Treasury yield.

The dynamics of the short-term interest rate are determined by a monetary policy rule specified as an inertial version of Taylor (1993):

$$R_t^* = \rho R_{t-1}^* + (1-\rho)\left(r_t^* + \pi_t^* + \alpha^\pi\left(\pi_t - \pi_t^*\right) + \alpha^y c_t\right) + \eta_t^{R^*}, \tag{10}$$

with $\alpha^\pi > 1, \alpha^y > 0, \rho \in [0,1), \eta_t^{R^*} \sim$ i.i.d. $N(0, \sigma^2_{\eta^{R^*}})$, and where R_t^* is the nominal interest rate that would be set by the monetary authority in the absence of a lower bound on the target federal funds rate, also called shadow rate. $r_t^* + \pi_t^*$ is a measure of the trend policy rate, when inflation is at its trend and the output gap is closed. This level of the short interest rate is called "neutral" because is neither expansionary nor contractionary (see Yellen, 2017). As a consequence, r_t^* is a measure of the trend real interest rate, also referred to as natural real interest rate (see Taylor, 1993). We also assume that the policymakers' inflation target is equal to the inflation trend.

In Taylor (1993)'s proposal, r^* was assumed to be constant and equal to 2%, close to the then-estimated steady-state growth rate of trend GDP. The choice of this value was supported at the time by the average historical value of the federal funds rate. However, the economic events that have taken place since the publication of the paper have led monetary policymakers and economists to reconsider the view and assumption of a constant level of r^*. For example, Yellen (2017) points out that a Taylor (1993) policy rule with r^* at 2% prescribes a path for the federal funds rate that is much higher than the median of Federal Open Market Committee (FOMC) participants' assessment of appropriate policy. Yellen mentions that, because overall growth has been quite moderate over the past few years, some recent estimates of the current value of r^* stand close to zero, citing HLW. Similarly, Bullard (2018) advocates for a modernized version of the Taylor (1999) rule in which r^* varies over time. Lower labor productivity growth, a slow pace of labor force growth, and a stronger desire for safe assets than in the past would be factors that currently imply a lower equilibrium real interest rate. In fact, Cúrdia et al. (2015) show that the data prefer an alternative specification of the Taylor (1993) rule, in which the federal funds rate tracks the Wicksellian efficient rate of return as the primary indicator of real activity.

In our benchmark model, we assume that r^* evolves as follows:

$$r_t^* = r_{t-1}^* + \eta_t^{r^*}, \tag{11}$$

with $\eta_t^{r^*} \sim$ i.i.d. $N(0, \sigma^2_{\eta^{r^*}})$. In this specification, we are agnostic about the structural factors that could move r^* over time. We are only interested in estimating the time-varying intercept of the monetary policy rule—that is, the value that would prevail when the output gap is closed and inflation is at its target. Hence, following Johannsen and Mertens (forthcoming), r^* is interpreted as the forecast of the real interest rate in the long run, or trend real interest rate.[5] Orphanides and Williams (2002) and Kiley (2015), among others, also use a random walk specification for r^*, as in equation (11). In fact, Hamilton et al. (2016) finds that the correlation between the real interest rate and output growth (as it is imposed in LW and HLW) can be weak and even switch signs depending on the period analyzed.

The inclusion of a monetary policy rule to improve the identification of r^* has also been investigated in Brand and Mazelis (2019), which they append to a version of the LW model. However, they ignore the matter of the ELB binding and the consequences of this omission for their estimation results. In contrast, we explicitly account for the ELB and specify the observed federal funds rate, denoted as R_t, as the maximum between a lower bound, denoted as $\underline{R}$, and the shadow rate, as follows:

$$R_t = \max\{R_t^*, \underline{R}\}. \tag{12}$$

Several papers in the literature have implemented the notion of the shadow rate to estimate the stance of monetary policy. Bauer and Rudebusch (2016) and Wu and Xia (2016), for instance, use shadow rate term structure models (SRTSM) to calculate the short interest rate during the zero lower bound episode of the U.S. economy. In the SRTSM, the short interest rate depends on latent factors extracted from yields at different maturities or from a combination of yields and macroeconomic variables. Our setup can be viewed as one in which the short interest rate depends on latent factors such as r^*, the inflation trend, and the output gap that are obtained from macroeconomic and financial variables.

To the best of our knowledge, Johannsen and Mertens (forthcoming) is the only study that incorporates the concept of r^* within the framework of a shadow nominal interest rate. The authors impose a long-run Fisher equation in which the shadow rate trend is decomposed into an inflation trend and a real-rate trend. The latter is modeled as in equation (11) and is interpreted as the median forecast of the real interest rate in the long run. Even though yields at different maturities are used to estimate the trends and cycles of the model, they do not impose any no-arbitrage condition.

In the spirit of Johannsen and Mertens (forthcoming), we include in our set of variables the 10-year Treasury yield, denoted as R_t^{10}, as in principle it provides information about the inflation trend and r^*, beyond that given by the short-term interest rate. We specify its dynamics as follows:

$$R_t^{10} = r_t^* + \pi_t^* + c_t^{10}, \tag{13}$$
$$c_t^{10} = p^{10} + \psi_1 c_{t-1}^{10} + \psi_2 c_{t-2}^{10} + \varepsilon_t^{10}, \tag{14}$$

with $\varepsilon_t^{10} \sim$ i.i.d. $N(0, \sigma^2_{\varepsilon^{10}})$ and where c_t^{10} is a stationary process that not only captures the dynamics of the term premium but also part of the short rate cycle—i.e., persistent deviations

[5]We point this out because in our benchmark specification, r^* is not an equilibrium concept, strictly speaking, since this particular specification does not allow for a feedback from the real interest rate to the output gap. That is, monetary policy does not play a role in closing the output gap over the long run. We investigate the statistical importance of this feedback in the specification presented in section 6.4.

of expected future short-term interest rates around the shifting endpoints $r_t^* + \pi_t^*$. While we assume that this process is stationary, we allow for a non-zero mean by introducing the constant p^{10}. With our focus on the estimation of macroeconomic and interest rate trends rather than the term premium and its dynamics, we are comfortable with conflating the cyclical dynamics of both the term premium and the short rate as a single process specified with an AR(2) structure. Johannsen and Mertens make a similar assumption, although they model the cyclical components of the yields with a more general VAR system. The inclusion of a long-term interest rate provides not only some signal about expected future variations in rates of shorter maturity, but also about shifts in their common low-frequency component, which follows from the cointegration relationship we impose. This information provided by the interest rate of longer maturity is particularly valuable when the short-term interest rate is at the ELB.[6]

4 Data

We use data on real GDP, the civilian unemployment rate, the PCE price deflator inflation excluding food and energy, the effective federal funds rate, the 10-year Treasury constant maturity rate, and the 10-year ahead PCE price deflator inflation expectations used in the FRB/US model (called "PTR").[7] All the variables come from the Federal Reserve Economic Data (FRED) database of the Federal Reserve Bank of St. Louis, except PTR, which comes from the publicly available FRB/US dataset. Our sample covers the period 1962:Q1 to 2020:Q1, except for the federal funds rate for which we use a sample that starts in 1988:Q1.[8] Appendix B details the data used.

5 Estimation

We estimate the model with Bayesian methods. The Gibbs sampler alternates sampling between coefficients and latent states. The explicit modeling of the ELB in the specification of the monetary policy rule implies that our state-space model is partially nonlinear. To deal with that situation, we embed the Bayesian estimation of Tobit models proposed by Chib (1995), called data augmentation, within the Gibbs sampler.

Broadly speaking, the procedure is as follows: First, given the (censored) data and initial latent states and parameters, we simulate the shadow rate, R_t^*, for the censored part of the sample from a truncated (from above) normal distribution with mean given by $\rho R_{t-1}^* +$

[6] Appendix A shows the state-space form of our semi-structural model of the U.S. economy in equations (1)—(14).

[7] The FRB/US model is a large-scale estimated general equilibrium model of the U.S. economy that has been in use at the Federal Reserve Board since 1996. The model is designed for detailed analysis of monetary and fiscal policies. More details can be found at the following webpage: https://www.federalreserve.gov/econres/us-models-about.htm

[8] Cúrdia et al. (2015) suggest using data from 1987:Q3 because this period coincides with the date in which Alan Greenspan became Chairman of the Federal Reserve, and monetary policy is generally viewed as have been relatively stable and consistent over time since then, and well-approximated by an interest rate rule.

$(1-\rho)\left(r_t^* + \pi_t^* + \alpha^\pi\left(\pi_t - \pi_t^*\right) + \alpha^y c_t\right)$ and variance $\sigma^2_{\eta^{R^*}}$. This is the data augmentation step suggested by Chib (1995). Second, we use the set of augmented data and obtain simulated states using the Durbin and Koopman (2002) simulation smoother from the state-space model. Third, we verify that the sampled states deliver a shadow rate below the ELB. Fourth, with the sampled states, we obtain draws of the parameters of the model using the conventional independent normal-inverse Gamma posterior scheme, including for the equation of the shadow interest rate. Finally, with the newly sampled parameters and states, we simulate the shadow rate as indicated before and repeat the steps.[9] Appendix C describes in more detail the sampler.[10]

5.1 Prior distributions

Table 1 presents the prior distributions and their hyper parameters in the second column. The hyperparameters of the prior distributions associated with output and the unemployment rate are informed by the relatively standard results in the literature of trend-cycle decompositions (see Clark, 1989; Gonzalez-Astudillo and Roberts, 2016, for example). With respect to inflation, Basistha and Nelson (2007) estimate the coefficient linked to inflation expectations to be between roughly 0.8 and 0.9, Chan and Grant (2017) estimate a posterior mean close to 0.7, and Blanchard (2016)—in a time-varying setting—estimate a sample average close to 0.6; we take a somewhat more conservative stance and set the prior mean of the persistence coefficient equal to 0.5. We also use the estimates from Blanchard to center our prior for the slope of the Phillips curve at 0.2. The variance of the inflation equation is centered at the estimated value in Basistha and Nelson (2007), whereas that of the inflation trend is centered close to the upper bound of the estimates in Stock and Watson (2007). The standard deviation of the measurement equation of inflation expectations is centered at 0.5 to allow for discrepancies between the data about inflation expectations and the inflation trend; we have not been able to find results in the literature that allow us to better inform our choice.

Regarding the monetary policy rule, we center the prior means of its parameters following the calibration of the FRB/US model (see Brayton, Laubach and Reifschneider, 2014), as well as parameter estimates of an inertial version of the Taylor (1993) rule that take into account the ELB and endogeneity, as in González-Astudillo (2018). The shock to r^* has a variance whose prior distribution is centered at a value close to the estimates in Kiley (2015). In terms of the long interest rate, we choose prior means such that the cycle has a hump shape and an average yield equal to that in the sample; the variance of the disturbance is centered at one, for the lack of information in the literature. Nevertheless, the hyper parameters of the inverse gamma prior distributions of the variances are such that only their means are

[9] Monte Carlo simulations confirm that this procedure produces unbiased trajectories of the latent variables.

[10] Our identification and sampling strategies are slightly different, although equivalent, to those suggested by Johannsen and Mertens (forthcoming). In their setup, censored data are treated as missing and drawn from the posterior distribution of the latent states. Only draws satisfying the censoring condition on the short-term interest rate are accepted. In our setup, draws of the shadow rate are simulated through an additional step in the Gibbs sampler by explicitly positing a Tobit regression for the monetary policy rule. This step involves data augmentation and sampling of the rule's coefficients.

well defined, whereas their variances are not, which allows the estimation to more freely pick up the posterior means of these coefficients. Finally, the means of the prior distributions of the initial values of the nonstationary latent factors are set in accordance with the initial values of the relevant variables in the sample.

6 Results

In this section, we present and discuss the results of our benchmark model. We also analyze the effects of imposing a censored specification for the federal funds rate and compare the results with those of a model without censoring. In addition, we analyze the results of a model in which the innovations of the trend growth rate of potential output are correlated with those of the r^* process, permitting but not imposing an economic relationship that is at the core of LW and HLW. Finally, we present the results of a model in which the interest rate gap affects the cyclical position of the economy, as it is the case in traditional IS curve specification of the business cycles.

In each of the models below, the results correspond to 2,000 draws from the posterior distribution after burning in 100,000 draws and thinning every 100th draw. The results have been checked for convergence and absence of autocorrelation of the posterior draws.

6.1 Model with shadow rate

This is our benchmark model and its parameter results appear in columns 3 and 4 of Table 1. The posterior mean estimates of the cycle imply that it is highly persistent with a hump-shaped dynamics and a half-life of about 20 quarters. The Okun's law coefficients indicate a slightly less significant relationship than the conventional 2-to-1 scaling between the output and unemployment gaps. The Phillips curve coefficients imply a somewhat weak link between actual inflation and its trend and a slope with a 68% credible interval between 0.04 and 0.08, which implies a low reaction of inflation to the output gap compared to historical estimates, as documented by Blanchard (2016).

Regarding the monetary policy rule, the posterior mean estimates indicate a relatively high degree of persistence in the rule and sensitivities to inflation and the output gap that are consistent with the literature and that obey the Taylor principle. However, the output gap coefficient is larger than other estimates that do not include the ELB period or that do not take into account the censored specification of the interest rate equation (see Brand and Mazelis, 2019, for example).

In addition, the estimated standard deviation of the innovation of the inflation trend is about 0.2%, whereas that of the measurement error associated to the inflation expectations equation is 0.1%. For r^*, the estimate of the variance of its perturbation implies a standard deviation around 0.2%, in the vicinity of the estimate in Kiley (2015).[11]

Finally, the parameters that characterize the cycle of the long-term interest rate—which picks up both movements of the term premium and future short rates—imply hump-shaped

[11] Kiley (2015) points out that the data provides little information to estimate the variance of the r^* shock in his version of the LW model. We find that the posterior distribution of this parameter has more variability than the prior and that its mean is higher, as can be seen in Appendix D.

dynamics, an average of 1.5%, and a standard deviation around 0.9%.

The results of the estimation with regard to the output gap, the growth rate of potential output, the natural unemployment rate, and r^* appear in figure 1. Our estimate of the output gap in figure 1a resembles those of the CBO—which is implied from their potential output estimation—and the staff of the Board of Governors of the Federal Reserve. Our estimate declines during NBER recession periods, but the magnitudes of the peak and troughs can occasionally differ. For example, both the Board's staff and the CBO estimated an output gap around -6% during the Great Recession, whereas our estimate is close to -8%. Nonetheless, these three estimates imply sweeping output losses relatively to its potential. In contrast, the output gap from LW casts the Great Recession as relatively shallow one. At the end of the sample, the available estimates for the CBO and LW have turned negative whereas our estimate has fallen by almost a full percentage point, but remains in positive territory.

Our estimate of the potential output growth rate, shown in figure 1b, has declined over the sample period, just as that of LW. However, our estimate initiates a decline toward the end of the 1990s that is more pronounced than shown by the estimate of LW. Our estimate stabilizes around 1.6% soon after the Great Recession, about 0.7 percentage point below that of LW. The inclusion of data through 2020:Q1 also results in our smoothed estimate to tick down toward the end of our sample.

The natural unemployment rate estimate in figure 1c shows some variation over time, fluctuating between 4.6% at the end of the sample and 7% during the 1970s; our estimate reached 5.7% during the Great Recession. We compare our measure with that from the CBO, which is lower in general throughout the sample. In 2020:Q1, the CBO estimate stands at 4.3%, within the 68% credible interval of our model, which covers the range 4.0%–5.1%.

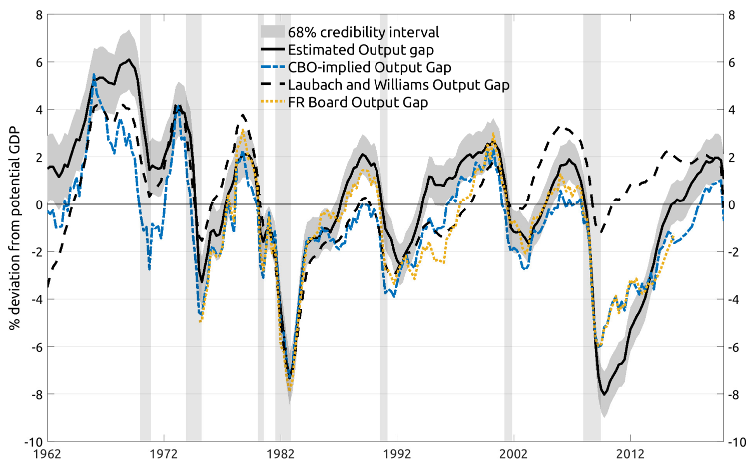
Finally, figure 1d depicts our smoothed estimate of r^* along with filtered estimates of other models in the literature and the smoothed estimated from LW. From the plot, it is apparent that in the period 1962–1982, the estimates that closely follow the approach of LW—in which r^* is explicitly linked to the growth rate of potential output (LW, HWL, and Lewis and Vazquez-Grande (2019))—are markedly above those that do not follow it (among those, our estimate). Higher-than-average economic growth during the 1960s and 1970s entails a similar pattern for the trend output growth rate, which, in turn, is more likely to hold for the equilibrium interest rate, unless the link is diminished through the contribution of the non-growth component and at a price, statistically speaking. Our model, which does not impose a relationship between these two variables, shows that the data prefer diverging patterns for the two trends over the first two decades of the sample. Our results suggest that the addition of a non-growth rate component, as in LW and Lewis and Vasquez-Grande, does not adequately account for the divergence implied by the data. Later in the sample, all the estimates in the existing literature trend down and have roughly stabilized in the last several years of our sample; they range between 0% and a bit above 2% in 2020:Q1. In contrast, our estimate shows a downward trend that has put its 68% credibility interval in negative territory in recent years; our estimate of r^* is -1.7% at the end of the sample. Only three estimates in the literature reach negative territory: that in Kiley (2015) does so after the Great Recession and those of Brand and Mazelis (2019) and Lopez-Salido et al. (2020) (not shown in the figure).

Table 1: Estimates of the model with shadow rate

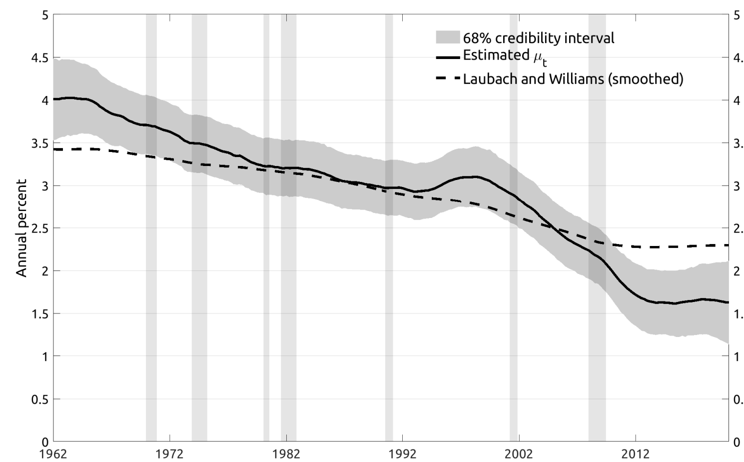
	Prior Distribution	Posterior Mean	68% Credibility Interval
ϕ_1	$\mathrm{N}(1.5, 1)$	1.65	[1.59 , 1.71]
ϕ_2	$\mathrm{N}(-0.6, 1)$	-0.68	[-0.74, -0.62]
σ^2_{ε}	$\mathrm{IG}(2, 0.36)$	0.19	[0.15, 0.22]
$\sigma^2_{\eta^y}$	$\mathrm{IG}(2, 0.49)$	0.29	[0.26, 0.33]
σ^2_{ν}	$\mathrm{IG}(2, 0.03^2)$	0.03^2	$[0.02^2, 0.03^2]$
θ_1	$\mathrm{N}(-0.25, 0.5)$	-0.36	[-0.41, -0.31]
θ_2	$\mathrm{N}(-0.25, 0.5)$	-0.18	[-0.23, -0.14]
$\sigma^2_{\eta^u}$	$\mathrm{IG}(2, 0.04)$	0.01	[0.01, 0.02]
σ^2_{υ}	$\mathrm{IG}(2, 0.01)$	0.003	[0.002, 0.003]
β	$\mathrm{N}(0.5, 1)$	0.24	[0.20, 0.29]
κ	$\mathrm{N}(0.2, 0.5)$	0.06	[0.04, 0.08]
$\sigma^2_{\eta^\pi}$	$\mathrm{IG}(2, 1)$	0.63	[0.58, 0.69]
$\sigma^2_{\eta^{\pi*}}$	$\mathrm{IG}(2, 1)$	0.03	[0.03, 0.04]
σ^2_{e}	$\mathrm{IG}(2, 0.25)$	0.01	[0.01, 0.01]
ρ	$\mathrm{N}(0.85, 0.1)$	0.70	[0.66, 0.74]
α^π	$\mathrm{N}(1.5, 0.5)$	1.33	[1.17, 1.50]
α^y	$\mathrm{N}(1, 0.5)$	1.06	[0.91, 1.20]
$\sigma^2_{\eta^{R*}}$	$\mathrm{IG}(2, 1)$	0.09	[0.08, 0.11]
$\sigma^2_{\eta^{r*}}$	$\mathrm{IG}(2, 0.01)$	0.05	[0.03, 0.07]
p^{10}	$\mathrm{N}(0.16, 1)$	0.22	[0.08, 0.35]
ψ_1	$\mathrm{N}(1.5, 1)$	1.11	[1.02, 1.20]
ψ_2	$\mathrm{N}(-0.6, 1)$	-0.26	[-0.35, -0.17]
$\sigma^2_{\varepsilon^{10}}$	$\mathrm{IG}(2, 1)$	0.13	[0.11, 0.16]
τ^y_0	$\mathrm{N}(816.1, 5)$	814.58	[813.15, 816.01]
μ_0	$\mathrm{N}(1.2, 0.5)$	1.00	[0.88, 1.12]
τ^u_0	$\mathrm{N}(5.8, 1)$	6.52	[5.76, 7.24]
r^*_0	$\mathrm{N}(1.2, 1)$	1.00	[0.03, 1.96]
π^*_0	$\mathrm{N}(1.7, 1)$	1.68	[1.59, 1.78]

Note: "N" stands for normal distribution whereas "IG" stands for inverse gamma distribution. In the former, the first parameter is the mean and the second is the standard deviation. In the latter, the first is the shape coefficient, denoted a, and the second is the scale, denoted b; the mean of the distribution is $b/(a-1)$ and the variance is $b^2/((a-1)^2(a-2))$.

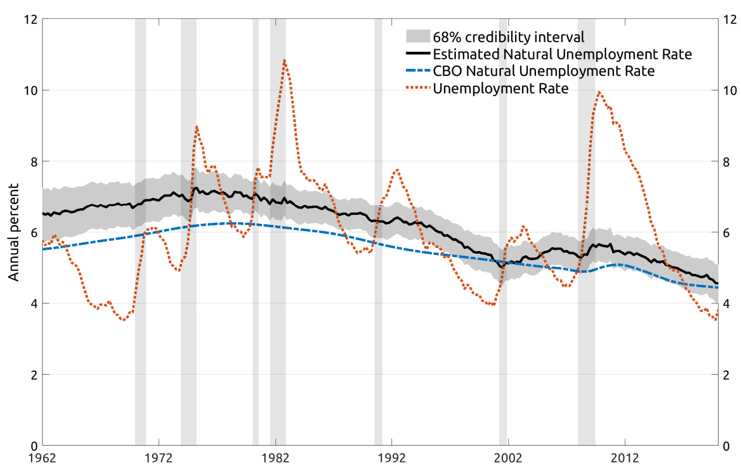
Figure 1: Results of the model with shadow rate



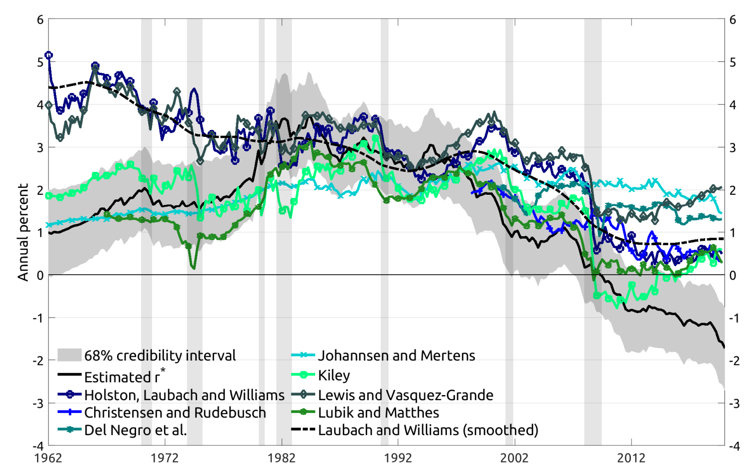
(a) Output gap



(b) Potential output growth rate



(c) Natural unemployment rate



(d) r^*

Note: Shaded vertical areas indicate NBER recession periods. Smoothed estimates are reported, except for the r^* estimates of other studies in the bottom right panel, which are the filtered estimates.

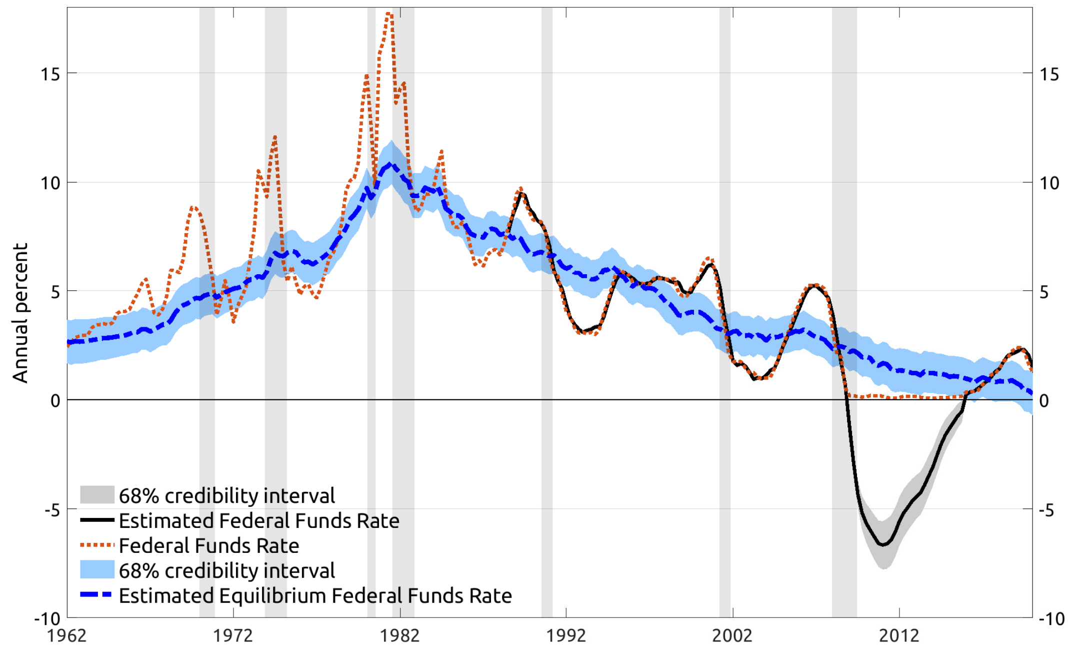
The trend federal funds rate, given by $r_t^* + \pi_t^*$, and the shadow value of the policy instrument (whenever the ELB binds) are shown in figure 2. Starting in 1988, the trend federal funds rate is shown to be a smoother counterpart to the observed rate, with the former above the later in the later stage of the expansions and below during or immediately after the recessions. Also, the estimate of the shadow interest rate reaches -6.8% at the trough of the Great Recession. The decline in the trend federal funds rate accelerates a bit at the end of our sample, and its estimate is slightly above zero with a 68% credibility interval between -0.7% and 1.2% at the beginning of 2020.

There are several candidate explanations for the difference between our estimate of r^* and those in the existing literature. We consider three of them in the coming sections. First, we estimate our model without imposing the ELB. Under this specification, we expect r^* to be higher. Second, we allow for correlated dynamics between r^* and potential output growth, as in the original work of LW. Lastly, we allow for the interest rate gap to affect the output gap dynamics, again as assumed by LW.

6.2 Model without shadow rate

To investigate the effect of ignoring the censored specification of the monetary policy rule, we estimate the model proposed using the observed federal funds rate. This alternative specification results in noticeable shifts in the posterior distributions of the coefficients of the monetary policy rule. The persistence of the rule is roughly unchanged but the reactions to inflation and the output gap are reduced by about 30% and 60%, respectively, whereas the standard deviation of the monetary policy shock increases by about 30%. Understandably, inference about the level of r^* is also influenced by the change in the specification of the model. The variability of the real interest rate trend is little affected by ignoring the censoring. However, as shown in figure 3, the level of the estimate has, on average, moved up compared with the benchmark specification. For instance, the mean estimate of its initial value is revised up by 60 basis points, which is also similar to the average revision over the whole sample now. One can better understand these differences by considering the implications of ignoring the censoring on the linear function that weights estimates of the inflation and output gaps, holding its coefficients constant over the whole sample. During ELB episodes, the outcome of monetary policy (including the contribution of its innovation) is constrained to be equal to the lower bound value, which puts the structure of the model under significant stress whenever it jointly infers large declines in actual output and its gap. That is, at the ELB and assuming for convenience that the inflation gap is unchanged, a large fall in the output gap must be offset either by an increase in the time-varying intercept (i.e. trend federal funds rate) and/or a residual of commensurate size. Given that the fluctuations in the output gap are large and persistent, relying only on policy residuals, which are *i.i.d* by construction and whose estimated variance is set with respect to the entire sample, is especially costly in terms of statistical fit. A mix of lifting the level of the trend federal funds rate, mitigating the declines in the output gap and imposing large negative residuals are ways for the model to satisfy the constraint that the monetary policy instrument must be equal to the ELB whenever it binds. Our results indicate that the estimation exercise relies in all of them.

Figure 2: Estimates related to the short-term nominal interest rate



6.3 Model with correlated trend output growth rate and r^*

In the original work of LW and several subsequent papers by different authors , r^* is explained by the trend growth rate of potential output and so-called other determinants. For instance, in HLW, r^* is given by the following equation (with the notation adapted to our paper):

$$\begin{aligned} r_t^* &= \mu_t + z_t, \\ \mu_t &= \mu_{t-1} + \nu_t, \\ z_t &= z_{t-1} + \eta_t^z, \end{aligned}$$

where z_t is meant to capture the net contribution of the other determinants of r^*.

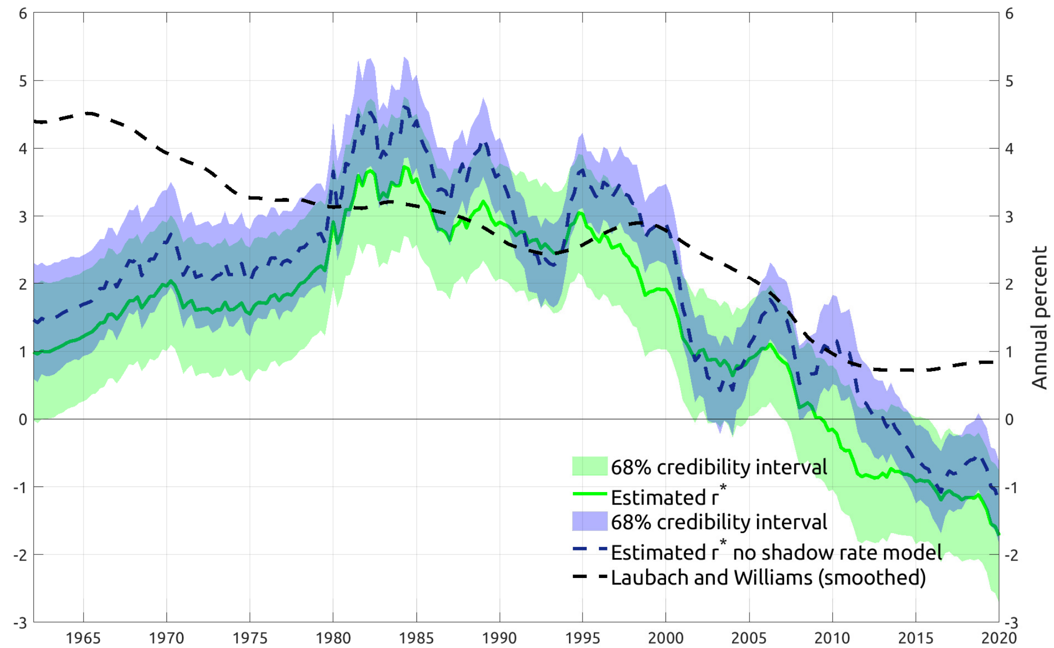
We introduce a dependence between r^* and the trend of potential output growth by allowing their respective error terms to be correlated, as follows:

$$\begin{aligned} r_t^* &= r_{t-1}^* + \eta_t^{r^*} \\ \mu_t &= \mu_{t-1} + \nu_t, \\ \operatorname{corr}\left(\eta_t^{r^*}, \nu_t\right) &= \omega. \end{aligned}$$

Notice that the correlation between changes in r^* and trend output growth rate in HLW is given by

$$\operatorname{corr}\left(\Delta r_t^*, \Delta \mu_t\right) = \operatorname{corr}\left(\eta_t^z + \nu_t, \nu_t\right) = \frac{\sigma_\nu}{\sqrt{\sigma_\nu^2 + \sigma_{\eta_z}^2}}.$$

Figure 3: r^* estimates of the model with and without a shadow rate



Hence, given the parameter estimates in their paper, the implied value of the correlation coefficient is 0.63.

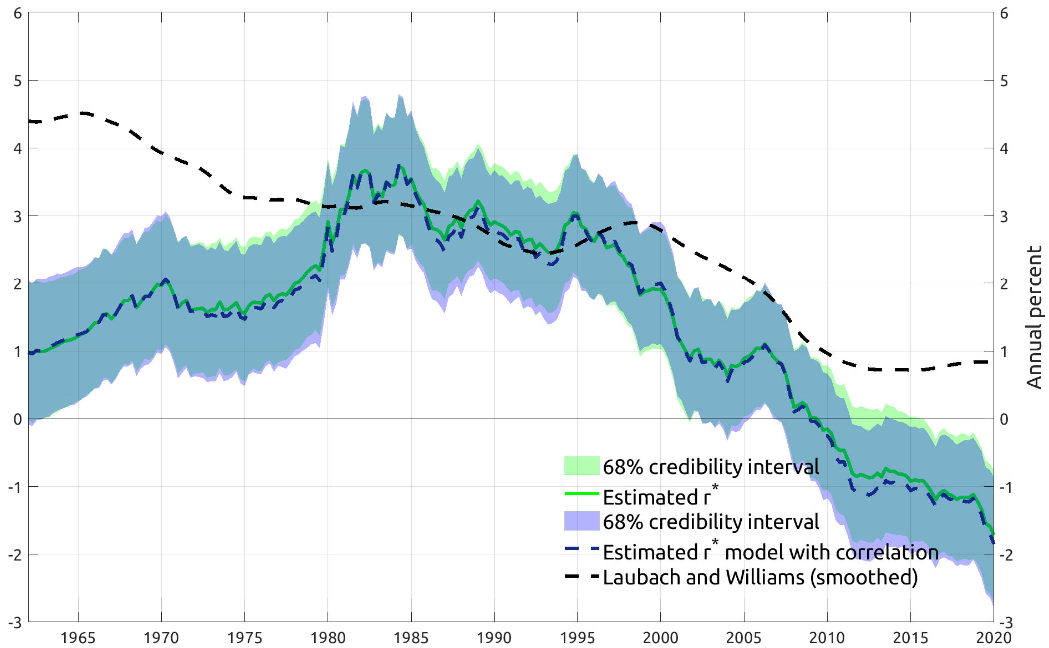
In our case, we assume a uniform prior distribution centered at zero on the correlation coefficient, ω, and embed its posterior sampler within the Gibbs steps.[12] Our mean estimate is slightly below 0.50, with a 68% credibility interval between 0.11 and 0.75, which includes the estimate in HLW. However, our r^* estimate—and of the model in general—does not materially change compared with the case in which the correlation between these two perturbations was assumed to be zero, as can be seen in Figure 4. This is further indication that assuming an exogenous process for r^*, as adopted by Kiley (2015), for example, should not be seen as an inferior choice.

6.4 Model with shadow rate and an IS curve

LW specified the output gap using a reduced-form IS equation where the variable is determined by its own lags and lags of the real funds rate gap, in addition to a serially uncorrelated error. We augment our model with this kind of specification. However, we follow Roberts (2018) in that it is the long-term real interest rate gap what affects the output gap. The reasons to use the long-rate gap—as opposed to the short-rate gap as in LW—are that (i) spending decisions more likely depend on the long-term interest rate gap than on the short interest rate one, (ii) monetary policy makers used balance sheet policies as well as

[12]Strictly speaking, we use an inverse-Wishart prior distribution with 2 degrees of freedom centered at zero to sample the posterior distribution of the correlation coefficient, given the data.

Figure 4: r^* estimates of the models with and without correlated disturbances



forward guidance to influence long rates during the financial crisis, and (iii) the relationship between short and long rates may have changed after the Great Recession. Importantly, in this setup, our estimate of r^* may have the interpretation of an equilibrium real interest rate.

Our specification of the output gap is as follows:

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \lambda_1 \tilde{c}^{10}_{t-1} + \lambda_2 \tilde{c}^{10}_{t-2} + \varepsilon_t,$$

where $\tilde{c}^{10}_t$ is the demeaned cyclical component of the long-term interest rate that appears in equation (14).[13]

Roberts (2018) discusses the parametrization of his model with respect to the sensitivity of the output gap to the interest rate gap. From the two calibrations chosen to match the dynamics of the FRB/US model and those of a Dynamic General Equilibrium Model (DSGE), one can infer that the long-run sensitivities are -3 and -14, respectively. Our estimates of the IS curve coefficients imply a long-run sensitivity of the output gap to the interest rate gap around -8 and a 68% credibility interval between -10.4 and -3.8. Hence, our results lie between those of a DSGE model (with a high interest rate elasticity) and a model like FRB/US (with a low interest rate elasticity).

Figure 5 shows the results of the model with an IS curve. The estimated output gap,

[13] Recall that we had allowed the cyclical component of the long-term interest rate, c^{10}_t, to have a nonzero mean.

trend growth rate of potential output, and natural unemployment rate (whose values in 2020:Q1 are 0.56%, 0.44%, and 4.2%, respectively) are roughly similar to those obtained from the model without the IS specification.

Figure 6 shows additional results from the estimation with an IS curve specification. In figure 6a, we compare the estimate of r^* in this section with that of the benchmark model. As mentioned before, the uncertainty around the estimate of r^* is smaller in the model with an IS curve specification. In this case, the natural interest rate is identified not only by the two interest rates, but also through the deviations of output from its trend, hence making, at least in principle, the estimate more precise.

Figure 6b shows the evolution of the estimated equilibrium federal funds rate. As can be seen, although it is a little more volatile and has less uncertainty around it than that of the benchmark specification (a featured inherited from the r^* estimate), both contours are broadly similar.

6.5 Why has r^* been negative since the Great Recession?

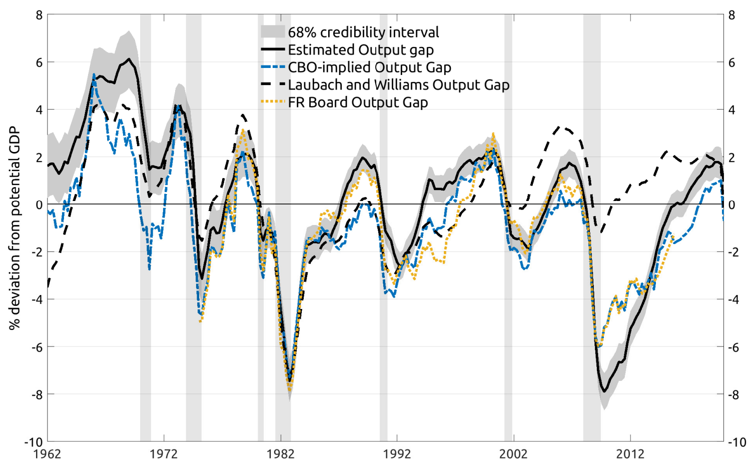
A common feature to all the model's specifications presented thus far is the negative estimates of the natural rate of interest during the Great Recession and since then. These results are in contrast with most of the alternative estimates from the literature shown in figure 1. The comparison begs the question: What aspects of the data and model's structures drive the natural rate of interest negative around the 2008—09 recession and keep it below zero thereafter?

Despite what the bottom-right panel of figure 1 may suggest, we are not one of the first studies to estimate negative values of r^* over the past 10 years. For instance, the model of Kiley (2015) shows a negative r^* between 2009 and 2015. Lopez-Salido et al. (2020) and Brand and Mazelis (2019), although not shown in the figure, also report negative values following the Great Recession, while Williams, Abdih and Kopp (2020) start showing negative estimates roughly around 2014.

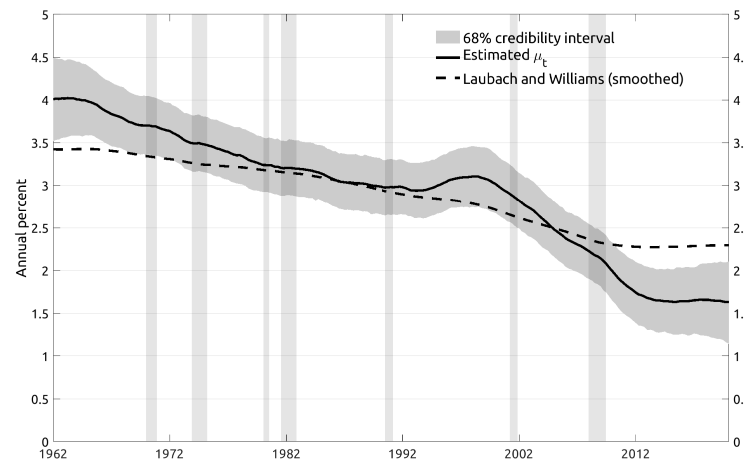
These four papers have in common a feature of our benchmark model that is absent from LW, HLW and the different variants reported in figure 1 (except for Kiley (2015)). They all specify, in contrast to LW and HLW, a Phillips curve in which current inflation is anchored to its trend and, more importantly, the latter is approximated with some measure of long-term inflation expectations. For instance, Lopez-Salido et al. (2020) use the Consensus Economics 10-year-ahead CPI inflation forecast extended back to 1961:Q2 by Blanchard, Cerutti and Summers (2015). Kiley (2015) and Williams, Abdih and Kopp (2020) use survey measures of long-run inflation expectations, as done in this paper. Lastly, Brand and Mazelis (2019) use an inflation trend equal to 2% after the early 1990s; it is also the value of the inflation target in their Taylor rule.

It turns out that using measures of long-run inflation expectations to approximate the inflation trend implies a negative average inflation gap during and following the Great Recession, as shown in figure 7. Lopez-Salido et al. (2020) offer an explanation of why a negative inflation gap can contribute to a lower-than-otherwise r^*: All else equal, a lower inflation gap leads to a lower output gap because of the link enforced by the Phillips curve. In turn, a bare-bone version of the IS curve equation compels a decline in the natural rate of interest to push up the interest rate gap (for a given observed real interest rate) to account for the

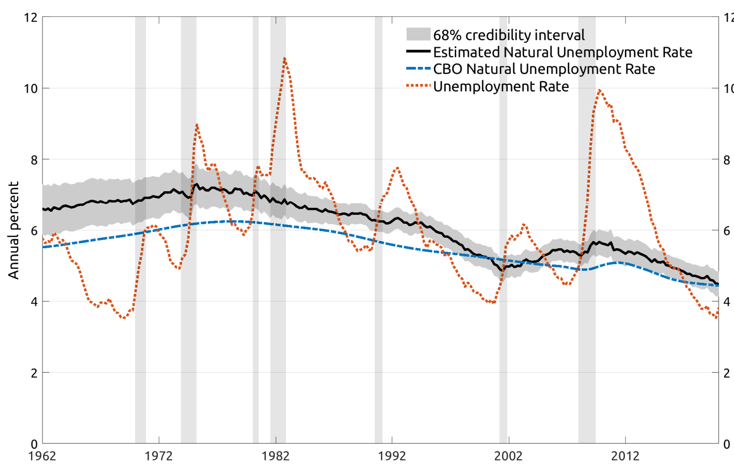
Figure 5: Results of the model with shadow rate and an IS curve



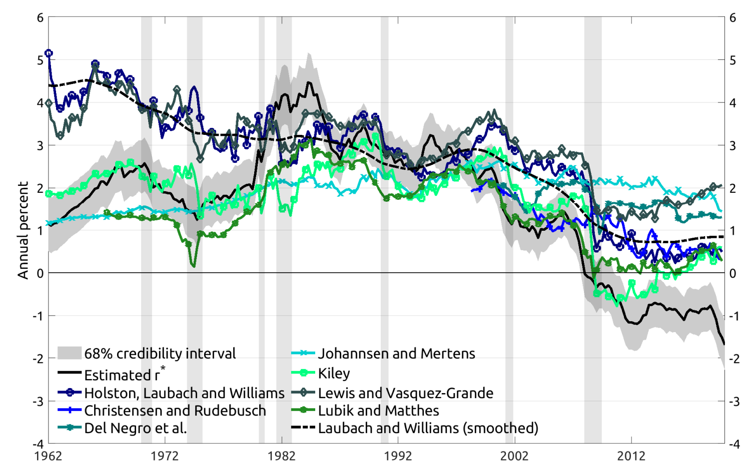
(a) Output gap



(b) Potential output growth rate



(c) Natural unemployment rate



(d) r^*

Note: Shaded vertical areas indicate NBER recession periods. Smoothed estimates are reported, except for the r^* estimates of other studies in the bottom right panel, which are the filtered estimates.

lower output gap on the left-hand side. While this logic accurately reflects the structures of the model in Lopez-Salido et al., our benchmark specification does not have the traditional linkage between the output gap and the short-term interest rate gap in its equation. So, where does the identification come from in our model?

To answer that question, we show the contributions of the observed variables to the path of the natural rate of interest in figure 8.[14] From this figure, we distinguish contributions to the low frequency and high frequency movements in r^*. Unsurprisingly, the evolution of the 10-year Treasury yield, net of the contribution from long-run inflation expectations and the average term premium, is the primary factor driving the lower frequency dynamics of r^*. We can think of this component as the expected real interest rate in the medium to long run.[15]

Another contributor to the low frequency movement of r^* displayed in figure 8 is the inflation rate. It should be noted that the explanatory power of inflation starts with the introduction of the policy rule in 1988, as indicated by the grey vertical line in the figure. Before that point, there is no channel for inflation to directly influence the estimate of r^*.[16] The largest contribution occurs when the policy rule is introduced, which also corresponds to a state of the economy where the level of inflation is at its highest relative to the rest of the sample. Given the policy rule, with everything else equal, higher inflation needs to be offset by a lower r^*. It is therefore not surprising that the filtering procedure assigns a large negative contribution to r^* during this high inflation period. As the inflation rate declines, so does the size of its contribution (in absolute value) to the natural rate of interest. The contribution of the inflation rate data remains negative, however, as realized inflation stabilizes, on average, slightly below its long-run expected value. This low frequency contribution to r^* can be motivated by the logic that a persistently (if not almost permanently insofar as the model can tell) declining and, more broadly, lower inflation are consistent, according to the policy rule, with a persistently lower federal funds rate (everything else equal), which in turn would be seen by the model as implying a lower r^*. Although through a different channel than Lopez-Salido et al. (2020), but with a similar outcome, persistently lower realized inflation—in a context in which it is anchored around stable long-run expectations—leads to noticeable negative contributions to the path of r^*.

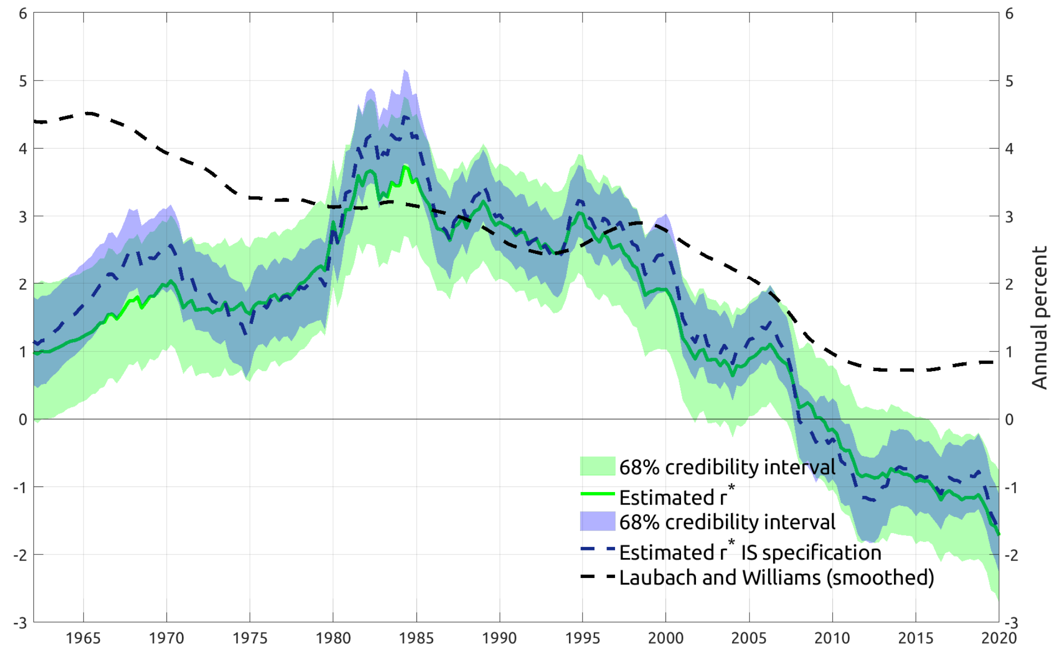
The deployment of the monetary policy rule equation in 1988 allows observations of the unemployment rate—a key factor driving the output gap—and, of course, the federal funds rate to inform the path of r^*. As can be seen in figure 8, the contribution of these two variables primarily relates to the higher frequency dynamics of the natural rate of interest. We interpret this joint contribution as the model's parsing of the stance of monetary policy (i.e., the federal funds rate) given the estimate of economic slack (whose business cycle

[14] This figure details the results of a historical data decomposition—i.e., a calculation of the contribution of each observed variable to the latent variables of the state equations of the model's state-space system. This kind of decomposition was proposed, building on the original work of Koopman and Harvey (2003), by Sander (2013) and Andrle (2013); these papers explain how to compute its elements by exploiting the linear structure of the model, as each observable variable has an independent effect on the smoothed estimates of a latent variable. We refer the readers interested in the more technical aspects of the decomposition to these papers as well as Chung et al. (2020).

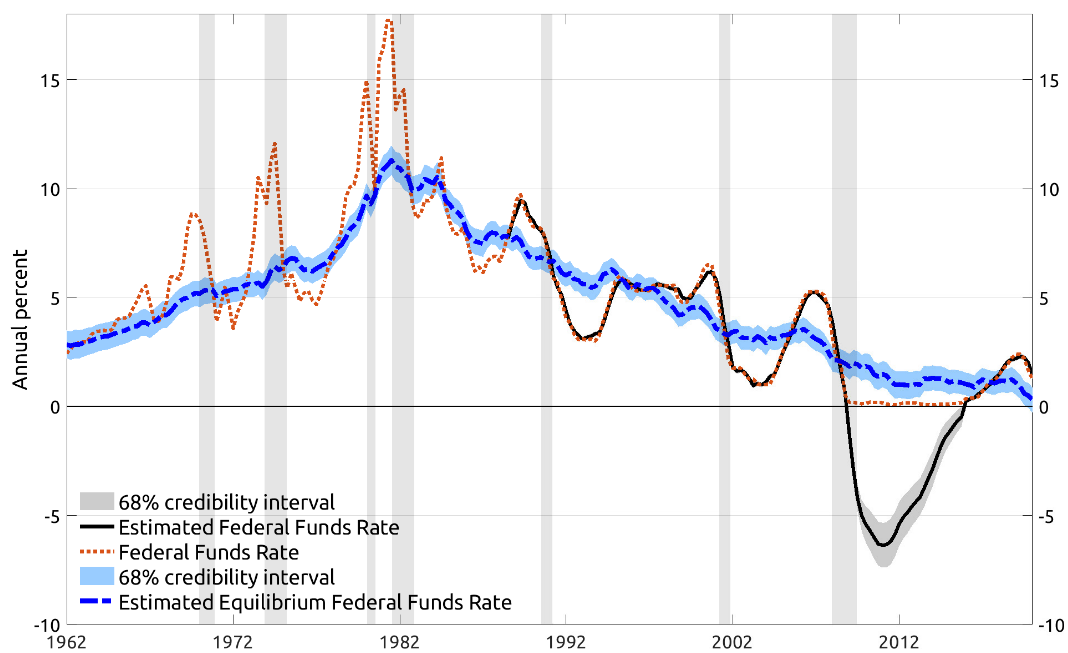
[15] Appendix E shows the disaggregated historical data decompositions for r^* and the output gap.

[16] The nonzero contributions before the implementation of the policy rule reflect the smoothed nature of the shown estimates.

Figure 6: r^* and equilibrium federal funds rate estimates of the model with an IS curve specification

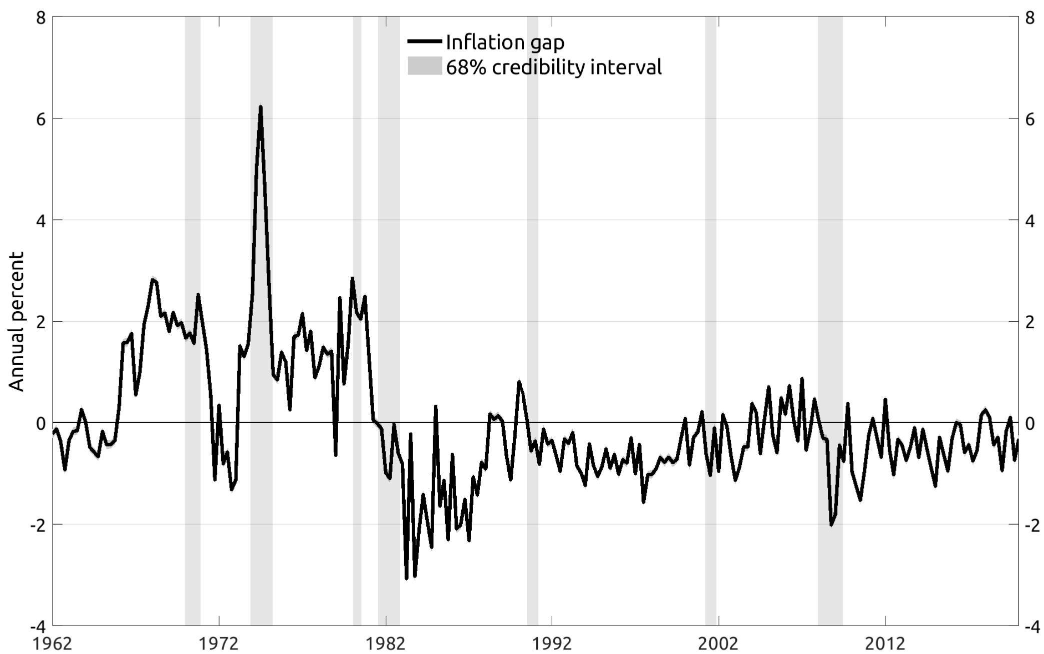


(a) r^* estimates from models with and without an IS curve specification



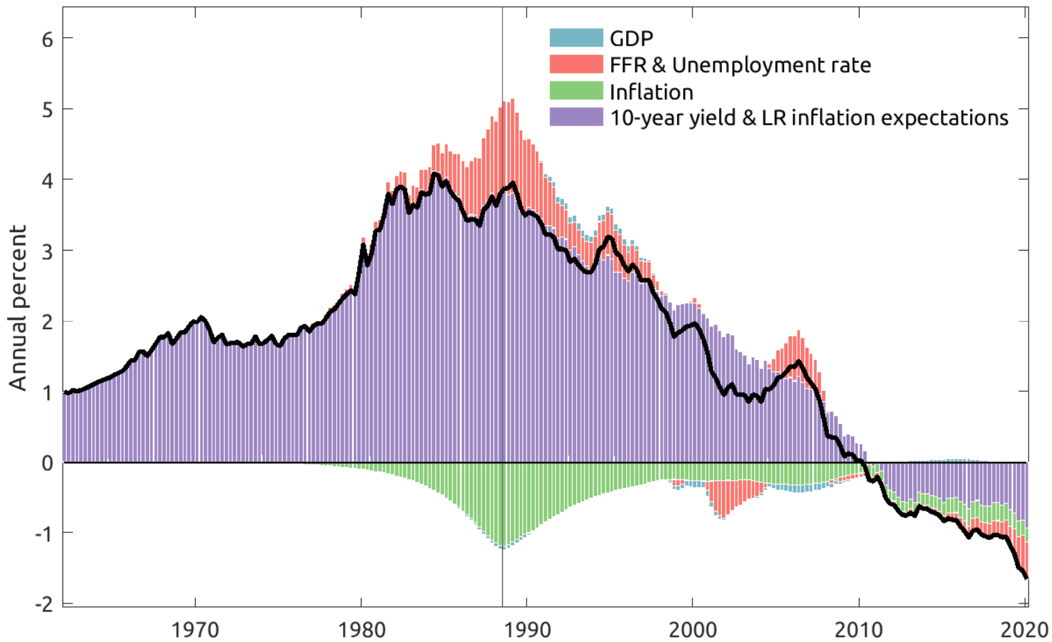
(b) Estimates related to the short-term nominal interest rate

Figure 7: Inflation gap



Note: Shaded vertical areas indicate NBER recession periods. Smoothed estimates are reported. The inflation gap is defined as the posterior mean of $\pi_t - \pi_t^*$.

Figure 8: Historical data decomposition of r^*



Note: The contributions of the federal funds rate (FFR) and the unemployment rate have been added together ('FFR Unemployment rate'). The contributions of the 10-year Treasury yield and long-run inflation expectations have been added together ('10-year yield Inflation expectations'). The grey vertical line indicates the period from which the Taylor rule was added to the system.

fluctuations are informed by the unemployment rate). All else equal, in an environment in which the monetary policy stance is tighter than what would be implied by the measure of the output gap, r^* would need to offset it. Similarly, a lower-than-expected federal funds rate will have to be squared with a higher r^*. For instance, when the policy rate lifted from the ELB at the end of 2015, the strength displayed by the output gap would have warranted a more aggressive policy action, given the rule and everything else. The model squared the observations by, among other things, inferring a more negative contribution to r^* than otherwise.

To summarize: We argue that the most important contributor to obtain a negative natural rate of interest at the onset of the Great Recession, its aftermath, and the recent past is the long-run real interest rate. In addition, the power of identification of a standard monetary policy rule—when realized inflation and a proxy for the policymakers' inflation objective are included in the information set—leads to a negative contribution to the level of r^* at business cycle frequencies.[17]

Finally, we also note the similarity of the results between our benchmark specification

[17] This result is consistent with that in Brand and Mazelis (2019). Starting with the LW model, they also find that the addition of a policy rule to the model's structure and the use of a survey measure of long-run inflation expectations entail a significant downward shift in the level of the natural rate of interest following the Great Recession.

and that with a channel similar to Lopez-Salido et al. (2020)'s, which corresponds to the specification with an IS curve described in section 6.4. Figure 6a indicates that either both identification structures strongly cohere with each other or the influence of the channel highlighted by Lopez-Salido et al. is small once the identification through the short and long interest rates is taken into account.

7 Model evaluations

With the exception of the one that ignores the censoring at the ELB, the different specifications investigated in the previous section appear to lead to rather similar conclusions from a narrative and qualitative perspective. In this section, we conduct a more formal comparison of the different model specifications to determine whether the additional features investigated in sections 6.3 and 6.4 or accounting explicitly for the ELB are quantitatively meaningful according to standard statistical measures and metrics. To that end, we first evaluate the out-of-sample forecasting capabilities of each specification and, second, compute the marginal data density of each model to assess the in-sample fit.

7.1 Pseudo out-of-sample forecasting exercises

To evaluate the forecasting performances of our models, we estimate them and generate projections in a pseudo real-time forecasting environment. More precisely, We begin with the initial sample spanning the period 1962:Q1 through 1989:Q4, estimate the models and jumping off from the last quarter of the aforementioned sample, produce four- to eight-quarter-ahead forecasts for all the observable variables of the model. We then roll forward the exercise by adding one quarter at the time and re-estimating the model every time.[18] Table 2 shows the root-mean-square forecast errors (RMSFE) of the four- and eight-quarter-ahead forecasts for the unemployment rate, the inflation rate, and the federal funds rate.

The results show that, overall, the models that incorporate censoring in the monetary policy rule equation perform better than the model that does not. The difference is especially large in the case of the policy instrument. This result is not necessarily surprising when we consider the inertial character of the policy rule and the inadequacy of the ELB value as jump off condition to this equation when the estimated r^* is significantly negative. Among the three models with a shadow rate, the benchmark specification seems to provide more accurate forecasts. However, the standard deviation of the distribution of RMSFEs does not allow us to conclude that the forecast errors are significantly different among the three models that include a censored specification. These results are consistent with the results discussed in the previous section.

As a final exercise, we also investigate the out-of-sample forecasting performance of a model that does not include interest rates, both short and long, as observable variables. The results show that the forecasting performance of the model deteriorates somewhat, in general, with respect to the models that do include an interest rate block.

[18]We consider 1,000 draws from the posterior distribution after burning in 6,000 draws and thinning every 12th draw. We initialize the Gibbs sampler at the posterior mean estimates obtained with the full sample.

Table 2: Root-mean-square forecast errors for the rates (p.p.)

Quarters ahead	Model	Unemployment	Inflation	Federal funds
Four	With shadow rate	0.75	0.87	1.36
	Without shadow rate	0.76	0.86	1.68
	With correlated disturbances	0.75	0.86	1.36
	With IS curve	0.83	1.23	1.49
	Without Taylor rule	0.78	1.28	
Eight	With shadow rate	1.40	1.02	2.27
	Without shadow rate	1.43	1.02	2.51
	With correlated disturbances	1.40	1.02	2.25
	With IS curve	1.47	1.47	2.42
	Without Taylor rule	1.45	1.55	

7.2 In-sample fit

We also evaluate the different models in terms of their in-sample fit. To that end, we compute the marginal data density of each model to establish the one that best fits the data according to this measure.[19] Table 3 reports this metric for our four models and the KR ratio of each model relative to the benchmark specification.[20]

A comparison of the marginal data densities shows our benchmark specification (first line) that accounts for the ELB and estimates a shadow rate is strongly preferred to the specification that ignores the issues and treats the observed federal funds rate as the relevant policy rate (line 2). Accounting for the correlation between the innovations of the trend rates of output growth and real interest (line 3) does not improve on the benchmark, but rather worsens the fit of the data, although not by very much. Finally, introducing a feedback channel between the long-term interest rate and business cycles (line 4) brings a more significant penalty, possibly by notably increasing the filtering uncertainty without commensurate improvement in reducing the RMSFE. All told, the benchmark specification seems to be preferred.

Table 3: Bayesian model comparison

	Model	Marginal data density	KR ratio
1	With shadow rate	-741.46	–
2	Without shadow rate	-752.76	22.6
3	With correlated disturbances	-742.24	1.6
4	With IS curve	-750.48	18.0

Note: KR ratio is the Kass and Raftery (1995) ratio with respect to the benchmark model.

[19] We use Chib (1995)'s procedure to compute the marginal likelihood of each model

[20] The KR ratio is defined as two times the log of the Bayes factor. Kass and Raftery (1995) suggest that values of KR above 10 can be considered "very strong" evidence in favor of a model. Values between 6 and 10 represent "strong" evidence, between 2 and 6, "positive" evidence, and values below 2 are "not worth more than a bare mention."

7.3 Pseudo real-time estimates

In the introduction of this paper, we cited Jerome Powell and his concern about the stability of the estimate of r^* over time. In this section, we examine the real-time properties of our r^*, u^*, and output gap estimates. Figure 9 plots the pseudo real-time against the final or ex-post estimates starting in 1990:Q1. The last inferred observations of r^*, u^*, and the output gap of each pseudo real-time sample are reported as a red circle.

The figures indicate that the rolling estimates of these three variables can be noticeably different from their final counterparts, as shown by the former positioning themselves at times outside the 68% credible sets of the final estimates.[21] One can observe the largest differences between the real-time and final estimates from the late 90s to most of the next decade. It appears that taking into account the information associated with the Great Recession and its aftermath leads the model to revise its inference on the levels of the latent variables over the decade that preceded these events. For instance, with the insight of the Great Recession and the recovery that followed, the model revised its estimate of r^* and u^* downward and upward, respectively, over the period ranging from 1998 to 2008.

The real-time and final estimates of r^* had been roughly consistent, hovering in negative territory around -1% for most of the last decade. However, the real-time and final estimates seem to have started to diverge again as the economy enters another recession at the end of our sample, in ways similar with previous ones in the period analyzed.

Regarding the natural rate of unemployment, the real-time estimates mostly remain within the 68% credibility interval of the final estimate since the Great Recession, which is an indication of relative stability, but deviate from the ex-post estimate by more than r^* does from its counterpart in the same period. Lastly, our real-time estimate of the output gap seems to be reasonably stable since 2008.

8 Conclusion

In this paper, we formulated and estimated a semi-structural model of the U.S. economy that provides measures of the natural rates of unemployment and interest. Estimates of these concepts can be valuable information into the process leading to decisions by monetary policymakers.

In addition to the estimates of key natural rates, our model also provides estimates of the output gap and potential output. The estimates of the output gap implied by our model are roughly consistent with institutional and judgmentally driven estimates, such as those produced by the CBO or the Federal Reserve Board's staff, in contrast to the estimates of LW and HLW.

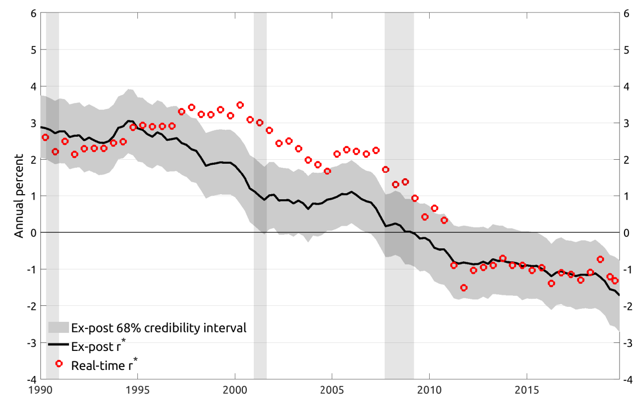
We note that introducing censoring in the monetary policy rule lowers significantly the estimate of r^* compared with a model in which censoring is ignored. This consideration also implies a significantly lower efficient federal funds rate, which is a benchmark recommended by economic theory to evaluate the stance of monetary policy.

[21] It is worth noting that these divergences occur without taking into account the uncertainty arising from data revisions (i.e., difference between early and final releases of the same data), which has been shown to be significant.

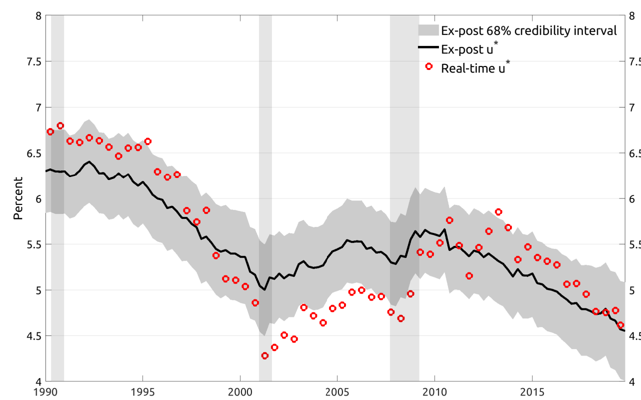
Finally, both in-sample and pseudo out-of-sample exercises suggest that a model specification with a shadow federal funds rate is preferred to a specification that uses the observed policy rate to infer the natural rate of interest. That model has offered a relatively stable real-time estimate of r^* over the recent past.

Figure 9: Pseudo real-time estimates

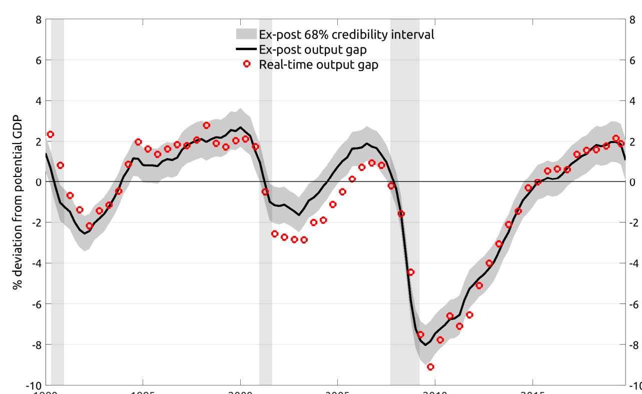
(a) Natural rate of interest



(b) Natural unemployment rate



(c) Output gap



Appendix

A Model in state-space form

The benchmark model is as follows:

$$\begin{bmatrix} y_t \\ u_t \\ \pi_t \\ R_t^* - \bar{\pi}_t \\ \pi_t^e \\ R_t^{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_1 & \theta_2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ z_t \\ c_t \\ c_{t-1} \\ c_{t-2} \\ \tau_t^y \\ \mu_t \\ \tau_t^u \\ r_t^* \\ c_t^{10} \\ c_{t-1}^{10} \\ c_{t-2}^{10} \\ \pi_t^* \end{bmatrix} + \begin{bmatrix} 0 \\ \upsilon_t \\ 0 \\ 0 \\ e_t \\ 0 \end{bmatrix}, \tag{A.1}$$

$$\begin{bmatrix} 1 & 0 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_1 \\ -(1-\rho)\alpha^\pi & 1 & -(1-\rho)\alpha^y & 0 & 0 & 0 & 0 & 0 & -(1-\rho) & 0 & 0 & 0 & (1-\rho)\alpha^\pi \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ z_t \\ c_t \\ c_{t-1} \\ c_{t-2} \\ \tau_t^y \\ \mu_t \\ \tau_t^u \\ r_t^* \\ c_t^{10} \\ c_{t-1}^{10} \\ c_{t-2}^{10} \\ \pi_t^* \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p^{10} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1-\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_1 & \psi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ z_{t-1} \\ c_{t-1} \\ c_{t-2} \\ c_{t-3} \\ \tau_{t-1}^y \\ \mu_{t-1} \\ \tau_{t-1}^u \\ r_{t-1}^* \\ c_{t-1}^{10} \\ c_{t-2}^{10} \\ c_{t-3}^{10} \\ \pi_{t-1}^* \end{bmatrix} + \begin{bmatrix} \eta_t^\pi \\ \eta_t^{R^*} \\ \varepsilon_t \\ 0 \\ 0 \\ \eta_t^y \\ \nu_t \\ \eta_t^u \\ \eta_t^{r^*} \\ \varepsilon_t^{10} \\ 0 \\ 0 \\ \eta_t^{\pi^*} \end{bmatrix}, \tag{A.2}$$

where $\bar{\pi}_t$ is the average of the inflation rate over the last four quarters.

B Data details

Our sample covers the period 1962:Q1 to 2020:Q1. The information about each variable appears below:

- Real GDP: Inflation-adjusted value of the goods and services produced by labor and property located in the United States, billions of chained 2012 dollars, seasonally adjusted, annual rate, quarterly frequency from the St. Louis Fed FRED database.
- Unemployment rate: Number of unemployed as a percentage of the labor force, percent, seasonally adjusted, monthly frequency from the St. Louis Fed FRED database, transformed to quarterly frequency by taking the average of the months in the quarter.
- Inflation rate: Annualized quarterly percentage change in the chain-type price index of the personal consumption expenditures excluding food and energy, percent, seasonally adjusted, quarterly frequency from the St. Louis Fed FRED database.
- Federal funds rate: Effective federal funds rate calculated as a volume-weighted median of overnight federal funds transactions reported in the FR 2420 Report of Selected Money Market Rates, percent, not seasonally adjusted, daily frequency from the St. Louis Fed FRED database, transformed to quarterly frequency by taking the average of the days in the quarter. We assume a lower bound equal to 0.25% that binds between 2009:Q1 and 2015:Q4.
- 10-year Treasury yield: Yield on the 10-year Treasury security at constant maturity, percent, not seasonally adjusted, daily frequency from the St. Louis Fed FRED database, transformed to quarterly frequency by taking the average of the days in the quarter.
- Inflation expectations: Survey of Professional Forecasters (SPF) 10-year-ahead inflation expectations from the Philadelphia Fed since 1991:Q4, first for expected the consumer price index (CPI) inflation and then, when it becomes available in 2007, for expected PCE price inflation. Data from 1981:Q1 to 1991:Q3 is primarily from a survey conducted by Richard Hoey and others called "Decision-Makers Poll." The Hoey and SPF CPI observations are reduced by 40 basis to account for the average difference between CPI and PCE inflation. Values before 1981 are constructed in a manner similar to the one described in Kozicki and Tinsley (2001). In percent, the frequency is quarterly, not seasonally adjusted from the FRB/US public database.
- Board of Governors of the Federal Reserve System output gap estimate: Real-time estimates and projections of the output gap used by the staff of the Board of Governors of the Federal Reserve System in constructing its Greenbook forecast. Obtained from the Philly Fed Greenbook Data Sets.
- CBO potential output: CBO's estimate of the output the economy would produce with a high rate of use of its capital and labor resources. The data is adjusted to remove the effects of inflation. Obtained from the St. Louis FRED database.

C Gibbs sampler details

Let $\Theta_y = \{\theta_1, \theta_2, \beta, \kappa, \rho, \alpha^\pi, \alpha^y, \sigma^2_\upsilon, \sigma^2_{\eta^\pi}, \sigma^2_{\eta^{R^*}}, \sigma^2_e\}$ be the parameters of the observation equations and $\Theta_x = \{\phi_1, \phi_2, p^{10}, \psi_1, \psi_2, \sigma^2_{\eta^y}, \sigma^2_\nu, \sigma^2_\varepsilon, \sigma^2_{\eta^u}, \sigma^2_{\eta^{\pi^*}}, \sigma^2_{\eta^{r^*}}, \sigma^2_{\varepsilon^{10}}\}$, the parameters of the transition equations. Let $\mathbf{y}_t$ be the vector of variables of the observation equation (A.1) and $\mathbf{x}_t$, the latent variables of the transition equation (A.2). The Gibbs sampler operates as follows:[22] The initialization of the Gibbs sampler consists in setting initial values for Θ_y and Θ_x. Moreover, the observations for the initialization of the shadow rate (R^*_t) are obtained by drawing from the model $R_t = \max\{R^*, \underline{R}\}$, with $R^*_t = \rho R^*_{t-1} + (1-\rho)\left(r^* + \pi^*_t + \alpha^\pi\left(\pi_t - \pi^*_t\right) + \alpha^y c_t\right) + \eta^{R^*}_t$ where r^*_t is a constant to be estimated, $\pi^*_t = \pi^e_t$, and the rest

[22]Whenever we obtain a posterior draw of the coefficients of the linear regression model

$$Y_t = X_t'\delta + \xi_t, \qquad \xi_t \sim i.i.dN(0, \sigma^2_\xi), \qquad , t = 1, 2, \ldots, T,$$

of the regressors are data on the federal funds rate, the PCE core inflation rate, and the CBO's estimate of the output gap.

1. Use the Durbin and Koopman (2002) simulator smoother to obtain a random draw of the latent variables, $\mathbf{x}_t$, using the state-space in Appendix A.

2. Generate a random draw of $\eta_t^{R^*}$ and verify that $R_t^* = \rho R_{t-1}^* + (1-\rho)\left(r_t^* + \pi_t^* + \alpha^\pi\left(\pi_t - \pi_t^*\right) + \alpha^y c_t\right) + \eta_t^{R^*} < \bar{R}$ during the ELB periods. If not, go back to step 1.

3. Using the simulated values of $Y_t = c_t$ and $X_t = [c_{t-1}, c_{t-2}]'$, sample ϕ_1, ϕ_2, and $\sigma^2_{\eta^{R^*}}$ from a truncated (to ensure covariance stationarity) independent normal-inverse Gamma posterior distribution.

4. Using the simulated values of $Y_t = c_t^{10}$ and $X_t = [1, c_{t-1}^{10}, c_{t-2}^{10}]'$, sample p^{10}, ψ_1, ψ_2, and $\sigma^2_{\varepsilon^{10}}$ from a truncated (to ensure covariance stationarity) independent normal-inverse Gamma posterior distribution.

5. Sample τ_0^y, μ_0^y, τ_0^u, r_0^*, and π_0^* using a normal distribution with posterior mean $\sigma^2_{x_0}(\bar{x_0}/s^2_{x_0} + x_1/\sigma^2_{\eta^x})$ and posterior variance $\sigma^2_{x_0} = 1/(1/s^2_{x_0} + 1/\sigma^2_{\eta^x})$, for $x = \tau^y, \mu^y, \tau^u, r^*$, and π^*, where $\bar{x_0}$ and $s^2_{x_0}$ are the prior mean and variance, respectively.

6. Sample $\sigma^2_{\eta^x}$ for $x = \tau^y, \mu^y, \tau^u, r^*$, and π^* from an inverse Gamma distribution with shape coefficient $\underline{a}_{\sigma^2_{\eta^x}} + 0.5 * T$ and rate coefficient $\underline{b}_{\sigma^2_{\eta^x}} + 0.5 * \hat{\eta}^x$, where $\hat{\eta}^x$ is the vector of residuals obtained from $x_t - x_{t-1}, \quad t = 1, 2, \ldots, T$, and $\underline{a}_{\sigma^2_{\eta^x}}$ and $\underline{b}_{\sigma^2_{\eta^x}}$ are the prior shape and rate coefficients.

7. Using the simulated values of π_t^*, make $\hat{e}_t = \pi_t^e - \pi_t^*$ to sample σ_e^2 from an inverse Gamma distribution with shape $\underline{a}_{\sigma^2_e} + 0.5 * T$ and rate $\underline{b}_{\sigma^2_e} + 0.5 * \hat{e}'\hat{e}$, where $\underline{a}_{\sigma^2_e}$ and $\underline{b}_{\sigma^2_e}$ are the prior shape and rate coefficients, respectively.

8. Using the observed and simulated values of $Y_t = u_t - \tau_t^u$ and $X_t = [c_t, c_{t-1}]'$, sample θ_1, θ_2, and σ_v^2 from an independent normal-inverse Gamma distribution.

9. Using the observed values for $Y_t = \pi_t - \pi_{t-1}$ and observed and simulated values for $X_t = [\pi_t^* - \pi_{t-1}, c_t]$, sample β, κ, and $\sigma^2_{\eta^\pi}$ from a truncated (to ensure homogeneity and positiveness) independent normal-inverse Gamma posterior distribution.

10. Using the observed and simulated values of $X_t = [R_{t-1}, r_t^* + \pi_t^*, \bar{\pi}_t - \pi_t^*, c_t]'$, generate R_t^* for t in the set of ELB periods from a truncated (from above at 0.25) normal distribution with mean $X_t'\delta$ and variance $\sigma^2_{\eta^{R^*}}$. Set $Y_t = R_t$ and put $Y_t = R_t^*$, $X_t = [R_{t-1}^*, r_t^* + \pi_t^*, \bar{\pi}_t - \pi_t^*, c_t]'$ for the ELB periods. Sample ρ, $(1-\rho)\alpha^\pi$, $(1-\rho)\alpha^y$, and $\sigma^2_{\eta^{R^*}}$ from a truncated independent normal-inverse Gamma posterior distribution.

---

we use an independent normal-inverse Gamma posterior distribution with mean

$$\left(\underline{\Sigma}^{-1} + \sum_{t=1}^{T} X_t X_t'/\sigma_\xi^2\right)^{-1}\left(\underline{\Sigma}^{-1}\underline{\mu} + \sum_{t=1}^{T} X_t Y_t/\sigma_\xi^2\right)$$

and variance

$$\left(\underline{\Sigma}^{-1} + \sum_{t=1}^{T} X_t X_t'/\sigma_\xi^2\right)^{-1},$$

with shape coefficient

$$\underline{a}_{\sigma_\xi^2} + 0.5 * T$$

and rate coefficient

$$\underline{b}_{\sigma_\xi^2} + 0.5 * \hat{\xi}'\hat{\xi},$$

where $\hat{\xi}$ is the vector of residuals conditional on the draw of δ, $\underline{\mu}$ and $\underline{\Sigma}$ are the prior mean and variance, respectively, of the normal prior distribution of δ, whereas $\underline{a}_{\sigma_\xi^2}$ and $\underline{b}_{\sigma_\xi^2}$ are the prior shape and rate coefficients of the prior inverse Gamma distribution of σ_ξ^2.

11. Initiate a new iteration by going back to step 1.

D Parameter diagnostics

Figures D.1 and D.2 show the prior and posterior distributions of the parameters of the benchmark model.

Figure D.1: Prior and posterior distributions - conditional mean parameters

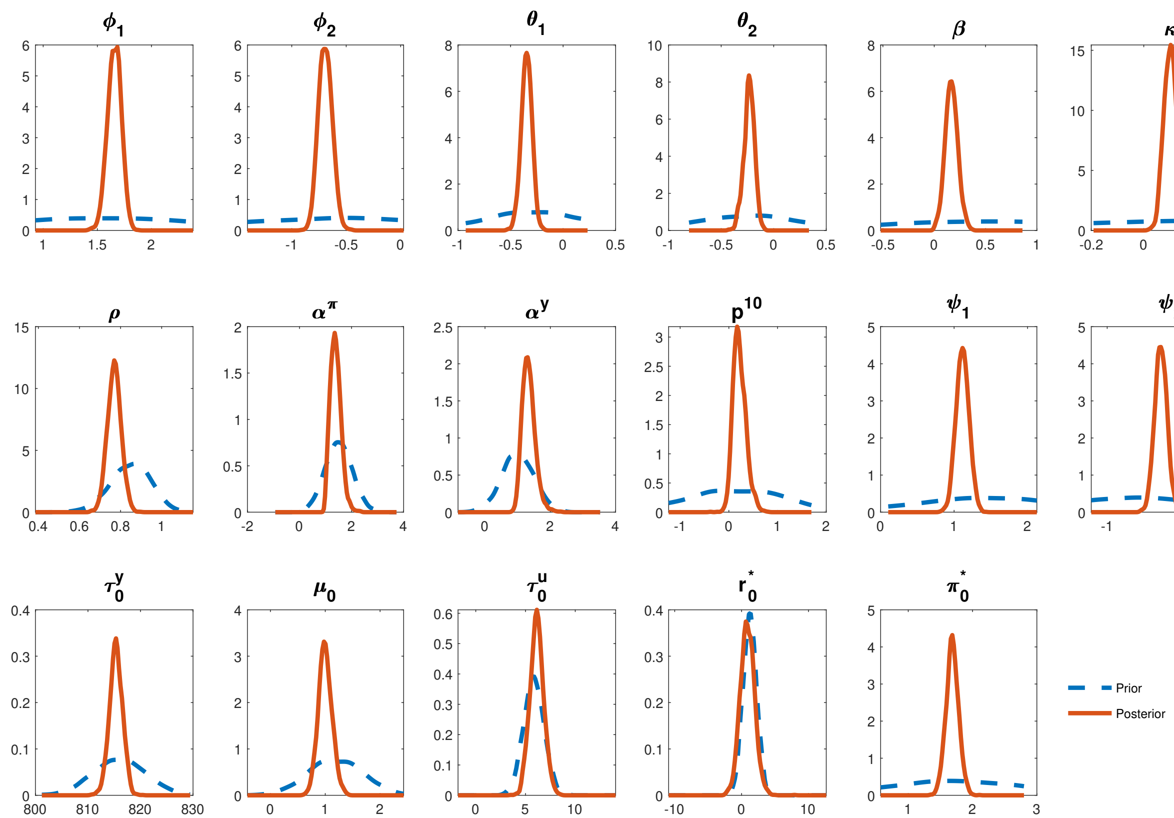
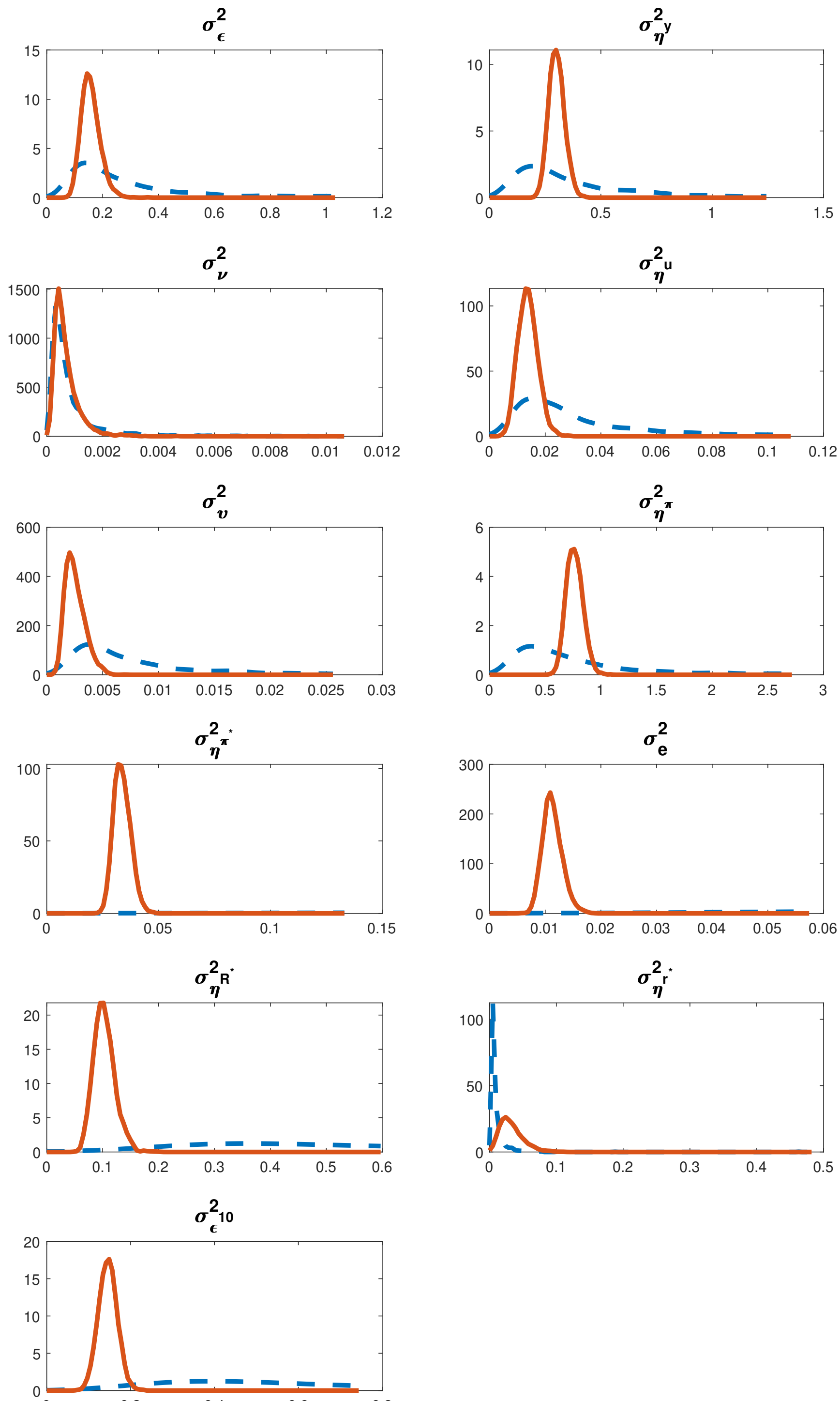


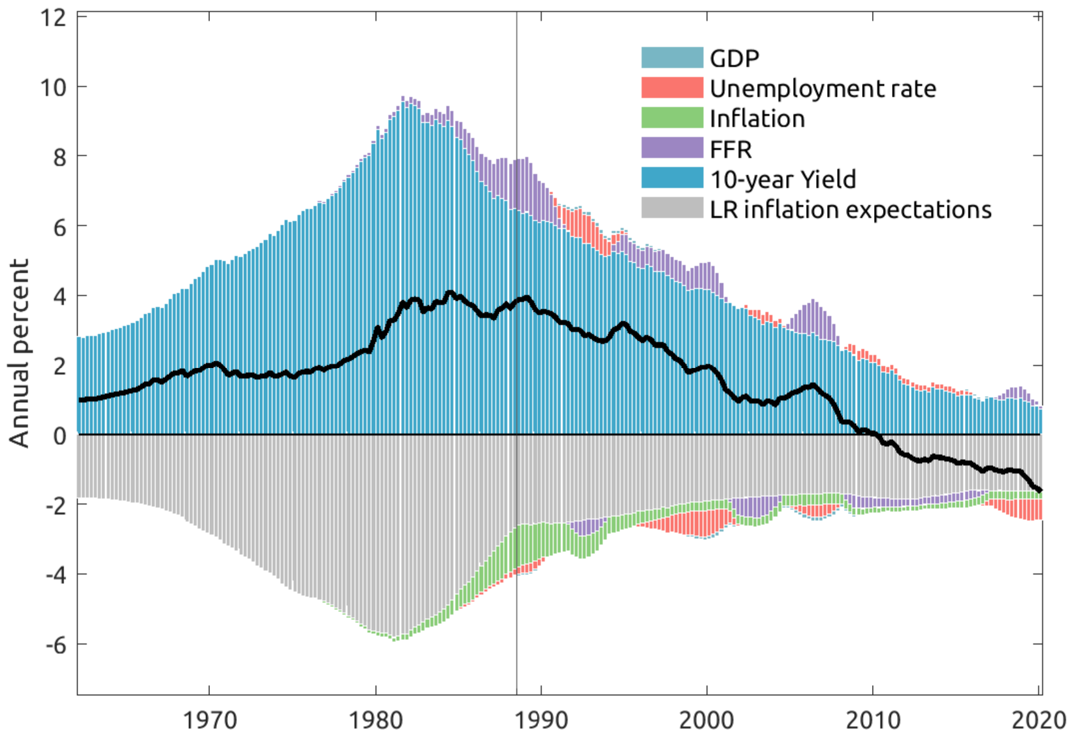
Figure D.2: Prior and posterior distributions - variance parameters



E Data decomposition

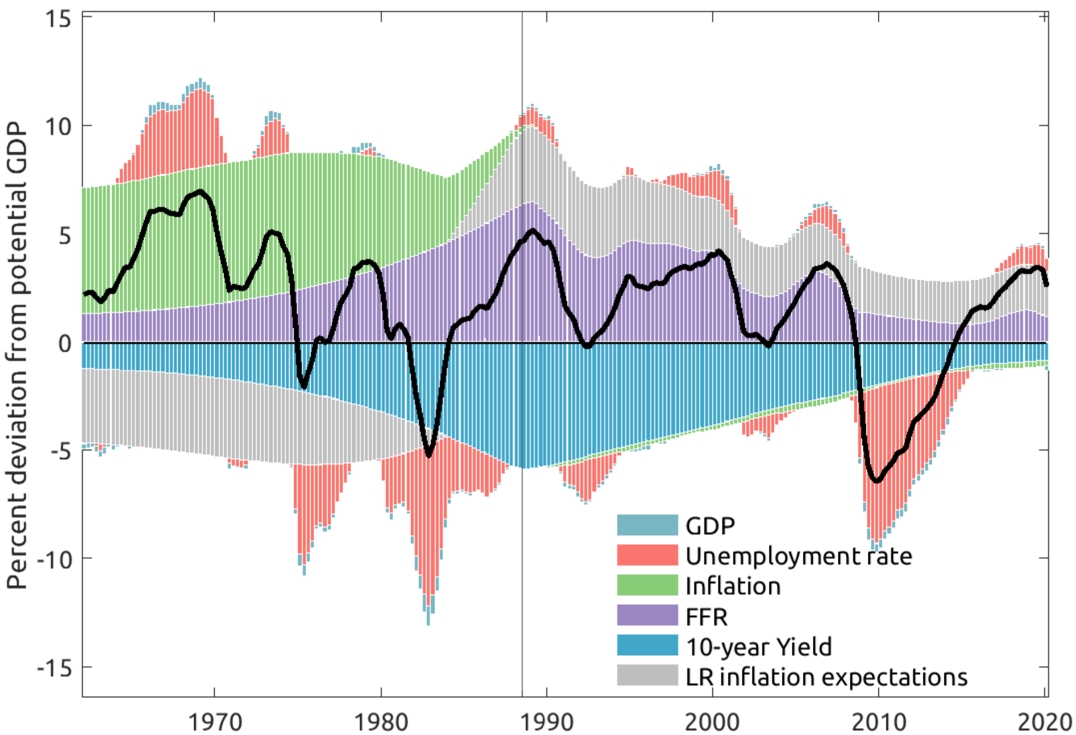
This appendix shows the historical data decompositions of r^* in figure E.1 and the output gap in figure E.2.

Figure E.1: Historical data decomposition of r^*



Note: FFR denotes the federal funds rate. LR denotes long run. The grey vertical line indicates the period from which the Taylor rule was added to the system.

Figure E.2: Historical data decomposition of the output gap



Note: FFR denotes the federal funds rate. LR denotes long run. The grey vertical line indicates the period from which the Taylor rule was added to the system.

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