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Optimal Bidder Selection in Clearing House Default Auctions

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Abstract

Central counterparties' ability to hold successful default auctions is critically important to financial stability. However, due to the unique features of these auctions, standard auction theory results do not apply. We present a model of CCP default auctions that incorporates both the vital, but non-standard, objective of minimizing the likelihood it suffers reputationally damaging losses and the potential for information leakage to affect CCP members' private portfolio valuations. This gives insight into the key question of how CCPs should select auction participants. In particular, we prove that an entry fee, by appropriately incentivizing some members not to enter the auction, can maximize the probability of auction success. The result is novel, both in auction theory and as a mechanism for CCP auction design.

Keywords: Auctions; Central counterparties; CCPs; Default; Derivatives; Entry mechanism; Financial stability; Systemic risk

JEL: D44; D47; G13; G23

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1 Introduction

Central clearing has long been a ubiquitous feature of exchange-traded derivatives. It is a critical market mechanism to mitigate counterparty credit risk: all large futures exchanges have an affiliated clearing house, and trades are novated to this central counterparty (or 'CCP') following their execution on the exchange. In response to the 2008 Global Financial Crisis, central clearing was mandated for over-the-counter (OTC) derivatives too, with the G-20 declaring that 'all standardized OTC derivative contracts should be... cleared through central counterparties by end-2012 at the latest'. OTC derivatives clearing mandates have now been enacted in most leading jurisdictions. As a result, major CCPs are critical to the functioning of financial markets and are often deemed to be systemically important financial infrastructures. These CCPs intermediate substantial amounts of risk, as consideration of the total initial margin they require indicates. The three largest CCPs for exchange-traded derivatives in the US and UK required over $300 billion in initial margin at the beginning of 2024, for instance, while the margin requirements of the three largest OTC derivatives CCPs topped $360 billion. Murphy (2012) and King et al. (2023) discuss the systemic importance and risks of central clearing in more detail.

The role of a CCP is to sit between market participants, guaranteeing their performance to each other and acting as the counterparty to all cleared trades. If one of their members defaults, the clearing house must close-out the defaulter's portfolio. Thus, they should make it far less likely that a failure of one market participant causes direct losses at another by acting as a buffer, preventing systemic runs as in Zawadowski (2013). In order to be able to do this, CCPs must have sufficient resources to absorb any plausible losses resulting from the default management process. A loss of confidence in a clearing house can cause or increase a loss of confidence in the financial system, as Bernanke (2015) discusses, so this is an important issue.

Different CCPs use different methodologies to size these resources, subject to applicable regulatory minima. This topic has received substantial attention in the literature, with a focus on the adequacy of resources in various situations. See, e.g., Antinolfi et al. (2022), Boissel et al. (2017), Capponi et al. (2017), Cerezetti et al. (2019), Kuong and Maurin (2023), Murphy and Nahai-Williamson

(2014), Rec (2019a,b). A particular issue is the potential clustering of defaults and episodic market illiquidity, as in Azizpour et al. (2018), Carlin et al. (2007). The related and important question of CCP (funding) liquidity risk has also been studied: see, e.g. Cont (2017) and King et al. (2023). In contrast, the close-out process, and particularly the design of CCP default auctions, is relatively less studied, despite it being crucial to the losses experienced by the clearing house in default management. Default actions are therefore the focus of this paper.

When the defaulter's portfolio is relatively small compared to available market liquidity, the default management problem is straightforward: in markets with central limit order books, the CCP can simply enter orders to liquidate the defaulter's portfolio. Similarly, in markets which trade largely on a request for quote basis, quotes will readily be available, and hence an auction may not be necessary.

The problem can become much more challenging with the default of larger or concentrated portfolios. Such defaults can create significant stress, as the CCP's ability to close out the defaulter near the market price is uncertain.[1] Moreover, unlike the problem in many trading situations, the CCP must trade a particular portfolio in a relatively short time frame: it has neither the mandate nor the loss-absorbing resources to bear market risk. Therefore, it is common for the CCP to auction the defaulter's portfolio either as is, or after it has been hedged. Both Lehman Brothers' $9 trillion interest rate swaps portfolio cleared by the London Clearing House, and their $2 billion exchange-traded futures and options portfolios cleared by Chicago Mercantile Exchange, were liquidated via multiple auctions, for instance.[2] The analysis of CCP default auctions is therefore practically important. It is also technically non-trivial, as these auctions have novel features.

CCPs have a fixed amount of funded resources to manage defaults, structured in what is known as their default waterfall. The first tranche of these resources are provided by the defaulter in the form of initial margin. This tranche is followed by the defaulter's contribution to a layer of

[1] A good example is the stress at the New Zealand clearing house caused by the default of Stephen Francis described in Cox et al. (2016).

[2] See Fleming and Sarkar (2014) and LCH.Clearnet (2008) for LCH and Valukas (2010) for CME. For further analysis of Lehman, see Wiggins and Metrick (2019)

mutualized resources: the CCP's guarantee or default fund. Then there is a (typically thin) layer of the CCP's own resources, known as skin in the game. After this comes the rest of the guarantee fund. Thus, first the defaulter pays, then the CCP does, then non-defaulting clearing members. Any resources from the defaulter that are not needed to manage the close-out are returned to the defaulter's estate. There are legal requirements in leading jurisdictions for CCPs to act reasonably in their close-out,[3] but these requirements leave substantial flexibility in default management. All of this means that CCPs are much more concerned with getting an acceptable bid in a default auction than in the price of that bid. If it cannot completely close-out the defaulter's portfolio without suffering a loss bigger than the resources that they have provided, not only will it face financial loss, but also it will almost certainly suffer reputational damage.

Obtaining an acceptable bid for the defaulter's portfolio is by no means guaranteed. For instance, in the 2018 default by Einar Aas at the Nasdaq Nordic clearing house, the first attempt at auctioning the portfolio failed, and the second attempt resulted in a loss of €114 million in excess of the resources Aas had provided to the CCP, resulting in losses to both the CCP's skin in the game and its guarantee fund. Subsequently, it was revealed that the second auction had not liquidated the portfolio, but rather had just established hedges. In both auctions, Nasdaq Nordic limited participation to four firms that it had selected.[4]

Standard auction theory suggests that revenue is increased by attracting as many bidders as possible to an auction.[5] However, this is not necessarily the best strategy in CCP default auctions due to a phenomenon known as information leakage. The trigger event for a CCP auction—the default of one or more clearing members—is public knowledge, as are the market movements associated with the default. However, before the auction begins, only the CCP has full details of

[3] See Braithwaite and Murphy (2017) for further details.

[4] For details on the default and auctions, see Clancy (2018b), Clancy (2018a), and Mourselas (2019). Mourselas (2021) covers the details that emerged later about the failure of the second auction and subsequent regulatory fine. Also, see Box A: Two defaults at CCPs, 10 years apart, by S. Bell and H. Holden (Faruqui et al., 2018, pp. 75–76 in) for more analysis, and McConnell and Saretto (2010) for a different example of auction failures.

[5] Auction theory is covered in Milgrom (2004) and Krishna (2010); both cover standard models and equivalence results.

the portfolio to be auctioned. Of course, it has to provide this information to bidders and, once they know it, they can either choose to bid, or to trade against the CCP. Thus, there is a concern that inviting more and more bidders will lead to an adverse change in market prices between the portfolio being revealed and bids being submitted. This period cannot be very short, as a defaulter's portfolio may be large, and bidders need some time to price it. A CCP default auction is thus analogous to a predictable trade, as discussed by Bessembinder et al. (2016), combined with fundamental participation questions. Nasdaq Nordic explicitly stated that it limited participation in its auction Aas' positions to avoid information leakage, as Mourselas (2019) discusses. The recent paper prepared by regulators, CPMI-IOSCO (2020), emphasizes both this topic and the questions of who to include in an auction and how to incentivize bidders, as part of its broader discussion of issues for CCPs to consider in default auctions. That paper raises these topics without providing guidance on how to simultaneously choose and incentivize bidders, limit information leakage, and successfully conduct an auction.

To summarise, clearing house default auctions have two salient features that sharply distinguish them from standard auctions. First, the CCP wants to minimize the probability of using its own or non-defaulting members' resources, not to maximize income from the auction. Second, the possible negative effects of information leakage means that it is not desirable to attract bidders who are unlikely to submit competitive bids. This later aspect bears some resemblance to the literature on endogenous auction participation (Lauermann and Wolinsky, 2017, Menezes and Monteiro, 2000). The combination of these two aspects is particularly novel from an auction theory perspective.

Our contribution is to model the salient features of a clearing house default auction and to design an optimal process for conducting it while allowing endogenous participation decisions. Different CCPs will have different rules for conducting auctions, and different strategies for recovering from an unsuccessful auction, so we focus on the common goals of the default management process across CCPs. We show how the opposing effects of information leakage—where having more bidders tends to lower the value of the object and hence leads to lower prices—and the need to maximize the probability of a successful bid can lead to an optimum number of bidders that is strictly less than the number of available bidders. The key aspect from a mechanism design perspective is the use of an optimally selected entry fee to get good bidders to self-select.

Bidders use commonly known information about the direction and likely size of the defaulted position, and private information about the value to them of adding the position to their own portfolio, to decide whether to pay the entry fee, become more informed about the actual size of the defaulted position, and then decide whether or not to submit a bid.[6] We prove that even with a reservation price, if the cost of information leakage is high enough, a CCP maximizes its probability of receiving a bid above its reservation price by setting a non-zero entry fee. Although the entry fee limits participation, it does so in an equitable fashion in that it is set before the default occurs. Furthermore, the entry fee excludes bidders who are not likely to bid high enough to win the portfolio and consequently whose main impact is to increase information leakage.

2 Literature Review

In addition to the scholarship cited above, various authors explore aspects of central clearing on market function. For example, Loon and Zhong (2014) find that central clearing improves market liquidity. Duffie et al. (2015) and Grothe et al. (2023) find that CCPs' impact on the demand for collateral depends on the market structure and that if margin is required for non-centrally cleared trades, then central clearing lowers the demand for collateral.

A different strand of the literature focuses more on clearing house default management. Cont (2015) and Armakolla and Laurent (2017) both focus on the impact of loss allocation in CCP default waterfalls. Cerezetti et al. (2019) is closer to the spirit of this paper as it looks at how to optimize CCP default processes, but it focuses on hedging and does not analyze auctions at all. Koeppl et al. (2012) considers the impact of default management on the market, demonstrating how concerns over default at a CCP can harm market liquidity. However, this research looks at the impact of central clearing prior to a default actually occurring.

[6] The decision by each potential bidder *whether* to pay the entry fee is a key part of this process. A pre-committed fee, such as a default fund contribution at risk from a failure to bid, would not achieve the same effect.

After default occurs, a CCP will likely need to liquidate the defaulter's positions and collateral. The default of a large institution can cause large disorderly collateral liquidations, as Oehmke (2014) discusses. Given that CCP default management is intended to make managing a default more orderly, this is unhelpful. The other liquidation—that of positions—has received less attention, which motivates our study of the problem.

There are, however, three papers that directly analyze CCP default auctions of positions. The closest paper to this one, Ferrara et al. (2019), considers various designs of CCP default auctions theoretically, but makes the standard assumption that the CCP seeks to maximize revenue. As we detail below, that assumption ignores crucial features of the CCP's payoff. Their paper also assumes that the number of bidders is fixed, so it is silent on the question of who should actual be invited to participate in the auction. The authors do examine the possibility that poor bids can face a negative externality due to low competitiveness in the bidding process, which has some similarities to the externality driven by information leakage in our model. However, there is no endogenous entry in their framework, so the implications are more muted.[7]

In related research, Oleschak (2019) considers first price single item CCP default auctions where bidders have private values and share eventual losses with the CCP. The author does look at the impact of being invited to the auction or not, finding that invited bidders are better off than those who are not invited to the auction. However, this mechanism depends on the CCP being able to pick bidders with high private values. The inability to do so is exactly what we seek to study: whether a CCP can include bidders with high valuations and conversely exclude those with low valuations without knowing or even having a signal about private valuations.

Lastly, there are similarities between this paper and Huang and Zhu (2024), which builds on Du and Zhu (2017). However, to avoid price impacts, the former paper assumes that bidders are

[7] This negative externality is enough however to break the revenue equivalence between first and second price auctions; the authors find that a second price auction with loss sharing, rather than first price auctions with or without penalty, increases the liquidation value of the portfolio. This paper also uses a second price auction framework, but focuses on designing an effective mechanism to maximize the chance of success with strategic participation, rather than the impact *ex post* loss sharing when the auction is not as successful under fixed participation.

infinitesimal and, similar to the other two papers on CCP auctions, focuses on analyzing what happens if an auction results in losses that must be absorbed by the guarantee fund. It examines the benefits of juniorization of guarantee fund contributions in such circumstances.

Interestingly, all three papers focus on the impact of loss sharing among clearing members when the default auction goes poorly. But in each case, there is no reputational cost to the CCP of such a poor auction outcome. In sharp contrast, we argue that CCPs are so strongly motivated by the reputational costs, which would result from needing to apportion losses to its members, that avoiding uncovered losses is their primary auction objective. Postulating loss-avoidance rather than revenue maximization as the auction objective leads to a novel auction problem, not just relative to the scant literature on CCP auctions, but also in the broader auction literature.

Besides specifying a realistic auction objective, we further tackle the related question of how to construct the pool of bidders. This is a key issue raised in CPMI-IOSCO (2020), and it is challenging due to the concerns raised about information leakage. As mentioned, there is a literature around predictable trades. Admanti and Pfleiderer (2015) argues for a positive view here. The pertinent concerns about information leakage are, however, much more consistent with a negative view, as in Brunnermeier and Pedersen (2005) and Carlin et al. (2007). Indeed, the worse-case scenario is that the liquidation of a defaulter's portfolio takes on the characteristics of a fire sale: see Coval and Stafford (2007) and Kuong (2020). Furthermore, limited participation is not merely a theoretical concern as it seems to be a common feature of the relatively few CCP auctions observed. Besides the prior discussion of Nasdaq Nordic, where the CCP selected four bidders only, it was reported in Sourbes (2015) that LCH's auctions of Lehman portfolios had (depending on the currency concerned) around five participants.

There also exists a related market microstructure literature on how information leakage, or front-running, affects trading. Burdett and O'Hara (1987) model how an institutional investor constrains the number of dealers it approaches to execute a large trade due to information leakage. In Hendershott and Madhavan (2015), the level of information leakage determines the venue that a dealer utilizes for a trade. More recently, Baldauf and Mollner (2024) endogenize the impact of information leakage in the subsequent on-market trading. Similarly to this last paper, we focus on information that is available to auction participants.

There are critical differences between this literature and the CCP problem, however. First, in the papers cited, the invitation to be in an auction reveals information to those invited, while a CCP's need to conduct an auction is public knowledge. Second, in previous information leakage papers, there are no consequences to an auction failing, while for CCPs an auction failure causes material loss and would significantly damage its reputation, potentially to the point that the CCP would no longer remain viable. Third, in a CCP member default, the direction the market moved to cause the clearing member default reveals the direction of the defaulting portfolio. Consequently, the optimal strategy in Baldauf and Mollner (2024) of requesting two-sided bids is not available. As the focus here is on the price impact of information leakage on the auction itself, we exogeneously specify the price impact, which resembles Baldauf and Mollner (2024) but do not model the mechanism through which it occurs like they do. In summary, the microstructure literature focuses on how and where to search for best execution of regular and repeated trades to minimize the impact of information leakage on transaction costs. In contrast, we focus on how to optimally endogneize bidders' participation to minimize the impact of information leakage on the success of the extremely irregular, but critically important, auction of a defaulter's portfolio.

The institutional characteristics of CCP default auctions mean that our analysis is non-standard and arguably novel. But it does connect to certain strands of the broader auction literature. Lou et al. (2013) finds that even in the liquid Treasury market, announced auctions can cause variations in valuations due to dealers' risk-capacity. Levin and Smith (1994) endogenizes entry and finds that the results of the auction can diverge from those predicted by an analysis that ignores the entry question. Furthermore Lauermann and Wolinsky (2017) and Menezes and Monteiro (2000) both prove that in circumstances with endogenous entry revenue can decrease with an increase in the number of participants. Similarly, in our paper, entry takes a centre stage in line with the discussion in CPMI-IOSCO (2020). Our model explicitly allows that more competition is not necessarily desirable. This is consistent with the empirical findings in Hong and Shum (2002) (although we use private valuations rather than common ones) and the model of Glebkin and Kuong (2023). A key difference is that here endogenous entry is considered in an auction where the CCP's objective is not maximizing revenue. In addition, potential auction participants are risk-averse rather than risk-neutral. Our setup also relates to Pinkse and Tan (2005), in that information leakage creates affiliation amongst independent valuations, although in a second-price auction rather than first price as in their analysis. Finally, Milgrom and Weber (1982) examine competition and entry fees, raising the possibility that monotonic equilibria might not exist. Landsberger (2007) extends this analysis and finds that such existence becomes increasingly unlikely as the number of bidders grows. Our analysis is in the same vein; our structure implies non-monotonicity, so adding more bidders is not optimal.

3 Derivatives Markets and CCPs

Modern derivatives markets are characterized by the clearing of standardized contracts at CCPs while other, potentially bespoke, contracts are cleared bilaterally.[8] Dealers, and perhaps some other market participants, are direct, or 'clearing' members of CCPs. Thus a dealer's net risk position is composed of its cleared position and its bilateral one, and only the former is typically known to the CCP.

CCPs require that their members post initial and variation margin at least daily. Variation margin on each cleared portfolio or 'account' is determined based on the current mark-to-market, so it can be thought of as settling the value of the portfolio every day. Initial margin is based on the risk of the portfolio: it is intended to cover its potential change in value over a fixed liquidation horizon, known as the margin period of risk, to a high degree of confidence. Regulation sets minimum standards for the margin period of risk and the confidence level of margin. For OTC derivatives, initial margin is required to cover at least the 99$^{\text{th}}$ percentile of potential changes in portfolio value over a five day period. Figure 1 on the following page illustrates the idea.

The margin period of risk is intended to be long enough that the non-defaulting party can determine that an event of default has occurred, begin default management, hedge the defaulter's

[8] Such contracts are properly considered to be non-centrally cleared, but are often referred to as uncleared.

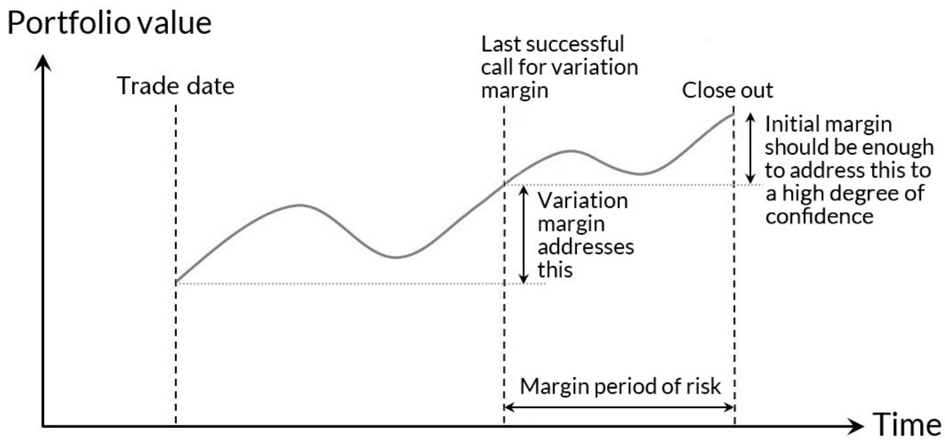

Figure 1: The roles of initial and variation margin

portfolio if necessary, and sell it. In our context, 'selling it' means conducting an auction, including determining who to invite to the auction, communicating the portfolio to them, receiving bids, deciding on a winner, and novating the defaulter's portfolio to them.

Bidders will use the market value of the portfolio at the point of bidding as a basis for their bids. Thus, the CCP is most at risk from losses in default management when the value of the portfolio has fallen between the last successful variation margin call with the defaulter, at $t = 0$ say, and when bids are submitted, at $t = T$ say.

We will model a modern futures market subject to initial and variation margin, reflecting these features. This means that market participants, when they enter into trades, only pay initial margin. Changes in value of their portfolios are settled day-by-day through variation margin. Before we develop our model of the derivatives market, the remainder of this section provides a concrete example of this situation.

Example. Suppose we are dealing with the front month London Metal Exchange ('LME') copper futures, and assume that a party bought 10 lots of this future on 19th May 2020. On this day the futures price was \$5,314. The LME copper future is the right to receive 25 tonnes of copper and is priced in US dollars per tonne. Therefore, this party locked in a price of \$5,314 $\times$ 10 $\times$ 25 $=$ \$1,328,500 for 250 tonnes of copper at the expiry of the future. The closing price of this future on Friday 19th June 2020 of the front month was \$5,855.50, meaning that someone who bought this future on that day, locked in a price of 25 $\times$ \$5,855.50 $=$ \$146,387.50 for 25 tonnes of copper at expiry. Suppose that our party defaulted at the close of business on Friday 19th June. The CCP will have paid a cumulative $(\$5{,}855.5 - \$5{,}314) \times 25 \times 10 = \$135{,}375$ of variation margin to the defaulter. Note that if the future expires with this level, 5,855.50 as its settlement price, longs pay this amount per tonne and shorts receive this amount: the reason that the defaulter locked in the lower level of $5,314 per tonne is that they have received the difference as variation margin.

The use of variation margin means that if the futures price goes up, long position holders receive money; if it goes down, they pay money. It is only at expiry that the obligation arises to pay the settlement price and receive the commodity.

Now suppose that the initial margin was 500 points or $12,500 per lot; the defaulter will have paid initial margin of $125,000. Further suppose that their guarantee fund contribution was $5,000 and that they default on Monday 22nd June, before the market opens. The CCP needs to auction the right to pay $5,855.50 per tonne for 250 tonnes of copper. If the bid is say a per-lot price of $5,200, then the CCP has lost $(\$5{,}855.50 - \$5{,}200) \times 25 \times 10 = \$163{,}875$ and it only has initial margin of $125,000 plus defaulter's guarantee fund contribution of $5,000 so there is a loss of $\$163{,}875 - \$125{,}000 - \$5{,}000 = \$33{,}875$ to be allocated first to the CCP's skin in the game ('SITG') then, if that is inadequate, to non-defaulter's guarantee fund contributions. The CCP's desired per-lot auction price is at least $\$5{,}855.50 - \frac{\$125{,}000+\$5{,}000}{25\times 10} = \$5{,}335.50$, as the defaulter's resources are sufficient to cover losses if the winning bid is at or above this level.

Note finally that the profits or losses of the auction are realized in the variation margin call at the end of the day of the auction. If the successful bid in an auction on Monday 22nd July was a per-lot price of $5,200 for a long position in 10 lots of futures, and the future closes at $5,800 (close to the previous Friday's close), then the CCP will pay the successful bidder $(\$5{,}800 - \$5{,}200) \times 25 \times 10 = \$150{,}000$. There is no cash flow in the auction itself—just like the initial purchase, what is being agreed is not a price to pay but a level from which to base future variation margin payments.

Similarly, if the defaulter was short 10 lots, then bidders would rationally bid above the current price of $5,855.50. If the winning bid was, say $5,950, and the future closes on the evening of the auction at $5,870, the CCP would pay the bidder $(\$5{,}950 - \$5{,}870) \times 25 \times 10 = \$20{,}000$.

4 A Model of a Derivatives Market

This section introduces the model of a derivatives market that we will use for the rest of the paper. We will model a single risk factor, to be thought of as the price of a commodity future.

There are two key features to this model. First, we assume that market participants are risk averse. This is modelled by a private value for position which reduces their value to the holder the bigger they are. Second, the model captures both OTC forwards and cleared futures positions, so the CCP does not know the net risk position of any market participant.[9] Without this, the problem of selecting auction market participants is trivial, as it simply invites those with positions closest to and opposite in sign from the defaulter. It is also realistic to assume that OTC positions can be significant, and can have a material effect on the exchange-traded market.[10]

4.1 Positions

The net risk position of each clearing member is defined in terms of a single risk factor that can take positive and negative integral values. The risk factor trades both as a cleared future and as uncleared forwards, so the CCP does not know any clearing member's net position. The position is expressed as a futures equivalent. We will write s_i for the position of clearing member $i \in I$, where $s_i < 0$ denotes a short position and $s_i > 0$ denotes a long position. Because we are dealing with a derivatives market, the sum of the longs equals the sum of the shorts:

$$S := \sum_{i \in I} \max(s_i, 0) = -\sum_{i \in I} \min(s_i, 0), \tag{1}$$

where S denotes the size of the market.

[9] The OTC market can be several times the size of the exchange-traded market, so this is a realistic assumption. See Oliver Wyman (2023) for an example of this situation.

[10] See, for instance, LME's Consultation 22/145, 2022, which notes that "Recent events in the LME Nickel market have demonstrated the effects that OTC activity can have on the wider LME market."

4.2 The Futures Price and Variation Margin

We will write $V(t)$ for the quoted price of one lot at time t: think of this as the futures price. Because of variation margin, the public mark-to-market of all positions cleared by non-defaulters at the end of each day is always zero. Let $t = 0$ denote the end of one day. If the market participant acquires a new position of s at the close, and $t = 1$ denotes the end of the next day, then the total variation margin paid to the market participant will be $s(V(1) - V(0))$ if this is positive, or from the market participant if it is negative.

After a default, the CCP has to continue to pay or receive variation margin on the other side of the defaulter's cleared position. We denote the defaulter by D and its cleared position by s_D.[11] Without loss of generality, denote the last time variation margin was exchanged prior to default by $t = 0$. Then if the per-lot price changes from $V(0)$ to $V(T)$ at the time T when the CCP novates the position to an auction participant, the mid-market profit or loss for the CCP on the defaulter's position is $(V(T) - V(0))s_D$.

4.3 Funded Resources and a Successful Auction

We will denote the initial margin posted by clearing member i by M_i, their guarantee fund contribution by G_i, and the CCP's skin in the game by SITG. The total guarantee fund G is $\sum_{i \in I} G_i$. The resources contributed by the defaulter are $M_D + G_D$. After a default, the CCP can use these resources to cover any losses it incurs in closing out the defaulter's position.

Figure 2 on the next page shows an example path of the mark-to-market of the defaulter's portfolio through the margin period of risk. It starts at zero by definition as we assume a successful variation margin call at $t = 0$.

We assume that at $t = T$ the CCP transfers the position at a price b which could be either positive or negative: positive b represents cash coming into the CCP from the next variation margin call and negative, cash leaving, as usual. The CCP's total profit or loss on the defaulter's portfolio, after

[11] The remainder of its portfolio, the bilaterally cleared part, would be managed during bankruptcy aside from the CCP auction, so it is not relevant here.

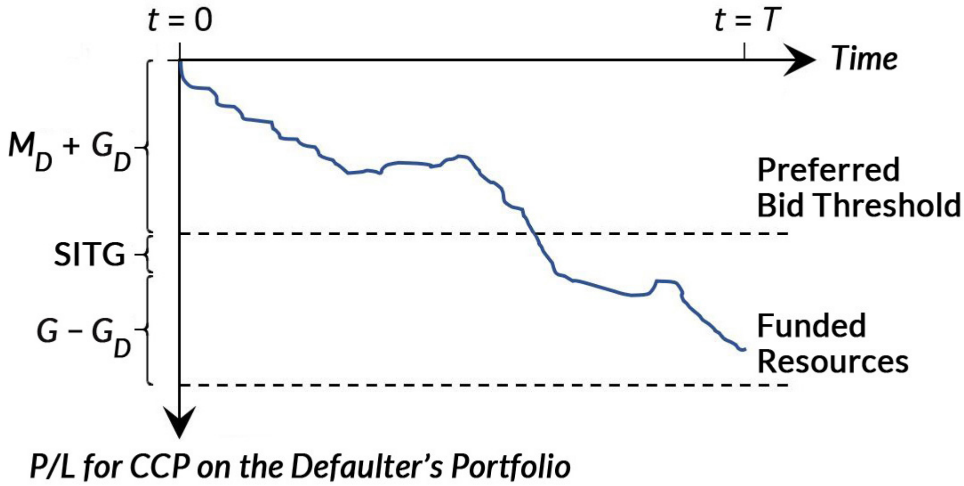

Figure 2: The value of the defaulter's portfolio through the margin period of risk and the use of resources in the default waterfall to absorb losses on it

using their resources, and including variation margin, is therefore:

$$P = (V(T) - V(0))s_D + b. \tag{2}$$

If the CCP has to pay out b in the auction, then the profit or loss is b below $(V(T) - V(0))s_D$. The total funded resources are $M_D + \text{SITG} + G$. In general $(V(T) - V(0))s_D$ could take either sign, but the auction is more difficult when it is negative, so this situation is illustrated as follows.

If the CCP does not make a loss on closeout, $P \geq 0$, it is obliged to return the defaulter's resources to the administrator of the estate. On the other hand, if it makes a loss, $P < 0$, it can absorb that loss with the available resources.[12] The CCP's total profit or loss on the defaulter's portfolio after using the defaulter's resources is therefore:

$$\begin{array}{ll} P & \text{if } P \geq 0, \\ 0 & \text{if } -M_D - G_D \leq P < 0, \\ P + M_D + G_D & \text{otherwise.} \end{array} \tag{3}$$

This can be simplified to

$$\max\left(P, \min(P + M_D + G_D, 0)\right). \tag{4}$$

[12] We assume that the CCP does not make any further recoveries in excess of the available collateral.

The CCP's own resources are at risk if the cost of default management is larger than $M_D + G_D$. For simplicity, we assume that there are no other costs—such as hedging costs—and that an auction succeeds if the CCP can liquidate the portfolio without paying more than this.

The payoff function in equation (4) is one of the things that makes CCP default auctions novel. A standard assumption in auction theory is that the seller seeks to maximize revenue, and most analysis of different auction characteristics focuses on the impact to expected revenue (Krishna, 2010). But in a default auction, the CCP does not actually profit from better bids b within the range

$$-M_D - G_D \leq P. \tag{5}$$

There is therefore little or no incentive for a CCP to care about increasing the bid within this range. In contrast, there is a considerable negative reputational impact associated with a default auction which eats into skin in the game. Furthermore, although it varies across CCPs, skin in the game is generally a relatively thin layer of resources. If the CCP's skin in the game is exhausted, the CCP starts mutualizing losses among the remaining clearing members. While unlikely, this situation could so severely damage the CCP's reputation that avoiding it is paramount to the CCP. Consequently, a CCP is much more concerned with maximizing the probability that it will receive a winning bid that is high enough to ensure that losses are covered by the defaulter's resources than with maximizing auction revenue. Put another way, the CCP can be viewed as largely indifferent to the level of revenue generated by the auction above the threshold illustrated in Figure 2, but it faces a discontinuous loss below it. This produces a particular form of risk aversion, so that CCPs focus on minimizing the downside risk rather than seeking to maximize revenues as a seller normally does. Rather than specifying some form of risk aversion that would incorporates these complicated threshold and reputational effects, we assume that the CCP seeks to maximize the probability that its loss on the portfolio is covered by the defaulter's resources $M_D + G_D$.

4.4 Private Values

Following the set-up in Du and Zhu (2017), suppose that market participants have an aversion to risk, which affects their private value of positions. In particular, the private mark-to-market of a position of size $s \in \mathbb{R}$ is

$$-\beta s^2 \tag{6}$$

for some positive β (i.e., the bigger positions become, the more holders (quadratically) discount them). For ease we will assume β is constant for all participants and this is known. This formulation is consistent with the findings of Lou et al. (2013) that dealer's risk capacity affects valuations around US Treasury auctions.

4.5 Private Values for Winning Bidders

The defaulting clearing member has a position s_D. If a clearing member wins the auction for this position at a price b, it will be netted with their existing position. If s_i was the clearing member's old position, the original position has a private value of $-\beta s_i^2$, and the new position is privately worth $-\beta(s_i + s_D)^2$. Hence, the clearing member is indifferent between buying the position s_D for b and not buying it when

$$-\beta(s_i + s_D)^2 - b = -\beta s_i^2.$$

We will write $\tilde{b}_i$ for this rational bid threshold for clearing member i:

$$\tilde{b}_i = -\beta(2s_i s_D + s_D^2).$$

Clearly it is irrational to bid above this level, as $-\beta(s_i + s_D)^2 - b < -\beta s_i^2$ when $b > \tilde{b}_i$.

4.6 Desirable and Undesirable Bidders

Suppose the defaulter's position is big, $s_D \gg 1$, and all the bidders are on the same side of the market as the defaulter and big too, $s_i \gg 1$ for all $i \in I$. Then $(2s_i s_D + s_D^2)$ is always positive and large. Hence, unless β is very small, for all $i \in I$, b_i will be negative and large, and the CCP will not get a bid above its threshold $-M_D - G_D$. Thus, only having bidders in the auction on the same side of the market as the defaulter is damaging for the CCP's objective of having a successful auction.

Conversely, if there is a bidder i with position opposite to the defaulter, $\text{sgn}(s_i) \neq \text{sgn}(s_D)$, and bigger than it in size, $|s_i| > |s_D|$. Then $2s_i s_D$ is negative and $s_i^2 > s_D^2$, so $(2s_i s_D + s_D^2) < s_i^2$, and hence winning the auction frees up private value. For this bidder, $\tilde{b}_i > 0$. Clearly, if instead $s_i = -s_D/2$, then $2s_i s_D + s_D^2 = 0$ and so $\tilde{b}_i = 0$.

Following on from this, if the number of auction participants is large and s_i is symmetrically distributed, then there will be some bidders for whom s_i and s_D are of opposite signs. Any of these participants who have $|s_i| > |s_D|$ will make positive bids, so unless $P < -M_D - G_D$, the probability that the CCP will get a bid above its threshold approaches 1 as the number of bidders increases. In this setting, inviting more participants to the auction is always better. This result is not surprising. Even with the change to the seller's objective, the folk theorem result that adding more bidders generally increases revenue intuitively suggests that adding more bidders will increase the probability that the CCP receives a bid above its threshold.

4.7 The Auction and Information Leakage

It is reasonable to assume that market participants know the overall direction of the defaulter's portfolio s_D—long or short—but not its precise size. The direction is revealed because the the direction of market moves can be easily compared to the time of default, which is known. The size is revealed to auction participants by the CCP just before the bidding process opens. Of course, actual portfolios are more complicated, and in a default auction the actual positions would be revealed to participants; revealing size stands as a good proxy for the revealed information in the simpler market model.

Thus far, the market value of defaulter's portfolio at the time of the auction has just depended on the drift in the futures price $V(t)$. We assume this movement is determined by a standard normal random variable $Z \sim \mathcal{N}(0, 1)$ so that

$$V(T) = V(0)\,(1 + \sigma Z)\,,$$

where σ is a volatility-like scaling parameter. In addition, we will assume that information leakage creates a risk that auction participants trade outside the auction against the defaulted position, which hurts its value.[13] As noted above, regulators recommend that CCPs should balance the risk of information leakage and the aim of obtaining a competitive price when deciding on the most appropriate execution method, highlighting the importance of modeling information leakage.

In order to do this, we assume that having N bidders with certainty reduces $V(0)$ by an amount $\gamma(N-1)^Q$ for fixed $\gamma > 0$ and Q. The constants $\gamma > 0$ and Q are known and abstractly account for the cost of information leakage. We assume that the impact of information leakage increases non-linearly in the number of auction participants. So for N bidders, the futures price at the moment bids are submitted is:

$$V(T) = V(0)\left(1 + \sigma Z - \gamma(N-1)^Q\right).$$

Note that the price move against the CCP depends on the number of (potential) bidders, so we are assuming that having more bidders means more information leakage. In their model of client trading, Baldauf and Mollner (2024) show how the possibility of information leakage, or front-running, can reduce dealer's competition for the client's trade. Although their mechanism endogenizes the impact of moving from one dealer to two, the actual price impact is exogenously assumed. As the focus here is on the price impact of information leakage on the auction itself, we exogeneously specify the price impact for a general increase in the number of auction participants.

The problem the CCP now faces is that there are two competing pressures: having more bidders increases the probability of receiving a good bid (*competition effect*), but it also increases the size of the price moving against the CCP (*information leakage effect*). The simple strategy of including everyone in the auction is no longer optimal. In the next section, we consider the CCP's auction

[13] The need to send portfolios, which potentially consist of tens of thousands of instruments, to multiple market participants and allow those participants time to price the portfolios and determine the impact winning them would have on their risk, capital and liquidity inevitably means that the minimum time between participants receiving details of the defaulter's portfolio and bids being due is measured in hours. This is ample time for information leakage to occur.

strategy in this situation. The CCP would rather like to be discriminating in who it invites. As the CCP does not have the information available to fully discriminate, we examine whether it can use an entry fee to endogenously encourage the right participants and discourage the wrong participants so that it can induce the participants to balance the competing pressures.

4.8 Bidder Values

Bidder values at time T depend upon the price of the future $V(T)$, the number of bidders N, and the defaulter's position s_D. We assume that the CCP truthfully reveals s_D to all the bidders that enter the auction.

Bidder i's maximum value of bidding v_i then is

$$v_i \equiv s_D V(T) - \beta s_D(2s_i + s_D) = s_D V(0)\left(1 + \sigma Z - \gamma(N-1)^Q\right) - \beta s_D(2s_i + s_D).$$

We can take v_i to represent bidder i's private value. Interestingly, information leakage has made bidders' private values affiliated, so that competition is less intense than a bidder would have thought before the auction; the mechanism is different but the result is similar to that in Pinkse and Tan (2005).

Despite the unusual event in the benchmark US oil futures market in March 2020 when the front month futures price briefly turned negative, we assume not just that $V(t) > 0$ but also that $V(t) - \beta s_i > 0$ for all t and any s_i, i.e., long positions always have positive values.

5 Self-Selecting Mechanism

Without loss of generality, we take $V(0) = 0$, i.e., all mark-to-market fluctuations on the portfolio are settled continuously before the auction. The maximum willingness to pay, or value, of bidder i for the default portfolio is then

$$v_i \equiv -\beta s_D(2s_i + s_D). \tag{7}$$

The timeline of the model is as follows:

- The CCP sets the entry fee $e \geq 0$ ex ante, before any default.[14]
- Default happens. The CCP announces the side of the default portfolio, $\text{sgn}(s_D)$, but not its magnitude. Bidders believe (correctly) that the magnitude of s_D is distributed according to a distribution with support from $[0, \infty)$.[15] Bidders decide whether they would pay e to observe s_D. Entry decisions are made simultaneously and only disclosed to the CCP.
- The CCP announces the number of bidders K who have paid the entry fee, entered the auction, and thus observed s_D.
- Bidders who enter the auction realize that their per-unit value of the default portfolio drops by

$$-\gamma(K-1)^Q, \tag{8}$$

  reflecting the cost of information leakage (as captured by $\gamma > 0$ and Q). Put differently, while the fair market value of the default portfolio remains $V = 0$ given the public information set, the private values of everyone in the auction drop by $\gamma(K-1)^Q$.[16] Bidders who do not enter the auction do not see K and therefore do not observe the resulting drop in value.

Let $R \equiv M_D + G_D > 0$ be the resources the CCP has from the defaulter. The CCP conducts a second-price auction with reservation price equal to $-(R + Ke)$. The CCP's objective function is

$$\max_e \ \text{Prob}(\mathcal{A}), \quad \text{where } \mathcal{A} = \{P + R + Ke \geq 0\}, \tag{9}$$

and K is a random variable that determines how many bidders enter the auction conditional on the entry fee e. The objective seeks to set the entry fee to maximize the probability that it avoids a loss

[14] Setting the entry fee ex ante ensures that the entry fee reveals no information about the size of the defaulter's position.

[15] The support could be restricted to have an upper limit smaller than S; the larger support just simplifies the notation as the point where the cumulative distribution reaches one does not affect the results.

[16] Another mechanism that could result in such reduced private values would be communication amongst the participants; Agranov and Yariv (2018) demonstrate experimentally that communication can reduce bids. The impact would be captured by costs increasing as the number of participants rises.

from the auction. Avoiding a loss is equivalent to ensuring that P (the cash flow from the auction including any variation margin on the defaulting portfolio paid at the end of the auction day) plus R (the available resources from the defaulter) plus Ke (the entry fees paid by the K bidders) is non-negative. Importantly, P declines as the number of bidders increases due to information leakage.

We can solve for the bidders' entry decisions. Conjecture that v_0 is the cutoff value, corresponding to a cutoff inventory level s_0 so that

$$v_0 \equiv -\beta s_D(2s_0 + s_D). \tag{10}$$

The bidder who is at the cutoff should be indifferent between entering and not entering the auction, so the expected profit of entering is equal to the entry fee e.

Note that besides the non-standard objective function, our model has two features that are absent in the conventional auction models. First, entry decisions are made without observing s_D, but the actual bidding depends on learning s_D after entry. Second, each entrant's value is lower if more bidders enter. These two features represent two shocks to bidders who enter the auction.

The threshold bidder who is indifferent wins if and only if she is the only one in the auction, in which case there is no cost of information leakage, i.e., $-\gamma(K-1)^Q = -\gamma(1-1)^Q = 0$. In the second-price auction, the winning price for the sole bidder is the CCP's reservation value, which is $-(R+e)$, i.e., the CCP gives $R+e$ to the sole bidder, who is the cutoff bidder, in return for exiting the defaulter's position. The sole bidder's ex post profit, if she bids, is

$$v_0 + (R+e) = -\beta s_D(2s_0 + s_D) + R + e. \tag{11}$$

If s_D turns out to be very positive, then this "profit" can be negative, so the optimal action for the sole entrant is not to bid in that case. Therefore, the indifference condition for the cutoff-type bidder is

$$e = (1 - F(s_0))^{N-1} \mathbb{E}\left[\max(-\beta s_D(2s_0 + s_D) + R + e, 0) \mid e\right], \tag{12}$$

where $F(\cdot)$ is the cumulative distribution function of s_i. Because the entry fee e is set in advance, it does not reveal s_D in equilibrium.

The CCP's objective is to minimize the probability that the auction fails, i.e., it receives no bids above the reservation price. Obviously, the auction fails if no one enters, which happens with probability $(1 - F(s_0))^N$. The auction may also fail if bidders who enter refuse to bid after observing the realizations of s_D and K.

If only the cutoff-type bidder enters, the auction succeeds if and only if

$$-\beta s_D(2s_0 + s_D) + R + e \geq 0. \tag{13}$$

Note that from the CCP's perspective, once e is set, the left-hand side of this condition is deterministic because e determines s_0 by equation (12).

We now proceed to develop a series of building blocks (stated as lemmas) that we use to establish our main result that characterizes the optimal CCP auction. Our first building block establishes an upper limit on the entry fee. Let $s_{\min}$ be the lowest inventory level of the K bidders. Then we can state the following:

Lemma 1. *If the entry fee e is sufficiently high, then no bidder enters and the auction fails.*

The result is not trivial since the winning bidder may be compensated in part by the entry fee(s). It is therefore necessary to prove that s_0 is decreasing in e, as we do in Appendix A. This result shows that increasing the entry fee will eventually push s_0 below $s_{\min}$, at which point $\text{Prob}(s_{\min} \leq s_0) = 0$, and no bidder enters.

In general, if $K \geq 1$ bidders enter, the auction succeeds as long as

$$\max_{i \in K} \underbrace{v_i}_{\substack{\text{bidder's} \\ \text{value}}} - \underbrace{\gamma(K-1)^Q s_D}_{\substack{\text{cost of information} \\ \text{leakage}}} \geq \underbrace{-(R + Ke)}_{\substack{\text{reservation} \\ \text{value}}}, \tag{14}$$

or equivalently,

$$-\beta s_D(2s_{\min} + s_D) - \gamma(K-1)^Q s_D + R + Ke \geq 0. \tag{15}$$

Therefore, the auction succeeds if and only if a high-value bidder, who has a high willingness to pay, enters and also bids, i.e., the CCP solves the following:

$$\max_{e} \ \text{Prob}(s_{\min} \leq s_0 \text{ and } -\beta s_D(2s_{\min} + s_D) - \gamma(K-1)^Q s_D + R + Ke \geq 0), \tag{16}$$

where:

- s_0 is determined by equation (12), i.e., the indifference condition.
- K is a random variable, determined by the rule that entry happens if a potential auction participant's inventory is lower than s_0.
- e depends on the distributions of s_D and s_j for $\forall j \in K$, where $\{s_j\}_{j \in K}$ represents the inventory levels of the participants, but does not depend on the realization of s_D.[17]

It will be helpful to rewrite the two inequalities in the Prob$(\cdot)$ of expression (16) as

$$s_{\min} \leq \min\left(s_0\,, \frac{R + Ke - \beta s_D^2 - \gamma(K-1)^Q s_D}{2\beta s_D}\right). \tag{17}$$

This formulation makes clear that the smaller one of the two terms in $\min(\cdot\,,\cdot)$ would be binding. Intuition suggests that we want the two terms to be "close" to make the probability large. In one extreme with $e = 0$, equation (12) implies that $F(s_0) = 1$, i.e., s_0 is the upper bound of the inventory distribution. As a result, everyone would enter the auction. However, in this case the second term becomes $\frac{R - \beta s_D^2 - \gamma(N-1)^Q s_D}{2\beta s_D}$ (as $K = N$), which is small if Q is large and γ is not too small. In the other extreme of setting e to be high, then K would be small, so $\frac{R + Ke - \beta s_D^2 - \gamma(K-1)^Q s_D}{2\beta s_D}$ is less binding but s_0 would be quite binding. Therefore, there should be some intermediate values of e that maximize the probability of auction success.

More formally, we first need to establish the existence of an optimal entry fee e^*. This result is stated in the following lemma:

[17] Recall that the potential for a large OTC market means that the CCP cannot use its information about the distribution of cleared positions to reliably infer the distribution of s_j.

Lemma 2. *There exists an optimal entry fee e^*, possibly 0, which maximizes the CCP's objective given in expression (16).*

The proof of this lemma is provided in Appendix B.

It is not surprising that the optimality of no entry fee cannot be ruled out in Lemma 2. Optimality of a zero entry fee would be the standard result without information leakage, as the CCP would want to maximize auction participation. What we wish to show is that under some circumstances a positive entry fee would be optimal, namely $e^* > 0$.

To do so, we need to derive the analytical expression of $\text{Prob}(\cdot)$ in expression (16). Note that the random variable K has a binomial distribution with probability of success $F(s_0)$. Let k denote a realization of K. Clearly, the condition $s_{\min} \leq s_0$ is equivalent to $k \geq 1$, i.e., at least one bidder has inventory below s_0. In addition, conditional on $k \geq 1$ bidders entering the auction, $s_{\min}$ is lower than s_0. Still conditioning on k, for any real value $x \geq s_0$, $\text{Prob}(s_{\min} < x \mid s_{\min} < s_0) = 1$; for $x < s_0$, we have

$$\begin{aligned}\text{Prob}(s_{\min} < x \mid s_{\min} < s_0) &= \frac{\text{Prob}(s_{\min} < x, s_{\min} < s_0)}{\text{Prob}(s_{\min} < s_0)} = \frac{\text{Prob}(s_{\min} < x)}{\text{Prob}(s_{\min} < s_0)} \\ &= \frac{1 - \text{Prob}(s_{\min} \geq x)}{1 - \text{Prob}(s_{\min} \geq s_0)} = \frac{1 - (1 - F(x))^k}{1 - (1 - F(s_0))^k}, \end{aligned} \tag{18}$$

where in the last step we use the fact that if k bidders enter the auction, then the minimum inventory of the N potential bidders is equal to the minimum inventory of the k bidders who have the k lowest inventories and who actually enter. The auction success probability, conditional on s_D, is written as

$$\text{Prob}(s_{\min} \leq s_0, -\beta s_D(2s_{\min} + s_D) - \gamma(K-1)^Q s_D + R + Ke \geq 0 \mid s_D). \tag{19}$$

Because K is binomial, this probability can be made explicit. In particular, let

$$\phi(s_D, K) := \frac{R + Ke - \beta s_D^2 - \gamma(K-1)^Q s_D}{2\beta s_D}. \tag{20}$$

Then we have the following

$$
\begin{aligned}
\text{Prob}&(s_{\min} \leq s_0, -\beta s_D(2s_{\min} + s_D) - \gamma(K-1)^Q s_D + R + Ke \geq 0 \mid s_D) \\
&= \text{Prob}\left(K \geq 1, s_{\min} < \underbrace{\frac{R + Ke - \beta s_D^2 - \gamma(K-1)^Q s_D}{2\beta s_D}}_{:= \phi(s_D, K)} \mid s_D \right) \\
&= \sum_{k=1}^{N} \text{Prob}(K = k) \,\text{Prob}(s_{\min} < \phi(s_D, k) \mid K = k \mid s_D) \qquad (21) \\
&= \sum_{k=1}^{N} \binom{N}{k} F(s_0)^k (1 - F(s_0))^{N-k} \\
&\quad \cdot \left[\mathbb{1}(\phi(s_D, k) \geq s_0) + \mathbb{1}(\phi(s_D, k) < s_0) \frac{1 - (1 - F(\phi(s_D, k)))^k}{1 - (1 - F(s_0))^k} \right],
\end{aligned}
$$

where $\mathbb{1}(\cdot)$ denotes an indicator function that takes the value one if the condition in the brackets is satisfied, and zero otherwise. Note, $\phi(s_D, k)$ is a decreasing function of s_D as

$$
\frac{\partial \phi(s_D, k)}{\partial s_D} = -\frac{\beta s_D^2 + R + ke}{2\beta s_D^2} < 0. \qquad (22)
$$

The final step to obtain the unconditional probability is to integrate the above conditional probability over s_D. Let $G(\cdot)$ denote the cumulative distribution function of s_D, then the probability of auction success, denoted as α, is

$$
\begin{aligned}
\alpha = &\sum_{k=1}^{N} \binom{N}{k} F(s_0)^k (1 - F(s_0))^{N-k} \\
&\cdot \int_0^{\infty} \left[\mathbb{1}(\phi(s_D, k) \geq s_0) + \mathbb{1}(\phi(s_D, k) < s_0) \frac{1 - (1 - F(\phi(s_D, k)))^k}{1 - (1 - F(s_0))^k} \right] dG(s_D). \qquad (23)
\end{aligned}
$$

The function $\phi(s_D, k)$ is decreasing in s_D, so $\phi(s_D, k) \geq s_0$ if and only if $s_D \leq \zeta(s_0)$, for some function $\zeta(s_0)$ that is decreasing in s_0.[18] Then we can write the probability of auction success as:

$$\alpha = \sum_{k=1}^{N} \binom{N}{k} F(s_0)^k (1 - F(s_0))^{N-k} \cdot \left[\int_0^{\zeta(s_0)} g(s_D)\, ds_D + \int_{\zeta(s_0)}^{\infty} \frac{1 - (1 - F(\phi(s_D, k)))^k}{1 - (1 - F(s_0))^k} g(s_D)\, ds_D \right]. \quad (24)$$

Based on equation (24), we have the following lemma, which is the key constructive result:

Lemma 3. *If γ is sufficiently high, a small increase of entry fee from zero raises the probability that the auction succeeds.*

The proof of this lemma is provided in Appendix C.

We can now state our main result as the following proposition, which follows immediately from Lemmas 1 through 3 (the first part follows from Lemmas 1 and 2 and the second part follows from Lemma 3):

Proposition 1. *There exists an optimal entry fee e^* that maximizes the CCP's objective given in expression* (16)*. If the impact of information leakage is sufficiently high, then the optimal entry fee is strictly positive.*

Our main result shows that, in the face of information leakage, a positive entry fee can be an effective mechanism to encourage potential bidders to self-select whether to enter an auction or not so that the probability of a successful auction is maximized from the CCP's point of view. To our knowledge this is a novel mechanism. Many CCPs require potential auction participants to incur

[18] The function $\zeta(s_0)$ is

$$\zeta(s_0) = \frac{-\gamma(k-1)^Q - 2\beta s_0 + \sqrt{\left[\gamma(k-1)^Q + 2\beta s_0\right]^2 + 4\beta(R + ke)}}{2\beta} > 0,$$

and so the first-order derivative with respect to s_0 is

$$\frac{d\zeta(s_0)}{ds_0} = -1 + \frac{\gamma(k-1)^Q + 2\beta s_0}{\sqrt{\left[\gamma(k-1)^Q + 2\beta s_0\right]^2 + 4\beta(R + ke)}} < 0,$$

which means that $\zeta(s_0)$ is decreasing in s_0.

ex ante costs, for example by participating in default exercises ex ante, which could be viewed as an entry fee, but such implicit entry fees do not help the CCP maximize the probability it avoids needing to use its skin in the game or non-defaulters' resources.

The advantage of the entry fee mechanism developed here is that it is effective with low information requirements. In particular, the CCP does not need to know the complete positions of each potential bidder. If it did, a more efficient mechanism likely could be constructed. But without such information, the entry fee mechanism avoids the CCP needing to use its own judgment on who to invite and subsequently being open to criticism for its decision, as Nasdaq Nordic was, after the auction.[19]

It should be clear that the result depends on an assessment of how costly information leakage will be. The model of information leakage cost is simple, but the key characteristic is that private values decrease as the number of participants increases. This characteristic seems intuitive and the results are likely to hold for other models of information leakage that maintain this feature. In our set-up, as the marginal cost of including participants in the auction increases, the optimal entry fee also weakly increases. We state this as the following proposition:

Proposition 2. *The optimal entry fee e^* is weakly increasing in the impact of information leakage.*

The proof of this proposition is provided in Appendix D.

The dependence on the cost of information leakage does, however, imply that whether or not an entry fee is an effective mechanism may vary from CCP to CCP. For example, CCPs clearing exchange-traded products might be less concerned about information leakage; more generally, information leakage costs are likely higher in less liquid markets.[20] But ignoring the potential impact of information leakage, and the consequent difficult question of who to invite to a default

---

[19] Such criticism was reported in Clancy (2018b) and Mourselas (2019).

[20] The impact of market liquidity is also apparent in the analysis of transaction costs on CCP hedging strategies during a close-out in Cerezetti et al. (2019) and in the general price impact of large trades even in liquid equity markets (Bouchaud et al., 2009, Eisler et al., 2012).

auction, would be inconsistent with CPMI-IOSCO (2020), where the regulators devote significant attention to these issues.

6 Numerical Example

In this section, we present a numerical example that illustrates the results of our model. We assume that $\{s_j\}_{j=1,2,\ldots,N}$ follows the Laplace distribution, i.e., the probability density function such that for $\lambda_F > 0$ and $s_j \in (-\infty, \infty)$,

$$f(s_j) = \frac{1}{2}\lambda_F e^{-\lambda_F |s_j|} = \begin{cases} \frac{1}{2}\lambda_F e^{-\lambda_F s_j}, & \text{if } s_j \geq 0 \\ \frac{1}{2}\lambda_F e^{\lambda_F s_j}, & \text{if } s_j < 0 \end{cases}. \tag{25}$$

In addition, we assume that s_D follows the exponential distribution, i.e., the probability density function such that for $\lambda_G > 0$ and $s_D \in [0, \infty)$,

$$g(s_D) = \lambda_G e^{-\lambda_G s_D}. \tag{26}$$

The algorithm for numerically calculating the probability of auction success as a function of e is detailed as follows:

- Set $R = 1$, $\beta = 0.1$, $\gamma \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, $N = 10$, $Q = 2$, and $\lambda_F = \lambda_G = 1$.
- Pick a non-negative e. We choose the grid such that $e \in [0, 0.01, 0.02, \ldots, 0.99, 1]$.
- Given e, numerically solve for s_0 from equation (12).[21]
- Draw $s_1, s_2, \ldots, s_N$ independently from $F(\cdot)$ (the Laplace distribution above). The number K is set to be the number of bidders whose inventory level s_j is lower than s_0 that we have just

[21] By plugging the exponential density function (26) into equation (12), solving equation (12) is equivalent to solving the following equation:

$$e = (1 - F(s_0))^{N-1}\left[2\beta s_0 e^{-b}(b+1) - 2\beta s_0 - 2\beta + \beta e^{-b}(b^2 + 2b + 2) + (R+e)(1 - e^{-b})\right],$$

where $b \equiv \frac{-2\beta s_0 + \sqrt{4\beta^2 s_0^2 + 4\beta(R+e)}}{2\beta}$. The derivation is demonstrated in Appendix E.

solved in the previous step. These bidders enter the auction and pay the entry fee e. Set $s_{\min}$ to be the lowest inventory of these bidders.

- Draw s_D from the distribution $G(\cdot)$ (the exponential distribution above).
- Check if both inequalities in Prob$(\cdot)$ in expression (16) hold. If both of them hold, then the auction is successful. Otherwise, the auction fails.
- Repeat the aforementioned three steps multiple times, set to be 100,000 times, to obtain the probability of auction success $\hat{\alpha}$ for each e:

$$\hat{\alpha} = \frac{1}{100000} \sum_{m=1}^{100000} \mathbb{1} \left(s_{\min}^m \leq s_0, -\beta s_D^m (2 s_{\min}^m + s_D^m) - \gamma (K^m - 1)^Q s_D^m + R + K^m e \geq 0 \right),$$

  where the superscript m denotes the m-th round of simulation and $\mathbb{1}(\cdot)$ denotes the indicator function.
- Plot the probability of auction success as a function of e.

The numerical result for the case when $\gamma = 0.5$ is shown in Figure 3 on the following page. Clearly, if the entry fee e increases a bit from zero, the probability that the auction succeeds increases. This finding confirms our idea that imposing a positive entry fee is an effective mechanism for CCP default auctions through endogenizing potential bidders' entry decisions.

Figure 4 on page 32 presents simulation results for different marginal costs of information leakage. Consistent with Proposition 2, we see that the optimal entry fee increases as γ increases. At the same time we also see that increasing γ reduces the probability of auction success. This is a general property of the model that follows from the fact that for any fixed e, through equation (12) s_0 is also fixed and invariant to changes in γ and so the first constraint in expression (16) is fixed. Consequently, the probability that the second constraint in expression (16) holds, i.e.,

$$-\beta s_D (2 s_{\min} + s_D) - \gamma (K - 1)^Q s_D + R + K e \geq 0, \tag{27}$$

is decreasing in γ (and Q).[22]

[22] For a given e, if this second constraint is binding, the probability of auction success will strictly decrease at that e for

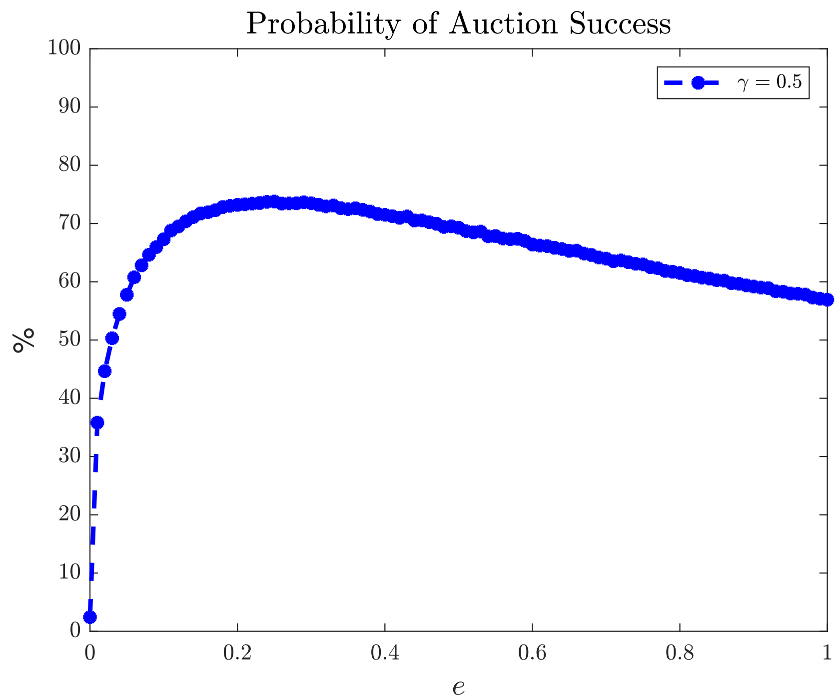

Figure 3: This figure illustrates the probability of auction success as a function of the entry fee e. Parameter values: $R = 1$, $\beta = 0.1$, $\gamma = 0.5$, $N = 10$, $Q = 2$, $\lambda_F = \lambda_G = 1$.

7 Conclusion

CCP default auctions are critically important. A CCP's main task is managing its risk such that a default does not spread. It is nearly a tautology to say that if a default auction is successful, the CCP's risk management is successful. Conversely, as suggested by Clancy (2018a), an unsuccessful default auction can cast doubt not only on the individual CCP but on central clearing more broadly. Nevertheless, CCP default auctions have received relatively little attention in the literature.

We have developed a simple but realistic model of the cleared market and the challenges a CCP faces in conducting a default auction. We paid particular attention to what information would be known to both the CCP and its clearing members at various points in the default management process. In particular, the occurrence of a default, the resulting need to conduct an auction, and the direction of the market move associated with the default are assumed to be common knowledge. The CCP is assumed to know members' cleared positions, but not their total market exposures. This assumption, and the associated opacity of clearing members' risk preferences, is realistic.

higher γ.

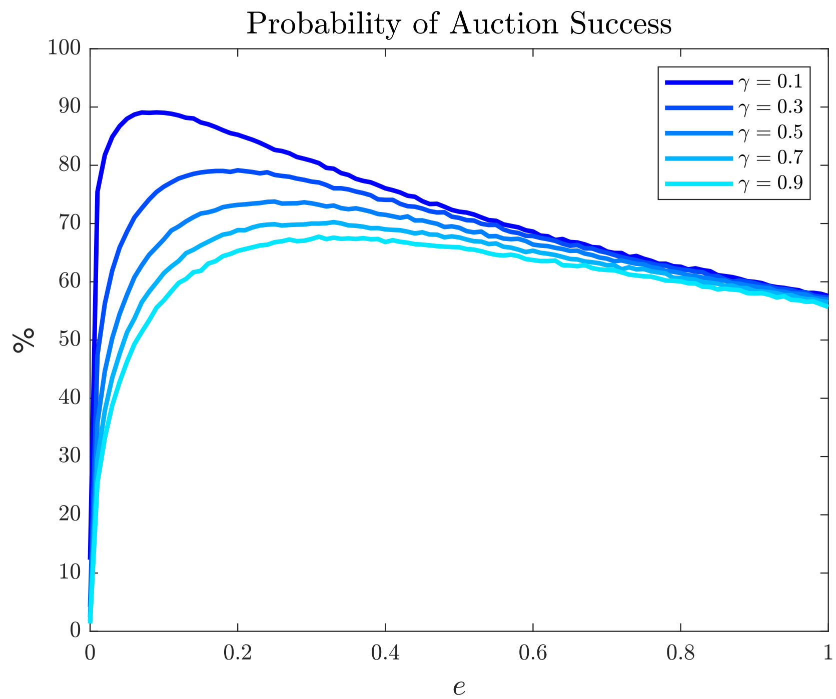

Figure 4: This figure illustrates the probability of auction success as a function of the entry fee e, with different values of γ. Parameter values: $R = 1$, $\beta = 0.1$, $\gamma \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, $N = 10$, $Q = 2$, $\lambda_F = \lambda_G = 1$.

Consequently, the CCP lacks the information to just specify who should be invited to an auction with any degree of reliability.

Within this model, we have addressed several unique aspects of CCP default auctions. First, that revenue maximization is not a reasonable objective for CCPs. Second, that one of the most common and fundamental questions CCPs face is who to include in the auction. Third, we have explicitly incorporated information leakage, which means inviting everyone is not optimal. These characteristics taken together result in a highly non-standard auction problem, so standard results in auction theory do not apply. Nevertheless, we are able to obtain a constructive analytical result.

In the CCP framework, the reserve price and the entry fee perform different functions, contrary to their equivalence in the standard auction model. The impact of this is that a positive entry fee can be optimal precisely because of its effect on endogenous entry decisions. To the authors' knowledge, this result is novel. It focuses attention on the key question of how CCPs decide which market participants to invite to an auction, and provides an effective mechanism for resolving it.

There are other characteristics of CCP default auctions that we have not modeled. For example, we have assumed that all participants are clearing members, thus ignoring the question of whether it would be advantageous to invite clients or others to bid. Although, clients are formally excluded, clearing members are modeled rather generally, suggesting that the results likely could incorporate clients in a similar way. In that sense, the entry fee mechanism may be even more useful as it may be able to effectively sort both clearing members and clients into participants and non-participants. Other characteristics, like the cost of preparing a bid, whether and how to split up the defaulting positions, the impact of hedging, or the chance for a limited number of participants to collude are not addressed. Hopefully, the model presented here, which incorporates some of the most unique characteristics of CCP auctions, will encourage more of these nuances to be addressed.

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Appendices

A Proof of Lemma 1

Proof. We formally prove that s_0 decreases as e increases, i.e., $\frac{ds_0}{de} < 0$. First, we need to obtain the mathematical expression for equation (12). We know that the first term in the $\max(\cdot, \cdot)$ operator is the following:

$$-\beta s_D(2s_0 + s_D) + R + e = -2\beta s_0 s_D - \beta s_D^2 + R + e.$$

Therefore, it is useful to solve the following quadratic equation with respect to s_D:

$$\beta s_D^2 + 2\beta s_0 s_D - (R + e) = 0, \tag{28}$$

where $\beta > 0$ and $R + e > 0$. Since the discriminant of equation (28) is

$$\Delta = (2\beta s_0)^2 - 4\beta[-(R + e)] = 4\beta^2 s_0^2 + 4\beta(R + e) > 0,$$

this quadratic equation has two different roots, denoted by a and b, which are

$$a = \frac{-2\beta s_0 - \sqrt{4\beta^2 s_0^2 + 4\beta(R + e)}}{2\beta},$$

and

$$b = \frac{-2\beta s_0 + \sqrt{4\beta^2 s_0^2 + 4\beta(R + e)}}{2\beta}.$$

Hence, $b > a$. Since $-\beta < 0$, $R + e > 0$, and the quadratic function of s_D, i.e., $h(s_D) \equiv -\beta s_D^2 - 2\beta s_0 s_D + (R + e)$, intersects with the vertical axis at the point $(0, h(0)) = (0, R + e)$, we have

$$a < 0 < b,$$

and thus,

$$-\beta s_D(2s_0 + s_D) + R + e \geq 0, \quad s_D \in [a, b],$$

$$-\beta s_D(2s_0+s_D)+R+e<0,\quad s_D\in(-\infty,a)\cup(b,+\infty).$$

In addition, because we require that $s_D\geq 0$, we then have

$$-\beta s_D(2s_0+s_D)+R+e\geq 0,\quad s_D\in[0,b],$$

and

$$-\beta s_D(2s_0+s_D)+R+e<0,\quad s_D\in(b,+\infty).$$

As a result, equation (12) can be expressed as

$$\begin{aligned}
e &= (1-F(s_0))^{N-1}\mathbb{E}\left[\max(-\beta s_D(2s_0+s_D)+R+e,0)\mid e\right]\\
&= (1-F(s_0))^{N-1}\int_0^{+\infty}\max(-\beta s_D(2s_0+s_D)+R+e,0)g(s_D)\,ds_D\\
&= (1-F(s_0))^{N-1}\int_0^{b}(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D. \qquad (29)
\end{aligned}$$

Next, taking the first-order derivatives with respect to e on both sides of equation (29), we can obtain

$$\begin{aligned}
1 = &-(N-1)(1-F(s_0))^{N-2}f(s_0)\frac{ds_0}{de}\left(\int_0^b(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D\right)\\
&+(1-F(s_0))^{N-1}\cdot\left[\frac{\partial}{\partial e}\int_0^b(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D\right.\\
&\left.+\left(\frac{\partial}{\partial s_0}\int_0^b(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D\right)\frac{ds_0}{de}\right]. \qquad (30)
\end{aligned}$$

By the Leibniz integral rule, we first get

$$\begin{aligned}
&\frac{\partial}{\partial e}\int_0^b(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D =\\
&\qquad\underbrace{(-\beta b(2s_0+b)+R+e)}_{=0}g(b)\frac{db}{de}+\int_0^b g(s_D)\,ds_D\\
&\qquad\qquad = G(b)-G(0)=G(b)\in(0,1).
\end{aligned}$$

We also get

$$\frac{\partial}{\partial s_0}\int_0^b (-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D = \underbrace{(-\beta b(2s_0+b)+R+e)}_{=0}g(b)\frac{db}{ds_0}+\int_0^b -2\beta s_D g(s_D)\,ds_D = -2\beta\int_0^b s_D g(s_D)\,ds_D,$$

where, in both equations, $g(\cdot)$ and $G(\cdot)$ denote the probability density function and the cumulative distribution function, respectively. Therefore, equation (30) can be written as

$$\begin{aligned} 0 &< 1-(1-F(s_0))^{N-1}G(b) \\ &= -(N-1)(1-F(s_0))^{N-2}f(s_0)\left(\int_0^b(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D\right)\frac{ds_0}{de} \\ &\quad -2\beta(1-F(s_0))^{N-1}\left(\int_0^b s_D g(s_D)\,ds_D\right)\frac{ds_0}{de}. \end{aligned} \tag{31}$$

We know that $\int_0^b(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D > 0$ and $\int_0^b s_D g(s_D)\,ds_D > 0$, so

$$-(N-1)(1-F(s_0))^{N-2}f(s_0)\left(\int_0^b(-\beta s_D(2s_0+s_D)+R+e)g(s_D)\,ds_D\right) < 0,$$

and

$$-2\beta(1-F(s_0))^{N-1}\left(\int_0^b s_D g(s_D)\,ds_D\right) < 0.$$

As a result, the condition

$$\frac{ds_0}{de} < 0$$

should hold in order to make the right-hand side of equation (31) be equal to the left-hand side of equation (31), i.e., $1-(1-F(s_0))^{N-1}G(b)$, which is positive. □

B Proof of Lemma 2

Proof. From Lemma 1, without loss of generality we can set an upper limit for the entry fee at $\bar{e}$ such that $\text{Prob}(s_{\min} \leq s_0) = 0$ at $\bar{e}$. Existence of an optimal entry fee is immediate from the

Extreme Value Theorem, by noting that the optimum is over a closed and bounded set $[0, \bar{e}]$, and that expression (16) is a continuous function because probabilities are continuous. Since we are only interested in the maximum, $\bar{e}$ can be ruled out as a candidate, and we have the existence of $e^* \geq 0$. □

C Proof of Lemma 3

Proof. We want to show that if we increase e from 0 to something positive, then α increases. Equivalently, we can show that α is decreasing in s_0 for sufficiently large s_0, i.e., if we reduce s_0 from ∞ to something smaller, α increases.

We repeat equation (24), which specifies the auction success probability, here for convenience,

$$\alpha = \sum_{k=1}^{N} \binom{N}{k} F(s_0)^k (1 - F(s_0))^{N-k} \cdot \underbrace{\left[\int_0^{\zeta(s_0)} g(s_D)\, ds_D + \int_{\zeta(s_0)}^{\infty} \frac{1 - (1 - F(\phi(s_D, k)))^k}{1 - (1 - F(s_0))^k} g(s_D)\, ds_D \right]}_{:= H_k(s_0)}. \tag{32}$$

Taking the first-order derivative with respect to s_0, we have

$$\begin{aligned} \frac{d\alpha}{ds_0} =& F(s_0)^N H_N'(s_0) + N F(s_0)^{N-1} f(s_0) H_N(s_0) + N F(s_0)^{N-1}(1 - F(s_0)) H_{N-1}'(s_0) \\ &+ [N(N-1)F(s_0)^{N-2}(1 - F(s_0)) - N F(s_0)^{N-1}] f(s_0) H_{N-1}(s_0) \\ &+ \sum_{k=1}^{N-2} \binom{N}{k} F(s_0)^k (1 - F(s_0))^{N-k} H_k'(s_0) \\ &+ \sum_{k=1}^{N-2} \binom{N}{k} \\ &\cdot \left[k F(s_0)^{k-1} (1 - F(s_0))^{N-k} - (N-k) F(s_0)^k (1 - F(s_0))^{N-k-1} \right] f(s_0) H_k(s_0). \end{aligned} \tag{33}$$

By the Leibniz integral rule, we can show that

$$H_k'(s_0) = -\frac{k(1 - F(s_0))^{k-1}}{[1 - (1 - F(s_0))^k]^2} f(s_0) \int_{\zeta(s_0)}^{\infty} [1 - (1 - F(\phi(s_D, k)))^k] g(s_D)\, ds_D < 0. \tag{34}$$

Therefore, to prove that $d\alpha/ds_0$ is negative as s_0 is sufficiently large, it is sufficient to show that all terms not involving $H_k'(s_0)$ are negative, i.e., we want to show that

$$\begin{aligned}0 > f(s_0)\Bigg[&NF(s_0)^{N-1}H_N(s_0)\\ &- NF(s_0)^{N-1}H_{N-1}(s_0) + N(N-1)F(s_0)^{N-2}(1-F(s_0))H_{N-1}(s_0)\\ &+\sum_{k=1}^{N-2}\binom{N}{k}[kF(s_0)^{k-1}(1-F(s_0))^{N-k} - (N-k)F(s_0)^k(1-F(s_0))^{N-k-1}]H_k(s_0)\Bigg] \quad (35)\end{aligned}$$

for sufficiently large s_0. Because density $f(s_0)$ is positive, we only need to show that the entire term within the square bracket is negative. But note that as s_0 becomes large, $(1-F(s_0))^{N-k}$ and $(1-F(s_0))^{N-k-1}$ both go to zero. So the sufficient condition for $d\alpha/ds_0$ becomes that

$$\lim_{s_0\to\infty}[H_{N-1}(s_0) - H_N(s_0)] > 0. \quad (36)$$

As $s_0 \to \infty$, we have $\zeta(s_0) \downarrow 0$ as $\frac{d\zeta(s_0)}{ds_0} < 0$, so the sufficient condition is that

$$\int_0^\infty (1-F(\phi(s_D,N)))^N g(s_D)\,ds_D > \int_0^\infty (1-F(\phi(s_D,N-1)))^{N-1} g(s_D)\,ds_D. \quad (37)$$

The sufficient condition for the above condition is that

$$\Gamma(k) \equiv \int_0^\infty (1-F(\phi(s_D,k)))^k g(s_D)\,ds_D \quad (38)$$

is strictly increasing in k. Ignoring the integral constraint on k and treating it as a real number, a sufficient condition is that $\Gamma'(k) > 0$ for $k \in [N-1, N]$. At $e = 0$, we need

$$\begin{aligned}0 < \Gamma'(k) = &\int_0^\infty [1-F(\phi(s_D,k))]^k \ln[1-F(\phi(s_D,k))] g(s_D)\,ds_D\\ &+ \frac{\gamma k Q(k-1)^{Q-1}}{2\beta}\int_0^\infty [1-F(\phi(s_D,k))]^{k-1} f(\phi(s_D,k)) g(s_D)\,ds_D, \quad (39)\end{aligned}$$

that is,

$$\frac{kQ(k-1)^{Q-1}}{2\beta}\gamma > \frac{-\int_0^\infty [1-F(\phi(s_D,k))]^k \ln[1-F(\phi(s_D,k))] g(s_D)\,ds_D}{\int_0^\infty [1-F(\phi(s_D,k))]^{k-1} f(\phi(s_D,k)) g(s_D)\,ds_D}. \quad (40)$$

Note that the left-hand side goes to ∞ as γ becomes large. On the right-hand side, the term $-[1 - F(\phi(s_D, k))]^k \ln[1 - F(\phi(s_D, k))]$ in the numerator is bounded above regardless of γ; label this upper bound $M_k > 0$. For the denominator, we want a lower bound. Choose a finite ϵ independent of k. Then

$$\begin{aligned}
\int_0^\infty [1 - F(\phi(s_D, k))]^{k-1} f(\phi(s_D, k)) g(s_D)\, ds_D \\
> \int_0^\epsilon [1 - F(\phi(s_D, k))]^{k-1} f(\phi(s_D, k)) g(s_D)\, ds_D \\
> [\min_{x \in [0,\epsilon]} f(x)] \int_0^\epsilon [1 - F(\phi(s_D, k))]^{k-1} g(s_D)\, ds_D. \quad (41)
\end{aligned}$$

Note that as $\gamma \to \infty$, $\phi(s_D, k) \to -\infty$. Thus, there exists an $\epsilon' < \epsilon$ and $\bar{\gamma}$ such that for all $\gamma > \bar{\gamma}$, $s_D \in [\epsilon', \epsilon]$, and $k \in [N-1, N]$, $(1 - F(\phi(s_D, k)))^{k-1} > 1/2$. So the lower bound of the denominator becomes

$$[\min_{x \in [0,\epsilon]} f(x)] \int_{\epsilon'}^{\epsilon} \frac{1}{2} g(s_D)\, ds_D. \quad (42)$$

The required inequality becomes

$$\frac{kQ(k-1)^{Q-1}}{2\beta}\gamma > \frac{\max_{k \in [N-1,N]} M_k}{[\min_{x \in [0,\epsilon]} f(x)] \int_{\epsilon'}^{\epsilon} \frac{1}{2} g(s_D)\, ds_D}, \quad (43)$$

which holds for sufficiently large γ. □

D Proof of Proposition 2

Proof. We first prove that for any entry fee e, the probability of auction success (weakly) decreases in γ. For any fixed e, through equation (12), s_0 is also fixed and invariant to changes in γ, so the first constraint in expression (16) is fixed. Consequently, the probability that the second constraint in expression (16) holds, i.e.,

$$-\beta s_D (2 s_{\min} + s_D) - \gamma (K-1)^Q s_D + R + Ke \geq 0, \quad (44)$$

is decreasing in γ (and Q) and all other variables are fixed. In addition, for a given e, if this second constraint is binding, the probability of auction success will strictly decrease at that e for higher γ.

Next, let e^* maximize expression (19). If γ increases, then based on the aforementioned result, the probability of auction success decreases at e^*. There are two possibilities. If the first constraint binds, then e^* is still the optimum and $\frac{\partial e^*}{\partial \gamma} = 0$. Alternatively, the second constraint binds. Then, we can take K to be fixed for small changes in γ. We then can restate the second condition as

$$e \geq \underbrace{\frac{1}{K}\left(\beta s_D(2s_{\min} + s_D) + \gamma(K-1)^Q s_D - R\right)}_{:= \mathcal{C}(\gamma)}. \tag{45}$$

Define γ^* such that

$$e^* = \mathcal{C}(\gamma^*). \tag{46}$$

This condition means that as γ increases to γ^*, the second constraint in expression (19) binds at e^*. In addition, define $\Delta\gamma > 0$ and e^{**} such that

$$e^{**} := e^* + \Delta e = e^* + \frac{\partial e^*}{\partial \gamma}\Delta\gamma, \tag{47}$$

and

$$\mathcal{C}(\gamma^* + \Delta\gamma) = \mathcal{C}(\gamma^*) + \frac{\partial \mathcal{C}(\gamma)}{\partial \gamma}\Delta\gamma = \mathcal{C}(\gamma^*) + \frac{(K-1)^Q s_D}{K}\Delta\gamma. \tag{48}$$

Since the condition (45) holds, we should have $e^{**} \geq \mathcal{C}(\gamma^* + \Delta\gamma)$. Therefore,

$$e^{**} = e^* + \frac{\partial e^*}{\partial \gamma}\Delta\gamma \geq \mathcal{C}(\gamma^* + \Delta\gamma) = \mathcal{C}(\gamma^*) + \frac{(K-1)^Q s_D}{K}\Delta\gamma \tag{49}$$

$$\implies \quad \frac{\partial e^*}{\partial \gamma}\Delta\gamma \geq \underbrace{\mathcal{C}(\gamma^*) - e^*}_{=0} + \frac{(K-1)^Q s_D}{K}\Delta\gamma, \tag{50}$$

and so

$$\frac{\partial e^*}{\partial \gamma} \geq \frac{(K-1)^Q s_D}{K} > 0, \tag{51}$$

for s_D held constant. □

E Derivation of the Analytic Expression for e in Sec. 6. Numerical Example

Proof. We want to derive the analytical expression for equation (12) when s_D follows the exponential distribution. The original equation is

$$e = (1 - F(s_0))^{N-1} \mathbb{E}\left[\max(-\beta s_D(2s_0 + s_D) + R + e, 0) \mid e\right].$$

Based on the results demonstrated in Section A of the Appendix, we know that

$$-\beta s_D(2s_0 + s_D) + R + e \geq 0, \quad s_D \in [0, b],$$

and

$$-\beta s_D(2s_0 + s_D) + R + e < 0, \quad s_D \in (b, +\infty).$$

Thus, we are able to calculate the expectation term. We have

$$\begin{aligned}
&\mathbb{E}\left[\max(-\beta s_D(2s_0 + s_D) + R + e, 0) \mid e\right] \\
&\quad = \int_0^{+\infty} \max(-\beta s_D(2s_0 + s_D) + R + e, 0) g(s_D)\, ds_D \\
&\quad = \int_0^{b} (-\beta s_D(2s_0 + s_D) + R + e) g(s_D)\, ds_D + \int_b^{+\infty} 0 \times g(s_D)\, ds_D \\
&\quad = \int_0^{b} (-\beta s_D(2s_0 + s_D) + R + e) g(s_D)\, ds_D \\
&\quad = \int_0^{b} (-\beta s_D(2s_0 + s_D) + R + e) \lambda_G e^{-\lambda_G s_D}\, ds_D \\
&\quad = 2\beta s_0 \frac{e^{-\lambda_G b}(\lambda_G b + 1) - 1}{\lambda_G} - \beta \frac{2 - e^{-\lambda_G b}(\lambda_G^2 b^2 + 2\lambda_G b + 2)}{\lambda_G^2} + (R + e)(1 - e^{-\lambda_G b}).
\end{aligned}$$

When $\lambda_G = 1$, then

$$\mathbb{E}\left[\max(-\beta s_D(2s_0 + s_D) + R + e, 0) \mid e\right]$$
$$= 2\beta s_0[e^{-b}(b+1) - 1] - \beta[2 - e^{-b}(b^2 + 2b + 2)] + (R + e)(1 - e^{-b})$$
$$= 2\beta s_0 e^{-b}(b+1) - 2\beta s_0 - 2\beta + \beta e^{-b}(b^2 + 2b + 2) + (R + e)(1 - e^{-b}).$$

Therefore, in order to numerically solve for s_0 from equation (12), we need to solve the following equation:

$$e = (1 - F(s_0))^{N-1}$$
$$\cdot \left[2\beta s_0 e^{-b}(b+1) - 2\beta s_0 - 2\beta + \beta e^{-b}(b^2 + 2b + 2) + (R + e)(1 - e^{-b})\right],$$

where $b \equiv \frac{-2\beta s_0 + \sqrt{4\beta^2 s_0^2 + 4\beta(R+e)}}{2\beta}$. □