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Contagion in Debt and Collateral Markets *

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Abstract

This paper investigates contagion in financial networks through both debt and collateral markets. We find that the role of collateral is mitigating counterparty exposures and reducing contagion but has a phase transition property. Contagion can change dramatically depending on the amount of collateral relative to the debt exposures. When there is an abundance of collateral (leverage is low), then collateral can fully cover debt exposures, and the network structure does not matter. When there is an adequate amount of collateral (leverage is moderate), then collateral can mitigate counterparty contagion, and having more links in the network reduces contagion, as interlinkages act as a diversifying mechanism. When collateral is not enough (leverage is high) and agents in the network are too interconnected, then the collateral price can plummet to zero and the whole network can collapse. Therefore, we show the importance of the interaction between the level of collateral and interconnectedness across agents. The model also provides the minimum collateral-to-debt ratio (haircut) to attain a robust macroprudential state for a given network structure and aggregate state.

Keywords: collateral, financial network, fire sale, systemic risk

JEL Classification Numbers: D49, D53, G01, G21, G33

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1. Introduction

This paper studies how initial shocks propagate through a network of counterparties in collateralized debt markets. The Global Financial Crisis (GFC) was exacerbated by a failure of collateralized debt markets, which is the most common source of short-term financing among financial institutions and includes repurchase agreement (repo) and asset-backed commercial paper (ABCP) markets (Gorton and Metrick, 2012; Covitz, Liang, and Suarez, 2013). The collapse in prices of subprime mortgages in 2008 had a direct effect on many financial institutions. However, the initial shock was exacerbated by the resulting bankruptcy of Lehman Brothers, which spread the initial losses to Lehman's counterparties (Singh, 2017). Currently, the volume of collateralized debt markets is very large. The average daily amount of cash and collateral traded in bilateral repo markets in the U.S. is more than \$4 trillion for 2019, 2020, and 2021.[1] Also, the global market for securities lending has more than \$3 trillion in outstanding contracts, with U.S. loans accounting for about half of the worldwide market in 2021 (Financial Stability Oversight Council, 2021). Hence, understanding how such markets can be vulnerable to contagion is important for both academics and policy makers.

Typical collateralized debt takes the form of a one-to-one relationship between a borrower and a lender because of customization (bespoke) and counterparty-specific contract terms, such as margins and rates. If the value of the collateral is greater than the face value of the debt, then the payment is always made in full, even if the borrower becomes insolvent. However, if the value of the collateral is less than the face value of the debt, then the payment depends on both the price of the collateral and the cash balance of the borrowing counterparty.

Therefore, a collateralized debt network has two transmission channels of shocks—the collateral price channel and the counterparty channel. If the asset price declines, then the net wealth of all agents decreases through the collateral price channel. If an agent defaults and spreads losses to its counterparties, then the counterparties' net wealth decreases because of the counterparty channel. The counterparty losses can continue to depress the asset price, resulting in further losses.

In this paper, we develop a network model with collateral featuring the two channels of propagation, which is the first attempt to endogenize asset prices while the debt contracts are full recourse.[2] Typical financial network models in the literature investigate only con-

[1] See https://www.sifma.org/wp-content/uploads/2022/02/SIFMA-Research-US-Repo-Markets-Chart-Book-2022.pdf.

[2] In Chang (2021), nonrecourse contracts are considered to solve for endogenous network formation. In this paper, we do not attempt to endogenize network formation, however we solve for the networks with full-recourse contracts. The full-recourse property complicates the propagation structure and makes the

tagion of liquidity shortages through the counterparty channel or price-mediated losses of common asset holdings, but not both. However, payments from collateralized debt contracts depend on the interaction between the counterparty channel and the collateral price channel because the collateral price changes endogenously and simultaneously. This paper explores the implications of this additional interaction of the two channels of propagation.

The model is based on an economy of n agents, who trade an asset that can be used as collateral in a competitive market. The price of the asset is endogenously determined in a competitive Walrasian market. The fundamental value of the collateralizable asset, which is common knowledge to everyone in the economy, is realized in the final period. In the initial period, agents borrow from each other using bilateral collateralized debt contracts, specifying the amount of debt and collateral. The structure of these liabilities can be summarized as the collateralized debt (financial) network. All the debt contracts mature in the interim period. Agents also have invested in a long-term project that generates a return at the final period, and the return from the long-term project is not pledgeable. Liquidating the long-term project is costly and thus, socially inefficient. However, if agents are under negative liquidity shocks (e.g., lower than expected short-term returns, a sudden increase in deposit withdrawals, wage expenses, taxes, or other senior creditors), then they may have to liquidate their long-term investment projects to pay their debt. If an agent's net wealth is still negative after completely liquidating the long-term project, then the agent defaults, which may trigger additional defaults through the network.

This model is consistent with the literature, as it extends and reproduces the results of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), who develop a model of unsecured debt networks. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) show that there exists a phase transition property of contagion, also called "robust yet fragile," depending on the size of the liquidity shocks. If the shocks are small, then the complete network is the most stable and resilient network because more lending relationships can diversify the spread of losses. However, if the shocks are large, each lending relationship becomes a source for contagion. Therefore, the complete network, having the most linkages, becomes the least stable and resilient network. The same result holds in our model when the collateral-to-debt ratio (the inverse of leverage) is at or close to zero.

The co-existence of collateral and bilateral debt relationships (financial network) in our model generates many different equilibrium outcomes depending on the amount of collateral and the network structure as well as the size of the shock. In particular, we find that the role of collateral shows a phase transition property, as it changes dramatically depending on the amount of collateral relative to the debt exposures.

problem highly intractable because of higher-degree of interactions across the network.

If the collateral-to-debt ratio is high enough, the equilibrium is in a fully insulated regime in which any borrower default can be fully covered by the market value of the collateral. The value of the collateral is transferred to the lenders of a defaulting agent, effectively transferring payments from liquidity shocks to the lenders in the network. This result is in line with real-world markets; for example, repo collateral is exempt from the automatic stay of bankruptcy provisions. Hence, a shock to an agent does not propagate to other agents regardless of the network structure and size of the liquidity shocks. Thus, any network is completely insulated from liquidity shocks of any size, as the collateral provides proper security of the market by adding a guaranteed payment amount regardless of the borrower's balance sheet.

Even if the collateral-to-debt ratio is not high enough to guarantee full payments, as long as the collateral-to-debt ratio is above a certain threshold, the collateral price is at its fundamental value regardless of the network structure and the size of the liquidity shocks. As in the previous case, collateral limits the drainage of cash flow from the network, so the remaining agents can buy the assets from a fire sale (from defaulting agents) at the asset's fundamental value. Even though the collateral price is at its fundamental value, the degree of contagion in the network depends on the network structure in this case. One can consider this regime to be robust because the complete network is the most stable and resilient network, implying that having more links is beneficial.

However, if the collateral-to-debt ratio is not high enough, a large, negative liquidity shock may trigger a cascade of defaults, and the collateral price could go well below its fundamental value. This can occur because there is not enough collateral to cover the initial default amount as a result of the large liquidity shock, causing additional agents to default. These heightened counterparty losses reduce available cash to buy the assets at a fire sale at the asset's fundamental value. Therefore, the collateral asset price declines, increasing the number of defaults and fire sales, which further decreases the asset price, and so on. Hence, unless the network reaches the threshold, collateral cannot serve its proper role of reducing counterparty exposures, and the recurring dynamic between counterparty contagion and collateral price declines from fire sales will permeate. Nevertheless, a network with limited interconnectedness across agents can prevent a full collapse of the network and collateral price by limiting the liquidity shock to only the portion of the network exposed to the shock.

Our results highlight the fragility of collateral's role in reducing counterparty exposures. When defaults occur, collateral can act as a buffer as long as the counterparty exposures themselves are limited. In most cases, either the collateral is enough or the external shocks are small, so the collateral price will be its fair value and no further contagion occurs. However, as soon as the counterparty exposures exceed the threshold and collateral is not enough, then the collateral price will plunge to zero, leaving all the contracts unsecured. Therefore, the equilibrium collateral price shows a phase transition (bang-bang) property.

Finally, the model provides insights on macro-prudential policy, such as leverage restrictions, as the economy faces aggregate shock. We obtain the levels of collateral-to-debt ratio required to attain a robust or fully insulated regime, depending on the fundamental value and the supply of the collateral asset. If the payoff of the asset decreases or the supply of the asset increases, the threshold level of the collateral-to-debt ratio required to achieve the robust or fully insulated regime increases.

Besides providing insights on financial contagion, this paper also provides insights on the role of explicit collateral and settlements in lending networks on mitigating contagion. Contrary to the many models in the macro literature that use the total amount of capital or going-concern value as collateral, contracts in our model specify explicitly designated collateral, which is consistent with the practices of collateralized debt markets in the real world. In our model, even if the total amount of assets or individual asset holdings are the same, an economy with higher collateral-to-debt ratios would have lower systemic risk than an economy with lower collateral-to-debt ratios. This is because explicit collateral acts as money and limits the transmission of idiosyncratic liquidity shocks to other agents in the network, minimizing the negative spillovers. Furthermore, our model implies that the reuse (rehypothecation) of collateral may improve financial stability for a given debt amount and asset holdings. However, reuse of collateral can alter the total cash holdings and network structure as well as the debt amount (Chang, 2021), which is outside the scope of this paper.

Our results have direct real-world and policy implications. One of the most relevant applications aside from the GFC is the contagion between the digital asset (e.g., crypto-asset) markets and the traditional financial markets. Decentralized finance (DeFi) and other lending platforms of digital asset markets typically use collateralized debt contracts (Azar et al., 2022). If issuers of stablecoins, which are often used as collateral for DeFi lending, have typical safe assets in the financial markets as their reserve assets, then both the financial markets and digital asset markets share the same underlying collateral base. Our results suggest that even if the two markets use the same assets as collateral, shocks to the digital assets do not necessarily spread to the financial markets as long as the interconnectedness between the two markets is small. However, if the two markets have more than a small degree of interconnectedness, a shock to the digital asset markets can also propagate to the financial market, resulting in a systemic event. Furthermore, our results imply that restricted leverage will prevent such contagion.

1.1. Relation to the Literature

The first contribution of this paper is developing a model with full-recourse contracts and both the debt and collateral channels of contagion, which is the first attempt in the literature. No major institution failed because of losses on its direct exposures to Lehman Brothers; thus, developing a model that combines different shock transmission channels in financial networks is important in understanding contagion through interconnectedness (Upper, 2011; Glasserman and Young, 2016). The model in this paper incorporates default cascades and price-mediated losses, and the interaction of the two channels leads to novel properties of contagion.

The literature on financial networks usually focuses on the tradeoff between diversification and the contagion channel from having more links. This paper suggests that the tradeoff can change depending on the collateral-to-debt ratio and the aggregate shocks, because the contracts are collateralized and the collateral asset price is endogenous. Eisenberg and Noe (2001) introduced an exogenous network model as a financial network with propagation through payments, which Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) extended with a debt financial network without collateral. The payment equilibrium concept employed in that literature is used in this paper as well. The model of this paper is based on Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) and extends their model further by incorporating collateral and collateral markets. The key difference is that we take the excess cash inflows of non-defaulting agents into account to track the changes in the collateral price that affect defaults through the collateralized debt contract structure. Elliott, Golub, and Jackson (2014) include discontinuous jumps in the payoffs of agents in the case of bankruptcy, which causes multiplicity of equilibria. They focus on the maximum equilibrium in which the payments are the highest among all possible equilibria. We also follow their method and focus on the maximum equilibrium in which the least number of defaults occur among all possible equilibria. In addition, multiplicity in this paper is mostly confined to non-generic cases. In the small shock case in our model, the linkages within networks function as liquidity coinsurance, which is studied by Allen and Gale (2000).

The feedback from agents' net wealth to collateral price is crucial in this paper. Other papers consider the interaction between counterparty and price channels, such as Capponi and Larsson (2015); Cifuentes, Ferrucci, and Shin (2005); Di Maggio and Tahbaz-Salehi (2015); Gai, Haldane, and Kapadia (2011); and Rochet and Tirole (1996). This paper differs by incorporating debt contracts with explicit collateral and an endogenous price channel of contagion for the underlying collateral.

The endogenous price determination in this paper is based on the literature on general equilibrium with collateralized debt, as in Geanakoplos (1997), Geanakoplos (2010), and Fostel and Geanakoplos (2015). This paper contributes to this literature by linking these features into the network contagion, which is crucial in bilateral repo and ABCP markets, and analyzing the effect of counterparty risks on prices.

Many financial network models have an equilibrium in which agents have overlapping asset and counterparty portfolios or a common correlation structure (Cabrales, Gottardi, and Vega-Redondo, 2017). Moreover, the literature documents fire sales in financial markets, which implies that sales of an asset manager or a bank can depress asset prices and lead to more sales from others, further depressing both the market price and balance sheets (Coval and Stafford, 2007; Chen, Goldstein, and Jiang, 2010; Jotikasthira, Lundblad, and Ramadorai, 2012; Greenwood, Landier, and Thesmar, 2015; Goldstein, Jiang, and Ng, 2017; Duarte and Eisenbach, 2021). However, these models typically assume linear price impacts from fire sales. Our model incorporates more complicated effects of fire sales by analyzing how asset prices can affect counterparty payments and vice versa. In addition, an extension of our model can provide a microfoundation of the linear price impact model in the literature.

Finally, this paper is also related to the literature on the role of collateral. Geanakoplos (2010) argues that collateralized debt makes the market more complete because it serves as an enforcement device. However, only the aggregate level of collateral matters in the general equilibrium literature because contracts are fully anonymized and diversified. This paper shows how individual collateral matters when counterparty risk is involved. Demarzo (2019) addresses that collateral can be a cost-efficient commitment device, and Donaldson, Gromb, and Piacentino (2020) argue that secured debt prevents debt dilution. This paper adds additional roles of collateral as providing mitigating and amplifying channels of counterparty contagion.

The two papers closest to this paper are Chang (2021) and Ghamami, Glasserman, and Young (2022), which analyze the interaction between the counterparty and price channel of spillovers. Chang (2021) simplifies the borrower default by assuming non-recourse contracts in order to focus on lender default and network formation. This paper focuses only on borrower default and exogenous networks while generalizing to full-recourse contracts in order to analyze more general and realistic borrower default contagion. Ghamami, Glasserman, and Young (2022) studies how different contract termination rules affect the spread of losses and defaults in financial networks with collateral. This paper differs by analyzing how network structures interact with collateral requirements.

This paper proceeds as follows. Section 2 introduces the model. Section 3 characterizes the mechanisms and resulting equilibrium of the model. Section 4 presents our main results related to contagion and systemic risk. Section 5 discusses extensions. Section 6 presents numerical simulation results. Section 7 concludes.

2. Model

The model builds on that of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). The key new feature is the existence of assets that can be used as collateral, whose prices are endogenously determined. The main heterogeneity of interest comes from how agents are connected to each other through collateralized debt relationships.

2.1. Agents and Goods

There are three periods $t = 0, 1, 2$ and two goods, cash and an asset, denoted as e and h, respectively. Cash is the only consumption good and is storable. The asset can be used as collateral at $t = 0$ and yields s amount of cash at $t = 2$. Agents gain no utility from just holding the asset. All agents learn the true value of the asset payoff s at $t = 1$; however, the asset payoff is realized at $t = 2$.

There are n different agents, and the set of all agents is $N = \{1, 2, \dots, n\}$. Agents are risk neutral, and their utility is determined by how much cash they consume at $t = 2$. Each agent is investing in a long-term investment project that will give ξ amount of cash at $t = 2$ if it is held until maturity. The payoff from this long-term investment project is not pledgeable. Agent j can partially liquidate the project by $l_j \in [0, \xi]$ amount at $t = 1$ to receive the scrap value of ζl_j in terms of cash, where $0 \leq \zeta < 1$ represents the liquidation inefficiency.

All information is common knowledge, and the markets for both goods are competitive Walrasian markets. Thus, agents are price-takers, and there is no asymmetric information. The price of cash is normalized to 1 at any period, and the price of the asset is p_t for $t = 0, 1, 2$. From now on, we use p instead of p_1 for the price of the asset at $t = 1$, as our main focus is analyzing the contagion in $t = 1$.

2.2. Collateralized Debt Network

At $t = 0$, each agent $j \in N$ holds e_j amount of cash and h_j amount of asset, which are exogenously given, until $t = 1$. At $t = 1$, agents can buy or sell the asset in a competitive market. Also at $t = 0$, agents borrow or lend cash using assets as collateral. All borrowing contracts are a one-period contract between $t = 0$ and $t = 1$ and are exogenously determined. A borrowing contract consists of the total promised cash payment, the ratio of collateral posted per one unit of promised cash, and the identities of the borrower and the lender. Denote d_{ij} as the total promised cash amount to pay at $t = 1$ to lender i by borrower j.

Denote c_{ij} as the collateral-to-debt ratio per one unit of promised cash, which we refer to as the collateral ratio. If borrower j pays back the full amount of promised d_{ij}, then the lender returns the collateral in the amount of $c_{ij}d_{ij}$. Otherwise, the lender keeps the collateral, and the cash value of the collateral is $c_{ij}d_{ij}p$. Normalize $c_{ii} = d_{ii} = 0$ for all $i \in N$ without loss of generality.

Define $C = [c_{ij}]$ and $D = [d_{ij}]$ as the matrices of collateral ratios and promised debt payment amounts, respectively. A collateralized debt network is a weighted, directed multiplex graph that is formed by the set of vertices N and links with two layers $\alpha = 1, 2$ defined as $\vec{\mathcal{G}} = (\mathcal{G}^{[1]}, \mathcal{G}^{[2]})$, where $\mathcal{G}^{[\alpha]} = (N, L^{[\alpha]})$, $L^{[1]} = C$, and $L^{[2]} = D$. A (collateralized) debt network can be summarized by a double (C, D) given at $t = 0$ with the set of vertices N. A debt network describes how much each agent borrows from or lends to other agents at what margin (collateral ratio). Denote the total inter-agent liabilities of agent j as $d_j \equiv \sum_{i \in N} d_{ij}$.

We assume that *collateral constraints* and *resource constraints* hold, which implies

$$\sum_{k \in N} c_{jk}d_{jk} + h_j \geq \sum_{i \in N} c_{ij}d_{ij} \qquad \forall j \in N \tag{1}$$

$$\sum_{i \in N} h_i \geq \sum_{i \in N} c_{ij}d_{ij} \qquad \forall j \in N. \tag{2}$$

The collateral constraint, (1), means that the total amount of collateral a borrowing agent j posts cannot exceed the amount of assets the agent has—either from other people's collateral that agent j has received as a lender or the amount of assets agent j purchased outright. This collateral constraint implies the model allows re-use (rehypothecation) of collateral.[3] The resource constraint, (2), means that the total amount of assets an agent is posting cannot exceed the total amount of assets in the economy.[4]

Each agent can be hit by a negative liquidity shock in cash at the absolute value of $\epsilon > 0$ at $t = 1$. Agents should pay the liquidity shock first before paying other agents. We interpret ϵ as a senior debt payment to external creditors, who also have linear utility.[5] A realized state of the liquidity shocks is $\omega \equiv (\omega_1, \omega_2, \ldots, \omega_n)$, and the set of all possible states is Ω. For example, $\omega_j = 1$ if agent j is under a liquidity shock and $\omega_j = 0$ otherwise. However, there can be a more general liquidity shock weight structure, such as $\Omega = [0, \infty]^n$.

[3] The same collateral can be reused an arbitrary number of times as in Chang (2021), which is in contrast to other models of reuse of collateral, as in Gottardi, Maurin, and Monnet (2019); Infante and Vardoulakis (2021); Infante (2019); and Park and Kahn (2019).

[4] If a resource constraint is not present, then there can be a spurious cycle of collateral justifying any arbitrary amount of collateral circulating in the economy. For example, $c_{21} = c_{32} = \cdots = c_{1n}$ can be a very large number and satisfy the collateral constraints while no one actually owns the asset as $\sum_i h_i = 0$.

[5] Alternative interpretations of negative liquidity shocks include a sudden increase in deposit withdrawals, wage expenses, taxes, and fines.

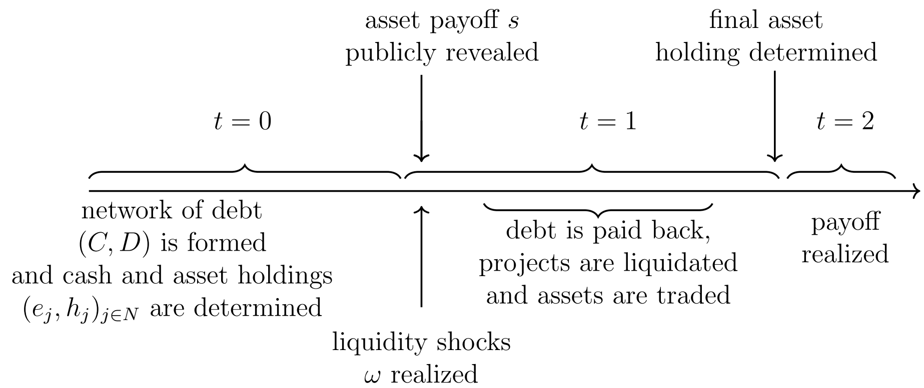


Figure 1: Timeline of the model

Liabilities other than the liquidity shock are all equal in seniority. Hence, any net wealth left after paying the liquidity shocks will be distributed across all agents on a pro rata basis. There are no additional endowments of goods at $t = 2$.

2.3. Timeline

The timeline of the model, depicted in Figure 1, is the following. Agents' cash and asset holdings are determined at $t = 0$, and agents form a collateralized debt network using the assets they own or received as collateral. The resulting network and cash and asset holdings are exogenously given. At the beginning of $t = 1$, asset payoff s is publicly revealed. Also at the beginning of $t = 1$, liquidity shocks (senior debt) of ϵ are realized for each agent. Agents may liquidate their long-term projects if they are short on liquidity. Each agent's debt is paid back, and collateral is returned to the borrower, if not defaulted. If an agent does not pay the debt in full, then the agent is defaulting on that contract, and any remaining assets in the agent's balance sheet will be distributed to all other creditors on a pro rata basis. At the end of $t = 1$, all agents' final asset holdings are determined. At $t = 2$, the payoff of the asset is realized and agents consume all the cash they have and gain utility from it.

3. Full Equilibrium

In this section, we define the equilibrium concept and its relevant elements.

3.1. Liquidation and Payment Rules

Let $x_{ij}(p)$ denote the actual payment net of collateral to agent i from agent j when the asset price is p at $t = 1$. This payment will be defined later in equation (6). The argument p is often omitted from now on. Denote

$$a_j(p) \equiv e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p - \sum_{i \in N} c_{ij} d_{ij} p + \sum_{k \in N} x_{jk}(p) \tag{3}$$

as the total *cash inflow* of agent j before liquidating the project, where the first term, e_j, is j's cash holding, the second term, $h_j p$, is the market value of j's direct asset holdings, the third term, $\sum_{k \in N} c_{jk} d_{jk} p$, is the market value of collateral assets posted by j's borrowers, the fourth term, $\sum_{i \in N} c_{ij} d_{ij} p$, is the market value of collateral assets posted to j's lenders, and the fifth term, $\sum_{k \in N} x_{jk}(p)$, is the actual payment net of collateral from j's borrowers. The total amount of liabilities net of collateral posted for agent j is

$$b_j(p, \omega) \equiv \sum_{i \in N} (d_{ij} - c_{ij} d_{ij} p) + \omega_j \epsilon, \tag{4}$$

which can be considered as the required total *cash outflow*. Note that the first term of the right-hand side can be negative if the contract is over-collateralized. The function argument ω is often omitted for simplicity from now on.

If $a_j(p) > b_j(p)$—that is, cash inflow exceeds cash outflow—then $x_{ij} = d_{ij} - c_{ij} d_{ij} p$ for any $i \neq j$. If $a_j(p) \leq b_j(p)$—that is, cash outflow exceeds inflow—then agent j liquidates the long-term project to meet liabilities to others. Moreover, if the price of the asset is very low, the return from purchasing the underpriced asset, s/p, can be greater than the long-term return of the project, $1/\zeta$, and all agents will liquidate all of their projects regardless of their obligations, as described in the agent's optimization problem in the next subsection.

Mathematically, agent j's liquidation decision, $l_j(p) \in [0, \xi]$, is

$$l_j(p) = \begin{cases} \left[\min \left\{ \frac{1}{\zeta} \left(b_j(p) - a_j(p) \right), \xi \right\} \right]^+ & \text{if} \quad p \geq s\zeta \\ \xi & \text{if} \quad p < s\zeta, \end{cases} \tag{5}$$

where $[\cdot]^+ \equiv \max\{\cdot, 0\}$, which guarantees that an agent cannot liquidate the long-term project for a negative amount if the agent can meet its liabilities with total cash inflow. The liquidation decision follows the *liquidation rule* if equation (5) holds. Early liquidation of a

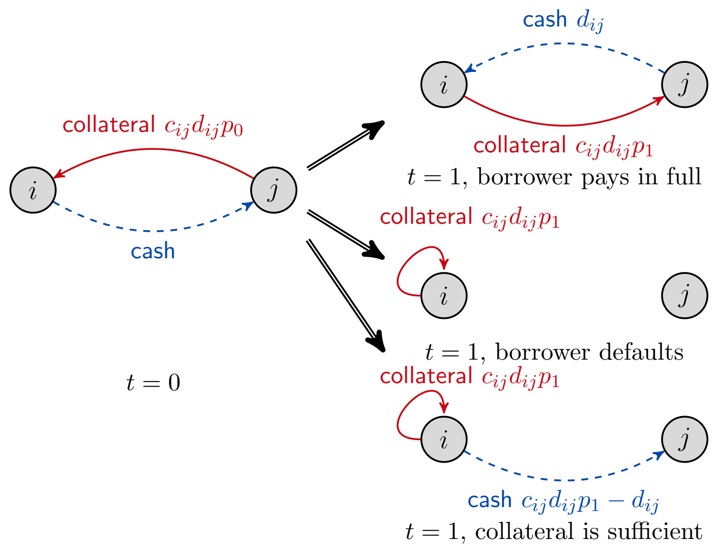


Figure 2: Flows of cash and collateral for three cases

Note: The two nodes, i and j, represent the lender and borrower of a contract, respectively. The blue dashed arrows represent flows of cash, and the red arrows represent flows of collateral. The left figure shows the flows in $t = 0$. The top right figure shows the flows in the case the borrower pays in full in $t = 1$, the middle-right figure shows the flows in the case with borrower default in $t = 1$, and the bottom right figure shows the flows in the case in which the collateral value exceeds the payment in $t = 1$.

long-term project is the primary source of inefficiency in the economy, as it is the only source of deadweight loss, whereas all other payments (including defaults) are zero sum.

Given the liquidation rule, the actual payment to lender i from borrower j is determined as $x_{ij}(p)$. If agent j can pay all of the obligations (possibly by liquidating all or part of the project), then j can pay the original promised amount such that $x_{ij}(p) = d_{ij} - c_{ij}d_{ij}p$ as the lender returns the collateral to the borrower, as depicted in the top right of Figure 2. Also, if the total value of collateral $c_{ij}d_{ij}p$ is greater than the face value of the debt d_{ij}, then the actual payment $x_{ij}(p)$ can be negative because the more valuable collateral sitting in the lender's balance is returned to the borrower, as depicted in the bottom right of Figure 2. Agent j *defaults* on inter-agent debt if the payment net of collateral is less than the promised payment—that is, $x_{ij}(p) < d_{ij} - c_{ij}d_{ij}p$—for some $i \in N$. In the extreme case, if agent j cannot pay the liquidity shock after full liquidation and the collateral value is less than the promised payment, then the actual payment will be $x_{ij}(p) = 0$ and the lender keeps the collateral, as depicted in the middle-right section of Figure 2. In an intermediate case, agent j can pay the liquidity shock in full but cannot pay the inter-agent debt in full. Under such a case, agent j's cash, after being used to pay off the senior debt of the liquidity shock, is paid out on a pro rata basis. This interaction is mathematically formulated as the following *payment rule*:

$$x_{ij}(p) = \min \left\{ d_{ij} - c_{ij} d_{ij} p \ , \quad q_{ij}(p) \left[a_j(p) + \zeta l_j + \sum_{i \in N} \left[c_{ij} d_{ij} p - d_{ij} \right]^+ - \omega_j \epsilon \right]^+ \right\}, \tag{6}$$

where $q_{ij}(p)$ is a weight under the *weighting rule*

$$q_{ij}(p) = \frac{\left[d_{ij} - c_{ij} d_{ij} p \right]^+}{\sum_{k \in N} \left[d_{kj} - c_{kj} d_{kj} p \right]^+} \tag{7}$$

for the pro rata basis of the actual payment. Note that if the weighting rule is not defined—that is, $\sum_{k \in N} \left[d_{kj} - c_{kj} d_{kj} p \right]^+ = 0$—then the weighting rule is never used in the payment rule, because any lender k will be paid in full, d_{kj}, as the market value of the collateral exceeds the promised payment—that is, $d_{kj} < c_{kj} d_{kj} p$.

3.2. Agent's Optimization Problem

Agent j would like to maximize long-term profit, π_j at $t = 2$, which is composed of cash holdings, asset holdings multiplied by the asset payoff, and the payoff from the long-term project net of the liquidation amount. The decision variables are how much cash (e) and assets (h) to hold and how much liquidation of the long-term investment project to make (l). All of these decisions are subject to wealth as well as paying off the inter-agent liabilities and liquidity shock, while taking the payments from other agents as given. Hence, agent j solves for the following optimization problem:

$$\begin{aligned} & \max_{e,h,l} \pi_j = e + hs + (\xi - l) \qquad (8) \\ \text{s.t.} \quad & e + hp = \left[a_j(p) - b_j(p) + \zeta l \right]^+ \\ & l \geq \underline{l} \equiv \left[\min \left\{ \frac{1}{\zeta} \left(b_j(p) - a_j(p) \right), \xi \right\} \right]^+ \\ & e \geq 0, h \geq 0, l \leq \xi, \end{aligned}$$

where the first constraint is the budget constraint, the second is the liquidation constraint to satisfy liabilities, the third and fourth are non-negativity constraints for cash and assets, respectively, and the last is the upper bound of the total liquidation.

There are four possible cases. First, suppose that agent j's available budget is zero even after liquidating the entire long-term project as $a_j(p) - b_j + \zeta \xi \leq 0$. Then, all the constraints are binding and agent j's portfolio is forced to be $(e, h, l) = (0, 0, \xi)$. Suppose that agent j has some budget available for the rest of the cases. Second, suppose that the asset price is $p = s$. Then, agent j is indifferent between holding more cash and holding more assets. Hence, j will divide the budget into any arbitrary combination of e and h. Third, suppose that the asset price is $s\zeta < p < s$. Then, agent j would prefer to buy more assets than cash because the return of buying an asset is s/p, which is greater than the cash return, 1. However, j does not liquidate any long-term project more than the required amount $\underline{l}$, as the long-term project return (when it is not liquidated) is $1/\zeta$, which is greater than the asset return, s/p. Fourth, suppose that the asset price is $p < s\zeta$. Then, the asset return is greater than the long-term project return because $s/p > 1/\zeta$. Hence, agent j will liquidate the entire long-term investment project (more than the necessary amount) to buy as much in assets as possible. See Appendix A.3 for a detailed derivation of the solution.

The solution to the optimization problem pins down the liquidation rule introduced in the previous subsection and the demand for assets with the market clearing condition in the next subsection.

3.3. Fire Sales and Market Clearing

For a given collateralized debt network and state realization $(N, C, D, e, h, s, \omega)$, where $e \equiv (e_1, e_2, \ldots, e_n)$ and $h \equiv (h_1, h_2, \ldots, h_n)$, the *net wealth* of agent j is

$$\begin{aligned} m_j(p) \equiv & \, a_j(p) + \zeta l_j(p) - b_j(p) \qquad (9) \\ = & e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p - \sum_{i \in N} c_{ij} d_{ij} p + \sum_{k \in N} x_{jk}(p) + \zeta l_j(p) - \omega_j \epsilon + \sum_{i \in N} (c_{ij} d_{ij} p - d_{ij}) \\ = & e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in N} d_{ij} + \sum_{k \in N} x_{jk}(p) \end{aligned}$$

under the liquidation and payment rules. Equation (9) consists of the following: cash holdings from $t = 0$, the market value of the asset holdings from $t = 0$, the market value of collateral received, cash from liquidating the long-term project, a negative liquidity shock, the total payment to be paid, and the actual net payment received. If $m_j(p) < 0$, then agent j defaults.

If an agent has to sell all or part of the asset holdings, then the agent has to fire-sell the asset. Denote the *fire-sale* amount of agent j as

$$\phi_j(p) = \min \left\{ [h_j p - m_j(p)]^+, h_j p \right\}. \qquad (10)$$

If agent j's net wealth subtracted by j's asset holdings, $m_j(p) - h_j p$, is enough to cover all of the payments (positive), then $\phi_j(p) = 0$—that is, no fire sales. If agent j's net cash flow is not enough without the sale of asset holdings, then $\phi_j(p) > 0$. If the cash shortage exceeds the total asset holdings (i.e., $h_j p - m_j(p) > h_j p$), then the fire-sale amount reaches its upper bound $\phi_j(p) = h_j p$. Note that a defaulting agent would always have $\phi_j(p) = h_j p$.

The market for the asset is a perfectly competitive Warlasian market. Unless there is not enough cash to purchase all of the asset sales in the market at the asset's fundamental value s, the market price will always be the fair value s. However, if there is not enough cash in the market, then the asset price can go below its fundamental value as $p < s$, which is a *liquidity constrained price*. Under such a case, the market clearing condition becomes a cash-in-the-market pricing condition. The *market clearing condition* can be summarized as

$$\begin{aligned} \sum_{j \notin \mathcal{D}(p)} [m_j(p) - h_j p]^+ &= \sum_{i \in N} \phi_i(p) \qquad \text{if } \ 0 \leq p < s \\ \sum_{j \notin \mathcal{D}(p)} [m_j(s) - h_j s]^+ &\geq \sum_{i \in N} \phi_i(s) \qquad \text{iff } \quad p = s, \end{aligned} \tag{11}$$

where $\mathcal{D}(p)$ is the set of agents who default under price p.

For the given rules, the definition of the equilibrium is as follows.

Definition 1. *For given $(N, C, D, e, h, s, \omega)$, if liquidation decisions $\{l_j(p)\}$ satisfy the liquidation rule (5), payments $\{x_{ij}(p)\}$ satisfy the payment rule (6), $\{m_j(p)\}$ is determined by net wealth equation (9), the fire-sale amount $\{\phi_j(p)\}$ is determined by equation (10), and price p clears the market as in (11), then $(\{x_{ij}\}, \{l_j\}, \{m_j\}, \{\phi_j\}, p)$ is a full equilibrium.*

The notion of this full equilibrium is a generalization of the payment equilibrium in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), which is based on Eisenberg and Noe (2001). In contrast to these papers, agents in our model not only have financial liabilities and liquidation of projects, but they also have posted collateral, and the price of the collateral asset is determined endogenously. Therefore, both the debt and collateral markets have spillovers to each other. Furthermore, unlike Chang (2021), the debt contract is full recourse, so the defaulting borrowers should still pay any discrepancy between the face value of the debt and the value of the collateral, making borrower default contagion more general and realistic.

In line with the literature, we refer to the interim equilibrium of our full equilibrium without fire sales and market clearing conditions—that is, the payment and liquidity decisions, $\{x_{ij}(p)\}$ and $\{l_j(p)\}$, which satisfy payment and liquidation rules, for a given asset price p—as the payment equilibrium of $(N, C, D, e, h, s, \omega)$ and p. Appendix A contains an alternative definition with matrix notation, which is useful for analysis.

Because the liquidation amount should cover the discrepancy $a_j(p) - b_j(p)$, if there is any, the payments should be between the collateral value and the full debt amount. Given that requirement and the collateral constraints, we can show that the sum of positive net wealth is increasing in the asset price p.

Lemma 1. *The aggregate positive net wealth* $\sum_{j\in N}[m_j(p)]^+$ *is increasing in the asset price* p. *Moreover, if* $\sum_{j\in N}[m_j(p)]^+ > 0$, *then it is strictly increasing in the asset price* p.

All proofs are relegated to the appendix. The main intuition of the proof is that either assets should be owned by the non-defaulting agents as confiscated collateral or the increase in p increases the cash inflow of defaulting agents, resulting in more payments to the non-defaulting agents.

The set of defaulting agents, $\mathcal{D}(p)$, becomes larger as p decreases by Lemma 1. The left-hand sides of the market clearing conditions are the total amounts of cash in the market net of marked-to-market values of asset holdings. Also by Lemma 1, the aggregate net wealth increases as p increases. The market clearing price can be simplified as the following lemma.

Lemma 2. *The market clearing asset price can be represented as*

$$p = \min\left\{\frac{\sum_{j\in N}[m_j(p)]^+}{\sum_{j\in N} h_j}, s\right\}. \tag{12}$$

Even though we have complex and realistic accounting for available cash and the required amount of assets on sales, the resulting computation of the market clearing price is surprisingly simple. This is because wealth that excludes the asset, $m_j(p) - h_j p$, will only contribute to the fire sales, $h_j p - m_j(p)$, and the market value of fire sales can simply be represented as negative demand $m_j(p) - h_j p < 0$.

The following proposition shows that the full equilibrium always exists.

Proposition 1. *For any given collateralized debt network, cash and asset holdings, asset payoff, and realization of shocks* $(N, C, D, e, h, s, \omega)$, *a full equilibrium always exists and is generically unique for a given equilibrium price. Furthermore, there exists a full equilibrium with the highest price among the set of full equilibria.*

The multiplicity of equilibria is similar to that of Elliott, Golub, and Jackson (2014). Because the market clearing price plummets after an additional default due to a jump in fire sales, the additional default could be caused by the decline in price followed by the default itself. Nevertheless, each equilibrium price has a (generically) unique payment equilibrium, and there exists a maximum full equilibrium that has the highest market clearing price among the set of full equilibria. From now on, we will focus on the results of the maximum full equilibrium as in Elliott, Golub, and Jackson (2014).

3.4. Discussion

The model incorporates the role of explicit collateral, as agents can settle the payments by giving up their collateral to their lenders. This is in line with the standard repo contracts such as the Securities Industry and Financial Markets Association's (SIFMA) Master Repurchase Agreement (MRA) used by most U.S. dealers and the SIFMA/International Capital Market Association (ICMA) Global Master Repurchase Agreement (GMRA) used for non-U.S. repos (Baklanova, Copeland, and McCaughrin, 2015). According to both the SIFMA MRA and SIFMA/ICMA GMRA, after determining the market value of the collateral, all repo exposures between the two counterparties are netted off, and whoever owns the residual sum must pay it by the next business day, including the interest on late payment.[6] Hence, the lender has recourse to the borrower's balance sheet and can claim any payment due net of the market value of the collateral (Gottardi, Maurin, and Monnet, 2019). The non-defaulting party may either immediately sell in a recognized market at prices the non-defaulting party reasonably deems satisfactory or give the defaulting party credit for collateral in an amount equal to the price obtained from a generally recognized source.[7] Hence, the default and settlement procedures in our model closely follow how they work in real-world financial markets.

The property of collateral directly covering the debt payment is crucial for our results. For example, if netting the debt with collateral was not possible, then after a negative liquidity shock to the system, all assets posted as collateral would be put on fire sale as all agents liquidate their collateral simultaneously to raise cash for payment obligations. This version of the model is equivalent to having no collateral at all, as in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), because collateral plays no direct role in shaping debt. Indeed, this is not the case in the real world, as financial institutions typically designate particular collateral and can effectively pay their obligations by giving up their collateral as described in the previous paragraph. In other words, collateral plays the role of money across the collateralized debt network when agents pay their liabilities.

This role of collateral as money highlights the importance of reuse (rehypothecation) of collateral. As long as the market value of collateral is large enough, agents can pay each other using the collateral. Hence, reuse of collateral can facilitate and secure more payments, enhancing the financial stability of the network. However, such a role for collateral exists only if the collateral price is high enough. In Section 4, we show that this role of collateral

[6] See https://www.sifma.org/wp-content/uploads/2017/08/MRA_Agreement.pdf for MRA and https://www.sifma.org/wp-content/uploads/2017/08/Global-Master-Repurchase-Agreement.pdf for GMRA.

[7] The Master Securities Loan Agreement for securities lending transactions in the U.S. also states similar procedures (Baklanova, Copeland, and McCaughrin, 2015).

as money can break down when the liquidity shock is large. Furthermore, reuse of collateral can also undermine financial stability because reusing collateral multiple times can increase the leverage of the system and the asset price in $t = 0$, as highlighted by Chang (2021). In this paper, we do not analyze this situation directly, as our networks are not endogenous.

4. Contagion and System Risk

In this section, we study how the structure of the collateralized debt network determines the extent of contagion and systemic risk of the market, which is the risk related to the total loss in values summed over all entities in the system (Glasserman and Young, 2016). For any given collateralized debt network and the corresponding full equilibrium, we define the utilitarian social surplus in the economy as the sum of the returns to all agents at $t = 2$ as

$$U = \sum_{i \in N} (\pi_i + T_i),$$

where $T_i \leq \epsilon$ is the transfer from agent i to its senior creditors (liquidity shock), which simply transfers to $t = 2$, and π_i is the agent's long-term profit evaluated at $t = 2$. This definition of social surplus is consistent with that of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015).

We focus on regular debt networks in which the total inter-agent claims and liabilities of all agents are equal. In other words, a debt network is *regular* if $\sum_{i \in N} d_{ij} = \sum_{i \in N} d_{ji} = d$ for all $j \in N$ and for some $d \in \mathbb{R}^+$. Also, assume that all agents hold the same amounts of cash and assets as $e_i = e_0$ and $h_i = h_0$ for all $i \in N$. This normalization guarantees that any variation in systemic risk is due to the interconnectedness of agents while abstracting away from potential effects from size, balance sheet heterogeneity, or hierarchical heterogeneity. Similarly, we assume that all agents have the same uniform collateral ratio—that is, $c_{ij} = c$ for all $i, j \in N$. For simplicity, and following the benchmark case in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), assume $\zeta = 0$, which is the limit case of $\zeta \to 0$ because, otherwise, the liquidation rule is not well defined.

Finally, we assume that the liquidity shocks are randomly received by κ number of agents, and the size of the shock is $\epsilon \in (e_0 + h_0 s + \zeta\xi, \infty)$. The lower bound on ϵ guarantees that, absent any payments from other agents, an agent under a liquidity shock is unable to pay senior debt from the liquidity shock even with fire sales of collateral at the best price.

The social surplus can be simplified as the following lemma.

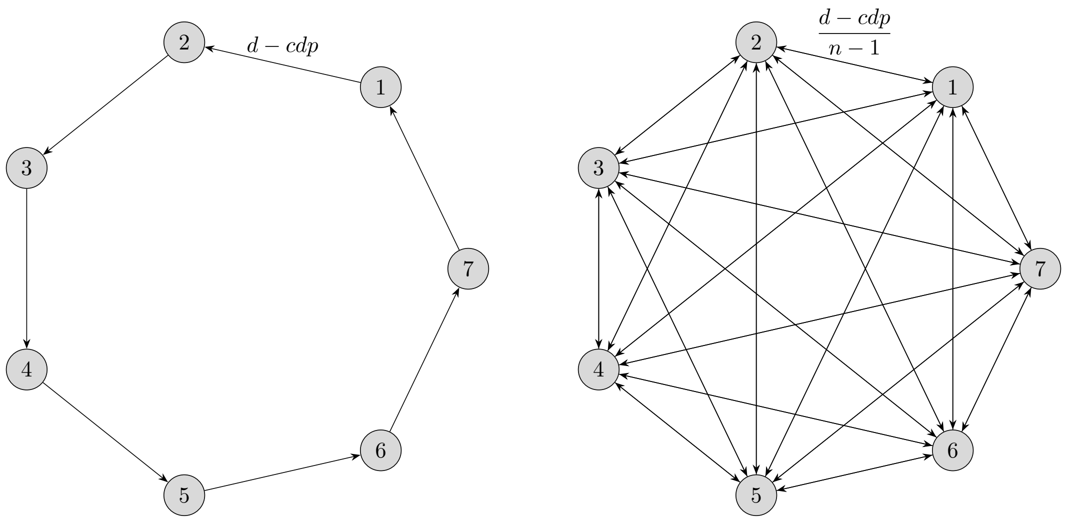


Figure 3: The ring network and the complete network

Lemma 3. *For any full equilibrium, the social surplus in the economy is equal to*

$$U = n(e_0 + h_0 s + \xi) - (1 - \zeta) \sum_{i \in N} l_i.$$

Lemma 3 clarifies that the source of social inefficiency comes from the early liquidation of the long-term project. This liquidation is due to either insufficient liquidity or a low asset price that makes asset purchase more profitable than the long-term project, as in Brainard-Tobin's Q-theory. Denote E as the expectation operator on ω.

Definition 2. *For a fixed (N, e, h, s, Ω), consider two debt networks (C, D) and $(\tilde{C}, \tilde{D})$.*

1. *(C, D) is more stable than $(\tilde{C}, \tilde{D})$ if $EU \geq E\tilde{U}$.*

2. *(C, D) is more resilient than $(\tilde{C}, \tilde{D})$ if $\min_{\omega \in \Omega} U \geq \min_{\omega \in \Omega} \tilde{U}$.*

The two notions compare the expected and worst-case social surplus of a given collateralized debt network. For simplicity of exposition, assume that $\omega_j \in \{0, 1\}$ and $\kappa \equiv \sum_{j \in N} \omega_j = 1$ for any $\omega \in \Omega$ unless noted otherwise. This simple setup allows us to examine the financial contagion over both collateral and debt markets for each network in the most intuitive way. We will discuss relaxing these assumptions in Section 5.

Before we describe the results, we define a few important, related concepts. First, we define the complete and ring networks. The complete network is one in which every agent owes the same amount to each other, $d/(n-1)$. Thus, the complete network contains the highest possible number of links. Second, the ring network is one in which every agent borrows all their debt from one other agent. For example, agent 1 has to pay d to agent 2, who has to pay d to agent 3, who has to pay d to agent 4, and so forth. Agent n has to pay d to agent 1 to make the ring network a regular network. The ring network has the lowest possible number of links for a connected regular network. Figure 3 illustrates the complete and ring networks. Note that the total payment after netting the total collateral posted is $d - cdp$.

Definition 3. *A (collateralized) debt network* (C, D) *is a* δ*-connected network if there exists* $\mathcal{S} \subset N$ *such that* $\max\{d_{ij}, d_{ji}\} \leq \delta d$ *for any* $i \in \mathcal{S}, j \notin \mathcal{S}$.

A δ-connected network implies that a network can be separated into two subsets of vertices with the cross-subset links being relatively small—δ or less.

Definition 4. *A debt network* $(C, \tilde{D})$ *is a* γ*-convex combination of two networks* $(C, D), (C, D')$ *if and only if* $\tilde{d}_{ij} = \gamma d_{ij} + (1 - \gamma) d'_{ij}$ *for any* $i, j \in N$.

The concept of a γ-convex combination of two networks is essentially equivalent to a typical convex combination of matrices. Note that the previous two networks are well defined under the uniform collateral ratio assumption.

4.1. Example

Suppose there are 5 agents that form a ring network as shown in Figure 4. Each agent is endowed with 1 asset ($h_0 = 1$) and 2 units of cash ($e_0 = 2$) at $t = 0$. The asset has a fair value of 1 ($s = 1$), and its price is determined as in (12) in Lemma 2. Each agent owes a total debt amount of 10 ($d = 10$) to the next agent along the ring as in Figure 4 and posts a collateral amount of 2, representing 20% of their debt. If the collateral is priced at its fair value, agents have to raise 8 units of cash to fulfill their debt obligation. Suppose that only agent 1 is under a liquidity shock in this example.

We examine four shock regimes to highlight the mechanisms of our model. When the liquidity shock is small and equal to 2, agent 1 uses cash to fulfill it and meets the debt obligation to agent 2 by reusing the payment received from agent 5. No agent defaults, and the asset price remains $p = s = 1$. If the liquidity shock is slightly larger, 3, agent 1 can no longer rely solely on cash holdings to meet the shock and must sell the asset. The total cash from potential buyers, the four other agents, is 8. Since only agent 1 needs to sell an asset, the supply is 1. Thus, the asset's price is its fair value, $p = s = 1$, and agent 1 receives 1 unit of cash from the fire sale. Agent 1 pays the liquidity shock and fulfills debt obligations to agent 2 by reusing agent 5's payment. No agent defaults, and the network remains stable.

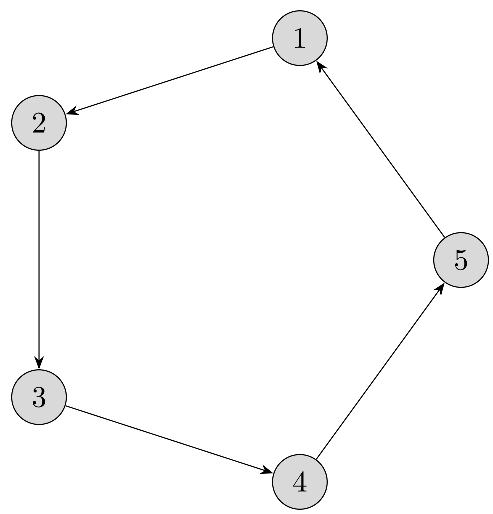


Figure 4: Ring network with 5 agents

If the liquidity shock is 10, assuming the asset is priced at s, agent 1 would have to pay the liquidity shock using a portion of agent 5's payment in addition to agent 1's cash holdings and payment from asset sales. Agent 1 defaults on the inter-agent debt obligation and pays agent 2 the remaining wealth of 1. Agent 2's wealth now consists of 1 unit of assets, 2 units of cash, and agent 1's payment of 1, which, in total, is smaller than 8, the required payment amount to agent 3. Therefore, agent 2 defaults, sells the asset holding, and pays agent 3 the remaining wealth of 4. With agent 2's partial payment, agent 3 cannot fulfill the debt obligation to agent 4 and therefore defaults, sells the asset holding, and pays agent 4 the remaining wealth of 7. Agent 4 uses agent 3's payment of 7 and 1 unit of cash holdings to pay agent 5 in full. Agent 5 can meet the debt obligation to agent 1 by reusing agent 4's payment. Since we assume the asset price is at its fair value, we check if that is indeed the case. There are 3 assets on sale, and the remaining collective cash is 3. Therefore, the price is sustained at 1, its fair value. The network remains partially solvent, with 3 agents defaulting.

If the liquidity shock is 15, assuming the asset is priced at s, the chain of events (defaults, selling assets) will remain the same as in the previous case, with the only difference being each agent's remaining net wealth. Agent 1 transfers a net wealth of 0 to agent 2. Agent 2 transfers a net wealth of 3 to agent 3. Agent 3 transfers a net wealth of 6 to agent 4. Using agent 3's payment of 6 and the cash holdings of 2, agent 4 fulfills the debt obligation to agent 5. Agent 5 pays agent 1 in full, reusing agent 4's payment. We check the asset price to see if our initial assumption of $p = s = 1$ holds. There are 3 assets on sale, and the total remaining cash in the economy is 2. Therefore, the price of the asset is not its fair value, and we recalculate the price to $\frac{2}{3}$. Now, the payments of agents who sold their assets and the value of collateral are smaller than previously calculated, which means the total wealth is smaller and each agent's debt obligation is larger. As a result, additional agents might have defaulted, and the price continues to diminish from $\frac{2}{3}$. In this example, the final asset price and total remaining wealth are 0. All agents default, and the social surplus becomes the minimum possible value.

4.2. Unsecured Debt Case

Suppose the uniform collateral ratio is $c = 0$, and $h_0 = 0$, so there is no collateral or asset in the market. This case encompasses the main model setting of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). When there are no collateral or asset holdings, the only remaining channel of contagion is the debt channel. The following result summarizes the main results of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) related to the phase transition property of financial contagion depending on the size of the shock. If the shock is small, the complete network is the most resilient and stable network (robust), but if the shock is large, the complete network is the least resilient and stable network (fragile).

Proposition 2. *(Acemoglu, Ozdaglar, and Tahbaz-Salehi, 2015) For $\epsilon^* = ne_0$, there exists $d^* = (n-1)e_0$ such that for any $\epsilon < \epsilon^*$ and $d > d^*$ the following holds:*

1. *The complete network is the most resilient and stable.*
2. *The ring network is the least resilient and stable.*
3. *The γ-convex combination of the ring and complete networks becomes more stable and resilient as γ increases.*

Furthermore, for any $\epsilon > \epsilon^*$ and δ sufficiently small,

1. *Both the complete and ring networks are the least resilient and stable.*
2. *A δ-connected network is more resilient and stable than the complete network.*

The results and proofs are almost identical to those in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015).

4.3. Insulation by Collateral

Suppose the uniform collateral ratio is now $c > 0$, and there is a positive amount of assets $h_0 > 0$. Note that, from now on, we fix each agent's endowment as e_0 of cash and h_0 of assets while changing the collateral ratio c, the network structure, and the size of the liquidity shock ϵ. Hence, we focus on the role of collateral and the network structure for a fixed amount of debt and endowments.

If the contracts are fully covered by collateral—that is, any face value of the debt is exceeded by the value of collateral—then all of the payments will be made in full, and no agent will default on their inter-agent debt. This full insulation property prevents network propagation, and any network becomes the most stable and resilient.

Proposition 3. *(Collateral Insulation)*
Suppose that $\kappa < n$ number of agents are hit with a liquidity shock. If $\kappa d \leq n(n-\kappa)e_0$ and

$$c^*(s,n) \equiv \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\} \leq \frac{s}{\zeta},$$

then, for any $c \geq c^*$, any network is the most resilient and stable network for any ϵ.

The first condition $\kappa d \leq n(n-\kappa)e_0$ is necessary to satisfy the economy's resource constraint. Otherwise, the network requires an exceedingly large amount of collateral circulating in the economy. One way to interpret this condition is that the leverage is at a realistic level. For example, if $\kappa = 1$, the condition implies that an agent's total liabilities cannot exceed $(n-1)$ times the total amount of cash in the economy, ne_0. The second constraint, $c^*(s,n) \leq s/\zeta$, is needed to prevent price-induced total fire sales stemming from the disproportional return of the asset compared with that of the long-term project. This is a realistic result if the long-term project is at least as profitable as the collateralizable asset. For example, yields of U.S. Treasury securities, which are commonly used as collateral, typically do not exceed the average yield of corporate equities.

In this case, there is enough collateral in the market relative to other cash sources. Therefore, all debt can be covered by the collateral, and the entire market is insulated from contagion, regardless of the network structure. From now on, assume that $\kappa d \leq n(n-\kappa)e_0$ holds.

4.4. Limited Collateral Contagion

If the collateral ratio is not enough to provide full insulation, then propagation still occurs as in the unsecured debt case. However, the implied network propagation changes as a result of the collateral contagion shifting the payments.

In this section, we show and analyze the cases in which collateral contagion is limited. First, we show that the upper bound of the number of defaulting agents is decreasing in the collateral ratio c for a given equilibrium price p. Then, we show that when the liquidity shock is small or the collateral ratio is high relative to certain thresholds, the asset price in $t = 1$ is always $p = s$, the fundamental value of the asset, so there is no fire-sale-induced liquidity constrained price. In other words, there is no contagion through the collateral price channel when the liquidity shock is small or the collateral ratio is high.

Lemma 4. *Suppose that κ number of agents are under a liquidity shock. Let $\mathcal{D}$ and p be the set of agents defaulting on their inter-agent debt and the price in full equilibrium, respectively, and $cp < 1$. Then, the number of defaults is bounded above and below as the following:*

$$\kappa \leq |\mathcal{D}| < \frac{\kappa \min\{\epsilon, e_0 + h_0 p + d - cdp\}}{e_0 + h_0 p}.$$

Therefore, the upper bound of the number of defaults is decreasing in the collateral ratio c and the equilibrium asset price p.

The more interesting part of the lemma is the upper bound of defaults. The numerator represents the total liquidity outflow from the system. If the liquidity shock is small, then an agent can pay this liquidity shock with the total inflow of cash for that agent. Then, the total outflow from the system is simply the size of the shock, ϵ. However, if the shock is large, the maximum total cash inflow, $e_0 + h_0 p + d - cdp$, will be drained from the system. The denominator represents the individual endowment of each agent. Therefore, if the total outflow from the system can be covered by individual endowments of defaulting agents, then there will be no further defaults. Hence, we obtain the upper bound of the total number of defaulting agents.

Lemma 4 highlights the relationship between the individual endowments ($e_0 + h_0 p$), total debt amount (d), and the value of collateral (cdp). The upper bound is decreasing in the endowments, as having more endowments implies that agents have more cash to pay for the liquidity shortfall. The upper bound is increasing in the total debt amount, as the liquidity shocks can trigger more inter-agent defaults. Finally, the upper bound is decreasing in the total value of the collateral, as the existence of collateral guarantees that at least part of the debt is paid through collateral. Hence, collateral is effectively transferring some amount of liquidity to the lenders in case of default.

The next proposition shows that the asset price, p, in $t = 1$ is always its fundamental value, s, whenever the collateral ratio is high enough or the liquidity shock is small.

Proposition 4. *(No Contagion through the Collateral Price Channel) Suppose that the collateral ratio c is $c < c^*(s, n)$ and $\kappa \in \{1, \ldots, n\}$. Let $\epsilon^* = ne_0/\kappa$. Then, if $\epsilon < \epsilon^*$, the equilibrium asset price in $t = 1$ is always $p = s$, regardless of the network structure. Furthermore, if $c \geq c_* \equiv \dfrac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds}$, then the equilibrium asset price in $t = 1$ is always $p = s$, regardless of the network structure for any ϵ. Finally, the threshold collateral ratio preventing the asset price from going below the fundamental value is (weakly) lower than the threshold collateral ratio for full insulation—that is, $c_* \leq c^*$.*

Interestingly, the threshold of the liquidity shock size is exactly the same as the threshold for the phase transition of contagion under the unsecured debt case. This is because every agent is indirectly connected to all other agents through the collateral market. Even if an agent is completely isolated from all other agents, the agent will participate in the asset market. Therefore, the asset market will remain intact—that is, agents trade assets at fair value—as long as the total liquidity in the market is sufficient. This coincides with the case in which all agents are connected to each other—that is, the complete network. Hence, when the shock is small, the collateral market prevents further contagion through diversification, as in the complete network.

In addition, a collateral ratio that is high yet below the full insulation level c^* can guarantee the asset price will remain $p = s$, regardless of the size of the liquidity shock. The main reason is that collateral can limit the drainage of liquidity, as discussed in the description of Lemma 4. Any collateral ratio $c \geq c_*$ limits the total liquidity outflow from the system, so the remaining agents can buy the assets on fire sale at the highest possible price s.

Combining Proposition 4 and Proposition 2 leads to the following result.

Proposition 5. *Suppose that the collateral ratio c is $c < c^*(s, n)$ and only one agent is hit with a negative liquidity shock, $\kappa = 1$. If either the size of the shock is $\epsilon < \epsilon^*$ or the collateral ratio is $c \geq c_*$, then the complete network is the most stable and resilient network, while the ring network is the least stable and resilient network. The number of defaults in the ring network decreases as c increases. Furthermore, the γ-convex combination of the complete and ring networks becomes more stable and resilient as γ increases.*

Because there is no contagion through the collateral channel, the only source of contagion is the debt channel. By using the technique of collateral netting, which we introduce in Appendix A, we can convert the collateralized debt network into an unsecured debt network after netting out all the collateral movements. Then, the same argument used by Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) applies to our setup. Furthermore, because the upper bound of defaults is decreasing in c by decreasing the total cash outflow from the network, the maximum number of defaults, which is the number of defaults in the ring network, is decreasing in c. In conclusion, if the liquidity shock is small or the collateral ratio is high, there is no collateral channel of contagion, and the debt channel of contagion is minimized by having the most links possible.

4.5. Active Collateral Contagion

If the liquidity shock is large and the collateral ratio is not high enough, the liquidity shock exceeds the total amount of available cash in the network, changing the nature of debt contagion, as in the unsecured debt case. Furthermore, there can be contagion through the collateral price channel, which adds an additional dimension to the overall contagion in the collateralized debt network. The following proposition highlights what happens when collateral contagion is active.

Proposition 6. *(Phase Transition under Collateralized Debt) Suppose that the collateral ratio c is $c < c_*(s, n)$, $\kappa = 1$, and $\epsilon > \epsilon^*$. Then, there exists d^* such that for any $d > d^*$ the following holds:*

1. *The complete and ring networks are the least stable and least resilient networks, and the corresponding equilibrium asset price is $p = 0$ for both networks.*

2. *For a small enough δ, a δ-connected network is more resilient and stable than the complete network, and the corresponding equilibrium asset price is $p > 0$.*

One interesting implication of Propositions 6 and 5 is that the equilibrium asset price is either 0 or s under the ring and complete networks. Since all agents are interconnected with each other, a liquidity shock to one agent will reach all agents in the network. If the price of collateral was high, then it would have mitigated the total outflow of cash (due to the liquidity shock) from the network. However, the remaining net wealth of the network is small after the outflow of cash, even when $p = s$, which depresses the price, as specified in the price equation (12). Declining prices increase every agent's need for additional liquidity, as the discrepancy between the debt and collateral, $d - cdp$, increases when p decreases. This effect further increases fire sales, and the price decreases even further. The dual loop of contagion between the collateral market and debt payment leads the asset price to $p = 0$, and all agents default.

Similar to the unsecured debt networks, a δ-connected network is more resilient and stable than the complete network when the collateral ratio is not high enough. The key is to have two or more separable components with limited connections to each other. Hence, the interconnectedness between the two components determines the extensiveness of contagion

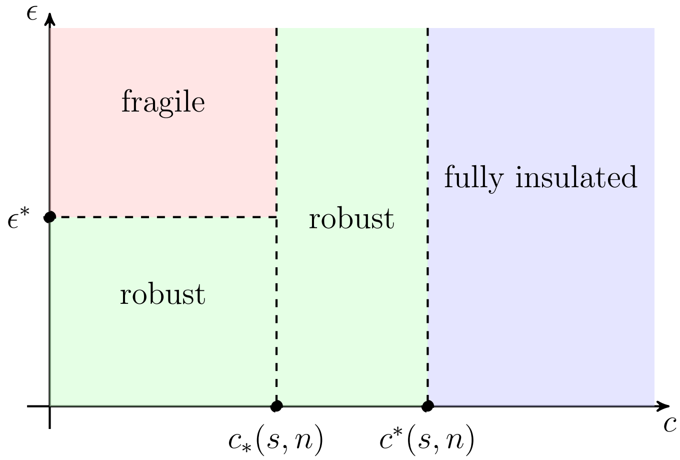


Figure 5: Different regimes depending on the collateral ratio and liquidity shock

within the network, not the collateral ratio, when $c < c_*$. In other words, assessing financial stability requires not only monitoring the average level of leverage, but also how counterparty exposures are distributed within the network as interconnections.

Lastly, recall that we are changing the amount of collateral while fixing agents' endowments and total debt amounts. Hence, our analysis highlights how specifying collateral explicitly could change the contagion drastically, ranging from full insulation to total market collapse. This is because collateral mitigates the effects of idiosyncratic shocks within the network by guaranteeing payments valued at the market price for collateral to surviving/non-defaulting agents. Such payments with collateral feed back into the network to support the high price of collateral. Therefore, our results highlight the importance of modeling explicit collateral as opposed to modeling collateral implicitly tied to an agent's total asset holdings or going-concern value.

Figure 5 summarizes our results. The horizontal axis represents the collateral ratio, c, and the vertical axis represents the size of the liquidity shock, ϵ. If c is above the threshold $c^*(s, n)$, then the equilibrium is under the fully insulated regime, which is shaded in blue. If c is above the threshold $c_*(s, n)$ but below $c^*(s, n)$, then the equilibrium is under the robust regime, which is shaded in green, and there is limited collateral contagion. Under the robust regime, having more links makes the network more stable and resilient. If c is below $c_*(s, n)$, then the regimes depend on the size of the liquidity shock. If $\epsilon < \epsilon^*$, then the equilibrium is under the robust regime. However, if $\epsilon > \epsilon^*$, the equilibrium is under the fragile regime, which is shaded in red, and there is active collateral contagion. Under the fragile regime, having more links can make the network less stable and resilient.

Our results highlight the fragility of collateral's role in reducing counterparty exposures.

When defaults occur, collateral can act as a buffer as long as the counterparty exposures are limited. However, as soon as the counterparty exposures exceed the threshold and collateral is not enough, then the collateral price can plummet to zero, leaving all contracts unsecured. Therefore, the equilibrium collateral price shows a phase transition (bang-bang) property.

One of the most relevant real-world implications of our results is the contagion between the digital asset (e.g., crypto-asset) markets and the traditional financial markets. Decentralized finance (DeFi) and other lending platforms of digital asset markets typically use collateralized debt contracts (Azar et al., 2022). If issuers of (asset-backed) stablecoins, which are often used as collateral for DeFi lending, have typical safe assets in the financial markets as their reserve assets, then both the financial markets and digital asset markets share the same underlying collateral. Our results suggest that even if the two markets use the same assets as collateral, a shock to the digital assets does not necessarily spread to the financial markets as long as the interconnectedness between the two markets is small. However, if the two markets have more than a small degree of interconnectedness, a shock to the digital asset markets can also propagate to the financial market, resulting in a systemic event. Furthermore, our results imply that restricted leverage will prevent such contagion.

4.6. Contagion in General Networks and Harmonic Distance

The results can be extended to contagion in general collateralized debt network structures. We use the concept of harmonic distance introduced by Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) to generalize the effect of interconnectedness on contagion and the systemic risk of a network. We define the distance between agents in the network following the definition of harmonic distance in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015).

Definition 5. *The harmonic distance from agent i to agent j is*

$$\mu_{ij} = 1 + \sum_{k \neq j} \left(\frac{d_{ik}}{d} \right) \mu_{kj}, \tag{13}$$

with the convention that $\mu_{ii} = 0$ for all i.

As noted in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), the harmonic distance from agent i to agent j depends not only on how far each of its immediate borrowers are from j, but also on the intensity of their liabilities to i, by d_{ik}/d. This debt-weighted notion for the unsecured debt network in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) is surprisingly useful in our model with secured debt, as we show in the following proposition.

Proposition 7. *Suppose that agent j is under a negative liquidity shock and defaults on it. Then, there exists $\mu^*(p) = (d - cdp)/(e_0 + h_0 p)$, which is a decreasing function of p, and the following holds:*

1. *If there is a nonempty set $\mathcal{S}$ such that agent $i \in \mathcal{S}$ does not default, then the equilibrium price is either $p = s$ or determined by*
$$\mathbb{1}' G \mu_{sj} = \frac{d - cdp}{e_0 + h_0 p} \mathbb{1}' G \mathbb{1} + \frac{n h_0 p}{e_0 + h_0 p}, \tag{14}$$
where μ_{sj} is the $|\mathcal{S}| \times 1$ vector of harmonic distances from agents in $\mathcal{S}$ to j, G is a $|\mathcal{S}| \times |\mathcal{S}|$ non-singular M-matrix,[8] and $\mathbb{1}$ is a vector of ones with the appropriate dimension for each case. Furthermore, if $\mu_{ij} < \mu^*(p)$, then agent i defaults.

2. *If all agents default, then the equilibrium price is $p = 0$ and $\mu_{ij} < \mu^*(0)$ for all i.*

3. *If $\mu^*(p) < 1$ for the equilibrium price p, then no other agents default.*

This result implies that the harmonic distance from the agent under a shock determines both the price of the collateral asset and the vulnerability of the remaining agents. This joint determination of price and defaulting agents is a simultaneous process, so it should be solved as a system of equations. Nevertheless, Proposition 7 implies that the pairwise harmonic distances determine the fragility of a network, generalizing the results in Proposition 6. For example, the harmonic distance between any pair of agents is minimized in the complete network (Acemoglu, Ozdaglar, and Tahbaz-Salehi, 2015). Also, in a δ-connected network, there always exists a pair of agents whose pairwise harmonic distance is greater than $\mu^*(0)$. Finally, $\mu^*(p)$ becomes small enough if c is large enough; hence, any pairwise harmonic distance is large enough to prevent any contagion stemming from agent j.

5. Extensions

In this section, we discuss extensions of the baseline model and the ways in which the results of the baseline model would change or remain.

[8] If matrix A can be expressed in the form
$$A = sI - B, \qquad s \geq \rho(B), \qquad B \geq 0$$
where $\rho(B)$ is the spectral radius of B, then A is called an M-matrix (Berman and Plemmons, 1979, p. 133). An M-matrix is non-singular if $s > \rho(B)$.

5.1. Aggregate Shock and Collateral Phase Transition

We have assumed a fixed fundamental value of the asset s so far. The payoff of the asset in the future can also fluctuate at $t = 1$. Changes in the value of s can be considered aggregate shocks to the economy because they change the return (productivity) of the entire economy.

Proposition 8. *(Aggregate Shock and Vulnerability) If $s > s'$, then the phase transition collateral ratios of the two cases are $c_*(s, n) < c_*(s', n)$, and the full insulation collateral ratios are $c^*(s, n) \leq c^*(s', n)$. Similarly, if $n > n'$, then $c_*(s, n) < c_*(s, n')$ and $c^*(s, n) \leq c^*(s, n')$. Thus, a given network with collateral ratio c under s' or n' is more vulnerable to liquidity shocks than that under s or n.*

The result implies that as the aggregate shock increases—that is, s decreases—then the overall safe region decreases as depicted in Figure 6. For the same collateral ratio, the complete network might be fully insulated, but after the aggregate shock, the network might become the least stable and resilient network if it is under the new threshold collateral ratio. Thus, an aggregate shock on s can entail different systemic risk levels for the same collateral amount and the same network structure. Therefore, Proposition 8 provides the required level of c for the given network structure, aggregate shock distribution, and the desired level of financial stability—whether to shut down collateral price contagion by setting $c > c_*(s, n)$ or to shut down any contagion by setting $c > c^*(s, n)$ with a certain probability. Furthermore, Proposition 8 implies that the threshold collateral ratios also increase when there are fewer agents in the economy.

5.2. Changes in the Total Supply of Assets

The results in Propositions 3 and 4 imply that an increase in h_0, the total supply of assets, would increase both c^* and c_*.

Corollary 1. *The threshold collateral ratios for the fully insulated regime and robust regime, $c^*(s, n)$ and $c_*(s, n)$, are increasing in the asset endowment h_0 and increasing in the total supply of assets nh_0.*

First, this relationship implies that the collateral ratio required to fully insulate contagion is higher under a greater supply of assets, holding all else equal. In other words, the total supply of assets would increase the amount of fire sales and put more downward pressure on prices. Thus, more collateral is needed to guarantee a contract will be fulfilled with collateral. Second, this relationship also implies that the ring network is more likely to be

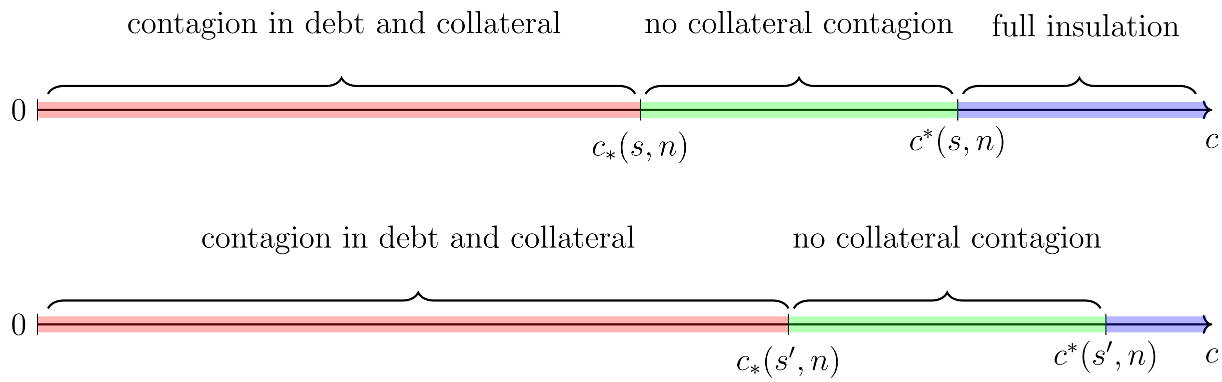


Figure 6: Aggregate shock and vulnerability

Note: Each line represents the collateral ratio. Red areas represent the region with both contagion in debt and collateral, green areas represent the region without collateral contagion, and blue areas represent the region with full insulation by collateral. The top line illustrates the three different regions under the baseline parameters (s, n). The bottom line illustrates the three different regions under the negative aggregate shock with $(s'n)$ such that $s' < s$. The figure shows that the required collateral ratios to attain the desired level of stability (no collateral contagion or full insulation) increase when there is a negative aggregate shock.

the least stable and resilient network as the required collateral to guarantee the payments within the ring network increases. Therefore, the leverage that makes a financial network stable would heavily depend on the total supply of assets due to the collateral price channel of contagion.

Technically, agents have more endowments with higher h_0. However, actual payments involve cash because agents under liquidity shocks must sell their assets to pay for the liquidity shock (senior debt). Since the total amount of cash remains the same, more assets in fire sales would only depress the price of assets, decreasing the value of collateral and increasing the outflow of cash from the network. Hence, the network suffers from more defaults, depressing the price further. This result again highlights the importance of explicit collateral in mitigating contagion in debt and collateral markets.

5.3. Iterative Fire Sales

The model assumes simultaneous fire sales, as all agents decide to sell their assets at the same time, and the market price is determined by market clearing conditions. Such instantaneous and collective adjustment may not occur in a very short period in the real world. Our model can be extended to incorporate an alternative model of fire sales, iterative fire sales, which is often used in empirical models of fire sales in the literature, such as in Greenwood, Landier, and Thesmar (2015) and Duarte and Eisenbach (2021). In an iterative model of fire sales, agents lower their leverage through fire sales in reaction to the initial fire sale amounts of other agents. The model in this paper can also be extended to this iterative fire-sale procedure by determining the fire-sale amount, while holding the previous price fixed:

$$(n-1)e_0 - d + cdp^0 = \sum_{i \in N} \frac{\phi_i^0(p^0)}{p^0} p^1$$

$$(n-1)e_0 - d + cdp^k = \sum_{i \in N} \frac{\phi_i^k(p^k)}{p^k} p^{k+1} \qquad \text{for any } k \leq K,$$

where the fire-sale amount is determined by the new price as

$$\phi_i^{k+1}(p^{k+1}) = \min\left\{ \left[h_j p^{k+1} - m_j(p^{k+1})\right]^+, h_j p^{k+1} \right\}.$$

This iterative procedure continues to generate the negative effects of fire sales on price p. The maximum number of iterations, K, can be different across different contexts. For example, Greenwood, Landier, and Thesmar (2015) assume $K = 1$.

5.4. Fire Sales with External Traders

In the baseline model, the only source of demand is the agents within the debt network. Now, suppose there exist external traders who are not directly involved with the debt network or its relevant payments but are buying and selling the assets. Indeed, the existence of external traders can mitigate or amplify the severity of the fire-sale externalities in the market depending on the parameters. Following the literature on fire sales, we assume that external traders can amplify the problem of fire sales, as their presence can exacerbate information asymmetry, liquidity hoarding, and margin spirals, which all accelerate the sales of the asset. Suppose that there are external traders with linear demand—that is, $\alpha - \beta p$. Without loss of generality, we focus on the case with $\alpha - \beta s < \sum h_i$, because otherwise, external demand can support the fair price of the asset even when every agent in the network defaults. Finally, assume that the fire-sale procedure goes through the iterative procedure discussed in the previous subsection, which follows the methods in Greenwood, Landier, and Thesmar (2015) and Duarte and Eisenbach (2021). The new market clearing condition becomes

$$\sum_{j \in N} [m_j(p)]^+ - \sum_{j \notin \mathcal{D}(p)} h_j p = \sum_{i \in N} \phi_i(p) - \alpha + \beta p \qquad \text{if} \quad 0 \le p < s \tag{15}$$

$$\sum_{j \in N} [m_j(p)]^+ - \sum_{j \notin \mathcal{D}(s)} h_j s \ge \sum_{i \in N} \phi_i(s) - \alpha + \beta s \qquad \text{iff} \quad p = s. \tag{16}$$

All the arguments in Proposition 1 hold under this setup. This extended model also provides a microfoundation of the dynamics of fire sales in Greenwood, Landier, and Thesmar (2015), which assume linear changes in net returns for fire sales—that is,

$$F_2 \equiv \frac{p_2 - p_1}{p_1} = L\phi,$$

where F_2 is the asset net return, p_2 and p_1 are asset prices before and after the sales, ϕ is the fire-sale amount, and L is the fire-sale parameter. This equation can be rearranged as

$$p_2 = L\phi p_1 + p_1.$$

Now consider our model in this extension as $p_1 = M/(\text{sales})$ and $p_2 = M/(\text{sales} + \beta)$, so that

$$\frac{p_2 - p_1}{p_1} = -\beta \text{sales},$$

which shows the linear price impact.

Thus, the two models are equivalent in the structure of the fire-sale effect. However, our model incorporates the endogenous fire-sale amount from the debt contagion rather than from a given fixed leverage target. Therefore, an interesting direction in which to extend the model in this paper would be combining the fire-sale spillovers to multiple markets of multiple assets with internal and external agents for the given collateralized debt network.

5.5. Other Generalizations

There are other ways to generalize the results. First, there are relatively straightforward ones. As in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), allowing partial liquidation of long-term projects as $\zeta > 0$ is relatively straightforward. All the main results of the baseline model hold with minor differences in conditions. Similarly, allowing for multiple shocks as $\kappa > 1$ is relatively straightforward, as in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), and, again, most results hold in this extension.

There are also more difficult ways to generalize the baseline model. First, allowing for

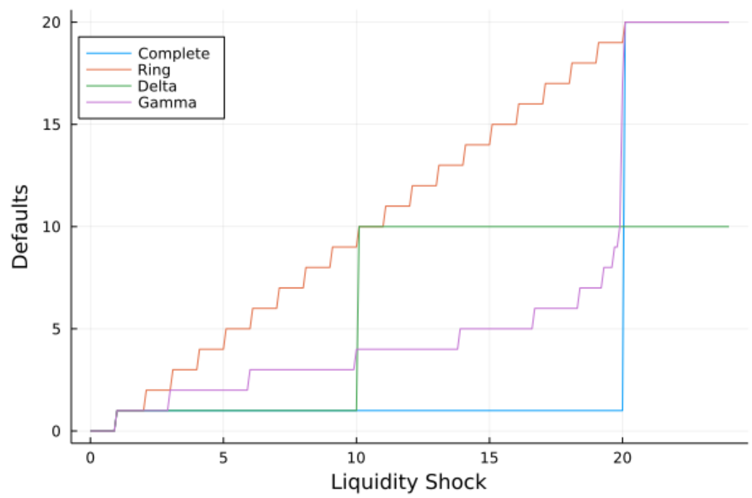


Figure 7: Unsecured debt

heterogeneous collateral ratios is challenging because the difference in payments depends on each individual collateral ratio for different price levels. Thus, there would be many different price regions of contagion depending on the shock size and network structure. Fortunately, the interaction between price and contagion is uni-directional: a lower price increases contagion, and vice versa, following Lemma 1. Therefore, a model with heterogeneous collateral ratios can certainly be solved numerically. Related to the numerical model, generalizing the framework with other forms of counterparty exposures is also possible by setting different c_{ij} depending on the contract structure. Second, analytical results with a more general shock distribution would be quite complex because of the exceedingly large number of dimensions to consider. Again, such a model can be solved numerically for any given distribution of ω.

6. Numerical Analysis

To highlight key attributes of the model, we conduct various numerical tests. First, we evaluate the number of defaulting agents for various network structures under different shock regimes when the collateral channel is excluded, confirming the main results from Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). When agents hold assets and post portions of them as collateral, we extend the results by including a third dimension for the collateral ratio. Lastly, we examine the δ-connected network by defining δ to be each agent's individual debt exposure to a secondary component in the network. We then compare levels of default and the equilibrium price of assets across changes in δ as a percentage and collateral ratio.

6.1. Network Contagion Patterns without the Collateral Channel

As shown by Proposition 2, our model extends Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). Therefore, our models' results should replicate Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) when the asset holdings (h_0) and collateral ratio (c) are 0 for all agents. To test this, we run a numerical simulation that compares the stability and resilience of the complete network, ring network, δ-connected network when $\delta = 0$, and a γ-convex combination of the complete and ring networks when $\gamma = 0.5$ by measuring the number of defaults across varying liquidity shocks (ϵ). The δ-connected network is structured as two discrete components of complete networks, each containing half of the agents in N. We assume homogeneity, where all agents have the same total debt (d) obligations and cash endowments (e_0), to ensure that the variation in our results is solely attributed to the variation in network structures. In this setup, with $n = 20$ agents with no assets and collateral, e_0 is 1 and d is $2d^* = 38$. Only one agent is under a liquidity shock at a given time.

As shown in Figure 7, we reaffirm Proposition 2 when counterparty debt exposure is the only source of contagion. The complete network is the most stable and resilient structure under the small shock regime and becomes as equally fragile as the ring and γ-convex networks under the large shock regime. The δ-connected network is more stable and resilient than the complete network, with only half of the agents defaulting under any shock. Lastly, the ring network is the least stable out of all four networks in all shock regimes.

6.2. Network Contagion Patterns with the Collateral Channel

Now, we evaluate how stability, resilience, and the collateral price change when agents can commit assets as collateral. Using the same setup as in the previous section, except that c can be greater than 0 and h_0 is equal to 2, we model each network's number of defaults under varying shocks (ϵ) and collateral ratios (c). The collateral ratio ranges from 0 to a value slightly larger than c^*, c^* being 1 and c_* being 0.55. The asset's fair value (s) is 1. The results depicting defaults and collateral prices are presented as 3-dimensional graphs in Figure 8 and Figure 9.

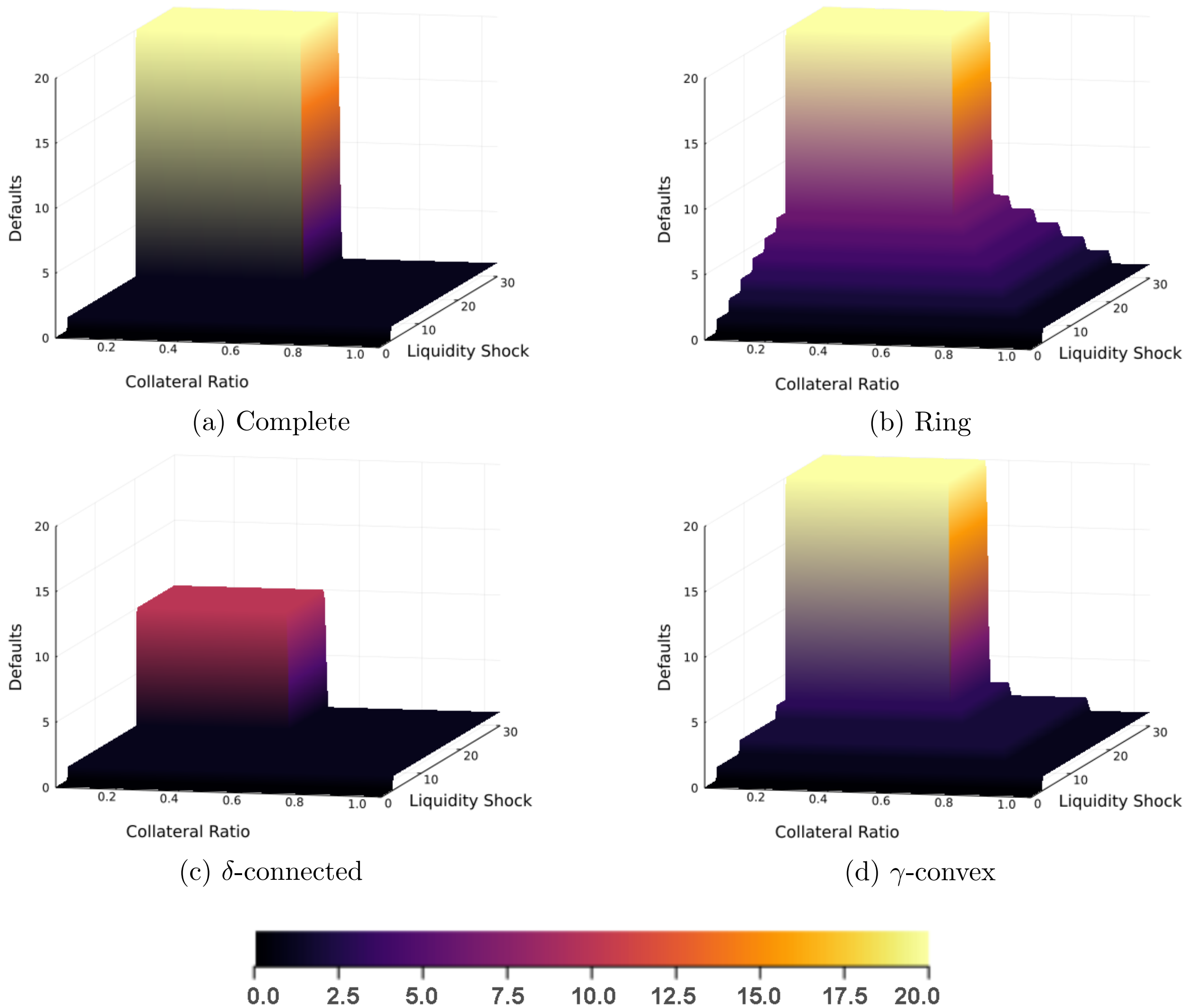
When $\epsilon < \epsilon^* = 20$ or $c \geq c_*$, the complete network is the most stable and resilient, the ring network is the least stable and resilient, and the γ-convex network remains in the middle. The price of collateral remains high, at its fair value. Both of these results reaffirm Proposition 5. When $\epsilon > \epsilon^*$ and $c < c_*$, all agents default in the complete, ring, and γ-convex networks, making the complete network also the least stable and resilient. The equilibrium price of collateral is 0, which maximizes agents' debt obligations ($d - cdp$). Therefore, contagion from the collateral price channel exacerbates contagion from the debt channel. With δ being sufficiently low, the δ-connected network is more resilient and stable than the complete network, and the collateral price is greater than 0, which is consistent with Proposition 6. When $\epsilon > \epsilon^*$ and $c > c_*$, the price of collateral remains its fair value s, and contagion is mitigated. An increase in the collateral ratio can reduce the number of defaulting agents for all networks, but most quickly for the complete network and most gradually for the ring network. When $c \geq c^*$ under any liquidity shock, we see collateral fully insulating all networks from debt contagion, resulting in only the shocked agent defaulting. Any network is the most stable and resilient for any ϵ as confirmed in Proposition 3.

Overall, these simulations highlight how collateral can be useful in enhancing stability because it lowers the spillovers through inter-agent liabilities, but only when agents commit a certain threshold of their assets, c_*. If the collateral ratio does not meet this threshold, the value of collateral cannot be supported, as agents are unable to pay their debt obligations, which diminishes their collective wealth and renders collateral ineffective in promoting network stability. Another important observation is that the contagion patterns between the four network structures are different when measuring defaults, yet similar when measuring equilibrium price. This shows how collateral price relies on collective wealth, which is the consistent factor across the different network structures. Having different levels of default despite maintaining the same collateral price shows that the network structure affects the final distribution of collective wealth across agents. For example, the ring network has the most defaulting agents, which means the final collective wealth is concentrated among fewer agents relative to the complete network under certain regimes. And because of this concentration, there are more inefficiencies due to costly liquidation of long-term investment projects that lowers the social surplus of the economy.

6.3. Delta-Connected Networks

In this experiment, we evaluate networks of 20 agents as in Figure 10, where each agent has 1 unit of cash, 2 assets, and a total liability of $d = 2d^* = 38$. One agent out of the 20 receives a large liquidity shock of $\epsilon > \epsilon^* = 20$. The network is segmented into two even components S and S^c, where each agent owes a large liability of $(d - 10\delta d)/9$ to every agent within the same component and a small liability of δd to every agent in the opposing component. Let δ^* be defined as the minimum δ value such that the δ-connected network represented in Figure 10 is the least stable network.

In this setup, $\delta^* = 0.0048$ and $c_* = 0.55$. We evaluate changes in δ represented as a percentage called intercomponent exposure against changes in the collateral ratio and measure the network's number of defaults as well as the equilibrium asset price, as shown in



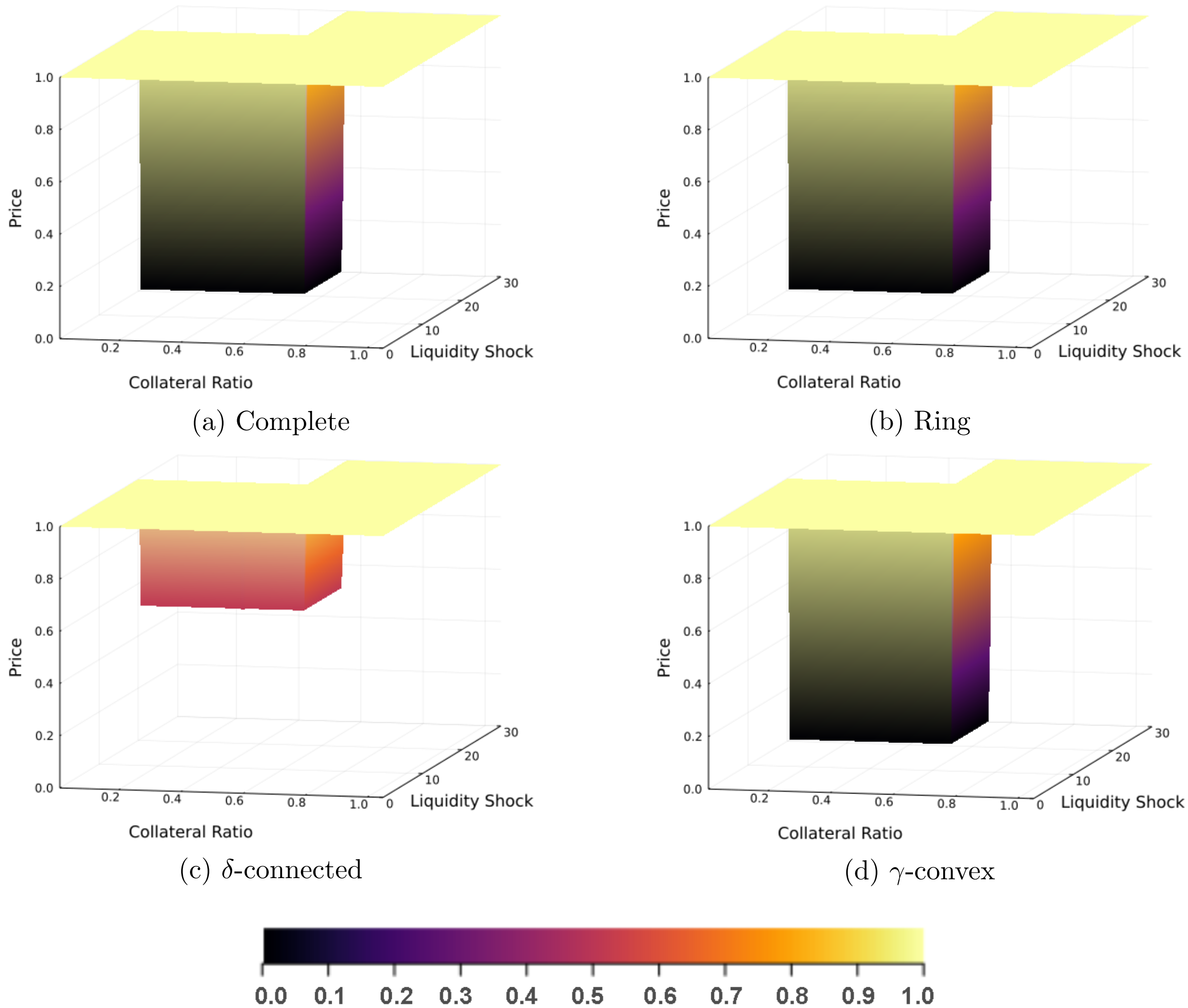
(a) Complete

(b) Ring

(c) δ-connected

(d) γ-convex

Figure 8: Contagion under collateralized debt



(a) Complete (b) Ring

(c) δ-connected (d) γ-convex

Figure 9: Price changes under collateralized debt

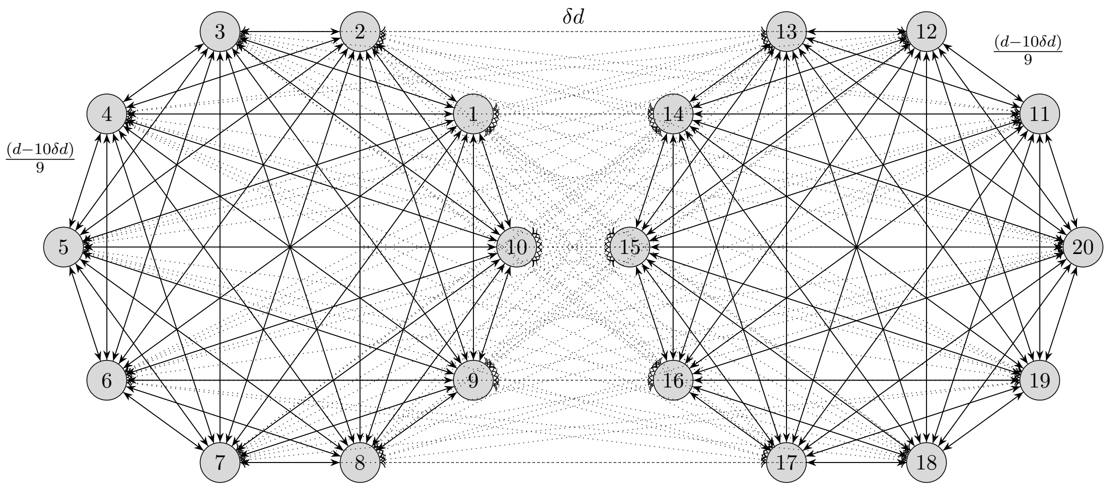


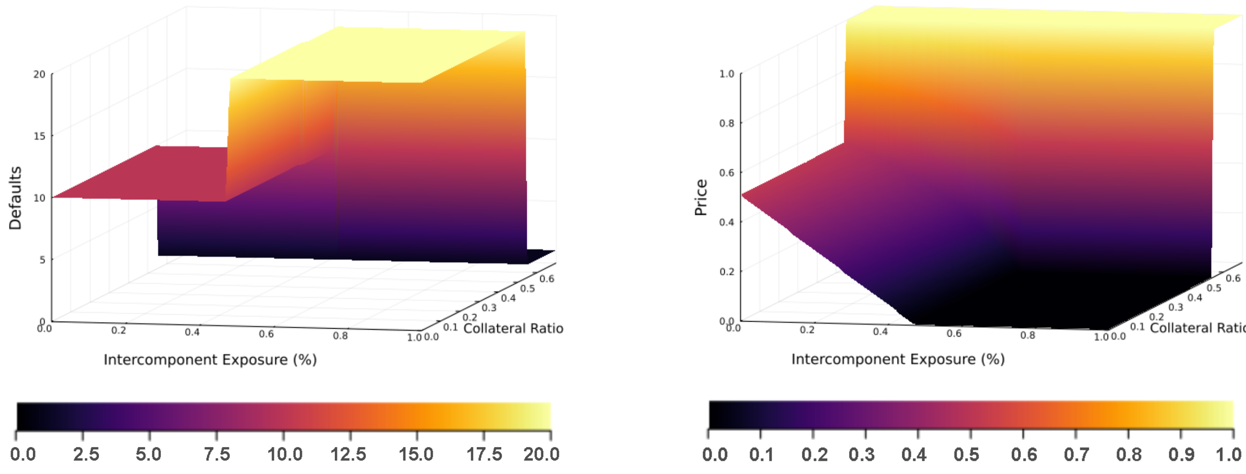
Figure 10: The δ-connected network for simulation

Figures 11a and 11b.

When $c < 0.55$ and $\delta < 0.0048$, defaults remain consistently at 10, while the price declines from 0.5. Even when the price is close to 0, the number of defaults can still remain at 10. When $\delta = 0.0048$, the number of defaults jumps from 10 to 20 and the price becomes 0. Defaults and the price remain at 20 and 0, respectively, when $\delta \geq 0.0048$. We find that δ^* is relatively small, meaning that it requires little exposure for the contagion to spread from one component to the other. Furthermore, contagion is minimized, contained to half the agents, when $\delta < 0.0048$ but is maximized, when $\delta \geq 0.0048$ exhibiting a quick phase transition. When $c \geq 0.55$, the network becomes safe and fully insulated, with the maximum price.

We numerically reinforce the notion of the δ-connected network being more stable and resilient than ring and complete networks conditional on the δ value being small by quantifying exactly how small it should be. We find that δ is minimal; only 4.8% of each agent's counterparty exposures to the secondary component is required for contagion to spread fully throughout the network. We also find that collateral plays a minimal role in shaping the network's stability when it is not enough and only makes a difference when c reaches the threshold c_*. Therefore, we find the phase transition property between collateral and the network architecture. When there is enough collateral, the network architecture becomes irrelevant. However, when there is not enough collateral, the network architecture becomes the dominant factor in determining the number of defaults in equilibrium. In a real world context, policymakers can increase the required collateral ratio (stricter leverage restriction) to ensure financial stability when the interconnectedness between two markets increases.

Finally, we replicate the experiment evaluating δ^*, labeled as the threshold intercomponent exposure, across various δ-connected networks where the two components are a γ-convex



(a) Defaults

(b) Price

Figure 11: Changes in intercomponent exposure (%) and collateral ratio (c)

combination of complete and ring networks. We keep the collateral ratio fixed to $c < c_*$ and the liquidity shock fixed to $\epsilon > \epsilon^*$. The results are displayed in Figure 12. We see that when $\gamma = 0$, the two components are ring networks and have the lowest threshold of intercomponent exposure. Therefore, this configuration is most prone to contagion relative to the other network structures. As γ increases, δ^* also increases and then remains constant at $\delta^* = 0.0048$. When $\gamma = 1$, we see the same result as in the previous experiment. The positive trend in δ^* implies that as γ increases for each component, the more stable the entire network becomes, as the threshold for the intercomponent exposure required for the network to fully default is higher.

7. Conclusion

This paper constructed a model with both debt and collateral market contagion with endogenous fire-sale prices. Collateral can mitigate debt contagion by guaranteeing the payment in case of borrower default. We find that the role of collateral shows a phase transition property, as it changes dramatically depending on the collateral ratio. If the collateral ratio is sufficiently large, the economy is fully insulated. Therefore, any network with any size liquidity shock will generate the same most stable and resilient outcome. If the collateral ratio is moderately high, then the collateral price is its fundamental value regardless of the network structure and the size of the negative liquidity shock. In this case, the economy is in a robust regime, as the complete network is the most stable and

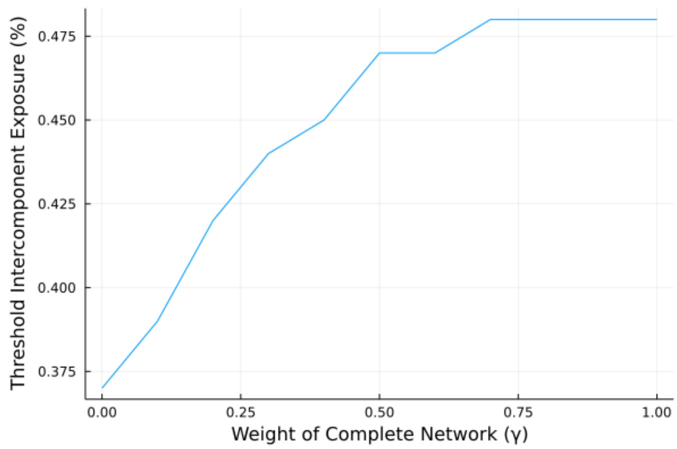


Figure 12: δ^* for varying δ-connected networks with γ-convex components

resilient network. However, if the collateral ratio is not very high and the liquidity shock is large, then all agents in the complete network default and the asset price goes to zero. The stark difference in collateral prices depending on the collateral ratio shows the fragility of collateral as a buffer for counterparty exposures. Since the payoff of collateral assets is public and fixed, our results also highlight the importance of liquidity flows in bilateral lending relationships during market stress and fire sales. In the last case, a network with two components with very limited interconnections with each other performs better, as the shock from one component does not spread to the other component. This result implies that the growing interconnectedness between the traditional financial markets and digital asset markets is concerning for financial stability.

Our results highlight both the importance of explicit collateral and the fragility of it. Contrary to the models using the total amount of capital or going-concern value as collateral, contracts in our model specify explicitly designated collateral, which is more in line with how collateralized debt markets operate in the real world. In our model, even if the total amounts of assets or individual asset holdings are the same, changes in collateral ratios can drastically change the contagion pattern. Also, this contagion pattern depends on the interconnectedness of agents.

The model also provides insights on macroprudential policy for financial stability. For the same network structure, if the value of the collateral asset decreases, the threshold levels that attain robust and fully insulated regimes increase. Similarly, if the total supply of the collateral asset increases, the required threshold levels also increase. Thus, as the aggregate economy changes, the model can provide the minimum collateral ratio required to attain a robust or fully insulated regime.

Finally, the model is general yet flexible enough to encompass and accommodate many types of extensions, including the recent literature on fire-sale spillovers. While the literature on financial networks with collateral is limited, the model in this paper can shed light on the framework that can be useful in empirical and numerical analysis for financial stability and systemic risk. Furthermore, this paper provides insights on the roles of collateral in financial markets in general.

Appendix: Omitted Proofs and Results

A. Preliminaries

A.1. Collateral Netting

We define a useful way of considering network contagion for a given asset price p, which is *collateral netting*. The role of collateral netting is to pre-calculate any over-collateralized payments, as they are guaranteed to be paid in full by the collateral posted. In addition, collateral netting also lumps the collateral of under-collateralized payments into the lender's asset holdings, as the collateral guarantees the market value of collateral even if the borrower pays nothing. Therefore, collateral netting will simplify the derivations while keeping the equilibrium payments the same.

For a given collateralized debt network and environment $(N, C, D, e, h, s, \omega)$ and asset price p, define debt obligations, cash holdings, and asset holdings after collateral netting as the following for any $i, j \in N$:

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij} d_{ij} p]^+ \tag{17}$$

$$\hat{e}_j(p) = e_j + \sum_{\substack{k \in N \\ c_{jk} p > 1}} d_{jk} - \sum_{\substack{i \in N \\ c_{ij} p > 1}} d_{ij} \tag{18}$$

$$\hat{h}_j(p) = h_j + \sum_{\substack{k \in N \\ c_{jk} p \leq 1}} c_{jk} d_{jk} - \sum_{\substack{i \in N \\ c_{ij} p \leq 1}} c_{ij} d_{ij}. \tag{19}$$

This collateral netting derives an interim network after netting out the collateral and payments across agents for a given price p. For example, if a contract d_{ij} is over-collateralized, $c_{ij} p > 1$, then $\hat{d}_{ij}(p)$ is 0, because full payment is guaranteed by the collateral posted. Collateral netting calculates the transfer of full payment d_{ij} to lender i, which is included in i's cash holdings $\hat{e}_i(p)$ and $\hat{e}_j(p)$, and the transfer of collateral $c_{ij} d_{ij}$ to borrower j, which is included in $\hat{h}_j(p)$. If a contract d_{jk} is under-collateralized, $c_{jk} p \leq 1$, then k owes j additional amount of $\hat{d}_{jk}(p) > 0$ on top of the collateral value. Because the market value of collateral does not depend on who owns the collateral, assume that the collateral is kept by lenders for under-collateralized debts, without loss of generality. Then, the collateral will be in j's balance sheet in the asset holdings $\hat{h}_j(p)$. Hence, collateral netting simplifies the cross-agent debt payments by taking care of payments related to collateral and the ownership of collateral, which do not depend on whether a borrower defaults or not.

A.2. Payment Equilibrium under Collateral Netting

We introduce a matrix notation for the payment equilibrium of a collateral-netting network that corresponds to the payment equilibrium of the original network. Let $Q(p) \in \mathbb{R}^{n \times n}$ be the matrix with its (i, j) element as q_{ij} defined in equation (7). Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)'$ be the vector of agents' total inter-agent liabilities and $\mathbf{l} = (l_1, l_2, \ldots, l_n)'$ be the vector of agents' liquidation decisions. Define

$$\hat{z}_j(p) \equiv \hat{e}_j + \hat{h}_j p - \omega_j \epsilon,$$

and $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_n)'$.

Equations (5) and (6) for every agent can be simplified in matrix notation as the system of equations below:

$$\hat{\mathbf{x}} = \left[\min\left\{Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}, \hat{\mathbf{d}}\right\}\right]^+ \tag{20}$$

$$\hat{\mathbf{l}}(p) = \begin{cases} \left[\min\left\{\frac{1}{\zeta}\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbb{1}\right\}\right]^+ & \text{if } p \geq s\zeta \\ \xi\mathbb{1} & \text{if } p < s\zeta, \end{cases} \tag{21}$$

where $\hat{x}_j = \sum \hat{x}_{ij}$, $\hat{\mathbf{x}}$ is the vector of $\hat{x}_j$'s, and $\mathbb{1}$ is a vector of ones for the appropriate dimension. Note that if Q is not defined, then the payments are trivially determined, as agents can pay their debt in full using collateral. The function entry p is omitted here and will be omitted from now on unless necessary for exposition. We define the payment equilibrium of a collateral-netting network, a debt network with no collateral after performing collateral netting on the original network, with these modified payment and liquidation rules.

Definition 6. *For a fixed price p and a collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is a payment equilibrium if it satisfies (20) and (21).*

We now show that the payment equilibrium under the *collateral-netting network* is equivalent to the payment equilibrium under the original network. In other words, if $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ satisfies the payment rule and the liquidation rule as above, then the corresponding $(\mathbf{x}, \mathbf{l})$ satisfies the payment and liquidation rules for the original network for the given price.

Lemma 5. *The net wealth and payments of agents in the payment equilibrium of the collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$ for a given full equilibrium price p is the same as the net wealth and payments of agents in the payment equilibrium of the original network $(N, C, D, e, h, s, \omega)$ for a given full equilibrium price p.*

Proof. The payment under the collateral-netting network is simplified as below for any $i, j \in N$,

$$\hat{x}_{ij}(p) = \min\left\{\hat{d}_{ij}, q_{ij}(p)\left[\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k\in N} \hat{x}_{jk}\right]^+\right\} \tag{22}$$

$$= \left[\min\left\{\hat{d}_{ij}, q_{ij}\left(\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k\in N} \hat{x}_{jk}\right)\right\}\right]^+,$$

where the second equality holds because $\hat{d}_{ij} \geq 0$ for any $i, j \in N$ by (17). If $c_{ij}p \leq 1$ and the net wealth for the second case is the same with $m_j(p)$, then the payments are the same. If $c_{ij}p > 1$, then $\hat{d}_{ij}(p) = 0$, but from $\hat{e}_j(p)$ and $\hat{h}_j(p)$, the payment $d_{ij} - c_{ij}d_{ij}p$ will be subtracted from j's net wealth. Therefore, the payments are equivalent to x_{ij} for any $i, j \in N$ as long as the net wealth is equivalent. The corresponding net wealth is

$$\begin{aligned}
\hat{m}_j(p) \equiv & \hat{e}_j + \hat{h}_j(p)p - \omega_j\epsilon + \zeta l_j(p) + \sum_{k\in N}\hat{x}_{jk} - \sum_{i\in N}\hat{x}_{ij} \\
= & e_j + \sum_{\substack{k\in N \\ c_{jk}p>1}} d_{jk} - \sum_{\substack{i\in N \\ c_{ij}p>1}} d_{ij} + h_j p + \sum_{\substack{k\in N \\ c_{jk}p\leq 1}} c_{jk}d_{jk}p - \sum_{\substack{i\in N \\ c_{ij}p\leq 1}} c_{ij}d_{ij}p \\
& + \sum_{\substack{i\in N \\ c_{ij}p\leq 1}} x_{jk} - \sum_{\substack{i\in N \\ c_{ij}p\leq 1}} (d_{ij} - c_{ij}d_{ij}p) - \omega_j\epsilon + \zeta l_j(p) \\
= & e_j + h_j p - \omega_j\epsilon + \zeta l_j(p) + \sum_{k\in N} c_{jk}d_{jk}p + \sum_{k\in N} x_{jk} - \sum_{i\in N} d_{ij} \\
= & m_j(p),
\end{aligned}$$

which implies the net wealth remains the same as in the original network. ■

Note that collateral netting is trivial when the asset price or collateral ratio is high. Therefore, the payment amount and the market value of the asset holding under the collateral-netting network are increasing in p.

The only thing left to check is the market clearing condition, equation (11), and if the market clearing condition holds, the given payment equilibrium values constitute a full equilibrium. The following lemma, which is a direct application of Lemma B2 in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), further simplifies the computation of the payment equilibrium and full equilibrium.

Lemma 6. *Suppose that p is a price from a full equilibrium. Suppose that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is from a* *collateral-netting network of a full equilibrium* $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$. *Then,* $\hat{\boldsymbol{x}}$ *satisfies*

$$\hat{\boldsymbol{x}} = \left[\min\left\{Q\hat{\boldsymbol{x}} + \hat{z} + \zeta\xi\mathbb{1}, \hat{\boldsymbol{d}}\right\}\right]^+. \tag{23}$$

Conversely, if $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ *satisfies (23), then there exists* $\hat{\boldsymbol{l}} \in [0, \xi]^n$ *such that* $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ *is a payment equilibrium and the corresponding* $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$ *is a full equilibrium.*

Proof of Lemma 6. Suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is from a full equilibrium and forms a payment equilibrium for the equilibrium price p. First, suppose that $p < s\zeta$. Then, every agent will liquidate its assets regardless of the payments so $\hat{\mathbf{l}} = \xi\mathbb{1}$ and the payment rule satisfies (23). Now suppose that $p \geq s\zeta$. By liquidation rule (21), $\zeta\hat{\mathbf{l}} = \left[\min\left\{\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \zeta\xi\mathbb{1}\right\}\right]^+$, which yields

$$\begin{aligned} Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\} \\ \Rightarrow \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\} &= \min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\}\right\} \\ &= \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}. \end{aligned}$$

Thus, $\hat{\mathbf{x}} = [\min\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\}]^+ = [\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}]^+$.

Now we consider the other direction. Again, if $p < s\zeta$, then agents will liquidate all of their projects. Therefore, if p is an equilibrium price, then there exists an equilibrium with $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$. Finally, suppose that $p \geq s\zeta$. Then, from (21) and (23), $\hat{\mathbf{l}}(p) = \left[\min\left\{1/\zeta\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbb{1}\right\}\right]^+$ is satisfied. Plugging this expression into the notation of $\mathbf{X}$, we get

$$\begin{aligned} Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\} \\ \Rightarrow \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\}\right]^+ &= \left[\min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right\}\right\}\right]^+ \\ &= \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right]^+ = \hat{\mathbf{x}}. \end{aligned}$$

as in the other direction. Therefore, the equilibrium payment rule is also satisfied, and $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is a payment equilibrium for price p. Because p is a full equilibrium price, the corresponding $(\mathbf{x}, \mathbf{l}, \mathbf{m}, p)$ is a full equilibrium. ■

A.3. Solution to the Agent's Optimization Problem

We solve for agent j's optimization problem (8). The first-order conditions (FOCs) of the optimization problem are

$$
\begin{aligned}
\partial e: &\quad 1-\lambda_w+\lambda_e=0\\
\partial h: &\quad s-\lambda_w p+\lambda_h=0\\
\partial l: &\quad -1+\lambda_w\zeta+\lambda_{\underline{l}}-\lambda_{\bar{l}}=0,
\end{aligned}
$$

where $\lambda_w, \lambda_e, \lambda_h, \lambda_{\underline{l}}$, and $\lambda_{\bar{l}}$ are the Lagrangian multipliers for the budget constraint, non-negativity constraint for e, non-negativity constraint for h, liquidation constraint for l, and the upper bound constraint for l, respectively.

Complementary slackness conditions are

$$
\begin{aligned}
&\lambda_{\underline{l}}(l-\underline{l})=0\\
&\lambda_e e=0\\
&\lambda_w h=0\\
&\lambda_{\bar{l}}(\xi-l_j)=0.
\end{aligned}
$$

Case 1. $\underline{l}=\xi$. This implies agent j is obligated to liquidate the long-term investment project in full to pay their payment obligations. Also, j does not have extra cash, as $a_j(p)-b_j(p)\leq\xi$. The budget constraint becomes

$$e+hp=0,$$

and by $e,h\geq 0$ and $p\geq 0$, $(e,h,l)=(0,0,\xi)$.

Case 2. $\underline{l}<\xi$. Under this case, only partial or no liquidation is required.

Case 2.1. $l_j=\xi$, $e>0$, $h>0$. FOCs for e and l imply

$$
\begin{aligned}
&\lambda_w=1\\
&\lambda_{\bar{l}}=-1+\lambda_w\zeta,
\end{aligned}
$$

and combining the two yields $\lambda_{\bar{l}}=-1+\zeta$, which contradicts $\lambda_{\bar{l}}>0$ because $\zeta<1$, so this case does not exist.

Case 2.2. $l_j = \xi$, $e > 0$, $h = 0$. FOCs imply

$$\lambda_w = 1$$
$$\lambda_h = \lambda_w p - s$$
$$\lambda_{\bar{l}} = -1 + \lambda_w \zeta,$$

implying $\lambda_{\bar{l}} = -1 + \zeta$, which contradicts $\lambda_{\bar{l}} > 0$ because $\zeta < 1$.

Case 2.3. $l_j = \xi$, $e = 0$, $h > 0$. FOCs imply

$$\lambda_w = \frac{s}{p}$$
$$\lambda_e = \frac{s}{p} - 1$$
$$\lambda_{\bar{l}} = -1 + \zeta \frac{s}{p},$$

which hold only if $p \leq s\zeta$.

Case 2.4. $\underline{l} < l_j \leq \xi$, $e = 0$, $h = 0$. This case does not exist because it violates the budget constraint.

Case 2.5. $l_j = \underline{l}$, $e > 0$, $h = 0$. FOCs imply

$$\lambda_w = 1$$
$$\lambda_h = p - s$$
$$\lambda_{\underline{l}} = 1 - \zeta,$$

which hold only when $p \geq s$ and $\underline{l} = 0$ because otherwise agent j's budget is 0, implying $e = 0$.

Case 2.6. $l_j = \underline{l}$, $e = 0$, $h > 0$. FOCs imply

$$\lambda_w = \frac{s}{p}$$
$$\lambda_e = \frac{s}{p} - 1$$
$$\lambda_{\underline{l}} = 1 - \zeta \frac{s}{p},$$

which hold only if $p \geq s\zeta$ and $\underline{l} = 0$ because otherwise agent j's budget is 0, implying $h = 0$.

Case 2.7. $l_j = \underline{l}$, $e > 0$, $h > 0$. FOCs imply

$$\lambda_w = 1 = \frac{s}{p}$$
$$\lambda_{\underline{l}} = 1 - \zeta,$$

which hold only if $s = p$ and $\underline{l} = 0$ because otherwise agent j's budget is 0, implying $e = h = 0$.

Case 2.8. $l_j = \underline{l}$, $e = 0$, $h = 0$. FOCs imply

$$\lambda_w = \frac{1}{\zeta}(1 - \lambda_{\underline{l}}) = 1 + \lambda_e = \frac{1}{p}(s + \lambda_h),$$

which happens only if $\frac{1}{\zeta} > \frac{s}{p}$ and $\underline{l} = \frac{b-a}{\zeta} < \xi$.

Case 2.9. $\underline{l} < l_j < \xi$, $e > 0$, $h > 0$. FOCs imply

$$\lambda_w = 1 = \frac{s}{p} = \frac{1}{\zeta},$$

which contradicts $\zeta < 1$, implying such case does not exist.

Case 2.10. $\underline{l} < l_j < \xi$, $e = 0$, $h > 0$. FOCs imply

$$\lambda_w = \lambda_e + 1 = \frac{s}{p} = \frac{1}{\zeta},$$

which hold only if $p = s\zeta$.

Case 2.11. $\underline{l} < l_j < \xi$, $e > 0$, $h = 0$. FOCs imply

$$\lambda_w = 1 = \frac{1}{\zeta}$$
$$\lambda_h = p - s,$$

which contradict $\zeta < 1$, so this case does not exist.

To summarize, agent j will liquidate only up to the required amount of the long-term project, $l = \underline{l}$, if $p \geq s\zeta$; be indifferent between cash and asset holdings only if $p = s$; buy assets using all the available budget if $p < s$ and $p \geq s\zeta$; and liquidate the long-term project in full to buy more assets if $p < s\zeta$.

A.4. Useful Lemmas

The following lemma, which is Lemma B1 from Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), is used for the proof of Proposition 1.

Lemma 7. *Suppose that* $\beta > 0$. *Then,*

$$|\,[\min\{\alpha, \beta\}]^+ - [\min\{\hat{\alpha}, \beta\}]^+\,| \leq |\alpha - \hat{\alpha}|. \tag{24}$$

Furthermore, the inequality is tight only if either $\alpha = \hat{\alpha}$ *or* $\alpha, \hat{\alpha} \in [0, \beta]$.

Proof of Lemma 7. **Case 1.** Suppose that $\beta < \alpha, \hat{\alpha}$. Then, inequality (24) becomes

$$0 \leq |\alpha - \hat{\alpha}|,$$

and the inequality is tight only if $\alpha = \hat{\alpha}$.

Case 2. Suppose that $0 \leq \alpha, \hat{\alpha} \leq \beta$. Then, inequality (24) becomes

$$|\alpha - \hat{\alpha}| = |\alpha - \hat{\alpha}|.$$

Therefore, the inequality is always tight if $\alpha, \hat{\alpha} \in [0, \beta]$.

Case 3. Suppose that either $\alpha < 0 \leq \hat{\alpha}$ or $\hat{\alpha} < 0 \leq \alpha$ holds. Then, the left-hand side of (24) is either $|\hat{\alpha}|$ or $|\alpha|$, which is less than the right-hand side of (24), $|\alpha - \hat{\alpha}|$, and the inequality is tight only if $\alpha = \hat{\alpha} < 0$.

Case 4. Suppose that $\alpha, \hat{\alpha} < 0$. Then, inequality (24) becomes

$$0 \leq |\alpha - \hat{\alpha}|,$$

which is tight only if $\alpha = \hat{\alpha}$.

Case 5. Suppose that either $\hat{\alpha} \leq \beta < \alpha$ or $\alpha \leq \beta < \hat{\alpha}$ holds. Then, the left-hand side of (24) is either $|\beta - \hat{\alpha}|$ or $|\alpha - \beta|$, which is less than the right-hand side of (24), $|\alpha - \hat{\alpha}|$, and the inequality can never be tight as $\alpha \neq \hat{\alpha}$. ∎

Lemma 8. *For a full equilibrium under the assumptions in Section 4 with* $\kappa = 1$, *define* $\epsilon^* \equiv ne_0$. *Then, the following statements are true:*

1. *If* $\epsilon < \epsilon^*$, *at least one agent does not default.*

2. *If* $\epsilon > \epsilon^*$ *and* $cp < 1$, *at least one agent defaults and cannot even pay the liquidity shock.*

3. If $cp \geq 1$, no agent defaults on inter-agent debt.

Proof of Lemma 8. For the first statement, suppose $\epsilon < \epsilon^*$ and use the collateral-netting network for the given equilibrium price. Suppose all agents default. Then, the only possible equilibrium price is $p = 0$ by (11). Because every agent defaults,

$$\hat{z}_j + \sum_{k \in N} \hat{x}_{jk} \leq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all $j \in N$ yields

$$ne_0 - \epsilon \leq 0,$$

which is a contradiction to $\epsilon < \epsilon^* = ne_0$.

For the second statement, suppose $\epsilon > \epsilon^*$ and $cp < 1$ and no one defaults. Then,

$$\hat{z}_j + \sum_{k \in N} \hat{x}_{jk} \geq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all the equations yields

$$n(e_0 + h_0 p) - \epsilon \geq 0, \tag{25}$$

and the only way to satisfy the inequality is for p to be large enough. However, because $ne_0 < \epsilon$, there will be no cash in the market to clear the market with $p > 0$, as

$$\begin{aligned} nh_0 p &= n(e_0 + h_0 p) - \epsilon \\ 0 &= ne_0 - \epsilon < 0, \end{aligned}$$

where the last inequality comes from $\epsilon > \epsilon^*$, so p becomes zero, and the above inequality (25) becomes

$$ne_0 - \epsilon \geq 0,$$

which is a contradiction.

For the third statement, recall that the payment under the collateral-netting network is

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij} d_{ij} p]^+ = 0$$

and any payment is fully covered by collateral. ■

B. Characteristics of Full Equilibrium

Proof of Lemma 1. First, for the given asset price p, denote the set of agents defaulting as $\mathcal{D}(p)$ and the complement set as $\mathcal{S}(p) \equiv N \backslash \mathcal{D}(p)$. If $\sum_{j \in N} [m_j(p)]^+ = 0$, then it cannot decrease further and it is trivially increasing in p. Thus, suppose that $\sum_{j \in N} [m_j(p)]^+ > 0$.

Recall that

$$\begin{aligned} m_j(p) =& e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in N} d_{ij} + \sum_{k \in N} x_{jk}(p) \\ =& e_j + h_j p + \sum_{k \in \mathcal{D}(p)} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in N} d_{ij} + \sum_{k \in \mathcal{S}(p)} d_{jk} + \sum_{k \in \mathcal{D}(p)} x_{jk}(p). \end{aligned}$$

Summing up the net wealth of non-defaulting agents implies

$$\sum_{j \in \mathcal{S}(p)} m_j(p) = \sum_{j \in \mathcal{S}(p)} \left(e_j + h_j p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in \mathcal{D}(p)} d_{ij} \right) + \sum_{j \in \mathcal{S}(p)} \sum_{k \in \mathcal{D}(p)} \left(c_{jk} d_{jk} p + x_{jk}(p) \right).$$

Note that the coefficients of p are positive. Also note that changes in the liquidation amount $l_j(p)$ do not decrease net wealth. If $p < s\zeta$, then $l_j(p) = \xi$, which is non-decreasing in p. If $p \geq s\zeta$, $l_j(p)$ does not decrease net wealth when p increases, because the liquidation amount should cover the discrepancy $b_j(p) - a_j(p)$, if there is any, and the liquidation amount should be just enough to maintain $m_j(p) = 0$. Hence, an increase in price does not decrease the net wealth of non-defaulting agents through the changes in $l_j(p)$. Therefore, $\sum_{j \in \mathcal{S}} m_j(p)$ is strictly increasing in p if

$$\sum_{j \in \mathcal{S}} \sum_{k \in \mathcal{D}} \left(x_{jk}(p) + c_{jk} d_{jk} p \right)$$

is strictly increasing in p. Recall that

$$\begin{aligned} x_{jk}(p) = \min \Bigg\{ & d_{jk} - c_{jk} d_{jk} p, q_{jk}(p) \Bigg[e_k + h_k p + \sum_{l \in N} c_{kl} d_{kl} p - \sum_{l \in N} c_{lk} d_{lk} p \\ & + \sum_{l \in N} [c_{lk} d_{lk} p - d_{lk}]^+ + \sum_{l \in N} x_{kl}(p) + \zeta \xi - \omega_k \epsilon \Bigg]^+ \Bigg\}, \end{aligned}$$

for any $k \in \mathcal{D}(p)$. We will focus on the case in which $\sum_{l \in N} [c_{lk} d_{lk} p - d_{lk}]^+ = 0$, which is for under-collateralized contracts. If $c_{lk} d_{lk} p - d_{lk} > 0$ for $l \neq j$, then $x_{jk}(p)$ will increase even further with the increase in p because an increase in $c_{lk} d_{lk} p - d_{lk}$ increases k's net wealth and payments to others, and the argument below still holds. Also, if $c_{jk}d_{jk}p - d_{jk} > 0$, then $x_{jk}(p) + c_{jk}d_{jk}p = d_{jk}$ will be non-decreasing in p. Therefore, it is enough to show that the statement is true in the case where there is no $l \in N$ such that $c_{lk}d_{lk}p - d_{lk} > 0$ for any $k \in \mathcal{D}(p)$.

Given that, we can simplify the expression as

$$\begin{aligned}
&\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p) + c_{jk}d_{jk}p) \\
&\quad= \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}\left(q_{jk}(p)\left[e_k + h_k p + \sum_{l\in N} c_{kl}d_{kl}p - \sum_{l\in N} c_{lk}d_{lk}p + \sum_{l\in N} x_{kl}(p) + \zeta\xi - \omega_k\epsilon\right]^+ + c_{jk}d_{jk}p\right) \\
&\quad= \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}\left(q_{jk}(p)\left[F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)} x_{kl}(p) - \sum_{l\in N} c_{lk}d_{lk}p\right]^+ + c_{jk}d_{jk}p\right),
\end{aligned} \tag{26}$$

where $F_k(p) = e_k + h_k p + \sum\limits_{l\in\mathcal{S}(p)} d_{kl} + \zeta\xi - \omega_k\epsilon$ is strictly increasing in p.

Case 1. Consider the case in which

$$F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)} x_{kl}(p) - \sum_{l\in N} c_{lk}d_{lk}p > 0, \qquad \forall k \in \mathcal{D}(p)$$

so no agents are defaulting on their liquidity shocks (senior debt). The weights should add up to 1, as $\sum_{j\in N} q_{jk} = 1$; therefore, the payments of defaulting agents should satisfy

$$\begin{aligned}
x_k(p) \equiv \sum_{l\in N} x_{lk}(p) &= \sum_{l\in\mathcal{S}(p)} x_{lk}(p) + \sum_{l\in\mathcal{D}(p)} x_{lk}(p) \\
&= F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p + \sum_{l\in\mathcal{D}(p)} x_{kl}(p) - \sum_{l\in N} c_{lk}d_{lk}p > 0
\end{aligned}$$

for any $k \in \mathcal{D}(p)$. Thus, (26) can be rearranged as

$$\begin{aligned}
\sum_{j\in\mathcal{S}(p)}\sum_{k\in\mathcal{D}(p)}(x_{jk}(p) + c_{jk}d_{jk}p) &= \sum_{k\in\mathcal{D}(p)} F_k(p) + \sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{D}(p)}(c_{kl}d_{kl}p - c_{lk}d_{lk}p) + \sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{D}(p)} x_{kl}(p) \\
&\quad - \sum_{k\in\mathcal{D}(p)}\sum_{l\in\mathcal{S}(p)} c_{lk}d_{lk}p - \sum_{l\in\mathcal{D}(p)}\sum_{k\in\mathcal{D}(p)} x_{lk}(p) + \sum_{j\in\mathcal{S}(p)}\sum_{k\in\mathcal{D}(p)} c_{jk}d_{jk}p \\
&= \sum_{k\in\mathcal{D}(p)} F_k(p),
\end{aligned}$$

which is strictly increasing in p.

Case 2. Now suppose that some agents default on their liquidity shocks. Denote the set of such agents as $\mathcal{B}(p)$, which implies $\forall k \in \mathcal{B}(p)$, $x_{jk}(p) = 0$, for any $j \in N$. We will often omit the argument p for the sets $\mathcal{B}(p), \mathcal{D}(p)$, and $\mathcal{S}(p)$ from now on for notational simplicity. Then, rearranging (26) yields

$$\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p) + c_{jk}d_{jk}p) = \sum_{k\in\mathcal{D}\backslash\mathcal{B}} F_k(p) + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl}p - c_{lk}d_{lk}p) - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}x_{lk}(p) + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{jl}d_{jl}p. \tag{27}$$

Case 2.1. If $x_{lk}(p)$ is zero—that is, $q_{lk}(p) = 0$—for all $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, then the right-hand side of (27) is trivially increasing in p by applying collateral constraints twice. The actual steps are similar to the steps shown in a more general case, Case 2.2., below.

Case 2.2. Suppose that $q_{lk}(p) > 0$ for some $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$. Recall that

$$x_k(p) = F_k(p) + \sum_{l\in\mathcal{D}}c_{kl}d_{kl}p - \sum_{l\in N}c_{lk}d_{lk}p + \sum_{l\in\mathcal{D}\backslash\mathcal{B}}x_{kl}(p)$$

for any $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$. Therefore, the matrix notation of the aggregate payments from $\mathcal{D}(p)\backslash\mathcal{B}(p)$ becomes

$$x_{\mathcal{D}\backslash\mathcal{B}} = G_{\mathcal{D}\backslash\mathcal{B}} + Q_{\mathcal{D}\mathcal{D}}x_{\mathcal{D}\backslash\mathcal{B}},$$

where $x_{\mathcal{D}\backslash\mathcal{B}}$ is $|\mathcal{D}(p)\backslash\mathcal{B}(p)|\times 1$ vector of $x_k(p)$ for each $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, $G_{\mathcal{D}\backslash\mathcal{B}}$ is $|\mathcal{D}(p)\backslash\mathcal{B}(p)|\times 1$ vector of $F_k(p) + \sum_{l\in\mathcal{D}(p)} c_{kl}d_{kl}p - \sum_{l\in N} c_{lk}d_{lk}p$ for each $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, and $Q_{\mathcal{D}\mathcal{D}}$ is a $|\mathcal{D}(p)\backslash\mathcal{B}(p)| \times |\mathcal{D}(p)\backslash\mathcal{B}(p)|$ matrix of weights $q_{ij}(p)$ for $i, j \in \mathcal{D}(p)\backslash\mathcal{B}(p)$. Note that the spectral radius of $Q_{\mathcal{D}\mathcal{D}}$ is less than 1 by assumption and $(I - Q_{\mathcal{D}\mathcal{D}})^{-1}$ exists by the property of the Neumann series. Hence,

$$x_{\mathcal{D}\backslash\mathcal{B}} = (I - Q_{\mathcal{D}\mathcal{D}})^{-1}G_{\mathcal{D}\backslash\mathcal{B}},$$

and the sum of payments to agents defaulting on senior debt is represented as

$$\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}x_{lk}(p) = \mathbb{1}'Q_{\mathcal{B}\mathcal{D}}x_{\mathcal{D}\backslash\mathcal{B}} = \mathbb{1}'Q_{\mathcal{B}\mathcal{D}}(I - Q_{\mathcal{D}\mathcal{D}})^{-1}G_{\mathcal{D}\backslash\mathcal{B}},$$

where $Q_{\mathcal{B}\mathcal{D}}$ is a $|\mathcal{B}(p)| \times |\mathcal{D}(p)\backslash\mathcal{B}(p)|$ matrix of weights $q_{lk}(p)$ for $l \in \mathcal{B}(p)$ and $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$.

Since all entries of $Q_{\mathcal{B}\mathcal{D}}$ are also less than 1, there exists $\eta < 1$ such that

$$\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}x_{lk}(p) = \eta\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}}F_k(p) + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl}p - c_{lk}d_{lk}p) - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{jk}d_{jk}p\right].$$

Thus, (27) implies

$$\begin{aligned}\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}}(x_{jk}(p) + c_{jk}d_{jk}p) =& (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}}F_k(p) + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl}p - c_{lk}d_{lk}p)\right]\\ &+ \eta\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{jk}d_{jk}p + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{jl}d_{jl}p.\end{aligned}$$

Adding $\sum_{j\in\mathcal{S}}h_jp$ from $\sum_{j\in\mathcal{S}}F_jp$ to the right-hand side makes the coefficient on p as follows:

$$\sum_{j\in\mathcal{S}}h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{jl}d_{jl} + \eta\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{jk}d_{jk} + (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}}h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}(c_{kl}d_{kl} - c_{lk}d_{lk})\right], \tag{28}$$

which is again positive by applying collateral constraints twice, as we show in the following. From the collateral constraints for $k \in \mathcal{D}(p)\backslash\mathcal{B}(p)$, we have

$$\begin{aligned}&\sum_{k\in\mathcal{D}\backslash\mathcal{B}}h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}c_{kl}d_{kl} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{k'\in\mathcal{D}\backslash\mathcal{B}}c_{kk'}d_{kk'} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{kj}d_{kj}\\ \geq& \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}c_{lk}d_{lk} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{k'\in\mathcal{D}\backslash\mathcal{B}}c_{kk'}d_{kk'} + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{jk}d_{jk}\\ \Rightarrow& \sum_{k\in\mathcal{D}\backslash\mathcal{B}}h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}c_{kl}d_{kl} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}}c_{lk}d_{lk} \geq \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{jk}d_{jk} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{kj}d_{kj}.\end{aligned} \tag{29}$$

Similarly, from the collateral constraints for $j \in \mathcal{S}(p)$, we have

$$\begin{aligned}&\sum_{j\in\mathcal{S}}h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{jl}d_{jl} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}}c_{jk}d_{jk} + \sum_{j\in\mathcal{S}}\sum_{j'\in\mathcal{S}}c_{jj'}d_{jj'}\\ \geq& \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}}c_{kj}d_{kj} + \sum_{j\in\mathcal{S}}\sum_{j'\in\mathcal{S}}c_{j'j}d_{j'j}\\ \Rightarrow& \sum_{j\in\mathcal{S}}h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{jl}d_{jl} \geq \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}}c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}}c_{kj}d_{kj} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}}c_{jk}d_{jk}.\end{aligned} \tag{30}$$

Hence, plugging (29) and (30) into (28) implies

$$\sum_{j\in\mathcal{S}} h_j + \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{jl}d_{jl} + \eta \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} + (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}} h_k + \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{l\in\mathcal{B}} (c_{kl}d_{kl} - c_{lk}d_{lk})\right]$$
$$\geq \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} - (1-\eta)\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk}$$
$$+ (1-\eta)\left[\sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{jk}d_{jk} - \sum_{k\in\mathcal{D}\backslash\mathcal{B}}\sum_{j\in\mathcal{S}} c_{kj}d_{kj}\right]$$
$$= \sum_{j\in\mathcal{S}}\sum_{l\in\mathcal{B}} c_{lj}d_{lj} + \eta\sum_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}\backslash\mathcal{B}} c_{kj}d_{kj} > 0,$$

and thus the coefficient on p is positive, implying that the aggregate positive net wealth is strictly increasing in p.

Finally, we consider the changes in the set of defaulting agents with an increase in p. Because the aggregate net wealth is increasing in p as in Case 1, an increase in p will only weakly decrease the number of defaulting agents. Therefore, there are no agents in $j \in \mathcal{S}(p)$ who will default due to an increase in p, and there can be agents $j \in \mathcal{D}(p)$ who will be solvent under higher p and their net wealth would be added to the sum of $\sum_{j\in\mathcal{S}(p)} m_j(p)$, increasing the aggregate positive net wealth further. ■

Proof of Lemma 2. Suppose that $p < s$ makes the market clear. Rearranging the first line of equation (11) yields

$$\sum_{j\notin\mathcal{D}(p)} [m_j(p) - h_j p]^+ = \sum_{i\in N} \min\left\{[h_j p - m_j(p)]^+, h_j p\right\}$$
$$\sum_{j\notin\mathcal{D}(p)} (m_j(p) - h_j p) = \sum_{i\in\mathcal{D}(p)} h_j p$$
$$\sum_{j\in N} [m_j(p)]^+ = \sum_{j\in N} h_j p$$
$$p = \frac{\sum_{j\in N} [m_j(p)]^+}{\sum_{j\in N} h_j},$$

which holds because $\phi_i(p) = h_i p$ for any $i \in \mathcal{D}(p)$.

Now suppose that $p = s$ satisfies the second line of equation 11, which implies

$$\begin{aligned}\sum_{j \notin \mathcal{D}(s)} [m_j(s) - h_j s]^+ &\geq \sum_{i \in N} \min\left\{[h_j s - m_j(s)]^+, h_j s\right\} \\ \sum_{j \notin \mathcal{D}(s)} (m_j(s) - h_s p) &\geq \sum_{i \in \mathcal{D}(s)} h_j s \\ \sum_{j \in N} [m_j(s)]^+ &\geq \sum_{j \in N} h_j s \\ s &\leq \frac{\sum_{j \in N} [m_j(s)]^+}{\sum_{j \in N} h_j}.\end{aligned}$$

Combining the two cases implies that the market price is bounded by s or the ratio between the aggregate positive net wealth and the total supply of assets; therefore, (12). ■

Proof of Proposition 1.

The first step, which is based on the proof of Proposition 1 in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), is to show that there exists a payment equilibrium that is generically unique for any given p. The second step is to show that there exists an equilibrium price p that satisfies the market clearing condition for the given payment and liquidation vectors of the corresponding payment equilibrium under p.

Existence of the payment equilibrium.

First, fix an asset price p. By Lemma 6, it is sufficient to show that there exists $\mathbf{x}^* \in \mathbb{R}^n_+$ that satisfies $\mathbf{x}^* = \left[\min\left\{Q\mathbf{x}^* + \hat{z} + \zeta\xi\mathbb{1}, \hat{\mathbf{d}}\right\}\right]^+$. Define the mapping $\Phi : \mathcal{X} \to \mathcal{X}$ as

$$\Phi(\mathbf{x}) = \left[\min\left\{Q\mathbf{x} + \hat{z} + \zeta\xi\mathbb{1}, \hat{\mathbf{d}}\right\}\right]^+,$$

where $\mathcal{X} = \Pi_{i=0}^n [0, d_i]$. This mapping is continuous, and its domain, which is the same as its range, is a convex and compact subset of the Euclidean space. Thus, there exists $\mathbf{x}^* \in \mathcal{X}$ such that $\Phi(\mathbf{x}^*) = \mathbf{x}^*$ by the Brouwer fixed-point theorem. The corresponding $\mathbf{l}^*$ can be obtained and the pair $(\mathbf{x}^*, \mathbf{l}^*)$ satisfies the payment and liquidation rules in the original network for any given price p.

Generic uniqueness of the payment equilibrium.

Assume that the financial network is connected without loss of generality, as we can apply the proposition for each component of a network that is not connected. Suppose that for the same equilibrium price p, there exist two distinct payment equilibria (X, l) and $(\tilde{X}, \tilde{l})$ such that $X \neq \tilde{X}$. Then, payments from each equilibrium should satisfy (23). Hence, for each agent j,

$$\begin{aligned}|\hat{x}_j - \hat{\hat{x}}_j| &= \left| \left[\min\left\{(Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi, \hat{d}_j\right\}\right]^+ - \left[\min\left\{\left(Q\hat{\hat{\mathbf{x}}}\right)_j + \hat{z}_j + \zeta\xi, \hat{d}_j\right\}\right]^+ \right| \\ &\leq \left|(Q\hat{\mathbf{x}})_j - \left(Q\hat{\hat{\mathbf{x}}}\right)_j\right|,\end{aligned}$$

where the last inequality is coming from the fact that both terms have the same upper bound and from the triangle inequality. Taking L^1 norm for the vector representation of both sides of the above inequality becomes

$$\begin{aligned}||\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}|| &\leq \left|\left|Q\left(\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}\right)\right|\right| \\ &\leq ||Q|| \cdot \left|\left|\left(\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}\right)\right|\right| \\ &= ||\hat{\mathbf{x}} - \hat{\hat{\mathbf{x}}}||,\end{aligned}$$

because Q is column stochastic from the weighting rule (7). Therefore, all the inequalities are binding and

$$|\hat{x}_j - \hat{\hat{x}}_j| = \left|(Q\hat{\mathbf{x}})_j - \left(Q\hat{\hat{\mathbf{x}}}\right)_j\right|$$

holds. Because $\hat{\mathbf{x}} = \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbb{1}\right\}\right]^+$ and by Lemma 7, either

$$(Q\hat{\mathbf{x}})_j = \left(Q\hat{\hat{\mathbf{x}}}\right)_j$$

or

$$\begin{aligned}0 &\leq (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi \leq d_j \\ 0 &\leq \left(Q\hat{\hat{\mathbf{x}}}\right)_j + \hat{z}_j + \zeta\xi \leq d_j.\end{aligned} \tag{31}$$

Therefore, the set of defaulting agents, $\mathcal{D}(p)$, is the same for the two different payment equilibria. For any agent satisfying (31)—that is, if $j \in \mathcal{D}(p)$—

$$(Q\hat{\mathbf{x}})_j - \left(Q\hat{\hat{\mathbf{x}}}\right)_j = \hat{x}_j - \hat{\hat{x}}_j.$$

For the other case, for all $j \notin \mathcal{D}(p)$, the other equality $(Q\hat{\mathbf{x}})_j = \left(Q\hat{\hat{\mathbf{x}}}\right)_j$ should hold. Because the collateral-netting network eliminates any idiosyncratic collateral ratio, the payment and weighting matrices are invariant to any permutation. Denote $\underline{Q}$ and $\underline{\hat{x}}$ as the weighting matrix and payment vector for collateral-netting matrix after a permutation of the order of agents by having $j \in \mathcal{D}(p)$ first and then $i \notin \mathcal{D}(p)$ later. Therefore,

$$\underline{Q}\left(\hat{\underline{\mathbf{x}}} - \hat{\hat{\underline{\mathbf{x}}}}\right) = \begin{bmatrix} \hat{\underline{\mathbf{x}}}_{\mathcal{D}} - \hat{\hat{\underline{\mathbf{x}}}}_{\mathcal{D}} \\ 0 \end{bmatrix}$$

where $\underline{x}_{\mathcal{D}}$ is the subvector of $\underline{x}$ including only the agents in $\mathcal{D}(p)$ and

$$\left|\left|\underline{Q}\left(\hat{\underline{\mathbf{x}}} - \hat{\hat{\underline{\mathbf{x}}}}\right)\right|\right| = ||\hat{\underline{\mathbf{x}}}_{\mathcal{D}} - \hat{\hat{\underline{\mathbf{x}}}}_{\mathcal{D}}||.$$

Thus, $\hat{x}_j = \hat{\hat{x}}_j$ for any $j \notin \mathcal{D}(p)$ and

$$\underline{Q}_{\mathcal{D}}(\hat{\underline{\mathbf{x}}}_{\mathcal{D}} - \hat{\hat{\underline{\mathbf{x}}}}_{\mathcal{D}}) = \hat{\underline{\mathbf{x}}}_{\mathcal{D}} - \hat{\hat{\underline{\mathbf{x}}}}_{\mathcal{D}} \tag{32}$$

where $\underline{Q}_{\mathcal{D}}$ is the submatrix of $\underline{Q}$ for the agents in $\mathcal{D}(p)$ and $\hat{\underline{\mathbf{x}}}_{\mathcal{D}}$ is a subvector of $\hat{\underline{\mathbf{x}}}$ for the agents in $\mathcal{D}(p)$. If the debt network is connected, then $\underline{Q}$ and $\underline{Q}_{\mathcal{D}}$ are irreducible non-negative matrices by construction. Then, by the Perron-Frobenius theorem, there exists a simple eigenvalue and right eigenvector whose components are all positive (Gaubert and Gunawardena, 2004).

If $\mathcal{D}(p)$ is a proper subset of N, then all of the column sums are less than one, and the spectral radii for Q and $\underline{Q}_{\mathcal{D}}$ are less than one. This result is due to $\lim_{k\to\infty} ||Q^k|| = 0$, which implies $0 = \lim_{k\to\infty} Q^k\mathbf{v} = \lim_{k\to\infty} \lambda^k\mathbf{v} = \mathbf{v}\lim_{k\to\infty} \lambda^k$, which leads to $\lim_{k\to\infty} \lambda^k = 0$, where λ and $\mathbf{v}$ are the eigenvalue and eigenvector, respectively. All of the eigenvalues of $\underline{Q}_{\mathcal{D}}$ have an absolute value less than one, and $\underline{Q}_{\mathcal{D}}\mathbf{v} = \mathbf{v}$ does not have a non-trivial solution. Hence, (32) cannot hold unless $\mathcal{D}(p) = N$. Then, $\hat{x}_j = (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi$ for all $j \in \mathcal{D}(p) = N$ and

$$\begin{aligned}\sum_{j\in N}\hat{x}_j &= \sum_{j\in N}\sum_{i\in N} q_{ij}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi \\ &= \sum_{i\in N}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi.\end{aligned}$$

Furthermore, the only asset price p under $\mathcal{D}(p) = N$ is $p = 0$ from (11). In other words, there cannot be multiple equilibria if the equilibrium price is $p > 0$. Finally, even if all the agents default and $p = 0$, the last equation implies

$$\sum_{j\in N}\hat{z}_j(p) = \sum_{j\in N}(e_j - \omega_j\epsilon) = -n\zeta\xi,$$

which holds only for a non-generic set of parameters, n, e, ω, ζ, ξ and ϵ, which is a line over a multidimensional Euclidean space. Thus, for the given equilibrium price p, the payment and liquidation pair (X, l) is generically unique.

Existence of the full equilibrium.

Now the only condition left for an equilibrium is the market clearing condition for price p. Suppose that there exists a unique (X, l) pair that satisfies the two equilibrium conditions for any given price $p \in [0, s]$.[9] If the resulting payment equilibrium $(\mathbf{x}^*, \mathbf{l}^*)$ generates m that satisfies the second line of equation (11), then $(X^*, \mathbf{l}^*, m^*, \phi^*, s)$ is a full equilibrium.

Now suppose the contrary, and only a price $p < s$ makes the market clear. From Lemma 2, we have

$$p = \frac{\sum_{j \in N} [m_j(p)]^+}{\sum_{j \in N} h_j}, \tag{33}$$

and by Lemma 1, the aggregate positive net wealth is continuously (and strictly) increasing in p (as long as $\sum_{j \in N} [m_j(p)]^+ > 0$). Therefore, the numerator of the right-hand side of (33) is continuously increasing in p. Also, $\sum_{j \in N} m_j(p)$ is increasing in p, as shown in the proof of Lemma 1. Thus, $\mathcal{D}(p) \subset \mathcal{D}(p')$ for any $p > p'$, and an increase in p will increase the price even further by including more agents on the numerator.

Define the mapping $\Psi : [0, s] \to [0, s]$

$$\Psi(p) = \frac{\sum_{j \in N} \left[m_j^*(p)\right]^+}{\sum_{j \in N} h_j},$$

where $m_j^*(p)$ is the corresponding net wealth for agent j under $(X^*, \mathbf{l}^*)$, which are derived from the corresponding payments and liquidation amounts after collateral netting for a given price p. Because $\Psi(p)$ is a continuously (and strictly) increasing function of p from $[0, s]$ to $[0, s]$ (in the region $\mathcal{P}$ such that for any $p \in \mathcal{P}$, $0 < \Psi(p) < s$), there exists a fixed point p^*, which is an equilibrium price. Therefore, a full equilibrium $(X^*, \mathbf{l}^*, m^*, \phi^*, p^*)$ exists, and there exists a maximum price $\bar{p}$, which is a full equilibrium price greater than any other full equilibrium prices. ■

[9] This is true for any price $p > 0$, as shown in the proof of the generic uniqueness of the payment equilibrium.

C. Characteristics of Contagion

Proof of Lemma 3. Suppose that (X, l, m, p) is a full equilibrium. Then any of the cash and assets will generate the given payoff to the whole economy, so $n(e_0 + h_0 s)$ should be part of the welfare. However, the total long-term projects $n\xi$ may not remain intact, as some or all of the project can be liquidated by l_j amount for each $j \in N$. The total liquidation amount will be $\sum_{i \in N} l_i$ while the cost of early liquidation is $(1-\zeta)$, as only the ζ proportion will be salvaged. Therefore, the social surplus of the economy is $U = n(e_0 + h_0 s + \xi) - (1-\zeta) \sum_{i \in N} l_i$. ■

Proof of Proposition 2. First, compute the collateral-netting network of the original network. The case of a fire-sale collapse under $p < s\zeta$ is irrelevant, as $nh_0 = 0$. Then, for this collateral-netting network, apply Propositions 4 and 6 of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) and the results follow. ■

Proof of Proposition 3. Without loss of generality, assume that

$$\min \left\{ 1, \frac{(n-\kappa)e_0}{\kappa s h_0} \right\} \geq \zeta,$$

which holds for $\zeta \to 0$. All the payments are covered by the collateral if $cp \geq 1$ by Lemma 8 in Appendix A.4. If all κ number of agents default on their senior debt, collateral covers all the payments, and no non-defaulting agent is liquidating the long-term project, then the total available cash in the economy is $(n - \kappa)e_0$. Also, as only κ number of agents are out of the market, from (12), we have

$$\begin{aligned} nh_0 p &\leq \sum_{j \notin \mathcal{D}} m_j(p) = (n-\kappa)(e_0 + h_0 p) \\ \kappa h_0 p &\leq (n-\kappa)e_0, \end{aligned}$$

which implies that the relevant amount of fire sales is κh_0. Therefore, the asset price is either the fundamental value s or the aggregate liquidity divided by the total amount of fire sales—that is,

$$p = \min \left\{ s, \frac{(n-\kappa)e_0}{\kappa h_0} \right\}.$$

If $s < \dfrac{(n-\kappa)e_0}{\kappa h_0}$ and $p = s$, then $c \geq 1/s$ will satisfy $cp \geq 1$.

If $s \geq \frac{(n-\kappa)e_0}{\kappa h_0}$, then $p = \frac{(n-\kappa)e_0}{\kappa h_0}$ and $cp \geq 1$ holds if $c \geq c^{\dagger} \equiv \frac{\kappa h_0}{(n-\kappa)e_0}$. Finally, the network should satisfy the resource constraints and $c \geq c^{\dagger}$; thus,

$$nh_0 \geq cd \geq \frac{\kappa h_0}{(n-\kappa)e_0} d,$$

which is possible only if $n(n-\kappa)e_0 \geq \kappa d$. Then, for $c^*(s,n) = \frac{1}{\min\left\{s, \frac{(n-\kappa)e_0}{\kappa h_0}\right\}}$, any $c \geq c^*$ will satisfy $cp \geq 1$. By the uniqueness of the (maximum) full equilibrium, this equilibrium is the only (maximum) equilibrium. In this equilibrium, all of the payments are made in full, and there will be no additional defaults. Thus, any network structure has the most stable and resilient results, as the collateral fully insulates any propagation. ■

Proof of Lemma 4. The lower bound is trivial because the size of the shock causes the agent under the liquidity shock to default regardless, as collateral does not cover the debt obligations by $cp < 1$. Now, consider the upper bound. For the collateral-netting network, recall $\mathcal{D}(p)$ is the set of agents that default under price p. Then, for each agent $j \in \mathcal{D}(p)$,

$$\sum_{i \in N} \hat{x}_{ij} = \left[e_0 + h_0 p + \sum_{k \in N} \hat{x}_{jk} - \omega_j \epsilon \right]^+,$$

where the agent can pay a positive amount only if the agent has enough cash inflows to cover the liquidity shock, and payments are zero otherwise. Note that the maximum payment amount an agent can receive as a lender is $d - cdp$. Therefore,

$$\sum_{i \in N} \hat{x}_{ij} \left[e_0 + h_0 p + \sum_{k \in N} \hat{x}_{jk} - \omega_j \min\left\{\epsilon, e_0 + h_0 p + d - cdp\right\} \right]^+.$$

Summing over all defaulting agents yields

$$\sum_{j \in \mathcal{D}(p)} \left[(e_0 + h_0 p) + \sum_{k \in N} \hat{x}_{jk} - \omega_j \min\left\{\epsilon, e_0 + h_0 p + d - cdp\right\} \right]^+ = \sum_{j \in \mathcal{D}(p)} \sum_{i \in N} \hat{x}_{ij}.$$

Because agents without liquidity shocks cannot have negative net wealth,

$$\sum_{j \in \mathcal{D}(p)} (e_0 + h_0 p) + \sum_{j \in \mathcal{D}(p)} \sum_{k \in N} \hat{x}_{jk} \leq \sum_{j \in \mathcal{D}(p)} \sum_{i \in N} \hat{x}_{ij} + \kappa \min\left\{\epsilon, e_0 + h_0 p + d - cdp\right\}, \qquad (34)$$

and by canceling out the payments among defaulting agents, we obtain the bound as

$$\kappa \min\{\epsilon, e_0 + h_0 p + d - cdp\} - (e_0 + h_0 p)|\mathcal{D}(p)| \geq \sum_{i \notin \mathcal{D}(p)} \sum_{j \in \mathcal{D}(p)} (\hat{d}_{ij} - \hat{x}_{ij}) > 0,$$

where the last inequality comes from the definition of the defaulting agents. Then, rearranging the inequality results in

$$|\mathcal{D}(p)| < \frac{\kappa \min\{\epsilon, e_0 + h_0 p + d - cdp\}}{e_0 + h_0 p}, \tag{35}$$

which is decreasing in the asset price p and collateral ratio c. ■

Proof of Proposition 4.

Case 1. $\epsilon < \epsilon^*$**.** The market clearing condition (12) implies that the aggregate net wealth of non-defaulting agents determines the asset price p. From (12) and (34), the aggregate net wealth of non-defaulting agents is bounded below by

$$\sum_{j \notin \mathcal{D}} (e_0 + h_0 p) - \sum_{i \in N} \sum_{j \in \mathcal{D}} \hat{x}_{ij} + \sum_{j \in \mathcal{D}} (e_0 + h_0 p) + \sum_{j \in \mathcal{D}} \sum_{k \in N} \hat{x}_{jk} - \kappa\epsilon = ne_0 + nh_0 p - \kappa\epsilon,$$

and the market clearing condition becomes

$$p \leq \left[\frac{ne_0 + nh_0 p - \kappa\epsilon}{nh_0}\right]^+.$$

Since $\epsilon < \epsilon^* \equiv \frac{ne_0}{\kappa}$, there exist agents with positive net wealth. Then, the market clearing condition implies

$$\underline{p} \leq \underline{p} + \frac{ne_0 - \kappa\epsilon}{nh_0},$$

so the maximum equilibrium would satisfy the market clearing condition with $p = s$.

Case 2. $c \geq c_*$**.** Suppose $c \geq c_* \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds}$. Also, assume $\epsilon > \epsilon^*$ because the result is trivially true by Case 1 otherwise. At least one agent under a liquidity shock defaults, even on the liquidity shock (senior debt) by Lemma 8 in Appendix A.4 and $cp < 1$, implying that all other agents would suffer the total default of the shocked agent in the amount of $\kappa(d - cdp)$ or less. Suppose that the total number of defaulting agents is $k = |\mathcal{D}| < n$. Then, even if all κ agents default on their senior debt, the market clearing condition is

$$\begin{aligned}(n-k)(e_0+h_0p)+(k-\kappa)(e_0+h_0p)-\kappa(d-cdp) &\geq nh_0p \\ (n-\kappa)e_0-\kappa(d-cdp) &\geq \kappa h_0 p \\ \kappa(cd-h_0)p &\geq \kappa d-(n-\kappa)e_0,\end{aligned} \tag{36}$$

where the left-hand side of the final inequality is increasing in p with $cd > h_0$, which holds when $c \geq c_*$. Therefore, if $c \geq c_*$, $p = s$ holds for the market clearing condition. Also, this implies the left-hand side of (36) is positive, implying that there are agents with positive net wealth—that is, there are solvent agents who can purchase the assets in the market. Therefore, the asset price is $p = s$ in the (maximum) equilibrium regardless of the network structure.

Finally, we show that $c_* \leq c^*$. The value of c^* can be either $1/s$ or $\kappa h_0/(n-\kappa)e_0$.

Case 2.1. Suppose that $\dfrac{(n-\kappa)e_0}{\kappa h_0} \geq s$; therefore, $c^* = 1/s$. Then, $c_* \leq c^*$ holds because

$$\begin{aligned} c_* \equiv \frac{d-((n-\kappa)/\kappa)\,e_0+h_0s}{ds} &\leq \frac{1}{s} \equiv c^* \\ d-((n-\kappa)/\kappa)\,e_0+h_0s &\leq d \\ h_0s &\leq ((n-\kappa)/\kappa)\,e_0 \\ s &\leq \frac{(n-\kappa)e_0}{\kappa h_0},\end{aligned}$$

which holds by the initial assumption.

Case 2.2. Suppose that $\dfrac{(n-\kappa)e_0}{\kappa h_0} < s$; therefore, $c^* = \dfrac{\kappa h_0}{(n-\kappa)e_0}$. Then, $c_* < c^*$ holds because

$$\begin{aligned} c_* \equiv \frac{d-((n-\kappa)/\kappa)\,e_0+h_0s}{ds} &< \frac{\kappa h_0}{(n-\kappa)e_0} \equiv c^* \\ d-((n-\kappa)/\kappa)\,e_0+h_0s &< \frac{d\kappa h_0 s}{(n-\kappa)e_0} \\ \left(d-\frac{n-\kappa}{\kappa}e_0\right)(n-\kappa)e_0 &< (d-\frac{n-\kappa}{\kappa}e_0)\kappa h_0 s \\ \frac{(n-\kappa)e_0}{\kappa h_0} &< s,\end{aligned}$$

which holds by the initial assumption. ■

Proof of Proposition 5. Suppose that $\epsilon < \epsilon^*$. Then, the setting becomes similar to that of Proposition 2 except that each agent is endowed with h_0 amount of assets, which is priced as $p = s$ by Proposition 4. Then, the same steps in Proposition 2 apply to this setup, with modified endowments for the collateral-netting network.

Now consider the case with $c > c_*$ and $\epsilon > \epsilon^*$. The equilibrium asset price is $p = s$ by Proposition 4. Therefore, each agent's required debt payment is $d - cds$. Because $c > c_*$,

$$d - cds < d - c_*ds = (n-1)e_0 - h_0 s,$$

which is the market clearing condition at $p = s$. Hence, even if the agent under a liquidity shock does not pay anything to the remaining agents, the remaining agents combined have sufficient cash to pay the debt to the agent under a liquidity shock and purchase the assets on fire sale at $p = s$. Because all agents are symmetric in the complete network, all remaining $n-1$ agents can pay their debt in full. Hence, the complete network is the most stable and resilient network.

Now consider the ring network. Without loss of generality, suppose that agent 1, who only owes to agent 2, is under a liquidity shock. By Lemma 8 in Appendix A.4 and $\epsilon > \epsilon^*$, agent 1 cannot even pay the liquidity shock. Then, agent 1 pays nothing to agent 2, and agent 2 only has the endowment $e_0 + h_0 s$ to pay to agent 3. Agent 3 will reuse this payment from agent 2 and add agent 3's own endowment, so the total payment from agent 3 to agent 4 is $2(e_0 + h_0 s)$. Similarly, agent 4 will pay agent 5 in the amount of $3(e_0 + h_0 s)$, and so forth. Then, agent $k+1$ would have total available cash of $k(e_0 + h_0 s)$, and agent $k+1$ is solvent only if

$$k(e_0 + h_0 s) > d - cds,$$

which implies

$$k + 1 > \frac{e_0 + h_0 s + d - cds}{e_0 + h_0 s},$$

which is above the upper bound of the number of defaults in Lemma 4 and (35), implying k reached the upper bound of the number of defaults when $p = s$. Also, note that the lowest k satisfying the above inequality is decreasing in c. Thus, the ring network is the least stable and resilient network, and the number of defaulting agents is decreasing in c.

Given the two results of the ring and complete network, we can use the same method of collateral netting as in the proof of Proposition 2 and apply Proposition 6 of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) to obtain the last result. ∎

Proof of Proposition 6. Suppose that $\epsilon > \epsilon^*$, $c < c_*$, and $d > d^* = (n-1)e_0$. First, note that $cp < 1$ and $p < s$ in any network structure because (36) in the proof of Proposition 4 does not hold. Consider the complete network. The agent under a liquidity shock defaults even on the senior debt (liquidity shock) by Lemma 8 in Appendix A.4 and defaults fully on the debt obligation of $(d - cdp)/(n-1)$ to all other agents. Suppose the contrary, that $n-1$ agents do not default. Then, each agent's net wealth equation should satisfy

$$(n-2)\frac{d - cdp}{n-1} + e_0 + h_0 p - (d - cdp) \geq 0. \tag{37}$$

The market clearing condition is

$$(n-1)e_0 - d + cdp \geq h_0 p, \tag{38}$$

where $p > 0$ by the assumption that there are surviving agents. Because $d > (n-1)e_0$, if $cd < h_0$, then there is no price $p > 0$ that can satisfy the market clearing condition, so $p = 0$. Then, (37) implies $(n-1)e_0 \geq d$, which is a contradiction. Now suppose that $cd > h_0$. This inequality implies (38) becomes

$$(cd - h_0)p \geq d - (n-1)e_0, \tag{39}$$

and the left-hand side is maximized when $p = s$, which is the case for the maximum equilibrium we are focusing on. Because $c < c_* \equiv \dfrac{d - (n-1)e_0 + h_0 s}{ds}$, (39) becomes

$$d - (n-1)e_0 \leq (cd - h_0)s < c_* ds - h_0 s = d - (n-1)e_0,$$

which is a contradiction. Therefore, $p = 0$, and, again, (37) implies $(n-1)e_0 \geq d$, which is a contradiction. Therefore, all agents default and the complete network is the least resilient and least stable network.

Now consider the ring network such that agent 1 borrows from agent 2, who borrows from agent 3, and so on. Without loss of generality, let agent 1 be the agent under a negative liquidity shock. Again, by Lemma 8 in Appendix A.4, agent 1 defaults the full amount, $d - cdp$. Agent 2 will pay agent 3 only with the endowment $e_0 + h_0 p$, then agent 3 will reuse this with her own endowment to pay agent 4 in the amount of $2(e_0 + h_0 p)$. Agent 4 will pay agent 5 in the amount of $3(e_0 + h_0 p)$, and so on. In order not to cascade after k length from agent 1,

$$k(e_0 + h_0 p) > d - cdp. \tag{40}$$

If agent $k+1$ pays the debt in full, then all the subsequent agents, $k+2, k+3, \dots, n$, can pay in full without selling their asset holdings. The total market clearing condition is

$$(n-k)(e_0+h_0p) - d + cdp + (k-1)(e_0+h_0p) \geq nh_0p$$
$$(cd-h_0)p \geq d-(n-1)e_0, \tag{41}$$

if $p > 0$, and the left-hand side is maximized at $p = s$. However, because $c < c_*$, (41) evaluated even at the highest $p = s$ implies

$$d-(n-1)e_0 \leq (cd-h_0)s < c_*ds - h_0s = d-(n-1)e_0,$$

which is a contradiction. Therefore, $p = 0$ is the market clearing price. Then, (40) becomes

$$ke_0 > d,$$

which implies $k \geq n$—that is, the first non-defaulting agent should be at the minimum distance of n or higher from agent 1, which exceeds the total number of agents n. Therefore, all agents default, and the ring network is the least resilient and least stable network.

Finally, consider a δ-connected network with $\delta < \dfrac{e_0}{(n-1)d}$ and the partition of agents sets being $(\mathcal{S}, \mathcal{S}^c)$ such that $d_{ij} \leq \delta d$ for any $i \in \mathcal{S}$ and $j \in \mathcal{S}^c$. Thus, $\sum_{j \notin S} d_{ij} \leq \delta d |\mathcal{S}^c|$ for any $i \in \mathcal{S}$. Therefore, for any $i \in \mathcal{S}$,

$$\sum_{j\in S}(d_{ij} - d_{ij}cp) \geq d - cdp - \delta(d-cdp)|\mathcal{S}^c| \geq d - cdp - e_0,$$

which implies

$$e_0 + h_0p + \sum_{j\in S}(d_{ij}-d_{ij}cp) \geq e_0 + \sum_{j\in S}(d_{ij}-d_{ij}cp) \geq d - cdp,$$

so agents in $\mathcal{S}$ can fulfill their debt even when all agents in $\mathcal{S}^c$ do not pay any amount to agents in $\mathcal{S}$. Note that the last inequality holds for any given collateral price p. In other words, all agents in $\mathcal{S}$ remain solvent when $\omega_j = 1$ for an agent $j \in \mathcal{S}^c$ regardless of the size of the shock. Therefore, the δ-connected network is more stable and resilient than the complete or ring networks. ■

Proof of Proposition 7. We prove the first statement first. Denote the set of defaulting agents other than the agent under the shock, agent j, as $\mathcal{D}$ and the set of non-defaulting agents as $\mathcal{S}$. First, note that the weights following the weighting rule do not change with the price p under the uniform collateral ratio because

$$q_{ij}(p) = \frac{d_{ij} - d_{ij}cp}{\sum_{k \in N} d_{kj} - d_{kj}cp} = \frac{d_{ij}(1-cp)}{\sum_{k \in N} d_{kj}(1-cp)} = \frac{d_{ij}}{d}.$$

Hence, we can represent the harmonic distance in (13) from agent i to agent j as

$$\mu_{ij} = 1 + \sum_{k \neq j} q_{ik}\mu_{kj}. \tag{42}$$

Using expression (42), the vectors of harmonic distances can be represented as

$$\mu_{dj} = \mathbb{1} + Q_{dd}\mu_{dj} + Q_{ds}\mu_{sj} \tag{43}$$

$$\mu_{sj} = \mathbb{1} + Q_{sd}\mu_{dj} + Q_{ss}\mu_{sj}, \tag{44}$$

where μ_{dj} is the $|\mathcal{D}| \times 1$ vector of harmonic distances from agents in $\mathcal{D}$ to j, μ_{sj} is the $|\mathcal{S}| \times 1$ vector of harmonic distances from agents in $\mathcal{S}$ to j, and $Q_{dd}, Q_{ds}, Q_{sd}, Q_{ss}$ are matrices of weights of liabilities for agents within $\mathcal{D}$, from $\mathcal{S}$ to $\mathcal{D}$, from $\mathcal{D}$ to $\mathcal{S}$, and within $\mathcal{S}$, respectively. Solving (43) for μ_{dj} and plugging it into (44) implies

$$\begin{aligned}
\mu_{sj} &= \mathbb{1} + Q_{sd}\left[(I - Q_{dd})^{-1}\mathbb{1} + (I - Q_{dd})^{-1} Q_{ds}\mu_{sj}\right] + Q_{ss}\mu_{sj} \\
\mu_{sj} &= \left[I + Q_{sd}(I - Q_{dd})^{-1}\right]\mathbb{1} + Q_{sd}(I - Q_{dd})^{-1} Q_{ds}\mu_{sj} + Q_{ss}\mu_{sj} \\
\left[I + Q_{sd}(I - Q_{dd})^{-1}\right]\mathbb{1} &= \left[I - Q_{ss} - Q_{sd}(I - Q_{dd})^{-1} Q_{ds}\right]\mu_{sj}.
\end{aligned} \tag{45}$$

The vector of total payments for defaulting agents $\mathbf{x}_d$ is determined as

$$\mathbf{x}_d = Q_{dd}\mathbf{x}_d + (d - cdp)Q_{ds}\mathbb{1} + (e_0 + h_0 p)\mathbb{1}$$

for a given price p, which can be solved as

$$\mathbf{x}_d = (I - Q_{dd})^{-1}\left[(d - cdp)Q_{ds}\mathbb{1} + (e_0 + h_0 p)\mathbb{1}\right]. \tag{46}$$

The vector of net wealth of non-defaulting agents is

$$m_s = Q_{sd}\mathbf{x}_d + Q_{ss}(d - cdp)\mathbb{1} + (e_0 + h_0 p)\mathbb{1} - (d - cdp)\mathbb{1},$$

where we use the fact that non-defaulting agents are paying their debt in full as $d - cdp$.

Plugging the payments in (46) into the net wealth vector implies

$$m_s = (e_0 + h_0 p)\left[I + Q_{sd}\left(I - Q_{dd}\right)^{-1}\right]\mathbb{1} - (d - cdp)\left[I - Q_{ss} - Q_{sd}\left(I - Q_{dd}\right)^{-1} Q_{ds}\right]\mathbb{1},$$

which can be simplified as

$$m_s = (e_0 + h_0 p)\left[I + Q_{sd}\left(I - Q_{dd}\right)^{-1}\right]\mathbb{1} - (d - cdp)G\mathbb{1}, \tag{47}$$

where $G = I - Q_{ss} - Q_{sd}\left(I - Q_{dd}\right)^{-1} Q_{ds}$ is a non-singular M-matrix because it is a Schur complement[10] of $I - Q_{dd}$ in the M-matrix,

$$M \equiv \begin{bmatrix} I - Q_{ss} & -Q_{sd} \\ -Q_{ds} & I - Q_{dd} \end{bmatrix},$$

based on Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) (Berman and Plemmons, 1979, p. 159). Combining (45) and (47) implies

$$m_s = (e_0 + h_0 p)G\mu_{sj} - (d - cdp)G\mathbb{1}. \tag{48}$$

If the market price is $p < s$, p should be a liquidity constrained price. Hence, the market clearing condition implies

$$\mathbb{1}' m_s = n h_0 p \tag{49}$$

from Lemma 2. Plugging (48) into the market clearing condition (49) yields

$$(e_0 + h_0 p)\mathbb{1}' G\mu_{sj} = (d - cdp)\mathbb{1}' G\mathbb{1} + n h_0 p,$$

which can be rearranged as (14).

We define the threshold harmonic distance, which is a function of equilibrium price p, as

$$\mu^*(p) \equiv \frac{d - cdp}{e_0 + h_0 p},$$

which is decreasing in p. Thus, it will be easier for an agent to have safe harmonic distance

[10] For a matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

the Schur complement of D in M is $M/D \equiv A - BD^{-1}C$ (Poole and Boullion, 1974).

from the agent under the shock under high p. By definition, non-defaulting agents should have $m_s \geq 0$, implying

$$(e_0 + h_0 p) G \mu_{sj} \geq (d - cdp) G \mathbb{1},$$

which is derived from (48). This implies

$$G\mu_{sj} \geq \mu^*(p) G \mathbb{1}. \tag{50}$$

Because G is a non-singular M-matrix, there exists G^{-1} such that $G^{-1} \geq 0$ (Poole and Boullion, 1974). Hence, G^{-1} is element-wise non-negative, implying that (50) yields

$$\mu_{sj} \geq \mu^*(p) \mathbb{1},$$

proving the contrapositive of the second part of the first statement. Thus, if $\mu_{ij} < \mu^*(p)$, then agent i defaults.

For the second statement, suppose that all agents default and $p = 0$. Then, for any $i \neq j$,

$$x_i = e_0 + \sum_{k \neq j} q_{ik} x_k < d,$$

which can be rearranged as

$$\frac{x_i}{e_0} = 1 + \sum_{k \neq j} q_{ik} \frac{x_k}{e_0} < \frac{d}{e_0} = \mu^*(0).$$

Thus, the expression coincides with the harmonic distance equation (13), using $\mu_{ij} = \dfrac{x_i}{e_0}$. Therefore, $\mu_{ij} < \mu^*(0)$ for all agents $i \neq j$.

For the third statement, suppose that p is the equilibrium price and

$$\mu^*(p) = \frac{d - cdp}{e_0 + h_0 p} < 1.$$

Hence, $d - cdp < e_0 + h_0 p$, implying that agents can pay their liabilities out of their endowments, as c is large enough. Therefore, no other agents default on their inter-agent liabilities.
■

D. Results from the Extended Models

Proof of Proposition 8. First, we show that the cutoff collateral ratio for full insulation, $c^*(s,n)$, is decreasing in s and n. Recall that $c^*(s,n) = \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\}$. If, $c^*(s,n) = 1/s$, then $c^*(s',n) = 1/s'$, so the cutoff for full insulation is higher at s'—that is, $c^*(s,n) < c^*(s',n)$. Under this case, $c^*(s,n) \leq c^*(s,n')$, as $c^*(s,n')$ is either $1/s$ or $\frac{\kappa h_0}{(n'-\kappa)e_0}$, which is greater than $1/s$. Otherwise, $c^*(s,n) = c^*(s',n) = \frac{\kappa h_0}{(n-\kappa)e_0}$ and $c^*(s,n) = \frac{\kappa h_0}{(n-\kappa)e_0} < c^*(s,n') = \frac{\kappa h_0}{(n'-\kappa)e_0}$ for $n > n'$.

Now we show that $c_*(s,n)$ is decreasing in s and n. Recall that

$$c_* \equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds},$$

which is trivially decreasing in n. The first-order derivative of c_* with respect to s is

$$\begin{aligned}\frac{\partial c_*}{\partial s} &= \frac{h_0 ds - d\left(d - ((n-\kappa)/\kappa)\, e_0 + h_0 s\right)}{(ds)^2} \\ &= \frac{d\left(\frac{n-\kappa}{\kappa}e_0 - d\right)}{(ds)^2} < 0,\end{aligned}$$

where the last inequality comes from $d > (n-1)e_0 > \frac{n-\kappa}{\kappa}e_0$. Therefore, the overall vulnerability region becomes smaller as s and n increase. ■

Proof of Corollary 1. From Proposition 3 and Proposition 4, we have

$$\begin{aligned}c_* &\equiv \frac{d - ((n-\kappa)/\kappa)\, e_0 + h_0 s}{ds} \\ c^* &\equiv \max\left\{\frac{1}{s}, \frac{\kappa h_0}{(n-\kappa)e_0}\right\},\end{aligned}$$

which are (weakly) increasing in h_0. ■

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