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James Hebden and Fabian Winkler

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Computation of Policy Counterfactuals in Sequence Space

James Hebden and Fabian Winkler*
Federal Reserve Board

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Abstract

We propose an efficient procedure to solve for policy counterfactuals in linear models with occasionally binding constraints in sequence space. Forecasts of the variables relevant for the policy problem, and their impulse responses to anticipated policy shocks, constitute sufficient information to construct valid counterfactuals. Knowledge of the structural model equations or filtering of structural shocks is not required. We solve for deterministic and stochastic paths under instrument rules as well as under optimal policy with commitment or subgame-perfect discretion. As an application, we compute counterfactuals of the U.S. economy after the pandemic shock of 2020 under several monetary policy regimes.

Keywords: Sequence Space; DSGE; Occasionally Binding Constraints; Optimal Policy; Commitment; Discretion

JEL: C61; C63; E52

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*Federal Reserve Board, 20th St and Constitution Ave NW, Washington DC 20551. Our email addresses are james.s.hebden and fabian.winkler, both @frb.gov. We thank Hess Chung for his work on FRB/US and the Board's suite of DSGE models that inspired the development of this project; Oliver de Groot, Etienne Gagnon, Tom Holden, Ben Johannsen, Alisdair McKay, Damjan Pfajfar, Christian Wolf, and participants at the 2022 CEF conference for helpful comments. The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or any other person associated with the Federal Reserve System. A previous version of this paper was titled "Impulse-Based Computation of Policy Counterfactuals."

1 Introduction

One key use of structural macroeconomic models is the construction of policy counterfactuals. How would the economy have behaved differently during some historical episode had some specific monetary policy been adopted? How would the economy likely behave in the future under such a policy?

The established procedure for constructing such counterfactuals in macroeconomics follows the logic of the state-space representation of a model. Given a model in state-space form, one filters latent structural states and shocks from observable data; then one rewrites the model to change the behavior of policy to the desired counterfactual; solves this new model; and computes the counterfactual equilibrium path using the filtered initial states and shocks obtained in the first step. One difficulty of this procedure is that it can quickly become computationally challenging when the model has many state variables and when non-linearities, such as a lower bound on interest rates, are present. Moreover, representing the state of economy by structural states and shocks is somewhat distant from the reality of policy analysis. Central banks and other policy institutions typically do not rely on any one model to inform their view of the economy, and construct their projections from a large amount of data, a variety of reduced-form and structural models, and judgment.

In this paper, we propose new algorithms for computing policy counterfactuals using the sequence-space representation of a model. We believe that in many cases, our procedures are computationally simpler and better adapted to the reality of policy analysis than state-space analysis. We only require a minimal amount of information about the model that is directly relevant to the problem at hand. Neither the structural or reduced-form equations of the model, its state variables, nor its shock processes need to be known. All that is required is a set of impulse responses of a few variables of interest (say, inflation, output, and interest rates) to anticipated future shocks about the policy instruments. These sets of impulse responses, also called sequence-space Jacobians, contain all the relevant information about the model. Also, rather than filtering structural shocks from many observables, the algorithms operate directly on forecasts of the few variables of interest, called projections. These impulse responses and projections are all that is required to compute numerically accurate counterfactual solutions.

We show how to compute solutions for instrument rules as well as optimal paths under commitment and discretion. Importantly, we are able to compute counterfactuals not only at one point in time but also over time as the economy is affected by shocks, even though these shocks need not be known explicitly. The sequences of projections contain all the necessary information about the shocks needed for the computation of policy counterfactuals. Our optimal commitment solution honors past commitments as time moves forward, because we establish the analogue of the Marcet and Marimon (2019) recursive form of the commitment problem directly for the sequence-space representation. For optimal discretionary policy, we compute subgame-perfect equilibria using a similar approach as the state-space algorithm in Dennis (2007). One key advantage of the sequence-space approach is that we can easily add occasionally binding constraints.

As an illustration, we discuss how the U.S. economy may have evolved after 2020 had the Federal Reserve adopted a Taylor-type interest rate rule or an optimal policy under commitment or discretion. We conduct this analysis using median projections of the economy made at that time by FOMC participants in the Survey of Economic Projections, and impulse responses obtained from the FRB/US model (Brayton, 2018; Erceg, Hebden, Kiley, Lopez-Salido, and Tetlow, 2018) and a modified version of the del Negro, Giannoni, and Schorfheide (2015) model, which have both been used previously for policy analysis at the Federal Reserve. We find that after the pandemic shock of 2020, optimal policy under commitment would have made a strong promise to keep interest rates low, which would have been honored even as inflation rose subsequently. The policy path would have been appeared to be "behind the curve" relative to simple Taylor-type rules, and this appearance would have reflected the time-inconsistency of the optimal policy.

We build on ideas from two strands in the literature. The first is the work by Svensson

(2005) and Svensson and Tetlow (2005), who show how to compute optimal commitment policies that accommodate a "judgmental" projection that originates outside of a particular model.[1] In this paper, we also place emphasis on the use of judgmental projections rather than filtering structural shocks, but go beyond these earlier contributions in several ways. First, we incorporate occasionally binding constraints efficiently. Second, we do not confine ourselves to optimal commitment policies, but also show how to solve for optimal discretionary policy and simple rules. Third, we can compute commitment policies with stochastic changes in the projections while honoring state-contingent commitments. Finally, we have found our procedure to be considerably faster because it is based on precomputed impulse responses and uses only a small subset of model variables.

The second strand of the literature relates to expessing models in sequence space. The utility of the sequence space was recognized several years ago by Holden (2016, 2023), who provided an efficient algorithm to compute solutions to forward-looking models with occasionally binding constraints. Auclert, Bardóczy, Rognlie, and Straub (2021) have shown how heterogenous-agent models with aggregate risk can be simulated very efficiently using a sequence-space representation. Our paper was developed in parallel to de Groot, Mazelis, Motto, and Ristiniemi (2021), Barnichon and Mesters (2023) and McKay and Wolf (2023). Like us, these studies also compute optimal policy in sequence space, and formally show that projections and impulse responses to anticipated policy shocks are sufficient statistics for doing so. All studies place emphasis on different aspects of the sequence-space representation. Our emphasis is on extending the capabilities of the algorithms. Our unique contributions among these studies are to compute optimal policies under commitment that handle stochastic shocks while keeping past promises, and to solve for subgame-perfect equilibria under discretion. Other algorithms cannot accommodate shocks under commitment and only compute time-consistent equilibria under discretion, which are generally not unique and different from what is commonly known as "discretion" in the literature.

[1] Algorithms that build on these contributions have been in continuous development at the Federal Reserve and other central banks, see e.g. Bersson, Hürtgen, and Paustian (2019) and Harrison and Waldron (2021).

Besides providing a simple way to compute policy counterfactuals, our procedure also facilitates the comparison of the effects of economic policies across different models, and can thus be used to address concerns of model uncertainty. All the information needed for such a comparison is contained in the impulse responses to anticipated shocks to the policy instruments. If these responses are identical for two models, then any choice of policy will yield the same outcomes (for the variables considered) in either model, thus providing a weaker form of the "principle of counterfactual equivalence" studied by Beraja (2021).

Lucas (1976) argued that one needs to understand fundamental economic relationships to conduct credible policy experiments. A practical insight that emerges from our analysis is that the entire model need not be correctly specified for such policy experiments to be valid. Our procedure (and, for that matter, any other solution method) can yield valid counterfactuals even when some aspects of a model are misspecified. What is crucial is that the impulse responses to anticipated monetary policy shocks are correct, since they completely summarize the ecomomy's response to changes in policy.

The remainder of this paper is structured as follows. Section 2 describes how to convert a model from state space to sequence space and introduces the notation used in the rest of the paper. Section 3 shows how to solve for policy counterfactuals in completely linear models. Section 4 describes how we approximate our solutions with finite computing horizons. In Section 5, we add occasionally binding constraints, and in Section 6 we extend our results to a tractable case of incomplete information that allows us to accommodate historical data revisions. Section 7 contains our application to the U.S. economy following the pandemic shock of 2020 and Section 8 concludes.

2 From state space to sequence space

In this section, we show how to move from state space to sequence space and introduce relevant notation. We start with a general dynamic macroeconomic model in state space and then show how the two inputs into our procedure, baseline projections and impulse responses, fit within the model. We then show that these two inputs are sufficient to compute any feasible equilibrium under alternative policy regimes.

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}$ the set of real-valued numbers. $\mathbb{R}^{\mathbb{N}\times n}$ denotes the space of sequences where each element is a real vector of length n. We consider the class of forward-looking stochastic models that are linear except for occasionally binding constraints that affect the conduct of policy. Time is discrete at $t \in \mathbb{N}$, so the model has a fixed initial period and an infinite horizon. The number of endogenous model variables is n. There are two types of variables: A set of p policy instruments $z_t \in \mathbb{R}^p$, the values of which are chosen by the policymaker, and a set of $n-p$ endogenous variables $\xi_t \in \mathbb{R}^{n-p}$. The endogenous variables depend on k exogenous shocks $u_t \in \mathbb{R}^k$ that are uncorrelated across time and have mean zero. We group all variables save for the exogenous shocks into one vector $y_t = (\xi_t', z_t')' \in \mathbb{R}^n$. Initial conditions y_{-1} are taken as given. To keep the notation light, we simply denote with y the stochastic process $(y_t)_{t=0}^{\infty}$. We define $\mathbb{F}$ as the natural filtration of the exogenous variables $(y_{-1}, u_0, u_1, \dots)$; that is, $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ with $\mathcal{F}_t$ the σ-algebra generated by $y_{-1}, u_{0,}, \dots, u_t$.

The endogenous variables y evolve according to the system of structural model equations in state space:

$$\Phi_{-1} y_{t-1} + \Phi_0 y_t + \Phi_1 E_t y_{t+1} + \Phi_u u_t = 0 \in \mathbb{R}^{n-p}. \tag{1}$$

The matrices have size $\Phi_{-1}, \Phi_0, \Phi_1 \in \mathbb{R}^{(n-p)\times n}$ and $\Phi_u \in \mathbb{R}^{(n-p)\times k}$. The expectations used throughout the paper will be defined under quasi-perfect foresight:

$$E_t y_{t+s} = \mathbb{E}\left[y_{t+s} \mid u_{t+s} = 0, \dots, u_{t+1} = 0, \mathcal{F}_t\right]. \tag{2}$$

The imposition of quasi-perfect foresight is innocuous when z is linear in u, because in that case certainty equivalence applies. But it becomes important when policy instruments are subject to non-linearities such as an effective lower bound (ELB) on interest rates. Our algorithms could be extended to approximate non-perfect foresight expectations and non-linear models using the techniques developed by Holden (2016).

The starting point for the analysis is a solution $\bar{y} = (\bar{y}_t)_{t=0}^{\infty}$ called a *baseline projection*: any stochastic process adapted to $\mathbb{F}$ that solves (1).[2] Uniqueness of this solution is not required. We will work with the perfect foresight expectations of $\bar{y}$, i.e. of $E_t \bar{y}_{t+s}$ for $s, t \geq 0$. The baseline projection is thus simply a sequence of forecasts of the economy.[3] Although we think of the baseline $\bar{y}$ as having been generated by particular realizations of the shocks u and under a particular policy regime, it is not necessary to know the structural determinants of $\bar{y}$. In particular, the baseline can be entirely judgmental, in the terminology of Svensson (2005), as long as there exist some shocks for which $\bar{y}$ solves (1).

Next, we introduce a linear policy rule $\Psi y_t = 0 \in \mathbb{R}^p$. The only purpose of Ψ is to derive a sequence-space Jacobian for the model. The choice of Ψ is arbitrary and does not have to coincide with the behavior of policy in the baseline projection or the desired policy counterfactuals. The only requirement for Ψ is that the rule in combination with (1) yields a unique, non-explosive solution of the model. A Taylor-type rule usually satisfies this requirement. To this rule, we append a set of anticipated shocks:

$$\Psi y_t - \sum_{s=0}^{\infty} \varepsilon_{t-s,t} = 0. \tag{3}$$

For $t, s \geq 0$, $\varepsilon_{t-s,t} \in \mathbb{R}^p$ is a zero-mean shock that is realized at time t but anticipated s periods in advance, i.e. $E_\tau \left[\varepsilon_{t-s,t}\right] = 0$ for $\tau < t - s$ and $E_\tau \left[\varepsilon_{t-s,t}\right] = \varepsilon_{t-s,t}$ for $\tau \geq t - s$. By assumption, the linear system of Equations (1) and (3) yields a unique solution for any realization of shocks. It is a standard computational exercise to find the impulse response of y_{t+s} to $\varepsilon_{t,t+\tau}$ for $s, \tau \geq 0$, which we denote $M_{s\tau} \in \mathbb{R}^{n \times p}$.[4] These impulse responses are the second input to our procedure. While computing $M_{s\tau}$ does require solving the state-space

[2] A process $\bar{y}$ is adapted to $\mathbb{F}$ if $\bar{y}_t$ is a function of $y_{-1}, u_0, \ldots, u_t$. In particular, it does not depend on other shocks such as sunspots and does not "see into the future". See e.g. Klenke (2008) for a more precise definition.

[3] This forecast may be conditional on some path for the policy instruments. Gali (2011) points out that such conditional forecasts can suffer from an indeterminacy problem. Even then, our procedure remains valid as long as the baseline projection is a valid solution of the model.

[4] Solvers like Gensys or Dynare can be used to compute these impulse responses.

model and its structural equations, this needs to be done only once and under an arbitrary policy regime. These impulse responses contain all the information about the model that is needed to accurately compute policy counterfactuals. Visualizing these impulse responses, as we do in our application later in the paper, can therefore be highly informative about which model properties matter for policy.

Because of the linearity of (1)–(3), there exist realizations of ε that reproduce the baseline projection $\bar{y}$. Taking expectations of (3), these *baseline shocks*[5] $\bar{\varepsilon}$ are computed as:

$$\bar{\varepsilon}_{t,t+s} = \Psi\left(E_t\bar{y}_{t+s} - E_{t-1}\bar{y}_{t+s}\right).$$

The baseline projection $\bar{y}$ solves (1) and (3) given $\bar{\varepsilon}$ and u and this solution is unique. To define the above shocks for $t = 0$, we employ the convention $E_{-1}\bar{y}_s = 0$ for all $s \geq 0$.

Next, we introduce a new set of *standardized policy instruments* x. For any process $(x_t)_{t=0}^{\infty} \in \mathbb{R}^{\mathbb{N}\times p}$ that is adapted to $\mathbb{F}$, we define corresponding shocks:

$$\varepsilon_{t,t+s} = \bar{\varepsilon}_{t,t+s} + E_t x_{t+s} - E_{t-1}x_{t+s} \tag{4}$$

with the convention that $E_{-1}x_s = 0$ for $s \geq 0$. This implies $x_t = \Psi\left(y_t - \bar{y}_t\right)$.

Denote with $y^{(t)} = (y_t, E_t y_{t+1}, E_t y_{t+2}, \ldots)'$ the expected path of model variables at time t. Also, let F be the forward-shift operator, i.e. $Fy^{(t)} = (E_t y_{t+1}, E_t y_{t+2}, E_t y_{t+3}, \ldots)'$. We can then express revisions to the expected path of the economy between $t-1$ and t as:

$$\hat{y}^{(t)} = y^{(t)} - Fy^{(t-1)}. \tag{5}$$

We will use the convention that $y^{(-1)} = 0$ so that $\hat{y}^{(t)} = y^{(0)}$. The expected paths $x^{(t)}$ and $\bar{y}^{(t)}$ and revisions $\hat{x}^{(t)}$ and $\hat{\bar{y}}^{(t)}$ can be defined analogously. We can now make use of the linearity of the model and express the solution to (1) and (3) in deviation from the baseline projection $\bar{y}$. This is the sequence-space representation of the model:

$$\hat{y}^{(t)} = \hat{\bar{y}}^{(t)} + M\hat{x}^{(t)} \tag{6}$$

[5]These baseline shocks are called "add factors" in Svensson and Tetlow (2005).

where the linear map $M : \mathbb{R}^{\mathbb{N}\times p} \to \mathbb{R}^{\mathbb{N}\times n}$ stacks the impulse responses $M_{s\tau}$ for $s, \tau \geq 0$. The map M is called the *sequence-space Jacobian*. For $x = 0$ we get back the baseline projection $y = \bar{y}$. By choosing an appropriate x, one can use (6) to obtain solutions to (1) under any counterfactual policy regime.

Proposition 1. *Consider the function F that maps stochastic processes $(x_t)_{t=0}^{\infty} \in \mathbb{R}^{\mathbb{N}\times p}$ to stochastic processes $(y_t)_{t=0}^{\infty} \in \mathbb{R}^{\mathbb{N}\times n}$ through Equation (6). For every x adapted to $\mathbb{F}$, $F(x)$ solves (1), and for every y that solves (1) and is adapted to $\mathbb{F}$, there exists an x such that $y = F(x)$.*

Proof. The first part of the proposition follows by construction of F. Let x be a stochastic process adapted to $\mathbb{F}$. Then we can construct shocks ε from x through (4), and then use (6) to recover a solution to (1). For the second part, let y be a process adapted to $\mathbb{F}$ that solves (1). Construct x as $x_t = \Psi(y_t - \bar{y}_t)$. For this x, $F(x)$ is a solution to (1). With this x and the corresponding shock ε obtained from (4), y jointly solves (1) and (3). Because we have assumed that (3) yields unique solutions for any combination of shocks, it has to be that $y = F(x)$. □

The proposition, which is similar to the independently derived Proposition 1 in McKay and Wolf (2023), implies that knowledge of the baseline projection $\bar{y}$ and the sequence-space Jacobian M is sufficient to compute any valid counterfactual model solution. It is neither necessary to know the structural equations of the model, the original policy instruments z, nor the exogenous shock processes and their realized values.

3 Linear policy rules and linear-quadratic optimal policy problems

In this section, we show all the basic insights of the paper using policy problems that imply completely linear solutions. All policy problems in this section reduce to finding a solution of the form $\Omega_y \hat{y}^{(t)} = 0$ for a linear map $\Omega_y : \mathbb{R}^{\mathbb{N}\times n} \to \mathbb{R}^{\mathbb{N}\times p}$. The solution is:

$$\begin{aligned} 0 &= \Omega_y \left(\hat{\bar{y}}^{(t)} + M\hat{x}^{(t)}\right) \\ \Rightarrow \hat{x}^{(t)} &= -\left(\Omega_y M\right)^{-1} \Omega_y \hat{\bar{y}}^{(t)} \\ \hat{y}^{(t)} &= \hat{\bar{y}}^{(t)} - M\left(\Omega_y M\right)^{-1} \Omega_y \hat{\bar{y}}^{(t)}. \end{aligned} \tag{7}$$

The map $\Omega_y M : \mathbb{R}^{\mathbb{N}\times p} \to \mathbb{R}^{\mathbb{N}\times p}$ has to be invertible to guarantee the existence of a solution.[6]

3.1 Linear policy rules

We start with linear policy rules of the form

$$Ay_t = 0. \tag{8}$$

where $A \in \mathbb{R}^{p\times n}$. As an example, suppose that there is only a single instrument ($p = 1$), the nominal interest rate i_t, and that we impose a Taylor rule that relates the nominal interest rate to inflation π_t and the output gap $ygap_t$ through the equation $i_t = \phi_\pi \pi_t + \phi_y ygap_t$. We can express this as $i_t - \phi_\pi \pi_t - \phi_y ygap_t = 0$.

We assume that agents know this condition to hold at all times in the future so that $AE_t y_{t+s} = 0$ as well. By linearity of expectations, $A\left(E_t y_{t+s} - E_{t-1} y_{t+s}\right) = 0$ as well and we can write:

$$\begin{pmatrix} A & 0 & \cdots \\ 0 & A & \\ \vdots & & \ddots \end{pmatrix} \hat{y}^{(t)} = \left(I_{\mathbb{N}} \otimes A\right) \hat{y}^{(t)} = 0. \tag{9}$$

This problem has the form in (7) with $\Omega_y = (I_{\mathbb{N}} \otimes A)$.[7]

To find the counterfactual evolution of the economy under rule (9), it is not necessary to know the baseline projection and sequence-space Jacobian for *all* model variables. It is sufficient to know these objects for the variables that enter the rule. For the Taylor rule

[6] We do not provide criteria to check for existence and uniqueness of a solution in sequence space. In ongoing work, Auclert, Rognlie, and Straub (2023) are making progress on this goal by building on the seminal contribution of Onatski (2006).

[7] The operator $\otimes$ denotes the Kronecker product.

above, only the baseline projection of, and impulse responses for inflation, the output gap and the nominal interest rate have to be known in order to compute the counterfactual model solution.

3.2 Optimal commitment policy

Next, we consider the problem of optimal policy under commitment. The objective of the policymaker is to minimize a quadratic loss function of the form

$$\min_{\left(\hat{y}^{(t)},\hat{x}^{(t)}\right)_{t=0}^{\infty}} \frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t y_t' W y_t$$

where $\beta \in (0,1)$ and the weighting matrix $W \in \mathbb{R}^{n\times n}$ is positive semi-definite. By Proposition 1, all feasible solutions to (1) available to the policymakers are given by (6) for some process for the standardized policy instruments $\hat{x}^{(t)}$. Therefore, we can write the constraints to the optimization problem as follows:

$$\text{s.t. } \hat{y}^{(t)} = \hat{\bar{y}}^{(t)} + M\hat{x}^{(t)} \tag{10}$$

$$E_t \hat{x}^{(t+1)} = 0. \tag{11}$$

The second constraint is necessary to ensure that $\hat{x}^{(t)}$ is indeed an unanticipated revision to the policy stance, which will end up to be a function of unanticipated baseline revisions.

We aim to obtain a recursive formulation of the optimal commitment policy by applying the Lagrangian method of Marcet and Marimon (2019) on the sequence-space form of the problem. To do so, we express the quadratic objective using more convenient notation. Let L be the lag or backward-shift operator, i.e. $Ly^{(t)} = (0, y_t, E_t y_{t+1}, E_t y_{t+2}, \dots)'$, and $B = \operatorname{diag}(1, \beta, \beta^2, \dots)$. Then we can express the Lagrangian as:

$$\mathcal{L} = \frac{1}{2}\left(\sum_{t=0}^{\infty} L^t \hat{y}^{(t)}\right)' (B \otimes W) \left(\sum_{t=0}^{\infty} L^t \hat{y}^{(t)}\right) + \sum_{t=0}^{\infty} \beta^t \left[\lambda^{(t)\prime}\left(\hat{\bar{y}}^{(t)} + M\hat{x}^{(t)} - \hat{y}^{(t)}\right) - \mu^{(t-1)\prime}\hat{x}^{(t)}\right].$$

The resulting first-order conditions for $\hat{y}^{(t)}$ and $\hat{x}^{(t)}$ are:

$$\left(L^t\right)' (B \otimes W) L^t y^{(t)} - \beta^t \lambda^{(t)} = 0$$

$$\beta^t M' \lambda^{(t)} - \beta^t \mu^{(t-1)} = 0$$

with the convention that $\mu^{(-1)} = 0$ (we do not optimize from a timeless perspective).[8] Combining these conditions yields:

$$M' (B \otimes W) y^{(t)} = \mu^{(t-1)}.$$

Because $\mu^{(t-1)}$ is $t-1$-measurable, we can subtract the time $t-1$-expectation of this equation and get:

$$M' (B \otimes W) \hat{y}^{(t)} = 0. \tag{12}$$

This now has the form in (7) with $\Omega_y = M' (B \otimes W)$.

Equation (12) constitutes a recursive formulation of the optimal commitment problem. Remarkably, it is not necessary to carry Lagrange multipliers for this linear-quadratic problem (although we will need to do so once we introduce occasionally binding constraints later on). Because of the linearity of the first-order conditions, the response of the optimal commitment to shocks is the same regardless of when the commitment started and what promises are being carried from the past.

We note again that in practice, only a small subset of model variables is required in these computations. If, for example, the weighting matrix W is such that policymakers are only concerned with deviations of an inflation and a measure of economic activity, as is commonly assumed in the literature, then only the baseline projection and sequence-space Jacobians for these two variables have to be known in order to be able to solve for the counterfactual under optimal policy.

3.3 Optimal discretionary policy

We now discuss a problem of optimal policy under discretion. We compute time-consistent optimal policy by assuming that at each point in time, a separate policymaker controls the value of the policy instruments at that time. This policymaker is only concerned with

[8] The transpose operator is defined canonically such that $M' : y \mapsto x$ with $x_s = \sum_{\tau=0}^{\infty} M'_{\tau s} y_\tau$.

present and future losses, but not past losses, and takes the decisions of future policymakers as given. Importantly, our solution yields a subgame-perfect equilibrium in the sequential game played by the sequence of discretionary policymakers.[9]

For each policymaker at time $h \geq 0$, we are looking for a set of equations that represent the solution of her optimal policy problem. This set of equations takes the form

$$d^{(h)} y^{(h)} = 0, \; h \geq 0. \tag{13}$$

The maps $d^{(h)} : \mathbb{R}^{\mathbb{N} \times n} \to \mathbb{R}^p$ can be used to back out the innovations to the policy instruments at time t through the basic relation (6).

Let's place ourselves in the shoes of a policymaker at time $t = h$. The policymaker at time h takes Equation (13) as given for $t = h+1, h+2, \ldots$ as she internalizes the behavior of future policymakers. It is in this sense that the equilibrium we construct is subgame-perfect: The policymaker takes into account the optimal response of future policymakers on every attainable path, not only on the equilibrium path. Her optimization problem can be written as follows:

$$\min_{\left(\hat{y}^{(t)}, \hat{x}^{(t)}\right)_{t=h}^{\infty}} \frac{1}{2} E_h \sum_{t=h}^{\infty} \beta^t y_t' W y_t$$

$$\text{s.t. } \hat{y}^{(t)} = \hat{\bar{y}}^{(t)} + M \hat{x}^{(t)} \tag{14}$$

$$E_t \hat{x}^{(t+1)} = 0, \; t \geq h \tag{15}$$

$$d^{(t)} y^{(t)} = 0, \; t \geq h+1. \tag{16}$$

This policymaker only cares about losses that accrue from period h onwards. In addition, note that there are no constraints placed on the innovations $\hat{x}^{(h)}$. This implies that the discretionary policymaker takes the state of the economy at time $t = h$ as given. She does not internalize that her actions in equilibrium will influence past expectations.

[9] Our solutions can be understood as Markov-perfect equilibria in sequence space, where the relevant state is the baseline projection $\bar{y}^{(t)}$.

The Lagrangean of this problem is:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}E_h\left(Fy^{(h-1)} + \sum_{t=0}^{\infty} L^t\hat{y}^{(h+t)}\right)'(B\otimes W)\left(Fy^{(h-1)} + \sum_{t=0}^{\infty} L^t\hat{y}^{(h+t)}\right)\\ &+ \sum_{t=h}^{\infty}\beta^{t-h}\left[\lambda^{(t)\prime}\left(\hat{\bar{y}}^{(t)} + M\hat{x}^{(t)} - \hat{y}^{(t)}\right) - \mu^{(t)\prime}\hat{x}^{(t+1)}\right]\\ &- \sum_{t=h+1}^{\infty}\beta^{t-h}\eta_t' d^{(t)}\left(F^{t-h+1}y^{(h-1)} + \sum_{s=h}^{t} F^{t-s}\hat{y}^{(s)}\right).\end{aligned}$$

For $t \geq h+1$, the first-order conditions with respect to $\hat{y}^{(t)}$ and $\hat{x}^{(t)}$ are:

$$\left(L^{t-h}\right)'(B\otimes W)\,L^{t-h}y^{(t)} - \beta^{t-h}\lambda^{(t)} - \sum_{s=t}^{\infty}\beta^{s-h}\left(d^{(s)}F^{s-t}\right)'\eta_s = 0 \tag{17}$$

$$\beta^{t-h}M'\lambda^{(t)} - \beta^{t-h}\mu^{(t-1)} = 0. \tag{18}$$

Defining $\eta^{(t)} = E_t\left(\eta_t', \eta_{t+1}', \dots\right)'$ and the map $D^{(t)} : y^{(t)} \mapsto \left(d^{(t)}y^{(t)}, d^{(t+1)}Fy^{(t)}, \dots\right) \in \mathbb{R}^{\mathbb{N}\times p}$, we can combine the first-order conditions into

$$M'\left((B\otimes W)\,y^{(t)} - D^{(t)\prime}\eta^{(t)}\right) = \mu^{(t-1)}. \tag{19}$$

This equation can in principle be used to solve for the multipliers $\eta^{(t)}$ for $t \geq h+1$ that describe the shadow value of the constraints of future policymakers' behavior on the current policymaker. But we are only interested in solving for the optimal policy at time $t = h$. To do so, we must derive the first-order conditions with respect to $\hat{y}^{(h)}$ and $\hat{x}^{(h)}$. They are:

$$(B\otimes W)\,y^{(h)} - \lambda^{(h)} - \beta\left(D^{(h+1)}F\right)'E_h\eta^{(h+1)} = 0 \tag{20}$$

$$M'\lambda^{(h)} = 0. \tag{21}$$

We will now split the Jacobian M as $M = (M_{\cdot 0}, M_{\cdot -0})$ where $M_{\cdot 0} : y^{(h)} \mapsto (M_{00}E_h y_h, M_{10}E_h y_h, \dots)$ is the first "column" of M and $M_{\cdot -0}$ stacks the remaining "columns." We can then combine the two first-order conditions to yield

$$(M_{\cdot 0})'\left((B\otimes W)\,y^{(h)} - \beta\left(D^{(h+1)}F\right)'E_h\eta^{(h+1)}\right) = 0 \tag{22}$$

$$(M_{\cdot -0})'\left((B\otimes W)\,y^{(h)} - \beta\left(D^{(h+1)}F\right)'E_h\eta^{(h+1)}\right) = 0. \tag{23}$$

The second equation (23) can be used to solve for $E_h \eta^{(h+1)}$:

$$E_h \eta^{(h+1)} = \left[\beta \left(M_{\cdot -0}\right)' \left(D^{(h+1)} F\right)'\right]^{-1} \left(M_{\cdot -0}\right)' \left(B \otimes W\right) y^{(h)}.$$

Substituting into (22) and solving for $y^{(h)}$ gives an equation of the form (13) with

$$d^{(h)} = \left(M_{\cdot 0} - M_{\cdot -0}\left[D^{(h+1)} F M_{\cdot -0}\right]^{-1} D^{(h+1)} F M_{\cdot 0}\right)' \left(B \otimes W\right) = 0. \tag{24}$$

This validates our guess that the optimal discretionary policy indeed takes the form in (13). We can also recursively update the matrix $D^{(h)}$as

$$D^{(h)} = \begin{pmatrix} d^{(h)} \\ \beta D^{(h+1)} F \end{pmatrix}. \tag{25}$$

From this formula, we can see that

$$D^{(h)} \hat{y}_t = D^{(h)} y^{(t)} - D^{(h)} F y^{(t-1)} = 0.$$

When there is an infinite sequence of discretionary policymakers, the derivations above suggest an iterative algorithm for determining the time-invariant policy $d^{(h)} = d^*$. The algorithm proceeds as follows:

1. Start with an initial guess for the matrix D (a good starting guess is the solution of optimal policy under commitment $D = M' \left(B \otimes W\right)$).

2. Compute d from D using the relation (24).

3. Update the guess for D using the relation (25).

4. Iterate until convergence.

This algorithm can be seen as the sequence-space analogue to the Dennis (2007) algorithm. The solution (d^*, D^*), which is the fixed point of (24) and (25), constitutes a problem of the form (7) with $\Omega_y = D$.

4 Finite-horizon approximation for computations

Due to the infinite horizon of the model, computing solutions requires manipulating infinite-dimensional objects which is not feasible on a computer. But the computations are straightforward to adapt to an arbitrarily distant finite horizon, as in Svensson (2005).

With some abuse of notation, we will now define $y^{(t)}$ as the *finite* vector of the expected path of the economy up to T periods ahead: $y^{(t)} = \left(y_t', E_t y_{t+1}', \ldots, E_t y_{t+T}'\right)' \in \mathbb{R}^{(T+1)n}$. We similarly define the vectors $\hat{y}^{(t)}, \bar{y}^{(t)}, \hat{\bar{y}}^{(t)} \in \mathbb{R}^{(T+1)n}$ and $x^{(t)}, \hat{x}^{(t)} \in \mathbb{R}^{(T-1)p}$. We can now approximate (5) as

$$\hat{y}^{(t)} \approx y^{(t)} - \tilde{F} y^{(t-1)} \tag{26}$$

where we define the finite-length forward shift operator $\tilde{F}$ as the linear map satisfying

$$\tilde{F} y_{0:T}^{(t-1)} = \begin{pmatrix} E_t \tilde{y}_{t+1} \\ \vdots \\ E_t \tilde{y}_{t+T-1} \\ E_t \tilde{y}_{t+T} \\ E_t \tilde{y}_{t+T} \end{pmatrix}.$$

The last available value is used twice.

Revisions in the expected model outcomes are related to the revisions in the standardized instruments through an approximation of (6):

$$\hat{y}^{(t)} \approx \hat{\bar{y}}^{(t)} + \tilde{M} \hat{x}^{(t)}. \tag{27}$$

The linear map $\tilde{M} : \mathbb{R}^{(T+1)p} \to \mathbb{R}^{(T+1)n}$ is now a finite-dimensional map consisting of the impulse responses of outcomes, and shocks that are anticipated to occur, up to T periods in the future:

$$\tilde{M} = \begin{pmatrix} M_{00} & \cdots & M_{0T} \\ \vdots & \ddots & \vdots \\ M_{T0} & \cdots & M_{TT} \end{pmatrix}.$$

All the problems studied in this paper can be approximated from this point onward. For example, linear simple rules of the form $(I_{\mathbb{N}} \otimes A)\, \hat{y}^{(t)} = 0$ studied in the previous section can be approximated with $(I_{T+1} \otimes A)\, \hat{y}^{(t)}_{0:T} = 0$, the optimal commitment problem with $\Omega = M' (B \otimes W)$ can be approximated with $\Omega = \tilde{M}' \left(\tilde{B} \otimes W\right)$ where $\tilde{B} = \text{diag}\left(1, \beta, \beta^2, \ldots, \beta^T\right)$, and so on.

5 Adding occasionally binding constraints

Occasionally binding constraints can be added to the problems in Section 3. Problems involving simple rules and optimal policy under commitment are relatively straightforward to handle because they can be expressed as linear complementarity problems (LCPs). As shown by Holden (2016), these problems can be solved efficiently as mixed-integer linear programming problems. We also document a heuristic to solve these problems, which not only works well in our experience but can also be applied to optimal policy problems under discretion with occasionally binding constraints, which cannot be expressed as LCPs. One limitation of our solution approach is that we assume expectations are formed under quasi-perfect foresight, i.e. as if all future shocks were zero. In the completely linear cases discussed thus far, certainty equivalence justified this assumption, but in this section it implies that the binding of constraints in the future can only be treated as an event of expected probability zero or one.

5.1 Policy rules

The most frequently encountered occasionally binding constraint in the context of simple policy rules is an effective lower bound (ELB) constraint. Adding such a constraint transforms the problem (9) into a linear complementarity problem of the form

$$\Omega_y \hat{y}^{(t)} = \Omega_u \hat{u}^{(t)} \tag{28}$$

$$u^{(t)} \geq 0 \tag{29}$$

$$\Theta_y y^{(t)} + \Theta_u u^{(t)} \geq 0 \tag{30}$$

$$\left\langle u^{(t)}, \Theta_y y^{(t)} + \Theta_u u^{(t)} \right\rangle = 0 \tag{31}$$

where $u_t \in \mathbb{R}^q$ is a set of auxiliary variables and $\langle \cdot, \cdot \rangle$ is the element-wise product $\langle x, y \rangle = (x_1 y_1, x_2 y_2, \dots)'$ that is used to describe complementary slackness conditions. The maps Ω_y and Ω_u map into $\mathbb{R}^{\mathbb{N} \times n}$ and the maps Θ_y and Θ_u map into $\mathbb{R}^{\mathbb{N} \times q}$.

As an example, consider the Taylor rule $i_t = \phi_\pi \pi_t + \phi_y ygap_t$, but modified to respect an ELB constraint $i_t \geq \underline{i}$. The rule is now $i_t = \max\{\underline{i}, \phi_\pi \pi_t + \phi_y ygap_t\}$. This can be rewritten as $\phi_\pi \pi_t + \phi_y ygap_t - i_t + u_t = 0$, $u_t \geq 0$, $i_t - \underline{i} \geq 0$, and a complementary slackness condition.[10] There is just one occasionally binding constraint so $q = 1$. Expressing the rule as $Ay_t + u_t = 0$ and the constraint as $Cy_t \geq 0$, we can set $\Omega_y = I_{\mathbb{N}} \otimes A$, $\Omega_u = -I_{\mathbb{N}}$ and $\Theta_y = I_{\mathbb{N}} \otimes C$.

To solve a problem of the form (28)–(31), we use (6) to write $\hat{x}^{(t)}$ as a function of $\hat{u}^{(t)}$:

$$\hat{x}^{(t)} = (\Omega_y M)^{-1} \left(\Omega_u \hat{u}^{(t)} - \Omega_y \hat{\bar{y}}^{(t)} \right). \tag{32}$$

Again, it is assumed that $\Omega_y M$ is invertible. We use this to express $y^{(t)}$ as a function of $u^{(t)}$:

$$\begin{aligned} y^{(t)} &= F y^{(t-1)} + \hat{y}^{(t)} \\ &= F y^{(t-1)} + \hat{\bar{y}}^{(t)} + M (\Omega_y M)^{-1} \left(\Omega_u \left(u^{(t)} - F u^{(t-1)} \right) - \Omega_y \hat{\bar{y}}^{(t)} \right). \end{aligned} \tag{33}$$

For the last line, we have used (6) and (32) and expressed $\hat{u}^{(t)} = u^{(t)} - F u^{(t-1)}$. The problem (29)–(31) has now been brought into the form of a standard LCP in $u^{(t)}$ of the form $u^{(t)} \geq 0$, $m + Qu^{(t)} \geq 0$, $\left\langle u^{(t)}, m + Qu^{(t)} \right\rangle = 0$ with $m \in \mathbb{R}^{\mathbb{N} \times q}$ and $Q : \mathbb{R}^{\mathbb{N} \times q} \to \mathbb{R}^{\mathbb{N} \times q}$. Given a solution for $u^{(t)}$, one can back out $\hat{x}^{(t)}$ from (32) and then $\hat{y}^{(t)}$ from (6).

The LCP can be solved using the mixed-integer linear programming methods developed in Holden (2016). In practice, we have found that a simple heuristic can often find a solution

[10] The constant $\underline{i}$ can be defined as an additional variable.

more quickly than a (standard-grade) MILP solver. Our heuristic iterates on guesses for when the occasionally binding constraints will be binding. We denote these guesses with $Z^{(t)} \in \{0,1\}^{\mathbb{N}\times q}$.

1. Start with an initial guess for $Z^{(t)}$, e.g. $Z^{(t)} = 0$.

2. Find the indices $\mathcal{I}_0 = \left\{ i \in \mathbb{N} \times \{1,\ldots,q\} \mid Z_i^{(t)} = 0 \right\}$ for which the guess is that the constraint is slack. Set $u_{\mathcal{I}_0}^{(t)} = 0$.

3. Find the indices $\mathcal{I}_1 = \left\{ i \in \mathbb{N} \times \{1,\ldots,q\} \mid Z_i^{(t)} = 1 \right\}$ for which the guess is that the constraint is binding. In those indices, the guess implies $m_{\mathcal{I}_1} + Q_{\mathcal{I}_1,\mathcal{I}_1} u_{\mathcal{I}_1}^{(t)} = 0$. Use this system of equations to solve for $u_{\mathcal{I}_1}^{(t)}$.

4. Now evaluate whether $u^{(t)} \geq 0$ and $m + Qu^{(t)} \geq 0$. If this is the case, $u^{(t)}$ is a valid solution. Stop.

5. Otherwise, modify the guess for $Z^{(t)}$. Let $\mathcal{I}_y = \left\{ i \in \mathbb{N} \times \{1,\ldots,q\} \mid m_i + Q_{i\cdot} u^{(t)} < 0 \right\}$ be the indices where the constraint was guessed to be slack but it is violated. Unless the set is empty, pick one element $i_1^* \in \mathcal{I}_y$ at random and set $Z_{i_1^*}^{(t)} = 1$. Similarly, let $\mathcal{I}_u = \left\{ i \in \mathbb{N} \times \{1,\ldots,q\} \mid u_i^{(t)} < 0 \right\}$ be the indices where the constraint was guessed to be binding but it is slack. Unless the set is empty, pick one element $i_2^* \in \mathcal{I}_u$ at random and set $Z_{i_2^*}^{(t)} = 0$.[11]

6. Iterate until a solution has been found.

More complex rules that involve non-linearities other than an ELB constraint can be accommodated in this way, including rules that respond only to negative output gaps. One example is an asymmetric Taylor rule with an ELB constraint $i_t = \max\{\underline{i}, \phi_\pi \pi_t + \phi_y \min\{ygap_t, 0\}\}$. We can introduce two auxiliary variables ($q = 2$) and write $i_t - \phi_\pi \pi_t + \phi_y u_{2t} + u_{1t} = 0$, $u_t \geq 0$, $i_t \geq \underline{i}$, $ygap_t - u_{2t} \geq 0$, as well as complementary slackness conditions, to express

[11] This step can be changed to modify several indices at once or to tweak the probability weights for randomization to be proportional to the severity of the violation of the non-negativity constraints.

it in the form (28)–(31). But there are other rules that can not readily be expressed in this form. Prominent examples are the "threshold" rules in Bernanke, Kiley, and Roberts (2019), where the policy rate is kept at the ELB until some condition is met, e.g. until the cumulative shortfall of inflation since the ELB became binding is made up. These rules consist of different regimes with switching conditions that depend on the endogenous model variables. Such a rule takes the form $A_+ y_t = 0$ if $Cy_t \geq 0$ and $A_- y_t = 0$ if $Cy_t < 0$. However, these kinds of problems can also be expressed as mixed-integer linear programming problems or solved using variations of our heuristic algorithm where $Z^{(t)}$ acts as an index of the regimes.

5.2 Optimal policy under commitment

Let us consider the optimal commitment problem with inequality constraints:

$$\min_{\left(y_t, \hat{x}^{(t)}\right)_{t=0}^{\infty}} \frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t y_t' W y_t$$

$$\begin{aligned} \text{s.t. } \hat{y}^{(t)} &= \hat{\bar{y}}^{(t)} + M \hat{x}^{(t)} \\ E_t \hat{x}^{(t+1)} &= 0 \\ C y_t &\geq 0 \end{aligned}$$

with $C \in \mathbb{R}^{q \times n}$. The Lagrangian of this problem is

$$\begin{aligned} \mathcal{L} =& \frac{1}{2} \left(\sum_{t=0}^{\infty} L^t \hat{y}^{(t)} \right)' (B \otimes W) \left(\sum_{t=0}^{\infty} L^t \hat{y}^{(t)} \right) + \sum_{t=0}^{\infty} \beta^t \lambda^{(t)\prime} \left(\hat{\bar{y}}^{(t)} + M \hat{x}^{(t)} - \hat{y}^{(t)} \right) \\ & - \sum_{t=0}^{\infty} \mu^{(t-1)\prime} \hat{x}^{(t)} - \eta' (B \otimes C) \sum_{t=0}^{\infty} L^t \hat{y}^{(t)}. \end{aligned}$$

The resulting first-order conditions for $\hat{y}^{(t)}$ and $\hat{x}^{(t)}$ are

$$(B \otimes W) y^{(t)} - \beta^t \lambda^{(t)} - (B \otimes C') \eta^{(t)} = 0 \tag{34}$$

$$\beta^t M' \lambda^{(t)} - \beta^t \mu^{(t-1)} = 0. \tag{35}$$

After subtracting $t-1$-expectations, we can combine these into:

$$M' (B \otimes W) \hat{y}^{(t)} - M' (B \otimes C') \hat{\eta}^{(t)} = 0. \tag{36}$$

In addition, we need $\eta^{(t)} \geq 0$, $(I_T \otimes C)\, y^{(t)} \geq 0$ and $\left\langle \eta^{(t)}, (I_T \otimes C)\, y^{(t)} \right\rangle$ to hold. We can express this problem in the form (29)–(31) with $\Omega_y = M' (B \otimes W)$, $\Omega_u = M' (B \otimes C')$, $\Theta_y = I_{\mathbb{N}} \otimes C$, and $\Theta_u = 0$. The algorithms described previously for simple rules can therefore also be used for the optimal commitment problem.

Simple inequality constraints like an ELB constraint can thus be added to the optimal commitment problem. But more complex policy problems can also be accommodated. Consider an asymmetric objective in which policy penalizes discounted deviations of inflation π_t from some target π_t^* and of shortfalls of output from potential output so that the loss function is $E_0 \sum_{t=0}^{\infty} \frac{1}{2} \beta^t \left[(\pi_t - \pi_t^*) + (\min \{ ygap_t, 0 \})^2 \right]$. This loss function is not quadratic, but the problem can nevertheless be rewritten with a quadratic objective. To do so, introduce an auxiliary variable aux_t and assume that the policymaker can control this variable, so that the number of policy instruments p is increased by one. The impulse responses of anticipated shocks to the additional instrument are given by the identity for aux_t and zero for all other variables. Now write the loss function as $E_0 \sum_{t=0}^{\infty} \frac{1}{2} \beta^t \left[\pi_t^2 + (aux_t)^2 \right]$ and add the additional inequality constraint $ygap_t - aux_t \geq 0$. Suppose $ygap_t \geq 0$. Then it is possible to set $aux_t = 0$ and therefore minimize the term $(aux_t)^2$ in the loss function. If $ygap < 0$, then the term $(aux_t)^2$ is minimized when $aux_t = ygap_t$. This new problem is therefore equivalent to the original one.

5.3 Optimal policy under discretion

We now add constraints of the form $Cy_t \geq 0$ to the optimal discretionary policy problem. Solving for an equilibrium in a sequential game with non-linear best responses is complex and we restrict ourselves to a subset of solutions that we are able to characterize.

Each policymaker has to internalize the effect of her actions on the behavior of future policymakers, but this behavior depends on which future constraints are binding. However, which constraints are binding also depends on the future state of the economy in the future, which is affected by the choices of the time-h policymaker. Solving this problem fully, even under the assumption of quasi-perfect foresight, is challenging. We simplify the problem by assuming that each policymaker takes the set of constraints that are expected to be binding in the future as given. Furthermore, we assume that all constraints are expected to be slack after at most H periods.

At each point in time t, we want to compute the expected path $y^{(t)}$ of the economy, assuming we have already computed the expected path $y^{(t-1)}$ in the previous period. We start with a guess $Z^{(t)} \in \{0,1\}^{\mathbb{N}\times q}$ of the binding constraints based on information available at time t. We then iterate backwards on the on the policymakers at time $h = t+H, t+H-1, \dots, t+1, t$. We set up a variant of the problem of the discretionary policymaker in the unconstrained case that introduces the inequality constraints:

$$\min_{y^{(h)}, \hat{x}^{(h)}} \frac{1}{2} y^{(h)\prime} (B \otimes W) \, y^{(h)}$$

$$\text{s.t. } y^{(h)} = F y^{(h-1)} + \hat{\tilde{y}}^{(h)} + M \hat{x}^{(h)} \tag{37}$$

$$d^{(s,t)} F^{s-h} y^{(h)} = 0, \; s > h \tag{38}$$

$$C y_h \geq 0 \tag{39}$$

where $d^{(s,t)}$ describes the reaction function of the policymaker at time s that is expected at time t. For $s > t+H$, this is the time-invariant unconstrained policy rule $d^{(s,t)} = d^*$ computed in Section 3.3. For $h < s \leq t+H$, $d^{(s,t)}$ has been computed in our backward iteration. To derive the policy path expected at time t, we will first solve this sequence of problems under the assumption that there are no further shocks to the baseline projection after time t, so we set $\hat{y}^{(h)} = 0$ and $y^{(h)} = F^{h-t} y^{(t)}$ for $h > t$. This way, we solve the discretionary problem under quasi-perfect foresight.

The Lagrangean of the above problem is

$$\mathcal{L}_h = \frac{1}{2} E_h y^{(h)\prime} (B \otimes W) \, y^{(h)} + \lambda^{(h)\prime} \left(F y^{(h-1)} + \hat{\tilde{y}}^{(h)} + M \hat{x}^{(h)} - y^{(h)} \right) - \eta^{(h)\prime} D^{(h+1,t)} F y^{(h)} - \mu_h' \tilde{C} y^{(h)}$$

where $\tilde{C} : \mathbb{R}^{\mathbb{N} \otimes n} \to \mathbb{R}^q$ is the mapping that encodes the constraint $C y_h = 0$. The first-order conditions with respect to $y^{(h)}$ and $\hat{x}^{(h)}$ can be combined to eliminate $\lambda^{(h)}$:

$$M' \left(B \otimes W \right) y^{(h)} = M' \left(D^{(h+1,t)} F \right)' \eta^{(h)} + M' \tilde{C}' \mu_h. \tag{40}$$

Analogously to the derivations in the unconstrained case, we can eliminate the multiplier $\eta^{(h)}$ to arrive at

$$\left(M_{\cdot 0} - M_{\cdot -0} \left[D^{(h+1,t)} F M_{\cdot -0} \right]^{-1} D^{(h+1,t)} F M_{\cdot 0} \right)' \left[\left(B \otimes W \right) y^{(h)} - \tilde{C}' \mu_h \right] = 0. \tag{41}$$

If we have guessed that all constraints in period h are slack ($Z_h^{(0)} = 0$), then the multipliers on the constraints should all be zero, $\mu_h = 0$, and Equation (41) constitutes the condition $d^{(h,t)} y^{(h)} = 0$ that characterizes the behavior of the time$h$ policymaker under the guess $Z^{(t)}$. But if we have guessed that some constraints are binding, we have to solve for the optimality condition and the value of the multiplier μ_h. For the constraints for which $Z_{h,i}^{(t)} = 1$, we impose the condition $\tilde{C}_{i \cdot} y^{(h)} = 0$, whereas for those constraints with $Z_{h,i}^{(t)} = 1$, we impose the condition $\mu_{hi} = 0$. The resulting ensemble of q equations together with (41) forms a system of equations of the form

$$\Omega_y y^{(h)} + \Omega_\mu \mu_h = 0 \in \mathbb{R}^{p+q}.$$

This system can be transformed into the form

$$\begin{aligned} d^{(h,0)} y^{(h)} &= 0 \\ \mu_h &= g^{(h,0)} y^{(h)} \end{aligned}$$

with $d^{(h,0)} : \mathbb{R}^{\mathbb{N} \times n} \to \mathbb{R}^p$ and $g^{(h,0)} : \mathbb{R}^{\mathbb{N} \times n} \to \mathbb{R}^q$.[12]

We can now lay out the recursion from the policymaker at time $h = t + H$ through to $h = t$. For $h = t + H$, the subsequent policymakers are unconstrained and we set $D^{(t+H+1,t)} = D^*$. We also initialize a map $G^{(t+H+1,t)} : \mathbb{R}^{\mathbb{N} \times n} \to \mathbb{R}^{\mathbb{N} \times q}$ to $G^{(t+H+1,t)} = 0$. For

[12]For finite-dimensional Ω_y and Ω_μ, a LU decomposition of yields $(\Omega_y, \Omega_\mu) = L \begin{pmatrix} U_{\mu\mu} & U_{\mu y} \\ 0 & U_{yy} \end{pmatrix}$ with L invertible, so that $\Omega_y y^{(h)} + \Omega_\mu \mu_h = 0$ is equivalent to $U_{\mu\mu} \mu_h + U_{\mu y} y^{(h)} = 0$ and $U_{yy} y^{(h)} = 0$. From there we can see that $d^{(h,0)} = U_{yy}$ and $g^{(h,0)} = -U_{\mu\mu} U_{\mu y}$.

each $h = t+H, t+H-1, \ldots, t+1, t$, we compute $d^{(h,t)}$ and $g^{(h,t)}$ as just described and set

$$D^{(h,0)} = \begin{pmatrix} d^{(h,t)} \\ \beta D^{(h+1,t)} F \end{pmatrix}, \quad G^{(h,t)} = \begin{pmatrix} g^{(h,t)} \\ \beta G^{(h+1,t)} F \end{pmatrix}. \tag{42}$$

At the end of this backward recursion, we end up with a set of conditions $D^{(t,t)} y^{(t)} = 0$ which we can use to solve for $\hat{x}^{(t)}$ and therefore $y^{(t)}$. We can then check whether our guess $Z^{(t)}$ of the constraints that are expected to be binding at time $t = 0$ is valid. The guess is valid if all inequality constraints are satisfied and all Lagrange multipliers on the constraints are non-negative, which is equivalent to

$$(B \otimes C)\, y^{(t)} \geq 0 \text{ and } G^{(t,t)} y^{(t)} \geq 0.$$

If the guess is valid, we can move on to solve for the expected trajectory at time $t = 1$ in the same way.

We now describe a heuristic similar to the one outlined earlier that is able to find solutions to the optimal policy problem under discretion. At each point in time $t = 0, 1, 2, \ldots$, run the following iterative algorithm:

1. Start with an initial guess for $Z^{(t)}$.[13]
2. For $h = t+H, t+H-1, \ldots, t+1, t$, compute $\left(D^{(h,t)}, G^{(h,t)}\right)$from $\left(D^{(h+1,t)}, G^{(h+1,t)}\right)$ as described above, starting from $D^{(t+H+1,t)} = D^*$ and $G^{(t+H+1,t)} = 0$.
3. Compute $y^{(t)}$ from $D^{(t,t)} y^{(t)} = 0$ and $y^{(t)} = F y^{(t-1)} + \hat{\tilde{y}}^{(t)} + M x^{(t)}$.
4. Now evaluate whether $(B \otimes C)\, y^{(h)} \geq 0$ and $G^{(0,0)} y^{(h)} \geq 0$. If this is the case, $y^{(t)}$ is a valid solution. Stop.
5. Otherwise, modify the guess for $Z^{(t)}$. Let $\mathcal{I}_C = \left\{ i \in \mathbb{N} \times \{1, \ldots, q\} \mid (B \otimes C)_{i\cdot}\, y^{(h)} < 0 \right\}$ be the indices where the constraint was guessed to be slack but it is violated. Unless the set is empty, pick one element $i_1^* \in \mathcal{I}_C$ at random and set $Z_{i_1^*}^{(t)} = 1$. Similarly,

[13]For $t = 0$, one can set $Z^{(0)} = 0$. For $t > 0$, $Z^{(t)} = F Z^{(t-1)}$ is a natural starting guess.

let $\mathcal{I}_\mu = \left\{ i \in \mathbb{N} \times \{1, \ldots, q\} \mid G_{i\cdot}^{(0,0)} y^{(t)} < 0 \right\}$ be the indices where the constraint was guessed to be binding but the corresponding multiplier is negative. Unless the set is empty, pick one element $i_2^* \in \mathcal{I}_\mu$ at random and set $Z_{i_2^*}^{(t)} = 0$.[14]

6. Iterate until a solution has been found.

If this algorithm finds a solution, it is a valid equilibrium path under optimal discretionary policy with quasi-perfect foresight. But there are certain types of solutions that we cannot discover this way. We have assumed that each policymaker takes the set of future constraints that are binding, encoded in $Z^{(t)}$, as given. Locally, this is true if the solution is an "interior point" in the sense that a marginal change in outcomes will not lead to any binding constraints becoming slack or slack constraints becoming binding. There can be, however, solutions which sit at a "corner" in the sense that at least one policymaker is exactly at the point where their constraint is binding, but the corresponding Lagrange multiplier is exactly zero. At this point, a marginal change in outcomes can flip constraints from binding to slack or vice-versa, leading to non-differentiability at the equilibrium point. Finding such corner solutions is beyond the scope of our algorithm.

6 Historic revisions and measurement error

So far, we have assumed that the current state of the economy y_t is perfectly observable to the policymaker. In practice, however, policymakers face a large amount of uncertainty about how to interpret current data and even how to interpret the past. Notably, economic data are subject to revisions that rewrite the path of history, which policymakers need to take into account.

In this section, we extend our procedure to a tractable case of imperfect information that accommodates historic revisions of the baseline projection. The economy continues to be

[14]This step can be changed to modify several indices at once or to tweak the probability weights for randomization to be proportional to the severity of the violation of the non-negativity constraints.

described by the model in (1), and solutions to the model continue to be adapted to the filtration $\mathbb{F}$ describing the information of the private sector whose behavior is described by the model. The policymaker, however, possesses more limited knowledge about the economy. Its information is described by a more restricted filtration $\mathbb{F}^* = (\mathcal{F}_t^*)_{t=0}^{\infty}$ for which $\mathcal{F}_t^* \subseteq \mathcal{F}_t$. Policymakers have to choose the instruments z_t such that they are adapted to $\mathbb{F}^*$. The policymaker's expectation is related to the full information expectation through the relation

$$E_t^* y_s = E_t y_s + e_{t,s},\ t, s \geq 0. \tag{43}$$

The above equation is just an identity that defines $e_{t,s}$ as a residual, but the term $e_{t,s}$ can be thought of as the measurement error of the policymaker. We will assume that error e is independent of policy: It is the same for every choice of the policy variables z. This assumption guarantees that the separation principle continues to apply (see Svensson and Woodford, 2004, for a detailed discussion). We also impose that there can be no uncertainty about current or past values of the policy instruments themselves: $E_t^* z_{t-s} = z_{t-s}$ for all $t, s \geq 0$.

Analogously to Section (2), we define $y_s^{*(t)} = E_t^* y_{t+s}$ and collect current and future states of the economy in $y^{*(t)} = \left(y_0^{*(t)\prime}, y_1^{*(t)\prime}, y_2^{*(t)\prime}, \dots\right)^\prime$. Because we now need to keep track of changes in history as well, we also introduce $y_-^{*(t)} = \left(y_0^{*(t)\prime}, \dots y_{t-1}^{*(t)\prime}\right)^\prime$ to denote the history of model outcomes at time t. As before, hats denote revisions and bars denote the baseline projection. With this notation, the evolution of the economy under the policymaker's and the private sector expectation is given by the following modification of (6):

$$\hat{y}^{*(t)} = \hat{\bar{y}}^{*(t)} + M\hat{x}^{*(t)} \tag{44}$$

$$\hat{y}_-^{*(t)} = \hat{\bar{y}}_-^{*(t)} \tag{45}$$

$$\hat{y}^{(t)} = \hat{\bar{y}}^{(t)} + M\hat{x}^{*(t)}. \tag{46}$$

To see this, note first that $E_t^* y_s - E_t^* \bar{y}_s = E_t y_s - E_t \bar{y}_s$ because of our assumption that e is independent of policy, so that it is the same under the baseline projection $\bar{y}$ and any counterfactual y. Thus, we have that $E_t^* y_s - E_t^* \bar{y}_s = E_t y_s - E_t \bar{y}_s$. Also, because $E_t^* x_s = \Psi E_t^* \left(y_s - \bar{y}_s\right)$, we have that $E_t^* x_s = E_t x_s$. The private sector expectation and the policymaker expectation of the standardized policy instruments are identical.

The computations for simple policy rules and optimal policy problems are preserved under this particular information structure. Consider, for example, the computation of outcomes under a simple policy rule with an occasionally binding constraint, as in Section (5.1). Under imperfect information, this requires

$$\begin{aligned} AE_t^* y_t &\geq 0 \\ CE_t^* y_t &\geq 0 \\ \langle AE_t^* y_t, CE_t^* y_t \rangle &= 0. \end{aligned}$$

Consider again the Taylor rule $i_t = \max\{\phi_\pi \pi_t + \phi_\pi ygap_t, \underline{i}\}$. The incomplete information version is $i_t = \max\{\phi_\pi E_t^* \pi_t + \phi_\pi E_t^* ygap_t, \underline{i}\}$, but modified to respect an ELB constraint $i_t \geq \underline{i}$. This can be expressed equivalently as $E_t^*[i_t - \underline{i}] \geq 0$, $E_t^*[i_t - \phi_\pi \pi_t - \phi_\pi ygap_t] \geq 0$ and $\mathbb{E}_t^*[i_t - \underline{i}]\,\mathbb{E}_t^*[i_t - \phi_\pi \pi_t - \phi_\pi ygap_t] = 0$. Taking expectations over the current interest rate is possible because under E_t^*, there is no uncertainty over current or past instruments.

This problem corresponds to the form

$$\begin{aligned} \Omega_y \hat{y}^{*(t)} &= \Omega_u \hat{u}^{(t)} \\ u^{(t)} &\geq 0 \\ \Theta_y y^{*(t)} + \Theta_u u^{(t)} &\geq 0 \\ \left\langle u^{(t)}, \Theta_y y^{*(t)} + \Theta_u u^{(t)} \right\rangle &= 0 \end{aligned}$$

with $\Omega_y = I_{\mathbb{N}} \otimes A$, $\Omega_u = I_{\mathbb{N}} \otimes I_p$, $\Theta_y = I_{\mathbb{N}} \otimes C$ and $\Theta_u = 0$. Using (44), the problem can be solved the same way as in the full information case. The only difference is that the baseline projection $\bar{y}^*$ can now change in history, as well. Under the simple rule, revisions in history are the same as under the baseline projection, and are given by (45). That is, historic revisions under a counterfactual policy regime move in lockstep with the corresponding revisions to the baseline projection.

Next, consider the problem of computing optimal commitment policies as in Section 3.2, but under incomplete information:

$$\min_{\left(\hat{y}^{(t)},\hat{x}^{*(t)}\right)_{t=0}^{\infty}} E_0^* \sum_{t=0}^{\infty} \frac{1}{2}\beta^t y_t' W y_t$$

$$\text{s.t. } \hat{y}^{(t)} = \hat{\bar{y}}^{(t)} + M\hat{x}^{*(t)}$$
$$E_t \hat{x}^{*(t+1)} = 0.$$

We have only replaced the policymaker expectations in the objective function. The first-order conditions for $\hat{y}^{(t)}$ and $\hat{x}^{(t)}$ are nevertheless

$$\left(L^t\right)' (B \otimes W)\, L^t E_t^* y^{(t)} - \beta^t \lambda^{(t)} = 0$$
$$\beta^t M' \lambda^{(t)} - \beta^t \mu^{(t-1)} = 0.$$

Subtracting the expectation under $\mathcal{F}_{t-1}^*$ of these equations and combining them yields Condition (12) with $\hat{y}^{(t)}$ replaced by $\hat{y}^{*(t)}$. The fact that there is uncertainty about the past does not enter the considerations of the optimizing policymaker. The reason is that, when history gets revised, the policymaker cannot affect the paths of history and the associated losses, and all the effects of the revision on current and future outcomes are contained in the forward-looking part of the baseline projection. Again, the policymaker's perception of the counterfactual equilibrium path under optimal policy can now change in history, but the revisions move in lockstep with the revisions to the baseline projection according to (45).

7 Application

We now present a practical application of our solution methods. We compute counterfactuals for the path of the U.S. economy between 2020 and 2023 under the assumption that monetary policy followed either a simple interest rate rule or optimal commitment or discretionary policy. Our simulations are carried out using either a linear version of the FRB/US model (Brayton, 2018) or a modified version of the del Negro, Giannoni, and Schorfheide (2015) model. Both models have been used previously for policy analysis at the Federal Reserve. For the baseline projections, we use the projections made in real time by members of the Federal Open Market Committee (FOMC) in their Summary of Economic Projections (SEP). Thus, our counterfactuals are conditioned on the information available to policymakers at that time rather than on ex-post realized and revised data. We achieve this without the need to filter structural shocks in either model, as the projections contain all the information about the economy that is relevant for the construction of valid counterfactuals.

The period between 2020 and 2023 spans the Covid-19 pandemic and the following rise of inflation. There are several reasons why this is an interesting episode for U.S. monetary policy. First, the FOMC had just adopted a new framework for its monetary policy strategy[15] which stipulated that "following periods when inflation has been running persistently below 2 percent, appropriate monetary policy will likely aim to achieve inflation moderately above 2 percent for some time" and that "policy decisions must be informed by assessments of the shortfalls of employment from its maximum level." Then, the pandemic shock of 2020 led to unprecedented disruptions in economic activity and a drop in inflation that prompted the FOMC to lower interest rates to the ELB. That was followed by a rise in inflation that was larger and more persistent than initially expected. The FOMC raised the federal funds rate to above five percent at a quick pace, but some outside observers argued that, based for example on the prescriptions of the Taylor rule, interest rates should have risen earlier. Our simulations will show how following such a rule, or following optimal policy with or without commitment, would have altered the federal funds rate path and economic conditions through the lens of structural macroeconomic models. Our main finding from these simulations is that optimal policy under commitment would have made a strong promise to keep interest rates low in 2020, which would have been honored even as inflation rose subsequently. The policy path would have been appeared to be "behind the curve" relative to simple Taylor-type rules, and this appearance would have reflected the time-inconsistency of the optimal policy.

[15] See the Statement on Longer-Run Goals and Monetary Policy Strategy, available at https://www.federalreserve.gov/monetarypolicy/files/FOMC_LongerRunGoals.pdf.

7.1 Baseline projections

We use 15 quarterly vintages of projections $\bar{y}^{(t)}$ for our simulation exercises, starting in 2020:Q2 and ending in 2023:Q4. Each baseline projection is based on the median SEP forecast released in that quarter.[16] In the SEP, participants provide yearly projections for the current and next two or three calendar years as well as for the "longer-run". These projections include real GDP growth, the unemployment rate, and headline and core PCE inflation projections, as well as participants' individual assumptions of the projected appropriate federal funds rate. The Federal Reserve's staff uses a model-guided interpolation and extrapolation procedure as well as current economic data to build quarterly series of these (and other) variables.[17] In particular, the paths of those variables available in the SEP are assumed to gradually converge to the median of the SEP longer-run projections. We will use these quarterly series as our baseline projections.[18]

Figure 1 shows the projected paths of the quarterly average of the federal funds rate, the four-quarter change in the (headline) PCE price index, the quarterly average of the civilian unemployment rate, as well as an unemployment gap measure, as projected at different points in time. The paths for the first three of these variables converge to the respective median longer-run SEP projections. The unemployment gap is constructed as the difference of the unemployment rate and an estimate of the natural rate of unemployment based on current and past median longer-run SEP projections of the unemployment rate; in particular, it converges to zero by construction.

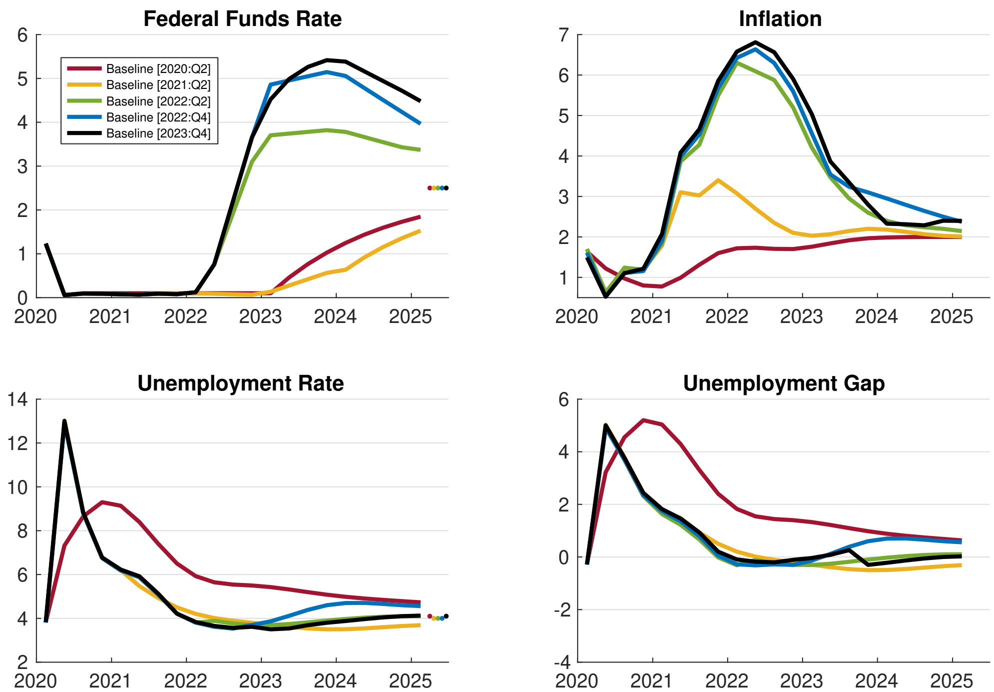
Since the start of the pandemic, FOMC participants revised their projections of the U.S. economy significantly. The top left panel shows that in the second quarter of 2020, the

[16] Economic projections are collected from each member of the Board of Governors and each Federal Reserve Bank president four times a year, in connection with the FOMC meetings in March, June, September, and December.

[17] The staff regularly publishes these time series along with further documentation as part of its FRB/US model package, available at https://www.federalreserve.gov/econres/us-models-package.htm.

[18] The resulting projections need not represent the economic projections of the Committee or of any Committee participant.

Figure 1: Baseline Projections.



Note: All projections are based on median SEP responses published in the respective quarter. "Federal Funds Rate" is the quarterly average of the nominal federal funds rate in percent. "Inflation" is the four-quarter change in the personal consumption expenditure (PCE) price index in percent. "Unemployment rate" is the quarterly average of the civilian unemployment rate in percent. "Unemployment gap" is the difference, in percentage points, of the unemployment rate to an estimate of the natural rate of unemployment constructed using a mechanical procedure based on current and past median longer-run SEP projections of the unemployment rate. For the federal funds rate and the unemployment rate, dots represent the median longer-run projected values.

federal funds rate was expected to stay at the ELB at least through 2022.[19] But the path of the federal funds rate was subsequently revised up considerably and the first rise in the policy rate in that rate cycle happened in March 2022. Subsequently, the projected funds rate path continued to be revised up as the actual federal funds rate was raised above five percent. These revisions were accompanied by increasingly large revisions to the inflation projections, shown in the upper right panel, as the rise in realized inflation was stronger and more persistent than the SEP median projections. At the same time, the unemployment rate projections also changed over this time. In the second quarter of 2020, unemployment was expected to be persistently high for several years.[20] But by the next year, this projection was already considerably revised as the labor market recovered more quickly than had been anticipated. At the end of 2022, unemployment was expected to increase again the following year, but this rise also did not materialize, as the strong labor market remained a defining feature of the post-pandemic economy.

7.2 Models

We use two different models for our simulations. The first is a linear version of the Federal Reserve's FRB/US model, a large-scale estimated general equilibrium model of the U.S. economy that has been in use at the Federal Reserve Board since 1996 and has been repeatedly adapted to the evolving the structure of the economy. A linearized version of that model was recently made publicly available and documented by Brayton and Reifschneider (2022).[21] For the purpose of our simulations, we only simulate three of these variables, namely the federal funds rate i_t, the four-quarter change in the headline PCE price index

[19]These projections do not capture the precise liftoff dates that were expected by FOMC participants because only year-end projections for a few years are provided. In particular, in 2020:Q2 and 2021:Q2 the SEP only elicited year-end projections through 2022 and 2023, respectively.

[20]Because SEP participants only project year-end values and the staff smoothly interpolates these values to create a baseline path, the unemployment projection of 2020:Q2 does not show a spike as in the realized data.

[21]The FRB/US model incorporates several alternative assumptions for expectation formation. We use model-consistent (i.e. rational) expectations for financial market participants and wage-price setters and VAR-based expectations for other sectors of the economy.

π_{4t}, and the unemployment gap $ugap_t$, which are natively defined in the model. Thus, we only need to compute impulse responses for these three variables.

The second model we use is the DGS-FHP model, a modified version of the dynamic general equilibrium model developed by del Negro, Giannoni, and Schorfheide (2015) that attenuates the forward-guidance puzzle through the introduction of finite-horizon planning for households. This model has previously been used in policy analysis at the Federal Reserve and is documented in Arias, Bodenstein, Chung, Drautzburg, and Raffo (2020). It features greater, yet empirically plausible, sensitivity of the economy with respect to interest rates than the FRB/US model. We make the following translations of our data series to this model: We equate the federal funds rate with the annualized quarterly nominal interest rate (that is, our i_t equals $4r_t$ in the model) and the four-quarter change in the headline PCE index with the sum of the current and last three quarters of the quarterly inflation rate (our π_{4t} equals $\sum_{s=0}^{3} \pi_{t-s}$ in the model). Because the DGS-FHP model does not feature unemployment, we approximate the unemployment gap with the model's output gap through a simple Okun's law and set $U_t - U_t^* = 0.5\,(y_t - y_t^*)$, where y_t and y_t^* are the log-levels of output in the model under sticky and flexible prices, respectively.

For both models, we compute impulse responses to anticipated shocks to the federal funds rate using Dynare. The choice of the equation that determines policy at this stage is irrelevant as long as it yields a unique solution. We use an unemployment gap version of the Taylor (1993) rule:

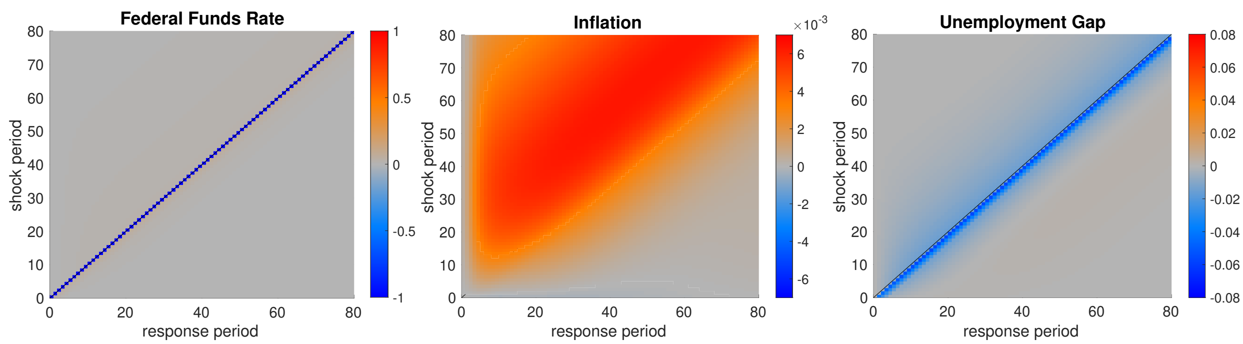
$$i_t = r_t^* + \pi_{4t} + 0.5\,(\pi_{4t} - 2) - (U_t - U_t^*)\,. \tag{47}$$

In addition to the three variables we simulate, the long-run level of the natural real interest rate r_t^* and the natural rate of unemployment U_t^* also appear in the rule. These two additional variables are independent of policy; thus, the impulse responses of these variables to anticipated policy shocks are zero everywhere.

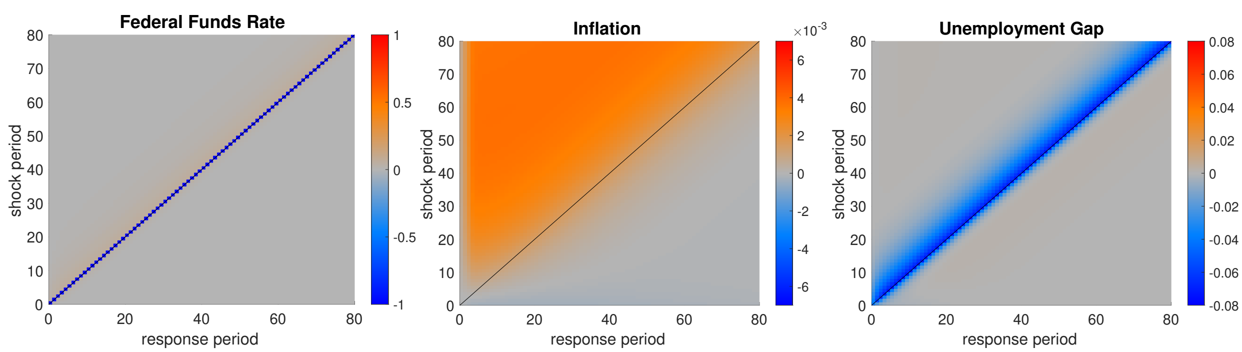
Figure 2 plots these impulse responses for both models. The upper panels show that in the FRB/US model, inflation increases and unemployment decreases at all lags and leads

Figure 2: Impulse Responses to Anticipated Monetary Policy Shocks.

(a) FRB/US Model.



(b) DGS-FHP Model.



Note: Each panel plots impulse responses $M_{\tau s}$, $0 \leq \tau, s \leq 80$, of a variable in period s (horizontal axis) to a shock occurring in period τ (vertical axis) and fully anticipated in period 0, when the federal funds rate is set according to the Taylor (1993) rule. Variable definitions as noted in the text. Shocks are normalized such that at time τ, the federal funds rate decreases by one percentage point.

following accommodative policy shocks. Inflation is very forward-looking: Inflation responds even to policy shocks that are expected to occur very far in the future. At the same time, the magnitude of the inflation response is modest due to a relatively flat Phillips curve. Unemployment, by contrast, responds more inertially to policy shocks, with the forward-looking effects before the realization of the shock (above the 45-degree line) less pronounced than the backward-looking effects after (below the 45-degree line).

The dynamics of the DGS-FHP model are noticeably different. Inflation responds even less strongly to policy, implying an even flatter Phillips curve, but the response is also more forward-looking with less lagged effects. The magnitude of the response of the unemployment gap to monetary policy is stronger and more forward-looking than in the FRB/US model, pointing to a higher sensitivity of real activity to changes in interest rates.

7.3 Interest rate rules

In our first set of simulations, we compute counterfactuals under the assumption that the federal funds rate is set according to one of two rules:

$$i_t = \max\{\underline{i},\, r_t^* + \pi_{4t} + 0.5\,(\pi_{4t} - 2) - (U_t - U_t^*)\} \tag{48}$$

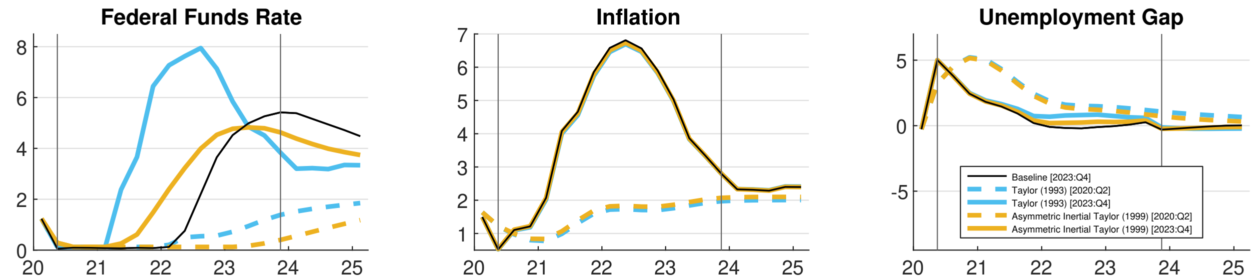
$$i_t = \max\{\underline{i},\, 0.85 i_{t-1} + 0.15\,(r_t^* + \pi_{4t} + 0.5\,(\pi_{4t} - 2) - 2\max\{U_t - U_t^*, 0\})\} \tag{49}$$

The first rule is the Taylor (1993) rule. The second rule is an asymmetric Taylor-type rule with inertia that responds only to shortfalls of employment. In both rules, the ELB is modeled as a lower bound $\underline{i}$ of 12.5 basis points.

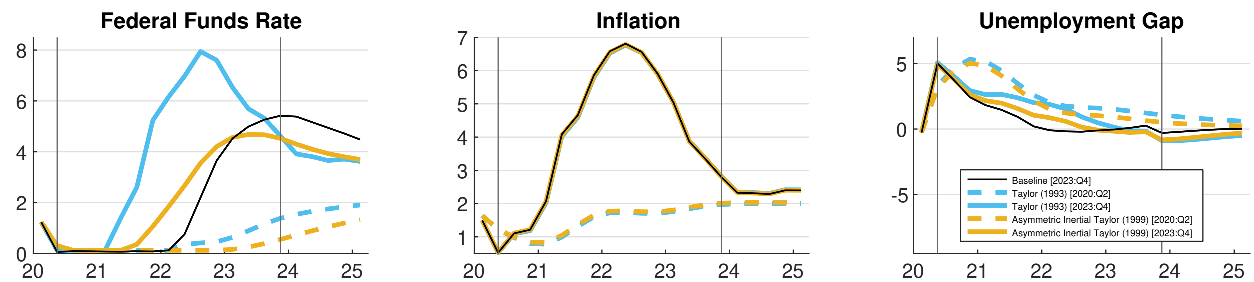
The upper three panels of Figure 3 show rules-based counterfactuals computed using the linearized FRB/US model. The economy is assumed to follow the baseline projection up to 2020:Q2. At that date, the respective rule starts to be followed; the path of the economy as it would have been projected in that quarter is represented by dashed lines. We then run the simulation using the sequence of baseline projections described above. Each

Figure 3: Rules-Based Counterfactuals.

(a) FRB/US Model.



(b) DGS-FHP Model.



Note: Counterfactual simulations all start in 2022:Q2 and continue through 2023:Q4. Each line represents past, current and future values of a variable as projected at a certain date, noted in brackets in the legend label. 2022:Q3–2023:Q3 counterfactual projections not shown.

quarter, the projected path of the counterfactual changes in response to the changes in the baseline projection. We show the final counterfactual projection as of 2023:Q4 with solid lines, omitting the projections in between.

The upper panels of Figure 3 show that, if the economy had evolved as projected by FOMC participants in 2020 following the initial pandemic shock, the Taylor (1993) rule (48) (blue lines) would have held the federal funds rate at the ELB through 2022 and subsequently lifted it to just below two percent in 2025. However, due to the subsequent upward surprises to inflation in the baseline projections and, to a lesser extent, downward surprises in the unemployment rate, the rule would have lifted the policy rate off the ELB by the first quarter of 2021 and would have quickly raised it to about eight percent. After inflation peaked in the middle of 2022, the rule would then have lowered the policy rate to below five percent by the end of 2023. Relative to the realized policy rate path, this counterfactual would therefore have had earlier and stronger tightening followed by quicker easing. Because this counterfactual tightening of policy is not very persistent, the Phillips curve in the model is quite flat, and long-run inflation expectations are implicitly anchored at 2 percent, the effects on inflation are modest, reducing it by less than 0.2 percentage point relative to the baseline. The effects on unemployment are more pronounced, with the unemployment rate up to one percentage point higher than in the baseline. The asymmetric inertial Taylor (1999) rule (49) (yellow lines) would have prescribed a slower pace of increases of the federal funds rate in 2021 and 2022, but also a slower pace of reductions thereafter. In the model, the net effect on the economy is similar to that of the non-inertial Taylor (1993) rule.

The lower three panels of Figure 3 repeat these simulations using the DGS-FHP model. All that is required for this change is to switch out the impulse responses M in the computations. Qualitatively, the counterfactuals retain the features discussed in the context of the FRB/US model above. The policy rate paths under the two rules shown are similar to those in the FRB/US model. The paths for inflation are also virtually the same as in the upper panels: Owing to the flat and forward-looking Phillips curve in both models, and anchored long-run inflation expectations, large changes to the policy rate have only small effects on inflation unless they persist for many years. However, the paths for the unemployment gap are noticably different because in the DGS-FHP model, unemployment reacts more strongly to contemporaneous changes in interest rates than in the FRB/US model. Under the Taylor (1993) rule, unemployment rises almost two percentage points above the baseline as a result of the additional policy tightening prescribed by that rule. Under the lower policy path prescribed by the asymmetric inertial Taylor (1999) rule, unemployment rises rises only about one percentage point above the baseline.

7.4 Optimal policy

We now turn to counterfactuals when the federal funds rate is set to minimize an intertemporal quadratic loss function under full discretion and full commitment. We start with an "equal-weights" loss function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[(\pi_{4t} - 2)^2 + (U_t - U_t^*)^2 + (i_t - i_{t-1})^2 \right]. \tag{50}$$

This loss function places equal weights on deviations of inflation from a target of 2 percent and deviations of the unemployment rate from the natural rate of unemployment. It also penalizes changes in the federal funds rate, which captures the desirability of gradualism (Woodford, 1999).

In addition, we also consider an asymmetric version of this loss function:

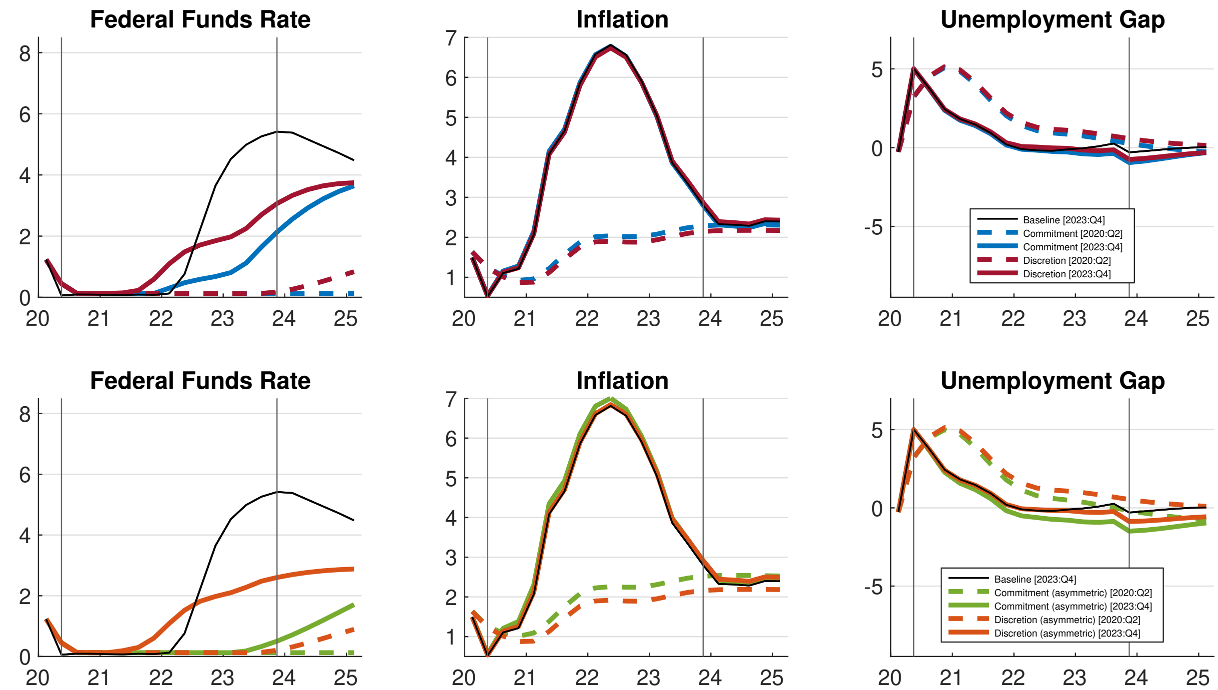
$$E_0 \sum_{t=0}^{\infty} \beta^t \left[(\pi_{4t} - 2)^2 + (\max\{U_t - U_t^*, 0\})^2 + (i_t - i_{t-1})^2 \right]. \tag{51}$$

This loss function differs from the first in that it only penalizes unemployment rate outcomes that are higher than the natural rate of unemployment. As in the simple rules simulations, we impose a lower bound on the nominal interest rate of 12.5 basis points.

Figure 4 shows optimal policy counterfactuals computed using the FRB/US model. As before, we start the counterfactuals at the 2020:Q2 baseline projection and let the respective policy regime stay in place through 2023:Q4. In particular, the commitment solution keeps honoring contingent promises as subsequent surprises to the baseline projection materialize each quarter.

The upper panels show optimal policy counterfactuals under the symmetric loss function (50). For the initial baseline projection in 2020:Q2, optimal policy would have kept the policy rate at the ELB through 2023 under discretion, and even longer under commitment, in order to counter a shortfall in projected inflation and an increase in projected unemployment (dashed lines). As these predictions failed to materialize and inflation instead rose substantially, the federal funds rate would have lifted off the ELB some time in 2021 under discretion and in 2022 under commitment. Interest rates would have been susbtantially lower under commitment than under discretion over the period shown, resulting in lower unemployment. This difference reflects the value of commitment: In 2020, the commitment policy embedded a promise to keep interest rates low in order to counter the expected shortfall in

Figure 4: Optimal Policy Counterfactuals in the FRB/US Model.

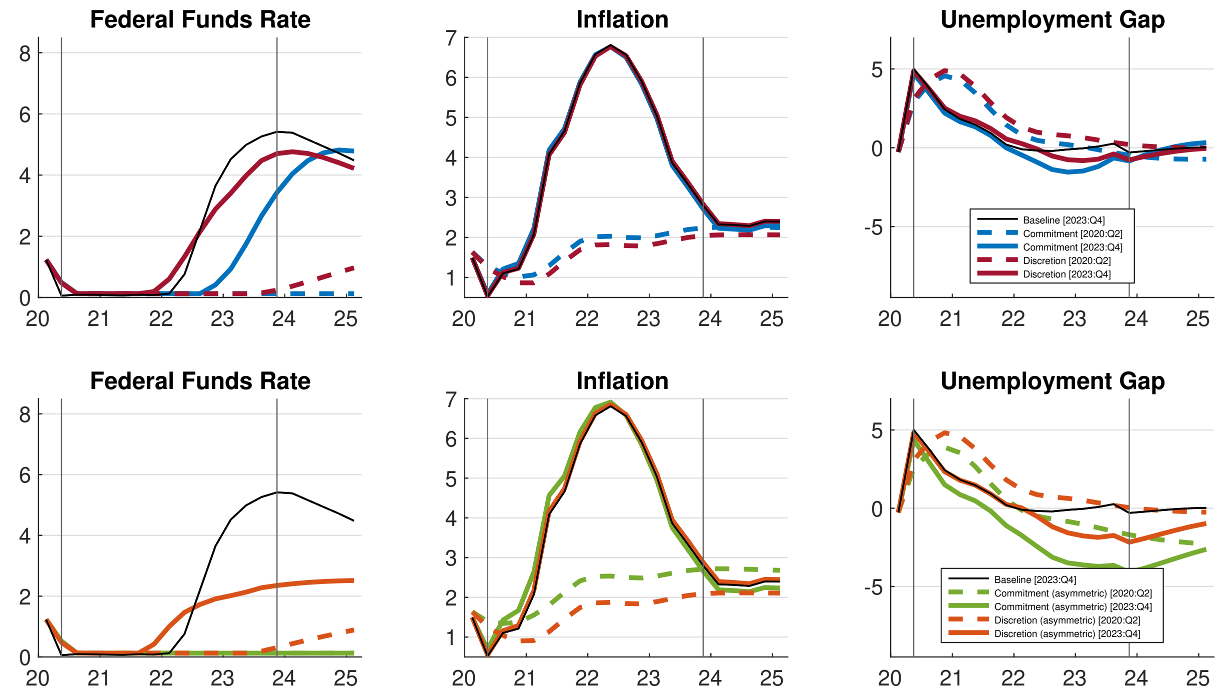


Note: Counterfactual simulations all start in 2022:Q2 and continue through 2023:Q4. Each line represents past, current and future values of a variable as projected at a certain date, noted in brackets in the legend label. 2022:Q3–2023:Q3 counterfactual projections not shown.

aggregate demand. In the simulation, this promise is honored even as the subsequent rise in inflation makes it appear as though this policy were "behind the curve": There would be an incentive to reoptimize and abandon the promises made during the pandemic. Finally, it is worth noting that the optimal policy path in the FRB/US model is shallower than the realized path. This result is somewhat sensitive to the amount of gradualism in the loss function (the weight on interest rate changes in (50)) but also reflects the fact that in this very forward-looking model, large immediate changes in the policy rate are less effective than gradual but more persistent anticipated changes.

The lower panels repeat the simulations under the asymmetric loss function (51). Under this loss function, policymakers do not see negative unemployment gaps as costly. As a result, they are willing to implement policies that lead to very low levels of unemployment if these policies also raise inflation from below 2 percent. Under discretion, this difference in preferences makes little difference in our simulation, as the unemployment rate barely falls

Figure 5: Optimal Policy Counterfactuals in the DGS-FHP Model.

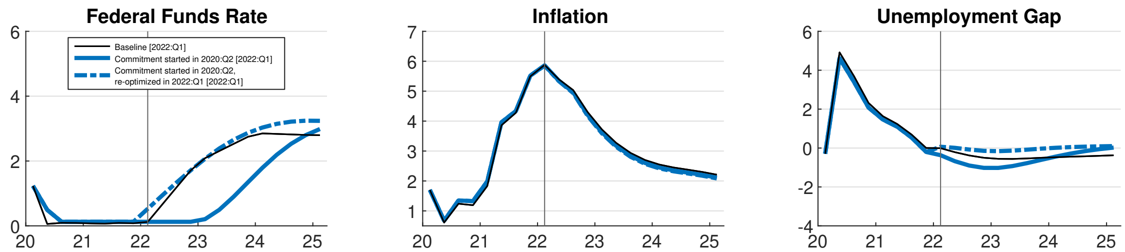


Note: Counterfactual simulations all start in 2022:Q2 and continue through 2023:Q4. Each line represents past, current and future values of a variable as projected at a certain date, noted in brackets in the legend label. 2022:Q3–2023:Q3 counterfactual projections not shown.

below the longer-run natural rate of unemployment. But under commitment, the difference is sizable. For the 2022:Q2 projection, the asymmetric commitment policy promises a policy path that manages to raise inflation by almost half a percentage point over several years and also narrows the projected unemployment gap (dashed lines). This strong promise of accommodative policy is honored even as the projections change over the next few years. As a result, the federal funds rate under this policy would have remained at the ELB during the entire rise in inflation and well into 2023.

Next, we repeat these simulations using the DGS-FHP model. The resulting counterfactual paths are displayed in Figure 5. The policy paths under under the symmetric loss function (50) are shown in the top panels. The projected paths for the initial baseline projection (dashed lines) are similar to those in the FRB/US model, but those in the final projection (solid lines) are noticeably steeper. The reason is that changes in interest rates have stronger contemporaneous effects in the DGS-FHP model so that optimal policy in

Figure 6: Honoring Past Commitments.



Note: Optimal policy simulations under commitment and the symmetric loss function (50). Each line represents past, current and future values of a variable as projected in 2022:Q1.

this model has stronger incentives to move rates quickly in response to changes in economic conditions. Under discretion, the policy path ends up quite similar to the realized path, lifting off the ELB in the first quarter of 2022. Under commitment, however, this lift-off is delayed until the last quarter of 2022, when inflation is already almost at its peak. Again, the reason is the initial promise made during the pandemic to keep policy accommodative that is being honored. This pattern is even more pronounced under the asymmetric loss function (51) as shown in the lower panels. With this loss function, the benefits of promising a lower-for-longer policy path are not tempered by costs arising from the unemployment rate falling below the longer-run natural rate. For the initial projection of 2022:Q2, the commitment solution anticipates keeping the policy rate at the ELB well beyond the period shown, resulting in an expected reduction in inflation of over half a percentage point despite the flat Phillips curve in the model. Even in the final projection of 2023:Q4, the commitment solution still anticipates staying at the ELB until 2026 as it still honors this past promise to some extent, resulting in lower inflation and much lower unemployment than under discretion.

We close this discussion by illustrating the ability of our sequence-space algorithm to honor past promises of optimal policy under commitment. Figure 6 shows the 2022:Q1 baseline projection (solid black lines) along with two different optimal policy simulations under commitment using the DGS-FHP model and the symmetric loss function.

In the first simulation (solid lines), policymakers start optimizing with commitment in 2020:Q2, and where we let policy respond to new information while keeping past promises.

It is the same simulation as that shown in Figure 5 (symmetric commitment, blue lines), but shown at a different point in time. Here, we show the projected counterfactual path in 2022:Q1, when four-quarter headline PCE inflation is at its peak of about seven percent. Because the commitment policy carries the past promise of keeping interest rates low, the projected policy path still implies keeping the policy rate at the ELB and incurring a significant undershoot of unemployment below the natural rate. Reneging on this promise is beneficial, which can be seen from the second simulation (dash-dotted lines). Here, we start with the same optimization with commitment in 2020:Q2, but in 2022:Q1 we restart the optimization at the current path.[22] When past promises are abandoned, it becomes optimal to immediately raise the federal funds rate and follow a path very similar to the baseline projected path. As a result of this less accommodative policy path, the unemployment gap is now projected to be closed and inflation falls further (if only slightly). The deviation between the two simulations illustrates the degree of time-inconsistenty of the initial commitment policy, which in the model is particularly strong around 2022.

8 Conclusion

In this paper, we have proposed a computational procedure to solve for policy counterfactuals in linear models with occasionally binding constraints in sequence space. The procedure requires only minimal knowledge of the structural model. The only two inputs are a projection, or sequences of projections, of the variables entering the policy problem; and impulse response functions of these variables to the monetary policy instruments under an arbitrary policy. We have shown how to compute solutions for instrument rules and optimal discretionary and commitment policies, as well as various extensions of practical relevance. A practical application to counterfactual paths of the U.S. economy after 2020 revealed that ex-ante optimal commitment policies to counter the pandemic shock would have appeared

[22] To keep the simulations comparable, we assume that policymakers drop their initial commitment unexpectedly and without suffering a loss of credibility.

to fall "behind the curve" as inflation rose because of the time-inconsistency of the lower-for-longer promises embedded in these strategies.

There are several directions in which our findings could be extended in future work. First, we are currently restricted to models that are linear up to occasionally binding constraints and quasi-perfect foresight expectations, but one can apply the methods described by Holden (2016) to extend our method to higher-order perturbation approximations of non-linear models and even incorporate genuine uncertainty about occasionally binding constraints. Second, we restrict ourselves to counterfactual policy regimes that yield unique equilibria, but the sequence space computations could also be used to find sunspot solutions. Finally, future research may address the use of the sequence-space approach to parameter or model uncertainty.

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A Appendix

A.1 MILP representation

Many problems with occasionally binding constraints can be written as linear complementarity problems of form $u^{(t)} \geq 0$, $m^{(t)} + Qu^{(t)} \geq 0$, $\langle u^{(t)}, m^{(t)} + Qu^{(t)} \rangle = 0$ with $m^{(t)} \in \mathbb{R}^{\mathbb{N}\times q}$ and $Q : \mathbb{R}^{\mathbb{N}\times q} \to \mathbb{R}^{\mathbb{N}\times q}$. The finite-dimensional approximation of this problem with up to T periods can be solved using mixed-integer linear programming (MILP) methods (Holden, 2016, 2023). There are several ways to express the LCP problem in a MILP representation. We choose the following representation:

$$\min_{\substack{u_{0:T}^{(t)} \in \mathbb{R}^{Tq} \\ Z \in \{0,1\}^{Tq}}} \quad \sum_{t=0}^{T} u_t^{(t)}$$

$$\begin{aligned} \text{s.t. } u^{(t)} &\geq 0 \\ Qu^{(t)} + m^{(t)} &\geq 0 \\ u^{(t)} &\leq \omega Z \\ Qu^{(t)} + m^{(t)} &\leq \omega \left(1 - Z\right). \end{aligned}$$

The constant ω has to be chosen large enough for the problem at hand. If there are multiple solutions to the LCP problem, this representation will choose the one for which the sum of the absolute values of $u_{0:T}^{(t)}$ is minimal. When the constraint is the ELB, this roughly represents the solution for which deviations from the unconstrained case are smallest.

A.2 Combining equality and inequality constraints

Some policy counterfactuals of interest may involve a mix of equality and inequality constraints. Such a situation can arise, for example, when there are multiple policy instruments and only a subset is subject to inequality constraints such as a lower bound constraint; when auxiliary variables need to be defined in order to express the policy problem; or when the optimal policy problem is subject to equality constraints, for example because one is interested in optimal interest rate policy subject to a simple rule for balance sheet policy. Here, we discuss a couple of cases that illustrate how to extend the computation of policy counterfactuals to such cases.

First, suppose we want to compute simple rules, where only some of them are subject to inequality constraints. We want to impose $Ay_t \geq 0$, $Cy_t \geq 0$, $\langle Ay_t, Cy_t \rangle = 0$, and $Dy_t = 0$. The matrices are $D \in \mathbb{R}^{n \times q_1}$ and $A, C \in \mathbb{R}^{n \times q_2}$ with $q_1 + q_2 = p$. Let $u_{1t} = Dy_t$ and $u_{2t} = Ay_t$. Then we can express the problem in the form:

$$\Omega_y \hat{y}_t = \Omega_u \hat{u}_t \tag{52}$$

$$u_1^{(t)} \geq 0 \tag{53}$$

$$\Theta_{y1} y^{(t)} + \Theta_{u1} u^{(t)} \geq 0 \tag{54}$$

$$\left\langle u_1^{(t)}, \Theta_{y1} y^{(t)} + \Theta_{u1} u^{(t)} \right\rangle = 0 \tag{55}$$

$$\Theta_{y2} y^{(t)} + \Theta_{u2} u^{(t)} = 0 \tag{56}$$

where $\Omega_y = I_{\mathbb{N}} \otimes \begin{pmatrix} A \\ D \end{pmatrix}$, $\Omega_u = I_{\mathbb{N}} \otimes I_p$, $\Theta_{y1} = I_{\mathbb{N}} \otimes C$, $\Theta_{u1} = 0$, $\Theta_{y2} = 0$, and $\Theta_{u2} = I_{\mathbb{N}} \otimes I_{q_2}$. To solve this problem, let S_1 and S_2 be the canonical mappings for which $\hat{u}^{(t)} = S_1 \hat{u}_1^{(t)} + S_2 \hat{u}_2^{(t)}$. Assume again that $\Omega_y M$ is invertible and write $R = M (\Omega_y M)^{-1}$. We make use of (32) to write

$$\hat{y}^{(t)} = (I - R\Omega_y) \hat{\bar{y}}^{(t)} + R\Omega_u \left(S_1 \hat{u}_1^{(t)} + S_2 \hat{u}_2^{(t)} \right). \tag{57}$$

We now let $\Pi_{ij} = (\Theta_{yi} R \Omega_u + \Theta_{ui}) S_j$ for $i = 1, 2$. We can substitute (57) into (56) and obtain:

$$0 = \Theta_{y2} (I - R\Omega_y) \hat{\bar{y}}^{(t)} + \Pi_{21} \hat{u}_1^{(t)} + \Pi_{22} \hat{u}_2^{(t)}.$$

This can be used to solve for $\hat{u}_2^{(t)}$:

$$\hat{u}_2^{(t)} = -\Pi_{22}^{-1} \left(\Theta_{y2} (I - R\Omega_y) \hat{\bar{y}}^{(t)} + \Pi_{21} \hat{u}_1^{(t)} \right).$$

Substituting this expression back into (57), we obtain:

$$\begin{aligned}\hat{y}^{(t)} &= \left(I - R\Omega_u S_2 \Pi_{22}^{-1}\Theta_{y2}\right)\left(I - R\Omega_y\right)\hat{\bar{y}}^{(t)} \\ &\quad + R\Omega_u\left(S_1 - S_2\Pi_{22}^{-1}\Pi_{21}\right)\hat{u}_1^{(t)}. \end{aligned} \tag{58}$$

Using the above expressions for $\hat{u}_2^{(t)}$ and $\hat{y}^{(t)}$, we can express the inequality (54) in the form $Qu_1^{(t)} + m \geq 0$ so that (53)–(55) form a standard LCP problem in $u_1^{(t)}$. The parameters Q and m are given by:

$$Q = \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} \tag{59}$$

$$\begin{aligned} m &= \Theta_{y1}Fy^{(t-1)} + \Theta_{u1}S_2Fu_2^{(t-1)} \\ &\quad + \left(\Theta_{y1} - \Pi_{12}\Pi_{22}^{-1}\Theta_{y2}\right)\left(I - R\Omega_y\right)\hat{\bar{y}}^{(t)} \\ &\quad - \left(\Theta_{y1}R\Omega_u S_1 - \Pi_{12}\Pi_2^{-1}\Pi_{21}\right)Fu_1^{(t-1)}. \end{aligned} \tag{60}$$

Second, consider an optimal policy problem under commitment with inequality and equality constraints:

$$\min_{\left(y_t, \hat{x}^{(t)}\right)_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \frac{1}{2}\beta^t y_t' W y_t$$

$$\begin{aligned} \text{s.t. } \hat{y}^{(t)} &= \hat{\bar{y}}^{(t)} + M\hat{x}^{(t)} \\ E_t\hat{x}^{(t+1)} &= 0 \\ Cy_t &\geq 0 \\ Dy_t &= 0. \end{aligned}$$

The Lagrangian of this problem is:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\sum_{t=0}^{\infty} L^t \hat{y}^{(t)}\right)' (B \otimes W)\left(\sum_{t=0}^{\infty} L^t \hat{y}^{(t)}\right) \\ &\quad + \sum_{t=0}^{\infty} \beta^t \lambda^{(t)\prime}\left(\hat{\bar{y}}^{(t)} + M\hat{x}^{(t)} - \hat{y}^{(t)}\right) - \sum_{t=0}^{\infty} \mu^{(t-1)\prime}\hat{x}^{(t)} \\ &\quad - \eta'(B \otimes C)\sum_{t=0}^{\infty} L^t \hat{y}^{(t)} - \phi'(B \otimes D)\sum_{t=0}^{\infty} L^t \hat{y}^{(t)}. \end{aligned}$$

After eliminating λ and μ in the usual way, the first-order conditions are:

$$M'(B \otimes W)\hat{y}^{(t)} = M'(B \otimes C')\hat{\eta}^{(t)} + M'(B \otimes D')\hat{\phi}^{(t)}.$$

In addition, a solution has to satisfy $(I_T \otimes C)\, y^{(t)} \geq 0$, $(I_T \otimes D)\, y^{(t)} = 0$, $\eta^{(t)} \geq 0$ and $\left\langle \eta^{(t)}, (I_T \otimes C)\, y^{(t)} \right\rangle$. Now we let $u_1^{(t)} = \eta^{(t)}$ and $u_2^{(t)} = \phi^{(t)}$. This has the form in (56)–(55) with

$$\Omega_y = M' \left(B \otimes W\right), \Omega_u = M' \left(B \otimes \begin{pmatrix} C' & D' \end{pmatrix}\right)$$
$$\Theta_1 = I_T \otimes C$$
$$\Theta_2 = I_T \otimes D.$$