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Information Externalities, Funding Liquidity, and Fire Sales[*]

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Abstract

We develop a theory of learning in a model of fire sales and collateralized debt to study how beliefs about fundamentals are shaped by market conditions. Agents exchange short-term debt contracts to invest in a long-term risky asset, and receive shocks to the opportunity cost of funds (cost shocks) and news about the fundamental of the asset, both of which are private information. Asset prices play a dual role of clearing markets and conveying agents' private information, but markets are informationally inefficient: Agents can partially, but never fully, infer their counterparties' private information from asset prices. The informational inefficiency of markets is more acute when liquidity conditions are especially tight or loose, as this impairs ability of prices to reveal private information about fundamentals. As a result, beliefs about fundamentals are shaped endogenously by cost shocks which are orthogonal to fundamentals, leading to socially costly booms and busts in asset prices. The equilibrium is constrained inefficient due to an information externality in which agents do not internalize how their choices affect the information set of other agents. Interventions in funding markets can stabilize asset prices by altering perceptions of risk.

JEL classification: D52, D53, E44, G28

Keywords: beliefs, learning, fire sales, liquidity, asset prices, information asymmetry

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[*]The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as the views of the Board of Governors of the Federal Reserve System or of anyone else associated with the Federal Reserve System.

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1. Introduction

Financial markets are inherently unstable and prone to episodes of panic. An important function of financial markets is aggregating private information about financial assets and facilitating price discovery. Yet in times of financial stress, when perhaps information matters the most, this function seems to break down, with sharp and temporary increases in uncertainty and excessive pessimism about fundamentals.[1] These episodes of financial panic are characterized by sudden and sharp declines in asset prices and a rise in risk premia, leading to credit and liquidity freezes which may come at great social costs. Moreover, a large literature has argued that asset price booms, in which spreads are compressed and valuations are stretched, are in part driven by investor exuberance, in which beliefs or expectations about asset returns become divorced from underlying fundamentals. Therefore, an understanding of how beliefs are shaped over the financial cycle is important to better understand the nature of financial panics and provide appropriate policy recommendations.

The literature has made substantial progress in understanding the role of beliefs in financial markets using models based on information and learning (Grossman and Stiglitz, 1980; Stiglitz, 1981; Morris and Shin, 2003; Gorton and Metrick, 2012; Babus and Kondor, 2018), sentiments and sunspots (Diamond and Dybvig, 1983; Goldstein and Pauzner, 2005; Allen et al., 2006; Angeletos and La'O, 2013), deviations from rational expectations (Brunnermeier et al., 2014; Barberis, 2018), among others. A more recent literature has sought to shed light how incomplete risk markets and information production interact (Gorton and Ordonez, 2014; Dang et al., 2020; Asriyan et al., 2021). In a similar spirit, we develop a model to understand how liquidity conditions in funding and asset markets affect beliefs about fundamentals, and the role that this plays in asset booms and busts.

Our model features collateralized debt, learning, and fire sales. A borrower issues short-term collateralized debt to a lender in order to finance investment in a long-term risky asset. Before the asset matures, agents are subject to an idiosyncratic shock to the opportunity cost of funds (cost shock) and may receive a private signal about the asset's future return. Both the cost shock and the signal received by an agent are private information. In response to these shocks, agents update their beliefs, exchange new debt contracts, and can adjust their portfolios before the asset matures. The price of debt and the price of the asset each play a dual role: they clear the market for debt and the asset, and they convey the lender and borrower's private information, respectively. As a result, agents can learn about the news that the other agent received by observing the price of debt or the asset. However, financial markets are informationally inefficient: An agent can never perfectly infer the extent to which these prices are driven by the other agent's private cost shock versus news

[1] For example, the severity of the liquidity freezes in debt and money markets in 2008 and 2019 suggest that these episodes were driven at least in part by temporary changes in the perceived riskiness of the underlying assets.

about the asset.[2]

This environment leads to the key mechanism of the model: Beliefs about fundamentals are endogenously shaped by the availability of liquidity in funding and asset markets. Namely, shocks to the opportunity cost of funds which are orthogonal to the asset's fundamental affect the price of debt and the asset, and therefore affect other agents' beliefs about the fundamental. Hence, the ability of prices to aggregate private information about fundamentals may be impaired by funding illiquidity, or by an abundance of liquidity. Moreover, this informational inefficiency may be more acute when liquidity conditions are especially tight or loose, as this impairs the ability of prices to reveal private information about fundamentals. This may cause agents' beliefs about the quality of assets to become systematically divorced from fundamentals in a manner which generates instability. As a result, liquidity conditions which are orthogonal to fundamentals can lead to socially costly asset price booms and busts through the evolution of beliefs about fundamentals.

Our paper makes three contributions. First, we show that liquidity shocks which are orthogonal to asset fundamentals can endogenously shape agents' beliefs about fundamentals. That is, pessimism and optimism about the fundamental value of assets arises endogenously in our model as a result of the availability of liquidity in funding and asset markets in a manner which can cause beliefs to become divorced from fundamentals. Second, we show that funding liquidity, market liquidity, and beliefs may interact and lead to asset price booms and busts. In particular, relatively loose liquidity conditions can lead agents to become overly optimistic about the fundamental value of an asset (relative to an appropriately defined counterfactual), leading to overinvestment in the asset and excess losses in bad states of the world. In contrast, tight liquidity conditions can cause agents to be overly pessimistic about asset fundamentals, leading to socially costly fire sales. In this way, the model sheds light on the inherent instability of financial markets.

Our normative contribution is to show that the competitive equilibrium is generically constrained inefficient due to the presence of a new externality we term the 'information externality', in which agents do not internalize how their decisions affect the information set and beliefs of other agents, in addition to a more standard pecuniary externality. We characterize optimal policy and show that government interventions in financial markets aimed at tightening funding liquidity ex ante (for example, through regulations which reduce lending in certain asset classes) or loosening funding liquidity ex post (for example, through asset purchases or lender-of-last-resort facilities), may work to stabilize in part through the new channel in our model: the effect of liquidity conditions on agents' beliefs about fundamentals.

The mechanism in our model derives from the informational inefficiency of financial markets,

[2]The underlying assumption is that agents can neither observe one another's private information, nor trade securities contingent on private information. This assumption is a stand-in for the various frictions, which imply that financial markets may not be informationally efficient and risk markets are incomplete.

which itself stems in part from an absence of complete risk markets. Indeed, an old literature has shown that prices may reveal information only imperfectly in the absence of complete risk markets.[3] In our paper, we show how the informational inefficiency of financial markets may vary according to liquidity conditions, and its implications for asset price booms and busts. Moreover, our paper contributes to the debate about the central bank interventions in securities markets in recent decades (Chen et al., 2020). In particular, our model suggests that, by acting as a 'dealer of last resort', a central bank can mitigate the informational inefficiency of markets during episodes of financial turmoil by reopening derivatives markets which would otherwise shut down.

In our model, there are three periods—dates 0, 1, and 2— and two risk-neutral agents—a lender and a borrower. There is a consumption good, which can be stored freely as cash. The borrower has access to a risky asset at date 0, which has a return at date 2 subject to a shock, and has constant returns-to-scale. New risky assets cannot be created after date 0. The lender cannot hold the risky asset directly, which creates gains from trade between the agents. The borrower can issue one-period debt to the lender in a competitive market, which is secured by its holdings of the risky asset.[4] Agents have a common prior belief about the asset's return—that is, the fundamental value of the asset.

At date 1, the borrower must either repay its date-0 debt or default. Since the return on the borrower's holdings of the risky asset are not realized until date 2, the borrower can finance the payment of its debt at date 1 out of its cash holdings or by raising new debt in a competitive market. Under the latter option, the borrower effectively rolls over a portion of its date 0 debt at date 1, potentially under new terms, depending on the equilibrium price of debt at date 1.[5]

Simultaneously at date 1, agents may receive private signals (news) about the risky asset's date 2 return. In addition, agents may receive an idiosyncratic shock to their opportunity cost of funds at date 1, a cost shock, which is uncorrelated with the asset return. For ease of exposition, we begin with the case in which only the lender receives a cost shock and news about the risky asset, while the borrower receives neither, and we discuss other cases later.

Both the cost shocks and signals are private information and cannot be verified by other agents. The willingness of the lender to lend at date 1 (the price of new debt at date 1) is affected by both the cost shocks and private signal: Cost shocks affect the willingness to lend by altering the lender's opportunity cost of lending, while the private signal alters the lender's beliefs about the fundamental value of the asset, which serves as collateral for the loan. Therefore, although the

[3]For example, Stiglitz (1981) showed that in the absence of complete risk market, prices must not only clear markets and aggregate information, but also allocate risk. As a result, asset prices may relay information only imperfectly.

[4]Although the lender cannot hold the risky asset directly, it cares about its return since it backs its holdings of the borrower's debt.

[5]The assumption that debt is one-period is meant to capture that, empirically, long-term assets are very often financed using short-term debt or contracts, which are subject to changing terms via margin requirements or covenants.

borrower cannot observe the lender's private information directly, the market price of date 1 debt is informative of this information through relationship between the lender's liquidity needs and beliefs and the market price of date 1 debt. Agents are Bayesian, have a common prior belief at date 0 about the asset return, and update their common prior beliefs about the asset's date 2 return at date 1 based on the variables they observe and their knowledge of the distribution of signals and cost shocks. In turn, agents adjust their portfolios at date 1 optimally given their posterior beliefs.

The borrower updates its prior beliefs about the fundamental value of the asset given its observation of the market price of debt at date 1 according to Bayes' Rule. Namely, the borrower computes the likelihood of all possible realizations of cost shocks and signals to the lender consistent with the observed value of the date 1 market price of debt, based on its knowledge of the distributions the shocks and signals, and forms an expectation over the lender's private signal accordingly.

Importantly, however, the borrower cannot perfectly disentangle how the market price of date 1 debt reflects the lender's private signal versus the lender's cost shock. This 'identification problem' arises because, in equilibrium, only one of the borrower's observables—the market price of date 1 debt—is informative about either of the two components of the lender's private information. That is, the borrower has an insufficient number of observables to disentangle the lender's private signal and cost shock. As a result, the equilibrium price of debt serves as a noisy signal to the borrower about the lender's private signal, where the noise is introduced by the lender's idiosyncratic cost shock. Thus, the borrower faces two layers of uncertainty about the asset's return: the market price of date 1 debt is a noisy signal to the borrower about the lender's private signal, while this signal itself is a noisy signal about the asset's return.[6]

The identification problem implies that liquidity conditions in funding markets affect perceptions about fundamentals. Loose liquidity conditions in funding markets, given by a high market price of date 1 debt, cause the borrower to become more optimistic about the quality of the asset. This happens regardless of whether the lender is willing to lend because of low cost shock or because it received positive news about the project. In either case, the borrower cannot identify how much of lender's demand for debt is driven by the lender's beliefs, ascribes a higher probability to the lender having received positive news, and becomes more optimistic about the asset's quality as a result. In contrast, tight funding liquidity, given by a low market price of date 1 debt, causes the borrower to become more pessimistic about the quality of the asset, since it ascribes a higher probability that the lender's low demand for debt is driven by the lender having received negative news about the asset. Thus, pessimism and optimism about the fundamental value of the risky

[6]The presence of two layers of uncertainty plays a crucial role in our model, as it gives rise to endogenous belief disagreement, and it implies that liquidity conditions will affect the borrower's beliefs about the fundamental value of the risky asset.

asset emerge endogenously as a result of funding liquidity—that is, the lender's demand for debt at date 1.

The model features the possibility of fire sales of the risky asset at date 1, similar to Lorenzoni (2008). In particular, the date 1 equilibrium features two regimes, which we call 'normal times' and 'fire sales'. In normal times, when lender's demand for debt is sufficiently high (which occurs when the cost shock to the lender are not too large and the signal is not too bad), the borrower has relatively optimistic beliefs about the risky asset's return. As a result, the price of the risky asset is high and the borrower holds all of the risky asset. At date 1, the borrower uses its cash holdings to repay any of its date 0 obligations which are not rolled over, and does not liquidate any of its holdings of the risky asset.

In the fire sale regime at date 1, when the lender's demand for debt is sufficiently low (which occurs when the lender's cost shock is sufficiently bad or when the lender receives sufficiently bad news about the asset), the borrower has relatively pessimistic beliefs about the risky asset's return. As a result, the borrower reduces its holdings of the risky asset by liquidating the asset to an outside traditional sector, who has a lower willingness to pay for the asset. Importantly, the nature of fire sales in our setting differs from that in much of the literature on fire sales (e.g. Lorenzoni (2008)): In response to tight funding liquidity conditions, the borrower liquidates the asset not simply because of its need for liquid funds, but rather because the borrower endogenously becomes more pessimistic about the fundamental value of the asset.

The model features asset price booms and busts due to an interaction between funding liquidity, beliefs, and market liquidity. In particular, relatively loose liquidity conditions in the funding market (i.e. a high market price of date 1 debt) leads the borrower to become overly optimistic about the fundamental value of an asset (relative to an appropriately defined counterfactual), leading to over-investment in the asset. This over-investment reflects a high asset price, and results in the borrower bearing excess losses in bad states of the world. In contrast, tight liquidity conditions in the funding market causes the borrower to be overly pessimistic about about the asset fundamental at date 1. As a result, the borrower may liquidate its holdings of the risky asset to the traditional sector, depressing the price of the asset below the value justified by its fundamentals (relative to a benchmark economy where all information is publicly observed). Thus, the model sheds light on the role that liquidity plays in asset price booms and busts, and how agents beliefs evolve endogenously during these episodes.

We also perform counterfactual exercises to shed light on how the information spillovers in our model affect the likelihood and severity of fire sale episodes in response to different shocks. In particular, we show that the information spillovers generated in our model amplify the likelihood and severity of fire sales driven by adverse liquidity shocks to the lender, and reduce the likelihood and severity of fire sales driven by bad news received by the lender, relative to a benchmark economy in which all information is commonly observed. Thus, the information spillovers may stabilize or destabilize financial markets depending on the shock.

We then analyze the normative implications of the model and show that the competitive equilibrium is constrained inefficient due to the presence of two externalities. In addition to a standard pecuniary externality along the lines of Lorenzoni (2008) and Dávila and Korinek (2018), the model features a new 'information externality' in which agents do not internalize how their decisions affect the information sets, and therefore the beliefs, of other agents. For instance, in choosing how much to lend to the borrower at a given price of debt, the lender does not internalize how its choice affects the borrower's perceptions of the asset's fundamental value due to the identification problem that the borrower faces. A social planner, who is subject to the same constraints as private agents—and who, accordingly, does not observe agents' private information—can engineer a Pareto improvement.

A government, which does not observe agents' private information, can implement a Pareto improvement using interventions in funding and asset markets. In the model, the prices that aggregate private information about fundamentals may be impaired during episodes of funding illiquidity and episodes when funding liquidity is abundant. This causes agents' beliefs to deviate systematically from fundamentals in a manner which generates instability. Government intervention in funding and asset markets, such as asset purchases or taxes on lending, can stabilize markets by exploiting the identification problem faced by agents and engineering a change in agents' perceptions of fundamentals. Our model thus shows that, in practice, the government interventions often implemented to stabilize financial markets, such as liquidity facilities and asset purchases, may operate through this additional channel not previously discussed in the literature. Indeed, policymakers have frequently cited the effect of the their interventions on investor confidence as a stabilizing force.

We also show that the government can also reduce the inefficiencies associated with the information externality using auctions of derivatives of the risky asset (such as put or call options). In equilibrium, only better-informed agents trade these securities, and therefore the market prices of these securities publicly reveal additional information about agents' private beliefs about the underlying asset. Essentially, by introducing new markets for derivatives of the risky asset, these policies mitigate the identification problem faced by agents and improve the information aggregation of financial markets. This is similar in spirit to the social value of public information, articulated in Allen et al. (2006), although the mechanism and source of inefficiency differ in our paper.

1.1. Related Literature

One of our main contributions is linking the imperfect information aggregation in Grossman and Stiglitz (1980) and its amplification through the feedback between market and funding liquidity in Brunnermeier and Pedersen (2009). Such interaction results in information externality on top of pecuniary externalities from fire sales as in Dávila and Korinek (2018).

Gale and Yorulmazer (2013) show that the hoarding of liquidity can shift the equilibrium allocation drastically. Similarly, we show that the liquidity hoarding due to asymmetric information can cause a huge decline in efficiency of the equilibrium allocation, but in the context of a market of collateralized debt contracts.

Garcia-Macia and Villacorta (2022) show how information frictions between banks can cause freezes in the interbank market and liquidity hoarding. Firms need liquidity for short-term investment opportunities. Banks have heterogeneous lending efficiency, which is private information. Interbank market facilitates the flow of funds from less efficient to more efficient banks, which ultimately lend to firms. However, when bank profitability is too low, interbank trade may not be incentive compatible due to information frictions, causing the interbank market freeze. Under interbank market freeze, firms have incentives to hoard liquid assets to be able to self-finance their liquidity needs. This liquidity hoarding reduces the demand for bank loans, which lowers bank profitability and making the interbank market freeze more likely. Because of such a feedback loop, there are multiple self-fulfilling equilibria in their model.

Unlike the model in Garcia-Macia and Villacorta (2022), our information friction is more direct, as the information asymmetry is between the borrower, who is also the investor, and the lender, who holds private information and collateral. Also, this information is about the fundamentals of the investment opportunity (asset), so there is no information friction that is amplified by coordination failure or multiple equilibria. Therefore, our model focuses on the direct effect of information asymmetry and imperfect learning through indirect signals, which can generate inefficiency in equilibrium.

A strand of literature on the asset misallocation due to downward-sloping demand curves for assets is based on Shleifer and Vishny (1992) and Kiyotaki and Moore (1997), followed by Lorenzoni (2008). Under the standard models of this literature, assets have different productivity depending on who holds them. If agents with higher productivity are financially constrained and sell assets to agents with lower productivity, this lowers the marginal product and the equilibrium price. This price decline would be amplified if agents have large leverage and more financially constrained. Agents don't internalize the pecuniary externality that falling asset prices impose on financially constrained agents. Therefore, collateral constraints can lead to an amplification of shocks and volatile real activity. The information externality in our model amplifies the pecuniary externality.

Kurlat (2016) proposes a model of fire sales with asymmetric information and difference in potential buyer's expertise in evaluating assets. Buyers with better expertise in detecting bad assets can refrain from buying such assets. Under certain conditions, the equilibrium features a downward-sloping relationship between asset sales and asset prices. This is because when the sale volume is high, the market clearing condition requires less-expert buyers to buy the asset, so the price has to fall. Our model also has a similar mechanism, with which financially constrained borrowers have to sell their assets to less productive agents. However, we separate the effect of asymmetric information and the inefficient fire sales. If there is a feedback mechanism as in Kurlat (2016), the equilibrium will have even larger fluctuations in our model.

Babus and Kondor (2018) propose a model with market clearing as competitive Walrasian auctioneer instead of separate decentralized platforms. Walrasian auctioneer in our model collects both the contract price and the appropriate amount of loans (or collateral) as well as the corresponding asset fire sales. Unlike in the model of Babus and Kondor (2018), we have a mixture distribution, binary state of the asset payoff and continuum of cost shocks, resembling the signal processing models. Also, our model incorporates collateral and simultaneous market clearing to understand the interaction between debt markets and collateral asset markets.

Our paper is also related to the literature on leverage cycles developed by Geanakoplos (1997), Geanakoplos (2003), Geanakoplos (2010), and Fostel and Geanakoplos (2015). We also use a general equilibrium model with heterogeneous agents and collateralized debt contracts following the literature. Unlike the models in this strand of literature, we highlight the role of asymmetric information in amplifying the leverage cycles. Also, our model has multiple states of uncertainty similar to Simsek (2013), who proposed a model with a continuum of states. Our model has another source of uncertainty on top of the asset return, which is the shock to the opportunity cost of funds (cost shock). In our model, the mixture distribution of the asset return and the continuum of cost shock realization generate the identification problem in learning as well as more richer interaction between prices and beliefs.

There is a large literature on the role of beliefs, sentiments, and learning in financial markets following Diamond and Dybvig (1983), who propose a model of financial fragility based on multiple equilibria from sunspots. Morris and Shin (2003), Goldstein and Pauzner (2005), Allen et al. (2006), and Angeletos and La'O (2013) develop this idea further to how the information and beliefs are endogenously determined, and how frictions in learning could prevent efficient information aggregation, leading to large swings and crises. Our model differs from many different models in this literature by providing the interaction between the formation of beliefs on fundamentals of the asset by financial conditions and fire sales of the asset.

Our modelling of fire sales as a change of the information regime is motivated by the literature on information production in credit markets as in Gorton and Ordonez (2014), in which lenders could produce information about collateral. Dang et al. (2020) show how collateralized debt can be optimal by alleviating problems of asymmetric information with information-insensitive debt. Dang et al. (2020) argue that short-term debt is designed to provide short-term stores of value by designing the debt such that it is not profitable for any agent to produce private information about the assets (collateral) backing the debt. A financial crisis is a switch of regime from information-insensitive debt regime to information-sensitive debt regime in which agents produce information about the collateral. Asriyan et al. (2021) also develop a model with costly information production and collateral. They show that high collateral values will crowd out information production and information on existing projects gets depleted. As a result, booms driven by collateral end in deep crises and slow recoveries. The crises caused by asymmetric information in this literature are similar to the crises in our model. However, the main driver of the crises in our model is the interaction between cost shocks and endogenous prices of debt and asset instead of information production. Moreover, the information on collateral and projects are interlinked in our model, as borrowers pledge their asset return directly as collateral.

Finally, our paper is related to the run on repo markets. The literature on repo runs by Gorton and Metrick (2012), Copeland et al. (2014), Martin et al. (2014), and Infante and Vardoulakis (2021) document and analyze the repo or collateral run dynamics. In most of the repo markets, in particular the repo markets analyzed in the literature, counterparty risks are minimal. Therefore, the fire sales dynamics in our model would be one of the more relevant mechanisms to analyze the fragility of the repo markets.

2. Model Setup

There are three dates indexed by $t \in \{0,1,2\}$. There are $N = 3$ types of agents: lender L , borrower B, and traditional sector T.[7] There is a continuum of agents of mass 1 for each type. Agents can observe each other's type. There is one consumption good (cash) which is storable that implies one unit of consumption good at t can be transformed into τ_t^i units of consumption good at $t+1$. For agent i, τ_t^i will control each agents opportunity cost of funds at t. For simplicity, assume $\tau_0^i = 1$ for each i. Each agent gets utility from date 2 consumption, C_2^i, according to $U_i(C_2^i) = C_2^i$.[8] Each agent is endowed at date 0 with e_0^i of the consumption good.

There is one risky asset a in positive net supply A with constant returns-to-scale.[9] An external un-modeled agent holds all the assets, sells the assets to B, and disappears at the end of date 0. The asset a transforms consumption goods invested at date 0 into goods at date 2, with a gross rate

[7] The traditional sector is introduced in order to have a notion of fire sales, similar to Lorenzoni (2008).

[8] Therefore, agents are risk-neutral and solving an investment problem.

[9] We can consider this as investment opportunity given by Bertrand competition in the economy with capacity constraint of the positive net supply.

of return of R. The return on the risky asset is subject to an aggregate shock at date 2, where is $R=\overline{R}>1$ with ex ante probability $(1-\pi)$ and $R=\underline{R}<1$ with probability π, where $0<\pi<1$ and $\underline{R}<\tau_1^B<\overline{R}$. We further assume that $\pi\underline{R}+(1-\pi)\overline{R}=1$ and $\tau_1^B=\tau_0^L\leq 1$ to make investment in the asset profitable, where τ_0^L is the expected value of τ_1^L. Uncertainty is fully resolved for all agents at date 2.

Importantly, investment in risky assets can be initiated only at date 0. Moreover, risky assets are illiquid in that they cannot be converted into the consumption good at date 1. In each period, there is a spot market for the risky asset in which the borrower and the traditional sector can participate, where p_t denotes the price of the risky asset at time t.

Markets for the risky asset are segmented: Only the borrower can invest in the risky asset at date 0. And only the borrower or the traditional sector can hold the risky asset at any date. The lender, by contrast, can only indirectly invest in risky assets by lending cash (consumption good) at date 0. The traditional sector has a date 1 reduced form inverse demand function of $F'(a_1^T)$, where a_1^T is its holdings of the risky asset at date 1. The traditional sector demand function is continuously differentiable with $F'>0$ and $F''<0$. We focus on the case in which the traditional sector always values the risky asset less than other agents do a priori – that is, the expected return of the asset is $E_0^B[R]/\tau_1^B>F'(0)$. Because the risky asset is illiquid at date 1, in equilibrium, the only way for the borrower to convert its holdings of the risky asset to the consumption good at date 1 is to sell it to the traditional sector in the spot market for the asset.

Cost shocks and private signals

The lender's opportunity cost of funds τ_1^L is a random variable which is drawn at date 1 from a continuous and smooth distribution with probability density function $\lambda_T(\tau)$ with mean τ_0^L and support $[\underline{\tau}^L,\bar{\tau}^L]$. We further assume that $\lambda_T(\tau)$ is decreasing in $|\tau-\tau_0^L|$. We refer to the realization of changes in τ_1^L as the cost shock to L. The cost shock is independent to $R\in\{\underline{R},\overline{R}\}$.

Agents have a common prior belief about R, where $R=\underline{R}$ with probability π_0. Each agent updates their beliefs π_1^L,π_1^B that $R=\underline{R}$ in response to new information at date 1 according to Bayes' rule. At date 1, L gets a private noisy signal about R. Let the signal be denoted as $s_1^L=R+\varepsilon$, where ε is drawn from a continuous and smooth distribution with cumulative density function $\Lambda_\varepsilon(\varepsilon)$, which has mean zero and variance σ_ε^2, and probability density function $\lambda_\varepsilon(\varepsilon)$ with support $(-\infty,\infty)$. We assume that $\lambda_\varepsilon(\varepsilon)$ is decreasing in $|\varepsilon|$ —that is, $\lambda_\varepsilon(\varepsilon)$ is single-peaked.

Both τ_1^L and s_1^L are neither observable nor verifiable to the borrower and cannot be contracted upon directly.

Contracting environment

Market segmentation implies that there are gains to trading financial contracts. More precisely, L cannot hold the risky asset at any point, but can gain exposure to the risky asset by lending to B through the use of financial contracts. We assume that, although the risky asset matures after two periods, the borrower finances its holdings of the risky asset using one-period collateralized debt contracts. By financing long-term risky assets with short-term debt, the borrower is exposed to liquidity risk at date 1. We formalize this environment and the nature of liquidity risk below.

Timeline

The timeline is as follows. At date 0, agents simultaneously enter into one-period contracts between date 0 and date 1, and make their investment decisions. At date 1, agents receive the cost shocks and signals about the quality of the risky asset. After the realization of the cost shocks and signals, agents simultaneously settle the date 0 contracts and enter into another one-period contract at date 1. At date 2, the return on the risky asset is realized, the date 1 contracts are settled, and agents consume.

Financial contracts

Formally, a date t contract is given by (c_t, f_t), which stipulates a transfer at date t of the risky asset of size c_t from the borrower to the lender as collateral with a promise for the borrower to repurchase the collateral at date $t+1$ at a unit price of f_t, normalized to $f_t = 1$. (That is, the borrower repays the lender $f_t c_t = c_t$ units of the consumption good at date $t+1$.) We assume that the collateral on the loan is held by some outside custodian, not modeled explicitly, who either returns the collateral to the borrower if the contract is honored or sells it and pays the proceeds to the lender if the borrower defaults. This ensures that the lender accepts the asset as collateral for the loan despite not being able to hold the asset itself.

These contracts are traded in a competitive market, where q_t denotes the market price of the contract at $t = 0, 1$. Therefore, the size of the loan from the lender to the borrower at t—that is, the quantity of the consumption good loaned at date t— is given by $q_t c_t$. At date $t+1$, the borrower decides whether to default on the contract and forgo the collateral or not. If the borrower chooses to default on the contract at date $t+1$, the value of this collateral $c_t p_t$ is transferred to the lender.

Note that both date 0 and date 1 contracts are one-period contracts and involve a repayment of c_t at date $t+1$. One may wonder why the borrower would be willing to enter into the date 0 contract in the first place, given that this will require the borrower to repay the lender c_0 units of the consumption good at date 1 despite the fact that the borrower has no income at date 1. In equilibrium, the borrower will be willing to borrow at date 0 because it anticipates that it can finance this repayment c_0 (at least partially) by refinancing (i.e. rolling over) its debt using the date 1 contract, albeit under different terms. One can interpret this contracting environment as involving long-term (two-period) debt which is renegotiated in the intermediate period.

Because the borrower must refinance its date 0 debt under potentially different terms, this renegotiation exposes the borrower to liquidity risk at date 1. If the loan amount $q_1 c_1$ of the new contract is more than the payment amount of initial loan c_0 in equilibrium, then the lender lends more consumption goods to the borrower for the same amount of collateral. In contrast, if the size of the loan under the date 1 contract $q_1 c_1$ is less than the the payment amount of initial loan under the date-0 contract c_0, then the borrower must either pay the difference in cash at date 1 or put up more collateral to borrow more at date 1.[10] In either case, the borrower has to pay the difference $c_0 - q_1 c_1$ in cash. In equilibrium, the borrower will try to pay the debt in full unless the borrower exhausts all of cash and asset holdings because the lender can require further payments from the borrower's cash holdings at date 1.

Since the borrower's risky assets are illiquid at date 1 and cannot be converted to the consumption good, the borrower only has two means of repaying the debt at date 1: the borrower must either use its own cash holdings to reduce the size of the loan, or it must sell some of its holdings of the risky asset in the date 1 spot market for the asset. The only way for the borrower or the lender to convert the risky asset to cash in aggregate at date 1 is to sell it to the traditional sector. Because the traditional sector's marginal valuation of the risky asset can be lower than that of the borrower, the borrower can sell the asset to the traditional sector at a loss. In this sense, tightening funding constraints may be associated with endogenous fire sales in which the borrower liquidates part of its holdings of the risky asset at a cost.

Note that the lender has full-recourse on the debt at date 1. Thus, the borrower will try to pay the debt in full at date 1 unless it exhausts all of cash and asset holdings. However, at date 2, there is no further recourse to the borrower's balance sheet as the borrower can simply walk away from the collateral. Under the date 1 terms of the contract, the borrower will decide whether to default on the contract and forgo the collateral or not, thus, the payment from the borrower to the lender at date 2 will be determined endogenously as $\min\{p_2, 1\}$, where 1 is the promised payment amount.

We also make the following assumptions:

Assumption 1. *$e_0^B \tau_1^B < A\left(E_0^B[R] - 1\right)$, $e_0^L > A\overline{R}/\underline{\tau}^L - e_0^B$, and $e_0^T > \max_{a\in[0,A]} aF'(a)$.*

The first part of the assumption ensures that borrowers cannot fund their asset purchases without debt. The second part ensures that lenders endowment is sufficiently large to satisfy borrowers' demand. Finally, the third part of the assumption ensures that the traditional sector has enough cash to purchase any arbitrary amount sold to them following their inverse demand. This assumption is

[10] One can consider this situation as a margin call.

almost exactly the same as the assumptions in Simsek (2013) and Gottardi et al. (2019), serving the same purposes.

Assumption 2. $F'(A) \geq E_0^B[R] - e_0^B/A$ *and* $E_0^B[R]/\tau_1^B > F'(0)$.

The first part of this assumption is to guarantee that the borrower can liquidate all their asset holdings A to the traditional sector at price of $p_1 = F'(A)$. Then, the total amount of asset fire-sales $AF'(A)$ would exceed the maximum borrowing amount $AE_0^B[R] - e_0^B$. Therefore, the total fire-sales of the borrower at date 1 is enough to repay all the initial borrowing amount c_0 at date 0.[11] The second part of the assumption is formally reiterating the case we are focusing in which the borrower always values the asset more than the traditional sector does at date 0.

Information sets

Let I_t^i denote the information set of agent $i \in \{L, B\}$ at date $t \in \{0, 1, 2\}$. We assume that, at date 0, $I_0^B = I_0^L$ and prior beliefs π_0 are the same for both agents. In addition, the date 0 opportunity costs of funds for each agent τ_0^B, τ_0^L are common knowledge. Both agents observe all prices and date 0 terms of the contract c_0, q_0.

At date 1, lender observes τ_1^L and s_1^L, but the borrower does not. Both agents observe prices and terms of contract c_1, q_1, p_1. Therefore, the lender's information set at date 1 is $I_1^L = \left\{\Theta_1, \tau_1^L, s_1^L\right\}$, while the borrower's information set is $I_1^L = \{\Theta_1\}$ where Θ_1 denotes the set of all variables observable to all agents, which includes prices p_1, q_1 and the terms of the contract c_1. Agents make decisions at each date conditional on their information set in that date. We denote agent i's expectation of variable x, conditional on its information set at time date t, as $E_t^i[x] \equiv E_t\left[x \,|\, I_t^i\right]$.

3. Lender's Problem

The (representative) lender behaves competitively. The lender solves a portfolio choice problem, deciding how much unit of collateral to lend, d_t^L for each date t, and how much cash to hold, κ_t^L for each date t, subject to its budget constraint, taking prices p_t, q_t and available contracts as given.

Lender's portfolio choice problem at date 1

At date 1, the lender's portfolio choice is to decide how much to lend versus how much cash to hold. The lender's opportunity cost of lending is its date 1 marginal return of cash (which is subject to the liquidity shock) τ_1^L, since this is the date 2 return that the lender gets on its holdings of cash at

[11] This is to avoid any pesky issues arising with the possibility of date 1 default of the borrowers in terms of violating their budget constraint. It could be possible that the borrower just wants to sell everything because it is not profitable to rollover their debt, but that liquidation amount $p_1 A$ can be less than $q_0 c_0$. Then, the budget constraint is violated.

date 1. If the borrower cannot finance the promised payment to the lender out of its cash holdings or with new borrowing at date 1, then either the borrower has to liquidate some of its holdings of the risky asset to the traditional sector for cash or the lender seizes the collateral and liquidates it to the traditional sector. Both of these actions are payoff-equivalent, so we ignore the confiscation of collateral by the lender and instead assume that the borrower liquidates to the traditional sector in this case.

The lender's problem at date 1 is to maximize its expected utility $E_1^L[C_2^L]$ from date 2 consumption by choosing its portfolio (cash versus a loan) subject to budget constraints at date 1 and date 2. Let d_1^L denote the lender's choice of how many units of loan to invest in (at price q_1) at date 1. (In equilibrium, the condition for market clearing for the loan will be $d_1^L = c_1$ if $c_1 > 0$. If $c_1 = 0$, then $d_1^L > c_1$ is possible because of the reason we discuss later in the the lender's optimal decision. By assumption 1, $d_1^L > 0$ is always possible even when $c_1 = 0$ as we show later.)

Lender's date 0 budget constraint At date 0, the lender allocates its cash endowment between date 0 cash holdings and the date 0 loan to the borrower.

$$\underbrace{\kappa_0^L}_{date\ 0\ cash\ holdings} + \underbrace{q_0 d_0^L}_{value\ of\ date\ 0\ loan} \leq \underbrace{e_0^L}_{date\ 0\ cash\ endowment}$$

Lender's date 1 budget constraint At date 1, the flow of payments received by the lender is simply $q_0 d_0^L - q_1 d_1^L$, the difference in the value of the loan under the contract terms agreed to at date 0 minus the value of the loan under the new contract terms.

$$\underbrace{\kappa_1^L}_{date\ 1\ cash\ holdings} + \underbrace{q_1 d_1^L}_{value\ of\ date\ 1\ loan} \leq \underbrace{\kappa_0^L}_{date\ 0\ cash\ holdings} + \underbrace{d_0^L}_{value\ of\ date\ 0\ loan}$$

i.e.

$$\underbrace{\kappa_1^L - \kappa_0^L}_{net\ increase\ in\ cash\ holdings} \leq \underbrace{d_0^L - q_1 d_1^L}_{net\ decrease\ in\ loan}$$

Lender's date 2 budget constraint At date 2, the contract is settled or defaulted upon by the borrower, the lender earns a return from its cash holdings, and consumes. Define the date 2 proceeds from the loan as $R_2^d \equiv \min\{R, 1\}$, which accounts for the possibility of default.

$$\underbrace{C_2^L}_{date\ 2\ consumption} = \underbrace{\tau_1^L \kappa_1^L}_{proceeds\ from\ cash\ holdings} + \underbrace{R_2^d d_1^L}_{proceeds\ from\ date\ 1\ loan}$$

Lender's problem at date 1

Taking as given the date 1 contract terms and prices q_1, and p_1, the lender decides how much of its funds to allocate to the loan or cash. The lender's optimization problem is

$$\max_{\kappa_1^L, d_1^L} \; E_1^L \left[C_2^L\right]$$

$$s.t. \quad C_2^L \leq \tau_1^L \kappa_1^L + R_2^d d_1^L$$

$$\kappa_1^L - \kappa_0^L \leq d_0^L - q_1 d_1^L$$

with non-negativity constraints:

$$\kappa_1^L \geq 0, \; d_1^L \geq 0.$$

Both budget constraints will bind at the optimum, so we can replace cash holdings κ_1^L using the date 1 budget constraint.

$$C_2^L = \tau_1^L \left(\kappa_0^L + d_0^L - q_1 d_1^L\right) + R_2^d d_1^L$$

Let $\xi_{d_1}^L$ and $\xi_{\kappa_1}^L$ denote the Lagrange multipliers on the respective non-negativity constraints. The Lagrangian for the lender's date 1 problem is

$$L_1^L = E_1^L \left[U_L \left(\tau_1^L \left(\kappa_0^L + d_0^L - q_1 d_1^L\right) + R_2^d d_1^L\right)\right] + \xi_{d_1}^L d_1^L + \xi_{\kappa_1}^L \left(\kappa_0^L + d_0^L - q_1 d_1^L\right)$$

The lender's first-order condition (FOC) for d_1^L is

$$q_1 = \frac{E_1^L \left[R_2^d\right] + \xi_{d_1}^L}{\tau_1^L + \xi_{\kappa_1}^L}. \tag{1}$$

This says that the lender chooses how much to lend to equalize the discounted marginal return from lending at date 1 to the discounted marginal return on its opportunity cost from holding cash at date 1.

We also have the two complementary slackness conditions:

$$\xi_{d_1}^L d_1^L = 0$$

$$\xi_{\kappa_1}^L \kappa_1^L = 0$$

Note that there is no d_1^L in (1) as the lender's problem is linear. The lender is indifferent across different values of d_1^L as long as the price of the contract is equal to the expected return of the loans, $E_1^L\left[R_2^d\right]/\tau_1^L$, the ratio of the discounted marginal return from lending at date 1 to the discounted marginal return on its opportunity cost from holding cash at date 1. Note that assumption 1 ensures that the lender always holds a strictly positive quantity of cash in equilibrium at date 1, $\kappa_1^L > 0$.[12] Therefore, the lender's Lagrange multiplier for the non-negativity constraint satisfies $\xi_{\kappa_1}^L = 0$ in equilibrium. Moreover, assumption 1 ensures that, lending takes place in equilibrium so that $d_1^L > 0$ and $\xi_{d_1}^L = 0$ in equilibrium.[13] Therefore, given these assumptions, the lender's optimality condition (1) reduces in equilibrium to

$$q_1 = \frac{E_1^L\left[R_2^d\right]}{\tau_1^L}. \tag{2}$$

Thus, in equilibrium, the competitive spot price of the contract at date 1, q_1, will ensure that this condition holds.[14]

4. Borrower's Problem

The (representative) borrower is competitive. At date 1, the borrower solves a portfolio choice and chooses how much to borrow, subject to its budget constraint, collateral constraint, and taking as given the price of the risky asset p_1 and the price of the contract q_1. At date 1, borrowers cannot initiate new risky assets. Moreover, borrowers can exchange their extant holdings of the risky asset to cash by selling it to the traditional sector, at the market price p_1 and investing the proceeds in cash for total date 2 return of $p_1\tau_1^B$. Hence, the borrower's opportunity cost of holding the risky asset is given by τ_1^B/p_1.

At date 1, the borrower must repay its date 0 debt by buying back from the lender the c_0 units of the risky asset that were posted at collateral. Recall that the repurchase price defined in the contract was $f_0 = 1$, so that the borrower must pay c_0 units of the consumption good to the lender

[12]The lender's optimality condition implies that, in equilibrium, the lender is just indifferent between holding cash and not, which implies that the lender's Lagrange multiplier from the non-negativity constraint satisfies $\xi_{\kappa_1}^L = 0$ in equilibrium. Otherwise the lender's expected discounted return from lending would exceed its opportunity cost of funds.

[13]Note that assumption 1 ensures that the lender's date 0 endowment e_0^L is large enough to cover all the loans even when the lender is maximally optimistic about the quality of the risky asset and believes $R = \overline{R}$ with probability 1.

[14]To see this intuitively, suppose that q_1 is less than the expected discounted return of a loan, $q_1 < \frac{E_1^L\left[R_2^d\right]}{\tau_1^L}$. Then the lender would want to maximize its holdings of the loans and hold no cash, so that $\kappa_1^L = 0$. But assumption 1 ensured that $\kappa_1^L > 0$, so that we have a contradiction. Suppose instead that we have $q_1 > \frac{E_1^L\left[R_2^d\right]}{\tau_1^L}$. Then, the lender would want to minimize d_1^L to negative infinity without the non-negativity constraint so that the non-negativity constraint of d_1^L would bind, $d_1^L = 0$. But assumption 1 ensured that $d_1^L > 0$ in equilibrium, so that we again have a contradiction.

at date 1. However, the borrower obtains no date-1 cash flow from its holdings of the risky asset at date 1. Therefore, the borrower can finance this repayment of c_0 in three ways: out of any cash holdings at date 1, by raising new debt at date 1, or by liquidating the risky asset in the date-1 spot market for the risky asset.

The borrower can raise new debt at date 1 (i.e. refinance its date-0 debt) using the date-1 contract—that is, by posting c_1 units of the risky asset as collateral in exchange for $q_1 c_1$ units of the consumption good as a loan, where q_1 is the competitive spot price of this contract. If the debt raised at date 1 $q_1 c_1$ is insufficient to cover the repayment c_0 the borrower must make at date 1, this difference must be financed either out of the borrower's cash holdings stored from date 0, or by selling some portion of its holdings of the risky asset in the spot market.

Define κ_0^B as the amount of cash the borrower carries into date 1 from date 0. Now this decision is relevant only in the date-0 optimization problem. For the date-1 problem, the borrower considers it as a predetermined amount of cash κ_0^B.[15]

Collateral constraint at date 1 Let a_0^B denote the borrower's total risky asset holdings brought to date 1 from date 0, and c_0 denote the borrower's date 0 risky asset holdings used as collateral at date 0. (We will later show that, in equilibrium, $c_0 = a_0^B$.) Let a_1^B denote the borrower's total risky asset holdings chosen at date 1, and c_1 denote the amount of the borrower's date 1 asset holdings used as collateral at date 1. (Note that the amount of asset sold by the borrower at date 1 is then $a_0^B - a_1^B$.) The 'collateral constraint' is then $c_1 \leq a_1^B$.

Borrower date 0 budget constraint At date 0, the borrower has a cash endowment and receives a cash loan from the lender, which it can allocate between date-0 cash holdings and investment in the risky asset.

$$\underbrace{q_0 c_0}_{date\ 0\ loan} + \underbrace{e_0^B}_{date\ 0\ cash\ endowment} \geq \underbrace{p_0 a_0^B}_{value\ of\ date\ 0\ asset\ holdings} + \underbrace{\kappa_0^B}_{date\ 0\ cash\ holdings}$$

Borrower date 1 budget constraint At date 1, if the borrower defaults on its date-0 debt obligations, it loses its collateral and has only its cash holdings to find its portfolio choices at date 1. In equilibrium, the borrower never defaults at date 1, because the selling all the collateral would be sufficient in repaying the debt by assumption 2. For this reason, we omit this case. In the event that the borrower does not default at date 1, its date 1 budget constraint is

[15] We will see in the date-0 problem that the borrowers financed their payment to purchase c_0 assets with the price p_0 at date 0 by their cash endowments e_0^B and the borrowing amount $c_0 p_0 (1 - m_0) = c_0 q_0$, where m_0 is the margin of the contract. Therefore, the leftover cash can be defined as $\kappa_0^B \equiv e_0^B + c_0 p_0 (1 - m_0) - c_0 p_0 = e_0^B - c_0 p_0 m_0$.

$$\underbrace{q_1 c_1}_{date\ 1\ loan} + \underbrace{p_1 a_0^B}_{value\ of\ date\ 0\ asset\ holdings} + \underbrace{\kappa_0^B}_{date\ 0\ cash\ holdings} \geq \underbrace{p_1 a_1^B}_{value\ of\ date\ 1\ asset\ holdings} + \underbrace{c_0}_{date\ 0\ loan} + \underbrace{\kappa_1^B}_{date\ 1\ cash\ holdings}$$

i.e.

$$\underbrace{q_1 c_1 - c_0}_{net\ increase\ in\ loan} \geq \underbrace{p_1\left(a_1^B - a_0^B\right)}_{net\ increase\ in\ asset\ holdings} + \underbrace{\kappa_1^B - \kappa_0^B}_{net\ increase\ in\ cash\ holdings}.$$

Borrower date 2 budget constraint The borrower's date 2 budget constraint limits date 2 consumption C_2^B by the return on the borrower's portfolio of assets held at date 1.

$$\underbrace{C_2^B}_{date\ 2\ consumption} \leq \underbrace{\tau_1^B \kappa_1^B}_{proceeds\ from\ cash\ holdings} + \underbrace{a_1^B R}_{proceeds\ from\ asset\ holdings} - \underbrace{c_1 R_2^d}_{repayment\ of\ debt},$$

where $R_2^d \equiv \min\{\underline{R}, 1\}$ denotes the realized date 2 repayment of the loan (including the possibility of default).

Each borrower takes the price q_1 of the contract at date 1 as given. In equilibrium, the price q_1 will be informative to the borrower about the lender's private information at date 1. We discuss how the borrower's beliefs evolve in section 6.2. We first characterize the borrower's optimality conditions, conditional on its information set at date 1.

Borrower's optimization problem at date 1

The borrower's optimization problem at date 1 is to make its portfolio and borrowing decision to maximize expected date 2 utility $E_1^B\left[C_2^B\right]$, conditional on its date 1 information set, taking prices as p_1 and q_1 as given.

$$\max_{c_1^B, a_1^B, \kappa_1^B} E_1^B\left[C_2^B\right] \tag{3}$$

$$\begin{aligned} \text{s.t. } & q_1 c_1 - c_0 \geq p_1\left(a_1^B - a_0^B\right) + \kappa_1^B - \kappa_0^B, \\ & C_2^B \leq \tau_1^B \kappa_1^B + a_1^B R - c_1 R_2^d \qquad (4) \\ & c_1 \leq a_1^B, \\ & c_1 \geq 0,\ a_1^B \geq 0,\ \kappa_1^B \geq 0 \end{aligned}$$

The first-order condition for c_1 is

$$\left(\tau_1^B + \xi_{\kappa_1}^B\right) q_1 - E_1^B\left[R_2^d\right] - \mu_1^B + \xi_{c_1}^B = 0, \tag{5}$$

while the first-order condition for a_1^B is

$$-\left(\tau_1^B + \xi_{\kappa_1}^B\right) p_1 + E_1^B[R] + \mu_1^B + \xi_{a_1}^B = 0. \tag{6}$$

The complementary slackness conditions are given by

$$\mu_1^B\left(a_1^B - c_1\right) = 0 \tag{7}$$

$$\xi_{c_1}^B c_1 = 0 \tag{8}$$

$$\xi_{a_1}^B a_1^B = 0 \tag{9}$$

$$\xi_{\kappa_1}^B \kappa_1^B = 0 \tag{10}$$

We can further characterize the borrower's behavior by combining the FOCs for the case with risk-neutrality. Suppose that the collateral constraint is binding in equilibrium so that $\mu_1^B > 0$. (We show later in Lemma 1 that this must be the case.) Then, the FOCs for c_1 and a_1^B hold with equality. Suppose also that $c_1 = a_1^B > 0$ in equilibrium. Combining these binding first order conditions yields

$$\underbrace{E_1^B\left[R_2^d\right] - q_1\tau_1^B}_{cost\ of\ financing\ asset\ holdings} + \underbrace{\xi_{\kappa_1}^B\left(p_1 - q_1\right)}_{shadow\ value\ of\ cash} = \underbrace{E_1^B[R] - \tau_1^B p_1}_{expected\ net\ return\ of\ risky asset} \tag{11}$$

It is convenient to define $\nu_1\left(\pi_1^B, q_1\right) \equiv E_1^B\left[R_2^d\right] - q_1\tau_1^B$ and $\eta_1\left(\pi_1^B; p_1\right) \equiv E_1^B[R] - \tau_1^B p_1$, so that (11) can be expressed as

$$\underbrace{\nu_1\left(\pi_1^B, q_1\right)}_{cost\ of\ financing\ asset\ holdings} + \underbrace{\xi_{\kappa_1}^B\left(p_1 - q_1\right)}_{shadow\ value\ of\ cash} = \underbrace{\eta_1\left(\pi_1^B; p_1\right)}_{expected\ net\ return\ of\ risky\ asset} \tag{12}$$

If $E_1^B[R] - \tau_1^B p_1 > E_1^B\left[R_2^d\right] - q_1\tau_1^B$, i.e. if the net return from buying asset exceeds the net return from offering collateral (balancing the benefit from a larger loan with the cost of a larger repayment), then we must have that $\xi_{\kappa_1}^B\left(q_1 - p_1\right) < 0$, which requires that $\xi_{\kappa_1}^B > 0$ (non-negativity constraint for κ_1^B binding) and $q_1 < p_1$, which we will show in Lemma 1 holds in equilibrium. In this case, the borrower is at a corner solution in its portfolio choice.

5. Traditional Sector's Problem

The traditional sector becomes relevant at date 1 only and enters the period with a new endowment of cash $\kappa_1^T = e_0^T$. It can store cash between dates 1 or 2, or buy existing risky assets at date 1 in the spot market. Existing assets need to be managed subject to some increasing costs. The date-2 return of holding a_1^T of the risky asset at date one, net of these management costs, is given by $F(a_1^T)$ units of the consumption good, where $F' > 0$ and $F'' < 0$. Without loss of generality, assume $\tau_1^T = 1$. The traditional sector gets utility only from date-2 consumption C_2^T with linear utility. The traditional sector's optimization problem is to choose how to allocate its endowment between cash or the risky asset to maximize its utility.

$$\begin{aligned} \max_{\kappa_2^T, a_1^T} \; & C_2^T \\ \text{s.t. } C_2^T &= \kappa_2^T + F(a_1^T) \\ \kappa_1^T &= \kappa_2^T + p_1 a_1^T \\ a_1^T &\geq 0 \\ \kappa_2^T &\geq 0 \end{aligned} \tag{13}$$

For simplicity, and without loss of generality, we assume the traditional sector's endowment of cash is sufficiently large that its non-negativity constraint on date-2 cash holdings is never binding, $\kappa_2^T > 0$, by assumption 1. This ensures that, in equilibrium, the traditional sector could buy all existing assets.[16] As a result, the traditional sector will always enter date 2 with positive cash holdings κ_2^T, so the non-negativity constraint $\kappa_2^T \geq 0$ is non-binding. Using binding date-1 and -2 budget constraints, the traditional sector's date-1 problem is to choose a_1^T to maximize $\kappa_1^T - p_1 a_1^T + F(a_1^T)$ subject to $a_1^T \geq 0$. the traditional sector's optimality condition is

$$p_1 = F'(a_1^T) + \xi_{a_1}^T. \tag{14}$$

[16]To see this, note that if the traditional sector holds all the assets, then $a_1^T = a_0^B$. So its date 1 expenditure is $p_1 a_0^B$. Note also that in the eq where the traditional sector is holding assets, we have $p_1 = F'(a_1^T)$, and so this expenditure is $p_1 a_0^B = F'(a_0^B) a_0^B$. Then the above assumption guarantees that the traditional sector has enough cash endowment at date one to make this purchase.

6. Equilibrium at Date 1

6.1. Market Clearing at Date 1

Recall that the date-1 contract is traded competitively in a market for claims at date 1. The market clearing for these claims at date 1 is given by[17]

$$d_1^L = c_1. \tag{15}$$

An analogous condition will hold for the date-0 market.

Recall that no new risky assets can be created after date 0. Therefore, the market clearing condition for the date-1 spot market for the risky asset implies that the total quantity of the asset held by the borrower and the traditional sector at date 1 is equal to the total amount of the asset created at date 0.

$$a_1^B + a_1^T = a_0^B \tag{16}$$

6.2. Beliefs at Date 1

The previous characterization of agents' optimal behavior is for given information set or beliefs. In this section, we discuss how agents' beliefs about the fundamental of the risky asset (the return R) evolve at date 1. Equilibrium will be pinned down by the joint determination of agents' actions and their beliefs. Because agents have private information, their beliefs about R evolve in different ways: the beliefs of the lender respond directly to the lender's private signal, while the beliefs of the borrower respond to the equilibrium prices and quantities that the borrower observes, to the extent that they are informative about the lender's private signal.

Lender's beliefs at date 1 Recall from section 3 that at date 1, the lender's information set consists of the observed prices and terms of the contract $\Theta_1 = \{c_1, q_1, p_1\}$ and its private cost shock and private signal. Note that the bounds of the posterior belief of L are $\pi_1^L \in (0,1)$.[18] the lender's posterior beliefs are derived using Bayes' Rule

$$\pi_1^L = \frac{\pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right)}{\pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right) + (1-\pi_0)\lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \bar{R}\right)}. \tag{17}$$

This is derived formally in Appendix A.1. Note also that π_1^L is decreasing in s_1^L by the assumption on the distribution from which ε is drawn is single-peaked. Denote the reduced form cumulative

[17] We have already derived this in the lender's problem in section 3.

[18] This follows from the fact that $s_1^L = R + \varepsilon$ with $\varepsilon \in (-\infty, \infty)$. Any realization of s_1^L is possible from both state of the world $R = \bar{R}$ and $R = \underline{R}$.

distribution function of π_1^L as $G_\pi(\cdot)$ and its probability density function as $g_\pi(\cdot)$.

Borrower's beliefs at date 1 The borrower observes all prices and contract terms, but does not observe the lender's liquidity shock or private signal. Therefore the borrower's date-1 information set consists only of the observed prices and terms of the contract $I_1^B = \{\Theta_1\}$. Recall that the borrower has an identical prior belief as the lender $\pi_0 \equiv Pr(R = \underline{R}|I_0)$. After observing equilibrium prices p_1 and q_1, the borrower updates its beliefs about the risky asset's return, where its posterior belief is denoted as π_1^B.[19]

Although the borrower itself does not receive a private signal about the asset's return, it can partially infer the lender's private signal based on equilibrium observables, in particular the contract price q_1. The observed price q_1 reflects the lender's willingness to lend, and through equation (17), is informative about the lender's private beliefs about the return on the risky asset. However, the borrower cannot separately identify s_1^L, τ_1^L based on the variables it observes, and thus cannot perfectly infer the lender's private signal. This is because q_1 is also affected by cost shocks to the lender, which are orthogonal to the private signal. Thus, q_1 serves as a noisy signal to the borrower about the lender's beliefs π_1^L, where the noise is given by the lender's cost shock τ_1^L. We refer to the inability of the borrower to perfectly disentangle the lender's private signal from its idiosyncratic liquidity needs as the 'identification problem' faced by the borrower.

Thus, the borrower faces two layers of uncertainty about the asset's return R: q_1 is a noisy signal to the borrower about the lender's private signal s_1^L, where the noise is introduced by the lender's idiosyncratic cost shock τ_1^L; and s_1^L itself is a noisy signal about R. As we will show, the presence of two layers of uncertainty plays a crucial role in our model: it will imply endogenous belief disagreement, and that liquidity conditions affect the borrower's beliefs about the fundamental value of the risky asset.

Identification problem To elaborate on the identification problem faced by the borrower, note that the borrower observes q_1, p_1, and c_1. The quantity of the asset invested in and collateralized c_1 is the borrower's choice, and reflects q_1 via market clearing $c_1 = d_1$, and therefore does not convey any additional information to the borrower beyond q_1. Similarly, p_1 is determined by the how the marginal buyer of the asset values the asset. Since the lender cannot hold the asset until date 2, there are only two potential marginal buyers, the borrower and the traditional sector. Hence, p_1 itself does not convey additional information about the lender's private signal or cost shock. Therefore, only q_1 contains information on the private information of lender τ_1^L, π_1^L through the equilibrium relationship (2).

[19] The assumption that atomistic lenders are competitive, plus the assumption that atomistic lenders are identical and receive the same shock and signal, imply that in equilibrium there is no strategic motive amongst the lenders to behave in a way inconsistent with its true values beliefs π_1^L or the true shock τ_1^L. Rather, the lender's behavior is always consistent with its true beliefs and liquidity shock. The borrower also knows this and updates its beliefs accordingly.

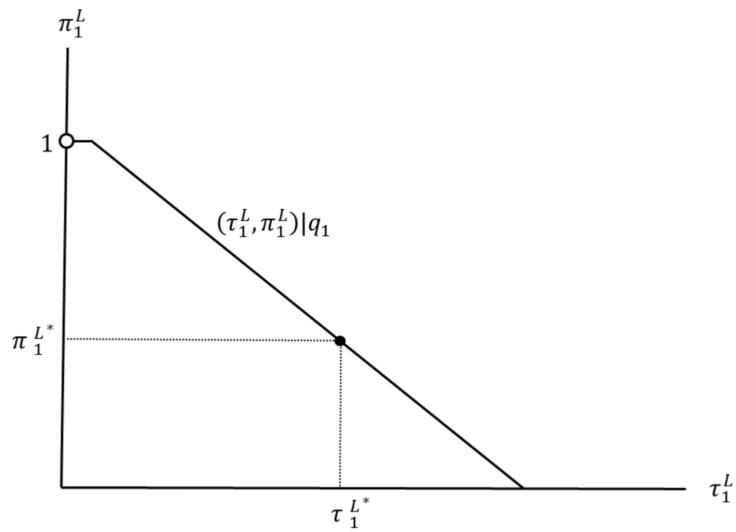

Figure 1: Identification Problem

Note: This figures illustrates the identification problem faced by borrowers at date 1. The curve in the figure plots the combinations of the lender's cost shock and beliefs, $\left(\tau_1^L, \pi_1^L\right)$, which are consistent with the equilibrium q_1, based on the lender's optimality condition for d_1^L. The borrower's observation of q_1 is insufficient to infer the true realizations $\left(\tau_1^{L*}, \pi_1^{L*}\right)$ separately.

$$q_1 = \frac{1 - \pi_1^L + \pi_1^L \underline{R}}{\tau_1^L}.$$

Moreover, the borrower knows how the lender's posterior beliefs depend on its private signal—that is, the borrower knows the mapping (17) from s_1^L to π_1^L.[20] This identification problem faced by the borrower is illustrated in Figure 1.

Since the borrower has only one observable, which is informative about the lender's private information, it cannot separately identify the lender's information set at date 1 (i.e. it's private signal) and its date 1 cost shock τ_1^L. Put differently, the borrower has only one equation (17) to infer two unobservables s_1^L, τ_1^L. Indeed, there is a continuum of pairs $\left(\tau_1^L, \pi_1^L\right)$ that satisfy the relation (17) for the observed q_1, as illustrated in Figure 1.

In light of the identification problem faced by the borrower, the borrower's posterior beliefs evolve according to

$$E_1^B\left[\tilde{\pi}_1^L\right] = \int_{\underline{\tau}}^{\bar{\tau}} \int_0^1 \pi \mathbb{1}\left\{\pi = \frac{1 - \tau q_1}{1 - \underline{R}}\right\} dG_\pi(\pi) d\Lambda_T(\tau), \tag{18}$$

[20]For a given value of q_1, the functional relationship between $\tilde{\tau}_1^L$ and $\tilde{\pi}_1^L$ is where $\tilde{\tau}_1^L \in [\underline{\tau}, \bar{\tau}]$ and $\tilde{\pi}_1^L \in (0,1)$.

where $\mathbb{1}\{\cdot\}$ is an indicator function, and $\Lambda_T(\cdot)$ is the distribution function of $\lambda_T(\cdot)$. By change of variables, the above equation can be rearranged as follows

$$\pi_1^B = \int_0^1 \pi \lambda_T \left(\frac{1-(1-\underline{R})\pi}{q_1} \right) \mathbb{1}\left\{ \underline{\tau} < \frac{1-(1-\underline{R})\pi}{q_1} < \bar{\tau} \right\} dG_\pi(\pi). \tag{19}$$

Intuitively, the borrower computes the likelihood of all possible realizations of (τ_1^L, s_1^L) consistent with the observed value of q_1, based on its knowledge of the joint distribution of τ_1^L and s_1^L, and given the mapping (17) between s_1^L and π_1^L. Based on the likelihood of these realizations, the borrower forms an expectation over the lender's private signal s_1^L and updates its own beliefs π_1^B accordingly.

Proposition 1. *Borrower's posterior belief π_1^B has the following properties in equilibrium:*

1. *Borrower's belief is the same as the expected posterior of lender, i.e. $\pi_1^B = E_1^B[\pi_1^L]$.*
2. *π_1^B is decreasing in q_1.*

Proof. See Appendix A.2 ■

Part 1 of the proposition shows that the borrower's posterior belief about the fundamental value of the asset is given by the borrower's belief about the lender's belief. This is simply a result of Bayesian updating. Part 2 of the proposition shows that if the borrower faces a tighter funding liquidity from the lender at date 1 (that is, a lower contract price $q_1 < q_0$), then the borrower becomes more pessimistic about the fundamental value of the asset. Importantly, this is regardless of whether the cause of the decline was an adverse cost shock τ_1^L or a bad signal s_1^L, because of the identification problem faced by B.

Cost shocks and beliefs An important implication of Proposition 1 is that pessimism and optimism and the fundamental value of the asset arises endogenously in our model as a result of funding liquidity. In particular, the borrower's beliefs are shaped also by the lender's cost shocks, despite the fact that these are orthogonal to the asset's return. As we will show, tight funding conditions (a low contract price q_1) leads to pessimism (a high π_1^B). By contrast, loose funding liquidity conditions (a high contract price q_1) leads to optimism (a low π_1^B).

6.3. Characterization of Equilibrium at Date 1

Definition of date-1 equilibrium An equilibrium at date 1 is a set of prices p_1, q_1, collateral amount c_1, quantities $d_1^L, \kappa_1^L, a_1^B, \kappa_1^B, a_1^T, C_2^L, C_2^B, C_2^T$, and posterior beliefs π_1^L, π_1^B satisfying the agents' optimality conditions and constraints, market clearing conditions, and the equations characterizing belief formation, taking as given variables determined at date 0 and the date-1 shocks

τ_1^L, s_1^L. Note that a unique equilibrium always exists because of the linearity of the agents' objective functions and the constraints of the borrower's optimization problem, and the smoothness of the inverse demand function of the traditional sector, which determines the market clearing condition.

The spot market for the risky asset is competitive, so the price p_1 will be pinned down in equilibrium at the marginal valuation of the marginal buyer. Recall that there can be two potential marginal buyers: (continuum of) borrowers, or the traditional sector. Let p_1^B denote the borrower's marginal valuation of the asset, which is given by the borrower's optimality condition (11):

$$p_1^B = \frac{1}{\tau_1^B}\left(E_1^B[R] - E_1^B\left[R_2^d\right] + q_1\tau_1^B\right). \tag{20}$$

Recall that the traditional sector has the inverse demand function $F'(a_1^T)$, so that it's marginal valuation is given by $F'(a_1^T)$. (Recall also that the traditional sector will buy any amount of assets up to its marginal valuation $F'(a_1^T)$ because the traditional sector has enough amount of cash endowments by Assumption 1.) Moreover, the price is characterized in equilibrium by

$$p_1 = \max\left\{\frac{1}{\tau_1^B + \xi_{\kappa_1}^B}\left(E_1^B[R] - E_1^B\left[R_2^d\right] + q_1\tau_1^B\right), F'(a_1^T)\right\}, \tag{21}$$

where the maximum willingness to pay determines the identity of the marginal buyer in equilibrium. Note that $p_1 < p_1^B$ can occur in equilibrium, if the borrower is liquidating assets for liquidity needs, which implies that the borrower is constrained in cash, i.e. $\xi_{\kappa_1}^B > 0$. It is convenient to derive some additional properties that are satisfied in equilibrium, summarized in the lemma below.

Lemma 1. *In equilibrium, the following holds:*

(A) The borrower's collateral constraint is generically binding, so that $c_1 = a_1^B$.

(B) The price of the asset exceeds the price of the contract, $p_1 > q_1$.

Proof. See Appendix A.4 ■

The date 1 equilibrium features two regimes, which we call 'normal times' and 'fire sales'. The partition of the state space into these two regimes can be characterized by the price p_1 and the identity of the marginal buyer in equilibrium. Part (A) of the proposition below describes the two regimes, while Part (B) shows that fire sales are more severe for worse private signals and also for worse liquidity shocks.

Proposition 2. *In equilibrium, the following holds:*

(A) ***Two regimes in date-1 equilibrium****: When* $p_1 \geq F'(0)$*, the economy is in the 'normal regime' in which* $p_1 = p_1^B$, $a_1^B = c_1 = a_0^B$*, and* $a_1^T = 0$*. When* $p_1 < F'(0)$*, the economy is in the 'fire sale regime' in which* $p_1 = F'(a_1^T)$, $a_1^B = c_1^B < a_0^B$*, and* $a_1^T > 0$*.*

(B) ***Fire sales are more severe for tighter funding liquidity****: In the fire sale regime, a lower realization of s_1^L or a higher realization τ_1^L, at the margin, is associated with larger a_1^T and lower p_1 in equilibrium.*

(C) ***Fire sales may be driven by liquidity needs or pessimism****: In a 'liquidity driven fire sale' in which the borrower is optimistic but is forced to liquidate assets to repay debt, we have $p_1^B > F'(0) > p_1$. In a 'belief driven fire sale' in which the borrower liquidates due to its pessimism, we have $p_1^B < F'(0)$.*[21]

Normal Regime In normal times, when cost shocks are not too large and the signal is not too bad, all of the risky asset is held by the borrower. In this situation, the borrower has relatively optimistic beliefs. As a result, the borrower is the marginal buyer of the asset and the asset price p_1 is relatively high. the borrower is at a corner solution in its portfolio choice and wants to hold only the risky asset, and so its holdings of the risky asset remain unchanged from date 0, $a_1^B = a_0^B$. (Moreover, since it is always optimal for the borrower to collateralize all its asset holdings, we have $c_1 = a_1^B = c_0$.) If the size of the loan under the date-1 contract $q_1 c_0$ is less than the promised payment of the loan under the date-0 contract c_0— that is, if the borrower faces a tightening funding liquidity from the lender—then, the borrower finances the difference $c_0 - q_1 c_0$ out of its cash holdings stored from date 0.

As q_1 falls, the borrower becomes more pessimistic, so that π_1^B falls in accordance with Proposition 1. However, in normal times, this pessimism is not sufficient to trigger a change in its portfolio. As a result, in equilibrium, the risky asset is held entirely by borrowers, $a_1^T = 0$. The marginal buyer of the asset at date 1 is still the borrower, and so p_1 is given by the borrower's marginal valuation of the asset p_1^B.

Fire Sale Regime In the fire sale regime, the equilibrium at date 1 features some strictly positive fraction of the risky asset held by the traditional sector, while the remainder is held by the borrower. In particular, when the cost shock is large (τ_1^L is high) or the signal is bad (s_1^L is low), the tightening of funding liquidity causes the borrower to become sufficiently pessimistic about the return on the asset that, and the borrower's portfolio decision switches: the borrower prefers to alter its portfolio from holding the risky asset to cash. As a result, the borrower meets the tighter funding liquidity by liquidating some of its holdings of the risky asset to the traditional sector at a fire-sale price.

To see this, recall the borrower's date 1 optimality condition (12)

$$\underbrace{v_1\left(\pi_1^B, q_1\right)}_{cost\ of\ financing\ risky\ asset\ holdings} = \underbrace{\eta_1\left(\pi_1^B; p_1\right)}_{expected\ net\ return\ of\ risky\ asset},$$

[21] The price p_1 can be less than p_1^B in this case, if the borrower is extremely pessimistic and $p_1^B < F'(a_0^B)$. See Appendix A.6 for the full characterization of this case.

where we have imposed the result that $\xi^B_{\kappa_1} = 0$, so that the borrower holds a positive amount of cash. At a given p_1, a tighter funding liquidity (lower q_1 relative to q_0) worsens the tradeoff to investing in the risky asset by making the borrower's beliefs more pessimistic. As a result ν_1 rises and η_1 falls by Proposition 1. If the shock or signal is sufficiently bad that the borrower's valuation is less than the traditional sector's (i.e. $p^B_1 < F'(0)$, then in order to meet the tighter funding liquidity, the borrower prefers to liquidate to the traditional sector).

Given this behavior of borrowers, how does the spot price p_1 of the asset change in equilibrium? Since all borrowers are equally pessimistic and trying to switch to holding cash, the only way for borrowers to convert asset to cash at date 1 in aggregate is to sell the asset to the traditional sector, who have higher marginal valuation. Therefore, the traditional sector, T, becomes the marginal buyer of the asset, and price is given by T's marginal valuation $p_1 = F'(a^T_1)$. Recall that $F''(\cdot)$ is a decreasing function, and so the price is then decreasing in quantity a^T_1 of the asset held by the traditional sector. Hence, equilibrium is restored when p_1 falls sufficiently to restore equality in the borrower's optimality condition at an interior solution in which the borrower holds both cash and the asset.

Part (B) of Proposition 2 indicates that a lower q_1 is associated with more severe fire sales—that is, higher a^T_1 and a lower p_1. Hence, more severe cost shocks and worse news about the quality of the asset lead to more of the risky asset being liquidated to the traditional sector by generating a larger decrease in funding liquidity and therefore a greater date-1 liquidity need for the borrower.

Part (C) of Proposition 2 describes two different cases of fire sales. In a liquidity driven fire sale, the borrower is still optimistic about the asset's fundamental value but the borrower is forced to liquidate assets to repay debt c_0. This fire sale is still inefficient because the borrower values the asset more than the traditional sector does, i.e. $p^B_1 > F'(0) \geq F'(a^T_1)$. On the contrary, in a belief driven fire sale, the borrower is pessimistic about the asset's fundamental value, which is even below the marginal valuation in the traditional sector, i.e. $p^B_1 < F'(0)$. In this case, the borrower is happy to sell the asset to the traditional sector, up to the point either the marginal valuation of the traditional sector decreases to $F'(a^T_1) = p^B_1$ or the borrower sells all the asset, i.e. $a^T_1 = a^B_0$.

We can partition the state space into the Normal and Fire Sale Regimes by defining the set of states $\left(\tau^L_1, \pi^L_1\right)$ consistent with $\hat{p}_1 \equiv F'(0)$, the threshold such that the economy enters the Fire Sale Regime when the asset price is below the threshold $\hat{p}_1$. In the Baseline Economy, which we constructed so far, there is a one-to-one mapping from q_1 to p_1, and hence, the threshold asset price $\hat{p}_1$ corresponds to a threshold value $\hat{q}_1$. The lender's optimality condition defines the set of $\left(\tau^L_1, \pi^L_1\right)$ consistent with $\hat{q}_1$, given by

$$\pi^L_1 = \frac{1 - \tau^L_1 \hat{q}_1}{1 - \underline{R}}. \tag{22}$$

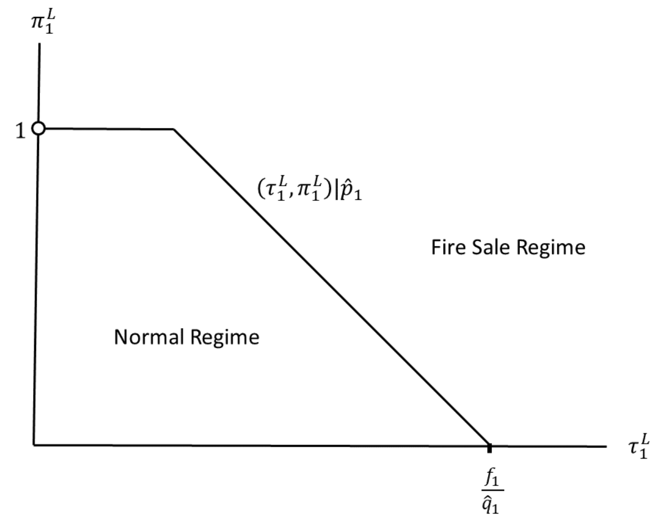

Figure 2: Two Regimes

Note: This figure illustrates the bisection of the state space into two regimes at date 1. The equilibrium is in the Fire Sale Regime if and only if the equilibrium value of p_1 is below a threshold value $\hat{p}_1$ (or, equivalently, the equilibrium value of q_1 is below a threshold value $\hat{q}_1$). The curve in the figure plots the combinations of the states $\left(\tau_1^L, \pi_1^L\right)$ consistent the threshold $\hat{p}_1$ and marks the frontier between the two regimes.

This equation (22) defines the frontier partitioning the state space into the Normal and Fire Sale Regimes, and is illustrated in Figure 2.

6.4. Downstream versus Upstream Information Spillovers

Thus far, we have focused on the case in which the lender gets the cost shock and private signal, but the borrower does not. This implies that information spillovers flow downstream from the lender to the borrower. Here, we briefly touch on the alternative case in which the borrower gets a cost shock and private signal, but the lender does not, leading to upstream information spillovers. Overall, while the mechanism differs slightly, the fundamental insights are the same.

Suppose the borrower gets a cost shock τ_1^B and private signal s_1^B, but the lender does not (i.e. τ_1^L is a fixed parameter known to both agents). How would the borrower's choices reflect its private information τ_1^B, s_1^B? In this case q_1 still only reflects the lender's first-order condition according to (2), and is therefore not informative about the borrower's private information. However, the asset price p_1 will reflect the borrower's private information through the equilibrium price (12). Using this equation, the spread between the borrower's valuation and the price of debt at date 1 is

$$\underbrace{p_1 - q_1}_{spread} = \frac{E_1^B[R] - E_1^B\left[R_2^d\right]}{\xi_{\kappa_1}^B + \tau_1^B} \tag{23}$$

(Note also that the quantity of the borrower's investment in the risky asset $a_1^B = c_1$ is just determined residually from this optimality condition and the market clearing condition, and hence contains no additional information about borrower's private information over the spread.) Therefore, the lender uses the observation of $p_1 - q_1$ to form its posterior belief π_1^L, but cannot separately infer π_1^B and τ_1^B due to a similar identification problem as what we had in the case with downstream information spillovers.

Suppose that the borrower receives a cost shock (high τ_1^B) or bad private signal so that the asset price is low. The lender then observes that the spread $p_1 - q_1$ is low and will update its belief, which becomes more pessimistic. As a result, as can be seen from the equilibrium expression for q_1, the equilibrium price of date-1 debt will fall—that is, there will be a tightening of funding liquidity. This worsens the borrower's liquidity position, and forces it to liquidate more of the risky asset, further lowering the asset price. Hence, the information spillover from the borrower to the lender still amplifies the fire sale through a feedback loop between market illiquidity and beliefs, in a manner which gives rise to asset price booms and busts.

7. Equilibrium at Date 0

Given the equilibrium at date 1, we can solve recursively for the date-0 equilibrium. For tractability, we assume here that $F'(a) = \alpha$ for all $a \geq 0$, although we did not assume this in characterizing the equilibrium at dates 1 and 2 above.[22] The date-0 equilibrium is derived in Appendix A.7, and the key properties are summarized in the following proposition.

Proposition 3. *Suppose that assumptions 1 and 2 hold. In date 0 equilibrium, the borrower holds zero amount of cash and full amount of asset, using all of them as collateral to borrow from the lender—that is,* $(c_0, a_0, \kappa_0^B) = (A, A, 0)$*— and the prices of the asset and contracts are* $(p_0, q_0) = (1 + e_0^B/A, 1)$.

Proof. See Appendix A.7. ■

The date-0 loan from the lender to the borrower is strictly positive, the borrower invests only in assets, and uses all of its holdings of the risky asset as collateral for the loan.

8. Information Contagion and Fire Sales

In the Fire Sale Regime, the evolution of agents' beliefs plays an important role in the dynamics at date 1. Moreover, the nature of fire sales here differs from that in much of the literature: in a fire

[22] Effectively, this assumption implies that, in a fire sale at date 1, the borrower liquidates all of its holdings of the risky asset.

sale, the borrower liquidates not simply because it is liquidity constrained (indeed, the borrower could finance some of the decrease in funding liquidity out of its cash holdings, as it does in the normal regime), but also because the borrower endogenously becomes more pessimistic about the fundamental value of the asset. Importantly, pessimism can occur even if the lender receives relatively good signal.

In this section, we show how misinformation affects the likelihood and severity of crises, and how it affects the distribution of states at date 1 across the two regimes. In our model, the interaction between beliefs and liquidity arises from two related but distinct features of the learning process:

1. (learning) In response to new information, agents update their prior beliefs about the fundamental value of the risky asset.
2. (heterogeneous beliefs) Because information is private, agents' beliefs evolve differently to new information.

To understand how each feature of beliefs interacts with market and funding liquidity, we conduct two separate exercises.

In the first exercise, we compare the date-1 equilibrium to that in an alternative benchmark economy in which agents do not update their beliefs, and we show that the endogenous response of beliefs to changes in funding liquidity exacerbates fire sales by worsening market liquidity. In the second set of exercises, we compare the date-1 equilibrium to another benchmark economy in which all information is publicly observed, so that beliefs are homogeneous. This counterfactual exercise allows us to understand how the heterogeneity in agents' beliefs affects outcomes through liquidity conditions.

8.1. Interaction between pessimism, funding liquidity, and market liquidity

To understand how learning interacts with funding and market liquidity, we first compare the response of our baseline economy to shocks to the response of an alternative benchmark economy in which beliefs are stale—that is, we assume agents' date-0 beliefs are never updated in response to new information. We measure the severity of a fire sale by the extent to which the borrower fire sells its holdings of the risky asset to the traditional sector: the greater (and, equivalently, the lower the asset price p_1), the more severe the fire sale at date 1.[23] This result is summarized in Lemma 2 below.

[23] As we discuss in the normative section of the paper, Section 9, the liquidation of the risky asset that occurs in a fire sale is not necessarily associated with constrained Pareto inefficiency.

Lemma 2 (Learning amplifies the severity of fire sales)**.** *For any set of shocks* (π_1^L, τ_1^L) *such that the equilibrium is in the Fire Sale Regime at date 1, the price of the risky asset* p_1 *is lower and the extent of misallocation of the asset* a_1^T *is higher in equilibrium compared to an alternative benchmark economy in which agents' posterior beliefs are exogenously set to their date-0 priors.*

Proof. See Appendix A.8. ■

We showed in Section 6 that pessimism and optimism and the fundamental value of the asset arises endogenously in our model as a result of funding liquidity. Lemma 2 above shows that this increased pessimism contributes to market illiquidity, by depressing asset prices and making it more costly to raise funds through liquidation. Therefore, the model yields an adverse feedback between pessimistic beliefs about fundamentals, funding liquidity, and market illiquidity—a dynamic, which exacerbates crises.

8.2. The interaction between belief disagreement and liquidity conditions

In order to isolate the role of belief disagreement, and misinformation in particular, in the equilibrium allocation at date 1, we compare our equilibrium in our Baseline Economy with a benchmark case without private information. We first briefly describe this benchmark.

Common Information Benchmark In the benchmark economy with common information, we assume that the borrowers can directly observe both the lender's signal s_1^L and the lender's shock to the opportunity cost of funds τ_1^L.[24] In this sense, neither agent has private information. A corollary of Proposition 1 is that, under this common information benchmark, the borrower and lender have identical posterior beliefs, $\pi_1^B = \pi_1^L$ in every state. The form of the equilibrium is otherwise the same as that in the baseline economy with private information.

By comparing the date-1 equilibrium allocation under the baseline economy with that under this benchmark (in both cases, taking date-0 variables as given), we can isolate the effect of misinformation on the equilibrium allocation at date 1.[25]

First, we characterize the frontier, which is partitioning the state space between Normal and Fire Sale Regimes under Baseline Economy and the Common Information Benchmark. $\hat{q}_1$ is the threshold q_1 separating the Normal and Fire Sale Regimes in the Baseline Economy, defined implicitly by $\left(1-\pi_1^B(\hat{q}_1)\right)\left(\overline{R}-1\right)+\tau_1^B\hat{q}_1 = \tau_1^B F'(0)$, and $\hat{p}_1$ is the corresponding asset price of the frontier. As in the Baseline Economy, the frontier under the Common Information Benchmark is defined by the set of states $\left(\tau_1^L, \pi_1^L\right)$ consistent with $p_1 = \tilde{p}_1 \equiv F'(0)$, which is given by

[24] We could equivalently assume that the date-1 signal about R is publicly observed while the lender's opportunity cost, τ_1^L, is not observed by the borrower. This is equivalent because, in equilibrium, the borrower would be able to perfectly infer τ_1^L given its observation of q_1 and its knowledge of π_1^L.

[25] In general, differences in the equilibrium allocations at date 0 across the baseline and benchmark may also manifest as differences in the date 1 allocation. We can show that the date-0 equilibrium allocations of the two cases would be the same under assumptions 1 and 2.

$$\left(\overline{R}-1\right)-\left[\left(\overline{R}-1\right)+\frac{\tau_1^B}{\tau_1^L}\left(1-\underline{R}\right)\right]\pi_1^L+\frac{\tau_1^B}{\tau_1^L}=\tau_1^B F'(0). \tag{24}$$

This condition defines the set of states, $\left(\tau_1^L,\pi_1^L\right)|\tilde{p}_1$, and characterizes the partition between the Fire Sale and Normal Regimes in the Common Information Benchmark. The following lemma shows how this frontier compares to that under the Baseline Economy.

Lemma 3 (Frontier in the Common Information Benchmark)**.** *The frontier $\left(\tau_1^L,\pi_1^L\right)|\tilde{p}_1$ in the Common Information Benchmark is strictly convex and has a negative slope in the domain for τ_1^L. Moreover, the y-intercept $\frac{1}{1-\underline{R}}$ of the frontier $\left(\tau_1^L,\pi_1^L\right)|\tilde{p}_1$ is the same as the y-intercept for the frontier $\left(\tau_1^L,\pi_1^L\right)|\hat{p}_1$ in the Baseline Economy, and the x-intercept $\frac{\tau_1^B}{\tau_1^B F'(0)-\overline{R}+1}$ in the Common Information Benchmark exceeds that in the Baseline Economy $\frac{1}{\hat{q}_1}$.*

Proof. See Appendix A.10. ■

For the given $\hat{q}_1$ of the frontier in the Baseline Case, there exists a point $\hat{\tau}_1^L$ such that the corresponding $\pi_1^L=1$ as

$$\pi_1^L=1=\frac{1-\hat{\tau}_1^L\hat{q}_1}{1-\underline{R}}.$$

Similarly, there exists a point $\tilde{\tau}_1^L$ such that the corresponding $\pi_1^L=1$ as

$$\pi_1^L=1=\frac{\tau_1^B f_1+\tilde{\tau}_1^L\left(\overline{R}-f_1-\tau_1^B F'(0)\right)}{\tilde{\tau}_1^L\left(\overline{R}-f_1\right)+\tau_1^B\left(f_1-\underline{R}\right)}.$$

Some of the results, which follow, depend on whether the frontier partitioning the state space into the Normal and Fire Sale Regimes in the Baseline Economy intersects the frontier in the Common Information Benchmark. If the following condition holds, then the frontier in the Baseline Economy is above the frontier under the Common Information Benchmark up to some point $(\tau_1^{L^*},\pi_1^{L^*})$ such that $\pi_1^{L^*}<1$, where the two frontiers meet.

Condition 1.

$$\hat{\tau}_1^L>\tilde{\tau}_1^L \tag{25}$$

Lemma 4. *The frontiers in the Baseline Case and the Common Information Benchmark never meet in the domain such that $\pi_1^L\leq 1$ if and only if $\hat{\tau}_1^L\leq\tilde{\tau}_1^L$.*

Proof. See Appendix A.11. ■

Condition 1 implies that the frontier in the Baseline case starts declining linearly at a point $(\hat{\tau}_1^L, 1)$, whereas the frontier in the Common Information Benchmark goes through the point $(\tilde{\tau}_1^L, 1)$. Because both frontiers started at the same y-intercept, the average slope of the frontier in the Common Information Benchmark is above that in the Benchmark case. Because the slope of the frontier in the Common Information Benchmark is increasing (by strict convexity shown in Lemma 3), the frontier in the Baseline Economy is always below the frontier in the Common Information Benchmark.

Figure 3 illustrates how the partitions of the state space between the Normal and Fire Sale Regimes compare across the Baseline Economy and the Common Information Benchmark, when Condition 1 holds.[26] Figure 6 in Appendix A.13 illustrates this for the case in which Condition 1 does not hold.

Proposition 4 summarizes how misinformation affects the severity of fire sales and the allocation in the Normal Regime in the baseline economy, relative to those of the Common Information benchmark, while taking date-0 variables as given.

Proposition 4 (Effect of misinformation on the allocation in the Normal and Fire Sale Regimes)**.** *Given the date-0 allocation, the following holds:*

(A) The allocation of the risky asset in the Normal Regime is identical in the Baseline Economy and the the Common Information Benchmark: $a_1^B = a_0^B$ in both economies.

(B) If Condition 1 holds in equilibrium, then misinformation amplifies the severity of liquidity driven fire sales and dampens severity of belief driven fire sales at date 1. Formally, for any state $\left(\tau_1^L, \pi_1^L\right)$ in the Fire Sale Regime, a_1^B is lower in the Baseline Economy compared to the Common Information Benchmark if τ_1^L is sufficiently high (exceeding some threshold $\overline{\tau}_1^L(\pi_1^L)$), and is lower if π_1^L is sufficiently high (exceeding some threshold $\overline{\pi}_1^L(\tau_1^L)$).

(C) If Condition 1 does not hold in equilibrium, then misinformation unambiguously amplifies the severity of fire sales. Formally, for every state $\left(\tau_1^L, \pi_1^L\right)$ in the Fire Sale Regime, a_1^B is always lower in the Baseline Economy compared to the Common Information Benchmark. (For the knife-edge cases in the Baseline Economy in which $\pi_1^B = \pi_1^L$, a_1^B is the same across the baseline and benchmark economies.)

[26] Note that, while the slope of $\left(\tau_1^L, \pi_1^L\right) | \hat{p}_1$ in the Baseline Economy is constant, the slope of $\left(\tau_1^L, \pi_1^L\right) | \tilde{p}_1$ in the Common Information Benchmark is not constant. To see why, note that, in the Baseline case, the frontier between the two regimes is determined entirely by a threshold q_1, and the the lender's FOC implies that the relationship between π_1^L and τ_1^L is linear. By contrast, in the Benchmark, q_1 is no longer sufficient to determine which regime the economy is in; the exact realization of shocks (because the exact realization determines π_1^B) also affects the determination of regime. Thus, the fact that there's more than one variable determining the equilibrium regime implies that the relationship between the π_1^L and τ_1^L) pairs consistent with the fact that the threshold p_1 is nonlinear. Relatedly, we should define the frontier not as $\hat{q}_1$ but rather as $\hat{p}_1$, since there is no such well-defined $\hat{q}_1$ in the Common Information Benchmark.

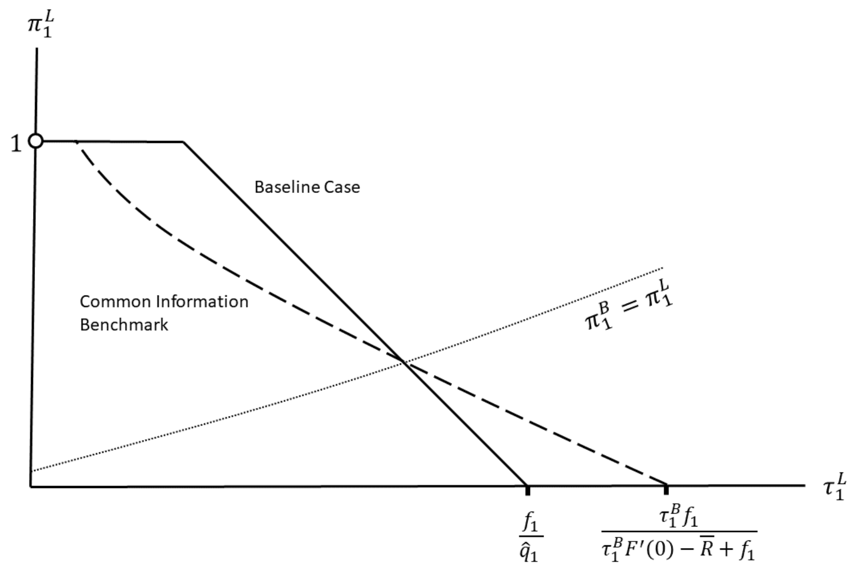

Figure 3: Two Regimes under the Baseline Case and the Common Information Benchmark

Note: This figure illustrates the bisection of the state space into two regimes at date 1 in the baseline case in which the lender's liquidity shock τ_1^L and beliefs π_1^L are private information, and under the Common Information Benchmark case in which this information is directly observable by the borrower. The equilibrium is in the Fire Sale Regime if and only if the equilibrium value of p_1 is below the threshold value $\hat{p}_1$. (For the baseline case, this corresponds to the threshold value of $\hat{q}_1$.) The solid curve in the figure plots the combinations of the states $\left(\tau_1^L, \pi_1^L\right)$ consistent the threshold $\hat{p}_1$, based on the lender's optimality condition for d_1^L, and denotes the frontier between the two regimes. The dashed curve plots the same frontier in the Common Information Benchmark in which the borrower directly observes the lender's private information, and hence $\pi_1^B = \pi_1^L$ all along this curve. For both cases, the region to the southwest of these curves is the Normal Regime, while the northeast is the Fire Sale Regime.

The dotted line is constructed by tracing out the borrower's posterior beliefs $\left(E_1^B\left[\tau_1^L\right], \pi_1^B\right)|q_1$ at different realizations of the equilibrium value q_1. For a given q_1, the borrower's posterior beliefs $\left(E_1^B\left[\tau_1^L\right], \pi_1^B\right)|q_1$ about the state are a point on the set $\left(\tau_1^L, \pi_1^L\right)|q_1$ of all possible states consistent with its observation of q_1 (illustrated in Figure 1). At a given q_1, all points on the curve $\left(\tau_1^L, \pi_1^L\right)|q_1$ to the northwest of the point $\left(E_1^B\left[\tau_1^L\right], \pi_1^B\right)|q_1$ feature $\pi_1^B > \pi_1^L$, while all points on the curve to the southeast feature $\pi_1^B > \pi_1^L$. Thus, the dotted curve in the figure demarcates the region of the state space in which $\pi_1^B < \pi_1^L$ in equilibrium to the northwest of the curve, and the region in which $\pi_1^B > \pi_1^L$ to the southeast. In the northwest region, the borrower is optimistic about the risky asset relative to the better-informed lender, while in the southeast, the borrower is relatively pessimistic about the risky asset.

Proof. See Appendix A.12. ■

Part (A) shows that misinformation does not affect the allocation of the risky asset in the Normal Regime at date 1. While the beliefs of the lender and borrower may diverge in a way which does not reflect fundamental shocks, this has no bite on the allocation of the risky asset in the Normal Regime as long as the risky asset is not sold to the traditional sector. (Nevertheless, the allocation of cash between borrowers and lenders may differ, in general.)

Part (B) indicates that, if Condition 1 holds, misinformation amplifies the severity of liquidity driven fire sales and dampens severity of belief driven fire sales at date 1. Therefore, the mechanism by which beliefs respond endogenously to conditions in funding markets implies that the economy features more severe fire sales when fire sales are driven by severe shocks to the opportunity cost of funds (high realizations of τ_1^L), while fire sales are less severe when driven by bad news. Intuitively, for fire sales driven by bad news about the asset's fundamental value, the borrower is more optimistic about the fundamental of the asset relative to the better-informed lender, and this relative optimism reduces the extent to which the borrower liquidates the asset to the traditional sector. In contrast, when the fire sale is driven by a negative cost shock to the lender, the borrower is more pessimistic about the asset since as the borrower cannot infer the lender's private information. As a result, this pessimism causes the borrower to liquidate more of the asset than it otherwise would. Note that the overall effect on severity of a fire sale relative to the Common Information Benchmark is ambiguous, as it depends on the relative importance of cost shocks versus news in driving fire sales. Part (C) indicates that, if Condition 1 does not hold, then misinformation features unambiguously more severe fire sales.

Effect of misinformation on the likelihood of fire sales

Next, we consider how misinfromation affects the distribution of the state space across the Normal and Fire Sale Regimes. Comparing the frontiers in the Baseline Economy and the Common Information Benchmark reveals that misinformation increases the likelihood of liquidity-driven fire sales and decreases the likelihood of signal-driven fire sales, if Condition 1 holds. However, if Condition 1 does not hold, the likelihood of fire sales is unambiguously higher in the Baseline Economy than that in the Common Information Benchmark. These results are summarized in Lemma 5.

Lemma 5 (Effect of misinformation on the likelihood of fire sales)**.**

(A) If Condition 1 holds, then in the Baseline Economy, the likelihood of entering a fire sale given a large cost shock is higher than in the Common Information Benchmark, while the likelihood of entering a fire sale given a bad news shock is lower in the Baseline Economy. Moreover, the overall effect on the unconditional probability of a fire sale in the Baseline Economy relative to the Common Information Benchmark is ambiguous.

(B) If Condition 1 does not hold, then the unconditional probability of a fire sale is unambiguously higher in the Baseline for any state $\left(\tau_1^L, \pi_1^L\right)$ in the Fire Sale Regime.

Proof. The results follow from Proposition 4. See Appendix A.13 for more details on Part (B). ■

Part (A) says that, if Condition 1 holds, misinformation increases the likelihood of fire sales in response to cost shocks, since cost shocks lead borrowers to become more pessimistic about the fundamental value of the asset, despite being orthogonal to this fundamental value. By contrast, misinformation reduces the likelihood of a fire sale in response to bad news about fundamentals. In this sense, the misinformation mechanism reduces the probability of fire sales in response to bad news about fundamentals, but increases the probability of fire sales in response to cost shocks. Thus, when Condition 1 holds, the effect of misinformation on the unconditional likelihood of crises is ambiguous and depends on the relative importance of cost shocks versus news in driving fire sales.

On the other hand, if Condition 1 does not hold, for any shock, the economy is more likely to enter the Fire Sale Regime in the Baseline Economy than in the Common Information Benchmark. See Figure 7 in Appendix A.13 for an illustration. As a result, the misinformation mechanism unambiguously increases the likelihood of fire sales when Condition 1 does not hold.

This characterizes how our mechanism by which funding liquidity, or conditions in funding markets, affects perceptions of risk and beliefs about fundamentals, affects the likelihood and severity of fire sales.

9. Normative Implications

In this section, we examine the normative implications of the model. For now, we focus on ex post efficiency—that is, we compare competitive equilibrium at date 1 to allocation at date 1 chosen by social planner, taking date-0 variables as given. We define the date-1 constrained efficient allocation as the solution to the problem of a constrained social planner who takes date-0 variables as given. The planner faces uncertainty about R at date 1 and forms beliefs about R through Bayesian updating in response to news at date 1. Although the planner observes the lender's private information (and hence has the same posterior beliefs as the lender), it respects that the borrower does not observe this private information but must infer this information imperfectly through observables.[27] In other words, we assume the planner cannot disclose the lender's private information to the borrower. And finally, the planner faces the borrowing constraint between the lender and borrower. However, the planner internalizes how choices about the terms under which to lend and borrow at date 1, and the choice to sell the risky asset, affect the allocation across the distribution of states.

We show that the competitive equilibrium at date 1 is generically constrained inefficient due to the presence of two externalities. In choosing how much to lend at date 1, lenders do not internalize how their choice affects the information set of borrowers through the price q_1. Therefore, the

[27] Section 10 analyzes the setup in which the government cannot observe the lender's private information.

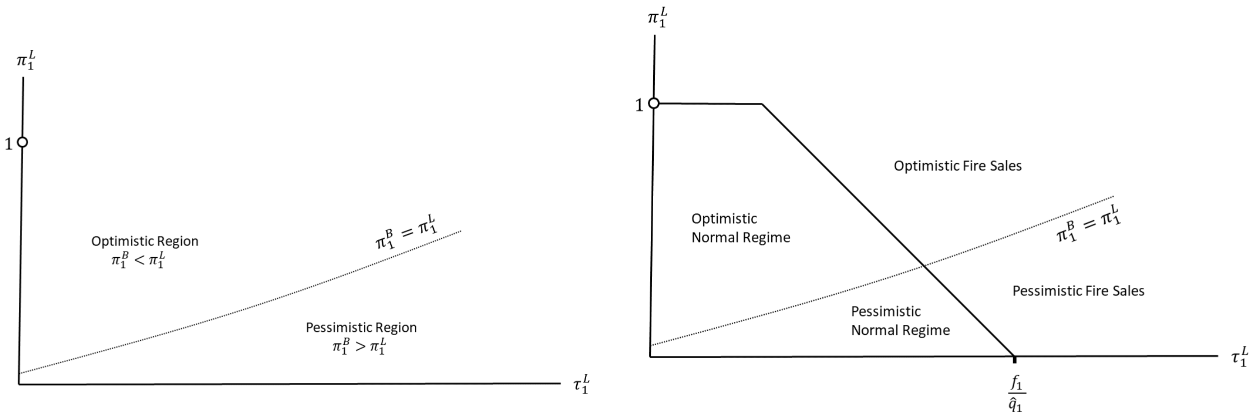

Figure 4: Optimistic/Pessimistic Region and Normal/Fire Sale Regime

Note: The left panel illustrates the bisection of the state space into regions based on the borrower's beliefs about the asset relative to those of the lender. The dotted curve demarcates the region of the state space in which $\pi_1^B < \pi_1^L$ in equilibrium to the northwest of the curve, and the region in which $\pi_1^B > \pi_1^L$ to the southeast. In the northwest region, the borrower is optimistic about the risky asset relative to the better-informed lender, while in the southeast, the borrower is relatively pessimistic about the risky asset.

The right panel overlays the left panel with Figure 2 and illustrates the demarcation of four regions of the state space by whether the equilibrium is in the Normal Regime or Fire Sale Regime, and by whether the borrower is optimistic or pessimistic about the risky asset relative to the better-informed lender.

economy features an *information externality*, which is new to the literature. In addition, there is a pecuniary externality which is essentially the same as in Lorenzoni (2008): atomistic borrowers do not internalize how their asset sales at date 1 affects the budget constraints of other borrowers through the asset price p_1. The planner can engineer a Pareto improvement over the competitive equilibrium by addressing both externalities.[28] Since policies designed to address pecuniary externalities have been analyzed extensively in the literature, we first focus on policies designed to address the information externality. Nevertheless, we show that a single policy tool alone cannot address both externalities simultaneously.

We can divide the state space (π_1^L, τ_1^L) at date 1 into two regions, illustrated in the left panel of Figure 4, in which we define the optimism of the borrower relative to the information set of the lender.

In the 'optimistic region' of the state space, the borrower is overly optimistic relative to the better-informed lender, i.e. $\pi_1^B < \pi_1^L$. The economy enters this optimistic region when the lender's

[28]Similarly, there are two broad sets of ex ante policies (policy interventions at date 0) which could implement a Pareto improvement: policies which reduce the likelihood and severity of fire sale (e.g. leverage restrictions, capital requirements), and policies which reduce opacity and improve information diffusion. In this section, we focus on date-1 interventions only.

cost shock is relatively good (low τ_1^L). In the 'pessimistic region', the borrower is overly pessimistic relative to the better-informed lender, i.e. $\pi_1^B \geq \pi_1^L$. The economy enters this pessimistic region when the lender's cost shock is relatively bad (high τ_1^L). The partition between these two regions, marked by the dotted line in both panels of Figure 4, is defined by tracing out the borrower's posterior beliefs $\left(E_1^B\left[\tau_1^L\right], \pi_1^B\right)|q_1$ at different realizations of the equilibrium value q_1. Since the social planner at date 1 observes the true state, the planner knows which region the economy is in at date 1. Note that the relative frequency with which the economy will end up in one region or the other depends on the distribution of shocks τ_1^L and news.

How do the planner's date-1 choices compare to those of agents in the competitive equilibrium? The social planner would make choices on behalf of agents in a way which increases the size of the total surplus and use transfers between agents to ensure that all agents remain at least as well off.

Fire Sale Regime First we focus on the Fire Sale Regime. Suppose that the economy happens to be in the pessimistic region of Figure 4. Then, the borrower would be better off in expectation if it held more of the asset (i.e. liquidated less of it), so that a_1^B is higher. This is for two reasons. First, in expectation, the borrower would be better off if it behaved as if $\pi_1^B = \pi_1^L$ since its portfolio choice would be making use of all information available to agents at date 1. Given that the borrower is risk-neutral, this implies that it would have a higher expected utility at date 1. Second, there is a pecuniary externality at date 1 in which borrowers do not internalize how selling the asset depresses the price of the asset, making it more costly to sell for other agents.

Therefore, in response to shocks, which leave the borrower relatively pessimistic, the planner would have the lender roll over more of the debt (i.e. the planner would choose a higher q_1 than what the lender would choose). This is for two reasons: the borrower would be less pessimistic and would sell less of the risky asset (thus mitigating the adverse effect of the information externality); and by increasing the price of the asset p_1, which would reduce the need for the borrower to sell (thus mitigating the adverse effect of the pecuniary externality). As a result, the borrower's expected date-2 return will be higher, since the borrower's holdings of the risky asset would now reflect the additional information available to the lender and planner.

For both of these reasons, the expected net return at date 2 on the borrower's portfolio will be higher, and the planner can transfer portion of this higher return from the borrower to the lender. Since, in expectation, date 2 output will be higher, this transfer from the borrower to the lender will be positive in expectation. Hence, the lender will also be better off at date 1. (To the extent that the planner would reduce fire sales at date 1 ex post rather than ex ante, this would be an additional way that planner would increase the total surplus.)

Suppose instead that the economy is in the optimistic region of Figure 4. Then the borrower would be better off in expectation if it held less of the asset (lower a_1^B). Therefore, in response to shocks, which leave the borrower relatively optimistic, the planner would choose a lower q_1. The planner would then transfer some of the increased surplus which accrues to the lender to the borrower to ensure the borrower is no worse off.

Normal Regime Now consider the Normal Regime. In the Normal Regime, the allocation of a_1^B is irrelevant since it is held fully by the borrower. However, q_1 affects the allocation of cash between the borrower and lender. By changing q_1 in a way which internalizes this choice on the borrower's beliefs, the planner can allocate cash more efficiently based on τ_1^L and τ_1^B (the return to cash of each agent). This will increase the total surplus. The planner can also design a lump-sum transfer Φ_2 between the borrower and lender to share this surplus in a way which leaves neither agent worse off in expectation. Thus, the date-1 competitive equilibrium is generically constrained inefficient.

Interpretation In the competitive equilibrium, the price of the contract q_1 (or equivalently the interest rate $1/q_1$) plays two roles: it facilitates a reallocation of liquidity between the borrower and lender in response to news and shocks at date 1, and it (imperfectly) diffuses new information about the fundamental value of the asset from the better-informed lender to borrower.[29] However, because q_1 diffuses information only imperfectly, it leads beliefs of the borrower to systematically diverge from fundamentals. Therefore, from a social perspective, there is a tradeoff associated with the price of lending q_1 between reallocating liquidity in response to shocks and facilitating information sharing.

In the competitive equilibrium, the equilibrium price q_1 facilitates the privately optimal solution to this tradeoff. However, the presence of information and pecuniary externalities means that these choices are not constrained efficient. In contrast, the planner internalizes the consequences of both roles of q_1 on the allocation. Effectively, the planner's solution decouples the two roles played by q_1 from one another: the planner's choice of q_1 internalizes the effect on the borrower's information set, while the date 2 transfer from the borrower to the lender is used to compensate the lender for the time and risk associated with lending to the borrower.

Pecuniary externality In addition to the information externality, the model features a pecuniary externality, similar to that outlined in Lorenzoni (2008), in which agents do not internalize how choices at date 1 affect the budget constraints of other agents through the asset price p_1. The planner can address this pecuniary externality using the tax γ^L on the lenders' return on cash holdings. By reducing the opportunity cost of lending, this tax forces lenders to internalize how their choices to lend affects liquidity needs of borrowers and therefore the effect on the price of the risky asset.

Much of the literature on financial crises and systemic risk has focused on the role of pecuniary

[29]More generally, q_1 serves as price of lending and compensates the lender for the opportunity cost of its funds and the risk associated with lending.

externalities and the policies needed to address them. In addition to a pecuniary externality, our setting features an information externality. Since the constrained inefficiency of the competitive equilibrium is driven by two externalities, we need two policy tools to implement the constrained efficient allocation. Moreover, using only one tool is not only insufficient, but also may even make the allocation worse in terms of Pareto efficiency.

Therefore, policies, which address only pecuniary externalities (such as leverage or margin requirements) without addressing the information externality cannot implement constrained efficiency, and they may even make agents worse off in some states due to information spillovers. For example, in states in which the borrower is optimistic relative to the better-informed lender, the borrower would be better off liquidating more of the asset and holding more cash. But a higher leverage or margin requirement would not incentivize this. Similarly, implementing a policy that addresses the information externality without also addressing the pecuniary externality may leave agents worse off in some states. For example, during optimistic fire sales, both agents may be made better off by the borrower's relative optimism, as this reduces the extent to which the borrower liquidates the asset, and thus partially offsets the negative effects associated with the pecuniary externality.

Thus, implementing the constrained efficient allocation requires a combination of policies designed to eliminate both externalities, such as the option facilities (discussed in Section 10) and Pigouvian taxes on liquidation mentioned above.

Liquidity facilities In practice, policymakers have implemented regulatory policies designed to reduce the inefficiencies associated with pecuniary externalities through ex post interventions such as liquidity facilities and asset purchases. Importantly, our model shows that these interventions may work to stabilize financial markets through an additional channel not discussed in the literature — the information channel.

Viewed through the perspective of our model, these episodes of financial turmoil are states of the world in which information becomes blurred by funding and market illiquidity. These dynamics increase pessimism about fundamentals. Government interventions, which directly increase market and funding liquidity, may boost investor confidence—that is, reduce pessimism about the fundamental value of financial assets—through the mechanisms we have outlined. Greater investor confidence, in turn, further increases funding and market liquidity. Indeed, several of the government interventions during periods of financial market turmoil appear to stabilize markets by restoring confidence in financial assets.[30] Our model suggests the interactions between beliefs

[30]For example, a notable feature of some of the US government interventions made during the financial turmoil in 2020, such as the Primary Market Corporate Credit Facility and the Secondary Market Corporate Credit Facility, is that they seemed to stabilize financial markets even with very little transaction volume. To explain this, some policymakers have suggested that these policies restored confidence in these financial assets, which then stabilized liquidity conditions and asset valuations.

about fundamentals and funding and market liquidity may play a role during such episodes.

10. Improving Informational Efficiency during Fire Sales

The inefficiencies in this model arise in part from the informational inefficiency of financial markets: prices cannot fully reveal agents' private information.[31] In our setting, this informational inefficiency stems in part from the incompleteness of the risk markets. Moreover, our model shows that this informational inefficiency may be more acute when liquidity conditions are especially tight or loose, as this impairs ability of prices to reveal private information.

In this section, we consider the effect of policies, which may improve the informational efficiency of financial markets during liquidity dry-ups by creating new markets. To that end, we introduce a government, who can offer insurance contracts, which are contingent on the asset's payoff. Unlike private agents, the government can commit to honoring its obligations and can enforce payment from its counterparties, even when the asset returns are low, giving it an advantage over the private sector in offering these contracts.

This exercise is motivated in part by policy interventions during recent episodes of financial turmoil in which central banks acted as a 'dealer of last resort' by acting as counterparty to securities trade when these markets froze, and effectively re-opening the markets for state-contingent securities, which shut down during the turmoil. In our setting, these policies may improve social welfare by improving the ability of prices to convey private information, rather than simply by reallocating risk from the private to the public balance sheets.[32] To highlight this belief channel, we focus on policies in which the quantity of these insurance contracts supplied by the government to the private sector is infinitesimally small, so that the quantity of risk transferred from the private sector to the government's balance sheet is negligible. Even these policies are sufficient to eliminate the inefficiencies associated with the information externality.

To proceed, we first modify the setup by introducing a government. The government does not observe agent's private information. At date 1, the government has access to lump-sum taxes and two distortionary taxes: the government can tax each unit revenue that the borrower obtains from liquidating the risky asset at date 1 at the rate γ^B; and the government can tax the date-2 return that the lender receives on its cash holdings from date 1 at the rate γ^L. In addition, the government can issue two securities with state-contingent payoffs, each in a competitive market. One contract

[31] It is well known, that in the absence of complete risk market, prices must not only clear markets and aggregate information, but also allocate risk. As a result, asset prices may reveal information imperfectly (e.g. Stiglitz (1981)).

[32] Note that, in general, the allocation implemented using this policy intervention will not coincide with the solution to the constrained planner's problem outlined in Section 9. This is because, while the planner's solution still features private information, the policy intervention outlined in this section publicly reveals all private information.

resembles a put option on the risky asset while the other resembles a call option.[33] This is a parsimonious way to capture the provision of insurance that many of the government interventions in credit and asset markets observed in the past brought about.[34] Modeling government intervention in this parsimonious way is a simple way to capture the insurance aspect of government intervention, which can play a role in stabilizing markets through agents' information and beliefs, a channel not heretofore identified in the literature.

The put option contract involves a transfer of T_2^p units of the consumption good at date 2 to the buyer of the contract if $R = \underline{R}$ and 0 otherwise. The contract is sold in a competitive market at date 1 at a spot price p_1^p. The call option contract involves a transfer of T_2^c units of the consumption good at date 2 to the buyer of the contract if $R = \overline{R}$ and 0 otherwise, and is sold in a competitive market at date 1 at a spot price p_1^c. We assume that the prices p_1^p, p_1^c are publicly observable. Therefore, if these securities are traded in positive supply, the equilibrium prices of the option will be informative to the borrower about the lender's private information.

Thus, the government has three types of tools to intervene in financial markets at date 1: the distortionary taxes γ^L, γ^B allow it to reallocate liquidity at date 1, the put and call options allow it to provide insurance at date 1, and the lump-sum transfers allow it to redistribute wealth. The government's budget constraint at date 1 is

$$\left(a_0^B - a_1^B\right)\gamma^B + p_1^p \omega_1^p + p_1^c \omega_1^c + \Gamma_1 = \kappa_1^G \tag{26}$$

where γ^B is the proportional tax on the borrower's liquidation of the asset at date 1, ω_1^p and ω_1^c is the total quantity of the put and call options sold to private agents (each of which is defined as $\omega_1^p \equiv \omega_1^{L,p} + \omega_1^{B,p}$ and $\omega_1^c \equiv \omega_1^{L,c} + \omega_1^{B,c}$), respectively, Γ_1 are lump-sum taxes at date 1, and κ_1^G is the government's date-1 cash holdings, Thus, the government's date-1 cash holdings consist of the proceeds of sales of the put and call options and taxes collected from the borrower's liquidation of the risky asset and lump-sum taxes.

The government's date-2 budget constraint is

$$\kappa_1^G + \Gamma_2 + \gamma^L \kappa_1^L = \omega_1^p T_2^p \mathbb{1}\left\{R = \underline{R}\right\} + \omega_1^c T_2^c \mathbb{1}\left\{R = \overline{R}\right\}. \tag{27}$$

[33] We assume that, due to limited enforcement, private agents cannot replicate these securities option facilities or trade other financial contracts (more than collateralized lending) in equilibrium. The government, by contrast, can enforce payment and so can offer these contracts, reducing the missing markets problem.

[34] For example, consider the credit and liquidity facilities implemented by the Federal Reserve and backed by the Treasury in 2020, such as the Secondary Market Corporate Credit Facility or the Commercial Paper Funding Facility, enacted during a period of severe market turmoil in which the prices of many financial assets fell dramatically. Backstopped by the Treasury, the Fed purchased qualifying securities from investors on the open market. If the value of these securities rose in the future, investors could buy them back at the market price, while if not, investors would be saved from incurring further losses. Therefore, at some level, the Fed and Treasury provided implicit insurance to investors.

Thus, the government's date 2 expenditure must be met out of its cash holdings stored from date 1, any lump-sum taxes at date 2, or tax revenue on the lender's cash holdings from date 1. The market clearing conditions for each option at date 1 are

$$\omega_1^p = \omega_1^{G,p}$$
$$\omega_1^c = \omega_1^{G,c},$$

where $\omega_1^{G,p}, \omega_1^{G,c}$ are the total quantities of each option supplied to the market by the government.

The lender's date 1 and date 2 budget constraints, respectively, adjusted for the lender's holdings of the option facility are

$$\kappa_1^L - \kappa_0^L + p_1^p \omega_1^{L,p} + p_1^c \omega_1^{L,c} \leq d_0^L - q_1 d_1^L$$

$$C_2^L \leq \tau_1^L \kappa_1^L + R_2^d d_1^L + \omega_1^{L,p} T_2^p \mathbb{1}\left\{R = \underline{R}\right\} + \omega_1^{L,c} T_2^c \mathbb{1}\left\{R = \overline{R}\right\}.$$

While the lender's first order condition for lending to the borrower is similar to (2), it reflects a higher opportunity cost of holding cash due to the unit tax γ^L on the return on cash at date 2:

$$q_1 = \frac{E_1^L\left[R_2^d\right]}{\tau_1^L - \gamma^L}. \tag{28}$$

The lender's first order condition for investing in the put and call options, respectively, are

$$p_1^p = \frac{T_2^p \pi_1^L}{\tau_1^L - \gamma^L + \xi_{\kappa_1}^L} \tag{29}$$

and

$$p_1^c = \frac{T_2^c\left(1 - \pi_1^L\right)}{\tau_1^L - \gamma^L + \xi_{\kappa_1}^L}. \tag{30}$$

Similarly, the borrower's first order conditions for each option are given by $p_1^p = \frac{T_2^p \pi_1^B}{\tau_1^B + \xi_{\kappa_1}^B}$ and $p_1^p = \frac{T_2^c\left(1-\pi_1^B\right)}{\tau_1^B + \xi_{\kappa_1}^B}$. In equilibrium, the lender will be the marginal buyer for exactly one of the two options, and the borrower will be the marginal buyer for the other, depending on the relative optimism and liquidity shocks of each agent.[35] Therefore, in equilibrium, either p_1^p or p_1^c will be priced by the

[35]In particular, if $\frac{\pi_1^L}{\tau_1^L} < \frac{\pi_1^B}{\tau_1^B}$, in which case the borrower's marginal valuation for the put option is higher than the lender's, then it must be that $\frac{\left(1-\pi_1^L\right)}{\tau_1^L} > \frac{\left(1-\pi_1^B\right)}{\tau_1^B}$, so that the lender's marginal valuation for the call option is higher than the borrower's (and vice versa). And as a result we will either have that $\omega^p = \omega^{L,p}$ and $\omega^c = \omega^{B,c}$, or that $\omega^p = \omega^{B,p}$

better-informed lender. Let $p_1^o \in \{p_1^p, p_1^c\}$ denote the price of the option, which is priced by the better-informed lender in equilibrium.

Note that the competitive equilibrium outlined in Section 6 is identical to the competitive equilibrium in this modified economy when the government is completely passive and issues 0 securities or taxes: $\omega_1^{G,p}, \omega_1^{G,c}, \Gamma_1, \Gamma_2, \gamma^L, \gamma^B, \kappa_1^G = 0$.

The government can engineer a Pareto improvement by supplying a strictly positive (but arbitrarily small) quantity of the the put and call options to the market, $\omega_1^{G,p}, \omega_1^{G,c} > 0$. As we show below, setting these quantities arbitrarily close to zero suffices to reap the full social benefits associated with these options by eliminating the inefficiencies associated with the information externality by incentivizing agents to credibly and publicly reveal their private information at date 1. As a result, the market prices of these options reveal agents' private information, thereby eliminating the identification problem, which gives rise to the information externality.

To see this, note from lender's optimality conditions (29) and (30) that the observed price p_1^o will be informative to the borrower about the lender's private information π_1^L, τ_1^L, over and above the information already provided by q_1 in (28). That is, because the payoff profile of the option differs than that the loan from the lender to the borrower, the lender's optimality conditions for the option and the loan are linearly independent. (In contrast, the price of the other option—for which the less-informed borrower is the marginal buyer—will be uninformative to the borrower, as it will simply reflect the borrower's beliefs.) Therefore, the borrower's knowledge of the lender's optimality conditions, together with the observations of q_1, p_1^o, can be used to separately identify τ_1^L and π_1^L. This is illustrated in Figure 5, where p_1^o is the price of the option priced by the better-informed lender.

Hence, the publicly observable price of the option facility p_1^o allows the borrower to perfectly identify the lender's private information. According to the characterization of the borrower's posterior belief (18), the borrower's posterior is then identical to that of the lender $\pi_1^B = \pi_1^L$. In this manner, by publicly revealing private information, the option facility eliminates the relative pessimism and optimism of the borrower associated with the information externality.

Discussion By selling a put option on the risky asset, the government is effectively bearing risk associated with the aggregate shock at date 2.[36] The extent to which the government bears this risk is proportional to the quantity of the option it supplies to the market ω^G. Importantly, however, the price p_1^o fully reveals the lender's private information even as the quantity ω^G of options sold is infinitesimally small (as long as $\omega^G > 0$). Therefore, the full social benefits of the option facility can be realized even if the government sells an arbitrarily small quantity of the

and $\omega^c = \omega^{L,c}$ in equilibrium, so that each option is held exclusively by one agent.

[36]Since the government bears risk through the option facility, this facility would be akin to the facilities operated by the Federal Reserve under section 13(3) of the Federal Reserve Act in which the Federal Reserve provides financing to the private sector but the Treasury provides a backstop for losses incurred on these option.

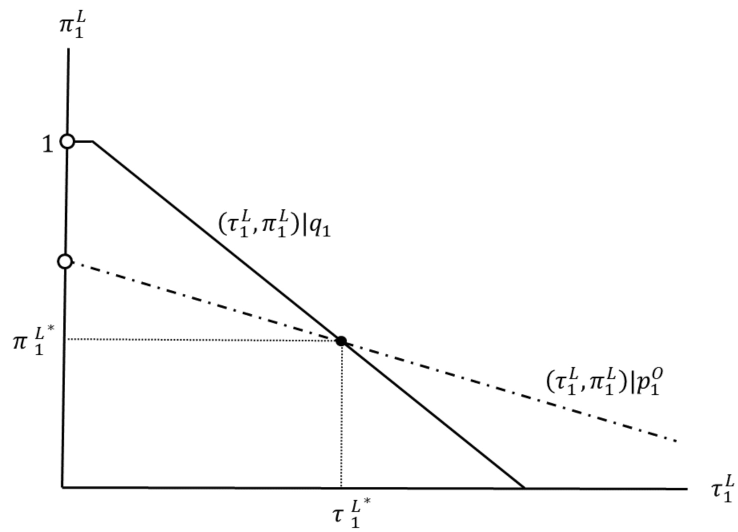

Figure 5: Effect of the option facility

Note: This figure illustrates the effect of the option facility on the borrower's identification problem at date 1. The solid curve in the figure plots the combinations of the lender's liquidity shock and beliefs, $\left(\tau_1^L, \pi_1^L\right)$, which are consistent with the equilibrium q_1, based on the lender's optimality condition for d_1^L. The dash-dotted curve plots the combinations of $\left(\tau_1^L, \pi_1^L\right)$ which are consistent with the equilibrium price of the option p_1^o, based on the lender's optimality condition for investment in the facility. The borrower's observations of both q_1 and p_1^o, together with knowledge of the lender's optimality conditions, are sufficient to infer the true realizations of τ_1^{L*} and π_1^{L*} separately.

option, and bears negligible risk. This is because the social benefit of the facility arises not from transferring risk to the government's balance sheet, but rather in the information provided by the price of the signal. When $\omega^G > 0$ is arbitrarily close to zero, the economy approaches the common information benchmark economy analyzed in Section 8.

The identification problem in our setting ultimately derives from a missing markets problem: Because of the absence of the insurance markets against cost shocks and the shock to the risky asset, there is only one price q_1, which conveys information about two orthogonal shocks.[37] By introducing a market for insurance against the shock to risky asset, the government adds a price signal which, together with q_1, fully reveals agents' private information.[38]

11. Conclusion

We developed a model of heterogeneous agents, collateralized debt, learning, and fire sales to shed light on how the availability of funding liquidity affects agents' perceptions about fundamentals, and the role that this may play in asset booms and busts. The model shows that beliefs about fundamentals are endogenously shaped by the availability of liquidity in funding and asset markets. As a result, cost (liquidity) shocks, which are orthogonal to asset fundamentals can cause beliefs to become systematically divorced from fundamentals. Moreover, this informational inefficiency may be more acute when liquidity conditions are especially tight or loose, as this impairs the ability of prices to reveal private information about fundamentals. As a result of this mechanism, loose funding liquidity can generate over-optimism about fundamentals leading to over-investment, while tight funding liquidity can lead to excessive pessimism about fundamentals and fire sales. We showed that the competitive equilibrium is generically constrained inefficient due to the presence of a new information externality, and we characterized interventions in funding markets, which can generate a Pareto improvement. Finally, we showed that, by acting as a dealer-of-last-resort, a central bank can mitigate the informational inefficiency of markets by reopening derivatives markets, which would otherwise shut down during episodes of financial turmoil.

[37] For simplicity, we assume that markets are incomplete in our setting and we leave aside the micro-foundation for this assumption. In practice, financial markets are indeed incomplete and private information cannot be fully inferred from price signals. We take this as given and draw out the implications of this for how beliefs are formed.

[38] In practice, there may be several reasons why the government is in a better position to offer this type of insurance. For example, the government, due to its ability to tax, may be able to absorb the risk associated with the aggregate shock better than private agents.

A. Appendix: Omitted Proofs

A.1. Derivation of the lender's posterior beliefs

First, note that the bounds of the posterior belief of the lender is $\pi_1^L \in (0,1)$. The bounds are derived from the fact that $s_1^L = R+\varepsilon$ with $\varepsilon \in (-\infty,\infty)$, so any realization of s_1^L is possible from both state of the world $R=\bar{R}$ and $R=\underline{R}$. The lender updates posterior beliefs as in (17)

$$\pi_1^L = \frac{\pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right)}{\pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right) + (1-\pi_0)\lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \bar{R}\right)}.$$

This expression of π_1^L is possible by using Bayes' rule for events with a positive measure, and then taking limits. For any $\delta > 0$, the events $\left\{s_1^L - \underline{R} \le \varepsilon_1^L \le s_1^L - \underline{R} + \delta\right\}$ and $\left\{s_1^L - \overline{R} \le \varepsilon_1^L \le s_1^L - \overline{R} + \delta\right\}$ are well defined and have positive probability. Therefore, Bayes' rule implies

$$\begin{aligned}
& P(R=\underline{R}|s_1^L - \underline{R} \le \varepsilon_1^L \le s_1^L - \underline{R} + \delta) \\
=& \frac{\pi_0 P\left(s_1^L - \underline{R} \le \varepsilon_1^L \le s_1^L - \underline{R} + \delta\right)}{\pi_0 P\left(s_1^L - \underline{R} \le \varepsilon_1^L \le s_1^L - \underline{R} + \delta\right) + (1-\pi_0) P\left(s_1^L - \overline{R} \le \varepsilon_1^L \le s_1^L - \overline{R} + \delta\right)} \\
=& \frac{\pi_0 \int_{s_1^L - \underline{R}}^{s_1^L - \underline{R}+\delta} \varepsilon' d\Lambda_\varepsilon(\varepsilon')}{\pi_0 \int_{s_1^L - \underline{R}}^{s_1^L - \underline{R}+\delta} \varepsilon' d\Lambda_\varepsilon(\varepsilon') + (1-\pi_0) \int_{s_1^L - \overline{R}}^{s_1^L - \overline{R}+\delta} \varepsilon' d\Lambda_\varepsilon(\varepsilon')},
\end{aligned} \tag{31}$$

where P is the probability function and both the denominator and numerator are positive. As $\delta \to 0$, we have

$$\begin{gathered}
P(R=\underline{R}|s_1^L - \underline{R} \le \varepsilon_1^L \le s_1^L - \underline{R} + \delta) \to P(R=\underline{R}|\varepsilon_1^L = s_1^L - \underline{R}) = \pi_1^L \\
\int_{s_1^L - \underline{R}}^{s_1^L - \underline{R}+\delta} \varepsilon' d\Lambda_\varepsilon(\varepsilon') \to \lambda_\varepsilon(\varepsilon_1^L = s_1^L - \underline{R}) \\
\int_{s_1^L - \overline{R}}^{s_1^L - \overline{R}+\delta} \varepsilon' d\Lambda_\varepsilon(\varepsilon') \to \lambda_\varepsilon(\varepsilon_1^L = s_1^L - \overline{R}).
\end{gathered}$$

Thus, taking the limit of $\delta \to 0$ on both sides of (31) results in (17).

A.2. Proof of Proposition 1

Proof.

1. B' s information at date 1, I_1^B, is given by the set of all variables observable to all agents, which are prices p_1 and q_1. In contrast, L has additional information of τ_1^L and s_1^L, denoted by I_1^L, on top of the publicly available information. By the law of iterated expectation

$$E_1^B\left[E_1^L[x]\right] = E\left[E\left[x \mid I_1^L\right] \mid I_1^B\right] = E\left[x \mid I_1^B\right] = E_1^B[x].$$

Applying the definition of posterior beliefs $\pi_1^i = E_1^i[\mathbb{1}\{R = \underline{R}\}]$ for $i = L, B$ leads to statement 1.

2. By the first statement of Proposition 1 and (19), we have

$$\pi_1^B(q_1) = \int_0^1 \pi\lambda_T\left(\frac{1-(1-\underline{R})\pi}{q_1}\right)\mathbb{1}\left\{\underline{\tau} < \frac{1-(1-\underline{R})\pi}{q_1} < \bar{\tau}\right\}dG_\pi(\pi),$$

for a given q_1.

First, note that there exists $\hat{\pi}(q)$ such that

$$\frac{1-(1-\underline{R})\hat{\pi}(q)}{q} = \tau_0$$

for any $q < q_0$. Also, note that $\hat{\pi}(q)$ is decreasing in q. From (2), we can rearrange the difference between the prior for τ, τ_0, and realized τ for a given π and equilibrium price q as

$$|\tau - \tau_0| = \left|\frac{1-(1-\underline{R})\pi}{q} - \tau_0\right|,$$

which can be further simplified as

$$\left|\frac{1-(1-\underline{R})\pi}{q} - \tau_0\right| = \frac{1}{q}\left|1-(1-\underline{R})\pi - 1 + (1-\underline{R})\hat{\pi}(q)\right| = \frac{1}{q}\left|(1-\underline{R})(\pi - \hat{\pi}(q))\right|. \tag{32}$$

We claim that for any π and $q' > q$,

$$\left|\frac{1-(1-\underline{R})\pi}{q} - \tau_0\right| < \left|\frac{1-(1-\underline{R})\pi}{q'} - \tau_0\right| \tag{33}$$

and therefore,

$$\lambda_T\left(\frac{1-(1-\underline{R})\pi}{q} - \tau_0\right) > \lambda_T\left(\frac{1-(1-\underline{R})\pi}{q'} - \tau_0\right) \tag{34}$$

by the functional form assumption on λ_T.

By equation (32), (33) is true if

$$\frac{1}{q}\left|(1-\underline{R})(\pi - \hat{\pi}(q))\right| < \frac{1}{q'}\left|(1-\underline{R})(\pi - \hat{\pi}(q'))\right|$$

holds. The previous equation holds if

$$\frac{q'}{q} < \frac{|\pi - \hat{\pi}(q')|}{|\pi - \hat{\pi}(q)|}$$

holds, which is equivalent to

$$\frac{\dfrac{1-(1-\underline{R})\hat{\pi}(q')}{\tau_0}}{\dfrac{1-(1-\underline{R})\hat{\pi}(q)}{\tau_0}} = \frac{\dfrac{1}{1-\underline{R}} - \hat{\pi}(q')}{\dfrac{1}{1-\underline{R}} - \hat{\pi}(q)} < \frac{|\pi - \hat{\pi}(q')|}{|\pi - \hat{\pi}(q)|} \tag{35}$$

by the definition of $\hat{\pi}(q)$.

First, consider the case in which π is either $\pi > \hat{\pi}(q)$ or $\pi < \hat{\pi}(q')$. Then, (35) becomes

$$\frac{\dfrac{1}{1-\underline{R}} - \hat{\pi}(q')}{\dfrac{1}{1-\underline{R}} - \hat{\pi}(q)} < \frac{\pi - \hat{\pi}(q')}{\pi - \hat{\pi}(q)}. \tag{36}$$

Note that for any x and $0 < b < c$, $\dfrac{x-b}{x-c}$ is decreasing in x, because

$$\frac{\partial \left(\dfrac{x-b}{x-c}\right)}{\partial x} = \frac{b-c}{(x-c)^2} < 0.$$

Thus, (36) holds because $\pi \leq 1 < \dfrac{1}{1-\underline{R}}$.

Finally, consider the case in which π is $\hat{\pi}(q') < \pi < \hat{\pi}(q)$. Then,

$$\frac{\pi - \hat{\pi}(q')}{\pi - \hat{\pi}(q)} < \frac{|\pi - \hat{\pi}(q')|}{|\pi - \hat{\pi}(q)|}$$

holds and by (36), (35) also holds. Therefore, (33) and (34) hold.

By the claim, the support of the conditional expectation is decreasing in q_1 for each given realization of π and for the given pdf λ_T as

$$\lambda_T\left(\frac{1-(1-\underline{R})\pi}{q}\right)\mathbb{1}\left\{\underline{\tau} < \frac{1-(1-\underline{R})\pi}{q} < \bar{\tau}\right\} > \lambda_T\left(\frac{1-(1-\underline{R})\pi}{q'}\right)\mathbb{1}\left\{\underline{\tau} < \frac{1-(1-\underline{R})\pi}{q'} < \bar{\tau}\right\}$$

for any $q < q'$. Thus, π_1^B is decreasing in q_1. ■

A.3. Solving the Borrower's Optimization Problem

$$\max_{c_1, f_1, a_1^B, \kappa_1^B} E_1^B\left[C_2^B\right] \tag{37}$$
$$\begin{aligned}\text{s.t. } & q_1 c_1 - c_0 \geq p_1\left(a_1^B - a_0^B\right) + \kappa_1^B - \kappa_0^B, \\ & C_2^B \leq \tau_1^B \kappa_1^B + a_1^B R - c_1 R_2^d(f_1) \\ & c_1 \leq a_1^B, \\ & c_1 \geq 0,\ a_1^B \geq 0,\ \kappa_1^B \geq 0\end{aligned}$$

Note that the date 2 budget constraint implies

$$C_2^B = \tau_1^B \kappa_1^B + a_1^B R - c_1 R_2^d(f_1),$$

while the date 1 budget constraint implies

$$\kappa_1^B = q_1 c_1 - c_0 - p_1\left(a_1^B - a_0^B\right) + \kappa_0^B.$$

Combining these two yields

$$C_2^B = \tau_1^B\left(q_1 c_1 - c_0 - p_1\left(a_1^B - a_0^B\right) + \kappa_0^B\right) + a_1^B R - c_1 R_2^d(f_1)$$

Therefore we can write the Lagrangian as

$$L_1^B = E_1^B\left[C_2^B\right] + \mu_1^B\left(a_1^B - c_1^B\right) + \xi_{c_1}^B c_1^B + \xi_{a_1}^B a_1^B + \xi_{\kappa_1}^B\left(q_1 c_1^B - c_0^B + p_1\left[a_0^B - a_1^B\right] + \kappa_0^B\right) \tag{38}$$

where we have $R_2^d(f_1) \equiv \min\{R, f_1\}$, and $E\left[R_2^d(f_1)\right] = (1-\pi_1)f_1 + \pi_1 \min\{\underline{R}, f_1\}$. Replacing q_1 yields

$$C_2^B = \tau_1^B\left(q_1 c_1^B - c_0^B - p_1\left(a_1^B - a_0^B\right) + \kappa_0^B\right) + a_1^B R - c_1^B R_2^d(f_1).$$

Then, the expectation is given by

$$\begin{aligned}E_1^B\left[C_2^B\right] &= E_1^B\left[\tau_1^B\left(q_1 c_1^B - c_0^B - p_1\left(a_1^B - a_0^B\right) + \kappa_0^B\right) + a_1^B R - c_1^B R_2^d(f_1)\right] \\ E_1^B\left[C_2^B\right] &= \tau_1^B\left(q_1 c_1^B - c_0^B - p_1\left(a_1^B - a_0^B\right) + \kappa_0^B\right) + a_1^B E_1^B[R] - c_1^B E_1^B\left[R_2^d(f_1)\right]\end{aligned}$$

Hence, the Lagrangian is:

$$L_1^B = E_1^B\left[C_2^B\right] + \mu_1^B\left(a_1^B - c_1^B\right) + \xi_{c_1}^B c_1^B + \xi_{a_1}^B a_1^B + \xi_{\kappa_1}^B\left(c_1^B q_1 - c_0^B + p_1\left[a_0^B - a_1^B\right] + \kappa_0^B\right) \tag{39}$$

where $E_1^B\left[C_2^B\right]$ is given by the above.

FOC for c_1^B:

$$\frac{dE_1^B\left[C_2^B\right]}{dc_1^B} - \mu_1^B + \xi_{c_1}^B + \xi_{\kappa_1}^B q_1 = 0$$
$$\tau_1^B q_1 - E_1^B\left[R_2^d(f_1)\right] - \mu_1^B + \xi_{c_1}^B + \xi_{\kappa_1}^B q_1 = 0$$
$$\left(\tau_1^B + \xi_{\kappa_1}^B\right) q_1 - E_1^B\left[R_2^d(f_1)\right] - \mu_1^B + \xi_{c_1}^B = 0$$

FOC for a_1^B:

$$\frac{dE_1^B\left[C_2^B\right]}{da_1^B} + \mu_1^B + \xi_{a_1}^B - \xi_{\kappa_1}^B p_1 = 0$$
$$-\tau_1^B p_1 + E_1^B[R] + \mu_1^B + \xi_{a_1}^B - \xi_{\kappa_1}^B p_1 = 0$$
$$-\left(\tau_1^B + \xi_{\kappa_1}^B\right) p_1 + E_1^B[R] + \mu_1^B + \xi_{a_1}^B = 0$$

A.4. Proof of Lemma 1

Proof.

Proof of Part (A):

By replacing μ_1^B in (5) with (6) and using the contract price q_1 that makes L to lend in a positive amount:

$$q_1\left(\tau_1^B + \xi_{\kappa_1}^B\right) + E_1^B[R] - E_1^B[R^d] - \tau_1^B p_1 + \xi_{a_1}^B - \xi_{\kappa_1}^B p_1 + \xi_{c_1}^B = 0,$$

which can be rearranged as

$$E_1^B\left[R - R^d\right] + \xi_{c_1}^B + \xi_{a_1}^B = \left(\tau_1^B + \xi_{\kappa_1}^B\right)(p_1 - q_1).$$

Consider the case in which B borrows a positive amount so $a_1, c_1 > 0$. Then, the above equality can be rearranged as

$$\tau_1^B + \xi_{\kappa_1}^B = \frac{E_1^B\left[R - R^d\right]}{p_1 - q_1} = \frac{(1-\pi_1^B)(\overline{R}-1)}{p_1 - q_1}, \tag{40}$$

which implies that the expected return of holding the asset with the collateralized debt net of B's shadow value of asset equals to the sum of cash return of B and B's shadow value of cash. The return of the leveraged asset holding on the right-hand side of (40) can be greater than return of B's storage technology, but then B has to exhaust all the cash, $\xi_{\kappa_1}^B > 0$, and B cannot pay for additional down payment (or cash collateral or variation margin), $p_1 - q_1(f_1)$. In addition, B might want to generate more asset because the return of the leveraged asset holding is exceedingly profitable. If this is the case, then the price will adjust to the point that the equality holds.

Now we show that the collateral constraint is generically binding. If B is borrowing zero amount—that is, $c_1 = 0$, B does not have enough cash to repay the loans to L unless B liquidates all the assets, implying $a_1^B = c_1 = 0$. B borrows a positive amount $c_1 > 0$ only if the return from rolling over the debt is greater than or equal to the cash return. Thus,

$$\tau_1^B + \xi_{\kappa_1}^B = \frac{E_1^B[R - R^d]}{p_1 - q_1} \tag{41}$$

holds in any equilibrium with $c_1 > 0$. If the collateral constraint is not binding, then by (6),

$$\tau_1^B + \xi_{\kappa_1}^B = \frac{E_1^B[R]}{p_1} \tag{42}$$

holds. (41) and (42) imply that

$$\tau_1^B + \xi_{\kappa_1}^B = \frac{E_1^B[R^d]}{q_1} = \frac{(1-\pi_1^B) + \pi_1^B \underline{R}}{q_1},$$

which holds only at non-generic realization of q_1 as π_1^B is decreasing in q_1. The return from leveraging the asset purchase, (41), should exceed the return from the asset purchase without leverage, (42), because otherwise B will not purchase with leverage—that is, $c_1 = 0$—which is a contradiction. Even if the previous equation holds, which is a not a generic case of parameters, B is indifferent between purchasing the asset with leverage and without leverage. Therefore, we can impose a tie-breaking rule for B, choosing to maximize c_1 when indifferent.[39] Hence, B leverages all of its asset purchase as $c_1 = a_1^B$, and the collateral constraint is binding.

Proof of Part (B):

[39]This tie-breaking rule can be justified by assuming that B obtains infinitesimally small utility of holding an asset as collateral.

The borrower's optimality condition (11), which can be rearranged as

$$\left(\xi_{\kappa_1}^B + \tau_1^B\right)(p_1 - q_1) = E_1^B[R] - E_1^B\left[R_2^d(f_1)\right]$$

$$p_1 - q_1 = \frac{1}{\xi_{\kappa_1}^B + \tau_1^B}\left[E_1^B[R] - E_1^B\left[R_2^d(f_1)\right]\right].$$

Since $\xi_{\kappa_1}^B \geq 0$ and $\tau_1^B > 0$, we have $p_1 > q_1$ if and only if

$$\begin{aligned} E_1^B[R] &> E_1^B\left[R_2^d(f_1)\right] \\ (1-\pi_1^B)\overline{R} + \pi_1^B\underline{R} &> (1-\pi_1^B)f_1 + \pi_1^B\underline{R} \\ (1-\pi_1^B)\left(\overline{R} - f_1\right) &> 0 \end{aligned}$$

This holds since $\overline{R} > 1$ by assumption. ■

A.5. Proof of Proposition 2

Proof. The proof is based on the full characterization of date-1 equilibrium in Proposition 5 in Appendix A.6.

Proof of Part (A):

As in Case 1 in Proposition 5, $p_1 > F'(0)$ implies that the borrower is not selling any assets to the traditional sector, and the asset price equals the fundamental value of the asset based on the borrower's beliefs. Whenever $p_1 < F'(0)$ in equilibrium, the asset is sold to the traditional sector in a positive amount, i.e. there are fire sales, as more sales to the traditional sector depresses prices as $F'(a_1^T) < F'(0)$.

Proof of Part (B):

By (17), π_1^L increases as s_1^L decreases. Also, q_1 is decreasing in π_1^L and τ_1^L by (2), implying the first half of the statement.

In Case 2 in Proposition 5, lower q_1 results in larger a_1^T and lower p_1, as

$$q_1 + \frac{c_0 - q_1 a_0^B - \kappa_0^B}{a_1^T}$$

is decreasing in q_1 because $a_0^B \geq a_1^T$, and

$$p_1 = F'(a_1^T) = q_1 + \frac{c_0 - q_1 a_0^B - \kappa_0^B}{a_1^T}.$$

Therefore, a decrease in q_1 will increase the liquidity shortage of the borrower, leading to a larger amount of fire sales and lower asset price.

Moreover, the borrower's posterior belief π_1^B is decreasing in q_1 by Proposition 1. In Case 3 in Proposition 5, higher π_1^B results in lower price through this belief channel. This is because the borrower's lower valuation of the asset should be met by the lower marginal valuation of the traditional sector through the decrease in $F'(a_1^T)$, i.e. larger a_1^T.

Proof of Part (C):

In Case 2 in Proposition 5, the borrower sells the asset because the borrower has to repay the date-0 debt contract and their cash holdings are not sufficient as $q_1 a_0^B < c_0^B - \kappa_0^B$. However, the borrower still believes the fundamental value of the asset is above the marginal valuation of the traditional sector evaluated at 0, $F'(0)$, which is always above the market price as $F'(0) > F'(a_1^T) = p_1$. Therefore, we have $p_1^B > F'(0) > p_1$ in a liquidity driven fire sale. In Cases 3 and 4 in Proposition 5, the borrower's valuation of the asset is below the marginal valuation of the traditional sector evaluated at 0, as $p_1^B < F'(0)$, which is why the borrower sells the asset, i.e. belief driven fire sale. In Case 3, the borrower has an interior solution, so that $p_1^B = F'(a_1^T) = p_1$. However, in Case 4, the borrower values the asset even less than the traditional sector does, implying that $p_1^B < F'(a_0^B) = p_1$. ■

A.6. Full Characterization of Date-1 Equilibrium

In the Normal Regime, we have

$$a_1^T, \xi_{d_1}^L, \xi_{c_1}^B, \xi_{a_1}^B, \xi_{\kappa_1}^L, \xi_{\kappa_1}^B = 0$$

$$q_1 = \frac{E_1^L\left[R_2^d(f_1)\right]}{\tau_1^L}$$

$$p_1 = \frac{E_1^B[R] - E_1^B\left[R_2^d(f_1)\right]}{\tau_1^B} + q_1$$

$$a_1^B = c_1^B = d_1^L = a_0^B$$

$$\kappa_1^B = q_1 a_0^B - c_0^B + \kappa_0^B$$

$$\kappa_1^L = \kappa_0^L + d_0^L - q_1 a_0^B$$

$$\kappa_1^T = \kappa_0^T - p_1 a_1^T$$

$$C_2^B = \tau_1^B \kappa_1^B + a_1^B R - a_1^B R_2^d(f_1)$$

$$C_2^L = \tau_1^L \kappa_1^L + R_2^d d_1^L$$

$$C_2^T = \kappa_1^T + F(a_1^T)$$

$$\mu_1^B = \left(\tau_1^B + \xi_{\kappa_1}^B\right) p_1 - E_1^B[R] - \xi_{a_1}^B$$

$$\mu_1^T = F'(0) - p_1$$

$$\pi_1^L = Pr\left(R = \underline{R}|s_1^L, I_0\right) = \frac{\pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right)}{(1-\pi_0)\lambda_\varepsilon\left(\varepsilon_1^L = s_1^L - \overline{R}\right) + \pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right)}$$

$$\pi_1^B = \int_0^1 \pi \lambda_T \left(\frac{1-(1-\underline{R})\pi}{q_1}\right) \mathbb{1}\left\{\underline{\tau} < \frac{1-(1-\underline{R})\pi}{q_1} < \bar{\tau}\right\} dG_\pi(\pi)$$

In the Fire Sale Regime, we have

$$\mu_1^T, \xi_{c_1}^B, \xi_{a_1}^B, \xi_{d_1}^L, \xi_{\kappa_1}^L, \xi_{\kappa_1}^B = 0$$

$$q_1 = \frac{E_1^L\left[R_2^d(f_1)\right]}{\tau_1^L}$$

$$p_1 = F'(a_1^T)$$

$$a_1^B = a_0^B - F'^{-1}\left(\frac{E_1^B[R] - E_1^B\left[R_2^d(f_1)\right]}{\tau_1^B} + \frac{E_1^L\left[R_2^d(f_1)\right]}{\tau_1^L}\right)$$

$$a_1^T = a_0^B - a_1^B$$

$$c_1^B = d_1^L = a_1^B$$

$$\kappa_1^B = q_1 a_1^B - c_0^B + \kappa_0^B - p_1\left(a_1^B - a_0^B\right)$$

$$\kappa_1^L = d_0^L - q_1 d_1^L + \kappa_0^L$$

$$\kappa_1^T = \kappa_0^T - p_1 a_1^T$$

$$C_2^B = \tau_1^B \kappa_1^B + a_1^B R - a_1^B R_2^d(f_1)$$

$$C_2^L = \tau_1^L \kappa_1^L + R_2^d d_1^L$$

$$C_2^T = \kappa_1^T + F(a_1^T)$$

$$\mu_1^B = \tau_1^B p_1 - E_1^B[R]$$

$$\pi_1^L = Pr\left(R = \underline{R} | s_1^L, I_0\right) = \frac{\pi_0 \lambda_\varepsilon\left(\varepsilon_1^L = s_1^L - \underline{R}\right)}{(1-\pi_0)\lambda_\varepsilon\left(\varepsilon_1^L = s_1^L - \overline{R}\right) + \pi_0 \lambda_\varepsilon\left(\varepsilon_1^L = s_1^L - \underline{R}\right)}$$

$$\pi_1^B = \int_0^1 \pi \lambda_T \left(\frac{1-(1-\underline{R})\pi}{q_1}\right) \mathbb{1}\left\{\underline{\tau} < \frac{1-(1-\underline{R})\pi}{q_1} < \bar{\tau}\right\} dG_\pi(\pi)$$

Below is the full solution of the date-1 equilibrium and derivation.

Proposition 5. *The equilibrium at date 1 can be characterized as the following:*

1. ***Normal Regime:*** *If* $F'(0) < \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}$ *and* $q_1 a_0^B \geq c_0^B - \kappa_0^B$*, then* $c_1 = a_1^B = a_0^B = d_1^L$*,* $a_1^T = 0$*,* $\kappa_1^B = q_1 a_0^B - c_0^B + \kappa_0^B$*, and there will be no fire-sales in the market and the asset price is*

$$p_1 = \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}$$

   or any number less than or equal to p_1 *if* $q_1 a_0^B - c_0^B = \kappa_0^B$*, which implies* $\kappa_1^B = 0$*.*

2. ***Fire Sale Regime with the borrower holding only the risky asset:*** *If* $F'(0) < \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}$ *but* $q_1 a_0^B < c_0^B - \kappa_0^B$*, then* $c_1 = a_1^B = a_0^B - a_1^T = d_1^L$*,* a_1^T

is determined by

$$F'(a_1^T) = q_1 + \frac{c_0 - q_1 a_0^B - \kappa_0^B}{a_1^T},$$

$\kappa_1^B = 0$*, and the asset price is* $p_1 = F'(a_1^T)$*.*

3. ***Fire Sale Regime with the borrower at interior solution in portfolio choice:*** *If* $F'(0) > \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L} > F'(a_0^B)$ *holds, then* $c_1 = a_1^B = d_1^L$ *is determined by*

$$F'(a_0^B - c_1) = \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L},$$

   with $a_1^T = a_0^B - c_1$*,* $p_1 = F'(a_0^B - c_1) = \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}$ *and* $\kappa_1^B = q_1 a_1^B - c_0^B + \kappa_0^B$*.*

4. ***Fire Sale Regime with market collapse (B holds only cash):*** *If* $F'(a_0^B) > \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}$*, then* $c_1 = a_1^B = d_1^L = 0$*,* $a_1^T = a_0^B$*, and* $\kappa_1^B = p_1 a_0^B - c_0^B + \kappa_0^B$ *with* $p_1 = F'(a_0^B)$*, so all the assets are sold to T and there will be no debt contract between B and L.*

Proof. From Lemma 1, we know that the collateral constraint is binding, and the asset price is greater than the contract price. Now we solve for the equilibrium price p_1.

Case 1. Consider the case in which B holds a positive amount of assets, $a_1 = c_1 > 0$. The reservation asset price p_1^B for B that makes B indifferent between purchasing the asset and holding cash is

$$p_1^B = \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}. \tag{43}$$

Case 1.1. If B is not selling any of its asset due to budget constraint, which is possible only if

$$q_1 a_0^B + \kappa_0^B \geq c_0,$$

then equilibrium price becomes $p_1 = p_1^B$, because otherwise T will be the only marginal buyer of the asset, which implies $a_1 = 0$, a contradiction.

Case 1.2. If B lacks cash to repay the debt as

$$q_1 a_0^B + \kappa_0^B < c_0,$$

then the asset price can be lower than p_1^B and $\xi_{\kappa_1}^B > 0$ as

$$p_1^B > p_1 = \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B + \xi_{\kappa_1}^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L} = F'(a_1^T),$$

because B liquidates some of the assets to T in order to pay the debt to L even though B prefers not to do so. Therefore, the asset price could be lower than the marginal return of the asset in such equilibrium and determined by the traditional sector's inverse demand. B will liquidate the assets up to the necessary amount needed to match the budget constraint with $\kappa_1^B = 0$,

$$a_1^T F'(a_1^T) = c_0 - q_1(a_0^B - a_1^T) - \kappa_0^B$$

in such equilibrium. Rearranging the terms yields

$$F'(a_1^T) = q_1 + \frac{c_0 - q_1 a_0^B - \kappa_0^B}{a_1^T}, \tag{44}$$

and $p_1 = F'(a_1^T)$. Note that there exists a unique a_1^T in this case, because F' is strictly decreasing.

Finally, the budget constraint determines the cash holdings as

$$\kappa_1^B = q_1 a_0^B - c_0^B + \kappa_0^B,$$

that pins down the optimal decision vector of B if $a_1^T = 0$. If $a_1^T > 0$, the marginal buyer will be T, and the price of the asset will be

$$p_1 = F'(a_0^B - c_1). \tag{45}$$

Combining (45) with (40) yields

$$F'(a_0^B - c_1) = \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}. \tag{46}$$

Therefore, if B is selling a positive amount of asset, the traditional sector's inverse demand function will pin down the asset price as well as the fire-sale amount. If either π_1^L is high or τ_1^L is high, then the right-hand side of (46) decreases and $a_0^B - c_1$ increases, meaning B sells more asset to T.

In the case that $F'(a_0^B) \geq \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}$, then B sells all the asset holdings to T and B does not borrow from L.

On the contrary, if

$$F'(0) < \frac{(1-\pi_1^B)(\bar{R}-1)}{\tau_1^B} + \frac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L},$$

then $c_1 = a_1^B = a_0^B$ assuming $c_0 = a_0^B$, and asset price p_1 is indeterminant as no agent is buying or selling in a positive amount and any value above $F'(0)$ is possible. ■

A.7. Proof of Proposition 3

Given the date-1 equilibrium, we can derive equilibrium at date 0. We assume $F'(a) = \alpha$ for all $a \geq 0$ from now on. This assumption is for tractability of the solution because otherwise the equilibrium is determined by implicit functions. The only case that disappears with this assumption is case 3 in Proposition 5, which is an intermediate equilibrium between full fire sales and zero fire sales.

Proof.

A.7.1. Lender's Problem at Date 0

In Section 6, we showed that when the lender, L, tightens funding liquidity, L's return for each unit of cash invested (either as cash or lending) is always τ_1^L regardless of the realization of (s_1^L, τ_1^L), because L always holds extra cash at date 1.

Taking date 1 equilibrium outcomes as given, L's problem at date 0 is

$$\max_{\kappa_0^L, d_0^L} \; E_0^L\left[\tau_1^L(\kappa_0^L + d_0^L)\right]$$

$$s.t. \quad \kappa_0^L + q_0 d_0^L \leq e_0^L$$

with non-negativity constraints,

$$\kappa_0^L \geq 0,\; d_0^L \geq 0.$$

Because L's budget constraint binds at the optimum, substitute $\kappa_0^L = e_0^L - q_0 d_0^L$. The first-order condition with respect to the only choice variable, d_0^L, is

$$E_0^L\left[\tau_1^L(1-q_0)\right] + \xi_{d_0}^L = 0. \tag{47}$$

By assumption 1, $d_0^L > 0$ in equilibrium. Therefore, the contract price that makes L indifferent across any amount of d_0^L is

$$q_0 = 1. \tag{48}$$

A.7.2. Borrower's Problem at Date 0

For each realization of q_1 at date 1, the borrower, B, takes the equilibrium market outcome, including B's own optimal decisions, as given. As we have seen in Proposition 5, there are different regimes under different realization of q_1. B accounts for that and solves the following optimization problem:

$$\begin{aligned}
\max_{c_0, a_0, \kappa_0^B} \quad & \underbrace{E_0^B \left[E_1 \left[C_2^B \middle| q_1 \right] \right]}_{expected\ utility\ for\ each\ realized\ q_1} \qquad (49) \\
= & \underbrace{E_0^B \left[\tau_1^B \left(\kappa_0^B + q_1 a_0 - c_0 \right) + \left(1 - \pi_1^B(q_1)\right) a_0 (\bar{R} - 1) \middle| q_1 \geq \bar{q}_1 \right] \Pr_0^B(q_1 \geq \bar{q}_1)}_{no\ liquidation\ case} \\
+ & \underbrace{E_0^B \left[\left(a_0 - \frac{c_0 - \kappa_0^B - q_1 a_0}{\alpha - q_1} \right) \left(1 - \pi_1^B(q_1)\right) (\bar{R} - 1) \middle| \underline{q_1} \leq q_1 < \bar{q}_1 \right] \Pr_0^B(\underline{q_1} \leq q_1 < \bar{q}_1)}_{sells\ the\ asset\ to\ pay\ debt\ but\ no\ sales\ due\ to\ pessimism} \\
+ & \underbrace{E_0^B \left[\tau_1^B \left(\kappa_0^B - c_0 + \alpha a_0^B \right) \middle| q_1 < \underline{q_1} \right] \Pr_0^B(q_1 < \underline{q_1})}_{liquidates\ all\ the\ assets}
\end{aligned}$$

$$\begin{aligned}
\text{s.t. } & e_0^B \geq p_0 a_0^B - q_0 c_0 + \kappa_0^B, \\
& a_0^B \geq c_0, \\
& \pi_1^B(q_1) = \int_0^1 \pi \lambda_T \left(\frac{1 - (1 - \underline{R})\pi}{q_1} \right) \mathbb{1} \left\{ \underline{\tau} < \frac{1 - (1 - \underline{R})\pi}{q_1} < \bar{\tau} \right\} dG_\pi(\pi), \\
& \underbrace{\bar{q}_1 = \frac{c_0 - \kappa_0^B}{a_0}}_{cutoff\ for\ liquidity\ induced\ fire\ sales}, \\
& \underbrace{\alpha = \frac{(1 - \pi_1^B(\underline{q_1}))(\bar{R} - 1)}{\tau_1^B} + \underline{q_1}}_{cutoff\ for\ belief\ induced\ fire\ sales},,
\end{aligned}$$

with non-negativity constraints,

$$c_0 \geq 0, a_0^B \geq 0, \kappa_0^B \geq 0.$$

There are three different regimes for B depending on the realization of q_1, which corresponds to each expected utility representation in the above optimization problem:

1. If $q_1 \geq \bar{q}_1$, then B will be able to roll over the debt from date 0 and keep all the assets.
2. If $\underline{q}_1 \leq q_1 < \bar{q}_1$, then B does not have enough cash to pay the promised debt amount but still optimistic enough to hold the assets as much as possible for the given prices. Therefore, B does liquidity-induced fire sales but sells no more than the necessary amount.
3. If $q_1 < \underline{q}_1$, then B is pessimistic about the asset payoff, so T values the asset more than B. Therefore, there will be belief-induced fire sales. B will sell all the assets to T.

We first confirm that the cutoff for belief-induced fire sales is above the level of q_1 that requires full sales of assets due to liquidity. This is because B can always sell all assets and repay the debt by assumption 2, and thus, any q_1 higher than $\underline{q}_1$ will be enough for B to repay the debt while holding a positive amount of asset.

Lemma 6. *If $q_1 \geq \underline{q}_1$, B can hold a positive amount of assets $c_1 > 0$ while repaying the debt to L in full.*

Proof. Recall that the amount of fire sales is

$$\frac{c_0 - \kappa_0^B - q_1 a_0}{\alpha - q_1},$$

which is decreasing in q_1 because

$$\frac{\partial \left(\dfrac{c_0 - \kappa_0^B - q_1 a_0}{\alpha - q_1} \right)}{\partial q_1} = \frac{-a_0(\alpha - q_1) + c_0 - \kappa_0^B - q_1 a_0}{(\alpha - q_1)^2} = \frac{-a_0 \alpha + c_0 - \kappa_0^B}{(\alpha - q_1)^2} < 0,$$

where the last inequality comes from assumption 2. ■

Denote the Lagrangian multiplier for the collateral constraint as μ. Substituting out κ_0^B and $\overline{q}_1$ using the binding budget constraint and the cutoff equation yields

$$
\begin{aligned}
L_0^B = &\int_{\frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0}}^{1/\underline{\tau}} \left[\tau_1^B \left(e_0^B + q_0 c_0 - p_0 a_0 + q_1 a_0 - c_0\right) + \left(1 - \pi_1^B(q_1)\right) a_0(\overline{R}-1)\right] dH(q_1) \\
&+ \int_{\underline{q}_1}^{\frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0}} \left(a_0 - \frac{c_0 - (e_0^B + q_0 c_0 - p_0 a_0) - q_1 a_0}{\alpha - q_1}\right)\left(1 - \pi_1^B(q_1)\right)(\overline{R}-1) dH(q_1) \\
&+ \int_{\underline{R}/\overline{\tau}}^{\underline{q}_1} \tau_1^B \left(e_0^B + q_0 c_0 - p_0 a_0 - c_0 + \alpha a_0^B\right) dH(q_1) \\
&+ \mu(a_0 - c_0) + \xi_{c_0} c_0 + \xi_{a_0} a_0 + \xi_{\kappa_0^B}(e_0^B + q_0 c_0 - p_0 a_0),
\end{aligned}
\tag{50}
$$

where $H(\cdot)$ is B's subjective distribution function of q_1. Then, we can apply the Leibniz integral rule for derivatives of the Lagrangian function. The first-order conditions of B's optimization problem are

FOC for c_0:

$$
\begin{aligned}
&-\frac{1-q_0}{a_0}\left[\tau_1^B\left(e_0^B + q_0 c_0 - p_0 a_0 + \frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0} a_0 - c_0\right)\right. \\
&\left. + \left(1 - \pi_1^B\left(\frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0}\right)\right) a_0(\overline{R}-1)\right] \\
&- \int_{\frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0}}^{1/\underline{\tau}} \tau_1^B (1-q_0) dH(q_1) \\
&+ \frac{1-q_0}{a_0}\left(a_0 - \frac{c_0 - (e_0^B + q_0 c_0 - p_0 a_0) - \frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0} a_0}{\alpha - \frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0}}\right) \\
&\times \left(1 - \pi_1^B\left(\frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0}\right)\right)(\overline{R}-1) \\
&- \int_{\underline{q}_1}^{\frac{(1-q_0)c_0 - e_0^B + p_0 a_0}{a_0}} \frac{(1-q_0)}{\alpha - q_1}\left(1 - \pi_1^B(q_1)\right)(\overline{R}-1) dH(q_1) \\
&- \int_{\underline{R}/\overline{\tau}}^{\underline{q}_1} \tau_1^B (1-q_0)\, dH(q_1) - \mu + \xi_{c_0} + \xi_{\kappa_0^B} q_0 = 0,
\end{aligned}
\tag{51}
$$

FOC for a_0:

$$
\begin{aligned}
&-\frac{p_0a_0-\left((1-q_0)c_0-e_0^B+p_0a_0\right)}{a_0^2}\left[\tau_1^B\left(e_0^B+q_0c_0-p_0a_0+\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}a_0-c_0\right)\right. \\
&\left.+\left(1-\pi_1^B\left(\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}\right)\right)a_0(\overline{R}-1)\right] \\
&+\int_{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}^{1/\underline{\tau}}\left[\tau_1^B(q_1-p_0)+\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)\right]dH(q_1) \\
&+\frac{p_0a_0-\left((1-q_0)c_0-e_0^B+p_0a_0\right)}{a_0^2}\left(a_0-\frac{c_0-(e_0^B+q_0c_0-p_0a_0)-\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}a_0}{\alpha-\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}\right) \\
&\times\left(1-\pi_1^B\left(\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}\right)\right)(\overline{R}-1) \\
&+\int_{\underline{q}_1}^{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}\left(1-\frac{p_0-q_1}{\alpha-q_1}\right)\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)dH(q_1) \\
&+\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\tau_1^B(\alpha-p_0)\,dH(q_1)+\mu+\xi_{a_0}-\xi_{\kappa_0^B}p_0=0,
\end{aligned}
\tag{52}
$$

A.7.3. Equilibrium Trade and Prices

FOC for c_0, (51), can be further simplified as

$$
\begin{aligned}
&-\frac{1-q_0}{a_0}\left(1-\pi_1^B\left(\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}\right)\right)a_0(\overline{R}-1)\\
&-\int_{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}^{1/\underline{\tau}}\tau_1^B(1-q_0)dH(q_1)-\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\tau_1^B(1-q_0)\,dH(q_1)\\
&+\frac{1-q_0}{a_0}\left(1-\pi_1^B\left(\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}\right)\right)a_0(\overline{R}-1)\\
&-\int_{\underline{q}_1}^{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}\frac{(1-q_0)}{\alpha-q_1}\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)dH(q_1)-\mu+\xi_{c_0}+\xi_{\kappa_0^B}q_0\\
=&-\int_{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}^{1/\underline{\tau}}\tau_1^B(1-q_0)dH(q_1)-\int_{0}^{\underline{q}_1}\tau_1^B(1-q_0)\,dH(q_1)\\
&-\int_{\underline{q}_1}^{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}\frac{(1-q_0)}{\alpha-q_1}\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)dH(q_1)-\mu+\xi_{c_0}+\xi_{\kappa_0^B}q_0\\
&=-\mu+\xi_{c_0}+\xi_{\kappa_0^B}=0,
\end{aligned}
$$

and the second to last inequality holds because of $q_0=1$, which is from (48).

If B does not borrow at all $c_0=0$, then the collateral constraint should be binding, implying $a_0=c_0=0$. If B is borrowing a positive amount as $c_0>0$, then $\xi_{c_0}=0$ and there can be two different cases. Consider the first case in which $\xi_{\kappa_0^B}$ positive, implying $\kappa_0^B=0$ and $\mu=\xi_{\kappa_0^B}q_0>0$. Thus, the collateral constraint is also binding and $c_0=a_0$. In the second case in which $\xi_{\kappa_0^B}$ is zero, μ should also be zero, implying that the collateral constraint is not binding and B purchases some assets without leverage as $a_0>c_0$.

The result holds because of the linearity of B's utility. If B holds a positive amount of cash, B should be indifferent between holding more or less cash. However, if the return of purchasing more assets by borrowing more from L at date 0 at the cost of less amount of cash at date 1 exceeds the cash return, B borrows with full capacity and spends all the cash to purchase the asset.

FOC for a_0, (52), can be further simplified as

$$
\begin{aligned}
&-\frac{p_0 a_0-\left((1-q_0)c_0-e_0^B+p_0 a_0\right)}{a_0^2}\left[\left(1-\pi_1^B\left(\frac{(1-q_0)c_0-e_0^B+p_0 a_0}{a_0}\right)\right)a_0(\overline{R}-1)\right] \\
&+\int_{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}^{1/\underline{\tau}}\left[\tau_1^B\left(q_1-p_0\right)+\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)\right]dH(q_1) \\
&+\frac{p_0 a_0-\left((1-q_0)c_0-e_0^B+p_0 a_0\right)}{a_0^2}\left(1-\pi_1^B\left(\frac{(1-q_0)c_0-e_0^B+p_0 a_0}{a_0}\right)\right)a_0(\overline{R}-1) \\
&+\int_{\underline{q}_1}^{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}\left(\frac{\alpha-p_0}{\alpha-q_1}\right)\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)dH(q_1) \\
&+\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\tau_1^B\left(\alpha-p_0\right)dH(q_1)+\mu+\xi_{a_0}-\xi_{\kappa_0^B}p_0 \\
=&\int_{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}^{1/\underline{\tau}}\left[\tau_1^B\left(q_1-p_0\right)+\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)\right]dH(q_1) \\
&+\int_{\underline{q}_1}^{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}\left(\frac{p_0-\alpha}{q_1-\alpha}\right)\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)dH(q_1) \\
&+\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\tau_1^B\left(\alpha-p_0\right)dH(q_1)+\mu+\xi_{a_0}-\xi_{\kappa_0^B}p_0=0.
\end{aligned}
$$

We finally derive the date-0 equilibrium allocation and prices.

From the FOCs for c_0 and a_0,

$$
\begin{aligned}
&\int_{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}^{1/\underline{\tau}}\left[\tau_1^B\left(q_1-p_0\right)+\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)\right]dH(q_1) \\
&+\int_{\underline{q}_1}^{\frac{(1-q_0)c_0-e_0^B+p_0a_0}{a_0}}\left(\frac{p_0-\alpha}{q_1-\alpha}\right)\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)dH(q_1) \\
&+\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\tau_1^B\left(\alpha-p_0\right)dH(q_1)+\xi_{c_0}+\xi_{a_0}-\xi_{\kappa_0^B}(p_0-1)=0.
\end{aligned}
$$

From the borrower's optimality condition under $a_0=c_0>0$, B holds zero amount of cash, $\kappa_0^B=0$.

Under this case, B's return from this leveraged asset purchase becomes

$$\begin{aligned}\xi_{\kappa_0^B} = \Bigg[& \int_1^{1/\underline{\tau}} \left[\tau_1^B (q_1 - p_0) + \left(1 - \pi_1^B(q_1)\right)(\overline{R} - 1)\right] dH(q_1) \\ & + \int_{\underline{q}_1}^1 \left(\frac{p_0 - \alpha}{q_1 - \alpha}\right)\left(1 - \pi_1^B(q_1)\right)(\overline{R} - 1) dH(q_1) \\ & + \int_{\underline{R}/\overline{\tau}}^{\underline{q}_1} \tau_1^B (\alpha - p_0)\, dH(q_1)\Bigg] \Big/ (p_0 - 1).\end{aligned} \tag{53}$$

It is sufficient to show that $\xi_{\kappa_0^B} > \tau_1^B$, so B prefers to purchase the asset with leverage to holding cash.

If B uses up all the cash endowments to purchase the asset with leverage, the asset price is

$$p_0 = \frac{A + e_0^B}{A}.$$

The price should be higher than T's willingness-to-pay for the asset, which is also above the debt payment amount at date 1, so B is able to repay the debt in full if B wants to liquidate the assets. Therefore,

$$p_0 = \frac{A + e_0^B}{A} > \alpha \geq 1. \tag{54}$$

From assumption 1,

$$\begin{aligned} & e_0^B \tau_1^B < A(E_0^B[R] - 1) \\ & \Rightarrow \tau_1^B < \frac{A}{e_0^B}(E_0^B[R] - 1) = \frac{E_0^B[R] - 1}{p_0 - 1},\end{aligned} \tag{55}$$

where the last equality comes from (54). Hence, if the marginal return of purchasing the asset with leverage, $\xi_{\kappa_0^B}$, is greater than or equal to $\dfrac{E_0^B[R] - 1}{p_0 - 1}$, then $a_0 = c_0 = A$ and $\kappa_0^B = 0$ is the equilibrium portfolio of B. The difference between the two returns is

$$\begin{aligned}\xi_{\kappa_0^B} - (E_0^B[R] - 1) = & \int_1^{1/\underline{\tau}} \left[\pi_1^B(q_1)(1 - \underline{R}) + \tau_1^B (q_1 - p_0)\right] dH(q_1) \\ & + \int_{\underline{q}_1}^1 \left[\pi_1^B(q_1)(1 - \underline{R}) - \left(\frac{p_0 - q_1}{\alpha - q_1}\right)\left(1 - \pi_1^B(q_1)\right)(\overline{R} - 1)\right] dH(q_1) \\ & + \int_{\underline{R}/\overline{\tau}}^{\underline{q}_1} \left[\pi_1^B(q_1)(1 - \underline{R}) + \tau_1^B(\alpha - p_0) - (1 - \pi_1^B(q_1))(\overline{R} - 1)\right] dH(q_1).\end{aligned} \tag{56}$$

We will show that this difference is positive. Because $\dfrac{(1-\pi_1^B(q))(\overline{R}-1)}{\tau_1^B}+q$ is increasing in q and $\alpha = \dfrac{(1-\pi_1^B(\underline{q}_1))(\overline{R}-1)}{\tau_1^B}+\underline{q}_1 \geq 1$,

$$\alpha - q < \frac{(1-\pi_1^B(q))(\overline{R}-1)}{\tau_1^B}$$

for any $q \in [\underline{q}_1, 1]$. Thus, we have

$$\begin{aligned}
&\int_{\underline{q}_1}^{1}\left[\pi_1^B(q_1)(1-\underline{R})-\left(\frac{p_0-q_1}{\alpha-q_1}\right)\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)\right]dH(q_1)\\
>&\int_{\underline{q}_1}^{1}\left[\pi_1^B(q_1)(1-\underline{R})-\left(\tau_1^B\frac{p_0-q_1}{\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)}\right)\left(1-\pi_1^B(q_1)\right)(\overline{R}-1)\right]dH(q_1)\\
=&\int_{\underline{q}_1}^{1}\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B(q_1-p_0)\right]dH(q_1).
\end{aligned} \tag{57}$$

Again, because $\alpha = \dfrac{(1-\pi_1^B(\underline{q}_1))(\overline{R}-1)}{\tau_1^B}+\underline{q}_1$ and $(1-\pi_1^B(q))$ is increasing in q,

$$\begin{aligned}
&\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B(\alpha-p_0)-(1-\pi_1^B(q_1))(\overline{R}-1)\right]dH(q_1)\\
>&\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B\left(\frac{(1-\pi_1^B(\underline{q}_1))(\overline{R}-1)}{\tau_1^B}+\underline{q}_1-p_0\right)-(1-\pi_1^B(\underline{q}_1))(\overline{R}-1)\right]dH(q_1)\\
>&\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B(q_1-p_0)\right]dH(q_1).
\end{aligned} \tag{58}$$

Combining (56), (57), and (58) implies

$$\begin{aligned}
\xi_{\kappa_0^B}-(E_0^B[R]-1) >& \int_1^{1/\underline{\tau}}\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B(q_1-p_0)\right]dH(q_1)\\
&+\int_{\underline{q}_1}^{1}\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B(q_1-p_0)\right]dH(q_1)\\
&+\int_{\underline{R}/\overline{\tau}}^{\underline{q}_1}\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B(q_1-p_0)\right]dH(q_1)\\
=&\int\left[\pi_1^B(q_1)(1-\underline{R})+\tau_1^B q_1-\tau_1^B p_0\right]dH(q_1)\\
=&\pi_0(1-\underline{R})+\tau_1^B E_0^B[q_1]-\tau_1^B p_0.
\end{aligned} \tag{59}$$

Recall that $q_1 = \dfrac{(1-\pi_1^L)+\pi_1^L \underline{R}}{\tau_1^L}$ from (2), which implies

$$E_0^B[q_1] = E_0^B\left[\frac{(1-\pi_1^L)+\pi_1^L\underline{R}}{\tau_1^L}\right] = E_0^B\left[(1-\pi_1^L)+\pi_1^L\underline{R}\right] E_0^B\left[\frac{1}{\tau_1^L}\right] = \left[(1-\pi_0)+\pi_1^0\underline{R}\right] E_0^B\left[\frac{1}{\tau_1^L}\right],$$

where the last two equalities hold because of the independence between τ_1^L and s_1^L and the law of iterated expectations, respectively. Because $f(x)=1/x$ is a convex function,

$$E_0^B\left[\frac{1}{\tau_1^L}\right] > \frac{1}{E_0^B\left[\tau_1^L\right]} = \frac{1}{\tau_0^L}$$

by Jensen's inequality. Finally, from (55), $E_0^B[R] > \tau_1^B p_0$.

Therefore, (59) becomes

$$\begin{aligned}\xi_{\kappa_0^B} - (E_0^B[R]-1) &> \pi_0(1-\underline{R}) + \tau_1^B E_0^B[q_1] - \tau_1^B p_0 \\ &> \pi_0(1-\underline{R}) + (1-\pi_0) + \pi_0\underline{R} - (1-\pi_0)\overline{R} - \pi_0\underline{R} = 0.\end{aligned}$$

Therefore, the return from investing in the asset with leverage $\xi_{\kappa_0^B} >$ is greater than the return from cash holdings τ_1^B. Finally, this also shows that the equilibrium with $\kappa_0^B = 0$ is the only equilibrium. If $\kappa_0^B > 0$, then the price of the asset p_0 will be even lower, increasing the asset return even higher. ■

A.8. Proof of Lemma 2

Proof. We show how learning amplifies misallocation during a fire sale for any q_1 (i.e. shock pair (π_1^L, τ_1^L)), relative to a benchmark in which beliefs don't change—that is, agents have private information but do not update their beliefs in response to new information. Note that in the Fire Sale Regime, we have

$$a_1^B = a_0^B - F'^{-1}\left(\frac{E_1^B[R] - E_1^B\left[R_2^d\right]}{\tau_1^B} + \frac{E_1^L\left[R_2^d\right]}{\tau_1^L}\right) \tag{60}$$

$$p_1 = F'(a_1^T)$$

Since $F'(\cdot)$ is monotonically decreasing, $F'^{-1}(\cdot)$ is also monotonically decreasing. Therefore, holding the borrower's beliefs constant for a moment, a lower $q_1 = \frac{E_1^L[R_2^d]}{\tau_1^L}$ implies that a_1^B is lower. Note that the spread $E_1^B[R] - E_1^B\left[R_2^d\right] = \left(1-\pi_1^B\right)\left(\overline{R}-1\right)$ is decreasing in π_1^B (since $\overline{R} > 1$ by assumption). A lower q_1 implies that π_1^B is higher (B is more pessimistic) by Proposition 1, which also makes a_1^B lower. Thus, a lower q_1 leads to a lower a_1^B in two ways: the direct effect of a lower q_1 on a_1^B and the effect of lower q_1 on a_1^B through greater pessimism and a lower spread.

Since a larger decrease in funding liquidity leads to lower a_1^B, this also implies that a_1^T is higher and hence p_1 is lower. Hence, the fire sale is more severe in the Fire Sale Regime when the funding illiquidity is more severe. Hence, pessimism/the information externality is associated with greater misallocation/worse fire sales during a fire sale. ■

A.9. Date-1 Equilibrium under Common Information Benchmark

In the Normal Regime under the Common Information Benchmark we have

$$a_1^T, \xi_{d_1}^L, \xi_{c_1}^B, \xi_{a_1}^B, \xi_{\kappa_1}^L, \xi_{\kappa_1}^B = 0$$

$$q_1 = \frac{E_1\left[R_2^d(f_1)\right]}{\tau_1^L}$$

$$p_1 = \frac{E_1[R] - E_1\left[R_2^d(f_1)\right]}{\tau_1^B} + q_1$$

$$a_1^B = c_1^B = d_1^L = a_0^B$$

$$\kappa_1^B = q_1 a_0^B - c_0^B + \kappa_0^B$$

$$\kappa_1^L = \kappa_0^L + d_0^L - q_1 a_0^B$$

$$\kappa_1^T = \kappa_0^T - p_1 a_1^T$$

$$C_2^B = \tau_1^B \kappa_1^B + a_1^B R - a_1^B R_2^d(f_1)$$

$$C_2^L = \tau_1^L \kappa_1^L + R_2^d d_1^L$$

$$C_2^T = \kappa_1^T + F(a_1^T)$$

$$\mu_1^B = \left(\tau_1^B + \xi_{\kappa_1}^B\right) p_1 - E_1[R] - \xi_{a_1}^B$$

$$\mu_1^T = F'(0) - p_1$$

$$\pi_1 := \pi_1^L = Pr\left(R = \underline{R} | s_1^L, I_0\right) = \frac{\pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right)}{\left(1-\pi_0\right) \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \overline{R}\right) + \pi_0 \lambda_\varepsilon \left(\varepsilon_1^L = s_1^L - \underline{R}\right)}$$

$$\pi_1^B = \pi_1$$

In the Fire Sale Regime under the Common Information Benchmark, we have

$$\mu_1^T, \xi_{c_1}^B, \xi_{a_1}^B, \xi_{d_1}^L, \xi_{\kappa_1}^L, \xi_{\kappa_1}^B = 0$$

$$q_1 = \frac{E_1\left[R_2^d(f_1)\right]}{\tau_1^L}$$

$$p_1 = F'(a_1^T)$$

$$a_1^B = a_0^B - F'^{-1}\left(\frac{E_1\left[R\right] - E_1\left[R_2^d(f_1)\right]}{\tau_1^B} + \frac{E_1\left[R_2^d(f_1)\right]}{\tau_1^L}\right)$$

$$a_1^T = a_0^B - a_1^B$$

$$c_1^B = d_1^L = a_1^B$$

$$\kappa_1^B = q_1 a_1^B - c_0^B + \kappa_0^B - p_1\left(a_1^B - a_0^B\right)$$

$$\kappa_1^L = d_0^L - q_1 d_1^L + \kappa_0^L$$

$$\kappa_1^T = \kappa_0^T - p_1 a_1^T$$

$$C_2^B = \tau_1^B \kappa_1^B + a_1^B R - a_1^B R_2^d(f_1)$$

$$C_2^L = \tau_1^L \kappa_1^L + R_2^d d_1^L$$

$$C_2^T = \kappa_1^T + F\left(a_1^T\right)$$

$$\mu_1^B = \tau_1^B p_1 - E_1\left[R\right]$$

$$\pi_1 := \pi_1^L = Pr\left(R = \underline{R}|s_1^L, I_0\right) = \frac{\pi_0 \lambda_\varepsilon\left(\varepsilon_1^L = s_1^L - \underline{R}\right)}{\left(1-\pi_0\right)\lambda_\varepsilon\left(\varepsilon_1^L = s_1^L - \overline{R}\right) + \pi_0 \lambda_\varepsilon\left(\varepsilon_1^L = s_1^L - \underline{R}\right)}$$

$$\pi_1^B = \pi_1$$

A.10. Proof of Lemma 3

Here we characterize the threshold between the Normal and Crisis Regimes in the Common Information Benchmark.

Proof.

Frontier in Baseline Economy

Recall the equilibrium asset price.

$$p_1 = \frac{E_1^B\left[R\right] - E_1^B\left[R_2^d(f_1)\right]}{\tau_1^B} + q_1.$$

The threshold price is defined by asset price at threshold of Normal and Fire Sale Regimes. This threshold asset price is $\hat{p}_1 = F'(0)$ satisfying

$$\begin{aligned}
\frac{E_1^B\left[R\right] - E_1^B\left[R_2^d(f_1)\right]}{\tau_1^B} + \hat{q}_1 &= F'(0) \\
\hat{q}_1 + \frac{\left(1-\pi_1^B(\hat{q}_1)\right)\left(\overline{R} - f_1\right)}{\tau_1^B} &= F'(0) \\
\left(1-\pi_1^B(\hat{q}_1)\right)\left(\overline{R} - f_1\right) + \tau_1^B \hat{q}_1 &= \tau_1^B F'(0).
\end{aligned} \tag{61}$$

This defines the threshold value $\hat{q}_1$. Given this, the set of $\left(\tau_1^L, \pi_1^L\right)$ consistent with $\hat{q}_1$ is given by

$$\frac{\left(1-\pi_1^L\right) f_1 + \pi_1^L \underline{R}}{\tau_1^L} = \hat{q}_1.$$

Solve for π_1^L :

$$\pi_1^L = \frac{f_1 - \tau_1^L \hat{q}_1}{f_1 - \underline{R}} = \frac{1 - \tau_1^L \hat{q}_1}{1 - \underline{R}}.$$

This defines the curve of the frontier partitioning the state space into the Normal and Crisis

Regimes.

The y-intercept of the curve (when $\tau_1^L = 0$, though never occurs) is $\frac{1}{1-\underline{R}} > 1$, while the x-intercept (when $\pi_1^L = 0$) is $\frac{1}{\hat{q}_1} > 0$.

The slope of the curve is negative and constant:

$$\frac{d\pi_1^L}{d\tau_1^L} = \frac{-\hat{q}_1}{1-\underline{R}} < 0.$$

Frontier in Common Information Benchmark

When $\pi_1^B = \pi_1^L$, the threshold in benchmark is defined by

$$\left(1-\pi_1^L\right)\left(\overline{R}-f_1\right) + \tau_1^B \hat{q}_1 = \tau_1^B F'(0)$$

i.e.

$$\left(\overline{R}-f_1\right) - \left[\left(\overline{R}-f_1\right) + \frac{\tau_1^B}{\tau_1^L}\left(f_1 - \underline{R}\right)\right]\pi_1^L + \frac{\tau_1^B}{\tau_1^L} f_1 = \tau_1^B F'(0) \tag{62}$$

We now trace out the frontier of all $\left(\tau_1^L, \pi_1^L\right)$ such that this is satisfied. First, we derive the x-intercept of the frontier by supposing that $\pi_1^L = 0$:

$$\begin{aligned}
\left(\overline{R}-f_1\right) + \frac{\tau_1^B}{\tau_1^L} f_1 &= \tau_1^B F'(0) \\
\left(\overline{R}-f_1\right)\tau_1^L + \tau_1^B f_1 &= \tau_1^L \tau_1^B F'(0) \\
\tau_1^B f_1 &= \tau_1^L\left(\tau_1^B F'(0) - \overline{R} + f_1\right) \\
\tau_1^L &= \frac{\tau_1^B f_1}{\tau_1^B F'(0) - \overline{R} + f_1}.
\end{aligned}$$

Claim 1. *This x-intercept is greater than the x-intercept in the Baseline case.*

Proof. We want to show that the x-intercept of the frontier in the Common Info benchmark is larger than that in the Baseline economy, i.e.

$$\frac{\tau_1^B f_1}{\tau_1^B F'(0) - \overline{R} + f_1} > \frac{f_1}{\hat{q}_1}$$

$$\hat{q}_1 > \frac{\tau_1^B F'(0) - \left(\overline{R}-f_1\right)}{\tau_1^B}$$

where $\hat{q}_1$ is defined by (61) in the Baseline case. Recall that $\hat{q}_1$ is defined by

$$\hat{q}_1 = F'(0) - \frac{\left(1-\pi_1^B(\hat{q}_1)\right)\left(\overline{R}-f_1\right)}{\tau_1^B}$$

So we want to show that

$$F'(0) - \frac{\left(1-\pi_1^B(\hat{q}_1)\right)\left(\overline{R}-f_1\right)}{\tau_1^B} > \frac{\tau_1^B F'(0) - \left(\overline{R}-f_1\right)}{\tau_1^B}.$$

The inequality is equivalent to

$$\begin{aligned} F'(0) - \frac{\left(1-\pi_1^B(\hat{q}_1)\right)\left(\overline{R}-f_1\right)}{\tau_1^B} &> F'(0) - \frac{\left(\overline{R}-f_1\right)}{\tau_1^B} \\ \left(1-\pi_1^B(\hat{q}_1)\right)\left(\overline{R}-f_1\right) &< \left(\overline{R}-f_1\right) \\ \left(1-\pi_1^B(\hat{q}_1)\right) &< 1, \end{aligned}$$

which holds because $\pi_1^B > 0$. ■

Hence, we have that the x-intercept is larger in the Common Info Benchmark compared to the Baseline case. (Note that this automatically implies that the x-intercept is positive, since it's obviously positive in the Baseline case.)

What is the y-intercept of the frontier in the Common Info Benchmark? (i.e. what happens to π_1^L as τ_1^L approaches zero?): The frontier equation (62) can be rearranged as

$$\begin{aligned} \left(\overline{R}-f_1\right) - \left[\left(\overline{R}-f_1\right) + \frac{\tau_1^B}{\tau_1^L}\left(f_1-\underline{R}\right)\right]\pi_1^L + \frac{\tau_1^B}{\tau_1^L} f_1 &= \tau_1^B F'(0) \\ -\left[\left(\overline{R}-f_1\right) + \frac{\tau_1^B}{\tau_1^L}\left(f_1-\underline{R}\right)\right]\pi_1^L &= \tau_1^B F'(0) - \frac{\tau_1^B}{\tau_1^L} f_1 - \left(\overline{R}-f_1\right) \\ \pi_1^L &= \frac{\frac{\tau_1^B}{\tau_1^L} f_1 + \left(\overline{R}-f_1\right) - \tau_1^B F'(0)}{\left(\overline{R}-f_1\right) + \frac{\tau_1^B}{\tau_1^L}\left(f_1-\underline{R}\right)} \\ \pi_1^L &= \frac{\tau_1^B f_1 + \tau_1^L\left(\overline{R}-f_1\right) - \tau_1^L\tau_1^B F'(0)}{\tau_1^L\left(\overline{R}-f_1\right) + \tau_1^B\left(f_1-\underline{R}\right)} \\ \pi_1^L &= \frac{\tau_1^B f_1 + \tau_1^L\left(\overline{R}-f_1-\tau_1^B F'(0)\right)}{\tau_1^L\left(\overline{R}-f_1\right) + \tau_1^B\left(f_1-\underline{R}\right)}. \end{aligned}$$

The y-intercept (when $\tau_1^L = 0$) is $\frac{f_1}{f_1-\underline{R}}$. Hence, this is the same y-intercept as in the Baseline case.

The slope of this frontier is defined by taking the derivative of π_1^L with respect to τ_1^L:

$$
\begin{aligned}
\frac{\partial \pi_1^L}{\partial \tau_1^L} =& \frac{\partial \left(\frac{\tau_1^B f_1 + \tau_1^L \left(\overline{R} - f_1 - \tau_1^B F'(0)\right)}{\tau_1^L \left(\overline{R} - f_1\right) + \tau_1^B (f_1 - \underline{R})} \right)}{\partial \tau_1^L} \\
=& \frac{\left(\overline{R} - f_1 - \tau_1^B F'(0)\right)\left(\tau_1^L\left(\overline{R} - f_1\right) + \tau_1^B\left(f_1 - \underline{R}\right)\right) - \left(\overline{R} - f_1\right)\left(\tau_1^B f_1 + \tau_1^L\left(\overline{R} - f_1 - \tau_1^B F'(0)\right)\right)}{\left(\tau_1^L\left(\overline{R} - f_1\right) + \tau_1^B\left(f_1 - \underline{R}\right)\right)^2} \\
=& \frac{\left(\overline{R} - f_1 - \tau_1^B F'(0)\right)\left(\tau_1^B\left(f_1 - \underline{R}\right)\right) - \left(\overline{R} - f_1\right)\left(\tau_1^B f_1\right)}{\left(\tau_1^L\left(\overline{R} - f_1\right) + \tau_1^B\left(f_1 - \underline{R}\right)\right)^2} < 0,
\end{aligned}
$$

which holds because $\left(\overline{R} - f_1 - \tau_1^B F'(0)\right) < \left(\overline{R} - f_1\right)$ and $\tau_1^B f_1 > \tau_1^B(f_1 - \underline{R})$. Denote $\Psi \equiv \left(\overline{R} - f_1 - \tau_1^B F'(0)\right)\left(\tau_1^B\left(f_1 - \underline{R}\right)\right) - \left(\overline{R} - f_1\right)\left(\tau_1^B f_1\right)$. Also, the second derivative becomes

$$
\frac{\partial^2 \pi_1^L}{\partial \left(\tau_1^L\right)^2} = \frac{-2\left(\overline{R} - f_1\right)\left(\tau_1^L\left(\overline{R} - f_1\right) + \tau_1^B\left(f_1 - \underline{R}\right)\right)\Psi}{\left(\tau_1^L\left(\overline{R} - f_1\right) + \tau_1^B\left(f_1 - \underline{R}\right)\right)^4} > 0,
$$

where the last inequality holds by $\overline{R} > f_1 > \underline{R}$ and $\Psi < 0$. Therefore, the frontier is strictly convex with a negative slope (in our relevant domain). ■

A.11. Proof of Condition for Intersection of the Frontiers

Using the results from the proof of Lemma 3 in Appendix A.10, we derive the necessary and sufficient condition for the existence of the intersection of the frontiers in the Baseline Economy and the Common Information Benchmark.

Proof. In the Baseline Economy, the slope is always $\frac{d\pi_1^L}{d\tau_1^L} = \frac{-\hat{q}_1}{f_1 - \underline{R}} < 0$. In the Common Info Benchmark, the slope is

$$
\frac{\left(\overline{R} - f_1 - \tau_1^B F'(0)\right)\left(\tau_1^B\left(f_1 - \underline{R}\right)\right) - \left(\overline{R} - f_1\right)\left(\tau_1^B f_1\right)}{\left(\tau_1^L\left(\overline{R} - f_1\right) + \tau_1^B\left(f_1 - \underline{R}\right)\right)^2}
$$

which is increasing as shown in Lemma 3. Therefore, once the frontier in the Common Information Benchmark is at the same point $(\tau_1^{L*}, \pi_1^{L*})$ or above that point as (τ_1^{L*}, π_1^L) with $\pi_1^L \geq \pi_1^{L*}$ and has the same slope as the frontier in the Baseline Case at τ_1^{L*}, the two frontiers would never meet for any $\tau_1^L > \tau_1^{L*}$ (see Figure 6 for a graphical example).

First, suppose that $\hat{\tau}_1^L \leq \tilde{\tau}_1^L$. We claim that the two frontiers will never meet for $\pi_1^L \leq 1$. Since the two frontiers have the same y-intercept, $\dfrac{1}{1-\underline{R}}$, the average slope of the frontier in the Common Information Benchmark within the interval $[0, \hat{\tau}_1^L]$ has to be greater than that of the frontier in the Baseline Case within the same interval. Because of strict convexity, the slope of the frontier in the

Common Information Benchmark at $\hat{\tau}_1^L$ has to be greater than that of the frontier in the Baseline Case as well. Thus, the two frontiers will never meet for the states with $\pi_1^L \leq 1$.

Now suppose that the two frontiers never meet for the states with $\pi_1^L \leq 1$. We claim that $\hat{\tau}_1^L \leq \tilde{\tau}_1^L$ should hold and prove this by contradiction. Suppose the contrary, $\hat{\tau}_1^L > \tilde{\tau}_1^L$. Because the x-intercept in the Common Information Benchmark is greater than the x-intercept in the Baseline case, the frontier in the Baseline case continuously goes from $\pi_1^L = 1$ to $\pi_1^L = 0$ within an interval that is contained in the interval between $\tilde{\tau}_1^L$ and the x-intercept in the Common Information Benchmark. Since the frontier is continuously decreasing and convex, there exists a point that the two frontiers meet by the intermediate value theorem. ■

A.12. Proof of Proposition 4

Proof.

Proof of Part (A): Effects of misinformation on the allocation in the Normal Regime

By comparing a Normal Regime equilibrium in baseline with the Common Information Benchmark, we can see that in both equilibria, $a_1^B = a_0^B$. So in the Normal Regime, beliefs have no effect on the allocation of the risky asset. What they have is an effect on the allocation of cash between the lender vs the borrower, via q_1, and the asset price p_1 (which itself doesn't matter in this regime beyond ensuring $a_1^B = a_0^B$, since pecuniary externality doesn't lead to misallocation here): $\kappa_1^B = q_1 a_0^B - c_0^B + \kappa_0^B$. In both versions, $q_1 = \frac{E_1^L\left[R_2^d(f_1)\right]}{\tau_1^L}$ determiend by beliefs of the lender, which is equivalent to that of the borrower in the common information benchmark. Hence, tighter q_1 is met with cash holdings in this regime, but belief disagreements (learning mechanism) don't affect tighter q_1.

Belief disagreement does affect p_1 (since $p_1 = \frac{E_1^B[R] - E_1^B\left[R_2^d(f_1)\right]}{\tau_1^B} + q_1$), but this has no allocative consequences in this regime. Thus, belief disagreement only affect equilibrium allocation (of asset or cash) at date 1 only to the extent that they affect allocation of asset. (Beliefs in general affect allocation of cash, but this is always determined by q_1 which is pinned down by the lender's belief. So, the borrower's belief matters only to the extent that it affects desired a_1^B).

Proof of Part (B): Effect of misinformation on the severity of fire sale when Condition 1 holds

As we outlined for the Normal Regime, belief disagreement affects the equilibrium allocation (of asset or cash) at date 1 only to the extent that they affect allocation of asset. (This is because in both the baseline case and the common info benchmarket, q_1 is pinned down by the lender's beliefs in equilibrium $q_1 = \frac{E_1^L\left[R_2^d(f_1)\right]}{\tau_1^L}$). So at the margin, belief disagreements can affect the equilibrium allocation only to the extent that it affects a_1^B and/or p_1. Recall that in the Fire Sale Regime, we have (60)

$$a_1^B = a_0^B - F'^{-1}\left(\frac{E_1^B\left[R\right] - E_1^B\left[R_2^d(f_1)\right]}{\tau_1^B} + \frac{E_1^L\left[R_2^d(f_1)\right]}{\tau_1^L}\right)$$

while for the benchmark with common information, we have

$$a_1^B = a_0^B - F'^{-1}\left(\frac{E_1\left[R\right] - E_1\left[R_2^d(f_1)\right]}{\tau_1^B} + \frac{E_1^L\left[R_2^d(f_1)\right]}{\tau_1^L}\right) \tag{63}$$

Since $F'^{-1}(\cdot)$ is monotonically decreasing, a_1^B is lower in the baseline case (i.e. the identification problem makes the fire sales more severe) only if the spread $E_1^B\left[R\right] - E_1^B\left[R_2^d(f_1)\right] = \left(1-\pi_1^B\right)\left(\overline{R}-f_1\right)$ is lower. This occurs iff $\pi_1^B > \pi_1^L$. Whether this is true in equilibrium or not depends on the actual realization of π_1^L, τ_1^L given the observed q_1. So, for a given q_1, the identification problem will amplify the fire sales relative to the common info benchmark (lower a_1^B) when π_1^L is low and τ_1^L is high; while it will dampen the fire sales relative to the common info benchmark (higher a_1^B) when π_1^L is high and τ_1^L is low.

The net effect of these two forces determines the overall effect of misinformation on the severity of fire sales.

Proof of Part (C): Effect of misinformation on the severity of fire sales when Condition 1 does not hold

As we established in the proof of Part (B), a_1^B is lower in the baseline case if and only if $\pi_1^B > \pi_1^L$. As we show in the the characterization of optimism and pessimism at date 1, if Condition 1 does not hold, then $\pi_1^B > \pi_1^L$ (except for knife-edge cases, in which $\pi_1^B = \pi_1^L$). ■

A.13. Figures for Lemma 5

Figure 6 illustrates the bisection of the state space into two regimes at date 1 under the case when Condition 1 does not hold. It plots this bisection for the baseline case in which the lender's liquidity shock τ_1^L and beliefs π_1^L are private information, and under the Common Information Benchmark case in which this information is directly observable by the borrower. The equilibrium is in the Fire Sale Regime if and only the equilibrium value of p_1 is below a threshold value $\hat{p}_1$. (For the baseline case, this corresponds to a threshold value of q_1.) The solid curve in the figure plots the combinations of the states $\left(\tau_1^L, \pi_1^L\right)$ consistent the threshold $\hat{p}_1$, based on the lender's optimality condition for d_1^L, and denotes the frontier between the two regimes. The dashed curve plots the same frontier in the Common Information Benchmark in which the borrower directly observes the lender's private information, and hence $\pi_1^B = \pi_1^L$ all along this curve. For both cases, the region to the southwest of these curves is the Normal Regime, while the northeast is the Fire Sale Regime.

The dotted curve in the figure demarcates the region of the state space in which $\pi_1^B = \pi_1^L$ on the

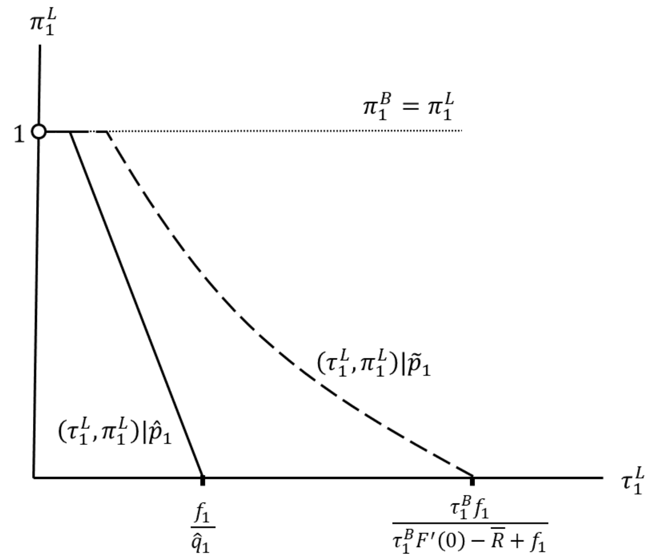

Figure 6: Two Regimes under the Baseline Case and the Common Information Benchmark when Condition 1 Does Not Hold

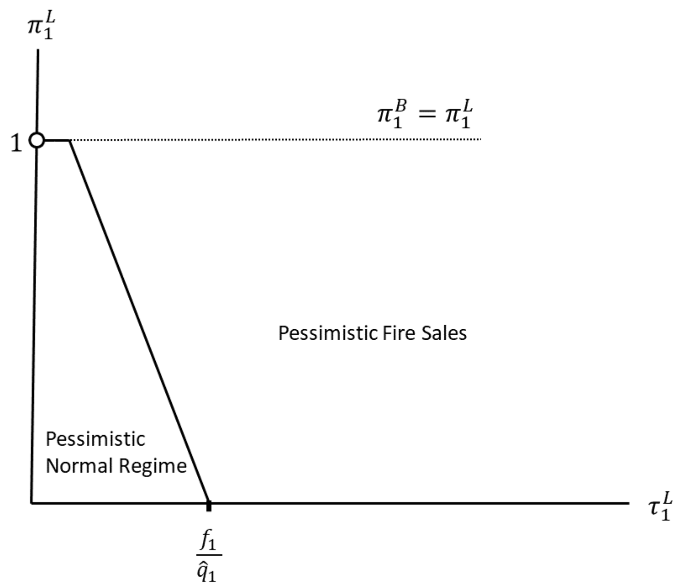

Figure 7: Four Regions of the State Space when Condition 1 Does Not Hold

curve itself, and the region in which $\pi_1^B > \pi_1^L$ below the curve. On the curve itself the borrower and lender have the same beliefs, while below the curve, the borrower is relatively pessimistic about the risky asset.

Figure 7 illustrates the demarcation of two regions of the state space under the case when Condition 1 does not hold. It divides the state space by whether the equilibrium is in the Pessimistic

Normal Regime or Pessimistic Fire Sale Regime. The dotted curve itself defines the region of the state space in which the lender and borrower have identical beliefs in equilibrium. Hence the solid (dotted) portion of the top line represents the region in which the Normal Regime (Fire Sale Regime) features neither optimism nor pessimism.

References

ALLEN, F., S. MORRIS, AND H. S. SHIN (2006): "Beauty Contests and Iterated Expectations in Asset Markets," *The Review of Financial Studies*, 19, 719–752.

ANGELETOS, G.-M. AND J. LA'O (2013): "Sentiments," *Econometrica*, 81, 739–779.

ASRIYAN, V., L. LAEVEN, AND A. MARTIN (2021): "Collateral Booms and Information Depletion," *Review of Economic Studies*, forthcoming.

BABUS, A. AND P. KONDOR (2018): "Trading and information diffusion in over-the-counter markets," *Econometrica*, 86, 1727–1769.

BARBERIS, N. (2018): "Psychology-based Models of Asset Prices and Trading Volume," in *Handbook of behavioral economics: applications and foundations 1*, Elsevier, vol. 1, 79–175.

BRUNNERMEIER, M. K. AND L. H. PEDERSEN (2009): "Market Liquidity and Funding Liquidity," *The Review of Financial Studies*, 22, 2201–2238.

BRUNNERMEIER, M. K., A. SIMSEK, AND W. XIONG (2014): "A Welfare Criterion for Models with Distorted Beliefs," *The Quarterly Journal of Economics*, 129, 1753–1797.

CHEN, J., H. LIU, A. SARKAR, AND Z. SONG (2020): "Dealers and the Dealer of Last Resort: Evidence from MBS Markets in the COVID-19 Crisis," *FRB of New York Staff Report*.

COPELAND, A., A. MARTIN, AND M. WALKER (2014): "Repo Runs: Evidence from the Tri-Party Repo Market," *The Journal of Finance*, 69, 2343–2380.

DANG, T. V., G. GORTON, AND B. HOLMSTRÖM (2020): "The Information View of Financial Crises," *Annual Review of Financial Economics*, 12, 39–65.

DÁVILA, E. AND A. KORINEK (2018): "Pecuniary Externalities in Economies with Financial Frictions," *The Review of Economic Studies*, 85, 352–395.

DIAMOND, D. W. AND P. H. DYBVIG (1983): "Bank Runs, Deposit Insurance, and Liquidity," *The Journal of Political Economy*, 401–419.

Fostel, A. and J. Geanakoplos (2015): "Leverage and Default in Binomial Economies: a Complete Characterization," *Econometrica*, 83, 2191–2229.

Gale, D. and T. Yorulmazer (2013): "Liquidity Hoarding," *Theoretical Economics*, 8, 291–324.

Garcia-Macia, D. and A. Villacorta (2022): "Macroprudential Policy with Liquidity Panics," *The Review of Financial Studies*, forthcoming.

Geanakoplos, J. (1997): "Promises Promises," in *The Economy as an Evolving Complex System II*, ed. by S. D. Brian Arthur and D. Lane, CRC Press, 285–320.

——— (2003): "Liquidity, Default, and Crashes Endogenous Contracts in General Equilibrium," in *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress*, Cambridge University Press, vol. 2, 170.

——— (2010): "The Leverage Cycle," *NBER Macroeconomics Annual*, 24, 1–65.

Goldstein, I. and A. Pauzner (2005): "Demand–deposit Contracts and the Probability of Bank Runs," *the Journal of Finance*, 60, 1293–1327.

Gorton, G. and A. Metrick (2012): "Securitized Banking and the Run on Repo," *Journal of Financial Economics*, 104, 425–451.

Gorton, G. and G. Ordonez (2014): "Collateral Crises," *American Economic Review*, 104, 343–78.

Gottardi, P., V. Maurin, and C. Monnet (2019): "A theory of repurchase agreements, collateral re-use, and repo intermediation," *Review of Economic Dynamics*, 33, 30–56.

Grossman, S. J. and J. E. Stiglitz (1980): "On the Impossibility of Informationally Efficient Markets," *The American Economic Review*, 70, 393–408.

Infante, S. and A. Vardoulakis (2021): "Collateral Runs," *The Review of Financial Studies*, 34, 2949–2992.

Kiyotaki, N. and J. Moore (1997): "Credit Cycles," *Journal of Political Economy*, 105, 211–248.

Kurlat, P. (2016): "Asset Markets with Heterogeneous Information," *Econometrica*, 84, 33–85.

Lorenzoni, G. (2008): "Inefficient Credit Booms," *The Review of Economic Studies*, 75, 809–833.

MARTIN, A., D. SKEIE, AND E.-L. VON THADDEN (2014): "Repo Runs," *Review of Financial Studies*, 27, 957–989.

MORRIS, S. AND H. S. SHIN (2003): "Global Games: Theory and Applications," in *Advances in Economics and Econometrics: Proceedings of the Eighth World Congress of the Econometric Society*, Cambridge University Press, vol. 1, 56–114.

SHLEIFER, A. AND R. W. VISHNY (1992): "Liquidation Values and Debt Capacity: A Market Equilibrium Approach," *The Journal of Finance*, 47, 1343–1366.

SIMSEK, A. (2013): "Belief Disagreements and Collateral Constraints," *Econometrica*, 81, 1–53.

STIGLITZ, J. E. (1981): "The allocation role of the stock market: Pareto optimality and competition," *The Journal of Finance*, 36, 235–251.