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Declining Search Frictions, Unemployment, and Growth Revisited*

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Abstract

This paper revisits the conditions under which search models generate balanced growth paths (BGPs)—equilibria where unemployment, vacancies, and job flows remain steady as search frictions decline. Martellini and Menzio (2020) claim that such paths exist only when matches are "inspection goods" and match quality follows a Pareto distribution. We show that these conditions are sufficient but not necessary. Their implementation assumes a strong form of stationarity—requiring the endogenous distribution of match qualities to remain invariant under proportional scaling. This restriction forces the reservation quality to grow at a constant, strictly positive rate, mechanically tying declining frictions to long-term growth and yielding counterfactual implications of eliminating search frictions—persistent unemployment and infinite welfare gains. Relaxing this restriction, balanced growth can arise under alternative forms of scaling, such as additive transformations that restore stationarity without Pareto tails or inspection. We further show that biased technological progress, when vacancies and unemployed workers are complementary inputs, also generates well-behaved BGPs with finite welfare gains and vanishing unemployment as search frictions disappear.

Keywords: search frictions; balanced growth; inspection models; Pareto tails; biased technological change

JEL Codes: E24; J64; O41

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”If search frictions in the labor market have diminished over the last 90 years, why do we not see a secular inward shift of the Beveridge curve, a secular negative trend in the unemployment rate, and a secular rise in the UE rate?... We seek a balanced growth path (BGP) for this economy, that is, an equilibrium along which unemployment, vacancies, UE, and EU rates are constant over time... A BGP exist iff (a) the quality of a firm-worker match is a sample from a Pareto distribution with some tail coefficient... and (b) the workers' benefit from unemployment and the firms' cost of maintaining a vacancy grow at the same rate as average productivity. The assumption that matches are inspection goods could be considered the third condition for the existence of a BGP.” (Martellini and Menzio, 2020, pp. 4392).

1. Introduction

Despite dramatic improvements in job-search technology over the past century—from newspaper classifieds to online platforms to algorithmic matching—aggregate labor market outcomes have remained remarkably stable. Unemployment rates show no secular decline, the Beveridge curve has not shifted inward over the long run, and job-finding and separation rates appear stationary. This apparent disconnect between technological progress and stable labor-market outcomes poses a fundamental puzzle for macroeconomic theory: how can declining search frictions coexist with steady employment outcomes?

Martellini and Menzio (2020) (hereafter MM) offer an influential answer. In the spirit of King, Plosser, and Rebelo (1988), they seek necessary and sufficient conditions for balanced growth in search-theoretic models of unemployment. Their solution is strikingly sharp: a balanced growth path exists if and only if firm–worker matches are “inspection goods” and match quality follows a Pareto distribution.[1] Under these assumptions, the long tail of the Pareto distribution induces increasingly selective matching that offsets the effects of declining frictions, thereby preserving the stability of labor market outcomes while generating long-run growth.

This paper revisits and challenges that characterization. We show that MM's conditions are *sufficient but not necessary*. The key lies in their implementation of the balanced-growth concept. In their definition—stated in the abstract and introduction—a BGP requires that aggregate labor variables such as unemployment, vacancies, and job flows remain constant over time, but says nothing about the underlying, unobservable distribution of match qualities. MM nevertheless impose a strong assumption on this latent distribution: it must remain invariant under *proportional scaling*. This restriction—common in growth models but untested and empirically unmotivated in

[1] Inspection means that when a worker and a vacancy meet, the match-specific productivity is revealed and the parties decide whether to form an employment relationship.

labor-market settings—forces both the sampling and the endogenous match-quality distributions to be Pareto and mechanically turns declining search frictions into a source of long-run growth. Yet proportional scaling is not required for a *stationary* BGP.[2] Once scaling is treated more generally, the Pareto structure is no longer necessary, and declining frictions need not affect the economy's growth rate.

We demonstrate that alternative time transformations can restore stationarity without invoking Pareto tails. For example, under *exponential sampling* of match quality, improvements in search efficiency cause the reservation quality to drift *linearly* over time, rather than multiplicative as in the Pareto case. This additive transformation of time ensures that the distribution of accepted matches remains stationary even as search efficiency improves exponentially. In equilibrium, self-selection adjusts just enough to offset the destabilizing effect of improved matching—restoring the time-invariance of labor market outcomes but without generating long-run growth.

The restrictive structure implied by multiplicative scaling also generates counterfactual implications. In the MM setup, unemployment persists even when posting vacancies is costless, and the welfare gains from eliminating search frictions are infinite. In contrast, the canonical Diamond–Mortensen–Pissarides (DMP) framework predicts that unemployment vanishes as posting costs fall and that welfare gains are finite, bounded by the forgone output of unemployed workers. Under exponential sampling, one core DMP prediction is restored—unemployment vanishes as posting costs fall—but, as in the Pareto inspection case, the welfare gains from eliminating frictions remain unbounded because ever-easier matching induces ever-higher reservation quality when match quality is unbounded above.

Balanced growth can also arise outside the inspection framework. When technological progress is input-biased and vacancies and unemployed workers are complementary inputs in the matching function, the economy converges to a well-behaved BGP with stationary unemployment, tightness, and job-finding rates. In this setting, unemployment vanishes as frictions disappear only if technological progress is worker-augmenting. Unlike inspection models, the welfare gains from eliminating frictions are finite. However, the resulting BGP is necessarily inefficient: the market equilibrium supports a stable labor market with declining frictions, but the planner's allocation does not. This inefficiency arises because the bargaining weight in the market equilibrium is fixed, whereas the planner's shadow bargaining power varies with tightness, preventing the Hosios condition (Hosios, 1990) from being satisfied. Thus, while the biased-technology DMP framework restores the core qualitative properties of DMP models and avoids the implausible welfare implications of inspection, it reveals an intrinsic inefficiency between market and planner allocations.

Together, these results show that MM's characterization overstates the conditions required for

[2] MM do not explicitly use the *stationary* terminology. We use it to clarity our contribution, as explained in Section 3.5.

balanced growth. Pareto distributions and inspection are sufficient under a particular (multiplicative) time transformation, but not necessary once alternative transformations are considered. In this article, we focus on the two most transparent cases—multiplicative and additive detrending—but the logic extends to more general transformations of time that can also restore stationarity. Our analysis clarifies the logical structure of balanced growth in search models and provides a broader foundation for understanding how declining frictions interact with stationarity and growth.

The rest of the paper proceeds as follows. Section 2 outlines the main mechanisms. Section 3 revisits the inspection model from the planner's perspective, defines the BGP, stationary BGP, and scale-invariant BGP, derives the efficiency conditions for the SI-BGP, and shows that the MM model is efficient under the Hosios condition and Cobb-Douglas matching. It further demonstrates that with exponential sampling, a BGP exists in which unemployment and vacancies remain stationary while declining frictions do not generate growth, and extends the analysis to more general cases. Section 4 analyzes a DMP model with homogeneous workers and biased technological progress in the matching function, showing that a well-behaved BGP arises when unemployment and vacancies are complementary inputs. Section 5 concludes.

2. Overview

Because the paper is technically detailed, it is useful to begin with an overview of the main ideas. MM analyze a two-stage matching process in the labor market. In the first stage, a worker meets a vacancy with arrival rate $A_t p(\theta_t)$, where θ_t is labor market tightness defined as vacancies over unemployed workers and A_t is a technological parameter growing exogenously at the constant rate g_A: $A_t = A_0 e^{g_A t}$. MM refer to this as the *meeting rate*. If a meeting occurs, the process moves to the second stage, where the productivity of the match is drawn from a distribution $F(z)$. The unemployment-employment (UE) rate is thus

$$h_{ue,t} = A_t p(\theta_t) \left(1 - F(R_t)\right), \tag{1}$$

where R_t is the reservation threshold of match quality.

MM seek a balanced growth path (BGP) in which both h_{ue} and θ remain constant, consistent with the empirical evidence of unemployment and vacancies discussed by MM. Since the matching technology improves over time, the reservation threshold must rise accordingly to satisfy the equation at all times. Time-differentiating yields

$$g_A = \frac{F'(R_t)}{1 - F(R_t)} \dot{R}_t, \quad \forall t, \tag{2}$$

and under the assumption that R_t grows at a constant rate, MM obtain their key condition (equation

(10)), which implies that F must be Pareto. They further show that R_t indeed grows at a constant rate as an equilibrium outcome, and that increasing selectivity adds to economic growth.

Our paper builds on three observations that qualify and extend this framework.

1. Pareto is not necessary. MM's equation (10) is a special case of the more general condition, equation (2). While Pareto sampling produces a constant UE transition rate, other distributions, such as exponential, can do so as well once the adjustment of reservation quality is specified differently. For instance, if $\dot{R}_t$ is constant (rather than $\dot{R}/R_t$), the reservation quality increases linearly rather than exponentially over time, and the solution to the differential equation (2) is exponential rather than Pareto. With exponential sampling, unemployment, tightness, and transition rates remain stationary, but output growth slows down and eventually stops. Thus, declining search frictions do not drive long-run growth, and stationarity does not uniquely require a Pareto distribution.

2. Inspection models have counterfactual implications. Introducing inspection fundamentally changes the standard DMP logic. In the canonical model, eliminating search frictions (making vacancies costless) drives unemployment to zero and yields finite welfare gains. In inspection models, by contrast, costless posting pushes tightness and reservation productivity to infinity. With Pareto sampling, both forces cancel each other out and unemployment persists even under costless posting. Moreover, the welfare gains from eliminating frictions become infinite, a problematic prediction as it overshadows the role of any other frictions known in economics.

3. Technological progress may be biased. MM assume Hicks-neutral progress, with A_t scaling the entire meeting rate. A more general formulation is

$$h_{ue,t} = p(\theta_t, A_t)(1 - F(R_t)),$$

where A_t affects the productivity of unemployment and vacancies differently. When these inputs are complements, biased progress in one input eventually faces diminishing returns: improvements in A_t no longer raise the meeting rate without bound, and the need for ever-rising reservation thresholds disappears. In the limit, search efficiency gains no longer fuel growth, undermining MM's claim that declining frictions necessarily generate long-run expansion.

3. Inspection

This section considers variants of the MM inspection model, with the aim of characterizing the necessary and sufficient conditions for the existence of a suitable balanced growth path. While the original framework analyzes a decentralized equilibrium, we study the corresponding social planner's allocation and show that the two coincide if the Hosios condition holds and the matching function is Cobb-Douglas. Our formulation generalizes the baseline model to incorporate both endogenous and exogenous job destruction, nesting the canonical case without declining search frictions as a special case.

3.1. Environment

The economy is populated by a continuum of workers of measure one and a continuum of firms of positive measure. At each date t, each worker is either unemployed, u_t, or employed in a job with firm-specific productivity z. Let $n_t(z)$ denote the measure of workers at time t employed in jobs with productivity z. Employed workers with productivity z produce $y_t z$, where $y_t = y_0 e^{g_y t}$ is an aggregate productivity term common to all jobs and grows at rate $g_y \geq 0$. Jobs are destroyed at an exogenously rate $\delta \geq 0$, or endogenously when either the worker or the firm chooses to separate. In the planner's allocation, endogenous separations are characterized by a productivity cutoff R_t: matches with $z < R_t$ are terminated, while those with $z \geq R_t$ continue. Unemployed workers produce b_t.

Unemployed workers can be assigned to jobs across different productivity levels, but doing so requires vacancy creation. Let v_t denote vacancies posted at cost k_t units of output per vacancy per period. These vacancies generate $A_t M(u_t, v_t)$ random matches between unemployed workers and vacant jobs, where A_t measures search efficiency, and M is a constant-returns-to-scale (CRS) matching function. M is increasing in each argument, concave, and satisfies the Inada conditions. Let M_1 and M_2 denote the corresponding partial derivatives with respect to the first and second arguments, respectively.

When a worker and firm meet, a match productivity z is drawn from a cumulative distribution $F(z)$ with density $f(z)$ and support $[z_l, \infty)$. The law of motion of $n_t(z)$ is:

$$\dot{n}_t(z) = A_t M(u_t, v_t) f(z) - \delta n_t(z) \text{ for } z \geq R_t \text{ and } t \geq 0.$$

3.2. Planner's Problem

Given an initial distribution of employment, $[n_0(z)]_0^\infty$, the planner solves the following problem:

$$\max_{\{v_t, u_t, R_t, \mathbf{n}_t\}_{t=0}^\infty} \int_{t=0}^\infty e^{-rt} \left[\int_{R_t} y_t z n_t(z) dz + u_t b_t - k_t v_t \right] dt \text{ subject to:}$$

$$\dot{n}_t(z) = A_t M(u_t, v_t) f(z) - \delta n_t(z) \text{ for } z \geq R_t \text{ for } t \geq 0, \text{and} \tag{3}$$

$$u_t = 1 - \int_{R_t}^\infty n_t(z) dz, \;\; R_t \geq z_l \text{ for } t \geq 0. \tag{4}$$

The planner selects vacancies, unemployment, and a productivity cutoff to maximize the discounted present value of net output at discount rate r, subject to the law of motion for employment across job productivities and the labor market resource constraint.

To connect this formulation with the MM decentralized equilibrium, it is useful to define the meeting rates implied by the matching function. An unemployed worker meets a vacancy at rate $m_t = A_t p(\theta_t)$ where $p(\theta) \equiv M(1, \theta)$ and $\theta_t = v_t/u_t$ is labor market tightness. Symmetrically, a vacancy meets an unemployed worker at rate $s_t = A_t q(\theta_t)$, where $q(\theta) = p(\theta)/\theta$.

3.3. Optimality Conditions

Let $e^{-rt}\lambda_t(z)$ and $e^{-rt}\eta_t$ be the Lagrange multipliers associated with equation (3) and equation (4), respectively, for $t \geq 0$. The conditions associated with the optimal choices of v_t, u_t, R_t, and $n_t(z)$ are:

$$k_t = A_t M_2(1, \theta) \int_{R_t} \lambda_t(z) f(z) dz = A_t q_t (1 - \mu_t) \int_{R_t} \lambda_t(z) f(z) dz, \tag{5}$$

$$\eta_t = b_t + A_t M_1(1, \theta) \int_{R_t} \lambda_t(z) f(z) dz = b_t + A_t p_t \mu_t \int_{R_t} \lambda_t(z) f(z) dz, \tag{6}$$

$$\eta_t = y_t R_t, \text{ and} \tag{7}$$

$$y_t z - \eta_t - \delta\lambda_t(z) = r\lambda_t(z) - \dot{\lambda}_t(z) \text{ for } z \geq R_t, \tag{8}$$

where $\mu_t \equiv \frac{\partial M_t}{\partial u_t}\frac{u_t}{M_t} = \mu(\theta_t)$ is the elasticity of matches with respect to unemployment.

Equation (5) states that the optimal mass of vacancies equates the marginal cost k_t to the marginal benefit: the additional matches created by an extra vacancy, $A_t \frac{\partial M_t}{\partial v_t} = A_t q_t (1 - \mu_t)$, multiplied by the expected shadow value of a filled job.

Equation (6) requires that the shadow flow value of an unemployed worker, η_t, equals the flow of output while unemployed, b_t, plus the additional matches generated by an unemployed worker, $A_t \frac{\partial M_t}{\partial u_t} = A_t p_t \mu_t$, times expected shadow value of a match.

Equation (7) implies that the optimal reservation productivity R_t is such that the production of the marginal worker, $y_t R_t$, equals the shadow flow value of unemployment, η_t. Finally, equation (8) characterizes the shadow value $\lambda_t(z)$ of a filled job of quality z.

We now rewrite these expressions in a form that will be useful later. Combining equation (6) and equation (7) yields:

$$y_t R_t - b_t = A_t p_t \mu_t \int_{R_t} \lambda_t(z) f(z) dz.$$

Substituting equation (5) into this expression leads to our first key relationship:

$$\frac{y_t R_t - b_t}{k_t} = \frac{p_t \mu_t}{q_t (1 - \mu_t)} = \frac{\mu(\theta_t)}{1 - \mu(\theta_t)} \theta_t. \tag{9}$$

Next, substituting equation (7) into equation (8) gives:

$$(r+\delta)\,\lambda_t(z) = y_t z - y_t R_t + \dot{\lambda}_t(z) \text{ for } z \geq R_t. \tag{10}$$

This is the familiar value function equation in which $\lambda_t(z)$ denotes the social value of a match. Solving this differential equation (see Appendix) yields:

$$\lambda_t(z) - e^{-(r+\delta)d}\lambda_{t+d}(z) = \int_0^d e^{-(r+\delta)\tau}\left(y_{t+\tau}z - y_{t+\tau}R_{t+\tau}\right)d\tau. \tag{11}$$

The transversality condition $e^{-(r+\delta)d}\lambda_{t+d} = 0$ must hold. If d is finite, this implies $\lambda_{t+d} = 0$ or, from equation (10),

$$z = R_{t+d_t(z)}. \tag{12}$$

Equation (12) defines $d_t(z)$, the optimal longevity of a match with productivity z at time t, absent an exogenous destruction shock. Substituting this into equation (11) gives:

$$\lambda_t\,(z) = \int_0^{d(z,t)} e^{-(r+\delta)\tau}\left(y_{t+\tau}z - y_{t+\tau}R_{t+\tau}\right)d\tau. \tag{13}$$

3.4. Aggregates

The measure of aggregate employment with match quality below z_t is:

$$N_t\,(z_t) \equiv \int_{R_t}^{z_t} n_t(x)dx. \tag{14}$$

Total employment is then:

$$N_t = N_t\,(\infty) = 1 - u.$$

From equation (3) and equation (14), we obtain:

$$\begin{aligned} \dot{N}_t\,(z_t) &= n_t(z_t)\dot{z}_t - n_t(R_t)\dot{R}_t + \int_{R_t}^{z_t} \dot{n}_t(x)dx \\ &= n_t(z_t)\dot{z}_t - n_t(R_t)\dot{R}_t + \int_{R_t}^{z_t} \left[A_t M(u_t, v_t) f(x) - \delta n_t(x)\right]dx \\ &= n_t(z_t)\dot{z}_t - n_t(R_t)\dot{R}_t + u_t A_t p_t \left(F\,(z_t) - F\,(R_t)\right) - \delta N_t\,(z_t)\,. \end{aligned} \tag{15}$$

This expression decomposes employment below z_t into net inflows from new matches and outflows from job destructions.

Similarly, the law of motion for total employment can be expressed as:

$$\begin{aligned} \dot{N}_t &= \int_{R_t}^{\infty} \dot{n}_t(z)dz - n_t(R_t)\dot{R}_t \\ &= u_t A_t p_t \left(1 - F(R_t)\right) - \delta N_t - n_t(R_t)\dot{R}_t. \end{aligned} \tag{16}$$

Here, total employment depends on the inflow from unemployed workers matched to jobs above R_t and the outflow from job destruction. Finally, define the cumulative distribution of match qualities as:

$$G_t(z) \equiv \frac{N_t(z)}{N_t}. \tag{17}$$

This is the fraction of employed workers in matches with productivity below z.

3.5. Balanced Growth

3.5.1. Definition

The definition of a balanced growth path (BGP) is central, as it imposes restrictions on endogenous variables and frees equations that help identify the exogenous forces—particularly the dynamics of k_t, b_t, and the function F—needed to sustain the path. We begin with the general definition.

Definition (BGP): A *Balanced Growth Path* (BGP) is an initial state $G_0(z)$ and an associated efficient allocation such that unemployment, tightness, and the employment-to-unemployment (EU) and unemployment-to-employment (UE) rates remain constant over time, while aggregate productivity and search efficiency grow at constant rates $g_y \geq 0$ and $g_A \geq 0$.

This is the definition that MM emphasize in the abstract and introduction of their paper. It guarantees constancy of aggregate labor-market variables but imposes no restriction on the evolution of the match-quality distribution $G_t(z_t)$. It is natural, however, to require that $G_t(z_t)$ exhibit some form of stationarity.

Definition (SBGP): A *Stationary Balanced Growth Path* (SBGP) is a BGP such that

$$G_t(T_t(z)) = G_0(z) \quad \text{for all } z \geq R_0,$$

where $T_t(z)$ is a time transformation of z capturing how the distribution evolves over time. The following, more restrictive version, is the one MM adopt to prove their main result, Theorem 1.

Definition (SI-BGP): A *Scale-Invariant Balanced Growth Path* (SI-BGP) is an SBGP for which $T_t(z) = ze^{g_z t}$.

An SI-BGP is therefore a special case of an SBGP—one that imposes a proportional (multiplicative) transformation on the endogenous distribution G_t. MM highlight the broader BGP definition in their abstract and introduction but adopt the more restrictive SI-BGP in their analytical derivations.

The general BGP concept requires only that aggregate labor-market variables remain stationary and imposes no structure on G_t, consistent with MM's stated objective: "*We seek conditions for the existence of a balanced growth path (BGP), where unemployment, vacancy, and worker's transition rates remain constant in the face of improvements in the production and search technologies.*"

The scale-invariance assumption on G_t follows related work in endogenous growth theory (e.g., Perla and Tonetti, 2014; Lucas and Moll, 2014; Buera and Oberfield, 2020; Benhabib et al., 2021), where it is purposeful because it generates growth. In the present context—focused on explaining labor-market stationarity—it inadvertently introduces growth even though it is not required for the stated objective. One implication is that the reservation productivity, the lower bound of support, must grow at rate g_z along an SI-BGP:

$$R_t = R_0 e^{g_z t}.$$

The broader BGP and SBGP definitions impose no such restriction. In our exponential example below, $T_t(z) = z + \phi t$, so the reservation productivity instead follows

$$R_t = R_0 + \phi t, \qquad \phi > 0.$$

3.6. BGP System of Equations

Along a BGP, the following versions of equations (5), (9), (12), (13), (15), and (16) must hold. First, the first order condition with respect to vacancies becomes

$$k_t = A_t M_2 (1, \theta) \int_{R_t} \lambda_t(z) f(z) dz, \tag{18}$$

while the reservation condition reads

$$\frac{y_t R_t - b_t}{k_t} = \frac{M_1 (1, \theta)}{M_2 (1, \theta)} = \frac{\mu (\theta)}{1 - \mu (\theta)} \theta. \tag{19}$$

Match longevity is defined by

$$z = R_{t+d_t(z)}, \tag{20}$$

and the shadow value of a filled job satisfies

$$\lambda_t (z) = \int_0^{d(z)} e^{-(r+\delta)\tau} \left(y_{t+\tau} z - y_{t+\tau} R_{t+\tau} \right) d\tau. \tag{21}$$

The dynamics of employment by productivity are given by

$$n_t(z_t)\dot{z}_t + uA_t p\left(\theta\right)\left(F\left(z_t\right) - F\left(R_t\right)\right) = n_t(R_t)\dot{R}_t + \delta N_t\left(z_t\right). \tag{22}$$

In steady state, inflows into unemployment must equal outflows, implying

$$uh_{ue} = \left(1 - u\right) h_{eu}, \tag{23}$$

where the job-finding and job-destruction rates are given by

$$h_{ue} \equiv A_t p\left(\theta\right)\left(1 - F(R_t)\right), \text{and} \tag{24}$$

$$h_{eu} \equiv \delta + \frac{n_t(R_t)}{N}\dot{R}_t, \tag{25}$$

respectively, with total employment $N = 1 - u$.

Given parameters, these conditions can be used to solve for $d_t(z)$, R_t, θ, $\lambda_t\left(z\right)$, $n_t(z)$, h_{ue}, and h_{eu}. Equation (22) corresponds to equation (15) when $\dot{N}_t\left(z_t\right) = 0$, and equation (23) corresponds to equation (16) when $\dot{N}_t = 0$. Finally, equation (24) and equation (25) provide explicit definitions for the steady-state job-finding and job-destruction rates.

As in MM, it is convenient to work with the distribution G_t defined in equation (17). To derive the implied restrictions on G, substitute equations (23)-(24) into equation (22):

$$\begin{aligned} n_t(z_t)\dot{z}_t + uA_t p(\theta)\left(F\left(z_t\right) - F\left(R_t\right)\right) &= h_{eu}N - \delta N + \delta N_t\left(z_t\right) \\ &= uh_{ue} + \delta\left(N_t\left(z_t\right) - N\right) \\ &= uA_t p(\theta)\left(1 - F(R_t)\right) + \delta\left(N_t\left(z_t\right) - N\right). \end{aligned}$$

Simplifying yields:

$$n_t(z_t)\dot{z}_t = uA_t p(\theta)\left(1 - F(z_t)\right) + \delta\left(1 - u\right)\left(\frac{N_t\left(z_t\right)}{N} - 1\right).$$

From equation (17) we have that $\left(1 - u\right) G_t'(z) = n_t(z)$. Substituting this into the previous expression gives:

$$\left(1 - u\right)\left(G_t'(z_t)\dot{z}_t + \delta\left(1 - G_t(z)\right)\right) = uA_t p(\theta)\left(1 - F(z_t)\right), \tag{26}$$

which provides the implied restriction on the evolution of the endogenous employment distribution. Moreover, the job destruction rate (25) becomes:

$$h_{eu} = \delta + G_t'(R_t)\dot{R}_t. \tag{27}$$

These last two equations can replace equation (22) and equation (25) in the definition of BGP. On a scaled invariant BGP (SI-BGP), productivity grows at a constant rate $\dot{z}_t/z_t = g_z$. Substituting this into equation (26) and equation (27) yields

$$(1-u)\left(G_t'(z_t)z_t g_z + \delta\left(1 - G_t(z)\right)\right) = uA_t p(\theta)\left(1 - F(z_t)\right), \tag{28}$$

$$h_{eu} = \delta + G_t'(R_t)R_t g_z. \tag{29}$$

These two conditions replace equation (26) and equation (27) in the definition of an SI-BGP.

Finally, average match quality satisfies

$$Z_t = \int_{R_t} zG_t'(z)dz, \tag{30}$$

which links the distribution of employed matches to aggregate productivity.

Decentralization and efficiency. A direct comparison shows the connection between the social planner's solution and the decentralized equilibrium in MM. Our system of equations defining an SI-BGP is identical to theirs if $\delta = 0$ (i.e., all job separations are endogenously determined by the reservation rule) and the matching elasticity $\mu(\theta)$ is replaced by a constant workers' bargaining power γ. In other words, when the Hosios condition holds and the matching function is Cobb–Douglas, MM's decentralized equilibrium is efficient.

Proposition 1 (Efficiency of the MM Equilibrium)**.** *Consider the decentralized equilibrium in Martellini and Menzio (2020). If job separations are entirely endogenous ($\delta = 0$) and the matching function is Cobb–Douglas with elasticity μ equal to the workers' bargaining power γ, then the decentralized equilibrium coincides with the social planner's allocation. Under these conditions, the equilibrium satisfies the Hosios efficiency criterion, and the scale-invariant balanced growth path (SI-BGP) is efficient.*

3.7. Characterization

We now present three main results. The first two parallel MM's Lemma 1 and Theorem 1, which establish necessary and sufficient conditions for the existence of a SI-BGP. Our results are novel in that they pertain to efficient rather than decentralized equilibrium allocations and allow for both endogenous and exogenous separations, whereas MM consider only the endogenous separations.

The third result is a counterexample: a SBGP that is not an SI-BGP, where the distribution F is exponential rather than Pareto, yet labor market statistics remain constant over time. This example demonstrates that search models need not be constrained by the narrow conditions of MM's Lemma 1 and Theorem 1.

Lemma 1 (Necessary conditions for a SI-BGP). Let $g_y \geq 0$ and $g_A > 0$ be arbitrary growth rates for the production and search technologies.

1. A SI-BGP may exist only if (a) the distribution F is Pareto with an arbitrary coefficient α; (b) the growth rate of the vacancy cost, g_k, and the growth rate of the unemployment benefit, g_b, are equal to $g_y + g_z$; and (c) the discount rate r is greater than $g_y + g_z$.
2. In any SI-BGP, the growth rate g_z of the distribution G_t is equal to g_A/α.

The proof of Lemma 1 follows MM closely and is therefore omitted. The need for a Pareto distribution follows from our discussion in Section 2. Intuitively, a Pareto distribution is required to ensure scale invariance of match quality, so that the reservation cutoff can grow proportionally over time. Moreover, equation (12) in MM, needed for the proof, is analogous to our equation (19), except that our expression involves the endogenous elasticity $\mu(\theta)$ rather than the exogenous bargaining weight γ. But since θ is constant along a BGP, this difference does not affect the proof of Lemma 1.

Assuming that the conditions of Lemma 1 are satisfied, the next step is to show that a SI-BGP exists and is unique. This amounts to verifying that, under those conditions, the system of equations defining a SI-BGP can be solved and yields a unique set of functions and values for $d_t(z)$, R_t, θ, $\lambda_t(z)$, $G_t(z)$, h_{ue}, and h_{eu}. Following MM's steps, this system can be reduced to two equations in the two unknowns R_0 and θ:

$$R_0 = \left(A_0 M_2(1,\theta)\,\Phi_1 y_0/k_0\right)^{1/(\alpha-1)}, \text{ and} \tag{31}$$

$$R_0 = \frac{b_0}{y_0} + \frac{M_1(1,\theta)}{M_2(1,\theta)}\frac{k_0}{y_0}, \tag{32}$$

where

$$\Phi_1 = \frac{z_l^{\alpha}}{(\alpha-1)\left((\alpha-1)\,g_z + \left(r+\delta-g_y\right)\right)}. \tag{33}$$

Given the assumed properties of M, this system has a unique solution. Equation (31) is a strictly decreasing function of θ, spanning from $+\infty$ to 0, while equation (32) is a strictly increasing function of θ, spanning from 0 to $+\infty$. By the intermediate value theorem, the two curves intersect at a unique θ, which binds down a unique solution for R_0 and θ. The solution for R_t is then $R_t = R_0 e^{(g_A/\alpha)t}$. A unique solution for $G_0(z)$ can then be found.

Given that F is Pareto, equation (28) reads:

$$(1-u)\left(G_t'(z) z g_z + \delta\left(1-G_t(z)\right)\right) = u A_t p(\theta)\left(\frac{z_l}{z}\right)^{\alpha}.$$

This is a differential equation for $G_t(z)$. A natural guess for the solution is $G_t(z) = 1 - \left(\frac{R_t}{z}\right)^{\alpha}$.

Substituting this into the differential equation yields

$$(1-u)\left(\frac{R_t}{z}\right)^{\alpha}(\alpha g_z+\delta)=uA_t p(\theta)\left(\frac{z_l}{z}\right)^{\alpha}.$$

This equation is satisfied if

$$(1-u)(\alpha g_z+\delta)=uA_t p(\theta)\left(\frac{z_l}{R}\right)^{\alpha}=uA_t p(\theta)(1-F(R))=uh_{ue},$$

or

$$u=\frac{\alpha g_z+\delta}{\alpha g_z+\delta+h_{ue}}=\frac{g_A+\delta}{g_A+\delta+h_{ue}}.$$

This condition is consistent with equation (23) and equation (29), together with the guessed solution for $G_t(z)$, thereby confirming its validity. Hence, an efficient SI-BGP exists, is unique and—importantly—is continuous at $g_A=0$ if $\delta>0$. This continuity is possible because $\delta>0$ allows for exogenous job destruction. By contrast, if $\delta=0$ and $g_A=0$, there would be no job destruction along a SI-BGP making full employment an absorbing state. The following theorem collects these results.

Theorem 1 (Existence and properties of a SI-BGP). Let $g_A>0$, $g_y\geq 0$, and $\delta>0$. An efficient SI-BGP exists if and only if (a) F is Pareto with coefficient $\alpha>1$; (b) g_b and g_k satisfy $g_b=g_y+g_A/\alpha$ and $g_k=g_y+g_A/\alpha$; and (c) $r>g_y+g_A/\alpha$.

If an efficient SI-BGP exists, it is unique, continuous at $g_A=0$, and has the following properties:
(i) u, θ, h_{ue}, and h_{eu} are constant, with $h_{ue}=A_0p(\theta)(1-F(R_0))$, $h_{eu}=g_A+\delta$;
(ii) $G_t(ze^{g_z t})=G_0(z)$ with $g_z=g_A/\alpha$ and $G_0(z)=1-\left(\frac{R_0}{z}\right)^{\alpha}$; and
(iii) labor productivity grows at the rate g_y+g_A/α.

The theorem confirms MM's findings for the planner. Under DMP decentralization, efficient and market allocations generally differ, except when the Hosios condition holds and the matching function is Cobb-Douglas. In contrast to MM, our SI-BGP is continuous at $g_A=0$.

We now present the main result of this subsection: an example of an SBGP that is not scale-invariant (SI-BGP). We refer to this case as an AI-BGP ("Additive Invariant"). As shown in Section 2, when the sampling distribution is exponential rather than Pareto, maintaining a constant UE rate requires the reservation productivity to rise linearly rather than exponentially over time. This linear drift preserves the stationarity of aggregate labor-market variables and prevents declining search frictions from generating long-run growth. The limit of our example therefore corresponds to an SI-BGP without growth. The following proposition establishes that such an equilibrium arises endogenously when unemployment benefits increase linearly over time while the cost of posting vacancies remains constant.

Proposition 2 (AI-BGP)**.** *Let $y = 1$, $g_A > 0$, $F(z) = 1 - e^{-\nu z}$, $k_t = k$, and $b_t = b + (g_A/\nu)\,t$. Then there exists a unique efficient BGP such that:*
(i) u, θ, h_{ue}, and h_{eu} are constant, with $h_{ue} = A_0 p(\theta)\,(1 - F\,(R_0))\,,\ h_{eu} = g_A + \delta$;
(ii) $R_t = R_0 + (g_A/\nu)\,t$;
(iii) $G_t(z + (g_A/\nu)\,t) = G_0(z) = 1 - e^{-\nu(z-R_0)}$;
(iv) average productivity satisfies $Z_t = 1/\nu + R_t$. Hence

$$\frac{\dot{Z}_t}{Z_t} = \frac{g_A}{1 + \nu R_0 + g_A t} \longrightarrow 0 \quad \text{as } t \to \infty.$$

Proof. Following MM's steps, the system reduces to two equations in the two unknowns R_0 and θ. Equation (24) becomes

$$h_{ue} = A_0 e^{g_A t} p\,(\theta)\,e^{-\nu R_t} = A_0 p\,(\theta)\,e^{-\nu R_0}.$$

From this expression, it follows that

$$R_t = R_0 + \frac{g_A}{\nu} t. \tag{34}$$

Substituting into equation (19) yields the first equation in two unknowns:

$$\frac{R_t - b_t}{k} = \frac{R_0 - b}{k} = \frac{M_1\,(1, \theta)}{M_2\,(1, \theta)}. \tag{35}$$

Next, using equation (34) in equation (20) gives $d(z) = (z - R_t)\,/(g_A/\nu)$. The appendix shows that

$$\int_{R_t} \lambda_t(z) f(z) dz = \Phi_2 e^{-\nu R_t}, \tag{36}$$

where $\Phi_2 = \frac{1}{r+\delta}\frac{1}{\nu}\left[\frac{r+\delta-g_A}{r+\delta} + \frac{1}{r+\delta}\frac{g_A^2}{r+\delta+g_A}\right]$. Plugging this and equation (34) into equation (18) gives the second equation:

$$k = A_t e^{-\nu R_t} M_2\,(1, \theta)\,\Phi_2 = A_0 e^{-\nu R_0} M_2\,(1, \theta)\,\Phi_2. \tag{37}$$

Equations (35) and (37) determine θ and R_0. Given the assumed properties of M, this system admits a unique solution. Now consider equation (26):

$$(1 - u)\left(G_t'(z)\frac{g_A}{\nu} + \delta\,(1 - G_t(z))\right) = u A_t p\,(\theta)\,e^{-\nu z}.$$

This is a differential equation for $G_t(z)$. Given the exponential form on the right-hand side, a natural conjecture is

$$G_t(z) = 1 - e^{-\nu(z-R_t)}.$$

Substituting yields

$$(1-u)\, e^{-\nu(z-R_t)}\, (g_A + \delta) = uA_t p\,(\theta)\, e^{-\nu z}.$$

This holds if and only if

$$(1-u)\,(g_A + \delta) = uA_t p\,(\theta)\, e^{-\nu R_t} = uA_t p\,(\theta)\,(1 - F\,(R_t)) = uh_{ue},$$

or equivalently,

$$u = \frac{g_A + \delta}{g_A + \delta + h_{ue}} = \frac{g_A + \delta}{g_A + \delta + A_t p\,(\theta)\, e^{-\nu R_t}},$$

which follows from equations (23), (24), (27), and the conjecture for G. Thus, the conjecture is verified. Finally, average match quality—equal to average labor productivity since $y = 1$— defined in equation (30) satisfies

$$Z_t = \frac{1}{\nu} + R_t.$$

Hence,

$$\frac{\dot{Z}_t}{Z_t} = \frac{\dot{R}_t}{Z_t} = \frac{g_A}{1 + \nu R_0 + g_A t} \to 0.$$

□

Comparing Theorem 1 and Proposition 1, we find that if the sampling distribution is Pareto, a SI-BGP requires both the cost of posting a vacancy and unemployment compensation to rise exponentially over time. By contrast, if the sampling distribution is exponential, the cost of posting a vacancy can remain constant, while unemployment compensation needs to grow only linearly. The underlying reason is that under an exponential distribution, small changes in the reservation threshold have large effects on the probability of drawing a high-quality match. Consequently, parameters do not need to adjust as much to sustain a BGP.

In summary, MM imposes an unnecessary extra assumption. They assume that in any BGP, the reservation productivity must grow at a constant rate. Proposition (2) removes this assumption. While it is true that an increase in reservation productivity is necessary to offset the persistent improvements in search technology, this increase does not need to be exponential over time.

3.8. The Role of Search Frictions

We now turn to the implications of eliminating search frictions. In the canonical DMP model, removing frictions—by making vacancy posting costless—drives unemployment to zero. In the MM inspection framework, by contrast, unemployment does not vanish: although jobs become easy to find, workers simultaneously become excessively selective, exactly offsetting the improved matching prospects. In our exponential benchmark, however, unemployment does vanish, since self-selection is too weak to neutralize the ease of finding work.

The welfare predictions differ just as sharply. In the DMP model, the gains from eliminating frictions are finite, bounded by the additional output that unemployed workers could contribute. In both the MM model and our exponential inspection variant, however, the welfare gains are unbounded. As matching becomes arbitrarily easy, it is optimal for workers to hold out for ever-better offers, and since match quality is unbounded above, expected gains diverge. This implausible implication highlights a core weakness of inspection frameworks. The present section illustrates these weaknesses, while Section 4 develops an alternative mechanism that avoids them.

For this analysis, we adopt the canonical Cobb-Douglas (CD) matching function, $M(v, u) = u^{\gamma} v^{1-\gamma}$, to facilitate comparison with MM's formulas.

3.8.1. The Underlying Source of Unemployment

In the two models we are considering, the unemployment rate satisfies

$$u = \frac{g_A + \delta}{g_A + \delta + h_{ue}}, \tag{38}$$

where h_{ue} satisfies equation (24). In this section, we are interested in the limit of this expression as $k \to 0$. In the DMP model, free vacancy posting would drive vacancies, tightness, θ, workers' matching probability, and the UE transition probability, h_{ue}, to infinite, thus eliminating unemployment.

3.8.2. MM's Model

In the MM model, the UE transition satisfies

$$h_{ue} \equiv A_0 p(\theta)(1 - F(R_0)) = A_0 \theta^{1-\gamma} z_l^{\alpha} R_0^{-\alpha}. \tag{39}$$

In order to characterize h_{ue}, we need to characterize θ and R_0.

Using the assumed matching function, equation (31) and equation (32) simplify to:

$$R_0 = \left(A_0 (1-\gamma) \Phi_1 \frac{y_0}{k_0^{1-\gamma}} \right)^{1/(\alpha-1)} (\theta k_0)^{-\gamma/(\alpha-1)}, \text{ and} \tag{40}$$

$$R_0 = \frac{b_0}{y_0} + \frac{\gamma}{1-\gamma} \frac{\theta k_0}{y_0}. \tag{41}$$

Equation (40) depicts R_0 as a decreasing function of θk_0 while equation (41) depicts R_0 as an increasing function of θk_0. A unique solution for R_0 and θk_0 exists. Moreover, as k_0 decreases, the curve from equation (40) shifts upwards and that from equation (41) remains unchanged, pushing the solution toward higher R_0 and θk_0, but also higher θ. As $k \to 0$ both R_0 and θ diverge to ∞. For k sufficiently low, the constant term $\frac{b_0}{y_0}$ in equation (41) becomes negligible so that

$$R_0 \approx \frac{\gamma}{1-\gamma}\frac{\theta k_0}{y_0}.$$

Using this and equation (40), closed form solution for R_0 and θ can be found as:

$$\theta_{k\to 0} = \Omega_1 k_0^{-\frac{\alpha}{\alpha-(1-\gamma)}}, \text{ and} \tag{42}$$

$$R_{k\to 0} = \frac{\gamma}{1-\gamma}\frac{\Omega_1}{y_0} k_0^{-\frac{1-\gamma}{\alpha-(1-\gamma)}}, \tag{43}$$

where $\Omega_1 = \left[\frac{1-\gamma}{\gamma}\left(A_0\left(1-\gamma\right)\Phi_1\right)^{1/(\alpha-1)} y_0^{\alpha/(\alpha-1)}\right]^{\frac{1}{1+\gamma/(\alpha-1)}}$. These solutions are valid in the limit, but if $b_0 = 0$, they are also valid for any k. Substituting these solutions into equation (39), the solution for h_{ue} is obtained. The solution then can be substituted into equation (38) to find unemployment. The following proposition summarizes the main results.

Proposition 3. *In a SI-BGP with $M(u, v) = u^\gamma v^{1-\gamma}$, and $k \to 0$,*

$$h_{ue} \to h_{ue}^* = (\alpha - 1)\left(\frac{\alpha-1}{\alpha} g_A + r + \delta - g_y\right)/\gamma,$$

$$u^* = \frac{g_A + \delta}{g_A + \delta + h_{ue}^*} > 0.$$

Proof. See Appendix. □

Discussion. Unlike the DMP model, unemployment persists in MM's model even when posting vacancies is costless. Firms post infinitely many vacancies, making matches certain, but workers become increasingly selective and wait for the best possible draws. Consequently, unemployment remains strictly positive and independent of level parameters such as A_0 or z_l. Moreover, unemployment decreases with the Pareto tail parameter: thinner tails induce workers to accept jobs more readily. The key driver of long run unemployment in MM's model is the random nature of match quality. If $\alpha = \infty$, so that match quality is deterministically z_l, then unemployment would fall to zero when $k = 0$.

3.8.3. SBGP with Exponential Sampling Distribution

The outcome changes when the sampling distribution is exponential rather than Pareto. Because the exponential distribution has much thinner tails, the selection motive is too weak to sustain positive unemployment in the presence of abundant job offers. As a result, unemployment vanishes once vacancies become costless, as we show.

Under the exponential distribution and CD matching, the UE transition rate is

$$h_{ue} = A_0 p\left(\theta\right) e^{-\nu R_0} = A_0 \theta^{1-\gamma} e^{-\nu R_0}.$$

To determine h_{ue}, we solve for (R_0, θ) using the efficient conditions (35) and (37). Under CD matching, they become:

$$k = A_0 e^{-\nu R_0} \left(1-\gamma\right) \theta^{-\gamma} \Phi_2, \text{ and} \tag{44}$$

$$\theta = \frac{1-\gamma}{\gamma} \frac{R_0 - b_0}{k}. \tag{45}$$

Substituting equation (45) into equation (44) gives a single equation in R_0:

$$LHS(R_0) := \frac{e^{-\nu R_0}}{\left(R_0 - b_0\right)^{\gamma}} = \frac{k^{1-\gamma}}{A_0 \left(1-\gamma\right)^{1-\gamma} \gamma^{\gamma} \Phi_2}. \tag{46}$$

Equation (46) admits a unique solution since $LHS(b_0) = \infty$, $LHS(\infty) = 0$, and $LHS'(R_0) < 0$. Moreover, as $k \to 0$, $R_0 \to \infty$. Once R_0 is determined, θ follows from equation (45).

Rewriting equation (44),

$$k = A_0 e^{-\nu R_0} \left(1-\gamma\right) \theta^{-\gamma} \Phi_2 = h_{ue} \left(1-\gamma\right) \theta^{-1} \Phi_2,$$

which implies, using equation (45),

$$h_{ue} = \frac{k\theta}{\left(1-\gamma\right) \Phi_2} = \frac{k \frac{1-\gamma}{\gamma} \frac{R_0 - b_0}{k}}{\left(1-\gamma\right) \Phi_2} = \frac{R_0 - b_0}{\gamma \Phi_2}.$$

Thus the UE rate is increasing in the reservation quality R_0. Since $\lim_{k \to 0} R_0 = \infty$, it follows that $h_{ue} \to \infty$. Consequently, unemployment converges to zero.

3.8.4. *Welfare Cost of Search Frictions*

We now highlight a problematic feature of inspection models: eliminating search frictions by making vacancy posting costless leads to unbounded productivity and infinite welfare gains. Using equation (35) and CD matching, social welfare can be written as

$$
\begin{aligned}
W &= \int_{t=0}^{\infty} e^{-rt} \left[\int_{R_t} y_t z n_t(z) dz + u b_t - k_t v \right] dt \qquad (47) \\
&= \int_{t=0}^{\infty} e^{-rt} \left[(1-u)\, y_t \int_{R_t} z_t g_t(z) dz + u b_t - k_t \theta u \right] dt \\
&= \int_{t=0}^{\infty} e^{-rt} \left[(1-u)\, y_t Z_t + u b_t - \frac{1-\gamma}{\gamma} \left(y_t R_t - b_t \right) u \right] dt,
\end{aligned}
$$

where $g_t(z)$ is the density corresponding to the distribution function $G_t(z)$.

In the MM model, Lemma 1 and Proposition 1 yield the closed-form expression

$$
\begin{aligned}
W &= \int_{t=0}^{\infty} e^{-(r-g_y-g_z)t} \left[(1-u) \frac{\alpha}{\alpha-1} y_0 R_0 + u b_0 - \frac{1-\gamma}{\gamma} \left(y_0 R_0 - b_0 \right) u \right] dt \\
&= \frac{\left[(1-u) \frac{\alpha}{\alpha-1} - \frac{1-\gamma}{\gamma} u \right] y_0 R_0 + \frac{1}{\gamma} u b_0}{r - g_y - g_z}. \qquad (48)
\end{aligned}
$$

Since $R_0 \to \infty$ as $k \to 0$, welfare diverges to $+\infty$ provided the coefficient on $y_0 R_0$ is positive—that is, when unemployment is not too high. Conditions (a) and (c) in Theorem 1 guarantee that it is, in fact, the case.

Proposition 4. *In an SI-BGP with* $M(u, v) = u^{\gamma} v^{1-\gamma}$, $\lim_{k \to 0} W = \infty$.

Proof. See Appendix. □

The implication of Proposition 4 is stark: eliminating search frictions delivers infinite welfare gains.

In our AI-BGP variant, unemployment vanishes as $k \to 0$, so welfare given in equation (47) simplifies to

$$
\begin{aligned}
\lim_{k \to 0} W &= \int_{t=0}^{\infty} e^{-rt} Z_t dt = \int_{t=0}^{\infty} e^{-rt} \left[\frac{1}{\nu} + R_0 + (g_A/\nu)\, t \right] dt \\
&= \frac{1}{r} \left(\frac{1}{\nu} + R_0 \right) + \frac{g_A/\nu}{r^2}.
\end{aligned}
$$

Since $R_0 \to \infty$, welfare again diverges to $+\infty$.

Conclusion. In both the Pareto and exponential inspection frameworks, eliminating search frictions leads to unbounded welfare gains. This outcome stands in sharp contrast to the DMP model, where welfare gains are finite, and highlights a fundamental weakness of inspection models: they predict implausibly large benefits from removing frictions.

3.9. Possible Generalizations

An SBGP requires that $G_t(T_t(z)) = G_0(z)$. So far, we have focused on two cases: the Pareto case, in which $T_t(z) = ze^{g_z t}$, and the exponential case, in which $T_t(z) = z + \phi t$. We refer to the corresponding SBGPs as the SI-BGP (Scale-Invariant) and the AI-BGP (Additive-Invariant), respectively.

We now outline a general methodology for determining the time-transformation function $T_t(z)$ for a given sampling distribution $F(z)$. Note first that by construction, $R_t = T_t(R_0)$. Along a BGP, equation (1) can be written as

$$R_t = H^{-1}\left(\phi_1 e^{-g_A t}\right),$$

where $H(R_t) = 1 - F(R_t)$ is the survival function and ϕ_1 is a constant. Because $\phi_1 = H(R_0)$, it follows that

$$R_t = T_t(R_0) \quad \text{where} \quad T_t(z) \equiv H^{-1}\left(H(z)e^{-g_A t}\right).$$

A few illustrative cases include:

1. **Pareto:** $F(z) = 1 - \left(\frac{z_l}{z}\right)^{\alpha}$, $H(R) = \left(\frac{z_l}{R}\right)^{\alpha}$, $H^{-1}(x) = z_l x^{-1/\alpha}$, $T_t(z) = ze^{g_A t/\alpha}$.
2. **Exponential:** $F(z) = 1 - e^{-\nu R}$, $H(R) = e^{-\nu R}$, $H^{-1}(x) = -\frac{1}{\nu}\ln x$, $T_t(z) = z + \frac{g_A t}{\nu}$.
3. **Gompertz:** $F(z) = 1 - e^{-\tau(e^{\nu z}-1)}$, $H(R) = e^{-\tau(e^{\nu R}-1)}$, $H^{-1}(x) = \frac{1}{\nu}\ln\left(1 - \frac{1}{\tau}\ln x\right)$, $T_t(z) = \frac{1}{\nu}\ln\left(e^{\nu z} + \frac{g_A t}{\tau}\right)$.
4. **Weibull:** $F(z) = 1 - e^{-(\nu R)^{\tau}}$, $H(R) = e^{-(\nu R)^{\tau}}$, $H^{-1}(x) = \frac{1}{\nu}(-\ln x)^{1/\tau}$, $T_t(z) = \frac{1}{\nu}((\nu z)^{\tau} + g_A t)^{1/\tau}$.

These examples illustrate that the detrending required to preserve stationarity depends sensitively on the underlying sampling distribution. Once $T_t(z)$ is determined, equation (19) can be used to back out the implied parameter restrictions for b_t and k_t:

$$b_t = y_t T_t(R_0) - k_t \frac{M_1(1,\theta)}{M_2(1,\theta)}.$$

Here, R_0 and θ are endogenous but constant.

The procedure above provides *candidates* for balanced growth paths. A complete characterization, however, requires verifying that all BGP-defining equations are jointly satisfied. Because some of the necessary integrals may not have closed-form solutions, numerical methods may be needed to verify existence and fully characterize the corresponding SBGP.

4. Biased Technological Change

This section develops an alternative to the inspection framework: a DMP model with homogeneous workers and biased technological progress in the matching function. We show that when technological change is biased and the inputs in the matching function are complements, a well-behaved limiting BGP exists. This BGP preserves the central properties of the standard DMP model under worker-augmenting technological progress: welfare gains from eliminating search frictions are finite, and unemployment vanishes when vacancies are costless.

The section proceeds in two parts. The first subsection formalizes the notion of biased technological progress in the matching function and provides a sharp characterization for the constant elasticity of substitution (CES) case. We show that, contrary to MM's claim, their results are not robust to the introduction of biased progress. In particular, the limiting growth rate of matches may converge to zero despite ongoing technological improvements, rendering their main theorem inapplicable in such cases. This analysis assumes a stationary tightness rate.

The second subsection embeds biased technological progress into the full DMP model. We show that a BGP with constant tightness emerges endogenously as a general equilibrium outcome—our main contribution. A key feature of this equilibrium is that it is necessarily inefficient, highlighting a sharp contrast between planner and market allocations.

4.1. Biased Technological Change in the Matching Function

Section 2 assumed a matching function of the form $A_t M(u_t, v_t)$. In this formulation, technological progress is Hicks-neutral. We now consider a more general specification,

$$M\left(A_t u_t, B_t v_t\right),$$

where A_t and B_t represent unemployment- and vacancy-augmenting technologies, growing at constant exogenous rates $g_A \geq 0$ and $g_B \geq 0$, respectively.

The job-finding rate is defined as

$$m_t \equiv \frac{M\left(A_t u_t, B_t v_t\right)}{u_t} = M\left(A_t, B_t \theta_t\right) =: m_t\left(\theta_t\right). \tag{49}$$

where $\theta \equiv \frac{v}{u}$ is the market tightness. For later purposes, it is convenient to define effective tightness as

$$\widehat{\theta}_t \equiv \frac{B_t v_t}{A_t u_t} = \theta_t \frac{B_t}{A_t}.$$

When $A_t = B_t$, the function simplifies to $A_t M(u_t, v_t)$, the Hicks-neutral case considered by MM. MM argue that their results extend beyond Hicks-neutral progress:

> ”In the case of input-augmenting search progress, the rate g_m converges to some g_m^*... In the limit as $g_m \to g_m^*$, our theorems hold with g_m^* replacing g_A.” (MM, footnote 10)[3].

However, for their results to hold, it is essential that $g_m^* > 0$. Otherwise, their main results do not apply. If $g_m = 0$, then a Pareto distribution cannot be derived from their equation (10). Moreover, with $g_A = 0$ and their assumption $\delta = 0$, job destruction disappears, and unemployment vanishes in the limit.

To see why $g_m^* = 0$ may naturally arise under biased technological progress, consider the CES matching function:

$$M(Au, Bv) = \left\{ \begin{array}{c} (\alpha (Au)^\sigma + (1-\alpha)(Bv)^\sigma)^{1/\sigma}, \ \sigma \leqq 1, \sigma \neq 0. \\ (Au)^\alpha (Bv)^{1-\alpha} \ \text{ if } \sigma = 0. \end{array} \right\}. \tag{50}$$

This specification has a long tradition in the search-and-matching literature (e.g., Den Haan et al., 2000; Hagedorn and Manovskii, 2008; Petrosky-Nadeau et al., 2018). The Cobb-Douglas case corresponds to $\sigma = 0$. Workers and vacancies are complements if $\sigma < 0$ and substitutes if $\sigma > 0$. Córdoba et al. (2024) discusses several advantages of the CES function with complementarity.

For the CES function, the growth rate of meetings is

$$g_{m,t} = \mu\left(\widehat{\theta}_t\right) g_A + \left(1 - \mu\left(\widehat{\theta}_t\right)\right) g_B, \tag{51}$$

where

$$\mu\left(\widehat{\theta}\right) = \frac{\alpha}{\alpha + (1-\alpha)\widehat{\theta}^\sigma}. \tag{52}$$

The formula confirms that when technological progress is Hicks-neutral ($g_A = g_B$), we obtain $g_{m,t} = g_A = g_B > 0$. The next proposition characterizes the limit behavior when technological progress is either worker or vacancy augmenting.

Proposition 5. *Suppose* $0 < \theta < \infty$, *and either (i)* $g_A > 0$ *and* $g_B = 0$*; or (ii)* $g_B > 0$ *and* $g_A = 0$.

[3]MM use the notation g_p for the growth rate of meetings; we use g_m.

Then

$$g_m^* = \lim_{t\to\infty} g_{m,t} = \left\{ \begin{array}{c} \max\{g_A, g_B\} > 0 \; if \, \sigma > 0 \\ \alpha g_A + (1-\alpha)\, g_B > 0 \; if \, \sigma = 0 \\ 0 \; if \, \sigma < 0 \end{array} \right\}.$$

Proof. As $t \to \infty$, effective tightness satisfies

$$\widehat{\theta}_t \to \left\{ \begin{array}{c} 0, \, g_A > 0, \\ \infty, \; g_B > 0 \end{array} \right\}.$$

From equation (52):

$$\lim_{t\to\infty} \mu\left(\widehat{\theta}_t\right) = \left\{ \begin{array}{ccc} & \sigma > 0 & \sigma < 0 \\ g_A > 0 & 1 & 0 \\ g_B > 0 & 0 & 1 \end{array} \right\}.$$

Substituting into equation (51) yields:

$$\lim_{t\to\infty} g_{m,t} = \left\{ \begin{array}{ccc} & \sigma > 0 & \sigma < 0 \\ g_A > 0 & g_A & 0 \\ g_B > 0 & g_B & 0 \end{array} \right\}.$$

□

The proposition shows that when inputs are substitutes, $g_m^* > 0$, which is necessary for MM's results to hold. The more interesting cases are when inputs are complements ($\sigma < 0$) and technological progress is biased, either worker-augmenting or vacancy-augmenting, in which cases $g_m^* = 0$. In these cases, MM's results do not hold. The reason is that biased progress runs into diminishing returns: the non-improving input becomes a bottleneck under strict complementarity, creating an upper bound on meetings even in the presence of continued technological progress.

We now explore the implications of this limit behavior within the DMP search-and-matching model with declining search frictions.

4.2. DMP Model with Biased Technological Change

There is a unit one of workers, of which n_t are employed and u_t are unemployed. Employed workers produce $y_t = ye^{gt}$, the unemployed workers produce $b_t = be^{gt}$, where $y > b$, and vacancy posting costs $k_t = ke^{gt}$ at time t. Given an initial employment level n_0, the social planner solves

$$\max_{\{n_t, v_t, u_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-(r-g)t} \left(n_t y + u_t b - k v_t\right) dt \text{ subject to}$$

$$\dot{n}_t = M\left(A_t u_t, B_t v_t\right) - \delta n_t, \forall t \geq 0, \tag{53a}$$

$$u_t = 1 - n_t, \ \forall t \geq 0, \tag{53b}$$

where $A_t = A_0 e^{g_A t}$ and $B_t = B_0 e^{g_B t}$, $r > g$, and $\delta > 0$. Let $\rho \equiv r - g$ denote the effective discount rate.

Optimality conditions. Let $e^{-\rho t}\lambda_t$ and $e^{-\rho t}\eta_t$ denote the Lagrange multipliers associated with constraints (53a) and (53b), respectively. The first-order conditions with respect to v_t, u_t, and n_t are:

$$k = \frac{\partial M_t}{\partial v_t}\lambda_t = s_t\left(1 - \mu_t\right)\lambda_t, \tag{54}$$

$$\eta_t = b + \frac{\partial M_t}{\partial u_t}\lambda_t = b + m_t \mu_t \lambda_t, \text{ and} \tag{55}$$

$$y - \eta_t - \delta\lambda_t = \rho\lambda_t - \dot{\lambda}_t. \tag{56}$$

Here, s_t is the job-filling rate and μ_t the elasticity of the matching function with respect to "effective" job seekers:

$$s_t \equiv \frac{M_t}{v_t} = m_t\left(\theta_t\right)/\theta_t \text{ and} \tag{57}$$

$$\mu_t \equiv \frac{\partial M_t}{\partial\left(A_t u_t\right)} \frac{A_t u_t}{M_t} = \mu\left(\widehat{\theta}_t\right).$$

Optimality also requires the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t n_t = 0$.

Equations (54) to (56) mirror those of the canonical DMP framework. From equation (54), the efficient number of vacancies equates the marginal cost k with the marginal new matches associated to a vacancy, $\frac{\partial M_t}{\partial v_t}$, multiplied by the shadow value of a match, λ_t. The marginal gain $\frac{\partial M_t}{\partial v_t}$ equals the average gain $s_t = \frac{M_t}{v_t}$, scaled down by the elasticity $1 - \mu_t$.

Equation (55) states that the shadow value flow of an unemployed worker, η_t, is equal the worker's own output, b, plus the expected contribution to new matches, $\frac{\partial M_t}{\partial u_t}$, weighted by λ_t. Here, the marginal gain $\frac{\partial M_t}{\partial u_t}$ equals the average gain, m_t, scaled by μ_t.

Finally, combining equation (55) and equation (56) yields the value of a match:

$$\rho\lambda_t = y - b - \left(\delta + m_t\mu_t\right)\lambda_t + \dot{\lambda}_t. \tag{58}$$

This expression shows that the net return of a match is the added output $y - b$ plus the the capital gains, $\dot{\lambda}_t$, offset by the effective depreciation rate. Depreciation includes both the exogenous job destruction rate, δ, and the endogenous effect $m_t\mu_t$, which captures the fact that a successful match reduces the pool of job seekers and thereby lowers future matching opportunities.

4.2.1. Balanced Growth Characterization

Consider balanced growth paths (BGPs) along which variables grow at constant rates. The following proposition shows that—despite improvements in the matching technology—labor-market variables remain stationary along any BGP.

Proposition 6. *Along a balanced growth path, the growth rates of n, u, m, μ, λ, s, and θ are zero.*

Proof. Since population is constant, both employment (n) and unemployment (u) must be constant along a BGP. Equation (53a) then reduces to

$$\delta n = m_t u = mu.$$

Thus $m_t = m$ along a BPG. Similarly, since $\mu_t \in [0, 1]$, we must have $\mu_t = \mu$. Substituting these results into equation (58) and imposing the transversality condition yields

$$\lambda_t = \lambda = \frac{y - b}{\rho + \delta + m\mu}.$$

Substituting into equation (54) gives

$$k = \frac{s_t (1 - \mu)}{\rho + \delta + m\mu} (y - b). \tag{59}$$

Equation (59) implies that $s_t = s$, and since $m = s\theta_t$ then $\theta_t = \theta$ along a BGP. □

Equation (59) determines θ, and the unemployment rate follows from

$$u = \frac{\delta}{m + \delta}. \tag{60}$$

Markets. Equation (59) parallels Pissarides (2000, Eq. 1.24) in a decentralized setting where firms post vacancies with success probability s^*, workers find jobs with probability $\theta^* s^*$, firms capture a fraction $1 - \gamma$ of the match surplus, and free entry holds. In our notation:

$$k = \frac{s^* (1 - \gamma)}{\rho + \delta + m^*\gamma} (y - b).^4 \tag{61}$$

[4] Pissarides (2000) assumes $g = 0$, unlike here.

Free entry further implies that the expected net return of posting a vacancy, $s^*(y - w^*)/k - \delta$, equals the effective market return ρ, where w which yields the wage equation:

$$w^* = y - \frac{(\rho + \delta)\, k}{s^*}.$$

The market equilibrium is generally inefficient because the worker's bargaining power is fixed at a constant value $\gamma > 0$, while in the planner's allocation the *effective* bargaining power is variable, $\mu\left(\widehat{\theta}_t\right)$. This permits the market equilibrium to sustain a BGP; the corresponding efficient allocation, by contrast, necessarily rules one out.

4.2.2. Equilibrium

Consider first the market solution. According to equation (61), a BGP with declining search costs exists if they do not affect m^* or $s^* = m^*/\theta^*$. Taking time derivatives of equation (49) yields

$$\begin{aligned} \frac{\dot{m}_t^*}{m_t^*} &= M_1(A_t, B_t\theta^*)\frac{A_t}{M}\frac{\dot{A}_t}{A_t} + M_2(A_t, B_t\theta^*)\frac{\theta B_t}{M}\frac{\dot{B}_t}{B_t} \\ &= \mu\left(\widehat{\theta}_t^*\right) g_A + \left(1 - \mu\left(\widehat{\theta}^*\right)\right) g_B. \end{aligned} \tag{62}$$

All terms in this expression are non-negative. The following proposition follows naturally:

Proposition 7. *There is no BGP in the market economy when $g_A > 0$ and $g_B > 0$. In particular, there is no BGP with Hicks-neutral technological progress in the matching function.*

This proposition confirms MM's result for the Hicks-neutral case, namely that there is no BGP in a model without inspection. The next lemma suggests that a BGP may exist when technological change is either vacancy or worker augmenting.

Lemma 2 Suppose $\theta = \theta^* \in (0, \infty)$. For $\frac{\dot{m}_t^*}{m_t^*} = 0$, one of the following two conditions must hold:
(i) $g_A > 0$, $g_B = 0$, and $\mu(0) = 0$, or
(ii) $g_B > 0$, $g_A = 0$, and $\mu(\infty) = 1$.

Proof. (i) If $g_A > 0$ and $g_B = 0$, then $\widehat{\theta}_t^* \to 0$ and $\mu\left(\widehat{\theta}_t^*\right) \to \mu(0) = 0$. Hence, $\frac{\dot{m}_t^*}{m_t^*} = \mu\left(\widehat{\theta}_t^*\right) g_A \to 0$. (ii) If $g_A = 0$ and $g_B > 1$, then $\widehat{\theta}_t^* \to \infty$ and $\mu\left(\widehat{\theta}_t^*\right) \to \mu(\infty) = 1$. Hence, $\frac{\dot{m}_t}{m_t} = \left(1 - \mu\left(\widehat{\theta}_t^*\right)\right) g_B \to 0$. □

At this point, it is convenient to focus on the CES matching function given in equation (50). Applying the lemma, we find that strict complementarity is a necessary condition for the existence of a BGP. In the CES case, the function $\mu\left(\widehat{\theta}\right)$ satisfies equation (52). When inputs are strict complements, $\sigma < 0$, we have $\mu(0) = 0$ and $\mu(\infty) = 1$, exactly as required by the lemma. In contrast, when inputs are substitutes ($\sigma > 0$), we obtain $\mu(0) = 1$ and $\mu(\infty) = 0$—the opposite of the condition required by the lemma. Therefore, strict complementarity is necessary, though not sufficient, for the existence of a BGP.

4.3. CES Matching

Proposition 8. *Suppose M is a CES matching function. An interior BGP of the market economy exists in the following two cases:*
(i) $g_A > 0$, $g_B = 0$, $\sigma < 0$, $\gamma > 0$, and

$$k < \frac{(1-\alpha)^{1/\sigma} B (1-\gamma)}{\rho+\delta} (y-b) ; \tag{63}$$

(ii) $g_A = 0$, $g_B > 0$, $\sigma < 0$, $\gamma < 1$, and $y > b$.

Proof. (i) Under the stated conditions, the matching function converges to $M(Au, Bv) = (1-\alpha)^{1/\sigma} Bv$. Hence, $s_1^* = (1-\alpha)^{1/\sigma} B$, $m_1^* = (1-\alpha)^{1/\sigma} B\theta_1^*$, $u_1^* = \frac{\delta}{m_1^*+\delta}$,

$$\theta_1^* = \frac{(1-\alpha)^{1/\sigma} B (1-\gamma)(y-b) - (\rho+\delta) k}{(1-\alpha)^{1/\sigma} B\gamma k}, \text{ and}$$

$$w_1^* = y - \frac{(\rho+\delta) k}{(1-\alpha)^{1/\sigma} B}.$$

Condition (63) guarantees that an interior solution for θ^* exists. (ii) Under the stated conditions, the matching function converges to $M(Au, Bv) = \alpha^{1/\sigma} Au$. Thus, $s_2^* = \alpha^{1/\sigma} A/\theta_2^*$, $m_2^* = \alpha^{1/\sigma} A$, $u_2^* = \frac{\delta}{m_2^*+\delta}$,

$$\theta_2^* = \frac{\alpha^{1/\sigma} A (1-\gamma)}{\rho+\delta+\alpha^{1/\sigma} A\gamma} \frac{y-b}{k}, \text{ and}$$

$$w_2^* = y - \frac{(\rho+\delta)(1-\gamma)(y-b)}{\rho+\delta+\alpha^{1/\sigma} A\gamma}.$$

An interior solution exist iff $0 < \gamma < 1$ and $y > b$. □

Discussion. The limit BGP characterized in Proposition 1 emerges because the CES matching function converges to a linear technology in which the sole effective input is the one not experiencing technological progress. With labor-augmenting progress, the matching function converges to a linear function of effective vacancies. Conversely, with vacancy-augmenting progress, it converges to a linear function of effective unemployed workers.

Despite these asymptotic linearities, the unemployment rate remains well behaved. For example, increases in the vacancy posting cost, unemployment benefits, or workers' bargaining power reduce market tightness and raise unemployment in the usual way.

Proposition 8 provides a counterexample to MM's claim—made in their footnote 10—that their model remains valid in the limit even under input-specific technological change. Not only the growth rate of the meetings rate goes to zero in these cases, but the canonical DMP model delivers a well-defined limit BGP without requiring heterogeneity, Pareto distributions, or inspection.

Of the two cases identified in Proposition 8, case (i) is the only one that delivers the standard result that unemployment vanishes when vacancy posting is costless. Furthermore, Córdoba et al. (2024) also show that labor-augmenting technological progress in the matching function can account for a significant share of the decline in the labor share and the fall in market tightness observed between 1980 and 2007.

4.3.1. *Welfare Cost of Search Frictions*

An important distinction between the DMP model analyzed in this section and the inspection models discussed previously lies in the potential welfare gains from eliminating search frictions. In a BGP, social welfare satisfies:

$$
\begin{aligned}
W(u) &= \frac{ny + ub - kv}{r-g} = \frac{y - uy + ub - k\theta u}{r-g} \\
&= \frac{y - (y-b)\left(\frac{\rho+\delta+m}{\rho+\delta+m\mu}\right)u}{r-g} \text{ (using equations (57) and (61))} \\
&= y\frac{1-(1-\varphi)\left(\frac{u(\rho+\delta)+(1-u)\delta}{u(\rho+\delta)+(1-u)\delta\mu}\right)u}{r-g} \text{ (using equation (60))},
\end{aligned}
$$

where $\varphi = b/y$.

The relative welfare costs of search frictions can then be defined as:

$$
\Psi(u) \equiv \frac{W(0) - W(u)}{W(0)} = (1-\varphi)\left(\frac{u(\rho+\delta)+(1-u)\delta}{u(\rho+\delta)+(1-u)\mu\delta}\right)u.
$$

This measure is relative to the ideal benchmark of full employment. Such benchmark is achieved when $k = 0$ in case (i) of Proposition 8 but not in case (ii). The key point is that the welfare costs of search frictions, or unemployment for short, is bounded above by $1 - \varphi$, which occurs when $\mu = 0$ and $\rho = 0$.

In practice, estimated welfare costs fall well below this upper bound. For example, under the parametrization employed by Shimer (2005), the welfare cost is $\Psi(u) = 6.5\%$.[5]

5. Conclusion

Martellini and Menzio (2020) pose a fundamental puzzle: how can technological progress in the matching function ("declining search frictions") be reconciled with the empirical stationarity of unemployment, tightness, and the Beveridge curve? In the spirit of King et al. (1988), they seek necessary and sufficient conditions under which balanced growth can coexist with those stationary labor-market facts.

This paper offers three main conclusions.

First, MM's characterization is too strong. Their conditions (inspection goods with Pareto-distributed quality) are sufficient but not necessary. Balanced growth paths arise outside their framework.

Second, the inspection approach has implausible implications in the cases we study. In the Pareto version, unemployment persists even when vacancies are free to post, and the welfare gains from eliminating search frictions are unbounded. In the exponential version, unemployment does vanish with costless posting, yet the welfare gains remain infinite because workers keep raising their reservation standards as matching becomes arbitrarily easy and the quality support is unbounded.

Third, a constructive alternative exists within a standard DMP environment once we allow for biased technological change and complementarity in matching. With complementary inputs, biased progress in one input makes the other input relatively scarcer, creating a bottleneck and hence diminishing returns to search improvements. The growth rate of meetings falls to zero, delivering a well-behaved BGP with stationary unemployment, tightness, and transition rates. In this setting, unemployment vanishes as frictions disappear but only if progress is worker-augmenting, and—crucially—welfare gains are finite. However, the BGP is necessarily inefficient: the market equilibrium admits a stationary path with declining frictions, whereas the planner's allocation does not, reflecting the failure of the Hosios condition when bargaining weights are fixed but the planner's shadow elasticity varies with tightness.

These results reframe the interpretations MM consider. They do not support the view that search frictions are irrelevant, nor that the historical decline in frictions has been too small. Instead, they point to a specific countervailing mechanism—endogenous bottlenecks from complementarity under biased progress—that can neutralize the growth effects of improved matching while preserving stationary labor-market variables. They also show that MM's sufficiency result does not pin down a unique path: stationarity can emerge without perpetual growth in reservation quality (as in the

[5] We use $\varphi = 0.4$, $\rho = 0.012$, $\mu = 0.72$, $\delta = 0.1$, and $u = 4\%$.

exponential case) and without attributing growth to declining frictions (as in the biased-technology DMP case).

Finally, this agenda opens clear avenues for future work. Empirically, measuring the bias in the matching progress (worker- vs. vacancy-augmenting) and the degree of complementarity is central to distinguishing between inspection and bottleneck mechanisms and to conducting credible welfare assessments. Theoretically, exploring policy in environments with biased progress and complementarity—where market BGPs are inefficient—can clarify the role of bargaining institutions as well as vacancy taxes or subsidies. Relatedly, Córdoba et al. (2024) show that CES matching with worker-augmenting progress can account for secular movements in the labor share and tightness, underscoring the empirical relevance of biased technological change in matching.

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Appendix

A.1. Proofs of Equations and Propositions

Proof of Equation (11): To solve this differential equation, write it as

$$e^{-(r+\delta)\tau}\left[\dot{\lambda}_{t+\tau}(z) - (r+\delta)\,\lambda_{t+\tau}(z)\right] = e^{-(r+\delta)\tau}\left[y_{t+\tau}R_{t+\tau} - y_{t+\tau}z\right].$$

Integrating yields:

$$\int_0^d e^{-(r+\delta)\tau}\left[\dot{\lambda}_{t+\tau}(z) - (r+\delta)\,\lambda_{t+\tau}(z)\right]d\tau = \int_0^d e^{-(r+\delta)\tau}\left[y_{t+\tau}R_{t+\tau} - y_{t+\tau}z\right]d\tau.$$

The integral on the left-hand side simplifies to:

$$\left[e^{-(r+\delta)\tau}\lambda_{t+\tau}(z)\right]_0^d = e^{-(r+\delta)d}\lambda_{t+d}(z) - \lambda_t(z).$$

Proof of Equation (36): Let a =: g_A/ν. Equation (13) becomes

$$\begin{aligned}
\lambda_t(z) &= \int_0^{d(z,t)} e^{-(r+\delta)\tau}\left(z-R_t-a\tau\right)d\tau \\
&= (z-R_t)\int_0^{d_t(z)} e^{-(r+\delta)\tau}d\tau - a\int_0^{d_t(z)} \tau e^{-(r+\delta)\tau}d\tau \\
&= (z-R_t)\left[-\frac{e^{-(r+\delta)\tau}}{r+\delta}\right]_0^{d_t(z)} - a\left[-e^{-(r+\delta)\tau}\left(\frac{\tau}{r+\delta}+\frac{1}{(r+s)^2}\right)\right]_0^{d_t(z)} \\
&= \frac{z-R_t}{r+\delta}\left[-e^{-(r+\delta)d_t(z)}+1\right] - \frac{a}{r+\delta}\left[-e^{-(r+\delta)d_t(z)}\left(d_t(z)+\frac{1}{r+\delta}\right)+\frac{1}{r+\delta}\right] \\
&= \frac{ad_t(z)}{r+\delta}\left[1-e^{-(r+\delta)d_t(z)}\right] + \frac{a}{r+\delta}\left[e^{-(r+\delta)d_t(z)}\left(d_t(z)+\frac{1}{r+\delta}\right)-\frac{1}{r+\delta}\right] \\
&= \frac{a}{r+\delta}\left[d_t(z)\left[1-e^{-(r+\delta)d_t(z)}\right]+e^{-(r+\delta)d_t(z)}\left(d_t(z)+\frac{1}{r+\delta}\right)-\frac{1}{r+\delta}\right] \\
&= \frac{a}{r+\delta}\left[d_t(z)\left[1-e^{-(r+\delta)d_t(z)}\right]+d_t(z)\,e^{-(r+\delta)d_t(z)}+e^{-(r+\delta)d_t(z)}\frac{1}{r+\delta}-\frac{1}{r+\delta}\right] \\
&= \frac{a}{r+\delta}\left[d_t(z)+\frac{e^{-(r+\delta)d_t(z)}}{r+\delta}-\frac{1}{r+\delta}\right].
\end{aligned}$$

We next need to calculate

$$\begin{aligned}
\int_{R_t}\lambda_t(z)f(z)dz &= \frac{a}{r+\delta}\int_{R_t}\left[d_t(z)+\frac{e^{-(r+\delta)d_t(z)}}{r+\delta}-\frac{1}{r+\delta}\right]f(z)dz \\
&= \frac{a}{r+\delta}\int_{R_t}\left[\frac{z-R_t}{a}+\frac{e^{-(r+\delta)\frac{z-R_t}{a}}}{r+\delta}-\frac{1}{r+\delta}\right]\nu e^{-\nu z}dz
\end{aligned}$$

$$\begin{aligned}
\int_{R_t}\lambda_t(z)f(z)dz &= (1-F(R_t))\int_{R_t}\lambda_t(z)\frac{f(z)}{1-F(R_t)}dz \\
&= (1-F(R_t))\,E\left[\lambda_t(z)|z>R\right] \\
&= (1-F(R_t))\,\frac{a}{r+\delta}E\left[d_t(z)+\frac{e^{-(r+\delta)d_t(z)}}{r+\delta}-\frac{1}{r+\delta}|z>R\right] \\
&= (1-F(R_t))\,\frac{a}{r+\delta}\left\{E_{z>R}\left[d_t(z)\right]+E_{z>R}\left[\frac{e^{-(r+\delta)d_t(z)}}{r+\delta}\right]-\frac{1}{r+\delta}\right\}.
\end{aligned}$$

Now,

$$E_{z>R}\left[d_t(z)\right]=E_{z>R}\left[\frac{z-R_t}{a}\right]=\frac{1}{a}E_{z>R_t}z-\frac{R_t}{a}=\frac{1}{a}\left(\frac{1}{\nu}+R_t\right)-\frac{R_t}{a}=\frac{1}{a\nu};\text{ and}$$

$$\begin{aligned} E_{z>R}\left[\frac{e^{-(r+\delta)d_t(z)}}{r+\delta}\right] &= \frac{1}{r+\delta}E_{z>R}\left[\frac{e^{-(r+\delta)d_t(z)}}{r+\delta}\right] \\ &= \frac{1}{r+\delta}\left[\frac{\nu a}{r+\delta+\nu a}\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{R_t} \lambda_t(z) f(z) dz &= (1-F(R_t))\frac{a}{r+\delta}\left\{\frac{1}{a\nu}+\frac{1}{r+\delta}\frac{\nu a}{r+\delta+\nu a}-\frac{1}{r+\delta}\right\} \\ &= e^{-\nu R_t}\frac{a}{r+\delta}\left[\frac{1}{a\nu}+\frac{1}{r+\delta}\frac{\nu a}{r+\delta+\nu a}-\frac{1}{r+\delta}\right] \\ &= e^{-\nu R_t}\frac{1}{(r+\delta)\nu}\left[1+\frac{1}{r+\delta}\frac{(\nu a)^2}{r+\delta+\nu a}-\frac{a\nu}{r+\delta}\right] \\ &= e^{-\nu R_t}\frac{1}{(r+\delta)\nu}\left[\frac{r+\delta-g_A}{r+\delta}+\frac{1}{r+\delta}\frac{g_A^2}{r+\delta+g_A}\right]. \end{aligned} \tag{64}$$

Proof of Proposition 3: Substituting equation (42) and equation (43) into equation (39):

$$\begin{aligned} h_{ue}^{\infty} &= A_0 z_l^{\alpha}\left(\Omega_1 k_0^{-\frac{\alpha}{\alpha-(1-\gamma)}}\right)^{1-\gamma}\left(\frac{\gamma}{1-\gamma}\frac{\Omega_1}{y_0}k_0^{-\frac{1-\gamma}{\alpha-(1-\gamma)}}\right)^{-\alpha} \\ &= \Omega_2 k_0^{\frac{\alpha(1-\gamma)-\alpha(1-\gamma)}{\alpha-(1-\gamma)}} = \Omega_2, \end{aligned}$$

where

$$\begin{aligned} \Omega_2 &= A_0 z_l^{\alpha}\Omega_1^{1-\gamma}\left(\frac{\gamma}{1-\gamma}\frac{\Omega_1}{y_0}\right)^{-\alpha} \\ &= A_0 z_l^{\alpha}\left(\frac{\gamma}{1-\gamma}\frac{1}{y_0}\right)^{-\alpha}\Omega_1^{1-\gamma-\alpha} \\ &= A_0 z_l^{\alpha}\left(\frac{\gamma}{1-\gamma}\frac{1}{y_0}\right)^{-\alpha}\left(\frac{1-\gamma}{\gamma}\left(A_0(1-\gamma)\Phi_1\right)^{1/(\alpha-1)} y_0^{\frac{\alpha}{\alpha-1}}\right)^{\frac{1-\gamma-\alpha}{1+\gamma/(\alpha-1)}} \\ &= A_0^{1+\frac{1}{\alpha-1}\frac{1-\gamma-\alpha}{1+\gamma/(\alpha-1)}} y_0^{\alpha+\frac{\alpha}{\alpha-1}\frac{1-\gamma-\alpha}{1+\gamma/(\alpha-1)}} \\ &\quad \times z_l^{\alpha}\left(\frac{\gamma}{1-\gamma}\right)^{-\alpha}\left(\frac{1-\gamma}{\gamma}\left((1-\gamma)\Phi_1\right)^{1/(\alpha-1)}\right)^{\frac{1-\gamma-\alpha}{1+\gamma/(\alpha-1)}} \\ &= z_l^{\alpha}\left(\frac{\gamma}{1-\gamma}\right)^{-\alpha}\left(\gamma^{-1}(1-\gamma)^{\alpha/(\alpha-1)}\right)^{\frac{1-\gamma-\alpha}{1+\gamma/(\alpha-1)}}\Phi_1^{-1} \\ &= z_l^{\alpha}\gamma^{-\alpha-\frac{1-\gamma-\alpha}{1+\gamma/(\alpha-1)}}(1-\gamma)^{\alpha}(1-\gamma)^{-\alpha}\Phi^{-1} \\ &= z_l^{\alpha}\gamma^{-1}\Phi_1^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} h_{UE}^{\infty} &= \Omega_2 = z_l^{\alpha}\gamma^{-1}\Phi_1^{-1} = \frac{z_l^{\alpha}\gamma^{-1}}{\frac{z_l^{\alpha}}{(\alpha-1)\left((\alpha-1)g_z+\left(r+\delta-g_y\right)\right)}} \\ &= (\alpha-1)\left(\frac{\alpha-1}{\alpha}g_A + r + \delta - g_y\right)/\gamma. \end{aligned}$$

Proof of Proposition 4: According to equation (48), $W \to \infty$ as $k \to \infty$ if $(1-u)\,\frac{\alpha}{\alpha-1} > \frac{1-\gamma}{\gamma}u$ or

$$\begin{aligned} \frac{\frac{\alpha}{\alpha-1}}{\frac{\alpha}{\alpha-1}+\frac{1-\gamma}{\gamma}} &= \frac{\frac{\alpha}{\alpha-1}}{\frac{\alpha\gamma+(1-\gamma)(\alpha-1)}{(\alpha-1)\gamma}} = \frac{\alpha\gamma}{\alpha\gamma+(1-\gamma)\,(\alpha-1)} \\ &> u^{\infty} = \frac{g_A+\delta}{g_A+\delta+h_{ue}}. \end{aligned}$$

This simplifies to:

$$\frac{\alpha}{\alpha-1}h_{ue}^{\infty} > \frac{1-\gamma}{\gamma}(g_A+\delta)$$

$$\frac{\alpha}{\alpha-1}h_{ue}^{\infty} = \frac{\alpha}{\gamma}(r+\delta-g_y+g_A-g_z) = \frac{\alpha}{\gamma}(g_A+\delta+r-g_y-g_z).$$

As long as $r-g_y-g_z>0$, then because $\alpha>1$ and $1-\gamma<1$

$$\frac{\alpha}{\alpha-1}h_{ue}^{\infty} = \frac{\alpha}{\gamma}(g_A+\delta+r-g_y-g_z) > \frac{1-\gamma}{\gamma}(g_A+\delta).$$