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Credit Surfaces and Economic Uncertainty[1]

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Abstract

The Credit Surface along the leverage dimension gives the bond spread as a function of the loan-to-value ratio. Empirically, we show that uncertainty shocks typically increase spreads and steepen the credit surface, profoundly affecting the supply of credit. Theoretically, we derive necessary and sufficient conditions for the convexity of the credit surface, and for changes in the anticipated distribution of collateral prices that lead to steepening of the credit surface. Finally, we show that the credit surface itself fully reveals the entire distribution of collateral prices, thus providing a new and vivid language with which to describe uncertainty and stochastic orders. Credit surface steepening itself is a new stochastic order that may better capture our intuitive notion of more uncertainty.

Keywords: Credit surfaces, economic uncertainty, bond pricing, credit spreads, Black-Scholes-Merton model, leverage.

JEL classification: E43, G12.

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1 Introduction

Credit is central to the macroeconomy and macroeconomic policy. Traditionally, credit conditions on a given credit market are described by one or two numbers: the level of the risk-free rate, possibly coupled with a *single* credit spread. A fuller description would be the entire mapping from the many contractual dimensions of a loan and the levels of each into different credit spreads (equivalently interest rates or prices), which Geanakoplos (2014) and we here call the Credit Surface. Credit conditions depend not only on the level of the surface, but also on its steepness and curvature.

We concentrate on analyzing credit surfaces along a single leverage dimension, holding all other credit terms fixed, over a continuum of possible values measured by loan to value, where bigger and bigger loans are taken out against the same collateral. The steepness indicates how the cost of credit rises as a borrower reaches for more leverage. The curvature indicates whether that rise accelerates with leverage.

Our analysis begins with evidence from corporate bond markets showing that the credit surface is convex, and that uncertainty shocks raise and steepen it. The rest of our paper aims for an exact characterization of the relationship between economic uncertainty and the pricing of collateralized debt—how uncertainty shapes, and is revealed by, the credit surface.

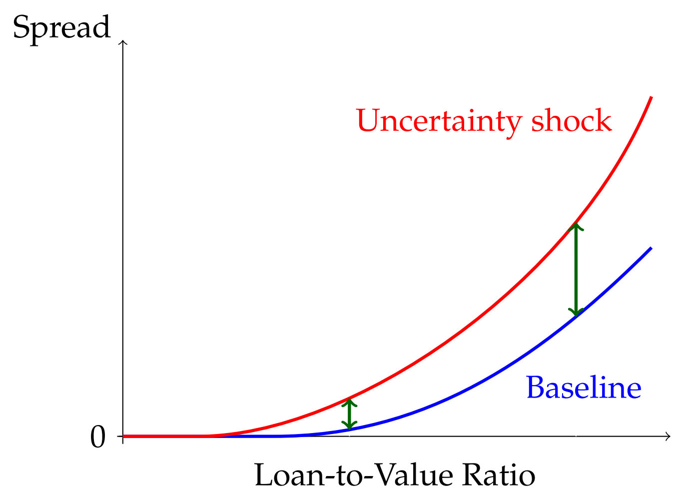
Our first goal is to prove that when the distribution of realizable collateral values is continuous, then under natural conditions, the credit surface is convex. As borrowers take out bigger loans with the same collateral, the promised rate of interest rises faster and faster; and credit conditions get tighter and tighter.

Our second and main goal is to prove that under plausible general conditions, uncertainty shocks push credit surfaces higher and steeper: the riskier the loan, the bigger the effect on spreads. Figure 1 illustrates the convexity of the credit surfaces, and the steepening that comes from increased uncertainty. This steepening has profound implications for the availability of credit to heterogeneous borrowers, especially in crises when uncertainty typically skyrockets: overall leverage likely goes down because borrowers seeking high leverage (who plan to repay) face prohibitively high spreads. For example, the credit surface dramatically rose and steepened in 2008 during the financial crisis.

Our last main goal is to define what "more uncertainty" means for credit markets. We argue that the right notion of "more uncertainty" for credit markets asks for more than a mean-preserving spread.

In Section 2 we present historical evidence that the credit surface is convex and does steepen with economic uncertainty, using data from corporate bonds of non-financial

Figure 1: Uncertainty Shocks and Credit Surfaces



Note: Representation of two credit surfaces: blue corresponds to baseline level of economic uncertainty and red corresponds to uncertainty shock. When the collateral payoff distribution is continuous and satisfies a natural condition, the credit surface is convex, and becomes steeper with uncertainty shocks.

firms. Our evidence consists of estimates of the credit surface in the dimension of leverage, for five different credit ratings, comparing a regime of low or normal volatility to a regime of very high volatility. In almost all rating groups, uncertainty shocks cause credit surfaces to rise and steepen, and almost all ten of the estimated credit surfaces are convex (see Figure 2). The few departures from convexity and steepening occur precisely where our theory predicts that data limitations on the market value of debt bias the estimation, strengthening the empirical case we make for convexity and steepening.

Our critical empirical assumption is that all non-financial firms with the same rating share the same distribution of returns, no matter what their size. An implication is that the spread depends only on the loan-to-value ratio (LTV), and not on the loan size and collateral value separately. Without this assumption we might observe the spread for a single bond issue for a single firm, giving us one point on the credit surface for that firm. Our assumption enables us to take observations from similar firms, with different LTV ratios, and plot all of them on the same one dimensional diagram with LTV on the horizontal axis and credit spread on the vertical axis, tracing out a clear picture of the credit surface as in Figure 1.

In Section 3 we present a standard model of collateralized bond prices. The firm or asset which serves as collateral for the bonds takes on values next period (the forward value or the forward price) given by a random variable X. We suppose first that the payoff next period from a bond promising j when the collateral X takes on the value x is $\min(j, x)$. The collateral is no-recourse in the sense that the lender, or bond holder, is entitled to seize the collateral but no more than the promise, and the bond issuer faces no further penalty beyond losing the collateral in case of default.[2]

We suppose second that the forward price of the collateral and all bonds backed by it are valued by their expected payoffs with respect to a distribution function F on the potential forward values of X.[3] The forward price p of the collateral and the forward price of a bond promising j are

$$p \equiv \mathbb{E}[X] = \int_0^\infty x dF(x) \qquad \text{and} \tag{1}$$

$$L(j) \equiv \mathbb{E}[\min(j, X)] = \int_0^\infty \min(j, x) dF(x) \ .$$

The function $L(j)$ will play a crucial role in our analysis; we refer to it as the (forward) *loan pricing function*.

Our third assumption is that there is a forward loan price as defined above for every $j > 0$. These pricing formulas are consistent with a wide range of models of credit risk, ranging from structural models of risky debt to general equilibrium models with complete and incomplete markets, as we describe in Section 3.

A New Language for Uncertainty. The only uncertainty in our model is over the future value of X, which is conventionally described by its cumulative distribution function F. The value $F(j)$ also gives the probability that a bond promising j renders the firm or the asset collateralizing it insolvent, so we refer to it as the *promise insolvency probability*.

The loan pricing function $L(j)$ satisfies various properties described in Section 3. Any function satisfying these properties arises from exactly one distribution F, so the loan pricing function is another way of faithfully describing all the uncertainty. In fact, the right-derivative of the loan pricing function equals the promise solvency probability, as the amount lent only increases by the probability the original promise is paid.

$$L'_+(j) = 1 - F(j) \ .$$

Since $L(j)$ is strictly increasing, we can define its inverse, the *promise function* J, where $J(\ell)$ is the promise that gives a forward loan value of ℓ dollars. The promise function is convex and satisfies several other properties immediately derivable from the properties

[2] The binomial tree model of Black-Scholes and Merton has exactly this flavor.

[3] The forward price is payable next period an instant before the collateral value is realized.

of L. Any function J satisfying those properties arises from exactly one distribution F; moreover,

$$J'_+(\ell) = \frac{1}{1 - F(J(\ell))} \,.$$

Thus the promise function is yet another way of faithfully describing all the uncertainty.

Most importantly, we can define the dual cumulative distribution $F^*(\ell)$, as the probability of insolvency from making the unique promise $J(\ell)$ just big enough to have a future loan value of ℓ

$$F^*(\ell) \equiv F(J(\ell)) \,.$$

We call it the *loan insolvency probability*.[4] Once again, any dual distribution F^* can be derived from exactly one distribution function F. Thus the dual distribution is yet another way of faithfully describing all the uncertainty. Restricting the dual distributions F^* to those that satisfy an easily understood property, such as an increasing hazard rate $h^*(\ell) = f^*(\ell)/(1 - F^*(\ell))$ in ℓ, automatically restricts the set of corresponding distributions F. Similarly, standard stochastic orders on dual distributions F^*, like the hazard rate order, carry over to novel stochastic orders on the original distributions F.

Leverage Ratios, Credit Surfaces and Dispersion. The ratio $j/\ell \equiv 1 + S$ of the loan promise to the forward loan value is called the credit spread or default premium. It is a single number indicating the riskiness of the loan that is accepted throughout the financial world. The spread is invariant to scaling the collateral values X and the promise j (or loan amount ℓ) by the same constant, so we can write it as a function of LTV (or the promise-to-value ratio PTV).

Measuring j and ℓ relative to $\mathbb{E}[X]$, or equivalently, under the normalization that the collateral has unit expected forward value ($p \equiv \mathbb{E}[X] = 1$), the forward loan price $\ell = L(j)$ equals the LTV ratio and the promise $j = J(\ell)$ equals the PTV ratio. We use this slight abuse of notation throughout: ℓ denotes both the loan price and the LTV; j denotes both the promise and the PTV. We will make the dependence on collateral value explicit when relevant.

The *loan credit surface* is defined by the function

$$S^*(\ell) \equiv \frac{J(\ell)}{\ell} - 1 \,. \tag{2}$$

This credit surface is the central object of our analysis: whenever we refer to the credit

[4] The dual distribution has a precedent in reliability theory: F^* is the law of $L(X)$, whose quantile function is the *total time on test* (TTT) transform of F, and $L(X)$ is the observed total time on test (Li and Shaked, 2004; Kochar, Li and Shaked, 2002; Barlow and Campo, 1975; Shaked and Shanthikumar, 2007). What is specific to our setting is the economic reading: $F^*(\ell)$ is the insolvency probability of a loan of value ℓ.

surface, we mean the loan credit surface.

The loan credit surface completely reveals the distribution of collateral *excess returns*—$X/\mathbb{E}[X] = X/p$, making it a picture of economic uncertainty. Just as Ross (1976) and Breeden and Litzenberger (1978) showed that option prices reveal the risk adjusted probabilities of outcomes, so does the credit surface. By examining the corporate bond credit surface in say 2006, one can infer not only the anticipated volatilities, but also the market risk adjusted probabilities of every outcome. Looking at the credit surface again in 2009, one can see how all these probabilities changed as a result of the financial crisis.[5]

Because the credit surface is invariant to a common scaling of collateral and promises, the surface cannot depend on the scale of the collateral at all. What the surface responds to is therefore not the level of collateral but the way its value is spread around its mean. Define the *dispersion* F^D of X as the distribution of excess returns $X/\mathbb{E}[X]$. It follows that the credit surface is determined by, and completely reveals, the dispersion F^D but not the distribution F. Our goal is to elucidate conditions on F^D, and changes in F^D, that guarantee the convexity and steepening, respectively, of the credit surface. As every distribution has a dual, every dispersion F^D has a dual dispersion F^{D*} that will give us the natural language in which to describe changes in risk. The standard stochastic orders, like hazard rate dominance, become trivial for dispersions, because the mean is fixed. Those same stochastic orders, and some new ones, applied to the dual dispersion F^{D*}, induce orders on F^D that capture subtler changes in risk than mean preserving spreads.

A Preview of our Convexity Results. In Section 4 we examine the shape of the credit surface. A natural economic question is whether the marginal cost of leverage rises with leverage—that is, whether the credit surface is convex. When the collateral takes on discrete values, the credit surface is concave, not convex, on each interval of ℓ requiring promises $J(\ell)$ between collateral payoffs (Fostel and Geanakoplos 2015). When these discrete collateral payoffs are pushed together and X takes on a continuum of values, it turns out that almost all of the standard parameterized families of continuous distributions give a convex loan credit surface. We explain why by giving a sequence of six progressively less demanding conditions on the density f of F that each guarantee convexity.[6] To illustrate the scope of these conditions we evaluate them for 12 standard distributions: beta, exponential, extreme value, gamma, Laplace, log-normal, logistic, normal, power function, power law, uniform, and Weibull.

The strongest sufficient condition, and an easy one to check, for convexity of the loan

[5]For some kinds of collateral, the option market is sparse but the bond market is deep, and so much more revealing about the underlying probabilities.

[6]Since the normalization by $\mathbb{E}[X]$ does not affect convexity, we can work directly with F instead of F^D.

credit surface is that the density f is log-concave. This holds for many standard families, such as the normal distributions, but not all.

The *promise hazard (default) rate* is the instantaneous fraction of non-defaults that turn into defaults as the promise increases, $h(j) = f(j)/(1 - F(j))$. A more general sufficient condition for convexity is that the distribution has increasing promise hazard rate (IPHR) as j increases, $h'(j) \geq 0$. Ten out of twelve standard distributions satisfy IPHR.

A strictly more general sufficient condition is that $h' + h^2 \geq 0$, or equivalently that the *loan hazard rate* $h^*(\ell) = f^*(\ell)/(1 - F^*(\ell))$ is increasing (ILHR). As more money is raised, the default rate increases. Eleven out of twelve standard distributions satisfy ILHR, and thus give rise to convex credit surfaces. The twelfth does too, for some parameter values.

The most permissive sufficient condition for a convex loan credit surface is that $h' + 2h^2 \geq 0$, or equivalently

$$\frac{\frac{df(x)}{dx}}{f(x)} \geq -\frac{3f(x)}{1 - F(x)} \,, \tag{3}$$

which holds automatically wherever f itself is rising, and allows f to be falling as long as it does not fall faster than three times the hazard rate. To see the implications of this, consider collateral values distributed according to a symmetric unimodal distribution. In this case, in the LTV interval associated with promises below the mode, which is where new loans are typically written, $\frac{df(x)}{dx} > 0$ and the credit surface is convex. A loan is called underwater when the collateral value has fallen below the promise—as happens in crises or when firms are downgraded. In these situations j is typically above the mode and $\frac{df(x)}{dx} < 0$ and the condition, and convexity, might fail. This last sufficient condition indicates why convexity fails for discrete distributions: the density drops infinitely fast to zero just after each mass point.

The most popular family of continuous distributions in finance, namely the log-normal distributions, corresponds to our 12^{th} standard continuous distribution. Surprisingly, it only satisfies ILHR for small enough values of the standard deviation parameter (σ). For larger standard deviations the credit surface has an interval of nonconvexity.

A Preview of our Steepening Results. In Section 5 we characterize the relationship between uncertainty shocks and credit surfaces. Geanakoplos (2003, 2010) emphasized scary bad news as the trigger for crises. The bad news lowers asset values, and the scary news tightens credit conditions. We use X_0 to denote collateral values in the baseline, and use X_1 to represent collateral values after the news. We can represent simple bad news as a proportional reduction in the value of collateral in every future state of the world $X_1 = \alpha X_0$, with $\alpha < 1$. Bad news impairs the collateral backing old loans, and presumably widens their spreads. However, the credit surfaces are invariant to this type of shock:

new borrowers who scale their promise by α obtain the same credit terms. The credit surface depends only on the shape of the collateral distribution, not its scale, that is, the credit surface depends on the dispersion F^D.

Just as Rothschild and Stiglitz (1970) showed that a mean-preserving spread is the right formalization of increased risk for a risk-averse utility maximizer, we identify the stochastic order that formalizes increased uncertainty for credit markets—the order under which the credit surface steepens. The first thing to note is that mean preserving spreads do not generally steepen the credit surface: contrary to an intuitive notion of increased risk, a mean preserving spread might make a bond less likely to default. We show that steepening is indeed a mean preserving spread, but of a special kind. It cannot correspond to a familiar stochastic order, because all of them are trivial on F^D, as there is no room to change the mean. We define a new but natural stochastic order on dual dispersions F^{D*} that completely characterizes surface steepening. We then give an equivalent ordering over differentiable dispersions F^D that is also equivalent to loan credit surface steepening.

According to the classic definition of first order stochastic dominance on the dual dispersions, $F_0^{D*} \gtrsim_1 F_1^{D*}$, the more secure dispersion F_0^{D*} dominates the riskier dispersion if and only if, as interpreted in the language of default, the solvency probability $1 - F_0^{D*}(\ell) \geq 1 - F_1^{D*}(\ell)$ no matter how much money ℓ is raised.[7] Intuitively, the solvency of a bond that raises ℓ is a signal that it is more likely to be the secure bond 0 than the riskier bond 1. The hazard rate ordering, $F_0^{D*} \gtrsim_H F_1^{D*}$, expresses a more demanding intuition about risk changes, namely that as the amount of money raised increases (and default becomes more likely), solvency becomes a stronger signal that the bond is the more secure bond 0.

In Theorem 4 we show that when a shock makes the loan insolvency distribution less favorable in the sense of the loan hazard rate order ($F_0^{D*} \gtrsim_H F_1^{D*}$), then the loan credit surface steepens. A slightly weaker new order that we call the harmonic hazard rate order $\gtrsim_{HH}$ is necessary and sufficient for steepening. Both are stronger than a mean-preserving spread, which only implies that the loan credit surface shifts up. Theorem 4 runs in both directions: a steeper surface not only follows from greater uncertainty but reveals it.

For all the standard continuous distributions we study, including the log-normal distribution, each has a natural parameter expressing uncertainty. We might ask whether increasing this parameter causes a mean preserving spread or more specifically a steepening of the credit surface. It turns out that in every case, increasing the uncertainty parameter satisfies the stronger loan hazard rate order and steepens the loan credit surface. We therefore propose loan credit surface steepening as a new stochastic order that

[7]First-order stochastic dominance on the duals is known in reliability theory as the *total time on test order* $\lesssim_{ttt}$ of Kochar, Li and Shaked (2002): $F_1 \leq_{ttt} F_0$ if and only if $F_1^* \leq_{st} F_0^*$ (Shaked and Shanthikumar 2007).

captures the idea of more uncertainty in credit markets. A worsening loan harmonic hazard rate is thus Geanakoplos's (2003, 2010) scary news, made precise.

The Promise Credit Surface. We can also define the promise credit surface by

$$S(j) \equiv \frac{j}{L(j)} - 1 \; . \tag{4}$$

The corresponding sufficient conditions for convexity of the promise credit surface (Theorem 3) require strictly more: ILHR is no longer sufficient, and one must assume the stronger IPHR property (or log-concavity of f, which implies IPHR). The reason is that $S^*(\ell) \equiv S(J(\ell))$ and J is convex, so if the promise credit surface S is increasing and convex, then the loan credit surface S^* must be convex.

Finally, we show that for low promise-to-value ratios, the promise credit surface steepens with a mean preserving spread, or any stronger deterioration in collateral returns. But for large promise-to-value ratios, the promise credit surface eventually flattens rather than steepening with any of these shocks. This hump-shaped response is one reason to prefer the loan credit surface when analyzing uncertainty shocks.

An appendix contains technical materials, including the formal proofs for our results.

Notes on Related Literature. Our paper is related to the literature that models credit contracts in general equilibrium.[8] The general equilibrium origin of the credit surface comes from Geanakoplos (1997). Within this literature, our immediate predecessor is Fostel and Geanakoplos (2015) which analyzed the shape of the loan credit surface and how it responds to uncertainty shocks in binomial economies, where collateral can take on only two values. They proved that it must always be concave, not convex, on the interval lying between the realizable collateral values. They showed that increases in downside risk always tighten credit so much that equilibrium leverage must go down. We take up that theme for general, continuous collateral distributions.

Our paper also relates to Simsek (2013), who derives comparative statics results about leverage choices with a single borrower and lender based on changes in the collateral payoffs that can be hazard rate ordered. Similarly, Dávila and Walther (2023) study optimal regulation when borrowers or lenders change their belief in a hazard rate sense. Our analysis makes use of first-order stochastic dominance, second-order stochastic dominance, and hazard rate dominance.

Our paper is also related to the literature that models credit contracts using option

[8]See also Geanakoplos (2003, 2010), David (2007), Fostel and Geanakoplos (2008), Ottonello and Winberry (2020), Diamond and Landvoigt (2022), among others.

pricing, pioneered by Black and Scholes (1973) and Merton (1974).[9] Their work established that when collateral returns are log-normal, volatility shocks raise all credit spreads; it soon became common knowledge that, for any distribution, mean-preserving spreads raise all credit spreads. We investigate two new properties: convexity of the surface and its steepening from uncertainty shocks.

Several authors have greatly advanced our understanding of credit by adding a single credit spread to the riskless interest rate. For instance, Gilchrist and Zakrajsek (2012) examine the relationship between credit conditions in corporate bond markets and economic activity using an average credit spread. Similarly, Ajello et al. (2023) in their new index of financial conditions include information from the level of the federal funds rate and the BBB spread to capture credit conditions in corporate bond markets. We argue that the steepness of the credit surface, and its steepening under uncertainty shocks, is a further determinant of credit conditions.

Bloom (2009) suggested that uncertainty shocks are a key driver of economic fluctuations, and Arellano, Bai and Kehoe (2019) and Alfaro, Bloom and Lin (2024) show that financial conditions amplify and increase the persistence of uncertainty shocks on economic activity. Because our results imply these shocks fall hardest on riskier, higher-leverage borrowers, the steepening channel offers a mechanism for that amplification and persistence.

More broadly, we work with a range of stochastic orders—and introduce a new one, which we call the harmonic hazard rate order, in the same spirit as the hazard rate—placing our analysis within the theory of stochastic orders and reliability (e.g., Shaked and Shanthikumar 2007; Chan, Proschan, and Sethuraman 1990). To the best of our knowledge, both the harmonic hazard rate order and Lemma C.1, which establishes the convexity of the mean residual life, are new to that literature. Methodologically, our contribution is the economic identification of the dual distribution, which coincides with the total-time-on-test transform of F known in reliability theory, and the demonstration that stochastic orders on the dual are exactly the orders that characterize credit surface steepening. The dual becomes especially powerful for dispersions, where the familiar orders are vacuous. Our comparative statics connect to the literature on robust comparative statics under uncertainty (Milgrom and Roberts 1994; Milgrom and Shannon 1994; Athey 2002), though our focus is on ranking debt prices rather than solutions to utility-maximization problems.

[9] See also, Longstaff and Schwartz (1995); Duffie and Lando (2001); and Collin-Dufresne and Goldstein (2001), among many others. Within the Black-Scholes-Merton model, Glasserman and Pirjol (2023) established relative-convexity properties of out-of-the-money call prices in strike and volatility.

2 Evidence from the Bond Credit Surface

This section presents our evidence on the relationship between credit surfaces in the dimension of leverage and economic uncertainty, using corporate bond market data. The focus on the corporate bond market is motivated by the availability of information and the importance of bonds as a source of non-financial corporate credit.[10] Our goal is to describe the observed interest rate spread of bonds associated with different leverage ratios, while other credit terms are held fixed. For that, we need information on bond spreads and characteristics of the bond issue and the issuer firm.

2.1 Data

We obtain bond spreads from 1998 to 2023 for the constituents of the investment grade (C0A0) and high yield (J0A0) bond indices from ICE Data Indices, LLC (formerly Bank of America Merrill Lynch Indices).[11] Using data for the constituents of these two widely followed bond indices has the advantage that the underlying bonds are expected to be traded more frequently and have less data quality issues. These data provide different measures of bond spreads, and we use option-adjusted spreads (OAS), allowing us to compare bonds with different structures and option features. In addition, these data provide the bond rating, term to maturity, and other characteristics that we use with the aim to isolate the role of leverage on bond spreads. We consider separately bonds with similar risk ratings and restrict our sample to bond issues by domestic nonfinancial firms, with a contractual remaining maturity of 7-10 years and rated CCC- or above. We discard observations with an OAS lower than 5 or in excess of 3,500 basis points. These filters yield a data set with 16,010 bonds and 4,967 firms.

To measure firms' leverage ratio we need information on the value of debt and assets. As our results point out, a more transparent picture of credit spreads and leverage is obtained when the latter is measured using the loan-to-value ratio: the ratio of the market value of debt to the market value of total assets. Although the market value of equity is readily available for publicly traded firms, marking to market other liabilities is more challenging. We follow the standard approach in the empirical corporate finance literature and combine the market value of equity with the book value of other balance sheet items, using the latest financial statement (e.g., Kahle and Stulz 2017 and He, Kelley and Manela

[10]According to the Financial Accounts of the United States, bonds surpassed loans as the largest source of credit for nonfinancial corporate firms in the early 1990s and remained the primary source of funds for this sector until the end of 2018.

[11]The ICE Data Indices, LLC data is used with permission and supplemented with bond pricing information from ICE Data Pricing Reference Data, LLC.

2017). Then, for a given firm, the *empirical leverage* (EL) ratio at day t is measured using

$$\mathrm{EL}_t = \frac{d_{B,t^*}}{a_{B,t^*} - e_{B,t^*} + e_{M,t}} , \tag{5}$$

where a denotes the value of total assets; e the value of equity; d the total value of debt; subscripts B and M are used to denote book and market values, respectively; and $t^* < t$ is the close date of the last financial-statement period.[12] We note that in previous studies the EL ratio in equation (5) is referred to as market leverage to emphasize the use of the market value of equity in the denominator. Our analysis emphasizes the different concepts used to value the debt.

The empirical measure of leverage uses the book value or carrying value of debt, corresponding to the historical cost of the debt minus the amortization of discounts (or premiums) instead of its market value. The book value of debt $d_{B,t}$ corresponds to the market value as long as credit conditions—interest rates and the distribution of collateral values—stay constant. So absent changes in credit conditions the EL ratio will equal the LTV ratio and both surfaces will be equal. If credit conditions suddenly change, then the market value of debt and the present value of promised payments could change, but the book value will be unaffected, causing possible discrepancies between the empirical leverage and the LTV and PTV ratios, as is well known.

To calculate the EL ratio in our sample, we use quarterly information from Compustat and daily information from CRSP. To merge these data sets with our data on bond spreads, we use the monthly links from Fang (2024), available at https://openbondassetpricing.com/. Using these links, we are able to match 67 percent of issuer firms and 80 percent of the bond constituents in our bond sample.The book value of total debt corresponds to long-term debt (dlttq) plus debt in current liabilities (dlcq). The denominator is measured as the sum of the book value of total assets (atq) minus book equity (ceq) on the closing date of the last financial statement plus the market value of equity on day t.[13]

[12] For simplicity, we assume investors learn about the financial statement the day after the reporting period closes, whereas this information is reported to investors later by the issuer, and we assume that investors observe daily changes in firms' market capitalization. The former is arguably an imperfect assumption, yet bond analysts pay close attention to the market moves and the finances of bond issuers and are likely to be able to closely track total debt and assets. In practice, these assumptions are expected to introduce noise to the data, biasing our estimates towards finding flat surfaces. Moreover, the results are robust to using the report date of quarterly earnings (rdq) for t^*. Our results are also robust to measuring the EL using $d_{B,t^*}/(d_{B,t^*} + e_{M,t})$.

[13] As recommended by Chen (2015), we measure the market capitalization of the firm on the closing date of the last financial statement using Compustat (csho*prcc_f) and update using the market capitalization of the firm on day t using CRSP (abs(prc)*shrout), i.e., $e_{M,t} = e_{CRSP,t}/e_{CRSP,t^*} e_{Compustat,t^*}$. When using CRSP, to account for splits, we use the adjusted stock price and shares outstanding (abs(prc)/cfacpr*shrout*cfacshr).

Our final sample comprises 11,774 bonds, issued by 3,100 different firms, for which we can calculate the empirical leverage ratio (Table 1).[14] Our sample includes bonds with different ratings from AAA (the highest) to CCC- (the lowest), so to isolate the effect of leverage on bond spreads, we consider separately bonds in the following rating groups: (*i*) bonds rated between AAA and A-; (*ii*) bonds rated between BBB+ and BBB-; (*iii*) bonds rated between BB+ and BB-; (*iv*) bonds rated between B+ and B-; and (*v*) bonds rated between CCC+ and CCC-.

With this data in hand, we inspect empirically the effect of uncertainty shocks. In the next sections we shall formally define uncertainty shocks; here we wish to present evidence from what would by any definition be regarded as an uncertainty shock. We separately consider months when the CBOE Volatility Index (VIX) is in the top decile of its distribution.[15] Table 1 presents descriptive statistics of our two main variables of interest: OAS and EL ratios, for the different rating groups during baseline months and months when an uncertainty shock is realized, defined as months where the average daily VIX was below or above 29.4, the 90$^{\text{th}}$ percentile between 1997 and 2023. By construction we have fewer observations in months when the VIX is above its 90$^{\text{th}}$ percentile, but even for the CCC+ to CCC- rating group, where we have the fewest number of observations, we have almost 10,000 bond-day observations in these months. In our baseline months, the average OAS for bonds in the AAA to A- group is 94 basis points, and this figure increases to 823 basis points for bonds in the CCC+ to CCC- group. Average spreads, as expected, increase in months with uncertainty shocks, with the average spreads for the same groups of bonds being 172 and 1,360 basis points. That is, uncertainty shocks increase credit spreads for bonds with similar risk ratings.

Next, we study the effect of uncertainty shocks on the steepness of the credit surface.

2.2 Estimation

Our goal is to estimate the functional relationship between credit spreads and leverage, measured by the empirical leverage ratio described in equation (5). We use a local linear kernel estimator that allows us to estimate this functional form nonparametrically, allowing us to be agnostic about the functional form of this relationship. At a given EL ratio, this estimator fits a weighted least-squares regression using weights calculated with a (normal) kernel function of the distance between EL ratios. That is, for any leverage ratio,

[14] To prevent the use of the wrong financial statements, we remove observations where the total value of bond debt in our sample is bigger than the total debt in the last financial statement.

[15] Chicago Board Options Exchange, CBOE Volatility Index: VIX [VIXCLS], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/VIXCLS.

Table 1: Descriptive Statistics by Rating Groups and Uncertainty Regime

Variable	Obs.	Mean	Std. Dev.	Min	p25	p75	Max
Ratings AAA to A-							
Baseline: 596 firms and 2,490 bonds							
Spread (OA, basis points)	1,058,107	94.1	42.0	5	68	109	986
EL Ratio (percent)	1,058,107	19.57	10.71	0.39	11.17	28.25	80.00
Uncertainty Shocks: 459 firms and 1,415 bonds							
Spread (OA, basis points)	150,335	171.6	118.5	5	99	205	1,426
EL Ratio (percent)	150,335	21.635	11.36	0.82	12.42	30.31	70.11
Ratings BBB+ to BBB-							
Baseline: 1,276 firms and 4,442 bonds							
Spread (OA, basis points)	1,664,297	167.9	83.1	6	116	196	3,396
EL Ratio (percent)	1,664,297	25.0	10.9	0.29	16.7	32.3	82.4
Uncertainty Shocks: 940 firms and 2,294 bonds							
Spread (OA, basis points)	222,807	306.7	217.1	7	165	381	3,303
EL Ratio (percent)	222,807	27.7	11.864	0.3	18.8	35.7	80.4
Ratings BB+ to BB-							
Baseline: 954 firms and 2,649 bonds							
Spread (OA, basis points)	652,323	317.6	147.2	5	231	368	3,482
EL Ratio (percent)	652,323	31.7	13.4	0.8	21.8	40.6	89.6
Uncertainty Shocks: 537 firms and 1,058 bonds							
Spread (OA, basis points)	86,373	557.6	345.5	6	352	642	3,499
EL Ratio (percent)	86,373	34.9	14.3	2.9	23.5	45.0	95.2
Ratings B+ to B-							
Baseline: 1,267 firms and 2,988 bonds							
Spread (OA, basis points)	610,014	453.2	280.5	5	302	510	3,500
EL Ratio (percent)	610,014	40.3	15.4	3.1	29.2	50.6	93.6
Uncertainty Shocks: 697 firms and 1,150 bonds							
Spread (OA, basis points)	77,448	726.0	448.3	8	455	820	3,499
EL Ratio (percent)	77,448	43.5	16.3	2.1	31.4	54.6	95.2
Ratings CCC+ to CCC-							
Baseline: 418 firms and 674 bonds							
Spread (OA, basis points)	96,030	822.5	555.2	6	465	973	3,499
EL Ratio (percent)	96,030	52.2	18.9	1.7	39.0	65.9	93.0
Uncertainty Shocks: 154 firms and 203 bonds							
Spread (OA, basis points)	9,714	1,359.6	786.7	32	767	1,735.8	3,498
EL Ratio (percent)	9,714	53.9	17.4	8.1	42.7	67.1	87.2
Total: 3,100 firms and 11,774 bonds							

Note: EL ratio corresponds to the empirical leverage ratio defined in equation (5).
Source: Own elaboration based on ICE Data Pricing & Reference Data, LLC; S&P Global, Compustat; and CRSP/Compustat Merged Database via WRDS. See Section 2.1 for details.

observations with similar ratios will have more influence in the estimated regression. We estimate this nonparametric regression separately for each rating group and uncertainty regime using daily bond observations.

Our daily bond observations can be represented by the vector (l_{it}, s_{it}), where s_{it} corresponds to the OAS of bond i on day t and l corresponds to the EL ratio of the issuer of bond i on day t. We denote with $\mathcal{I}_w$ the set of bonds with rating w, T_0 the set of days with baseline uncertainty, and T_1 the set of days with uncertainty shocks. Then, for every rating group w and uncertainty regime $v = 0, 1$, we fit the nonparametric model

$$s_{it} = m_{wv}(l_{it}) + \varepsilon_{it} \quad \text{with} \quad i \in \mathcal{I}_w \quad \text{and} \quad t \in T_v \ ,$$

where m_{wv} is the nonparametric function to be estimated and ε_{it} are zero mean disturbances. These nonparametric models are fitted using a kernel bandwidth of 20 and at least 9,000 observations from 200 different bonds, and in some cases using more than 1 million observations from more than 2,500 bonds (Table 1). Our estimate of the empirical-credit surface for rating group, w, under uncertainty regime, v, is $\hat{s}_{it} = \hat{m}_{wv}(l_{it})$.

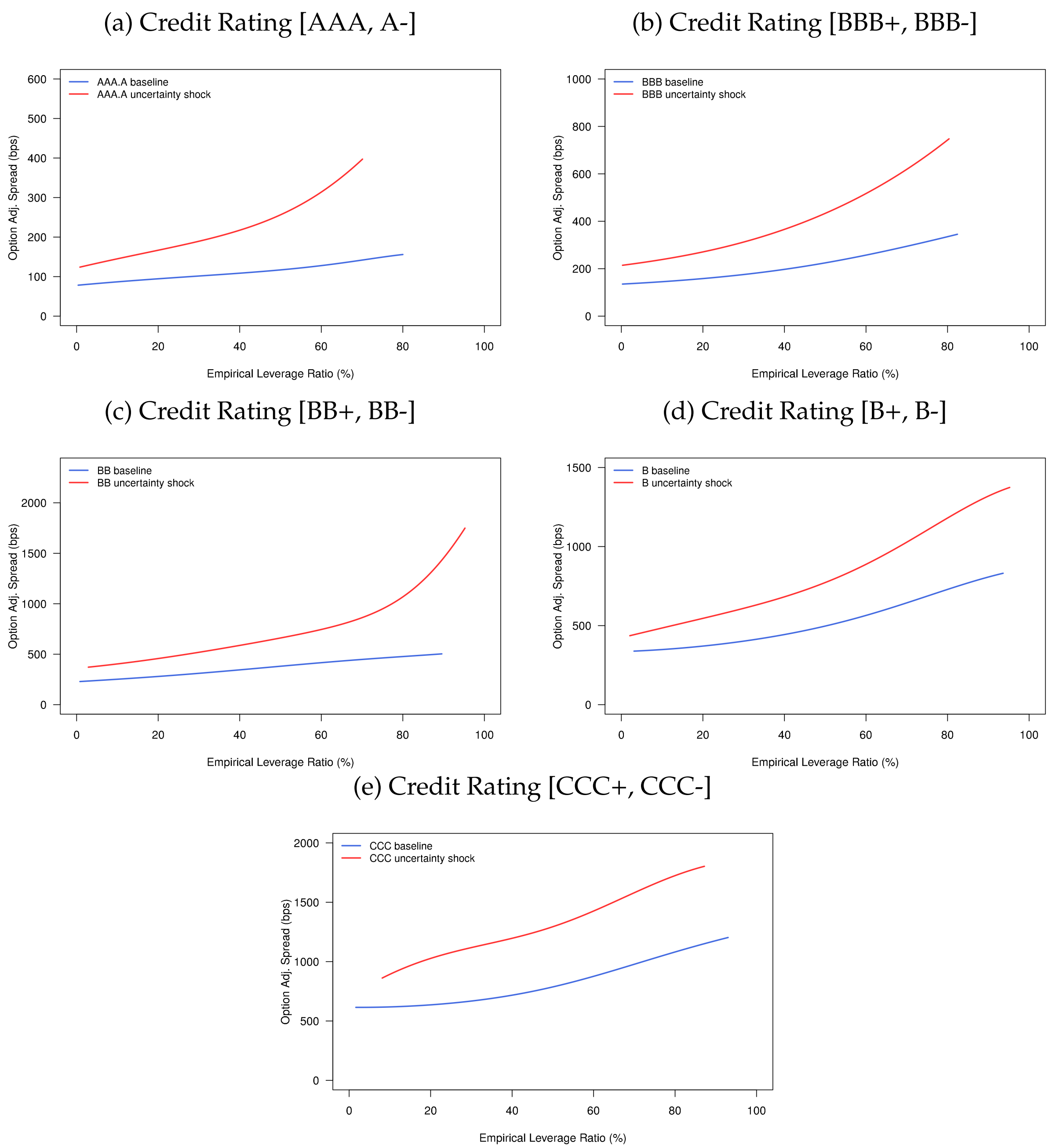
Figure 2 presents our estimates of these empirical-credit surfaces. The figure presents two credit surfaces for each of the five rating groups considered in our analysis. The two credit surfaces correspond to the fitted nonparametric regression for the days in the baseline regime and days when uncertainty shocks are realized. The estimated empirical credit surfaces in the baseline are depicted in blue and the surfaces corresponding to uncertainty shocks are depicted in red.

Our estimated credit surfaces appear convex. In the baseline uncertainty regime (blue curves), the convex pattern is apparent for bonds rated BBB+ to BBB-; B+ to B-; and CCC+ to CCC-. The convex pattern of the empirical-credit surface is most salient in the period identified as uncertainty shocks (red curves); however, for the riskier rating groups the fitted functions appear concave for large enough EL ratios. As the lowest rated firms have more volatile assets, our evidence suggests that for the highest volatility (low rated bonds during periods of uncertainty shocks) the empirical-credit surface might not be convex.[16]

Our estimated credit surfaces get steeper with uncertainty shocks, as depicted by the red curves. This pattern is very stark for bonds rated B- or above. For bonds rated CCC+ to CCC- the pattern is less stark, but these surfaces are estimated with less precision given the use of relatively fewer observations compared to the other rating groups. Even in this case, uncertainty shocks seem to widen credit spreads more for bonds with higher

[16]For estimates of the volatility of assets for firms conditional on their rating, see Choi and Richardson (2016).

Figure 2: Uncertainty Shocks and Bond-Credit Surfaces by Rating Groups



(a) Credit Rating [AAA, A-]

(b) Credit Rating [BBB+, BBB-]

(c) Credit Rating [BB+, BB-]

(d) Credit Rating [B+, B-]

(e) Credit Rating [CCC+, CCC-]

Notes: Empirical leverage (EL) ratio defined in equation (5). Uncertainty shocks correspond to months when the average level of the VIX is in the top decile of its historical distribution.
Source: Own elaboration based on ICE Data Pricing & Reference Data, LLC; S&P Global, Compustat; CRSP and Fang (2024).

leverage.

Our empirical analysis uses the EL ratio defined in equation (5), as opposed to the LTV or PTV ratios used in our theoretical analysis. To understand the bias introduced by our use of the EL ratio consider the stylized case of a firm financed with only debt and equity. In this case, we can write the theoretical ratios, as in the definition of the EL ratio in equation (5), using the market value of equity

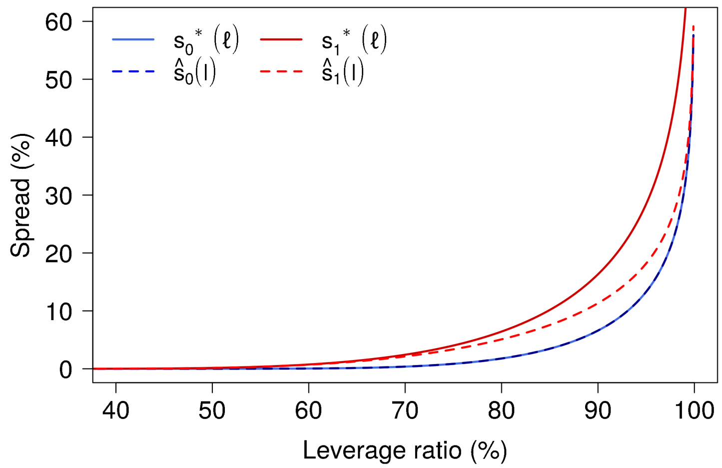
$$\mathrm{LTV}_t = \frac{d_{M,t}}{d_{M,t} + e_{M,t}} \qquad \text{and} \qquad \mathrm{PTV}_t = \frac{j_t}{d_{M,t} + e_{M,t}},$$

where j_t corresponds to the present value, in period t, of the promised payments.

Our theory makes two precise predictions: the loan credit surface is convex and it steepens for all LTV ratios with higher uncertainty. Neither prediction holds as generally or at all, for the PTV credit surface. We are only able to estimate the empirical credit surface, which may not have those properties to the extent that it differs from the loan credit surface.

When the market value of debt is close to its book value, then the empirical credit surface would be a good approximation of the loan credit surface in our illustrative example. In normal times, we might expect that to be the case. When uncertainty rises, the market value of debt falls below its book value, so the EL ratio overstates the true LTV ratio. Plotting the credit spread against EL rather than LTV therefore stretches each observation toward higher leverage, making the empirical-credit surface flatter at high leverage compared to the loan credit surface described in our theoretical results. The upshot is that our empirical test is conservative: had we been able to measure leverage at the market value of debt, our estimated credit surfaces would reveal more steepening under an increase in uncertainty than the EL-based evidence. We illustrate this with a numerical example, in which collateral is log-normally distributed with log-volatility σ_v. Figure 3 plots the LTV- and empirical-credit surfaces when uncertainty increases from $\sigma_0 = 0.2$ to $\sigma_1 = 0.5$. Our estimated credit surfaces lie between the baseline and stressed loan credit surfaces when uncertainty is high, with the wedge between the two becoming important only for high leverage. Therefore, our evidence understates the steepening the theory predicts for the loan credit surface.

Figure 3: LTV- and Empirical-Credit Surfaces and Uncertainty Shocks (Log-Normal Distributions)



Notes: Log-normal corresponds to $\log X_0 \sim N(-0.02, 0.2^2)$ and $\log X_1 \sim N(-0.045, 0.3^2)$. See Appendix A.

3 Uncertainty, Bond Pricing, and Credit Surfaces

In this section, we formally describe our simple model of collateral and bond pricing. In the model, uncertainty about collateral values is described by a probability distribution about those payoffs, denoted by F.

The economy has two periods $t = 0, 1$. In period $t = 0$, agents trade financial assets and risky credit contracts. Each contract is described by a promised amount $j \geq 0$ to be paid in period $t = 1$ and by the collateral backing the promise. These contracts can represent secured bonds or unsecured corporate bonds that are senior to equity. In general, financial contracts can specify other credit terms that influence their pricing. But we fix all these dimensions to focus on different levels of borrowing against the same collateral, representing different leverage ratios. We suppose that the collateral is the only enforcement mechanism on the contract, so the bond delivery or payment is always the minimum of the promise j and the collateral payoff x.

We assume that there is a single risk-neutral measure that determines the forward price of the collateral and all bonds backed by it. Further, we assume that all these bonds (but not necessarily the collateral) are valued today using the same discounting, denoted with r. In this section, we make no assumptions about the distribution of collateral values, F. It can be any nonnegative function defined on $[0, \infty)$ that is weakly increasing and continuous from the right, with $\lim_{x\to\infty} F(x) = 1$.[17]

Contracts that promise more or less than the possible values of collateral in any state are allowed, but to characterize them analytically it will be convenient to specify the support of the distribution F. We denote with $M_F = \inf\{x : F(x) = 1\}$ the upper bound of the support of F, which might be infinite. All contracts $j \geq M_F$ will have the same payout and price. In addition, we denote the lower bound of the support of F with $m_F = \inf\{x \geq 0 : F(x) > 0\}$. All contracts $j \leq m_F$ will be overcollateralized, so they will always deliver the promised amount. Since F is a key primitive of the model we use a subscript F to indicate the variables that are pinned down by F, but to avoid the notation burden whenever there is no ambiguity we omit this subscript in what follows.

In models with complete markets, like the archetypal model of debt contracts and collateral given by Merton's adaptation of the Black-Scholes option model, we shall interpret the risk neutral measure as the probability used to price both the bond and the collateral cash flows, r would be the common discount rate used to price all assets, and the price of the collateral would equal $p/(1 + r)$. In general equilibrium models with incomplete markets, like Geanakoplos (1997, 2010) or Fostel and Geanakoplos (2008), or Simsek (2013), the discounting and probability assessments may not coincide across different investors. In these models, for instance, the probability assessments pricing the bonds can reflect the belief of lenders, whereas the probability assessments pricing collateral assets might reflect the beliefs of asset buyers. In these models, we shall interpret the risk neutral measure in our analysis as the lender's probability assessments of the future cash flows of the bonds and the collateral. The price of the collateral could be higher than $p/(1 + r)$, reflecting a premium that may include risk premia and the collateral value of the asset.[18] By LTV we mean the ratio of the forward value of the bond to the forward value of the collateral, both evaluated with respect to the lender's beliefs. When lenders and borrowers discount those same forward values with the same interest rate, then LTV is also equal to the ratio of the current price of the loan to the current price of its collateral.

The Loan Pricing Function. The loan function introduced in equation (1) gives the forward loan amount (forward price) of a bond that promises j.[19] A direct property of

[17] It should be noted that throughout this paper "increasing" means nondecreasing and "decreasing" means nonincreasing, and we will say when increasing and decreasing relationships are strict. Finally, $c/0$ is taken to equal ∞ when $c > 0$.

[18] See Fostel and Geanakoplos (2008) for the definition of collateral value.

[19] We focus on the forward price instead of the current price for two reasons. First, the analysis is independent of the interest rate at which agents might discount the forward price down to the current price. Second, the forward values x and j are here treated as cash flows, so their forward valuations are not affected by other considerations. By contrast, the current value of the asset X might be increased because of the service it provides to agents as collateral for borrowing.

the loan function is that the $\lim_{j\to M} L(j) = p$. The next lemma describes the rich structure of the loan pricing function. Moreover, together with theorem below, these results show that the loan function fully reveals the distribution F.

Lemma 1 (Properties of the Loan Function) *Given any distribution F, with support in (m, M), the loan function L on $[0, \infty)$ is*

i. continuous and bounded, with $0 \le L(j) \le j$ for all $j \ge 0$ and $L(j) = j$ on $[0, m]$;

ii. concave;

iii. strictly increasing on $[0, M)$, and flat on $[M, \infty)$, with its right derivative $L'_+(0) \le 1$. If $M = \infty$ then $L'_+(j) \to 0$ as $j \to \infty$; and

iv. differentiable from the right (and the left), with its right derivative being right continuous and equal to $L'_+(j) \equiv 1 - F(j)$

The function L in equation (1) is increasing, as the sum of nonnegative numbers, and concave, as the sum of concave functions. The concavity of this function implies that its right (and left) derivatives exists everywhere and that the right-derivative is right continuous. Part *(iv)* formalizes the intuition that as the borrower increases the promise, the amount lent only increases by the probability the original promise is paid, i.e., the solvency probability.

Theorem 1 (Loan Function and Probability Distribution) *Any function L on $[0, \infty)$ satisfying (i)-(iii) in the Lemma is the loan function for collateral X that has a unique distribution F, recoverable from the formula*

$$F(j) = 1 - L'_+(j) \, .$$

The previous lemma allow us to provide sharp characterizations of bond prices and spreads and the promise function. Integrating condition *(iv)* in Lemma 1, and using that $L(0) = 0$, allows us to write the loan function directly in terms of F as

$$L(j) = \int_0^j (1 - F(x))\, dx \ = j - \int_0^j F(x)\, dx \, . \tag{6}$$

It follows that the *expected loss* of a contract that promises j can be written in terms of F as

$$\Lambda(j) \equiv \mathbb{E}[j - \min\{j, X\}] = \int_0^j F(x)\, dx \, . \tag{7}$$

Equation (7) is a "layer-cake" identity: $\Lambda(j) = \int_0^j F(x)\, dx$ accumulates the distribution function F horizontally, the exact counterpart of $L(j) = \int_0^j [1 - F(x)]\, dx$. Since $\min\{j, X\} +$

$(j - X)^+ = j$, these two horizontal integrals split the promise, $L(j) + \Lambda(j) = j$. This identity gives the credit surface an immediate interpretation. The spread is simply the expected loss per unit of loan value,

$$S(j) = \frac{j}{L(j)} - 1 = \frac{j - L(j)}{L(j)} = \frac{\Lambda(j)}{L(j)} \, .$$

The expected loss is thus the object that governs the shape of the credit surface: its level sets the spread, and, as we show in Section 5, the way its derivative responds to greater collateral uncertainty is what steepens the surface across the leverage dimension.

From Lemma 1 we have that, for any distribution of collateral values F, the loan function is strictly increasing on $[0, M)$ and thus it admits an inverse that we denote with $J_F : [0, p] \to [0, M]$, or simply J. This inverse function is continuous and bounded below with $J(0) = 0$, $J(\ell) \geq \ell$, for all $\ell \in [0, p)$ and $J(\ell) = \ell$, for all $\ell \in [0, L(m))$. It is strictly increasing and convex, with $J'_+(0) \geq 1$ and $J'_+(\ell) \to \infty$ as $\ell \to p$. Moreover, the promise function shall encode all the information in F about collateral payoffs.

Using the properties of L and its inverse J, we can say much about the credit surfaces. The promise credit surface is well defined in $(0, \infty)$ given that $L(j) > 0$ for all $j > 0$; continuous and non-negative, with $S(j) = 0$ for $0 \leq j \leq m$, $S(j) > 0$ for $j > m$ and $S(j) \to \infty$ as $j \to \infty$; strictly increasing on (m, M); and $S'_+(j) \to 1$ as $j \to M$. The promise credit surface is increasing because it is the ratio of a variable to a concave function of that variable that starts at the origin.

The loan credit surface is continuous and non-negative, with $S^*(\ell) = 0$ for $0 \leq \ell \leq L(m)$ and $S^*(\ell) > 0$ for $\ell > L(m)$; increasing on $(L(m), 1)$; and $S^{*\prime}_+(\ell) \to \infty$ as $\ell \to 1$. The ratio of a convex function that starts at the origin to the argument of the function is increasing, making the loan credit surface increasing.

We now state the credit surface counterpart of Theorem 1.

Corollary 1 (Credit Surfaces and Dispersion)

i. ***Promise Credit Surface.*** *Any nonnegative, right differentiable and increasing function S on $[0, \infty)$ such that $(1 + S(j))/j$ is decreasing, $j^2 S'_+(j)/(1 + S(j))^2$ is increasing and $S'_+(j) = 1$ for all $j > M < \infty$ or $S'_+(j) \to 1$ as $j \to \infty$ is the promise credit surface for collateral X that has a unique dispersion F^D, recoverable from the formula*

$$F^D(j) = 1 - \frac{1}{1 + S(j)} + \frac{j S'_+(j)}{(1 + S(j))^2} \, .$$

ii. ***Loan Credit Surface.*** *Any nonnegative, absolutely continuous, right differentiable and* *increasing function S^* on $[0,1)$ such that $\ell^2 S^{*\prime}_{+}(\ell)$ is increasing and $S^{*\prime}_{+}(\ell) \to \infty$ as $\ell \to 1$ is the loan credit surface for collateral X that has a unique dispersion F^D, recoverable from*

$$F^D(J(\ell)) = F^{D*}(\ell) = 1 - \frac{1}{1 + S^*(\ell) + \ell S^{*\prime}_{+}(\ell)} .$$

Corollary 1 is a direct consequence of Theorem 1 and the one-to-one relationship between F^D and F^{D*}.

Equity and Solvent Equity Pricing. The equity pricing function is the complement of the loan function: it gives the expected payoff to the equity holder after the bondholder is paid. Given a bond promise j, the forward equity price is given by the expression[20]

$$e(j) \equiv \mathbb{E}[(X-j)_+] = \int_j^\infty (x-j)\, dF(x) = \int_0^\infty x\, dF(x) - L(j) = p - L(j) .$$

If the expectation of F, $p_F \equiv \int_0^\infty x\, dF(x)$, is finite, then the equity pricing function also reveals the loan function, and therefore is another equivalent way of describing the same information as F. That is, any decreasing and convex equity function on $[0,\infty)$ such that $e'_+(0) \geq -1$ and $e'_+(j) \to 0$ as $j \to M$ is the equity pricing function for collateral that has a unique distribution function, which can be recovered from this pricing function.

Another function that will play a central role in our analysis is the *solvent equity value*

$$n(j) \equiv n_F(j) \equiv \begin{cases} \mathbb{E}[X-j \mid X>j] & \text{if } j<M \\ 0 & \text{otherwise} \end{cases} . \tag{8}$$

In reliability theory, $n(j)$ is called the mean residual life. For many common distributions, $n(j)$ is convex. It is constant for the exponential and linear for the uniform and power-law distributions (e.g., $n(j) = (T-j)/2$ for the uniform distribution on $[0,T]$), so convexity is immediate. For the normal and other distributions it is convex but not obviously so, and we establish this in Appendix C.

4 Leverage and the Shape of the Credit Surface

Section 3 established that credit surfaces are increasing in leverage: spreads rise as borrowers take on more risk. The question we address here is whether they are also convex,

[20] In incomplete market environments, the following expression corresponds to the pricing of equity using the risk neutral measure associated with F, or the lenders' probability assessments.

so that the cost of credit rises at an increasing rate with leverage. When collateral values are discrete, Fostel and Geanakoplos (2015) showed that the credit surface must be concave between collateral payoffs, not convex. The situation is very different for continuous distributions. We give sufficient conditions for the convexity of the credit surface that hold across almost all parameter values within twelve standard families of continuous distributions of collateral payoffs. Curiously, the workhorse log-normal distribution is the only one where convexity fails, when volatility is very high.

The curvature of the credit surface depends on the financial ratio used to measure leverage, and there is a natural hierarchy: the loan credit surface is convex under milder assumptions than other leverage ratios. In the case of the promise credit surface, this follows because $S^*(\ell) = S(J(\ell))$, with the promise function $J(\ell) = L^{-1}(\ell)$ convex in ℓ (Lemma 1). If S is increasing and convex, then S^* is convex, as the composition of an increasing convex function with a convex function.[21]

For this section we assume that F is continuously differentiable and that f' exists on (m, M), though f' need not be continuous. Because the support of X, $(m, M) \subseteq (0, \infty)$ is a convex set, the space of continuously differentiable functions $C^1((m, M))$ is well defined. These assumptions impose more regularity than is needed; we extend the main results below to the case of piecewise continuous (but not necessarily differentiable) density functions, without mass points in the appendix. To describe the more general case, the notation burden increases significantly, clouding the exposition, so for clarity we work with the stronger assumption and relegate the generalization to Appendix D.

4.1 The Shape of the Loan Credit Surface

The following theorem provides a list of progressively weaker conditions that are sufficient for S^* to be convex. Condition (*iv*) is a familiar and intuitive condition of increasing hazard rate on the dual distribution F^* (ILHR): each successive dollar raised reduces the remaining chance of solvency by a bigger percentage. It can be applied to all 12 standard continuous distributions that we consider.

Theorem 2 (The Convexity of the Loan Credit Surface) *Suppose that $F \in C^1((m, M))$, and that $f'(x)$ exists for all $x \in (m, M)$. Then, the loan credit surface $S^* : [0, 1) \to \mathbb{R}_+$, is convex provided that one of the following sufficient conditions holds:*

i. The density function f is log concave;

[21] Similarly, the mapping from LTV to the asset-to-equity ratio is (increasing and) convex, so increasing and convexity with respect to the asset-to-equity ratio implies convexity with respect to the LTV ratio. Thus, convexity of the loan credit surface obtains under milder assumptions than using the other leverage ratios.

ii. (IPHR) The promise hazard rate, $h(x) = f(x)/(1-F(x))$, is increasing in x,

$$h'_+(x) = \frac{1}{1-F(x)}\left(f'(x) + \frac{[f(x)]^2}{1-F(x)}\right) \geq 0 \; ;$$

iii. The solvent equity value, $n(j) = \mathbb{E}[X - j | X > j]$, is convex in j;

iv. (ILHR) The loan hazard rate, $h^*(\ell) = f^*(\ell)/(1-F^*(\ell))$, is increasing in ℓ, or equivalently for all $x \in (m, M)$;

$$h'_+(x) + [h(x)]^2 \geq 0 \quad \Leftrightarrow \quad f'_+(x) + \frac{2\,[f(x)]^2}{1-F(x)} \geq 0 \; ; \tag{9}$$

v. For all $x \in (m, M)$,

$$h'_+(x) + 2\,[h(x)]^2 \geq 0 \quad \Leftrightarrow \quad f'_+(x) + \frac{3\,[f(x)]^2}{1-F(x)} \geq 0 \; ; \tag{10}$$

vi. For all $\ell \in (0,1)$,

$$h^{*\prime}_+(\ell) + [h^*(\ell)]^2 \geq 0 \quad \Leftrightarrow \quad f^{*\prime}_+(\ell) + \frac{2\,[f^*(\ell)]^2}{1-F^*(\ell)} \geq 0 \quad \Leftrightarrow \quad J'''_+(\ell) \geq 0 \, . \tag{11}$$

Moreover, condition (i) implies (ii), (ii) and (iii) each imply (iv); condition (iv) implies (v) ⇔ (vi).

Conditions (*v*) and (*vi*) are the most general conditions; the former stated on the original distribution, the latter on the dual distribution, equivalently the promise function. Because they are strictly weaker than (*i*)–(*iv*), they admit cases such as the log-normal with high volatility, which violates the stronger conditions. Inequality (10) carries a direct economic reading: wherever the density is non-decreasing the inequality is automatic, since the lower bound on f'_+ is negative. Where the density is decreasing and positive, it requires that the elasticity of f exceeds minus three times the hazard rate; in short, the probability cannot decay faster than three times the insolvency rate.

The main steps of the proof are as follows. For LTV ratios $\ell \leq L(m)$, i.e., risk-free promises $J(\ell) \leq m$, it is easy to see that the loan credit surface S^* is flat (at zero) since $J(\ell) = \ell$, $S^*(\ell) = \frac{J(\ell)}{\ell} - 1 = \frac{\ell}{\ell} - 1 = 0$. Convexity then is immediate in this region.

For LTV values $\ell > L(m)$, i.e., risky promises $J(\ell) > m$, we work backward. Taking derivatives, it is straightforward to verify that

$$S^{*\prime\prime}_+(\ell) = \frac{J''_+(\ell)}{\ell} - \frac{2J'_+(\ell)}{\ell^2} + \frac{2J(\ell)}{\ell^3} \, .$$

After cancellations,

$$\frac{d^+}{d\ell}\left(\ell^3\, S_+^{*\prime\prime}(\ell)\right) = \ell^2\, J_+^{\prime\prime\prime}(\ell)\ .$$

It is easy to verify the boundary condition $b^3 \cdot S_+^{*\prime\prime}(b) \geq 0$ at $b = L(m)$;[22] together with condition (*vi*), $J_+^{\prime\prime\prime}(\ell) \geq 0$ for all $\ell \in (0,1)$, these conditions assure that $\ell^3\, S_+^{*\prime\prime}(\ell) \geq 0$ for all $\ell \in [L(m), 1)$. This guarantees that $S_+^{*\prime\prime}(\ell) \geq 0$ for all $\ell \in (0,1)$, that is the convexity of S^*.

The equivalences within (*iv*) and between (*v*) and (*vi*) let us recast conditions on the dual distribution F^* as conditions on the original distribution F. Both rest on the fact that the density $f^*(\ell)$ is given by the hazard rate $h(J(\ell))$ of F

$$f^*(\ell) = F_+^{*\prime}(\ell) = F'_+(J(\ell))J'_+(\ell) = \frac{f(J(\ell))}{1 - F(J(\ell))} = h(J(\ell))\ . \tag{12}$$

Two consequences follow. First, differentiating gives

$$h_+^{*\prime}(\ell) = \frac{h'_+(J(\ell))J'(\ell)}{1 - F(J(\ell))} + \frac{h(J(\ell))^2}{(1 - F(J(\ell)))^2} = \frac{h'_+(J(\ell)) + h(J(\ell))^2}{(1 - F(J(\ell)))^2}\ ,$$

so $h_+^{*\prime}(\ell) \geq 0$ for all $\ell \in (0,1)$ iff $h'_+(x) + h(x)^2 \geq 0$ for all $x \in (m, M)$, establishing the equivalence within (*iv*). Second, adding $h^*(\ell)^2$ to the expression above yields

$$h_+^{*\prime}(\ell) + h^*(\ell)^2 = \frac{h'_+(J(\ell)) + 2\, h(J(\ell))^2}{(1 - F(J(\ell)))^2}\ ,$$

so $h_+^{*\prime}(\ell) + h^*(\ell)^2 \geq 0$ for all ℓ iff $h'_+(x) + 2h(x)^2 \geq 0$, establishing (*v*)⇔(*vi*).

In addition, it follows from the previous results that

$$J_+^{\prime\prime\prime}(\ell) = \frac{1}{(1 - F^*(\ell))^2}\left(f_+^{*\prime}(\ell) + 2\frac{[f^*(\ell)]^2}{(1 - F^*(\ell))}\right) = \frac{1}{1 - F^*(\ell)}\left(h_+^{*\prime}(\ell) + [h^*(\ell)]^2\right)\ ,$$

giving the equivalences in (*vi*).

It is evident that condition (*iv*) is strictly more inclusive than condition (*ii*) and (*vi*) is more inclusive than (*iv*). That (*i*) implies (*ii*) is standard, though we give the proof in the appendix. The other implications and equivalences are handled in similar fashion in the

[22] Note that $J_+^{\prime\prime}(\ell) = f^*(\ell)/(1 - F^*(\ell))^2$. The boundary condition obtained by differentiating S^* from the right at $\ell = b \equiv L(m) > 0$, is then

$$S_+^{*\prime\prime}(b) = \frac{J_+^{\prime\prime}(b)}{b} - \frac{2J'_+(b)}{b^2} + \frac{2J(b)}{b^3} = \frac{f^*(b)}{b(1 - F^*(b))^2} - \frac{2}{b^2(1 - F^*(b))} + \frac{2b}{b^3} = \frac{f^*(b)}{b} - \frac{2}{b^2} + \frac{2}{b^2} = \frac{f^*(b)}{b} \geq 0\ ,$$

where we used that $J(b) = b$ and $F^*(b) = 0$. If $m = 0$, the same boundary condition holds in the limit, $\lim_{\ell \to 0^+} \ell^3 S_+^{*\prime\prime}(\ell) = 0$.

Table 2: Distributions Used in the Analysis, Their Densities, and Key Properties

Distribution	**Support**	**Parameters**	**Density** $f(x)^{\dagger}$	**Log-concave**	**IPHR**	**ILHR**	$n(j)$ **convex**
Normal†	(m,∞)	$\mu,\sigma>0$	$\frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$	✓	✓	✓	✓
Logistic†	(m,∞)	$\mu,s>0$	$\frac{e^{-(x-\mu)/s}}{s[1+e^{-(x-\mu)/s}]^2}$	✓	✓	✓	✓
Extreme Value†	(m,∞)	$\mu,\beta>0$	$\frac{1}{\beta}e^{-(x-\mu)/\beta}\exp\{-e^{-(x-\mu)/\beta}\}$	✓	✓	✓	✓
Laplace†	(m,∞)	$\mu,b>0$	$\frac{1}{2b}e^{-\lvert x-\mu\rvert/b}$	✓	✓	✓	✓
Exponential	$(0,\infty)$	$\lambda>0$	$\lambda e^{-\lambda x}$	✓	✓	✓	✓
Gamma	$(0,\infty)$	$\alpha,\beta>0$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$	$\alpha\geq 1$	$\alpha\geq 1$	$\alpha\geq 1$	$\alpha\geq 1$
Weibull	$(0,\infty)$	$k,\lambda>0$	$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k}$	$k\geq 1$	$k\geq 1$	$k\geq 1$	$k\geq 1$
Uniform	(m,M)	$0\leq m<M$	$\frac{1}{M-m}$	✓	✓	✓	✓
Beta (scaled)	$(0,M)$	$a,b>0;M>0$	$\frac{1}{MB(a,b)}\left(\frac{x}{M}\right)^{a-1}\left(1-\frac{x}{M}\right)^{b-1}$	$a,b\geq 1$	$a,b\geq 1$	$a,b\geq 1$	$a,b\geq 1$
Power Function	$(0,M)$	$\alpha>0;M>0$	$\frac{\alpha}{M}\left(\frac{x}{M}\right)^{\alpha-1}$	$\alpha\geq 1$	$\alpha\geq 1$	$\alpha\geq 1$	$\alpha\geq 1$
Power Law	(m,∞)	$m>0;\beta>1$	$(\beta-1)m^{\beta-1}x^{-\beta}$	×	×	$\beta>2$	$\beta>2$
Log-Normal	$(0,\infty)$	$\mu\in\mathbb{R},\sigma>0$	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-(\log x-\mu)^2/(2\sigma^2)}$	×	×	$\sigma<1.52$	×

†Truncated densities are left-truncated at $m\geq 0$ and normalized by $1-F(m)$ to integrate to 1 over (m,∞). Truncation preserves all four properties listed. Here φ is the standard normal density. IPHR = Increasing Promise Hazard Rate; ILHR = Increasing Loan Hazard Rate.

appendix.

For many statistical distributions it is possible to verify one of the conditions of Theorem 2. Table 2 presents a list of twelve standard distributions indicating which satisfies the four key properties of Theorem 2: log-concavity of the density, IPHR, ILHR, and convexity of the solvent equity value $n(j)$. Note that all of them satisfy ILHR with the parameters in the ranges reported there. Thus we have:

Corollary 2 (Standard Distributions and Convexity of the Loan Credit Surface) When collateral values are distributed according to any of the distributions listed in Table 2, with their indicated parameter restrictions, the loan credit surface is convex.

Log concavity of the density can be easily checked using the functional form of the density functions listed in Table 2. For the gamma and Weibull families, log-concavity holds when the shape parameter satisfies $\alpha\geq 1$ and $k\geq 1$, respectively; these conditions are equivalent to unimodality of the density, which in turn is equivalent to log-concavity. By contrast, the power law distribution is log-convex and has decreasing promise hazard rate, $h(x) = (\beta - 1)/x$. Yet for $\beta > 2$, it has increasing loan hazard rate because $h_+^{*\prime}(\ell) \geq (\beta - 1)(\beta - 2)/m^2 > 0$,[23] and so is convex, illustrating that ILHR does not require log-concavity or IPHR. The log-normal is the only family whose restriction bounds volatility.

We show in Appendix C that $n(j)$ is convex in nearly all the cases described in Table 2. For six of these distributions—normal, logistic, extreme value, Laplace, gamma ($\alpha \geq 1$), and Weibull ($k \geq 1$)—convexity of $n(j)$ follows from Lemma C.1, which establishes that IPHR together with a convex inverse hazard rate $1/h(j)$ implies $n''(j) \geq 0$ for distributions with unbounded support.[24] The remaining cases—uniform, exponential, power law, power function, and beta—are verified directly.

The corollary shows that for continuous distributions, convexity of the credit surface is to be expected. This stands in sharp contrast to Fostel and Geanakoplos (2015), who showed that the loan credit surface is concave between mass points of a discrete distribution. Our framework makes this discrepancy easy to explain. Any discrete distribution leaves an interval between two mass points on which the density vanishes, and on each such interval the surface is concave. Suppose F has no mass in (a_2, a_3) between points a_2 and a_3, but some mass on $(0, a_2]$ to the left of a_2 and some mass on (a_3, ∞) to the right of a_3. Then, for all $\ell \in (L(a_2), L(a_3))$ we have $f^*(\ell) = 0$ and $0 < F^*(\ell) < 1$ and $J(\ell)(1 - F^*(\ell)) < \ell$. In this region,

$$\ell^3 S_+^{*\prime\prime}(\ell) = \frac{f^*(\ell)\ell^2}{[1 - F^*(\ell)]^2} - \frac{2\ell}{1 - F^*(\ell)} + 2J(\ell) = -\frac{2[\ell - J(\ell)(1 - F^*(\ell))]}{1 - F^*(\ell)} < 0\,.$$

Thus, the loan credit surface is strictly concave, not convex, on the interval $(L(a_2), L(a_3))$.

When F has positive density throughout (m, M), such intervals of vanishing density do not arise; combined with $J'''(\ell) \geq 0$ and the boundary behavior at $\ell = m$—the conditions of Theorem 2—global convexity follows.

The parameter bounds on the first 11 distributions are set at the natural boundaries of the families, not fiddled to ensure convexity. The gamma and Weibull restrictions bind at the exponential distribution, the beta and power function restrictions at the uniform, and the power law restriction at finite variance. The log-normal is the only family in Table 2 whose restriction is not a natural boundary of the family. As the most famous distribution in finance, it bears investigation into exactly where it generates non-convexity. The next corollary of Theorem 2 describes this subset.

[23] To see this, note that $h_+^{*\prime}(\ell) = \frac{(\beta-1)(\beta-2)J(\ell)^{2(\beta-2)}}{m^{2(\beta-1)}} = \frac{(\beta-1)(\beta-2)}{m^2}\left(\frac{J(\ell)}{m}\right)^{2(\beta-2)}$ and use that $J(\ell) \geq m$.

[24] In the reliability-theory literature, the IPHR property is known as the Increasing Failure Rate (IFR) property. A well-known result in this literature, is that IFR implies decreasing mean residual life (DMRL), meaning $n'(j) \leq 0$, but not convexity of $n(j)$.

Corollary 3 (Log-Normal Distributions and Convexity of the Loan Credit Surface) *The loan credit surface, S^*, is convex everywhere, when X is distributed according to a log-normal distribution, with its logarithm having mean $\mu \in \mathbb{R}$ and standard deviation $0 < \sigma \leq \inf_{z\in\mathbb{R}} \{3\, h_N(z) - z\} \approx 1.86$.*[25] *Moreover, when $\sigma >\approx 1.86$, the loan credit surface is convex at every ℓ with $(\log J(\ell) - \mu)/\sigma \notin (-\sigma, \sigma/2)$.*

Corollary 3 shows that, when collateral values are assumed to be log-normally distributed, convexity fails when $(\log(J(\ell)) - \mu)/\sigma \in (-\sigma, \sigma/2)$ and $\sigma >\approx 1.86$. The critical value for σ is larger than the value reported in Table 2, reflecting the strictly stronger nature of the ILHR condition. The failure of convexity is of limited practical relevance. For one, the range of empirical estimates for the volatility of assets at annual frequency for public corporations, $[0.21, 0.66]$, lies well below the critical volatility $\sigma \approx 1.86$.[26] For another, even when we increase the volatility of assets to a level of $3 >> 1.86$, the non-convexity appears to be minimal, as shown in Panel (b) of Figure 4.

To illustrate the previous and subsequent results, we present numerical examples, using different parametric families of distributions to describe the uncertain collateral value, X. Our simulations consider four families of distributions: log-normal, normal, power-law and uniform. For each family we fix the mean at 1 and choose the remaining parameters so that the standard deviation is approximately $\sigma \approx 0.2$, facilitating comparison across distributions.[27] Panel (a) of Figure 4 depicts the loan credit surface for these four distribution functions. As predicted by Theorem 2 and Corollary 3 these credit surfaces are convex.

4.2 The Shape of the Promise Credit Surface

The promise credit surface is also of direct interest. Borrowers and lenders negotiate over the promised repayment, and the promise—not the loan—is the object of the structural credit-risk literature. We now turn to the convexity of this credit surface. The approach parallels the LTV case, now with $S(j) = j/L(j) - 1$.

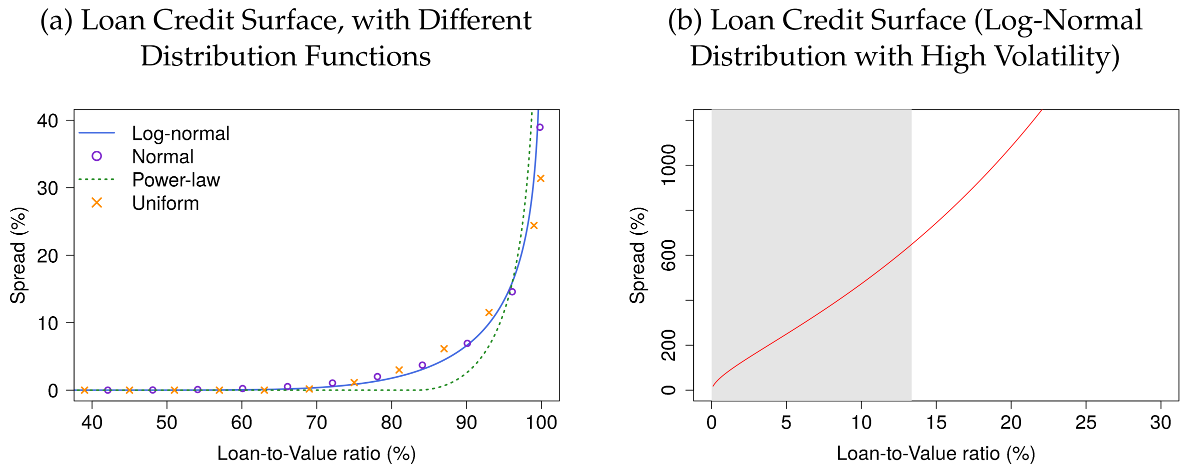
Theorem 3 (The Convexity of the Promise Credit Surface) *Suppose that $F \in C^1((m, M))$, and that $f'(x)$ exists for all $x \in (m, M)$. Then, the promise credit surface is convex provided that one of the following sufficient conditions holds:*

[25]The function h_N corresponds to the hazard rate of the standard normal. We obtain $\inf_{z\in\mathbb{R}} \{3\, h_N(z) - z\} \approx 1.86$ numerically, see Appendix F.1.

[26]For example, Choi and Richardson (2016) report estimates for the volatility of assets at the monthly frequency between 0.06 and 0.19 using different approaches and groups of firms by market leverage and rating groups. Adjusting those estimates to annual frequencies by multiplying by $\sqrt{12}$, yields estimates between 0.21 and 0.66.

[27]See Appendix A for additional details.

Figure 4: Numerical Simulations: Loan Credit Surfaces



Notes: Left panel distributions: log-normal with $\log X \sim N(-0.02, 0.2^2)$; normal with $X \sim N(0.9991, 0.2^2)$, truncated to the left at 3 standard deviations from the mean; power-law with $X \sim F^D$ induced by $F(x) = 1 - x^{-(\beta-1)}$ for $x \geq 1$ and zero otherwise with $\beta = 7$; and uniform with $X \sim U[0.65, 1.35]$. Right panel presents an example when the LTV-credit surface is not convex, using a log-normal with $\log X \sim N(-2.3, 3^2)$ (high volatility). Shaded area corresponds to LTV ℓ such that $(\log(J(\ell)) - \mu)/\sigma \in (-\sigma, \sigma/2)$. See Appendix A for additional details.

i. The density function f is log concave;

ii. (IPHR) The promise hazard rate, $h(x) = f(x)/(1 - F(x))$, is increasing in x;

iii. For all $j \in (m, M)$,

$$3\, f(j)\ [L(j) - j\,[1 - F(j)]] + j\, L(j)\, f'(j) \geq 0\,; \tag{13}$$

iv. For all $x \in (m, M)$,

$$h'(x) + \frac{1}{2}\,[h(x)]^2 = \frac{1}{1 - F(x)} \left[f'(x) + \frac{3\,[f(x)]^2}{2(1 - F(x))} \right] \geq 0\,. \tag{14}$$

Moreover, condition (i) implies (ii), and (ii) and (iii) imply (iv).

Following the same logic as for the loan credit surface, condition (*iv*) is the weakest sufficient condition for the convexity of the promise credit surface and is implied by IPHR. IPHR ensures that the density cannot decrease too fast, which in turn guarantees condition (*iii*).

In contrast to Theorem 2, the ILHR condition (increasing loan hazard rate, $h^{*\prime}(\ell) \geq 0$) is not sufficient for convexity of the promise credit surface. The key condition (*iv*) requires that $h' + \frac{1}{2}h^2 \geq 0$. IPHR ($h' \geq 0$) guarantees this is positive, but ILHR ($h' + h^2 \geq 0$) only yields $h' + \frac{1}{2}h^2 \geq -\frac{1}{2}h^2$, which may be negative. Thus the stronger IPHR condition is needed for the promise credit surface, while ILHR suffices for the loan credit surface.

The next corollary of Theorem 3 characterizes the region of the parameter space where the promise credit surface is convex for distributions that do not satisfy the IPHR condition.

Corollary 4 (Non-IPHR Distributions and Convexity of the Promise Credit Surface) *The promise credit surface S is convex everywhere, when X is distributed according to:*

i. A log-normal distribution, with its log having mean $\mu \in \mathbb{R}$ and standard deviation $\sigma \leq \inf_{z \in \mathbb{R}} \left\{ \frac{3}{2} h_N(z) - z \right\} \approx 1.19$;

ii. A Power-Law distribution, with density function $f(x) = (\beta - 1)m^{\beta-1}x^{-\beta}$ for $x \in (m, \infty)$, with $m > 0$ and $\beta \geq 3$.

Moreover, in the log-normal case, when $\sigma >\approx 1.19$, the promise credit surface is convex at every x with $z = (\log(x) - \mu)/\sigma \notin (-\sigma, 2\sigma)$.

Corollary 4 shows that the promise credit surface remains convex over a large region of the parameter space, though a strictly smaller one than for the loan credit surface. The requirements for log-normal and power-law families are more restrictive. For the log-normal, convexity fails only when $\sigma > 1.19$, as Panel (b) of Figure 5 illustrates. Panel (a) of Figure 5 depicts promise credit surfaces for volatilities close to 0.2, which are convex for all four families of distributions in our simulations, as Theorem 3 and Corollary 4 predict.

Finally, to facilitate the comparison of our results with the existing literature, we describe how our model nests the model presented in Merton (1974). Assume that $p = 1$ and X is distributed according to a log-normal distribution, with $\log(X) \sim N(-\sigma^2/2, \sigma^2)$, and denote by $z(x) = (\log(x) - \mu)/\sigma$. Then, a simple integration, yields a closed-form expression for the contract effective payout when there is default, $\mathbb{E}[X \mathbb{1}_{\{X \leq j\}}] = \Phi(z(j) - \sigma)$ (see equation (E.1)). Thus, we obtain the following closed-form expression for the loan pricing function[28]

$$L(j) = \mathbb{E}[X \mathbb{1}_{\{X \leq j\}}] + j[1 - F(j)] = \Phi(z(j) - \sigma) + j\,\Phi(-z(j)) \ , \tag{15}$$

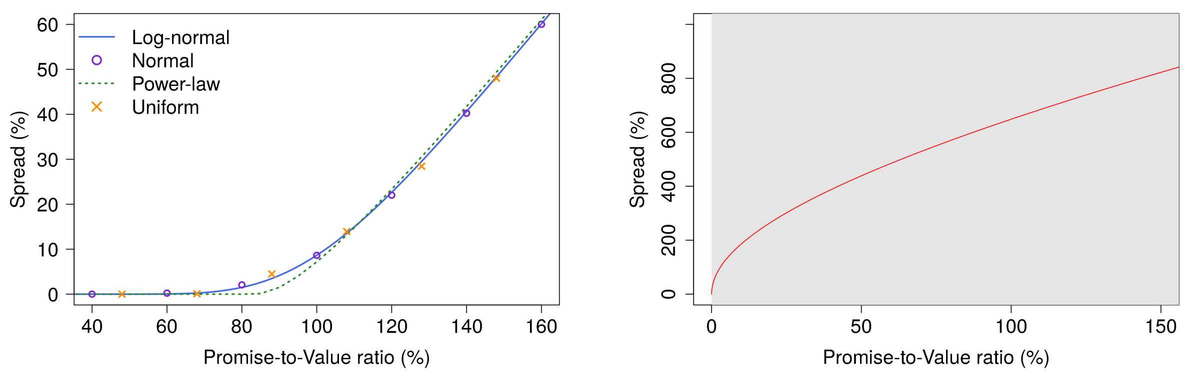
[28]Equation (15) recovers the bond price equation (13) of Merton (1974). In fact, considering a maturity $\tau = 1$, the functions h_1 and h_2 corresponds here to

$$h_1 = \frac{1}{\sigma}\left[\log(j) - \frac{\sigma^2}{2}\right] = z(j) - \sigma \quad \text{and} \quad h_2 = \frac{1}{\sigma}\left[-\log(j) - \frac{\sigma^2}{2}\right] = -z(j) \ .$$

Figure 5: Numerical Simulations: Promise-Credit Surfaces

(a) PTV-Credit Surface, with Different Distribution Functions

(b) Promise-Credit Surface (Log-Normal Distribution with High Volatility)



Notes: Left panel, represents log-normal with $\log X \sim N(-0.02, 0.2^2)$; normal with $X \sim N(0.9991, 0.2^2)$, truncated to the left at 3 standard deviations from the mean; power-law corresponds to $X \sim F(x) = 1 - x^{-(\beta-1)}$ for $x \geq 1$ and zero otherwise with $\beta = 7$; and uniform with $X \sim U[0.65, 1.35]$. Right panel, presents an example when the promise-credit surface is not convex, using a log-normal with $\log X \sim N(-2.3, 3^2)$ (high volatility). Shaded area corresponds to promises j such that $(\log(j) - \mu)/\sigma \in (-\sigma, 2\sigma)$. See Appendix A for additional details.

Merton (1974) never looked at the loan credit surface. He studied the logarithm of the promise credit surface, defining the "risk structure of interest rates," as $\log(j/L(j)) = \log(1 + S(j)) \approx S(j)$, and displaying a picture of it that was neither convex nor concave.

5 Uncertainty Shocks and Credit Surfaces

We characterize the uncertainty shocks that make the loan credit surface steeper. Steepening is transitive, so it defines a (new) stochastic order on dispersions F^D. We shall show that the steepening order is equivalent to a new but easily interpreted stochastic order on the dual distributions F^{D*}, which we call the harmonic hazard rate ordering.

The stochastic orders on F^{D*} that capture different kinds of uncertainty shocks are transparent, but sometimes difficult to verify from the given F^D, because F^{D*} is often a complicated transformation of F^D. However, we show that when a dispersion F_0^D (and its dual F_0^{D*}) can be differentiably transformed into another distribution F_1^D (and its dual F_1^{D*}), along a path F_v^D, F_v^{D*}, $0 \leq v \leq 1$, then these transparent orderings of the F_v^{D*} correspond to easy to verify properties of $\frac{\partial}{\partial v} F_v$.

Using this technique, we verify that the familiar hazard rate ordering on F_v^{D*} holds for all the standard parameterized continuous dispersions F_v^D, where in each case, v is the natural uncertainty parameter. Since the familiar hazard rate ordering is stronger than our new harmonic hazard rate ordering, increasing uncertainty steepens the credit surface for all the standard continuous distributions.

To formalize the differences in uncertainty we use the following definitions of stochastic orders.

Definition 1 (Hazard Rate Order) *The distribution F_0, is said to be larger than the distribution F_1 in the hazard rate order, written $F_0 \gtrsim_H F_1$, iff $[1 - F_0(x)]/[1 - F_1(x)]$ is increasing for all x in $(0, \max\{M_0, M_1\})$.*

When the distributions are absolutely continuous, this is equivalent to $f_1(x)/[1-F_1(x)] \geq f_0(x)/[1 - F_0(x)]$, for all x, hence the name of this order.

Definition 2 (Harmonic Hazard Rate Order) *The distribution F_0 is said to be larger than the distribution F_1 in the harmonic hazard rate order, written $F_0 \gtrsim_{HH} F_1$, iff $\int_0^x [1-F_1(u)]^{-1}\, du \Big/ \int_0^x [1-F_0(u)]^{-1}\, du$ is increasing for all x in $(0, \min\{M_0, M_1\})$.*

The harmonic hazard rate order asserts that the harmonic average solvency probability (over bond promises $u \in [0, x]$) of the secure collateral becomes progressively higher relative to the risky collateral, that is, $[\int_0^x [1 - F_0(u)]^{-1}\, du]^{-1} \Big/ [\int_0^x [1 - F_1(u)]^{-1}\, du]^{-1}$ is increasing in x.

Definition 3 (First Order) *The distribution F_0, is larger than the distribution F_1 in the first order, written $F_0 \gtrsim_1 F_1$, iff for any $x \in \mathbb{R}_+$, $F_0(x) \leq F_1(x)$.*

Definition 4 (Second Order and Mean-Preserving Spread) *The distribution F_0, is larger than the distribution F_1 in the second order, written $F_0 \gtrsim_2 F_1$, iff $\int_0^x F_0(z)\, dz \leq \int_0^x F_1(z)\, dz$ for all $x \in \mathbb{R}_+$.*

When $p_{F_0} = p_{F_1} = 1$, $F_0 \gtrsim_2 F_1$ is equivalent to $F_0 \gtrsim_{MPS} F_1$, where $\gtrsim_{MPS}$ denotes a mean preserving spread.

The introduction of our language of default enables us to interpret each in terms of solvency probabilities. For example, first order stochastic dominance $F_0(u) \leq F_1(u)$ means that secure (dominant) collateral 0 is more likely to remain solvent $1-F_0(x) \geq 1-F_1(x)$ than the risky (dominated) collateral no matter what bond it backs. Second order stochastic dominance means that the arithmetic average solvency probability (over bond promises $u \in [0, x]$) of the secure collateral is higher than the risky collateral $\int_0^x [1 - F_0(z)]\, dz \geq \int_0^x [1 - F_1(z)]\, dz$ for all $x \in \mathbb{R}_+$. The hazard rate ordering means that the ratio of these solvency probabilities increases in x; thus solvency is a stronger signal of a secure collateral the higher the promise x.

It is well known that[29]

$$\gtrsim_H \implies \gtrsim_{HH} \implies \gtrsim_1 \implies \gtrsim_2 .$$

As we said in the introduction, the first three stochastic orders defined above are useless in characterizing uncertainty shocks to dispersions F^D, because all of them become trivial since they all reduce the expectation. We therefore turn to the same stochastic orders on F^{D*}.

5.1 Loan Credit Surfaces and Uncertainty Shocks Characterized via the Dual Distribution F^{D*}

The next theorem characterizes how changes in uncertainty affect the loan credit surfaces $S_1^*(\ell)$ and $S_0^*(\ell)$ for high- and low-uncertainty environments. Each change in uncertainty corresponds to a stochastic order on the dual distribution F^{D*}, and we present these orders from weakest to strongest. In the next section we describe directly the stochastic orders on F^D that are induced by these orders on F^{D*}.

Theorem 4 (Uncertainty and Loan Credit Surfaces) *Suppose F_0^D and F_1^D represent two dispersion functions of collateral values, with duals F_0^{D*} and F_1^{D*}, then*

i. $F_1^D \lesssim_{MPS} F_0^D$ if and only if $J_1(\ell) - J_0(\ell)$ is positive if and only if $S_1^*(\ell) - S_0^*(\ell)$ is positive.

ii. $F_1^{D*} \lesssim_1 F_0^{D*}$ if and only if $J_1(\ell) - J_0(\ell)$ is positive and increasing. This implies that $S_1^*(\ell) - S_0^*(\ell)$ is positive.

iii. $F_1^{D*} \lesssim_{HH} F_0^{D*}$ if and only if $J_1(\ell)/J_0(\ell) = (1 + S_1^*(\ell))/(1 + S_0^*(\ell))$ is increasing (multiplicative steepening). This implies that $S_1^*(\ell) - S_0^*(\ell)$ is positive and increasing (additive steepening).

iv. $F_1^{D*} \lesssim_H F_0^{D*}$ if and only if $J_1'(\ell)/J_0'(\ell)$ is increasing. This implies multiplicative and additive steepening and that $J_1(\ell) - J_0(\ell)$ is positive, increasing, and convex.

Moreover, $F_1^{D*} \lesssim_H F_0^{D*} \Rightarrow F_1^{D*} \lesssim_{HH} F_0^{D*} \Rightarrow F_1^{D*} \lesssim_1 F_0^{D*} \Rightarrow F_1^D \lesssim_{MPS} F_0^D$.

The main conclusion of the theorem is that an increase in uncertainty according to the harmonic hazard rate ordering on the dual distribution F^{D*} is equivalent to a multiplicative steepening of the credit surface. For higher LTV loans, uncertainty pushes the spread up by

[29] The result that $\gtrsim_{HH}$ implies $\gtrsim_1$ might be less known, but is direct. The harmonic hazard rate order implies that the ratio of harmonic averages is increasing from 1, and thus at least 1 everywhere. Since the pointwise ratio $[1 - F_0(x)]/[1 - F_1(x)]$ must exceed the ratio of harmonic averages for the latter to be increasing, the pointwise ratio is also at least 1 everywhere.

more. The proof is that since $J_i(\ell) = \int_0^\ell [1-F_i^{D*}(u)]^{-1}\,du$, the ratio $J_1/J_0 = (1+S_1^*(\ell))/(1+S_0^*(\ell))$ is increasing precisely when $F_1^{D*} \lesssim_{HH} F_0^{D*}$.

The last part of the theorem reminds us that we have ordered the hypotheses (and therefore the consequences) of (*i*)-(*iv*) from weakest to strongest. The last comparison $F_0^{D*} \gtrsim_1 F_1^{D*} \Rightarrow F_0^D \gtrsim_{MPS} F_1^D$ shows that first order dominance on F^{D*} induces second order dominance on F^D. First order dominance gives $J_1'(\ell) = \frac{1}{1-F_1^{D*}(\ell)} \geq \frac{1}{1-F_0^{D*}(\ell)} = J_0'(\ell)$ (which proves (*ii*)), so $J_1(\ell) - J_0(\ell)$ is increasing from 0, hence nonnegative, which is the MPS condition (part (*i*)). This in turn implies that both orders $\lesssim_H$ and $\lesssim_{HH}$ on the dual F^{D*} induce mean preserving spreads on X.

Conditions (*ii*)-(*iv*) in Theorem 4 holds for all our parametric families, and they say something intuitive about which collateral is more likely to remain solvent. Let X_0 and X_1 share the same expected forward value, with X_1 the more uncertain in the sense of (*ii*)-(*iv*). Raise the same amount ℓ against each, i.e., the borrower promises $J_0(\ell)$ or $J_1(\ell)$ and defaults whenever the collateral falls short of the promise. Draw one of the two loans, equally likely, and learn only that it did not default. The probability that it is the safe loan, backed by X_0, is

$$\frac{1-F_0^{D*}(\ell)}{[1-F_0^{D*}(\ell)]+[1-F_1^{D*}(\ell)]} = \frac{1}{1+[1-F_1^{D*}(\ell)]/[1-F_0^{D*}(\ell)]}\,. \tag{16}$$

Condition (*ii*) says (16) is greater than 1/2: the safe collateral is more likely to remain solvent, because the risky collateral defaults more often at every loan size. Condition (*iv*) says (16) rises with ℓ: at higher leverage, solvency is a clearer signal that the collateral is the safer one. Condition (*iii*) is the intermediate, multiplicative version. All three conditions are stronger than $F_1^D \lesssim_{MPS} F_0^D$, condition (*i*). Under (*i*) alone the riskier collateral can be more likely to remain solvent, since spreading both tails might lower the default rate. Conditions (*ii*)-(*iv*) rule this out: more uncertainty always raises the chance of bankruptcy, the more so as the loan grows.

5.2 Credit-Surface Orders as Stochastic Orders on Dispersions

We have established that the loan credit surfaces correspond 1-1 with dispersions F^D. Any transitive partial order on credit surfaces therefore induces a potentially new stochastic order on dispersions.[30] For example, if one credit surface always lies above another credit

[30] The closest antecedent to the stronger orders we introduce for dispersions is the generalized TTT family of Li and Shaked (2007), which compares $\int^{F^{-1}(q)} \phi(F(x))dx$ pointwise across quantiles q for a fixed weight ϕ. The choice $\phi(u) = 1/(1-u)$ reproduces the integrand $\int dx/(1-F)$ of our new harmonic hazard rate order,

surface, then they are ordered in the *credit surface level order*. Theorem 4 part (*i*) says that the loan credit surface shifts up if and only if the dispersion of collateral values spreads in a mean-preserving way; in other words, the credit surface level order is identical to the mean preserving spread order on F^D. So the credit surface gives a vivid picture of a well known stochastic order.

The *multiplicative credit surface steepening order* in Theorem 4 part (*iii*) is obviously transitive, so it induces a new order on F^D, and hence a new definition of more uncertainty on dispersions. Since all credit surfaces start at the same point (0 credit spread when LTV is 0), a steepening necessarily raises the credit surface. We see vividly that credit surface steepening must be a special kind of mean preserving spread. In Section 5.4 we shall show that when the uncertainty parameter is increased on all the standard parameterized continuous distributions, the credit surface steepens, and not just rises. Hence steepening, more precisely than mean preserving spread, may be the natural definition of more uncertainty for continuous distributions.

There are to be sure variants of multiplicative steepening, such as additive steepening, or accelerating multiplicative steepening. Each of these induce a (slightly different) stochastic order on F^D.

5.3 Characterizing Differentiable Orders on F^{D*} in terms of F^D

Theorem 4 characterized the steepening of the loan credit surface via stochastic orders on F^{D*}. We now find equivalent conditions on F^D that yield the same steepening conclusions, provided that we can represent changes in uncertainty "differentiably". We assume that we can write $F^D_v(x)$ as $F^D(x, v)$, where $v \in [0, 1]$ represents the degree of uncertainty, F^D is differentiable in its second argument, and x varies over an interval (m_v, M_v) with end-points that are continuously differentiable in v. We call this a *differentiable family of dispersions*. All the standard parameterized continuous distributions satisfy these differentiability conditions, and they all have a natural parameter v that corresponds to uncertainty.

Define the function

$$K(j, v) \equiv \frac{1}{1 - F^D(j, v)} \int_{m_v}^{j} \frac{\partial F^D(x, v)}{\partial v}\, dx\ . \tag{17}$$

The function K depends on the family of distributions $F^D(x, v)$ directly, and not on the dual distributions. Properties of K can be verified directly from the functions F^D, as

but their comparison is pointwise at common quantiles, whereas ours is a monotone-ratio comparison at common loan values ℓ.

we shall see in the next subsection. Recall that $\int_{m_v}^{j} F^D(x,v)dx$ is the expected loss from promising j when uncertainty is at v. The derivative $\frac{\partial}{\partial v}\int_{m_v}^{j} F^D(x,v)dx = \int_{m_v}^{j} \frac{\partial F^D(x,v)}{\partial v}dx$ gives the loss in value when uncertainty increases.[31] The expression $1 - F^D(j,v)$ represents the gain in bond value per unit increase in the bond promise j. Thus changing the promise by an infinitesimal $K(j,v)$ in response to an infinitesimal unit increase in v leaves the bond value unchanged. In other words

$$\frac{\partial J(\ell,v)}{\partial v} = K(J(\ell,v),v)\ .$$

Theorem 5 (Return Uncertainty and Loan Credit Surfaces) *Let F_v^D be a differentiable family of dispersions. If the conditions below must hold for all $0 \le t \le v \le u \le 1$, then*

i. $F_u^D \lesssim_{MPS} F_t^D$, if and only if $K(j,v) \ge 0$ for all j, if and only if $J_u(\ell) - J_t(\ell)$ is positive for all ℓ, if and only if $S_u^*(\ell) - S_t^*(\ell)$ is positive for all ℓ.

ii. $F_u^{D*} \lesssim_1 F_t^{D*}$ if and only if $K(j,v) \ge 0$ is increasing in j if and only if $J_u(\ell) - J_t(\ell)$ is positive and increasing in ℓ. This implies that $S_u^*(\ell) - S_t^*(\ell)$ is positive for all ℓ.

iii. $F_u^{D*} \lesssim_{HH} F_t^{D*}$ if and only if $K(j,v)/j$ is increasing in j if and only if $J_u(\ell)/J_t(\ell) = (1 + S_u^*(\ell))/(1+S_t^*(\ell))$ is increasing in ℓ (multiplicative steepening). This implies that $S_u^*(\ell)-S_t^*(\ell)$ is positive and increasing in ℓ (additive steepening).

iv. $F_u^{D*} \lesssim_H F_t^{D*}$ if and only if $K(j,v)$ is convex in j if and only if $J_u'(\ell)/J_t'(\ell)$ is increasing in ℓ. This implies multiplicative and additive steepening and that $J_u(\ell) - J_t(\ell)$ is positive, increasing, and convex in ℓ.

To give an idea of the proof, consider first the central case of multiplicative steepening in part (*iii*). Using subscripts to denote partial derivatives to save on space, observe that

$$\frac{d}{d\ell}[\partial_v log(J(\ell,v))] = \frac{d}{d\ell}\left[\frac{J_v}{J}\right] = \frac{d}{d\ell}\left[\frac{K}{J}\right] = \frac{K_j J_\ell}{J} - \frac{K J_\ell}{J^2} = \frac{J_\ell}{J}\left[K_j - \frac{K}{J}\right]\ .$$

Note that $[K_j - K/J] \ge 0$ if and only if K/J is increasing in j. Furthermore, $J_u(\ell)/J_t(\ell)$ is increasing in ℓ for all $u > t$ if and only if $\frac{d}{d\ell}[\partial_v log(J(\ell,v))] = \partial_v[\frac{d}{d\ell}log(J(\ell,v))] \ge 0$.

Next consider the case (*iv*). Observe that

$$\frac{d}{d\ell}[\partial_v log(J_\ell(\ell,v))] = \frac{d}{d\ell}\left[\frac{J_{\ell v}}{J_\ell}\right] = \frac{d}{d\ell}\left[\frac{K_j J_\ell}{J_\ell}\right] = K_{jj}J_\ell\ .$$

[31] We use the fact that $F^D(m_v,v) = 0$.

This is nonnegative if and only $K_{jj} \geq 0$, that is if and only if K is convex in j. Also $\frac{d}{d\ell}[\partial_v log(J_\ell(\ell, v))] \geq 0$ if and only if $J'_u(\ell)/J'_t(\ell)$ is increasing in ℓ for all $u > t$.

5.4 Uncertainty Steepens the Loan Credit Surface for the Standard Distributions

We now apply Theorem 5 to the same families of distributions analyzed in Section 4. All of them satisfy K-convexity, and hence the natural increase in uncertainty creates a new distribution in which $F_0^{D*} \gtrsim_H F_1^{D*}$, so the credit surface steepens in every case. To verify the K-convexity, we divide the parametric distributions into three types, as summarized in Table 3. The first type corresponds to location-scale families, like the normal, logistic, extreme value, and Laplace, and uniform distributions with expanding support. In these cases, increasing the scale parameter while preserving the mean produces a proportional mean-preserving spread.

Definition 5 (Proportional Mean-Preserving Spreads) *Let $X \sim F \in C^1((m, M))$ be an integrable random variable with no atoms and mean $\mathbb{E}[X] = p > 0$, and let $v > 0$ be a scale parameter. We say that X_v is a proportional mean-preserving spread (PMPS) of X of scale v if $X_v = p+v(X-p)$.*

It is straightforward to verify that a PMPS $X_v = p + v(X - p)$ of X is a MPS.[32] Moreover, its distribution satisfies $F_v(x) = F((x-p)/v+p)$, which is smooth in v for all x in the interior of the support.

The next theorem shows that for PMPS, the K-convexity condition of Theorem 5 reduces to a simple and interpretable property of the distribution: convexity of the solvent equity value $n(j)$.

Theorem 6 (Proportional Mean-Preserving Spreads Steepens the Loan Credit Surface) *Suppose that X_v is a PMPS of X. Given X, let $n(x)$ denote the solvent value of equity. Then $v\, K_{jj}(j) = n''(p + (j - p)/v)$. Thus, when n is convex, the loan credit surface steepens everywhere when the scale v increases.*

Theorem 6 says that when the solvent equity value $n(j)$ is convex, a PMPS steepens the loan credit surface. For location-scale families, increasing the scale parameter while adjusting the location to preserve the mean produces a PMPS. Theorem 6 then applies directly: because each of these distributions has a convex solvent equity value $n(j)$ (Table

[32]Define the random variable Y, where conditional on $X = x$, with probability $\frac{1}{v}$,$Y = v(x-p)+p$, and with probability $(1 - \frac{1}{v})$, Y is distributed like X_v. Then $\mathbb{E}[Y|X = x] = \frac{1}{v}[v(x - p) + p] + (1 - \frac{1}{v})p = x$. And clearly with probability $\frac{1}{v}$, Y is distributed like X_v, and with probability $1 - \frac{1}{v}$, Y is also distributed like X_v. Thus Y and $X_v$are mean preserving spreads of X.

Table 3: Distributions for which uncertainty steepens the credit surface.

Distribution	Uncertainty parameter	Direction	Mean preservation	PMPS	K convex
Uniform $[m_v, M_v]$	$M_v - m_v$	$M_0 - m_0 < M_1 - m_1$	$(m_v + M_v)/2 = p$	✓	✓
Normal $(\mu_v, \sigma_v)^\dagger$	σ_v	$\sigma_0 < \sigma_1$	$c_F(\delta) = \varphi(\delta)/\Phi(\delta)$	✓	✓
Logistic $(\mu_v, s_v)^\dagger$	s_v	$s_0 < s_1$	$c_F(\delta) = (1+e^\delta)\log(1+e^\delta)/e^\delta - \delta$	✓	✓
Extreme Value $(\mu_v, \beta_v)^\dagger$	β_v	$\beta_0 < \beta_1$	$c_F(\delta) = c_{EV}(\delta)$	✓	✓
Laplace $(\mu_v, b_v)^\dagger$	b_v	$b_0 < b_1$	$c_F(\delta) = (1+\delta)/(2e^\delta - 1)$	✓	✓
Log-Normal (μ, v)	v	$v_0 < v_1$	$\mu(v) = \log p - \frac{1}{2}v^2$	×	✓
Power Law (β_v)	β_v	$\beta_0 > \beta_1 > 2$	adjust m_β s.t. $\mathbb{E}[X_{\beta_v}] = p$	×	✓
Gamma (α_v, β_v)	α_v	$\alpha_0 > \alpha_1 \geq 1$	$\beta_v = \alpha_v/p$	×	$\alpha_v \geq 1$
Weibull (k_v, λ_v)	k_v	$k_0 > k_1 \geq 1$	$\lambda_v = p/\Gamma(1 + 1/k_v)$	×	$k_v \geq 1$
Power Function (α_v, M_v)	α_v	$\alpha_0 > \alpha_1 \geq 1$	$M_v = p(\alpha_v + 1)/\alpha_v$	×	$\alpha_v \geq 1$
Beta (a_v, b_v, M_v)	a_v, b_v	$a_0 > a_1 \geq 1$, $b = b_0 = b_1$	$M_v = p(a_v + b)/a_v$	×	$a_v, b \geq 1$

†Truncated location-scale families. The left truncation point is $a_v = \mu_v - \delta\,\sigma_v$, held at δ standard deviations below the mean (σ_v here denotes the family's scale parameter: σ_v for Normal, s_v for Logistic, β_v for Extreme Value, b_v for Laplace). Mean preservation requires $\mu_v = p - \sigma_v\, c_F(\delta)$, where $c_F(\delta) = \mathbb{E}[W \mid W > -\delta]$ is the standardized truncated mean of the base variable W. For Extreme Value, $c_{EV}(\delta)$ has no elementary closed form.
For details about these distributions see Table 2. Direction gives the inequality on the uncertainty parameter that yields an MPS, holding the mean fixed at p. ✓ = property satisfied for all parameter values; × = property fails.

2), the loan credit surface steepens.[33] The uniform distribution likewise admits a PMPS (widening the support symmetrically about the mean) and has a convex solvent equity value, so it falls into the same category.

The second type covers other important families—the log-normal, power law, Gamma,

[33]Since collateral values must be non-negative, these distributions are truncated on the left at $a_v = \mu_v - \delta\sigma_v$, where δ is held fixed so that the truncation point remains a constant number of standard deviations below the mean. Because the truncation point moves with the scale parameter, the truncated family remains a location-scale family with base distribution $W = Z \mid Z > -\delta$, so increasing σ_v is an exact PMPS of the truncated distribution. The location parameter μ_v is then adjusted to preserve the mean p. For the normal, this requires $\mu_v = p - \sigma_v\,\varphi(\delta)/\Phi(\delta)$, where φ and Φ are the standard normal density and CDF; analogous adjustments apply to the other location-scale families.

Weibull, power function, and Beta. In these cases, there is a natural parameter associated with uncertainty as listed in Table 3, but the resulting mean-preserving spread is not proportional. We therefore verify K-convexity directly, case by case in the next corollary.[34]

Corollary 5 (Standard Distributions Steepen the Loan Credit Surface) *The loan credit surface steepens when the uncertainty parameter is increased in the natural way to hold the mean constant, for the following families of distributions, because each of them displays K-convexity:*

i. Beta distribution on $[0, M]$ *with parameters* $a, b \geq 1$ *and* $M = p(a+b)/a$*, for* $a_0 > a_1 \geq 1$ *at fixed b;*

ii. Gamma distribution, $X \sim \Gamma(\alpha, \alpha/p)$ *so that* $\mathbb{E}[X] = p$*, for* $\alpha_0 > \alpha_1 \geq 1$*;*

iii. Log-normal distribution, with $\log X_v \sim N(\mu(v), v^2)$ *and* $\mu(v) = \log p - \frac{1}{2}v^2$*, for* $v_0 < v_1$*;*

iv. Power function distribution on $[0, M]$ *with shape* α *and* $M = p(\alpha+1)/\alpha$*, for* $\alpha_0 > \alpha_1 \geq 1$*;*

v. Power law distribution, with density $f(x) = (\beta - 1)m_\beta^{\beta-1}x^{-\beta}$ *for* $x \geq m_\beta$*, mean-preserving normalization* $\mathbb{E}[X_\beta] = p$*, for* $\beta_0 > \beta_1 > 2$*; and*

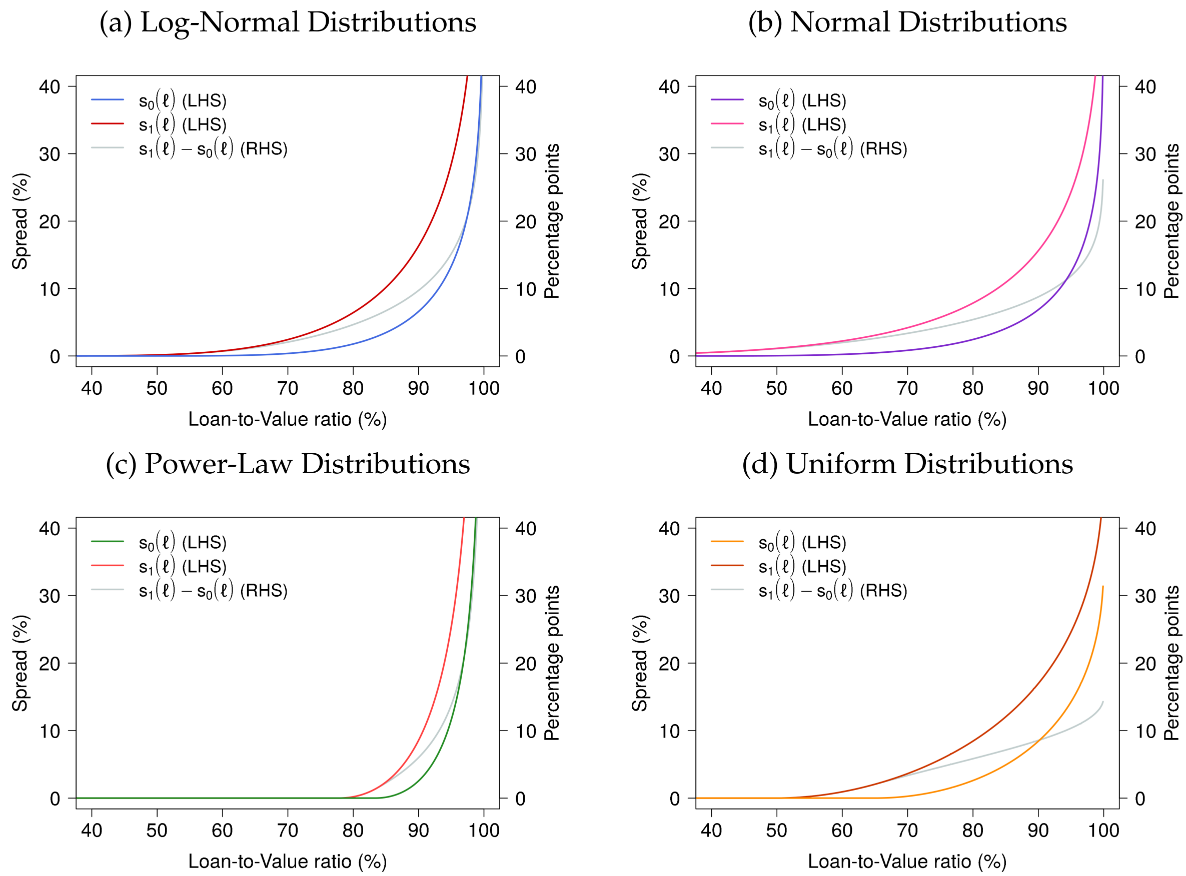
vi. Weibull distribution, $X \sim$ *Weibull*(k, λ) *with* $\lambda = p/\Gamma(1 + 1/k)$ *so that* $\mathbb{E}[X] = p$*, for* $k_0 > k_1 \geq 1$.

Finally, for the exponential distribution (the third type), fixing the mean determines the single free parameter, so no mean-preserving spread is admissible within the family. It is a pure scale family; changing λ changes the expected collateral value $p = 1/\lambda$ but leaves the dispersion distribution—and hence the credit surface—unchanged. The exponential can be viewed as the limiting case of the gamma family (shape $\alpha \to 1^+$) or the Weibull family (shape $k \to 1^+$), representing the minimum-uncertainty member within each. Within both families, moving the shape parameter away from this boundary increases uncertainty and steepens the credit surface.

To illustrate the previous statements, we continue to use the numerical examples described in more detail in Appendix A. In the case of the log-normal family of distributions, we consider an increase in the volatility of its logarithm from $\sigma_0 = 0.2$ to $\sigma_1 = 0.3$. In the case of the normal family of distributions, we consider an increase in volatility from $\sigma_0 = 0.2$ to $\sigma_1 = 0.3$. In the case of the power-law family, we use X_0 and X_1, with support in $[1, \infty)$ and coefficients $\beta_0 = 7$ and $\beta_1 = 5.5$. Finally, in the case of the uniform family of distributions, we consider an increase in the support of the distribution with $X_0 \sim U[0.65, 1.35]$ and

[34]We note that we also established the steepening result for some families of distributions by establishing the log-submodularity of the dual dispersion $F^D(\ell, v)$. Once a parametric family is chosen, explicit differentiation allows us to establish the log-submodularity.

Figure 6: Loan Credit Surfaces and Uncertainty Shocks with Different Distribution Families



Notes: Log-normal corresponds to $\log X_0 \sim N(-0.02, 0.2^2)$ and $\log X_1 \sim N(-0.045, 0.3^2)$; normal corresponds to $X_0 \sim N(0.9991, 0.2^2)$ and $X_1 \sim N(0.9987, 0.3^2)$, truncated to the left at 3 standard deviations from the mean; power-law corresponds to X_0 and X_1, where $X_v \sim F_v(x) = 1 - x^{-(\beta_v - 1)}$ for $x \geq 1$ and zero otherwise and $\beta_0 = 7$ and $\beta_1 = 5.5$; and uniform corresponds to $X_0 \sim U[0.65, 1.35]$ and $X_1 = U[0.5, 1.5]$. See Appendix A.

$X_1 \sim U[0.5, 1.5]$. Figure 6 shows that for the four families of distributions the difference between the LTV-surfaces, $S_1(\ell) - S_0(\ell)$, is increasing.

5.5 Promise Credit Surfaces and Uncertainty Shocks

The next theorem characterizes the relationship between uncertainty shocks and the promise credit surface.

Theorem 7 (Uncertainty and Promise Credit Surfaces) *Suppose F_0 and F_1 (F_0^D and F_1^D) represent two distributions (dispersions) of collateral values, then*

i. $F_1^D \lesssim_{MPS} F_0^D$ if and only if $L_0(j) - L_1(j)$ is positive if and only if $S_1(j) - S_0(j)$ is positive;

ii. $F_1 \lesssim_1 F_0$ *if and only if* $L_0(j) - L_1(j)$ *is positive and increasing in* j*. This implies* $S_1(j) - S_0(j)$ *is positive for all* j*;*

iii. $F_1 \lesssim_H F_0$ *if and only if* $L_0'(j)/L_1'(j)$ *is increasing in* j*. This implies* $S_1(j) - S_0(j)$ *is positive for all* j *and* $L_0(j) - L_1(j)$ *is positive, increasing, and convex in* j*.*

iv. If $j \to \infty$ *then* $S_1(j)/S_0(j) \to 1$*.*

Moreover, when $F_1^D \lesssim_{MPS} F_0^D$ *or* $F_1 \lesssim_1 F_0$ *or* $F_1 \lesssim_H F_0$ *then* $S_1(j) - S_0(j)$ *is hump shaped.*

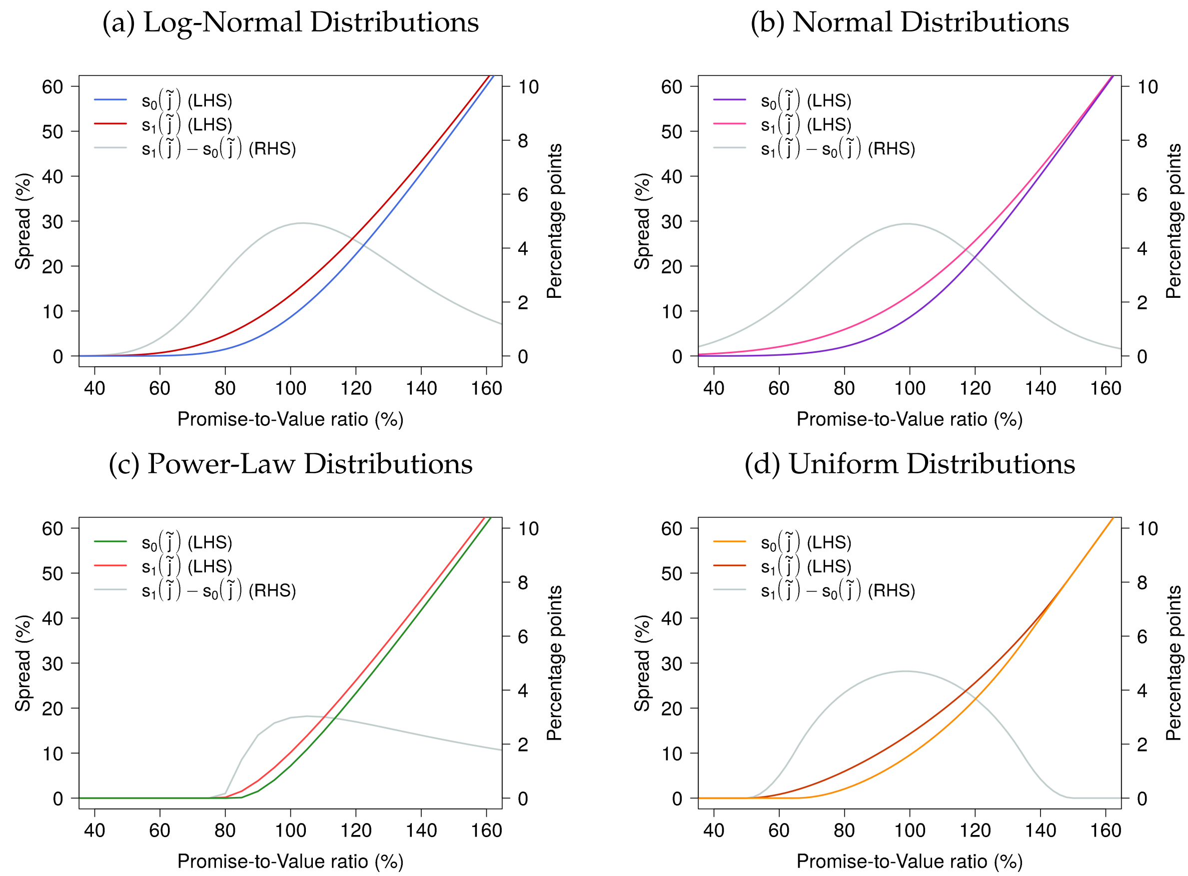
The comparison between Theorems 4 and 7 reveals that both surfaces move up in response to an uncertainty shock, but there is a stark difference in the effect of uncertainty shocks on the steepness of the surface, depending on the ratio used to measure leverage. The loan credit surface steepens for any LTV ratio, whereas the promise credit surface steepens only for small enough promises and for large enough promises it flattens, as the promise credit surface is unaffected for large enough promises. In other words, the change in the promise credit surface generated by uncertainty shock is hump-shaped.

Note that when X_v is distributed according to any of the standard distributions listed in Table 3, like the log-normal, normal, or power-law distributions, we have that $F_1^D \lesssim_{MPS} F_0^D$ and so by Theorem 7 the change in the promise credit surface is hump shaped. Figure 7 presents for the four families of distributions considered in our simulations the difference between the promise credit surfaces, $S_1(y) - S_0(y)$. Our simulations show that this difference is hump-shaped, increasing for low PTV ratios and then decreasing for PTV ratios above 1.

6 Conclusions

In this paper, we present historical evidence showing that the bond-credit surface not only moves up, but also steepens with economic uncertainty. In addition, we present a simple model of collateral and equilibrium to explore the general shape of the credit surface along the leverage dimension and its relationship with economic uncertainty. We show that bond prices fully reveal the distribution of excess returns to collateral, so the credit surface is a sufficient statistic for collateral uncertainty. The credit surface is convex across almost all parameter values within twelve standard continuous distributions, meaning the marginal cost of leverage rises with leverage. We show that for continuous distributions of collateral values, uncertainty shocks increase and steepen the loan credit surface, but not the promise credit surface. Our results are robust to changes in the model specification

Figure 7: Promise-Credit Surfaces and Uncertainty Shocks
with Different Distribution Families



Notes: Log-normal corresponds to $\log X_0 \sim N(-0.02, 0.2^2)$ and $\log X_1 \sim N(-0.045, 0.3^2)$; normal corresponds to $X_0 \sim N(0.9991, 0.2^2)$ and $X_1 \sim N(0.9987, 0.3^2)$, truncated to the left at 3 standard deviations from the mean; power-law corresponds to X_0 and X_1, where $X_v \sim F_v(x) = 1 - x^{-(\beta_v - 1)}$ for $x \geq 1$ and zero otherwise and $\beta_0 = 7$ and $\beta_1 = 5.5$; and uniform corresponds to $X_0 \sim U[0.65, 1.35]$ and $X_1 = U[0.5, 1.5]$. See Appendix A.

and, as such, will be valid on a wide range of models of credit risk, ranging from structural models of risky debt to general equilibrium models with multiple credit terms.

Our results describe a new steepening channel through which uncertainty shocks affect the supply of credit and the macroeconomy. The effect of uncertainty on credit conditions is heterogeneous along the leverage dimension, with riskier borrowers facing disproportionately larger increases in spreads. The steepening channel can contribute to the amplification and persistence of economic fluctuations, and to the adverse consequences of financial crises. As such, our results suggest that macroeconomic policy could be more effective if it targets the entire credit surface, rather than its riskless perimeter.

In practice, financial contracts specify multiple credit terms, spanning other dimensions of the Credit Surface. For instance, the contract can specify the number of periods until the payment of the promise, or different contracts can be offered to different borrowers depending on their creditworthiness. In fact, lenders use different observable borrower characteristics, or borrower credit histories, to assess the likelihood of borrowers' default, which are typically summarized in credit ratings or scores and which condition the credit offering for these borrowers (see, for instance, Chatterjee et al. 2023). That is the essence of the credit surface. To focus on differences in leverage, we have deliberately abstracted from these other dimensions that are relevant in practice. Future research should explore the properties of the Credit Surface and the effect of uncertainty shocks along these additional dimensions. Our analysis might provide a point of departure for future research in this area.

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Appendix

A Simulations

This appendix provides details of the numerical simulations used in the paper. To illustrate our results, we consider the following four families of distributions: (1) log-normal; (2) normal; (3) power-law ; and (4) uniform.

For each family of distributions, we compare two distributions for the value of collateral, describing the random variables X_0 and X_1. The first random variable, X_0 represents the baseline, and X_1 represents the high uncertainty environment, with $F_1^D \lesssim_{MPS} F_0^D$, as in Theorem 4. To facilitate the comparison between families of distributions, we either fix $\mathbb{E}[X_0] = \mathbb{E}[X_1] = 1$ or consider $X_0/\mathbb{E}[X_0]$ and $X_1/\mathbb{E}[X_1]$, so the mean is also equal to 1.

In all these cases, our numerical approximation of the promise as a function of LTV, $J_v(\ell)$, is obtained using $\ell = L_F(J(\ell))$ and a numerical root finder. Given ℓ and the distribution of the value of the collateral, F, we the previous equation and the value of the integrals in that expression to solve for $J_F(\ell)$.

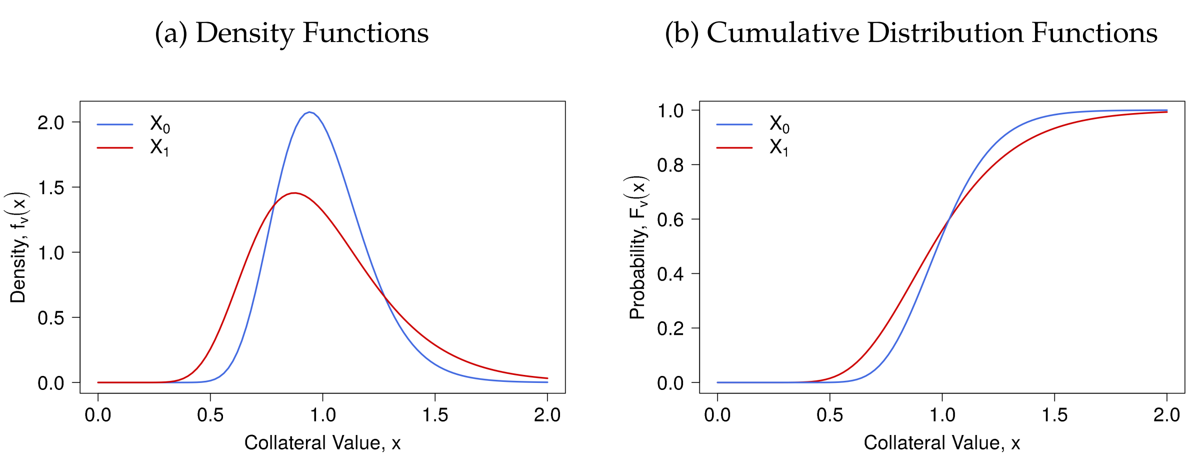
A.1 Log-Normal Distributions

In this case, the value of the risky collateral is log-normally distributed, i.e., $X_v \sim$ Log-normal$(\mu_{X,v}, \sigma^2_{X,v})$. To obtain log-normal distributions that are ordered by SSD we fix the mean of the random variables and increase the variance of the logarithm of the random variable (Levy 1973).

In our numerical illustration, we fix $\mu_{X,v} = 1$, thus $\mu_v = -\sigma_v^2/2$. In particular, in our numerical simulations, we consider $\sigma_0 = 0.2$ and $\sigma_1 = 0.3$. Figure A.1 presents the density and cumulative distribution of these log-normally distributed random variables.

To illustrate the statement of Corollaries 3 and 4 we use a log-normal distribution with a high

Figure A.1: Log-Normal Distributions: Density and Cumulative Distribution Functions



Notes: Two log-normally distributed variables used in the numerical illustrations, corresponding to $\log X_0 \sim N(-0.02, 0.2^2)$ and $\log X_1 \sim N(-0.045, 0.3^2)$.

standard deviation of its logarithm, $\sigma_H = 3$. In this case, we set $\mu_{X,H} = 9$, so $\mu_H = \log(\mu_{X,H}) - \sigma_H^2/2 = -2.3028$.

A.2 Normal Distributions

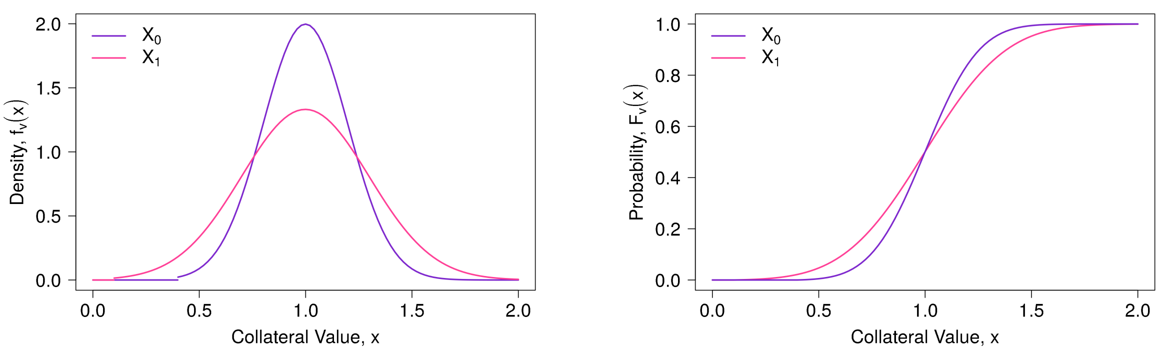
In this case, the value of the risky collateral is normally distributed, i.e., $X_v \sim N(\mu_v, \sigma_v^2)$. We truncate these distributions on the left to prevent negative values for the value of collateral, but leave the right tail unbounded as our results, in the normal case, depend on having an unbounded distribution on the right. We truncate these distributions on the left at $\delta > 0$ standard deviations from the mean to ensure that the distributions are ordered according to SSD, as proposed by Levy (1982). We denote the truncation point on the left with $a_v = \mu_v - \delta\sigma_v$, and choose the parameters μ_v, σ_v and δ such that $\delta < \mu_v/\sigma_v$ to ensure $a_v > 0$.

To fix the mean of X_v at 1, for a given standard deviation, σ_v, we set $\mu_v = 1 - \sigma_v \phi(\delta)/\Phi(\delta)$, where ϕ and Φ are the density and cumulative distribution of the standard normal.

In particular, in our simulations we set $\delta = 3$, $\sigma_0 = 0.2$ and $\sigma_1 = 0.3$. These yield values for $\mu_0 = 0.9991$ and $\mu_1 = 0.9987$, and $a_0 = 0.3991$ and $a_1 = 0.0987$. Figure A.2 presents the density and cumulative distribution for the distributions of X_0 and X_1, under the aforementioned conventions.

Figure A.2: Normal Distributions: Density and Cumulative Distribution Functions

(a) Density Functions (b) Cumulative Distribution Functions



Notes: Two normally distributed variables used in the numerical illustrations, corresponding to $X_0 \sim N(0.9991, 0.2^2)$ and $X_1 \sim N(0.9987, 0.3^2)$, truncated to the left at 3 standard deviations from the mean.

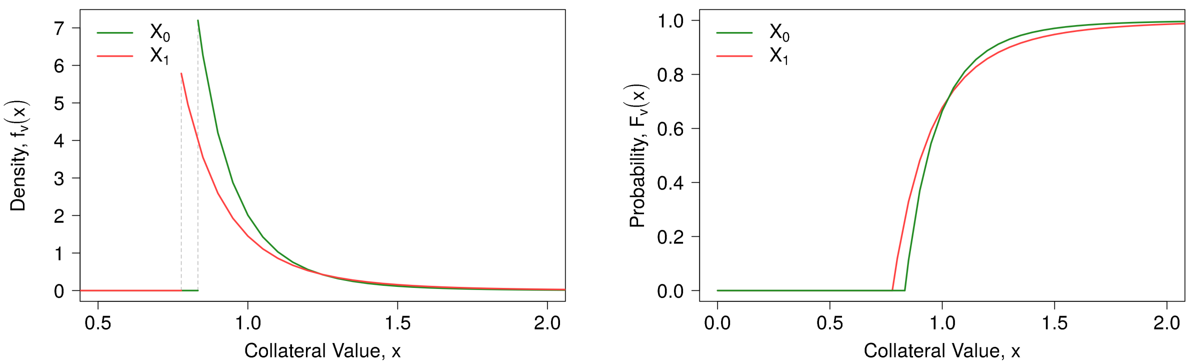
A.3 Power-Law Distributions

In this case, X_v follows a power-Law distribution, with density function $f_v(x) = (\beta_v - 1)x^{-\beta_v}$ when $x \geq 1$, and zero otherwise. We restrict $\beta_v > 3$ so the expected value and the variance are finite, $\mathbb{E}[X_v], \mathrm{Var}(X_v) < \infty$. To obtain power-law distributions that are ordered by SSD we fix the mean of the random variables by normalizing by $\mathbb{E}[X_v] = (\beta_v - 1)/(\beta_v - 2)$ and decreasing the power-law coefficient β_v. In this case, when $\beta_v > 3$, we have that $\mathrm{Var}(X_v/\mathbb{E}[X_v]) = (\beta_v - 2)^2/(\beta_v - 1)/(\beta_v - 3)$.

In our numerical illustration, we use $\beta_0 = 7$ and $\beta_1 = 5.5$, so $\sqrt{\mathrm{Var}(X_0/\mathbb{E}[X_0])} = 0.2041$ and $\sqrt{\mathrm{Var}(X_1/\mathbb{E}[X_1])} = 0.2981$. Figure A.3 presents the density and cumulative distribution of these power-law random variables.

Figure A.3: Power-Law Distributions: Density and Cumulative Distribution Functions

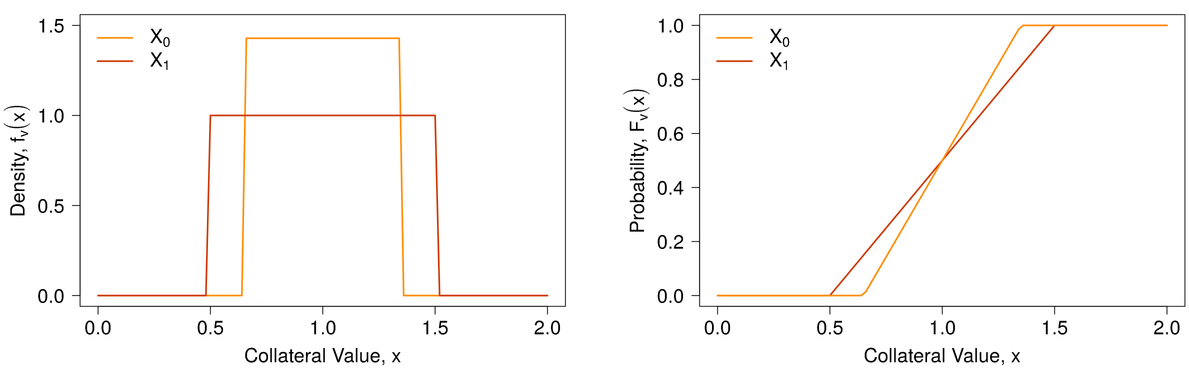
(a) Density Functions (b) Cumulative Distribution Functions



Notes: Two power-law distributions used in the numerical illustrations, corresponding to $X_0/\mathbb{E}[X_0]$ and $X_1/\mathbb{E}[X_0]$, where X_v has a density $f_v(x) = (\beta_v - 1)x^{-\beta_v}$ and $\beta_0 = 7$ and $\beta_1 = 5.5$.

Figure A.4: Uniform Distributions: Density and Cumulative Distribution Functions

(a) Density Functions (b) Cumulative Distribution Functions



Notes: Two uniformly distributed variables used in the numerical illustrations, corresponding to $X_0 \sim U[0.65, 1.35]$ and $X_1 = U[0.5, 1.5]$.

A.4 Uniform Distributions

In the paper, the baseline distribution is uniform in [0.65, 1.35] that we denote $X_0 \sim \text{Uni}[0.65, 1.35]$. Using the formula for the variance of a uniform random variable, $X_v \sim \text{Uni}[a_v, \ell_v]$, $\text{Var}(X_v) = (\ell_v - a_v)^2/12$, it follows that the standard deviation of X_0, $\sigma_0 = (1.35 - 0.65)/\sqrt{12} = 0.20$. We consider uniform distributions with expanding support to model SSD. How much the support expands, will determine the increase in volatility, and to describe the high uncertainty environment, we choose $X_1 \sim \text{Uni}[0.5, 1.5]$ so $\sigma_1 = (1.5 - 0.5)/\sqrt{12} = 0.29$.

B Proofs

Proof of Lemma 1:
i. *From the definition $L(j) = \mathbb{E}[\min\{j, X\}]$: since $0 \le \min\{j, x\} \le j$ for all $x \ge 0$, we have $0 \le L(j) \le j$. Since $\min\{j, x\} \le x$, taking expectations gives $L(j) \le p_F$. Hence $L(j) \le \min\{j, p_F\}$. When $j \le m_F$, we have $P(X \ge j) = 1$, so $\min\{j, X\} = j$ a.s. and $L(j) = j$. In particular, L is bounded.*

To establish continuity, we first show that L is concave. For any $x \ge 0$, the map $j \mapsto \min\{j, x\}$ is concave (it is the minimum of the linear function j and the constant x). Hence $L(j) = \mathbb{E}[\min\{j, X\}]$ is concave as the expectation of concave functions. Since a concave function on an interval is continuous on its interior (Rockafellar 1970, Theorem 10.1), L is continuous on $(0, \infty)$; continuity at 0 follows from $0 \le L(j) \le j$.

ii. *Since $j_1 < j_2$ implies $\min\{j_1, x\} \le \min\{j_2, x\}$ for all $x \ge 0$, L is increasing. Moreover, if $j_1 < M_F$ then $\min\{j_1, x\} < \min\{j_2, x\}$ for all $x > j_1$, and $P(X > j_1) > 0$, so $L(j_1) < L(j_2)$. Thus L is strictly increasing on $[0, M_F)$. When $M_F < \infty$, $P(X \le M_F) = 1$, so $\min\{j, X\} = X$ a.s. for all $j \ge M_F$, giving $L(j) = p_F$; thus L is flat on $[M_F, \infty)$.*

Concavity implies the right derivative $L'_+(j)$ exists everywhere and is right-continuous and decreasing (Rockafellar 1970, Theorem 24.1). Since $L(j) \le j$, we have $L'_+(0) = \lim_{h \downarrow 0} L(h)/h \le 1$. When $M_F = \infty$, L is bounded above by p_F and increasing, so $L'_+(j) \to 0$ as $j \to \infty$ (otherwise $L(j) \to \infty$, contradicting boundedness).

iii. *Concavity was established in part (i).* ■

Proof of Theorem 1: *Since F is monotone and bounded, the integral $\int_0^j F(x)\,dx$ exists for all $j \ge 0$. Because F is right-continuous, the right-hand derivative of $\int_0^j F(x)\,dx$ exists and equals $F(j)$. From Lemma 1, L is concave, so its right derivative $L'_+(j)$ exists everywhere and is right-continuous (Rockafellar 1970, Theorem 24.1). To derive the integral representation, observe that for any $j,\ x \ge 0$,*

$$\min\{j, x\} = \int_0^j \mathbb{1}_{\{x>t\}}\,dt \ .$$

Taking expectations and applying Tonelli's theorem to swap the order of integration,

$$L(j) = \mathbb{E}[\min\{j, X\}] = \int_0^j \mathbb{P}(X > t)\,dt = \int_0^j [1 - F(t)]\,dt = j - \int_0^j F(t)\,dt \ .$$

Since F is right-continuous, differentiating gives $L'_+(j) = 1 - F(j)$. The expected loss representation follows immediately: $\Lambda(j) = j - L(j) = \int_0^j F(t)\,dt$.

For the converse: suppose L is an increasing, concave function with $L(0) = 0$, $L'_+(0) \le 1$, and either $L'_+(j) = 0$ for all $j \ge M$ (some $M < \infty$) or $L'_+(j) \to 0$ as $j \to \infty$. By concavity, the right-hand derivative $L'_+(j)$ exists everywhere, is right-continuous and nonincreasing, and satisfies $L'_+(j) \in [0, 1]$. Define

$$F_L(j) = 1 - L'_+(j) \ .$$

This is a right-continuous, nondecreasing function with $F_L(0) \ge 0$. If $M < \infty$, then $L'_+(j) = 0$ for all $j \ge M$, so $F_L(j) = 1$ for all $j \ge M$. If $M = \infty$, then $L'_+(j) \to 0$, so $F_L(j) \to 1$ as $j \to \infty$. Hence F_L is a valid cumulative distribution function on $\mathbb{R}_+$, uniquely determined by L. ■

Proof of Corollary 1:
i. Promise Credit Surface. *Suppose S is nonnegative and increasing on $[0,\infty)$ and $(1+S(j))/j$ is decreasing, with the stated boundary conditions. Define $L(j) = j/(1+S(j))$. Then $L(0) = 0$ and L is increasing, since $j/(1+S(j))$ is increasing. To see that L is concave, note that $L(j) - jL'_+(j) = j^2 S'_+(j)/(1+S(j))^2$ and*

$$\frac{d}{dj}\left(\frac{1}{1+S(j)}\right) = -\frac{S'_+(j)}{(1+S(j))^2} \ .$$

Then, for any $0 < a < b < \infty$

$$\begin{aligned} L'_+(b) - L'_+(a) &= -\int_a^b \frac{x^2 S'_+(x)}{x^2(1+S(x))^2}dx - \frac{bS'_+(b)}{(1+S(b))^2} + \frac{aS'_+(a)}{(1+S(a))^2} \\ &\leq \frac{a^2 S'_+(a)}{(1+S(a))^2}\left(\frac{1}{b} - \frac{1}{a}\right) - \frac{bS'_+(b)}{(1+S(b))^2} + \frac{aS'_+(a)}{(1+S(a))^2} = \frac{1}{b}\left[\frac{a^2 S'_+(a)}{(1+S(a))^2} - \frac{b^2 S'_+(b)}{(1+S(b))^2}\right] \leq 0 \ , \end{aligned}$$

where we used that $j^2 S'_+(j)/(1+S(j))^2$ is increasing, and we conclude L'_+ is decreasing and thus L is concave. The boundary conditions on S'_+ translate into boundary conditions on L'_+ required by Theorem 1. Hence L recovers a unique dispersion F^D via $F^D(j) = 1 - L'_+(j)$, which gives the stated formula.

***ii.* Loan Credit Surface.** *Suppose S^* is nonnegative and increasing on $[0,1)$ with the stated conditions. Define $J(\ell) = \ell(1+S^*(\ell))$. Then $J(0) = 0$ and J is increasing. To see that J is convex, note that $\ell J'_+(\ell) - J(\ell) = \ell^2 S^{*\prime}_+(\ell)$ and we can write S^* as the integral of its right derivative. Then, for any $0 < a < b < 1$*

$$\begin{aligned} J'_+(b) - J'_+(a) = \int_a^b \frac{u^2 S^{*\prime}_+(u)}{u^2}du + bS^{*\prime}_+(b) - aS^{*\prime}_+(a) \geq a^2 S^{*\prime}_+(a)\left(\frac{1}{a} - \frac{1}{b}\right) + bS^{*\prime}_+(b) - aS^{*\prime}_+(a) \\ = \frac{1}{b}\left[b^2 S^{*\prime}_+(b) - a^2 S^{*\prime}_+(a)\right] \geq 0 \ , \end{aligned}$$

where we used that $\ell^2 S^{*\prime}_+(\ell)$ is increasing, and we conclude J'_+ is increasing and thus J is convex. The boundary condition $S^{*\prime}_+(\ell) \to \infty$ translates into $J'_+(\ell) = 1 + S^*(\ell) + \ell S^{*\prime}_+(\ell) \to \infty$ as $\ell \to 1$. Hence J recovers a unique insolvency dispersion F^{D*} via $F^{D*}(\ell) = 1 - 1/J'_+(\ell)$, which gives the stated formula. Finally, F^D and F^{D*} determine one another through $F^{D*} = F^D \circ J$. ■

Proof of Theorem 2: *By Lemma 1, $J(\ell)$ is strictly increasing and convex, with $J(0) = 0$. Since $J(\ell)$ is convex and $J(0) = 0$, the ratio $J(\ell)/\ell$ is increasing, so $S^*(\ell) = J(\ell)/\ell - 1$ is increasing for $0 \leq \ell < 1$. To see that $S^*(0) = 0$, note that when $m > 0$, $J(\ell) = \ell$ for all $\ell < m/p$, so $S^*(\ell) = 0$ in this range. When $m = 0$, as $\ell \to 0$, $J(\ell)/\ell \to J'_+(0) = 1/[1 - F(0)] = 1$, thus $S^*(0) = 0$.*

(v) implies S^* is convex. *Given that $F \in C^1((m, M))$, then $f = F'$ is continuous and taking derivatives, we obtain*

$$\frac{d^2 J(\ell)}{d\ell^2} = \frac{f(J(\ell))}{[1 - F(J(\ell))]^3} \ .$$

First, consider the case when $m = 0$. Then,

$$\lim_{\ell \to 0^+} \ell^3 S^{*\prime\prime}(\ell) = \lim_{\ell \to 0^+} \ell^2 J''(\ell) - 2\ell J'(\ell) + 2J(\ell) = \frac{0^2 f(0^+)}{[1 - F(0)]^3} - \frac{2 \cdot 0}{1 - F(0)} + 2 \cdot 0 \geq 0 \ ,$$

where the inequality follows from $f(0^+) \geq 0$. Alternatively, when $m > 0$, we have that for all $0 < \ell < m/p$, $f(J(\ell)) = F(J(\ell)) = 0$ and $\ell J'(\ell) = \ell = J(\ell)$, so we conclude that $\ell^3 S^{*\prime\prime}(\ell) = \ell^2 J''(\ell) - 2\ell J'(\ell) + 2J(\ell) = 0$,

so S^* is convex in $[0, m/p)$.
Moreover, since for all $x \in (m, M)$, f' exists, then

$$\frac{d^3 J(\ell)}{d\ell^3} = \frac{1}{[1 - F(J(\ell))]^4} \left\{ \frac{df(J(\ell))}{dx} + \frac{3[f(J(\ell))]^2}{1 - F(J(\ell))} \right\} \geq 0 \,,$$

where the inequality follows from condition (10). Since $\frac{d}{d\ell}(\ell^3 S^{*\prime\prime}(\ell)) = \ell^2 J'''(\ell) \geq 0$, the boundary condition holds, and f is continuous, it follows that $S^{*\prime\prime}(\ell) \geq 0$, hence S^* is convex on $(m/p, 1)$, and thus on $[0, 1)$.

(i) implies (ii). *Let $h(x)$ denote the insolvency rate of the distribution of collateral values, i.e., $h(x) = f(x)/[1 - F(x)] \geq 0$. If the density function $f(x)$ is log concave, then $1 - F(x)$ is log concave (e.g., Bagnoli and Bergstrom 2005; Marshall and Olkin 2007).By definition $h(x) = -(\log(1 - F(x)))'$, thus $h'(x) = -(\log(1 - F(x)))'' \geq 0$, where the inequality holds because $1 - F(x)$ is log concave. So h is non-decreasing, i.e., IPHR holds.*

(ii) implies (iv). *The implication is direct since $h'(x) \geq 0$, implies $h'(x) + [h(x)]^2 \geq 0$. Now we show the equivalences within **(iv)**. By the chain rule applied to $F^* = F \circ J$ and $J'(\ell) = 1/(1 - F(J(\ell)))$,*

$$h^{*\prime}(\ell) = \frac{h'(J(\ell)) + h(J(\ell))^2}{(1 - F(J(\ell)))^2} \,,$$

so since $(1 - F(J(\ell)))^2 > 0$, the ILHR condition $h^{*\prime}(\ell) \geq 0$ holds iff $h'(x) + h(x)^2 \geq 0$. Moreover, using that $h'(x) = f'(x)/(1 - F(x)) + [f(x)]^2/[1 - F(x)]^2$, we conclude that

$$0 \leq h'(x) + [h(x)]^2 = \frac{1}{1 - F(x)} \left[f'(x) + \frac{2[f(x)]^2}{1 - F(x)} \right] .$$

(iii) implies (iv). *It follows from equation (6) and $L(j) \to \mathbb{E}[X]$ as $j \to \infty$ that $\mathbb{E}[X] = \int_0^\infty [1 - F(u)]\, du$. It follows that $\mathbb{E}[(X - j)_+] = \mathbb{E}[X] - L(j) = \int_j^\infty [1 - F(u)]\, du$; and we can write the solvent value of equity as*

$$n(j) = \frac{\mathbb{E}[(X - j)_+]}{1 - F(j)} = \frac{1}{1 - F(j)} \int_j^\infty 1 - F(u)\, du \,. \tag{B.1}$$

Taking derivatives in equation (B.1), for all $x \in (m, M)$, we obtain

$$n'(x) = h(x)n(x) - 1 \qquad \text{and} \qquad n''(x) = \left(h'(x) + [h(x)]^2\right) n(x) - h(x) \,. \tag{B.2}$$

Since $n''(x) \geq 0$ by hypothesis, then $\left(h'(x) + [h(x)]^2\right) n(x) \geq h(x) \geq 0$. Since $n(x) > 0$ for $x < M$, we conclude that $h'(x) + [h(x)]^2 \geq 0$, which by the equivalence within (iv) is ILHR.

(iv) implies (v). *This is direct as $[h(x)]^2 \geq 0$ implies $h'(x) + 2[h(x)]^2 \geq h'(x) + [h(x)]^2 \geq 0$.*

Equivalences within (v) and (vi). *It remains to verify the equivalences displayed inside conditions (v) and (vi), and to show (v)⇔(vi). Using $f = h(1 - F)$ and $f' = (h' - h^2)(1 - F)$,*

$$f'(x) + \frac{3[f(x)]^2}{1 - F(x)} = (1 - F(x))\left(h'(x) + 2[h(x)]^2\right) ,$$

and since $1 - F(x) > 0$, the two forms in (v) are equivalent. The same algebra applied to F^* gives

$$\frac{df^*(\ell)}{d\ell} + \frac{2[f^*(\ell)]^2}{1 - F^*(\ell)} = (1 - F^*(\ell))\Big(h^{*\prime}(\ell) + [h^*(\ell)]^2\Big) .$$

Using $h^*(\ell) = h(J(\ell))/(1 - F(J(\ell)))$ and the chain-rule identity for $h^{*\prime}$ derived above,

$$h^{*\prime}(\ell) + [h^*(\ell)]^2 = \frac{h'(J(\ell)) + h(J(\ell))^2}{(1 - F(J(\ell)))^2} + \frac{h(J(\ell))^2}{(1 - F(J(\ell)))^2} = \frac{h'(J(\ell)) + 2[h(J(\ell))]^2}{(1 - F(J(\ell)))^2} ,$$

so the f^*, F^* form in (vi) holds iff $h'(x) + 2[h(x)]^2 \geq 0$. For the J'''-form, starting from $J''(\ell) = f(J(\ell))/(1 - F(J(\ell)))^3$,

$$J'''(\ell) = \frac{f'(J(\ell)) + 3[f(J(\ell))]^2/(1 - F(J(\ell)))}{(1 - F(J(\ell)))^4} = \frac{h'(J(\ell)) + 2[h(J(\ell))]^2}{(1 - F(J(\ell)))^3} ,$$

so $J'''(\ell) \geq 0$ also iff $h'(x) + 2[h(x)]^2 \geq 0$. The two forms inside (vi) are therefore equivalent to each other and to condition (v). ■

Proof of Corollary 3: *In this case it will be useful to define*

$$z(x) = \frac{\log(x) - \mu}{\sigma} . \tag{B.3}$$

Then $F(x) = \Phi\left(z(x)\right)$ and $f(x) = \phi\left(z(x)\right)/(\sigma x)$, so that

$$\frac{df(x)}{dx} = -\frac{\phi\left(z(x)\right)}{\sigma^2 x^2}\left[z(x) + \sigma\right] , \qquad \text{and} \qquad x\,h(x) = \frac{x\,f(x)}{1 - F(x)} = \frac{h_N\left(z(x)\right)}{\sigma} , \tag{B.4}$$

where h_N is the standard normal hazard rate.

Condition (10) of Theorem 2 is the member $c = 3$ of the family (F.3). By Appendix F.1, for the log-normal that condition holds at every $x > 0$ if and only if $\sigma \leq \sigma_3^*$, where $\sigma_3^* = \inf_{z\in\mathbb{R}}\{3h_N(z) - z\} \approx 1.86$, as in (F.4). Theorem 2 then delivers convexity of the loan credit surface whenever $0 < \sigma \leq \sigma_3^*$.

Suppose now that $\sigma > \sigma_3^*$. Recall that $\ell^3 S_+^{*\prime\prime}(\ell) = \ell^2 J_+''(\ell) - 2\ell J_+'(\ell) + 2J(\ell)$, from where we get

$$[1 - F(J(\ell))]^3\,\ell^3 S^{*\prime\prime}(\ell) = \ell^2 f(J(\ell)) - 2\ell\,[1 - F(J(\ell))]^2 + 2J(\ell)\,[1 - F(J(\ell))]^3 .$$

Evaluating at $\ell = L(j)$, we obtain

$$[1 - F(j)]^3\,[L(j)]^3\;S^{*\prime\prime}(L(j)) = f(j)\,[L(j)]^2 - 2\,[1 - F(j)]^2\,[L(j) - j(1 - F(j))] .$$

Because the arithmetic mean is larger that the geometric mean, $(a + b)^2 \geq 4ab$. Applying this inequality to

$$[L(j)]^2 = [\{L(j) - j(1 - F(j))\} + j(1 - F(j))]^2 \geq 4\,\{L(j) - j(1 - F(j))\}\,j(1 - F(j)) .$$

It follows that

$$\begin{aligned}[1 - F(j)]^3\,[L(j)]^3\;S^{*\prime\prime}(L(j)) &\geq 4f(j)\,\{L(j) - j(1 - F(j))\}\,j(1 - F(j)) - 2\,[1 - F(j)]^2\,[L(j) - j(1 - F(j))] \\ &\geq 4\,\{L(j) - j(1 - F(j))\}\,[1 - F(j)]^2\left[j\,h(j) - \tfrac{1}{2}\right] ,\end{aligned}$$

which is nonnegative when $j\,h(j) \geq \frac{1}{2}$.

If $z(j) \geq \sigma/2$, we use that $h_N(z) > z$ for every $z \in \mathbb{R}$ by Lemma F.2, so (B.4) gives $j\,h(j) = h_N\left(z(j)\right)/\sigma > z(j)/\sigma \geq 1/2$, and we conclude that $S^{*\prime\prime}(\ell) \geq 0$.

If instead $z(j) \leq -\sigma$, then $z(x) + \sigma \leq 0$ for every $x \in (0, j]$, so (B.4) gives $f'(x) \geq 0$ on $(0, j]$: the density is non-decreasing there. Condition (10) therefore holds on $(0, j]$. The argument in the proof of Theorem 2 then applies unchanged on $(0, L(j)]$: there $\ell^3 S^{*\prime\prime}(\ell)$ is non-decreasing and vanishes as $\ell \to 0^+$, so $S^{*\prime\prime}(\ell) \geq 0$ for every $\ell \in (0, L(j)]$.

■

Proof of Theorem 3: *When $F \in C^1((m, M))$ and f' exists, from Lemma 1, $L'(j) = 1 - F(j)$, and when $j \in (m, M)$, $L''(j) = -f(j)$, $L'''(j) = -f'(j)$. Define*

$$\phi(j) \;\equiv\; -j\,L(j)\,L''(j) - 2L'(j)\,[L(j) - j\,L'(j)] \;=\; [L(j)]^3\,S''(j)\,, \tag{B.5}$$

so that, since $L(j) > 0$ for $j > 0$, the promise credit surface is convex if and only if $\phi \geq 0$ on (m, M). Differentiating (B.5) and using $L'' = -f$ and $L''' = -f'$,

$$\phi'(j) = -j\,L(j)\,L'''(j) - 3L''(j)\,[L(j) - j\,L'(j)] = 3\,f(j)\,[L(j) - j\,[1 - F(j)]] + j\,L(j)\,f'(j)\,. \tag{B.6}$$

Because $L(m) = m$, $L'(m) = 1 - F(m) = 1$ and $L''(m) = -f(m)$,

$$\phi(m) = m^2 f(m) \geq 0\,. \tag{B.7}$$

(iv) implies convexity. *Suppose $h' + \frac{1}{2}h^2 \geq 0$ on (m, M). Writing $h = f/(1-F) = -L''/L'$, (B.5) and (B.6) give the identity*

$$\phi'(j) + \tfrac{3}{2}\,h(j)\,\phi(j) \;=\; j\,L(j)\,L'(j)\left[h'(j) + \tfrac{1}{2}h(j)^2\right] \;\geq\; 0\,. \tag{B.8}$$

Multiplying by the strictly positive integrating factor $[1 - F(j)]^{-3/2}$, whose derivative is $\frac{3}{2}h(j)[1 - F(j)]^{-3/2}$, yields

$$\frac{d}{dj}\left\{[1 - F(j)]^{-3/2}\phi(j)\right\} = [1 - F(j)]^{-3/2}\left(\phi'(j) + \tfrac{3}{2}h(j)\phi(j)\right) \;\geq\; 0\,,$$

so $[1 - F]^{-3/2}\phi$ is nondecreasing on (m, M). Since $F(m) = 0$, its limit as $j \downarrow m$ is $\phi(m) = m^2 f(m) \geq 0$ by (B.7). Hence $[1 - F(j)]^{-3/2}\phi(j) \geq 0$, so $\phi(j) \geq 0$ and S is convex.

(ii) implies (iv). *If $h' \geq 0$ then $h' + \frac{1}{2}h^2 \geq \frac{1}{2}h^2 \geq 0$.*

(iii) implies (iv). *If $\phi' \geq 0$ on (m, M) then $\phi(j) \geq \phi(m) \geq 0$ by (B.7). As $h \geq 0$, the left-hand side of (B.8) is then nonnegative, and $j\,L(j)\,L'(j) > 0$ on (m, M); hence $h' + \frac{1}{2}h^2 \geq 0$.*

(i) implies (ii). *When condition (i) holds, log-concavity of f implies IPHR by Bagnoli and Bergstrom (2005) .* ■

Proof of Corollary 4:
i. Log-Normal Distributions. *It will be useful to work with $z(x)$ defined in equation (B.3), and with the identities in (B.4).*

Condition (14) of Theorem 3 is the member $c = 3/2$ of the family (F.3). By Appendix F.1, for the log-normal that condition holds at every $x > 0$ if and only if $\sigma \leq \sigma^*_{3/2}$, where $\sigma^*_{3/2} = \inf_{z\in\mathbb{R}}\left\{\frac{3}{2}h_N(z) - z\right\} \approx 1.19$. Theorem 3 then delivers convexity of the promise credit surface whenever $0 < \sigma \leq \sigma^*_{3/2}$.

Suppose now that $\sigma > \sigma^*_{3/2}$. From (B.5),

$$[L(j)]^3\, S''(j) = L(j)\, [j\, f(j) - 2\,[1 - F(j)]] + 2j\,[1 - F(j)]^2 \ ,$$

which is nonnegative when $j\,h(j) \geq 2$.

If $z(j) \geq 2\sigma$, we use that $h_N(z) > z$ for every $z \in \mathbb{R}$ by Lemma F.2, so (B.4) gives $j\,h(j) = h_N\,(z(j))\,/\sigma > z(j)/\sigma \geq 2$, and we conclude that $S''(j) \geq 0$.

If instead $z(j) \leq -\sigma$, then $z(x) + \sigma \leq 0$ for every $x \in (0, j]$, so (B.4) gives $f'(x) \geq 0$ on $(0, j]$: the density is non-decreasing there. Condition (13) therefore holds on $(0, j]$. The argument in the proof of Theorem 3 then applies unchanged on $(0, j]$: there $[L(y)]^3\, S''(y)$ is non-decreasing, by (B.6), and vanishes as $y \to 0^+$, by (B.7) with $m = 0$, so $S''(y) \geq 0$ for every $y \in (0, j]$.

***ii. Power-Law Distribution.** We have that*

$$1 - F(j) = \int_j^\infty cx^{-\beta} dx = \frac{c}{\beta - 1}\,[j]^{-(\beta-1)} \ . \tag{B.9}$$

In addition, from equation (B.6), we know that it suffices to show that $jf(j) - 2[1 - F(j)] \geq 0$. It follows from (B.9) that

$$jf(j) - 2[1 - F(j)] = c\,[j]^{-(\beta-1)} - 2\frac{c}{\beta - 1}\,[j]^{-(\beta-1)} = c\,[j]^{-(\beta-1)}\left[1 - \frac{2}{\beta - 1}\right] = c\,[j]^{-(\beta-1)}\,\frac{\beta - 3}{\beta - 1} \geq 0 \ ,$$

where the inequality follows from $\beta \geq 3$.

■

Proof of Theorem 4:

***i.** In this case, the definition of LTV implies that*[35]

$$\mathbb{E}\,[\min\{J_1(\ell), X_1\}] = \mathbb{E}\,[\min\{J_0(\ell), X_0\}] \geq \mathbb{E}\,[\min\{J_0(\ell), X_1\}] \ ,$$

where the last inequality follows from $F_1^D \lesssim_{MPS} F_0^D$. So we conclude that $J_1(\ell) \geq J_0(\ell)$.
When $J_1(\ell) - J_0(\ell)$ is positive, then $S_1^*(\ell) - S_0^*(\ell) = (J_1(\ell) - J_0(\ell))/\ell \geq 0$.

***ii.** Note that $j_0(0) = j_1(0) = 0$, so it suffices to show that $dJ_1(\ell)/d\ell \geq dJ_0(\ell)/d\ell$. Since $F_1^{D*} \lesssim_1 F_0^{D*}$ we have that $F_1(J_1(\ell)) \geq F_0(J_0(\ell))$. Then, our hypothesis implies that for $0 \leq l < 1$*

$$\frac{dJ_1(\ell)}{d\ell} = \frac{1}{1 - F_1(J_1(\ell))} \geq \frac{1}{1 - F_0(J_0(\ell))} = \frac{dJ_0(\ell)}{d\ell} \ .$$

***iii.** By definition, $F_1^{D*} \lesssim_{HH} F_0^{D*}$ means $\int_0^\ell [1 - F_1^{D*}(u)]^{-1}\, du \Big/ \int_0^\ell [1 - F_0^{D*}(u)]^{-1}\, du$ is increasing. Since $J_i(\ell) = \int_0^\ell [1 - F_i^{D*}(u)]^{-1}\, du$, this ratio equals $J_1(\ell)/J_0(\ell) = (1 + S_1^*(\ell))/(1 + S_0^*(\ell))$, giving the equivalence.*

***iv.** First, note that $F_1^{D*} \lesssim_H F_0^{D*} \Rightarrow F_1^{D*} \lesssim_1 F_0^{D*}$ so from part (ii) we have that the difference $J_1(\ell) - J_0(\ell)$ is positive and increasing and we are only left to show that the difference is convex.*

[35] This result uses the characterization of second order as $F_1 \lesssim_2 F_0$ iff $\mathbb{E}[\phi\,(X_1)] \leq \mathbb{E}[\phi\,(X_0)]$ for every increasing and concave function ϕ. In other words, the order relationship $\lesssim_2$ is equivalent to $\lesssim_{ICV}$, where the latter denotes the increasing concave order (e.g., Shaked and Shanthikumar 2007).

Since $F_1^{D*} \precsim_H F_0^{D*}$ we have that $[1-F_0(J_0(\ell))]/[1-F_1(J_1(\ell))]$ is increasing, and since F_0 and j_0 are increasing $1/[1-F_0(J_0(\ell))]$ is increasing, so we conclude that the derivative wrt ℓ of the difference $J_1(\ell)-J_0(\ell)$ is increasing. In fact,

$$\frac{dJ_1(\ell)}{d\ell}-\frac{dJ_0(\ell)}{d\ell}=\frac{1}{1-F_1(J_1(\ell))}-\frac{1}{1-F_0(J_0(\ell))}=\frac{1}{1-F_0(J_0(\ell))}\left[\frac{1-F_0(J_0(\ell))}{1-F_1(J_1(\ell))}-1\right].$$

Thus, $J_1(\ell)-J_0(\ell)$ is convex in ℓ.

When $F_1^{D*} \precsim_H F_0^{D*}$ then $g(\ell)=J_1(\ell)-J_0(\ell)$ is positive, increasing and convex, with $g(0)=0$. Then, $S_1^*(\ell)-S_0^*(\ell)=g(\ell)/\ell \geq 0$, where the inequality follows from the positivity of $g(\ell)$. Moreover, given that $g(\ell)$ is convex and $g(0)=0$, since g is convex and $g(0)=0$, the ratio $g(\ell)/\ell$ is increasing, so $S_1^*(\ell)-S_0^*(\ell)$ is increasing.

$\boldsymbol{F_0^{D*} \gtrsim_H F_1^{D*} \Rightarrow F_0^{D*} \gtrsim_{HH} F_1^{D*}}$**.** *The insolvency rate order means the pointwise survival ratio $\bar{F}_0^{D*}(u)/\bar{F}_1^{D*}(u)=[1-F_0^{D*}(u)]/[1-F_1^{D*}(u)]$ is increasing. The second hazard rate order means the ratio of integrals $\int_0^\ell [1-F_1^{D*}]^{-1}\,du \big/ \int_0^\ell [1-F_0^{D*}]^{-1}\,du$ is increasing. When the pointwise ratio of the integrands is increasing, the ratio of their integrals is also increasing; this is the monotone ratio lemma of Anderson, Vamanamurthy and Vuorinen (2006).*

$\boldsymbol{F_0^{D*} \gtrsim_{HH} F_1^{D*} \Rightarrow F_0^{D*} \gtrsim_1 F_1^{D*}}$**.** *The second hazard rate order means the ratio $R(\ell)=J_1(\ell)/J_0(\ell)$ is increasing from $R(0^+)=1$, so $R(\ell)\geq 1$ for all ℓ. Moreover, for R to be increasing the pointwise survival ratio $\bar{F}_0^{D*}(\ell)/\bar{F}_1^{D*}(\ell)$ must exceed $R(\ell)\geq 1$, giving $\bar{F}_0^{D*} \geq \bar{F}_1^{D*}$, which is the insolvency probability order.*

$\boldsymbol{F_0^{D*} \gtrsim_1 F_1^{D*} \Rightarrow F_0^{D} \gtrsim_{MPS} F_1^{D}}$**.** *The insolvency probability order gives $J_1'(\ell)\geq J_0'(\ell)$ (from part (ii)), so $J_1(\ell)-J_0(\ell)$ is increasing from 0, hence nonnegative, which is the MPS condition (part (i)).* ■

Proof of Theorem 5: *Recall from the derivation preceding the theorem that, for every loan-to-value ratio ℓ and every v, the promise function satisfies*

$$\frac{\partial J(\ell,v)}{\partial v}=K(J(\ell,v),v), \qquad \text{and} \qquad K(m_v,v)=0, \tag{B.10}$$

with $\partial J(\ell,v)/\partial v=0$ on the region where $J(\ell,v)\leq m_v$. Differentiating (B.10) with respect to ℓ, and using $J_\ell>0$,

$$J_{v\ell}(\ell,v)=K_j(J(\ell,v),v)\,J_\ell(\ell,v). \tag{B.11}$$

We establish the equivalences between the properties of K and the properties of J. Their equivalence with the stochastic orders on the dual distribution F_v^{D*} and the properties of S^* are given by Theorem 4.

i. *By (B.10), $K(j,v)\geq 0$ for all j if and only if $J_v(\ell,v)\geq 0$, and integrating over $v\in[t,u]$,*

$$J_u(\ell)-J_t(\ell)=\int_t^u J_v(\ell,v)\,dv\geq 0.$$

ii. *By (B.11) and $J_\ell>0$, $K(j,v)$ is increasing in j (i.e. $K_j\geq 0$) if and only if $J_{v\ell}\geq 0$, that is J_v is increasing in ℓ. Since $K(m_v,v)=0$, $K_j\geq 0$ also gives $K\geq 0$, so $J_u(\ell)-J_t(\ell)$ is nonnegative and increasing in ℓ.*

iii. *Since $\partial_v \log J=J_v/J=K(J,v)/J$,*

$$\frac{d}{d\ell}\Big[\partial_v \log J(\ell,v)\Big]=\frac{K_jJ_\ell J-K\,J_\ell}{J^2}=\frac{J_\ell}{J}\left[K_j-\frac{K}{J}\right]. \tag{B.12}$$

Now $K(j,v)/j$ is increasing in j if and only if $K_j - K/j \geq 0$, which at $j = J$ is the bracket in (B.12); as $J_\ell/J > 0$, this is equivalent to $\partial_v\big[\partial_\ell \log J\big] \geq 0$. Integrating over $v \in [t,u]$,

$$\partial_\ell \log \frac{J_u(\ell)}{J_t(\ell)} = \int_t^u \partial_v\big[\partial_\ell \log J(\ell,v)\big]\,dv \geq 0,$$

so $J_u(\ell)/J_t(\ell)$ is increasing in ℓ (multiplicative steepening).

iv. *Since $\partial_v \log J_\ell = J_{v\ell}/J_\ell = K_j(J,v)$ by (B.11),*

$$\frac{d}{d\ell}\big[\partial_v \log J_\ell(\ell,v)\big] = K_{jj}(J,v)\,J_\ell. \tag{B.13}$$

As $J_\ell > 0$, $K(j,v)$ is convex in j (i.e. $K_{jj} \geq 0$) if and only if $\partial_v\big[\partial_\ell \log J_\ell\big] \geq 0$. Integrating over $v \in [t,u]$,

$$\partial_\ell \log \frac{J_u'(\ell)}{J_t'(\ell)} = \int_t^u \partial_v\big[\partial_\ell \log J_\ell(\ell,v)\big]\,dv \geq 0,$$

so $J_u'(\ell)/J_t'(\ell)$ is increasing in ℓ. ■

Proof of Theorem 6: *In the case of a PMPS, for every $x \in (\underline{x}_v, M_v)$, since F is differentiable*

$$F(x,v) = F\left(p + \frac{x-p}{v}\right) \qquad \text{thus} \qquad \frac{\partial F(x,v)}{\partial v} = -f\left(p + \frac{x-p}{v}\right)\frac{x-p}{v^2} = -f(x,v)\left(\frac{x-p}{v}\right).$$

Thus, we have that

$$K(y,v) = \frac{1}{1-F(y,v)}\int_{m_v}^{y} \frac{\partial F(x,v)}{\partial v}\,dx = \frac{1}{1-F(y,v)}\int_{m_v}^{y} f(x,v)\frac{(p-x)}{v}\,dx$$

For any $v > 0$, using the change of variables $u = p + (x-p)/v$ we get

$$K(y,v) = \frac{1}{1-F(y,v)}\int_{m_v}^{y} f(x,v)\frac{(p-x)}{v}\,dx = \frac{v}{1-F(u(y))}\int_{m}^{u(y)} f(u)(p-u)\,du \equiv v\tilde{K}(u(y)).$$

But $u(y)$ is an affine transformation of y, so $K(y,v)$ is convex iff $\tilde{K}(u)$ is convex.

Let $\underline{u} = u(m)$ and integrating by parts, we get

$$\tilde{K}(u) = \frac{1}{1-F(u)}\int_m^u f(x)(p-x)\,dx = (p-u) - \frac{(p-m)}{1-F(u)} + \frac{1}{1-F(u)}\int_m^u 1-F(x)\,dx.$$

Now note that

$$\int_m^u 1-F(x)\,dx = \int_m^\infty 1-F(x)\,dx - \int_u^\infty 1-F(x)\,dx = e - m - \int_u^\infty 1-F(x)\,dx,$$

where in the last equality we used the tail integral formula:

$$\int_m^\infty 1-F(y)\,dy = \mathbb{E}[X-m] = e-m. \tag{B.14}$$

Then, using equation (B.1), we obtain

$$\tilde{K}(u) = e - u + \frac{1}{1-F(u)} \int_u^\infty 1 - F(x)\,dx = e - u + n(u)\;.$$

Then, taking derivatives we obtain the desired result.

$$\tilde{K}'(u) = -1 + n'(u) = h(u)n(u) - 2 \qquad \text{and} \qquad \tilde{K}''(u) = n''(u)\;.$$

It follows that the map $y \mapsto K(y,v)$ is convex iff n is convex. ■

Proof of Corollary 5:
i. Beta Distribution. *The Beta distribution with parameters $a, b \geq 1$ scaled to $[0, M]$ has CDF given by*

$$F_G(x,a) = F(\omega,a) = \int_0^\omega f(t)\,dt\;, \qquad \omega = \frac{x}{M}\;, \qquad f(t) = \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}\;, \qquad 0 \leq x \leq M\;.$$

Fixing b, mean preservation at $p = 1$ requires $M(a) = p(a+b)/a$. The variance is given by $p^2 b/[a(a+b+1)]$, so at fixed b the natural increase in uncertainty is to decrease the parameter a, accordingly set $v = -a$. All functions of the scaled coordinate $\omega \in (0,1)$ refer to the standard Beta distribution: the density f, its logarithmic derivative $d_1 = f'/f = (a-1)/\omega - (b-1)/(1-\omega)$, the CDF F, the survival function $\bar{F} = 1 - F$, and the reciprocal hazard rate $\varphi = \bar{F}/f$; primes denote derivatives in ω.

Now we differentiate F_G wrt a at fixed x. We use $\frac{d\log M}{da} = \frac{1}{a+b} - \frac{1}{a} = \frac{-b}{a(a+b)} = -\beta_0$, so that $\frac{d\omega}{da} = \beta_0\,\omega$ with $\beta_0 = b/[a(a+b)]$; and $\frac{\partial \log B(a,b)}{\partial a} = \psi(a) - \psi(a+b)$ so $\frac{\partial \log f(t)}{\partial a} = \log t + \psi(a+b) - \psi(a)$, where ψ is the digamma function.

$$\frac{\partial F_G(x,a)}{\partial a} = \int_0^\omega f(t)\,(\log t + \psi(a+b) - \psi(a))\,dt + \beta_0\,\omega f(\omega)\;.$$

Recall that the digamma function is strictly increasing on $(0,\infty)$ so $\psi(a+b)-\psi(a) > 0$. By the definition of the digamma function, the previous integral vanishes over the full support: using $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$,

$$\int_0^1 f(t)\,(\log t + \psi(a+b) - \psi(a))\,dt = \frac{1}{B(a,b)}\frac{\partial B(a,b)}{\partial a} + \psi(a+b) - \psi(a) = 0\;.$$

Hence the integral from 0 to ω equals $-G(\omega)$, where

$$G(\omega) \equiv \int_\omega^1 f(t)\,(\log t + \psi(a+b) - \psi(a))\,dt\;, \qquad G(0) = G(1) = 0\;.$$

By definition $G'/f = -(\log\omega + \psi(a+b) - \psi(a))$, so $(G'/f)' = -1/\omega$. With $v = -a$,

$$\frac{\partial F_G(x,v)}{\partial v} = G(\omega) - \beta_0\,\omega f(\omega)\;.$$

Substituting $x = M\omega$, K becomes a function of $z = j/M$ alone:

$$K(j) = \frac{1}{1-F_G(j)} \int_0^j \frac{\partial F_G(x,v)}{\partial v}\,dx = M\,g(z)\;, \qquad g \equiv \frac{A}{\bar{F}}\;, \qquad A(z) \equiv \int_0^z (G(t) - \beta_0\,t f(t))\,dt\;.$$

Because $j \mapsto z = j/M$ is affine, K is convex on $(0, M)$ iff g is convex on $(0, 1)$.

Differentiating twice, using $\bar{F}' = -f$, $f' = d_1 f$, and $1 + \varphi' = -d_1\varphi$,

$$g' = \frac{A'\bar{F} + Af}{\bar{F}^2} = \frac{f}{\bar{F}^2} H \,, \qquad H \equiv \varphi A' + A \,, \qquad \text{and} \qquad g'' = \frac{f^2}{\bar{F}^3} \left[(1 - \varphi') H + \varphi H'\right] \,.$$

The Beta density is log-concave for $a, b \geq 1$, so the hazard rate is nondecreasing and $\varphi' \leq 0$, whence $1 - \varphi' \geq 1$. Thus, $g'' \geq 0$ wherever H and H′ are both nonnegative, and where $H' < 0$ it suffices that $(1 - \varphi') H \geq -\varphi H'$.

Now $H(0^+) = 0$: $A(0) = 0$, and $\varphi A' = \varphi G - \beta_0 \omega \bar{F} \to 0$ since $\varphi G \sim \omega \log(1/\omega)/a$. Indeed, near the origin $\bar{F} \to 1$ and $f(\omega) \sim \omega^{a-1}/B(a, b)$, so $\varphi \sim B(a, b)\, \omega^{1-a}$; while $\int_0^\omega t^{a-1} \log t \, dt = \omega^a \left[\frac{\log \omega}{a} - \frac{1}{a^2}\right]$ gives $G(\omega) = -\int_0^\omega f(t) \left(\log t + \psi(a + b) - \psi(a)\right) dt \sim \omega^a \log(1/\omega)/\left[a\, B(a, b)\right]$. The powers of ω and the factors $B(a, b)$ cancel in the product. Using $A' = G - \beta_0 \omega f$ and $A'' = G' - \beta_0 \left(\omega f\right)'$,

$$H' = (1 + \varphi')A' + \varphi A'' = \varphi \left(A'' - d_1 A'\right) = -\bar{F}\, \Sigma \,, \qquad \Sigma \equiv \beta_0 + \frac{d_1 G - G'}{f} \,,$$

so $H(\omega) = -\int_0^\omega \bar{F}\Sigma \, dt$. Moreover,

$$\int_0^1 \bar{F}\, \Sigma \, dt = \beta_0 \int_0^1 \bar{F} \, dt + \int_0^1 \varphi \left(d_1 G - G'\right) dt = \beta_0\, p_0 - \int_0^1 G \, dt = 0 \,,$$

where $p_0 = a/(a + b) = \int_0^1 \bar{F} \, dt$: the middle step integrates by parts using $\varphi d_1 = -(1 + \varphi')$ and $[\varphi G]_0^1 = 0$, and $\int_0^1 G \, dt = \beta_0 p_0$ follows from Fubini, the same digamma differentiation applied to $B(a + 1, b)$, and the recurrences $\psi(a + 1) = \psi(a) + 1/a$, $\psi(a + b + 1) = \psi(a + b) + 1/(a + b)$. Hence also $H(\omega) = \int_\omega^1 \bar{F}\Sigma \, dt$.

Next we show that Σ is nondecreasing on $(0, 1)$ for every $a, b \geq 1$. Differentiating, using $G'' = d_1 G' - f/\omega$,

$$\Sigma' = \frac{1}{\omega} + \frac{d_1 G' - c_2 G}{f} = \frac{c_2}{f} D \,, \qquad D \equiv \frac{f/\omega + d_1 G'}{c_2} - G \,,$$

where

$$c_2 \equiv d_1^2 - d_1' = d_1^2 + \frac{a - 1}{\omega^2} + \frac{b - 1}{(1 - \omega)^2} \,,$$

which is positive unless $a = b = 1$ (the uniform case, treated below); so $\Sigma' \geq 0$ iff $D \geq 0$. A direct computation gives the exact identity

$$D' = -\frac{2(a - 1)(b - 1)\, \omega G' + P_3(\omega)\, f}{c_2^2\, \omega^4 (1 - \omega)^3} \,,$$

$$P_3(\omega) = (a + b - 1)(a + b - 2)\, \omega^3 - (3a - b - 2)(a + b - 1)\, \omega^2 + 3a(a - 1)\, \omega - a(a - 1) \,.$$

Dividing the numerator by $\omega f > 0$ and using $(G'/f)' = -1/\omega$, we obtain

$$\frac{d}{d\omega} \left[2(a - 1)(b - 1)\, \frac{G'}{f} + \frac{P_3}{\omega}\right] = (2\omega + 1)(1 - \omega)^2 c_2 \geq 0 \,.$$

Thus the numerator changes sign at most once, from negative to positive, so D changes sign at most once, from positive to negative. When it does change sign $\{D' \leq 0\}$ is an interval with right endpoint 1: D rises and then falls. For $a, b > 1$, at the endpoints $G \to 0$, while $f/\omega + d_1 G' > 0$ near each end (near 0, $d_1 > 0$

and $G' > 0$; near 1, $d_1 < 0$ and $G' < 0$), so $\liminf D \geq 0$ at both ends. A function that rises and then falls attains its infimum at the boundary, hence $D \geq 0$ on $(0, 1)$. In the boundary cases the sign of D' can be read off directly: for $a = 1$, the G'-term vanishes and $P_3 = b(b-1)\omega^2(1+\omega) \geq 0$, so $D' \leq 0$ and D decreases to $\liminf_{\omega\to 1} D \geq 0$; for $b = 1$, $P_3 = -a(a-1)(1-\omega)^3 \leq 0$, so $D' \geq 0$ and D increases from $\liminf_{\omega\to 0} D \geq 0$. (For $a = b = 1$, $\Sigma = \log\omega + 3/2$, increasing.) In all cases Σ is nondecreasing.

Since Σ is nondecreasing with $\int_0^1 \bar{F}\Sigma\,dt = 0$ and $\bar{F} > 0$, either $\Sigma \equiv 0$ (then $H \equiv 0$, g is constant, and there is nothing to prove) or there is $\omega_H \in (0, 1)$ with $\Sigma \leq 0$ on $(0, \omega_H]$ and $\Sigma \geq 0$ on $[\omega_H, 1)$. So H rises from $H(0^+) = 0$ and falls back to $H(1^-) = 0$; in particular $H \geq 0$ throughout. Below ω_H, $H(\omega) = -\int_0^{\omega} \bar{F}\Sigma\,dt \geq 0$ and $H' = -\bar{F}\Sigma \geq 0$, so $g'' \geq 0$. Above ω_H, the monotonicity of Σ gives

$$H(\omega) = \int_{\omega}^{1} \bar{F}\Sigma\,dt \geq \Sigma(\omega)\int_{\omega}^{1} \bar{F}\,dt = \Sigma(\omega)\,\bar{F}(\omega)\,n(\omega)\;,$$

where $n(\omega) = \int_{\omega}^{1} \bar{F}\,dt/\bar{F}(\omega)$ is the solvent equity value of the standard Beta distribution. The identity $n' = n/\varphi - 1$ implies $n'' = (n/\varphi)'$ and $(1-\varphi')\,n - \varphi = \varphi^2\,(n/\varphi)'$, so with $-\varphi H' = \varphi\bar{F}\Sigma$,

$$(1-\varphi')\,H + \varphi H' \geq \bar{F}\,\Sigma(\omega)\,[(1-\varphi')\,n - \varphi] = \bar{F}\,\Sigma(\omega)\,\varphi^2\,n'' \geq 0\;,$$

where the last inequality is the convexity of the solvent equity value for the Beta family with $a, b \geq 1$, established in Appendix C. Therefore $g'' \geq 0$ on $(0, 1)$ and $j \mapsto K(j, v)$ is convex on $(0, M)$ for every $a, b \geq 1$: the Beta family displays K-convexity along the mean-preserving path that decreases a at fixed b, and the loan credit surface steepens.

ii. Gamma Distribution. *The Gamma distribution with shape $\alpha \geq 1$ and rate α/p has CDF given by*

$$F_G(x, \alpha) = F(u, \alpha) = \int_0^u f(t)\,dt\;, \qquad u = \frac{\alpha x}{p}\;, \qquad f(t) = \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}\;, \qquad x \geq 0\;.$$

Mean preservation holds by construction: $\mathbb{E}[X] = p$ for every α. The variance is given by p^2/α, so the natural increase in uncertainty is to decrease the parameter α, accordingly set $v = -\alpha$. All functions of the standardized coordinate $u \in (0, \infty)$ refer to the standard Gamma distribution $\Gamma(\alpha, 1)$: the density f, its logarithmic derivative $d_1 = f'/f = (\alpha-1)/u - 1$, the CDF F, the survival function $\bar{F} = 1 - F$, and the reciprocal hazard rate $\varphi = \bar{F}/f$; primes denote derivatives in u.

Now we differentiate F_G wrt α at fixed x. We use $\frac{du}{d\alpha} = \frac{x}{p} = \frac{u}{\alpha}$; and $\frac{\partial \log\Gamma(\alpha)}{\partial\alpha} = \psi(\alpha)$ so $\frac{\partial \log f(t)}{\partial\alpha} = \log t - \psi(\alpha)$, where ψ is the digamma function.

$$\frac{\partial F_G(x, \alpha)}{\partial\alpha} = \int_0^u f(t)\,(\log t - \psi(\alpha))\,dt + \frac{u}{\alpha}\,f(u)\;.$$

By the definition of the digamma function, the previous integral vanishes over the full support: using $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$,

$$\int_0^\infty f(t)\,(\log t - \psi(\alpha))\,dt = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \psi(\alpha) = 0\;.$$

Hence the integral from 0 to u equals $-G(u)$, where

$$G(u) \equiv \int_u^\infty f(t)\,(\log t - \psi(\alpha))\,dt\;, \qquad G(0) = G(\infty) = 0\;.$$

By definition $G'/f = -(\log u - \psi(\alpha))$, so $(G'/f)' = -1/u$. With $v = -\alpha$,

$$\frac{\partial F_G(x,v)}{\partial v} = G(u) - \frac{u}{\alpha} f(u) \; .$$

Substituting $x = pu/\alpha$, K becomes a function of $z = \alpha j/p$ alone:

$$K(j) = \frac{1}{1-F_G(j)} \int_0^j \frac{\partial F_G(x,v)}{\partial v}\, dx = \frac{p}{\alpha}\, g(z) \; , \qquad g \equiv \frac{A}{\bar F} \; , \qquad A(z) \equiv \int_0^z \Big(G(t) - \frac{t}{\alpha} f(t) \Big) dt \; .$$

Because $j \mapsto z = \alpha j/p$ is affine, K is convex on $(0,\infty)$ iff g is convex on $(0,\infty)$.

Differentiating twice, using $\bar F' = -f$, $f' = d_1 f$, and $1 + \varphi' = -d_1\varphi$,

$$g' = \frac{A'\bar F + Af}{\bar F^2} = \frac{f}{\bar F^2}\, H \; , \qquad H \equiv \varphi A' + A \; , \qquad \text{and} \qquad g'' = \frac{f^2}{\bar F^3}\left[(1-\varphi')\, H + \varphi H'\right] \; .$$

The Gamma density is log-concave for $\alpha \geq 1$, so the hazard rate is nondecreasing and $\varphi' \leq 0$, whence $1 - \varphi' \geq 1$. Thus, $g'' \geq 0$ wherever H and H' are both nonnegative, and where $H' < 0$ it suffices that $(1-\varphi')\, H \geq -\varphi H'$.

Now $H(0^+) = 0$: $A(0) = 0$, and $\varphi A' = \varphi G - \frac{u}{\alpha}\bar F \to 0$ since $\varphi G \sim u \log(1/u)/\alpha$. Using $A' = G - \frac{u}{\alpha} f$ and $A'' = G' - \frac{1}{\alpha}(uf)'$,

$$H' = (1+\varphi')A' + \varphi A'' = \varphi\,(A'' - d_1 A') = -\bar F\, \Sigma \; , \qquad \Sigma \equiv \frac{1}{\alpha} + \frac{d_1 G - G'}{f} \; ,$$

so $H(u) = -\int_0^u \bar F \Sigma\, dt$. Moreover,

$$\int_0^\infty \bar F\, \Sigma\, dt = \frac{1}{\alpha}\int_0^\infty \bar F\, dt + \int_0^\infty \varphi\,(d_1 G - G')\, dt = 1 - \int_0^\infty G\, dt = 0 \; ,$$

where the middle step uses $\int_0^\infty \bar F\, dt = \alpha$, the mean of the standard Gamma distribution, and integrates by parts using $\varphi d_1 = -(1+\varphi')$ and $[\varphi G]_0^\infty = 0$; and $\int_0^\infty G\, dt = 1$ follows from Fubini, the same digamma differentiation applied to $\Gamma(\alpha+1)$, and the recurrence $\psi(\alpha+1) = \psi(\alpha) + 1/\alpha$. Hence also $H(u) = \int_u^\infty \bar F \Sigma\, dt$.

Next we show that Σ is nondecreasing on $(0,\infty)$ for every $\alpha \geq 1$. Differentiating, using $G'' = d_1 G' - f/u$,

$$\Sigma' = \frac{1}{u} + \frac{d_1 G' - c_2 G}{f} = \frac{c_2}{f}\, D \; , \qquad D \equiv \frac{f/u + d_1 G'}{c_2} - G \; ,$$

where

$$c_2 \equiv d_1^2 - d_1' = d_1^2 + \frac{\alpha - 1}{u^2} \; ,$$

which is positive for every $\alpha \geq 1$ (at $\alpha = 1$, $c_2 \equiv 1$); so $\Sigma' \geq 0$ iff $D \geq 0$. A direct computation gives the exact identity

$$D' = -\frac{2(\alpha-1)\, uG' + P(u)\, f}{c_2^2\, u^4} \; , \qquad P(u) = u^2 - \alpha(\alpha-1) \; .$$

Dividing the numerator by $uf > 0$ and using $(G'/f)' = -1/u$, we obtain

$$\frac{d}{du}\left[2(\alpha-1)\, \frac{G'}{f} + \frac{P}{u} \right] = c_2 \geq 0 \; .$$

Thus the numerator changes sign at most once, from negative to positive, so D changes sign at most once, from positive to negative. When it does change sign, $\{D' \le 0\}$ *is an interval extending to the right end of the support: D rises and then falls. For* $\alpha > 1$*, at the endpoints* $G \to 0$*, while* $f/u + d_1 G' > 0$ *near each end (near 0,* $d_1 > 0$ *and* $G' > 0$*; for u large,* $d_1 < 0$ *and* $G' < 0$*), so* $\liminf D \ge 0$ *at both ends. A function that rises and then falls attains its infimum at the boundary, hence* $D \ge 0$ *on* $(0, \infty)$*. In the boundary case* $\alpha = 1$*, the* G'*-term vanishes and* $P = u^2 \ge 0$*, so* $D' \le 0$ *and D decreases to* $\liminf_{u\to\infty} D \ge 0$*. In all cases* Σ *is nondecreasing.*

Since Σ *is nondecreasing with* $\int_0^\infty \bar{F}\Sigma\, dt = 0$ *and* $\bar{F} > 0$*, either* $\Sigma \equiv 0$ *(then* $H \equiv 0$*, g is constant, and there is nothing to prove) or there is* $u_H \in (0, \infty)$ *with* $\Sigma \le 0$ *on* $(0, u_H]$ *and* $\Sigma \ge 0$ *on* $[u_H, \infty)$*. So H rises from* $H(0^+) = 0$ *and falls back to* $H(\infty) = 0$*; in particular* $H \ge 0$ *throughout. Below* u_H*,* $H(u) = -\int_0^u \bar{F}\Sigma\, dt \ge 0$ *and* $H' = -\bar{F}\Sigma \ge 0$*, so* $g'' \ge 0$*. Above* u_H*, the monotonicity of* Σ *gives*

$$H(u) = \int_u^\infty \bar{F}\Sigma\, dt \ge \Sigma(u) \int_u^\infty \bar{F}\, dt = \Sigma(u)\, \bar{F}(u)\, n(u)\ ,$$

where $n(u) = \int_u^\infty \bar{F}\, dt / \bar{F}(u)$ *is the solvent equity value of the standard Gamma distribution. The identity* $n' = n/\varphi - 1$ *implies* $n'' = (n/\varphi)'$ *and* $(1-\varphi')\, n - \varphi = \varphi^2\, (n/\varphi)'$*, so with* $-\varphi H' = \varphi\bar{F}\Sigma$*,*

$$(1-\varphi')\, H + \varphi H' \ge \bar{F}\, \Sigma(u)\, [(1-\varphi')\, n - \varphi] = \bar{F}\, \Sigma(u)\, \varphi^2\, n'' \ge 0\ ,$$

where the last inequality is the convexity of the solvent equity value for the Gamma family with $\alpha \ge 1$*, established in Appendix C. Therefore* $g'' \ge 0$ *on* $(0, \infty)$ *and* $j \mapsto K(j, v)$ *is convex on* $(0, \infty)$ *for every* $\alpha \ge 1$*: the Gamma family displays K-convexity along the mean-preserving path that decreases* α*, and the loan credit surface steepens.*

iii. Log-Normal Distribution. *For a log-normal distribution with mean e and scale v*

$$F(x, v) = \Phi(z(x, v)) \qquad \text{with} \quad z(x, v) = \frac{\log(x) - \mu(v)}{v} = \frac{\log(x/e)}{v} + \frac{v}{2}$$
$$\frac{\partial F(x, v)}{\partial v} = \phi\left(\frac{\log(x/e)}{v} + \frac{v}{2}\right)\left(-\frac{\log(x/e)}{v^2} + \frac{1}{2}\right)\ .$$

With the change of variables $u = \frac{\log(x/e)}{v} + \frac{v}{2}$ *so that* $x = e\, e^{v(u - \frac{v}{2})}$ *and* $dx = vx\, du$*, and using* $-\frac{\log(x/e)}{v^2} + \frac{1}{2} = 1 - \frac{u}{v}$*, we obtain*

$$\int_0^y \frac{\partial F(x, v)}{\partial v}\, dx = \int_{-\infty}^{z(y,v)} \phi(u)\Big(1 - \frac{u}{v}\Big)\, v\, x(u)\, du = e \int_{-\infty}^{z(y,v)} \phi(u - v)(v - u)\, du\ ,$$

where in the last equality we used that $\phi(u - v) = \phi(u)\ \exp\Big(vu - \frac{v^2}{2}\Big)$ *and* $vu - \frac{v^2}{2} = \log(x/e)$ *so* $\phi(u - v) = \phi(u)\, (x/e)$*. Using this relationship again and the change of variables* $w = u - v$ *we obtain*

$$\int_0^y \frac{\partial F(x, v)}{\partial v}\, dx = -e \int_{-\infty}^{z(y,v)-v} \phi(w) w\, dw = e\, \phi\Big(z(y, v) - v\Big) = y\, \phi\Big(z(y, v)\Big)\ .$$

Hence,

$$K(y, v) = \frac{1}{1 - F(y, v)} \int_0^y \frac{\partial F(x, v)}{\partial v}\, dx = y\, h\Big(z(y, v)\Big)\ .$$

Differentiating

$$K_y(y,v) = h\big(z(y,v)\big) + y\,h'\big(z(y,v)\big)\,\frac{\partial z(y,v)}{\partial y} = h\big(z(y,v)\big) + \frac{1}{v}\,h'\big(z(y,v)\big)$$
$$K_{yy}(y,v) = \frac{1}{vy}\left[h'\big(z(y,v)\big) + \frac{1}{v}\,h''\big(z(y,v)\big)\right] \geq 0\ .$$

The last inequality verifies convexity of $K(y,v)$ in y and follows from $h', h'' \geq 0$ (Lemma F.2).

iv. Power Function Distribution. *The power function distribution with shape $\alpha \geq 1$ on $[0, M]$, has CDF given by*

$$F(x,\alpha) = \left(\frac{x}{M}\right)^{\alpha}\ , \qquad 0 \leq x \leq M\ .$$

Mean preservation at $p = 1$ requires $M(\alpha) = p(\alpha+1)/\alpha$. The variance is given by $p^2/\alpha/(\alpha+2)$, so the natural increase in uncertainty is to decrease the parameter α, accordingly set $v = -\alpha$. Differentiating $F = \exp\{\alpha\,[\log x - \log M(\alpha)]\}$, and using $\frac{d}{d\alpha}\log M = \frac{1}{\alpha+1} - \frac{1}{\alpha} = -\frac{1}{\alpha(\alpha+1)}$,

$$\frac{\partial F(x,\alpha)}{\partial \alpha} = F(x,\alpha)\left[\log\frac{x}{M} + \frac{1}{\alpha+1}\right]\ .$$

Write $z = j/M \in (0,1)$ and substitute $x = Mt$. Using $\int_0^z t^{\alpha}\log t\,dt = z^{\alpha+1}\left[\frac{\log z}{\alpha+1} - \frac{1}{(\alpha+1)^2}\right]$ and $\int_0^z t^{\alpha}dt = \frac{z^{\alpha+1}}{\alpha+1}$, the $(\alpha+1)^{-2}$ terms cancel exactly and

$$K(j) = \frac{-1}{1-F(j)}\int_0^j \frac{\partial F(x,\alpha)}{\partial\alpha}\,dx = \frac{M}{\alpha+1}\left[\frac{z^{\alpha+1}(-\log z)}{1-z^{\alpha}}\right] \equiv \frac{M}{\alpha+1}g(z) > 0\ .$$

Then, because $y \mapsto z = j/M$ is affine and $M/(\alpha+1) > 0$, K is convex on $(0, M)$ iff g is convex on $(0,1)$. Let $A(z) = z^{\alpha+1}(-\log z)$ and $B(z) = 1 - z^{\alpha}$, then $g(z) = A(z)/B(z)$ and $B^3 g'' = A''B^2 - 2A'B'B - ABB'' + 2A\,(B')^2$ and after substitution and canceling terms

$$B^3 g'' = z^{\alpha-1}\left\{(-\log z^{\alpha})\,[\alpha+1+(\alpha-1)z^{\alpha}] - \Big[(2\alpha+1) - 2(\alpha+1)z^{\alpha} + z^{2\alpha}\Big]\right\} \equiv z^{\alpha-1}G(z^{\alpha})\ .$$

So g is convex iff $G > 0$. To see that $G > 0$ note that $G(1) = 0$. In addition, $G'(1) = 0$ and $G'' > 0$ in $(0,1)$, so G is decreasing in $(0,1)$ and thus positive in that interval. In fact, let $q = z^{\alpha} \in (0,1)$

$$G'(q) = -\frac{\alpha+1}{q} - (\alpha-1) + (-\log q)(\alpha-1) + 2(\alpha+1) - 2q\ .$$

Then, $G'(1) = 0$ and

$$G''(q) = \frac{\alpha+1}{q^2} - \frac{\alpha-1}{q} - 2 = \frac{1}{q^2}(1-q)(\alpha+1+2q) > 0\ ,$$

where we used that $\alpha \geq 1 > -1$ and $q \in (0,1)$.

v. Power-law Distribution. *To alleviate the notational burden we write simply β for $\beta(v)$ and note $\beta'(v) < 0$. With this convention, we can write the CDF of the normalized power-law distribution as follows.*

For $x \geq m_v = \frac{\beta-2}{\beta-1}$ the CDF is

$$F(x,v) = 1 - c(\beta)\, x^{-(\beta-1)} \;, \qquad \textit{with} \quad c(\beta) = \left(\tfrac{\beta-2}{\beta-1}\right)^{\beta-1} \;,$$

and $F(x,v) = 0$ for $x < x_v$.
For $y \geq x_v$,

$$1 - F(y,v) = c(\beta)\, y^{-(\beta-1)} \qquad \textit{and} \qquad f(y,v) = \frac{\partial F(y,v)}{\partial y} = c(\beta)(\beta-1)y^{-\beta} \;,$$

so the hazard is

$$h(y,v) = \frac{f(y,v)}{1-F(y,v)} = \frac{\beta-1}{y} \qquad \textit{and} \qquad h_y(y,v) = -\frac{\beta-1}{y^2} \;.$$

By the chain rule

$$\frac{\partial F(y,v)}{\partial v} = -\frac{\partial}{\partial \beta}\left(c(\beta) y^{-(\beta-1)}\right) \cdot \beta'(v) \;.$$

Now we compute the derivative of each term inside the parentheses

$$\frac{\partial}{\partial \beta}\left(y^{-(\beta-1)}\right) = y^{-(\beta-1)} \log(y^{-1}) = -y^{-(\beta-1)} \log(y)$$

$$c'(\beta) = c(\beta) \frac{d}{d\beta}\left[(\beta-1)\log\left(\frac{\beta-2}{\beta-1}\right)\right] \;.$$

With

$$\frac{d}{d\beta}\left[(\beta-1)\log\left(\frac{\beta-2}{\beta-1}\right)\right] = \log\left(\frac{\beta-2}{\beta-1}\right) + (\beta-1)\frac{\beta-1}{\beta-2}\left[\frac{1}{\beta-1} - \frac{\beta-2}{(\beta-1)^2}\right] = \log\left(\frac{\beta-2}{\beta-1}\right) + \frac{1}{\beta-2} \equiv a(\beta) \;.$$

Therefore:

$$\begin{aligned}\frac{\partial F(y,v)}{\partial v} &= -\beta'(v)\left[c'(\beta)y^{-(\beta-1)} - c(\beta)y^{-(\beta-1)}\log(y)\right] \\ &= -\beta'(v)c(\beta)y^{-(\beta-1)}\left[\log\left(\frac{\beta-2}{\beta-1}\right) + \frac{1}{\beta-2} - \log(y)\right] \\ &= \beta'(v)(1-F(y,v))\left(\log(y) - a(\beta)\right) \;.\end{aligned}$$

It will convenient to define

$$A(y,v) = \int_0^y \frac{\partial F(x,v)}{\partial v} dx \;.$$

Moreover, we can calculate the antiderivative of

$$\int_{m_v}^{y} (1-F(x,v))(\log x - a)\, dx = \int_{m_v}^{y} (1-F(x,v)) \log x\, dx - a \int_{m_v}^{y} (1-F(x,v))\, dx \;.$$

First, lets compute the first integral, via integration by parts with $u = \log(x)$ and $dv = x^{-(\beta-1)}dx$ (so

$du = dx/x$ and $v = -x^{-(\beta-2)}/(\beta-2)$)

$$\begin{aligned}
\int_{m_v}^{y}(1-F(x,v))\log x\,dx = c(\beta)\int_{m_v}^{y} x^{-(\beta-1)}\log x\,dx \\
&= -c(\beta)\,\log(x)\frac{x^{-(\beta-2)}}{\beta-2}\Big|_{m_v}^{y} - c(\beta)\int_{m_v}^{y}\frac{x^{-(\beta-1)}}{\beta-2}\,dx \\
&= -c(\beta)\log(y)\frac{y^{-(\beta-2)}}{\beta-2} + \frac{m_v\log(m_v)}{\beta-2} - \frac{c(\beta)}{(\beta-2)^2}\,x^{-(\beta-2)}\Big|_{m_v}^{y} \\
&= -c(\beta)\log(y)\frac{y^{-(\beta-2)}}{\beta-2} + \frac{m_v\log(m_v)}{\beta-2} - \frac{c(\beta)}{(\beta-2)^2}y^{-(\beta-2)} + \frac{m_v}{(\beta-2)^2}\ .
\end{aligned}$$

And

$$\int_{m_v}^{y}(1-F(x,v))\,dx = c(\beta)\int_{m_v}^{y} x^{-(\beta-1)}\,dx = -\frac{c(\beta)}{\beta-2}\,x^{-(\beta-2)}\Big|_{m_v}^{y} = -\frac{c(\beta)}{\beta-2}y^{-(\beta-2)} + \frac{m_v}{\beta-2}\ .$$

Thus,

$$\int_{m_v}^{y}(1-F(x,v))(\log x - a)\,dx = \frac{c(\beta)y^{-(\beta-2)}}{\beta-2}\left(-\log y - \frac{1}{\beta-2} + a\right) + \frac{m_v}{\beta-2}\left(\log m_v + \frac{1}{\beta-2} - a\right)\ .$$

But by definition $a = \log\left(\frac{\beta-2}{\beta-1}\right) + \frac{1}{\beta-2}$ so the last term in brackets if zero and we obtain

$$\int_{m_v}^{y}(1-F(x,v))(\log x - a)\,dx = \frac{c(\beta)y^{-(\beta-2)}}{\beta-2}\left(-\log y - \frac{1}{\beta-2} + a\right)\ .$$

Therefore,

$$K(y) = \frac{A(y,v)}{1-F(y,v)} = \frac{\beta'(v)c(\beta)y^{-(\beta-2)}}{(\beta-2)c(\beta)y^{-(\beta-1)}}\left(-\log y - \frac{1}{\beta-2} + a\right) = \frac{\beta'(v)y}{\beta-2}\left(-\log y - \frac{1}{\beta-2} + a\right)\ .$$

Differentiating,

$$K'(y) = \frac{\beta'(v)}{\beta-2}\left(-\log y - \frac{1}{\beta-2} + a\right) - \frac{\beta'(v)}{\beta-2}$$

$$K''(y) = -\frac{\beta'(v)}{(\beta-2)y} > 0\ ,$$

where the inequality follows from $\beta'(v) < 0$ and $y \geq m_v > 0$.

vi. Weibull Distribution. *The Weibull distribution with shape $k \geq 1$ and scale $\lambda > 0$ has CDF given by*

$$F(x,k) = 1 - e^{-(x/\lambda)^k}\ , \qquad x \geq 0\ .$$

Mean preservation at $p = 1$ requires $\lambda(k) = p/\Gamma(1+1/k)$. The variance is given by $p^2\left[\Gamma(1+2/k)/\Gamma(1+1/k)^2 - 1\right]$, decreasing in k, so the natural increase in uncertainty is to decrease the parameter k, accordingly set $v = -k$. Let $\alpha = 1/k \in (0,1]$ and $\psi = \psi(1+\alpha)$ denote the digamma function evaluated at $1+\alpha$, and let $z = (j/\lambda)^k$ denote the exponential coordinate, so that $1 - F(j) = e^{-z}$ and $\frac{dz}{dj} = h(j)$, *the hazard rate.*

Recall $\psi = \Gamma'/\Gamma$. *Differentiating* $F = 1 - \exp\{-\exp(k[\log x - \log\lambda(k)])\}$, *and using* $\frac{d\log\lambda}{dk} = -\psi\,\frac{d}{dk}\frac{1}{k} = \frac{\psi}{k^2}$,

$$\frac{\partial F(x,k)}{\partial k} = -e^{-t}\,\alpha t\,(\psi - \log t)\ , \qquad t = (x/\lambda)^k\ .$$

With $v = -k$, *substituting* $t = (x/\lambda)^k$, *so that* $dx = \lambda\alpha t^{\alpha-1}dt$ *and* $x : 0 \to j$ *maps to* $t : 0 \to z = (j/\lambda)^k$,

$$\int_0^j \frac{\partial F(x,v)}{\partial v}\,dx = \lambda\alpha^2 \int_0^z t^\alpha\,(\psi - \log t)\,e^{-t}\,dt\ .$$

The full integral vanishes as a consequence of the definition of the digamma function. Using that $\Gamma(1+\alpha) = \int_0^\infty t^\alpha e^{-t}\,dt$, *we have*

$$\int_0^\infty t^\alpha\,(\log t - \psi)\,e^{-t}\,dt = \Gamma'(1+\alpha) - \psi\,\Gamma(1+\alpha) = 0\ ;$$

so $\int_0^z t^\alpha\,(\psi - \log t)\,e^{-t}\,dt = \int_z^\infty t^\alpha\,(\log t - \psi)\,e^{-t}\,dt$ *and*

$$K(j) = \frac{1}{1-F(j)}\int_0^j \frac{\partial F(x,v)}{\partial v}\,dx = \lambda\alpha^2\,e^z G(z)\ , \qquad G(z) = \int_z^\infty t^\alpha\,(\log t - \psi)\,e^{-t}\,dt\ .$$

Let $B(t) = 1 + \alpha\,(\log t - \psi)$, *strictly increasing with unique zero* $z^* = e^{\psi - k}$. *Since* $\frac{d}{dt}\left[t^\alpha\,(\log t - \psi)\right] = t^{\alpha-1}B(t)$, *integrating* G *by parts gives*

$$G(z) = z^\alpha e^{-z}\,(\log z - \psi) + H(z)\ , \qquad H(z) = \int_z^\infty e^{-t}t^{\alpha-1}B(t)\,dt\ ,$$

and letting $z \to 0$ *(the boundary term vanishes) shows* $H(0) = G(0) = 0$. *Note also that* $H = G + G'$, *since* $G'(z) = -z^\alpha\,(\log z - \psi)\,e^{-z}$. *Then, using* $\frac{dz}{dj} = h$,

$$K'(j) = \lambda\alpha^2\,e^z\,[G(z) + G'(z)]\,h(j) = \lambda\alpha^2\,e^z H(z)\,h(j)\ .$$

Differentiating once more, using $H'(z) = -e^{-z}z^{\alpha-1}B(z)$ *and* $h' = (1-\alpha)h^2/z$ *(from* $h(j) = (k/\lambda)\,z^{1-\alpha}$*),*

$$K''(j) = \lambda\alpha^2 h^2\,e^z\left[H + H' + \frac{1-\alpha}{z}H\right] = \lambda\alpha^2 h^2\,\frac{e^z}{z}\left[(z+1-\alpha)\,H(z) - z^\alpha e^{-z}B(z)\right]\ ,$$

so K *is convex if and only if* $(z+1-\alpha)\,H(z) \geq z^\alpha e^{-z}B(z)$ *for all* $z > 0$.

Below the barrier, $z \leq z^*$, *the right side is nonpositive, while* $B < 0$ *on* $(0,z)$ *gives* $H(z) = H(0) - \int_0^z e^{-t}t^{\alpha-1}B(t)\,dt > 0$, *so the left side is positive.*

Above the barrier we use a sharp bound for the upper incomplete gamma function: for $\alpha \in (0,1]$ *and all* $z > 0$,

$$\int_z^\infty t^{\alpha-1}e^{-t}\,dt\ \geq\ \frac{z^\alpha e^{-z}}{z+1-\alpha}\ .$$

This is the first Padé lower bound for the upper incomplete gamma function, see Eq. 8.10.5 in Olver, Lozier, Boisvert, and Clark (2010), and Luke (1969).(At $k = 1$, *where* $1-\alpha = 0$, *it holds with equality.) To establish the inequality, let* $\varphi(t) = t^\alpha e^{-t}/(t+1-\alpha)$. *A direct computation gives the pointwise identity*

$$t^{\alpha-1}e^{-t} + \varphi'(t) = \frac{(1-\alpha)\,t^{\alpha-1}e^{-t}}{(t+1-\alpha)^2}\ \geq\ 0\ ,$$

and integrating it over (z, ∞), using $\varphi(z) = -\int_z^\infty \varphi'(t)\,dt$, yields the inequality. Then, for $z \geq z^*$, since $B(z) \geq 0$ and B is increasing,

$$H(z) \;\geq\; B(z) \int_z^\infty t^{\alpha-1} e^{-t}\,dt \;\geq\; B(z)\,\frac{z^\alpha e^{-z}}{z+1-\alpha}\,,$$

which is precisely the required inequality. Therefore $j \mapsto K(j, v)$ is convex on $(0, \infty)$ for every $k \geq 1$: the Weibull family displays K-convexity, and the loan credit surface steepens. ■

Proof of Theorem 7:
i. When $F_1^D \lesssim_{MPS} F_0^D$ then $L_0(j) = \mathbb{E}[\min\{j, X_0\}] \geq \mathbb{E}[\min\{j, X_1\}] = L_1(j)$. Thus $S_1(j) - S_0(j) \geq 0$.

ii. When $F_1 \lesssim_1 F_0$ we have that for any $x \geq 0$, $F_1(x) \geq F_0(x)$, thus

$$\frac{dL_0(j)}{dj} = 1 - F_0(j) \geq 1 - F_1(j) = \frac{dL_1(j)}{dj}\,.$$

Moreover, $L_0(0) = L_1(0) = 0$, so we conclude that $L_0(j) - L_1(j) \geq 0$.

iii. When $F_1 \lesssim_H F_0$ then $F_1 \lesssim_1 F_0$, so from part (ii) we conclude $L_0(j) - L_1(j)$ is positive and increasing. In addition, since $F_1 \lesssim_H F_0$ then $[1 - F_0(j)]/[1 - F_1(j)]$ is increasing and we conclude that

$$\frac{\frac{dL_0(j)}{dj}}{\frac{dL_1(j)}{dj}} = \frac{1 - F_0(j)}{1 - F_1(j)} \quad \textit{is increasing}\,.$$

So we conclude that L_0 is more convex than L_1.

iv. *Note that*

$$\lim_{j\to\infty} \frac{S_0(j)}{S_1(j)} = \lim_{j\to\infty} \frac{j\,L_1(j)}{j\,L_0(j)} = \lim_{j\to\infty} \frac{L_1(j)}{L_0(j)} = 1\,.$$

Finally, taking derivative in equation (4) we obtain

$$\frac{dS_v(j)}{dj} = \frac{1}{L_v(j)} - \frac{j[1 - F_v(j)]}{[L_v(j)]^2} = \frac{\mathbb{E}[\tilde{X}_v \mathbb{1}_{\{\tilde{X}_v \leq j\}}]}{[L_v(j)]^2}\,.$$

Moreover, if either $F_1^D \lesssim_{MPS} F_0^D$ or $F_1 \lesssim_1 F_0$ it must be that $f_1(x) - f_0(x) > 0$ for $x \leq x^\star$ for some $x^\star > 0$ and we conclude that $\mathbb{E}[X_1 \mathbb{1}_{\{X_1 \leq j\}}] > \mathbb{E}[X_0 \mathbb{1}_{\{X_0 \leq j\}}]$. In addition, since $L_1(j) \leq L_0(j)$, thus, $dS_1(j)/dj \geq dS_0(j)/dj$ for $x \leq x^\star$. Furthermore, from part (i) we know that as $j \to \infty$, $S_1(j) - S_0(j) \to 0$, so we conclude that $S_1(j) - S_0(j)$ is hump shaped. ■

C Convexity of the Solvent Equity Value

This appendix proves that the solvent equity value (or mean-residual life) $n(j) = \mathbb{E}[X - j \mid X > j]$ is convex ($n''(j) \geq 0$) for each distribution listed in Table 2. Two methods are used. The first method applies the Lemma C.1, established below for distributions with increasing hazard rate h

and convex reciprocal hazard rate $1/h$. The second method establishes convexity directly for the specific distribution.

Lemma C.1 (Convexity of the Solvent Equity Value) *Let X have support $(0, \infty)$, density f, and hazard rate $h = f/(1-F)$. If (i) h is non-decreasing and (ii) $1/h$ is convex, then $n(j) = \mathbb{E}[X - j \mid X > j]$ is convex.*

Proof of Lemma C.1: *Define $k(j) = n(j)h(j)$. From equation (B.2) $n' = nh - 1 = k - 1$, from which $k' = n''$. In addition, we have that*

$$k' = n\left[h^2 + h'\right] - h \qquad \text{and} \qquad k'' = n\left[h^3 + 3hh' + h''\right] - \left[h^2 + 2h'\right]$$

At any critical point j^* of k (where $k'(j^*) = 0$), we have that $n(j^*) = h(j^*)/[(h(j^*))^2 + h'(j^*)]$ and thus

$$k''(j^*) = \frac{-2(h'(j^*))^2 + h(j^*)h''(j^*)}{(h(j^*))^2 + h'(j^*)}.$$

Since $(1/h)'' = [2(h')^2 - hh'']/h^3$, condition (ii) implies $-2(h')^2 + hh'' \leq 0$ at j^*, so $k''(j^*) \leq 0$: every critical point of k is a local maximum. Using that $h(x) = -[\log(1 - F(x))]'$, we have $[1 - F(j + y)]/[1 - F(j)] = e^{-\int_j^{j+y} h(z)\,dz}$. And using equation (B.1) and integrating by substitution using $y = t/h(j)$ we obtain

$$k(j) = h(j) \int_0^\infty e^{-\int_j^{j+y} h(u)\,du}\,dy = \int_0^\infty e^{-\int_j^{j+t/h(j)} h(u)\,du}\,dt\,.$$

Since h is nondecreasing, for all $u \in [j, j + t/h(j)]$, $h(j) \leq h(u) \leq h(j + t/h(j))$. Thus $\int_j^{j+y} h(u)\,du \geq y\,h(j)$, so

$$\int_0^\infty e^{-t/h(j)\,h(j+t/h(j))}\,dt \leq k(j) \leq \int_0^\infty e^{-t/h(j)\,h(j)}\,dt = -e^{-t}\Big|_0^\infty = 1\,.$$

As $j \to \infty$, $h(j + t/h(j))/h(j) \to 1$, whence $k(j) \to 1$, as $j \to \infty$; and $k(j) \leq 1$. The bound is strict at $j = 0$—giving $k(0) < 1$—because h is strictly increasing on part of the support. A function with no local minimum, starting below 1 and tending to 1, must be increasing. Therefore $n'' = k' \geq 0$. ∎

We now verify conditions (*i*) and (*ii*) for the six distributions covered by Lemma C.1. All six have log-concave densities above the lower truncation point m, which implies $h' \geq 0$. We verify $(1/h)'' \geq 0$ for each.

Normal (truncated at m). $h' \geq 0$ holds by log-concavity. Writing $z = (j - \mu)/\sigma$, the inverse hazard is $1/h = \sigma\bar{\Phi}(z)/\varphi(z)$, and one verifies $(1/h)'' = \sigma^2[\varphi(z)\bar{\Phi}(z) + \varphi(z)^2]/\bar{\Phi}(z)^3 \geq 0$.

Logistic (truncated at m). $h' \geq 0$ holds by log-concavity. Writing $z = (j - \mu)/s$, we have $h(j) = e^{-z}/(s(1 + e^{-z}))$, and a direct calculation confirms $(1/h)'' \geq 0$.

Extreme Value (Gumbel) (truncated at m). The Gumbel density is log-concave, giving $h' \geq 0$, and $(1/h)'' \geq 0$ holds by direct computation.

Laplace (truncated at $m \geq \mu$). For $j \geq \mu$, the tail is $\bar{F}(j) \propto e^{-(j-\mu)/b}$ with constant hazard $h = 1/b$, so $1/h = b$ is constant and $(1/h)'' = 0$. IPHR holds trivially.

Gamma ($\alpha \geq 1$). The Gamma density is log-concave for shape parameter $\alpha \geq 1$, giving $h' \geq 0$. Convexity of $1/h = \bar{F}/f$ follows from the log-convexity of $\bar{F}/f$ as a function of j, verified via the ratio of incomplete gamma functions.

Weibull ($k \geq 1$). The hazard rate is $h(j) = (k/\lambda)(j/\lambda)^{k-1}$, non-decreasing for $k \geq 1$. The inverse hazard is $1/h(j) = (\lambda/k)(j/\lambda)^{1-k}$. For $k = 1$, $1/h$ is constant. For $k > 1$, $(1/h)''(j) = (\lambda/k)(1-k)(-k)\lambda^{k-1}j^{-(k+1)} > 0$ since $(1-k)(-k) = k(k-1) > 0$.

For three distributions, $n(j)$ is linear or constant, so $n'' = 0$ immediately.

Uniform. For $X \sim \mathrm{Uni}(m, M)$, $\bar{F}(x) = (M-x)/(M-m)$, so $n(j) = (M-j)/2$ is linear and $n'' = 0$.

Exponential. The memoryless property gives $n(j) = 1/\lambda$ (constant), so $n'' = 0$.

Power Law. For $f(x) = cx^{-\beta}$ on (m, ∞) with $\beta > 2$, we have $\bar{F}(x) = (m/x)^{\beta-1}$, and integrating gives

$$n(j) = \frac{1}{1-F(j)} \int_j^\infty 1 - F(u)\, du = \frac{\beta-1}{c} j^{\beta-1} \frac{c}{(\beta-1)(\beta-2)} j^{-(\beta-2)} = \frac{j}{\beta-2},$$

which is linear and thus convex ($n'' = 0$).

For the last two distributions we provide algebraic arguments.

Power Function. Let $f(x) = (\alpha/M)(x/M)^{\alpha-1}$ on $(0, M)$ with $\alpha \geq 1$. Setting $u = j/M$, integration gives the closed form

$$n(j) = M \cdot \frac{\alpha - (\alpha+1)u + u^{\alpha+1}}{(\alpha+1)(1-u^\alpha)}.$$

A direct calculation yields

$$n''(j) = \frac{\alpha^2}{(\alpha+1)M} \cdot \frac{u^{\alpha-2}\hat{Z}(u)}{(1-u^\alpha)^3},$$

where $\hat{Z}(u) = (\alpha-1)(1-u^{\alpha+1}) - (\alpha+1)u(1-u^{\alpha-1})$. The prefactor is positive for $u \in (0, 1)$, so it suffices to show $\hat{Z}(u) \geq 0$.

Factor $\hat{Z}(u) = (1-u^{\alpha+1})[(\alpha-1) - (\alpha+1)G(u)]$ where $G(u) = u(1-u^{\alpha-1})/(1-u^{\alpha+1})$. Since $(1-u^{\alpha+1}) \geq 0$, it suffices to show $G(u) \leq (\alpha-1)/(\alpha+1)$ for all $u \in [0, 1)$. By L'Hôpital's rule, $G(1^-) = (\alpha-1)/(\alpha+1)$. We show G is non-decreasing, so $G(u) \leq G(1^-)$. Letting $B = 1 - u^{\alpha+1}$, $G'B^2 \geq 0$ reduces to showing $N(u) = (1-u^{2\alpha}) - \alpha u^{\alpha-1}(1-u^2) \geq 0$.

Setting $t = u^2$, $N \geq 0$ is equivalent to $(1-t^\alpha)/(1-t) \geq \alpha t^{(\alpha-1)/2}$. Substituting $t = e^{-s}$ and $x = s/2$, this becomes $\sinh(\alpha x) \geq \alpha \sinh(x)$. To prove this, let $g(\alpha) = \sinh(\alpha x) - \alpha \sinh(x)$. Then $g(1) = 0$ and $g'(\alpha) = x\cosh(\alpha x) - \sinh(x)$. For $\alpha \geq 1$, $\cosh(\alpha x) \geq \cosh(x)$, so $g'(\alpha) \geq x\cosh(x) - \sinh(x)$, which equals zero at $x = 0$ and has derivative $x\sinh(x) \geq 0$. Hence $g(\alpha) \geq 0$, giving $\hat{Z}(u) \geq 0$ and $n''(j) \geq 0$.

Beta Distribution. Let X follow a Beta (a, b) distribution scaled to $(0, M)$ with $a, b \geq 1$. The argument uses a bounded-support adaptation of Lemma C.1: conditions (*i*)–(*ii*) imply every critical point of $p = nh$ is a local maximum. Since $p(0^+) = 0 < b/(b+1) = p(M^-)$ (for $a > 1$), p is non-decreasing, giving $n'' = p' \geq 0$. (For $a = 1$, p is identically constant and n is linear.) The Beta density is log-concave for $a, b \geq 1$, giving IPHR. It remains to establish $(1/h)'' \geq 0$.

Let $u = x/M$ and write $\varphi(u) = 1/h(u) = \bar{F}(u)/f(u)$. Setting $\rho = (1-u)/u$, one shows $\varphi(u) = (1-u)F(\rho)$ where $F(\rho) = \int_0^1 (1+w\rho)^{a-1}(1-w)^{b-1}\, dw$. A change-of-variables calculation then gives

$$\varphi'' \geq 0 \iff \frac{d}{d\rho}\left[\rho^2 F'(\rho)\right] \geq 0.$$

Differentiating, with $F'(\rho) = (a-1)\int_0^1 w(1+w\rho)^{a-2}(1-w)^{b-1}\, dw$:

$$\frac{d}{d\rho}\left[\rho^2 F'(\rho)\right] = (a-1)\int_0^1 w\rho(2 + aw\rho)(1+w\rho)^{a-3}(1-w)^{b-1}\, dw.$$

Every factor in the integrand is nonneg for $a, b \geq 1$ and $\rho > 0$, so $(1/h)'' \geq 0$. Therefore $n''(j) \geq 0$.

D Piece-wise Continuously Differentiable CDFs

Our results apply to situations where f is continuous and f' exists in the support of X, but can be generalized to situations where f is continuous and f' exists except possibly at isolated points $A \subseteq \mathbb{R}_+$. The set A can have infinitely countable points but has no accumulation in (m, M). The following definition will help us to describe the general case.

Definition 6 (Absolutely Continuous Distributions) *We say that F_0 is absolutely continuous with respect to F_1, written $F_0 << F_1$, if the probability measure $\mathbb{P}_0$ is absolutely continuous with respect to the measure $\mathbb{P}_1$. That is, for every measurable set $M \in \mathcal{F}$, $\mathbb{P}_1(M) = 0$ implies $\mathbb{P}_0(M) = 0$.*

The probability density function is well defined when $\mathbb{P}_v$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}_+$, which will be assumed in what follows. In these cases, the probability density f_v will be continuous almost everywhere in $\mathbb{R}_+$. Therefore, the set of points A, where f_v is not continuous, is of measure zero. Our assumptions amount to strengthen the absolute continuity requirement. The no-accumulation requirement rules out cases in which A is dense in some sub-interval—for instance, the endpoints of the deleted intervals in a fat Cantor set, where f_v exists as a function and F_v is absolutely continuous but the discontinuity points cluster everywhere on the Cantor set. In what follows, it will be convenient to include $a_0 = 0$ in the set A and denote with $a_0 < a_1 < \ldots$ the points in the set A. Finally, we add to the sequence the supremum of the possible values of collateral, $\sup_{\omega\in\Omega} X_v(\omega)$, so that regardless of the number of discontinuity points, we can recover the set of feasible promises as the union of intervals of the form $[a_i, a_{i+1})$, i.e., $\bigcup_{i\geq 0}[a_i, a_{i+1}) = [0, \sup_{\omega\in\Omega} X_v(\omega))$. Finally, recall that we assume that the forward expected value of collateral p is normalized to 1.

Theorem 2′ (The Shape of the Loan Credit Surface). *Suppose that F is continuous, f is continuous except possibly at a set of isolated points $A = \{a_0 < a_1 < \cdots\}$ and f'_+ exists for all $x \in (m, M)$. Let $\ell_i = L(a_i)$. Suppose that for some i*

$$\lim_{\ell\to\ell_i^+} \ell^3 S^{*\prime\prime}(\ell) = \frac{f(a_i^+)\ell_i^2}{[1 - F(a_i)]^3} - \frac{2\ell_i}{1 - F(a_i)} + 2a_i \geq 0 \,, \tag{D.1}$$

and for all $x \in (a_i, a_{i+1})$

$$\frac{df(x)}{dx} + \frac{3\,[f(x)]^2}{1 - F(x)} \geq 0 \,. \tag{D.2}$$

Then, the loan credit surface, $S^*(\ell)$, is convex on $[\ell_i, \ell_{i+1}]$. In particular, when the above conditions are satisfied for every interval (a_i, a_{i+1}), then the loan credit surface is convex on $[0, 1)$.

The theorem establishes that conditions (D.1) and (D.2) are sufficient for the loan credit surface to be convex on $[\ell_i, \ell_{i+1}]$. Condition (D.1) guarantees that the loan credit surface is convex to the right of the discontinuity point $a_i = J(\ell_i)$. This condition corresponds to a lower bound for $f(a_i^+)$ that limits how much the density can jump down at the discontinuity points $a_i \in A$.[36] Equation (D.2) corresponds to a lower bound for the derivative of the density function, df/dx, that guarantees that the second derivative of the credit surface with respect to the LTV ratio is increasing. Note

[36]The existence of $f(a_i^+)$ need not be assumed: (D.2) makes J'' non-decreasing on (ℓ_i, ℓ_{i+1}), so $J''(\ell_i^+)$ — and hence $f(a_i^+) = J''(\ell_i^+)[1 - F(a_i)]^3$ — exists and is finite as the infimum of J'' on the open interval.

that here the derivative of the density function is evaluated between discontinuity points, which are associated to the promises required to achieve an LTV, $\ell \in (\ell_i, \ell_{i+1})$.

Without the boundary condition (D.1), the inequality $J'''(\ell) \geq 0$ on each piece guarantees only that $\ell^3 S^{*\prime\prime}(\ell)$ is non-decreasing on the piece — not that it is non-negative; $S^{*\prime\prime}$ could turn positive only after spending time negative. The boundary condition rules this out by requiring $\ell^3 S^{*\prime\prime}$ to enter each piece at a non-negative value. Continuity of f on each open interval (a_i, a_{i+1}) enters here: it makes $\ell^3 S^{*\prime\prime}(\ell)$ continuous on the piece, so the right-limit value supplied by (D.1) propagates as a lower bound across the whole piece once (D.2) makes $\ell^3 S^{*\prime\prime}$ non-decreasing.

Proof of Theorem D:
i. *By F absolutely continuous, $L(j) = j - \int_0^j F(x)\,dx$ is continuous and concave with $L(0) = 0$; hence $J = L^{-1}$ is continuous and convex with $J(0) = 0$, so $S^*(\ell) = J(\ell)/\ell - 1$ is positive and increasing on $[0, 1)$ with $S^*(0) = 0$.*
ii. *By inverse-function differentiation, $J'(\ell) = 1/[1 - F(J(\ell))]$; on each open piece (ℓ_i, ℓ_{i+1}), where f is continuous and f' exists, $J''(\ell) = f(J(\ell))/[1-F(J(\ell))]^3$ and $J'''(\ell) = \{f'(J(\ell)) + 3f(J(\ell))^2/[1-F(J(\ell))]\}/[1-F(J(\ell))]^4$. Expanding $S^*(\ell) = J(\ell)/\ell - 1$ yields $\ell^3 S^{*\prime\prime}(\ell) = \ell^2 J''(\ell) - 2\ell J'(\ell) + 2J(\ell)$, so $\frac{d}{d\ell}[\ell^3 S^{*\prime\prime}(\ell)] = \ell^2 J'''(\ell)$. Condition (D.2) gives $J'''(\ell) \geq 0$ on each open piece, hence $\psi(\ell) := \ell^3 S^{*\prime\prime}(\ell)$ is non-decreasing on the open piece.*
iii. *Condition (D.1) is exactly $\psi(\ell_i^+) \geq 0$. Combined with step ii, $\psi \geq 0$ on (ℓ_i, ℓ_{i+1}), hence $S^{*\prime\prime} \geq 0$ there, so $S^{*\prime}$ is non-decreasing on the open piece. Although $S^{*\prime\prime}$ may itself jump down at ℓ_i, the slope $J'(\ell) = 1/[1 - F(J(\ell))]$ is continuous (since F and J are continuous), so $S^{*\prime}(\ell) = J'(\ell)/\ell - J(\ell)/\ell^2$ is continuous on $(0, 1)$; therefore $S^{*\prime}$ is non-decreasing across each break-point and hence on $[0, 1)$, so S^* is convex on each closed sub-interval $[\ell_i, \ell_{i+1}]$.*
iv. *By the no-accumulation hypothesis, A is locally finite in (m, M), so any compact $[x, y] \subset (m, M)$ contains finitely many a_i. Since $S^{*\prime}$ is continuous on $[0, 1)$ and non-decreasing on each $[\ell_i, \ell_{i+1}]$, it is non-decreasing on $[0, 1)$, hence S^* is convex on $[0, 1)$.* ■

An analogous piecewise generalization of Theorem 3 holds. The boundary condition at each jump a_i^+ is $\phi(a_i^+) \geq 0$, where ϕ is the function appearing in condition (iii) of Theorem 3 — equivalently a lower bound on $f(a_i^+)$. The proof parallels the above and is omitted.

E Alternative Proof

In this section, we provide an alternative proof for the steepening result for some family of distributions by establishing the log-submodularity of the density of F^{D*} as a function of ℓ and v.

Alternative Proof. *In this proof, we use the fact that the likelihood ratio order is stronger than the hazard rate order, i.e., $X_1 \lesssim_{LR} X_0 \Rightarrow X_1 \lesssim_H X_0$. Moreover, the dual distributions $F_1^{D*} \lesssim_{LR} F_0^{D*}$ iff*

$$\frac{\frac{\partial F(J(\ell,v_0),v_0)}{\partial \ell}}{\frac{\partial F(J(\ell,v_1),v_1)}{\partial \ell}} = \frac{f(J(\ell,v_0),v_0)\frac{\partial J(\ell,v_0)}{\partial \ell}}{f(J(\ell,v_1),v_1)\frac{\partial J(\ell,v_1)}{\partial \ell}} \quad \text{is increasing in } \ell\ .$$

This is equivalent to $f(J(\ell,v),v)dJ(\ell,v)/d\ell$ being log-submodular, which in turn is equivalent to the cross-derivative of the following function being negative[37]

[37] Note that our convention is that a *greater* uncertainty index corresponds to a *smaller* random variable in the LR order; therefore, we need log-submodularity as opposed to log-supermodularity.

$$Q(\ell,v)=\log\left(f(J(\ell,v),v)\frac{\partial J(\ell,v)}{\partial \ell}\right) .$$

i. Log-Normal Distributions. *In this case,* $\log(X_v)\sim N(\mu_v,v^2)$*. Note that* $\mu_v=-v^2/2$ *so* $\mathbb{E}[X_v]=\exp(\mu_v+v^2/2)=1$*. Levy (1973) showed that two log-normal distributions can be ordered by SSD if the mean of the logarithm decreases, the variance of the logarithm increases, and the mean of the random variable decreases. That is, if* $\mu_0\geq\mu_1$*,* $\sigma_0\leq\sigma_1$ *and* $\mathbb{E}[X_0]\geq\mathbb{E}[X_1]$*, which are verified if* $\sigma_0\leq\sigma_1$*. In this case,* $z_v(j)=\log(J_v(\ell))/v+v/2$*. Then,* $F_v(x)=\Phi(z_v(x))$ *and* $f_v(x)=\phi(z_v(x))/(vx)$*.*

It will be useful to express $L_v(j)$ *in closed form (equation (15)). Using that* $\mathbb{E}[X_v]=e^{-\sigma_v^2/2+\sigma_v^2/2}=1$*, we can write*

$$L_v(j)=\int_0^j xf_v(x)dx+j\left[1-F_v(j)\right] .$$

We calculate the first term, the truncated expectation of X_v*, integrating by substitution twice*

$$\int_0^j xf_v(x)dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_v(j)}e^{\frac{-(z-v)^2}{2}}dz=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z_v(j)-v}e^{\frac{-w^2}{2}}dz=\Phi\left(z_v(j)-v\right) , \tag{E.1}$$

where $z_v(j)=\log(J_v(\ell))/v+v/2$*. Finally, using equation (E.1) and* $1-F_v(j)=1-\Phi(z_v(j))=\Phi(-z_v(j))$*, we obtain equation (15).*

In what follows, it will be useful to denote $z(\ell,v)=\log(J(\ell,v))/v+v/2$*. To establish the likelihood ratio order, we need to establish that the cross derivative of the following function is negative*

$$\begin{aligned}Q(\ell,v)&=\log\left(\phi(z(\ell,v))\right)-\log(v)-\log(J(\ell,v))-\log\left[1-\Phi(z(\ell,v))\right]\\&=\log\left(\phi(z(\ell,v))\right)-\log(v)-vz(\ell,v)+\frac{v^2}{2}-\log\left[1-\Phi(z(\ell,v))\right] .\end{aligned}$$

Taking derivative wrt ℓ *we get*

$$\frac{\partial Q(\ell,v)}{\partial\ell}=\frac{1}{\phi(z(\ell,v))}\frac{d\phi(z(\ell,v))}{dz}\frac{\partial z(\ell,v)}{\partial\ell}-v\frac{\partial z(\ell,v)}{\partial\ell}-\frac{1}{1-\Phi(z(\ell,v))}\left(-\phi(z(\ell,v))\right)\frac{\partial z(\ell,v)}{\partial\ell} .$$

And using that $d\phi(z(\ell,v))/dz=-z(\ell,v)\phi(z(\ell,v))$*,* $\partial z(\ell,v)/\partial\ell=1/(vJ(\ell,v))\partial J(\ell,v)/\partial\ell$ *and* $J(\ell,v)=e^{vz(\ell,v)-v^2/2}$ *yields*

$$\frac{\partial Q(\ell,v)}{\partial\ell}=\frac{e^{-vz(\ell,v)+v^2/2}}{v[1-\Phi(z(\ell,v))]}\left[\frac{\phi(z(\ell,v))}{1-\Phi(z(\ell,v))}-(z(\ell,v)+v)\right] .$$

Taking derivatives wrt v *we get*

$$\begin{aligned}\frac{\partial^2 Q(\ell,v)}{\partial v\partial\ell}=\frac{e^{-vz(\ell,v)+v^2/2}}{v[1-\Phi(z(\ell,v))]}\Bigg\{&\\&\left[-z(\ell,v)-v\frac{\partial z(\ell,v)}{\partial v}+v-\frac{1}{v}+\frac{\phi(z(\ell,v))}{1-\Phi(z(\ell,v))}\frac{\partial z(\ell,v)}{\partial v}\right]\left[\frac{\phi(z(\ell,v))}{1-\Phi(z(\ell,v))}-z(\ell,v)-v\right]\\&+\frac{1}{[1-\Phi(z(\ell,v))]}\frac{d\phi(z(\ell,v))}{dz}\frac{\partial z(\ell,v)}{\partial v}+\frac{\left[\phi(z(\ell,v))\right]^2}{[1-\Phi(z(\ell,v))]^2}\frac{\partial z(\ell,v)}{\partial v}-\frac{\partial z(\ell,v)}{\partial v}-1\Bigg\}\end{aligned}$$

To calculate $\partial z(\ell, v)/\partial v$ we substitute $J(\ell, v) = e^{vz(\ell,v)-v^2/2}$ and take derivatives in equation (15)

$$0 = \phi\left(z(\ell, v) - v\right)\left[\frac{\partial z(\ell, v)}{\partial v} - 1\right] \\ + e^{vz(\ell,v)-v^2/2}\left[1 - \Phi\left(z(\ell, v)\right)\right]\left[z(\ell, v) + v\frac{\partial z(\ell, v)}{\partial v} - v\right] - e^{vz(\ell,v)-v^2/2}\phi\left(z(\ell, v)\right)\frac{\partial z(\ell, v)}{\partial v}$$

Note that

$$e^{vz-v^2/2}\phi\left(z(\ell, v)\right) = e^{vz-v^2/2}\frac{1}{\sqrt{2\pi}}e^{-z^2/2} = \frac{1}{\sqrt{2\pi}}e^{-(-2vz+v^2+z^2)/2} = \frac{1}{\sqrt{2\pi}}e^{-(z-v)^2/2} = \phi(z(\ell, v) - v)\ . \quad \text{(E.2)}$$

Then, we obtain

$$\frac{\partial z(\ell, v)}{\partial v} = \frac{\phi(z(\ell, v))}{v\left[1 - \Phi\left(z(\ell, v)\right)\right]} - \frac{z(\ell, v) - v}{v}\ . \quad \text{(E.3)}$$

Using equation (E.3) and denoting the hazard rate of the standard normal $h(z) = \phi(z)/[1 - \Phi(z)]$, we get that

$$\frac{\partial^2 Q(\ell, v)}{\partial v \partial \ell} = \frac{e^{-vz(\ell,v)+v^2/2}}{v[1 - \Phi(z(\ell, v))]}\Bigg\{ \\ \left[-h(z(\ell, v)) - \frac{1}{v} + h(z(\ell, v))\left(\frac{h(z(\ell, v)) - z(\ell, v) + v}{v}\right)\right]\left[h(z(\ell, v)) - z(\ell, v) - v\right] \\ + h(z(\ell, v))\left[h(z(\ell, v)) - z(\ell, v)\right]\left[\frac{h(z(\ell, v)) - z(\ell, v) + v}{v}\right] - \frac{h(z(\ell, v)) - z(\ell, v) + v}{v} - 1\Bigg\}\ .$$

Rearranging terms

$$\frac{\partial^2 Q(\ell, v)}{\partial v \partial \ell} = \frac{e^{-vz(\ell,v)+v^2/2}}{v[1 - \Phi(z(\ell, v))]}\Bigg\{ \\ \frac{2h(z(\ell, v))\left[h(z(\ell, v)) - z(\ell, v)\right]\left[h(z(\ell, v)) - z(\ell, v) + v\right]}{v} - 2h(z(\ell, v))\left[h(z(\ell, v)) - z(\ell, v)\right] \\ - \frac{2\left[h(z(\ell, v)) - z(\ell, v)\right]}{v} - 1\Bigg\}\ .$$

Then,

$$\frac{\partial^2 Q(\ell, v)}{\partial v \partial \ell} < \frac{2e^{-vz(\ell,v)+v^2/2}}{v^2[1 - \Phi(z(\ell, v))]}\left[h(z(\ell, v)) - z(\ell, v)\right]\left\{h(z(\ell, v))\left[h(z(\ell, v)) - z(\ell, v)\right] - 1\right\} < 0\ ,$$

where the inequality follows from Lemma F.2, since for all $z \in \mathbb{R}$, $h(z) - z > 0$ and $h(z)[h(z) - z] < 1$.

ii. Normal Distributions. *For convenience we denote $v = \sigma_v$ and a_v the truncation point on the left, which is characterized by $\Phi(z_v(a_v)) = \Phi(-\delta)$, or simply $z_v(a_v) = -\delta$. In this case, $F(x, v)$ and $f(x, v)$ are given by*

$$F(x, v) = \frac{\Phi(z_v(x)) - \Phi(-\delta)}{1 - \Phi(-\delta)} \qquad \text{and} \qquad f(x, v) = \frac{\phi(z_v(x))}{v\left[1 - \Phi(-\delta)\right]}\ .$$

Note that since $v_1 > v_0$ then $a_1 < a_0$ and $F(x, v_1) - F(x, v_0)$ single-crosses from above. Then, since $\mathbb{E}[X_0] = \mathbb{E}[X_1]$ we conclude that $F_0 \gtrsim_2 F_1$.
As was the case for the log-normal distribution, we can write $L_v(j)$ in terms of the standard normal distribution. Using integration by substitution, with $z = z_v(x)$, $dz = dx/v$ and $x = vz + \mu_v$ and using that $d\phi(z)/dz = -z\phi(z)$ we get

$$\int_{a_v}^{j} x f_v(x)dx = \frac{1}{1-\Phi(-\delta)} \int_{-\delta}^{z_v(j)} (vz+\mu_v)\phi(z)dz = \frac{\mu_v[\Phi(z_v(j)) - \Phi(-\delta)]}{1-\Phi(-\delta)} - \frac{v[\phi(z_v(j)) - \phi(-\delta)]}{1-\Phi(-\delta)} .$$

Finally, using that $\mathbb{E}[X_v] = 1$, we can write the LTV as

$$L_v(j) = \frac{\mu_v[\Phi(z_v(j)) - \Phi(-\delta)]}{1-\Phi(-\delta)} - \frac{v[\phi(z_v(j)) - \phi(-\delta)]}{1-\Phi(-\delta)} + j\left[\frac{1-\Phi(z_v(j))}{1-\Phi(-\delta)}\right] . \tag{E.4}$$

In what follows, it will be convenient to denote $z(\ell, v) = (J(\ell, v) - \mu_v)/v$. We need to establish that the cross derivative of the following function is negative

$$Q(\ell, v) = \log\left(f(J(\ell, v), v)\frac{\partial J(\ell, v)}{\partial \ell}\right) = \log\left(\phi\left(z(\ell, v)\right)\right) - \log(v) + \log\left(\mathbb{E}[X_v]\right) - \log\left(1 - \Phi\left(z(\ell, v)\right)\right) .$$

Taking derivatives wrt to ℓ we obtain

$$\frac{\partial Q(\ell, v)}{\partial \ell} = \frac{1}{\phi\left(z(\ell, v)\right)} \frac{d\phi(z(\ell, v))}{dz} \frac{\partial z(\ell, v)}{\partial \ell} + \frac{\phi\left(z(\ell, v)\right)}{1 - \Phi\left(z(\ell, v)\right)} \frac{\partial z(\ell, v)}{\partial \ell} .$$

Using that $\mathbb{E}[X_v] = 1$, $d\phi(z)/dz = -z\phi(z)$ and $\partial z(\ell, v)/\partial \ell = (1/v)\partial J(\ell, v)/\partial \ell$ we get that

$$\frac{\partial Q(\ell, v)}{\partial \ell} = \frac{1-\Phi(-\delta)}{v\left(1 - \Phi(z(\ell, v))\right)} \left[\frac{\phi\left(z(\ell, v)\right)}{1 - \Phi\left(z(\ell, v)\right)} - z(\ell, v)\right] .$$

Taking derivatives wrt to v

$$\begin{aligned} \frac{\partial^2 Q(\ell, v)}{\partial v \partial \ell} = \frac{1-\Phi(-\delta)}{v\left(1 - \Phi(z(\ell, v))\right)} \Bigg\{ & \left[-\frac{1}{v} + \frac{\phi\left(z(\ell, v)\right)}{1 - \Phi\left(z(\ell, v)\right)} \frac{\partial z(\ell, v)}{\partial v}\right] \left[\frac{\phi\left(z(\ell, v)\right)}{1 - \Phi\left(z(\ell, v)\right)} - z(\ell, v)\right] \\ & + \left[\frac{-z(\ell, v)\phi\left(z(\ell, v)\right)}{1 - \Phi\left(z(\ell, v)\right)} + \frac{\left[\phi\left(z(\ell, v)\right)\right]^2}{\left(1 - \Phi\left(z(\ell, v)\right)\right)^2} - 1\right] \frac{\partial z(\ell, v)}{\partial v} \Bigg\} . \end{aligned}$$

To calculate $\partial z/\partial v$, we take derivative wrt v in equation (E.4)

$$\begin{aligned} 0 = & \frac{d\mu_v}{dv}[\Phi(z(\ell, v)) - \Phi(-\delta)] + \mu_v \phi(z(\ell, v)) \frac{\partial z(\ell, v)}{\partial v} - [\phi(z(\ell, v)) - \phi(-\delta)] \\ & + vz(\ell, v)\phi(z(\ell, v)) \frac{\partial z(\ell, v)}{\partial v} + \frac{\partial J(\ell, v)}{\partial v}(1 - \Phi(-z(\ell, v))) - J(\ell, v)\phi(z(\ell, v)) \frac{\partial z(\ell, v)}{\partial v} . \end{aligned}$$

Note that

$$\mathbb{E}[X_v] = 1 = \mu_v + v\frac{\phi(-\delta)}{1-\Phi(-\delta)} \qquad \Rightarrow \qquad \frac{d\mu_v}{dv} = -\frac{\phi(-\delta)}{1-\Phi(-\delta)} .$$

Substituting this equation, reordering terms, and using that $vz(\ell,v) = J(\ell,v) - \mu_v$, we get

$$\frac{\partial J(\ell,v)}{\partial v} = \frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)} - \frac{\phi(-\delta)}{1-\Phi(-\delta)} \; .$$

Then, we have that

$$\frac{\partial z(\ell,v)}{\partial v} = \frac{1}{v}\left[\frac{\partial J(\ell,v)}{\partial v} + \frac{\phi(-\delta)}{1-\Phi(-\delta)} - z(\ell,v)\right] = \frac{1}{v}\left[\frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)} - z(\ell,v)\right] \; .$$

It follows that

$$\begin{aligned}
&\frac{\partial^2 Q(\ell,v)}{\partial v \partial \ell} = \\
&\quad \frac{1-\Phi(-\delta)}{v^2\left(1-\Phi(z(\ell,v))\right)}\left\{\left[\frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)}\left[\frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)} - z(\ell,v)\right] - 1\right]\left[\frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)} - z(\ell,v)\right]\right. \\
&\qquad \left. + \left[\frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)}\left[\frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)} - z(\ell,v)\right] - 1\right]\left[\frac{\phi\left(z(\ell,v)\right)}{1-\Phi\left(z(\ell,v)\right)} - z(\ell,v)\right]\right\} \\
&\qquad\qquad = \frac{2\left(1-\Phi(-\delta)\right)}{v^2\left(1-\Phi(z(\ell,v))\right)}\left[h\left(z(\ell,v)\right)\left[h\left(z(\ell,v)\right) - z(\ell,v)\right] - 1\right]\left[h\left(z(\ell,v)\right) - z(\ell,v)\right] < 0 \; ,
\end{aligned}$$

where the inequality follows from Lemma F.2, for all $x \in \mathbb{R}$, $h(z)\left[h(z) - z\right] < 1$ and $h(z) - z > 0$.

iii. Power-Law Distributions. *Note that when $\beta > 2$, the mean of X_β is given by*

$$\mathbb{E}[X_\beta] = \int_1^\infty (\beta-1)x^{-\beta}x dx = (\beta-1)\int_1^\infty x^{-(\beta-1)}dx = -\frac{\beta-1}{\beta-2}\left[x\right]^{-(\beta-2)}\Big|_1^\infty = \frac{\beta-1}{\beta-2} > 1 \; .$$

So $\mathbb{E}[X_\beta]$ is decreasing in β and $\mathbb{E}[X_1] > \mathbb{E}[X_0]$ for $\beta_1 < \beta_0$. Moreover, because both normalized random variables have the same mean, equal to 1, the lower bound of the support of $X_1/\mathbb{E}[X_1]$ is smaller than $X_0/\mathbb{E}[X_0]$, and the difference of their CDFs, $F_1(x\mathbb{E}[X_1]) - F_0(x\mathbb{E}[X_0])$ single-crosses from above, then $X_1/\mathbb{E}[X_1] \lesssim_2 X_0/\mathbb{E}[X_0]$.
It will be useful to denote $J(\ell,\beta) = J_\beta(\ell)$, $F(x,\beta) = F_\beta(x)$ and $f(x,\beta) = f_\beta(x)$.

When ℓ and β are such that $J(\ell,\beta) \leq 1$, we have that $F(J(\ell,\beta),\beta) = f(J(\ell,\beta),\beta) = 0$ or for some combinations of ℓ and β such that $J(\ell,\beta) = 1$, $F(J(\ell,\beta),\beta) = 0$ and $f(J(\ell,\beta),\beta) > 0$. Since j is increasing in ℓ, for all $\ell \leq \ell'$ and $\beta > 2$, $f(J(\ell',\beta),\beta) = 0$ imply $f(J(\ell,\beta),\beta) = 0$. In addition, since $X_1/\mathbb{E}[X_1] \lesssim_2 X_0/\mathbb{E}[X_0]$ from Theorem 4 we have $J(\ell,\beta_1) \geq J(\ell,\beta_0) \geq J(\ell,\beta_0)/\mathbb{E}[X_1]$ for $\beta_1 \leq \beta_0$. So we conclude that $1 \geq J(\ell,\beta_1) \geq J(\ell,\beta_0)$. Then, if $f(J(\ell,\beta_1),\beta_1) = 0$ it follows that $f(J(\ell,\beta_0),\beta_0) = 0$ and we verify that $f(J(\ell',\beta_0),\beta_0)f(J(\ell,\beta_1),\beta_1) \geq f(J(\ell,\beta_0),\beta_0)f(J(\ell',\beta_1),\beta_1)$ for all $\ell \leq \ell'$ and $\beta_1 \leq \beta_0$, and thus $F_1^{D*} \lesssim_{LR} F_0^{D*}$.

When ℓ and β are such that $J(\ell,\beta) > 1$, we need to establish that the cross derivative of the following function is positive

$$Q(\ell,\beta) = \log\left(f(J(\ell,\beta),\beta)\frac{\partial J(\ell,\beta)}{\partial \ell}\right) = \log\left(f(J(\ell,\beta),\beta)\right) + \log\left(\mathbb{E}[X_\beta]\right) - \log\left(1 - F(J(\ell,\beta),\beta)\right) \; .$$

Using that

$$1 - F(J(\ell,\beta),\beta) = \left[J(\ell,\beta)\right]^{-(\beta-1)} \qquad \text{and} \qquad f(J(\ell,\beta),\beta) = (\beta-1)\left[J(\ell,\beta)\right]^{-\beta} \; .$$

Then,

$$Q(\ell,\beta) = 2\log(\beta-1) - \log(\beta-2) - (2\beta-1)\log\left(J(\ell,\beta)\right) \ .$$

Taking derivative wrt ℓ

$$\frac{\partial Q(\ell,\beta)}{\partial \ell} = -\frac{2\beta-1}{J(\ell,\beta)}\frac{dJ(\ell,\beta)}{d\ell} = -\frac{(2\beta-1)(\beta-1)}{(\beta-2)\left[J(\ell,\beta)\right]^{-(\beta-2)}} \ .$$

And taking derivative wrt β

$$\begin{aligned}\frac{\partial^2 Q(\ell,\beta)}{\partial\beta\partial\ell} = -\frac{2(\beta-1)\left[J(\ell,\beta)\right]^{(\beta-2)}}{\beta-2} + \frac{(2\beta-1)\left[J(\ell,\beta)\right]^{(\beta-2)}}{\beta-2} \\ -(2\beta-1)(\beta-1)\left[J(\ell,\beta)\right]^{-(\beta-3)}\frac{\partial J(\ell,\beta)}{\partial\beta} + \frac{(2\beta-1)(\beta-1)\left[J(\ell,\beta)\right]^{(\beta-2)}}{(\beta-2)^2} \ .\end{aligned}$$

From the definition of LTV we have

$$\frac{\beta-1}{\beta-2}\ell = \int_1^{J(\ell,\beta)} xf(x,\beta)dx + J(\ell,\beta)[1-F(J(\ell,\beta),\beta)] \ .$$

The integral is given by

$$\int_1^{J_\beta(\ell)} xf_\beta(x)dx = (\beta-1)\int_1^{J_\beta(\ell)} x^{-(\beta-1)}dx = -\frac{\beta-1}{\beta-2}\,x^{-(\beta-2)}\Big|_1^{J_\beta(\ell)} = -\frac{\beta-1}{\beta-2}\left[J_\beta(\ell)^{-(\beta-2)}-1\right] \ .$$

Then,

$$\begin{aligned}\frac{\beta-1}{\beta-2}\ell &= -\frac{\beta-1}{\beta-2}\left[J_\beta(\ell)^{-(\beta-2)}-1\right] + J_\beta(\ell)[J_\beta(\ell)]^{-(\beta-1)} \\ 1 &= (\beta-1)(1-\ell)J_\beta(\ell)^{(\beta-2)} \ .\end{aligned}$$

It follows that

$$(\beta-1)\left[J(\ell,\beta)\right]^{-(\beta-3)}\frac{\partial J(\ell,\beta)}{\partial\beta} = \frac{\left[J(\ell,\beta)\right]^{(\beta-2)}}{\beta-2} - \frac{(\beta-1)\left[J(\ell,\beta)\right]^{(\beta-2)}}{(\beta-2)(1-\ell)} \ .$$

Substituting the previous expression and rearranging terms we have

$$\frac{\partial^2 Q(\ell,\beta)}{\partial\beta\partial\ell} = \frac{(\beta-1)\left[2\beta(\beta-2)+3(1-\ell)\right]\left[J(\ell,\beta)\right]^{(\beta-2)}}{(\beta-2)^2(1-\ell)} \geq 0 \ ,$$

where the inequality follows from $0 \leq \ell < 1$ and $\beta > 2$.

iv. Uniform Distributions. *In this case, for any $x \in [0, m_v)$, $F_v(x) = f_v(x) = 0$; and for any $x \in [m_v, \bar{x}_v]$, $F_v(x) = (x-m_v)/(\bar{x}_v - m_v)$ and $f_v(x) = 1/(\bar{x}_v - m_v)$.*
For any ℓ and v such that $J(\ell,v) \leq m_v$, we have $F(x,v) = f(x,v) = 0$ or for some combinations of ℓ and v $F(x,v) = 0$ and $f(J(\ell,v),v) > 0$. Since j is increasing in ℓ, the support expands with v and $J(\ell,v_1) \geq J(\ell,v_0)$ for any ℓ and $v_0 \leq v_1$ (Theorem 4), then for all ℓ and $v_0 \leq v_1$ we have $f(J(\ell,v_1),v_1) = 0$ implies $f(J(\ell,v_0),v_0) = 0$ and for all v and $\ell \leq \ell'$ we have $f(J(\ell',v),v) = 0$ imply $f(J(\ell,v),v) = 0$. Thus,

we verify that $f(J(\ell,v_1),v_1)f(J(\ell',v_0),v_0) \geq f(J(\ell',v_1),v_1)f(J(\ell,v_0),v_0)$ *for all* $\ell \leq \ell'$ *and* $v_0 \leq v_1$*, and so* $F_1^{D*} \lesssim_{LR} F_0^{D*}$*.*

When ℓ *and* v *are such that* $J(\ell,v) > m_v$*, then, we can write*

$$Q(\ell,v) = \log\left(f(J(\ell,v),v)\frac{\partial J(\ell,v)}{\partial \ell}\right) = \log\left(f(J(\ell,v),v)\right) + \log\left(\mathbb{E}[X_v]\right) - \log\left(1-F(J(\ell,v),v)\right) \ .$$

Taking derivatives wrt ℓ *we get*

$$\frac{\partial Q(\ell,v)}{\partial \ell} = \frac{f(J(\ell,v),v)}{1-F(J(\ell,v),v)}\frac{\partial J(\ell,v)}{\partial \ell} = \frac{f(J(\ell,v),v)\mathbb{E}[X_v]}{(1-F(J(\ell,v),v))^2} = \frac{\mathbb{E}[X_v]\,(\bar{x}_v - m_v)}{(\bar{x}_v - J(\ell,v))^2} \ .$$

Taking derivate wrt v *we get*

$$\frac{\partial^2 Q(\ell,v)}{\partial v \partial \ell} = \frac{\mathbb{E}[X_v]}{(\bar{x}_v - J(\ell,v))^2}\left[\frac{d\bar{x}_v}{dv} - \frac{dm_v}{dv}\right] - \frac{2\mathbb{E}[X_v]\,(\bar{x}_v - m_v)}{(\bar{x}_v - J(\ell,v))^3}\left[\frac{d\bar{x}_v}{dv} - \frac{\partial J(\ell,v)}{\partial v}\right] \ .$$

Since the expected value of collateral is constant, we have that

$$\frac{d\mathbb{E}[X_v]}{dv} = \frac{1}{2}\left[\frac{d\bar{x}_v}{dv} + \frac{dm_v}{dv}\right] = 0 \qquad \Rightarrow \qquad \frac{d\bar{x}_v}{dv} = -\frac{dm_v}{dv} \ . \tag{E.5}$$

In addition, we can differentiate the definition of LTV, keeping ℓ *fix, to obtain an expression for* $\partial j/\partial v$*. When* ℓ *is such that* $J_v(\ell) > m_v$ *we have*

$$\mathbb{E}[X_v]\ell = \int_{m_v}^{J(\ell,v)} x f(x,v)dx + J(\ell,v)[1-F(J(\ell,v),v)] = \frac{[J(\ell,v)]^2 - m_v^2}{2\,(\bar{x}_v - m_v)} + \frac{J(\ell,v)\,(\bar{x}_v - J(\ell,v))}{\bar{x}_v - m_v} \ .$$

Taking derivatives wrt v

$$\begin{aligned} 0 = {} & \frac{1}{2\,(\bar{x}_v - m_v)}\left[2J(\ell,v)\frac{\partial J(\ell,v)}{\partial v} - 2m_v\frac{dm_v}{dv}\right] - \frac{\left([J(\ell,v)]^2 - m_v^2\right)}{2\,(\bar{x}_v - m_v)^2}\left[\frac{d\bar{x}_v}{dv} - \frac{dm_v}{dv}\right] \\ & \frac{\partial J(\ell,v)}{\partial v}\frac{(\bar{x}_v - J(\ell,v))}{\bar{x}_v - m_v} + \frac{J(\ell,v)}{\bar{x}_v - m_v}\left[\frac{d\bar{x}_v}{dv} - \frac{\partial J(\ell,v)}{\partial v}\right] - \frac{J(\ell,v)\,(\bar{x}_v - J(\ell,v))}{(\bar{x}_v - m_v)^2}\left[\frac{d\bar{x}_v}{dv} - \frac{dm_v}{dv}\right] \ . \end{aligned}$$

Multiplying by $(\bar{x}_v - m_v)$*, using equation (E.5), canceling terms and rearranging we obtain*

$$0 = \left[-[J(\ell,v)]^2 + J(\ell,v)\,(\bar{x}_v + m_v) - \bar{x}_v m_v\right]\frac{d\bar{x}_v}{dv} - (\bar{x}_v - J(\ell,v))\,(\bar{x}_v - m_v)\,\frac{\partial J(\ell,v)}{\partial v}$$

$$\frac{\partial J(\ell,v)}{\partial v} = \left[\frac{-[J(\ell,v)]^2 + J(\ell,v)\,(\bar{x}_v + m_v) - \bar{x}_v m_v}{(\bar{x}_v - J(\ell,v))\,(\bar{x}_v - m_v)}\right]\frac{d\bar{x}_v}{dv} \ . \tag{E.6}$$

Using equations (E.5) and (E.6) we obtain that

$$\begin{aligned} \frac{\partial^2 Q(\ell,v)}{\partial v \partial \ell} &= \frac{2\mathbb{E}[X_v]}{(\bar{x}_v - J(\ell,v))^2}\frac{d\bar{x}_v}{dv}\left[1 - \frac{\bar{x}_v - m_v}{\bar{x}_v - J(\ell,v)} + \frac{-[J(\ell,v)]^2 + J(\ell,v)\,(\bar{x}_v + m_v) - \bar{x}_v m_v}{(\bar{x}_v - J(\ell,v))^2}\right] \\ &= -\frac{2\mathbb{E}[X_v]\bar{x}_v J(\ell,v)}{(\bar{x}_v - J(\ell,v))^4}\frac{d\bar{x}_v}{dv} < 0 \ . \end{aligned}$$

■

F Properties of the Hazard Rate of the Standard Normal Distribution

Let $\phi : \mathbb{R} \to \mathbb{R}_+$ and $\Phi : \mathbb{R} \to [0,1]$ denote the standard normal density function and the standard normal cumulative distribution function, respectively. Then, for finite $\delta > 0$, we define the truncated standard normal density function, $\bar{\phi} : [-\delta, \infty) \to \mathbb{R}_+$, and the truncated standard cumulative distribution function $\bar{\Phi} : [-\delta, \infty] \to [0,1]$ as

$$\bar{\Phi}(z) = \frac{\Phi(z) - \Phi(-\delta)}{1 - \Phi(-\delta)} \qquad \text{and} \qquad \bar{\phi}(z) = \frac{\phi(z)}{1 - \Phi(-\delta)} \, . \tag{F.1}$$

Lemma F.2. *Let $h(z) = \phi(z)/[1 - \Phi(z)]$, denote the hazard rate of the standard normal, and $\bar{h}(z) = \bar{\phi}(z)/[1 - \bar{\Phi}(z)]$, denote the hazard rate of the truncated standard normal, then*

i. $\bar{h}(z) = h(z)$, for all $z \in [-\delta, \infty)$,

ii. $0 < h'(z) = h(z)\,(h(z) - z) < 1$, for all $z \in \mathbb{R}$, and

iii. $h''(z) = h(z)\,[(h(z) - z)\,(2h(z) - z) - 1] > 0$, for all $z \in \mathbb{R}$.

Parts (*ii*) and (*iii*) of Lemma F.2 were established by Sampford (1953), here we present a similar proof for completeness.

Proof of Lemma F.2:
i. Observe that for $z \in [-\delta, \infty)$

$$1 - \bar{\Phi}(z) = 1 - \frac{\Phi(z) - \Phi(-\delta)}{1 - \Phi(-\delta)} = \frac{1 - \Phi(z)}{1 - \Phi(-\delta)} \, .$$

Thus,

$$\bar{h}(z) = \frac{\bar{\phi}(z)}{1 - \bar{\Phi}(z)} = \frac{\phi(z)}{1 - \Phi(z)} = h(z) \, .$$

ii. Taking derivative, and using that $\phi'(z) = -z\phi(z)$, we obtain

$$h'(z) = -\frac{z\phi(z)}{1 - \Phi(z)} + \frac{\left[\phi(z)\right]^2}{[1 - \Phi(z)]^2} = h(z)\,[h(z) - z] \, . \tag{F.2}$$

To show that $h'(z) > 0$, we only need to show that $h(z) > z$, since by definition $h(z) > 0$. For $z \le 0$, this is clearly true as $h(z) > 0 \ge z$. For $z > 0$, we proceed as follows. Let $g(z) = \phi(z) - z[1 - \Phi(z)]$. Then, $g(0) = \phi(0) > 0$. Recall also that $\lim_{z\to\infty} z[1 - \Phi(z)] = 0$, because for $z \ge 1$

$$1 - \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-t^2/2} dt \le \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-t/2} dt = -\frac{2}{\sqrt{2\pi}} e^{-t/2}\Big|_z^\infty = \frac{2}{\sqrt{2\pi}} e^{-z/2} \, ,$$

and $ze^{-z/2} \to 0$ as $z \to \infty$. In addition, for $z > 0$, $g(z)$ is monotonic, since

$$g'(z) = -z\phi(z) - [1 - \Phi(z)] + z\phi(z) = -[1 - \Phi(z)] < 0 \, .$$

Hence, for all $z \ge 0$, $g(z) > 0$, and thus $h(z) > z$.

To establish that $h'(z) < 1$, we proceed as follows. Consider the random variable $Y \sim \Phi$, truncated to the left at $z \in \mathbb{R}$. Then, Y has pdf $\phi(y)/[1-\Phi(z)]$ and mean $\mathbb{E}[Y|Y > z]$. To calculate the truncated expectations of a random variable distributed according to the standard normal, we use that $\frac{d\phi(z)}{dz} = -z\phi(z)$, and obtain

$$\mathbb{E}[Y] = \mathbb{E}[Y|Y > z] = \frac{-1}{1-\Phi(z)} \int_z^\infty -y\phi(y)dy = \frac{-1}{1-\Phi(z)}\, \phi(y)\Big|_z^\infty = \frac{\phi(z)}{1-\Phi(z)} \, .$$

Similarly, using integration by parts we get

$$\begin{aligned}\mathbb{E}[Y^2] &= \frac{1}{1-\Phi(z)} \int_z^\infty y^2\phi(y)dy = \frac{1}{1-\Phi(z)} \int_z^\infty (-y)[-y\phi(y)]dy \\ &= \frac{1}{1-\Phi(z)} \left[-y\phi(y)\Big|_z^\infty - \int_z^\infty -\phi(y)dy \right] = \frac{z\phi(z)}{1-\Phi(z)} + \frac{1}{1-\Phi(z)} \left[\Phi(y)\Big|_z^\infty \right] = \frac{z\phi(z)}{1-\Phi(z)} + 1\end{aligned}$$

So we conclude that for any $z \in [-\infty, \infty)$

$$1 - h'(z) = 1 + z\frac{\phi(z)}{1-\Phi(z)} - \left(\frac{\phi(z)}{1-\Phi(z)}\right)^2 = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \mathrm{Var}(Y) > 0 \, .$$

iii. *Taking derivatives in equation (F.2) we obtain*

$$\begin{aligned} h''(z) &= h'(z)\,[h(z) - z] + h(z)\,[h'(z) - 1] \\ &= h(z)\left([h(z) - z]^2 + h(z)\,[h(z) - z] - 1\right) \\ &= h(z)\,([h(z) - z]\,[2h(z) - z] - 1) \ . \end{aligned}$$

Given that by definition $h(z) > 0$ it suffices to show that

$$\gamma(z) = (h(z) - z)\,(2h(z) - z) > 1 \, .$$

To establish this we will use the limiting properties of γ as $z \to \pm\infty$. Let $k = 1/\sqrt{2\pi}$, then we can write

$$1 - \Phi(z) = k \int_z^\infty e^{-x^2/2}dx = k \int_z^\infty \frac{-xe^{-x^2/2}}{-x}dx \, .$$

Integrating by parts, we get

$$1 - \Phi(z) = k \int_z^\infty \frac{-xe^{-x^2/2}}{-x}dx = -\frac{ke^{-x^2/2}}{x}\Bigg|_{x=z}^{x=\infty} - k\int_z^\infty \frac{e^{-x^2/2}}{x^2}dx = \frac{\phi(z)}{z} + k\int_z^\infty \frac{-xe^{-x^2/2}}{x^3}dx \, .$$

Integrating by parts the last term,

$$k\int_z^\infty \frac{-xe^{-x^2/2}}{x^3}dx = k\,\frac{e^{-x^2/2}}{x^3}\Bigg|_{x=z}^{x=\infty} + 3k\int_z^\infty \frac{e^{-x^2/2}}{x^4}dx = -\frac{\phi(z)}{z^3} + \rho(z) \, .$$

So we conclude that

$$1 - \Phi(z) = \frac{\phi(z)}{z} - \frac{\phi(z)}{z^3} + \rho(z) \qquad \Leftrightarrow \qquad h(z) = \frac{\phi(z)}{1-\Phi(z)} = \frac{z}{1 - \frac{1}{z^2} + \frac{z\rho(z)}{\phi(z)}} \, .$$

Note that

$$\rho(z) = 3k \int_z^\infty \frac{e^{-x^2/2}}{x^4} dx \longrightarrow 0 \quad as \quad z \longrightarrow \infty \qquad and \qquad \rho'(z) = -\frac{3\phi(z)}{z^4} \longrightarrow 0 \quad as \quad z \longrightarrow \infty \,.$$

Then,

$$\gamma(z) = \frac{z\left(\frac{1}{z^2} - \frac{z\rho(z)}{\phi(z)}\right) z\left(1 + \frac{1}{z^2} - \frac{z\rho(z)}{\phi(z)}\right)}{\left(1 - \frac{1}{z^2} + \frac{z\rho(z)}{\phi(z)}\right)^2} = \frac{\left(1 - \frac{z^3\rho(z)}{\phi(z)}\right)\left(1 + \frac{1}{z^2} - \frac{z\rho(z)}{\phi(z)}\right)}{\left(1 - \frac{1}{z^2} + \frac{z\rho(z)}{\phi(z)}\right)^2} \longrightarrow 1 \quad as \quad z \longrightarrow \infty \,,$$

since, using the L'Hoptal rule

$$\lim_{z\to\infty} \frac{z\rho(z)}{\phi(z)} = \lim_{z\to\infty} \frac{\rho(z)}{z^{-1}\phi(z)} = \lim_{z\to\infty} \frac{\rho'(z)}{-z^{-2}\phi(z) - \phi(z)} = \lim_{z\to\infty} \frac{3}{z^2(z^2+1)} = 0 \qquad and$$
$$\lim_{z\to\infty} \frac{z^3\rho(z)}{\phi(z)} = \lim_{z\to\infty} \frac{\rho(z)}{z^{-3}\phi(z)} = \lim_{z\to\infty} \frac{\rho'(z)}{-3z^{-4}\phi(z) - z^{-2}\phi(z)} = \lim_{z\to\infty} \frac{3}{3+z^2} = 0 \,.$$

In addition, $\gamma(z) \to \infty$ as $z \to -\infty$, since $h(z) \to 0$ as $z \to -\infty$.

Now, suppose that there exists z_0 such that $\gamma(z_0) = 1$. Since γ is continuous and $\gamma(z) \to \infty$ as $z \to -\infty$ and $\gamma(z) \to 1$ as $z \to \infty$, then there exists a point $z_1 \in \mathbb{R}$ such that $\gamma(z_1) \leq 1$ and $\gamma'(z_1) = 0$. But, given the results from part (ii), this is a contradiction since for any $z \in \mathbb{R}$

$$\begin{aligned}\gamma'(z) &= (h'(z) - 1)(2h(z) - z) + (h(z) - z)(2h'(z) - 1) \\ &= (h'(z) - 1)(h(z) + h(z) - z) + (h(z) - z)(h'(z) + h'(z) - 1) \\ &= h(z)(\gamma(z) - 1) + 2(h'(z) - 1)(h(z) - z) < h(z)(\gamma(z) - 1) \,.\end{aligned}$$

So we conclude that $\gamma(z) > 1$ for any $z \in \mathbb{R}$. ■

F.1 Log-Normal Distributions and the Bounds on σ

Several of the sufficient conditions used in the paper belong to the one-parameter family

$$f'(x) + \frac{c\,[f(x)]^2}{1 - F(x)} \;\geq\; 0 \qquad \Longleftrightarrow \qquad h'(x) + (c-1)\,[h(x)]^2 \;\geq\; 0\,, \tag{F.3}$$

where the equivalence follows from $f' = (h' - h^2)(1 - F)$. The cases used in the paper are $c = 1$, increasing promise hazard rate (IPHR, $h' \geq 0$); $c = 3/2$, convexity of the promise credit surface ($h' + \frac{1}{2}h^2 \geq 0$); $c = 2$, increasing loan hazard rate (ILHR, $h' + h^2 \geq 0$); and $c = 3$, convexity of the loan credit surface ($h' + 2h^2 \geq 0$).

Let $h_N(z) = \phi(z)/[1 - \Phi(z)]$ denote the hazard rate of the standard normal of Lemma F.2, substitution into (F.3) gives

$$f'(x) + \frac{c\,[f(x)]^2}{1 - F(x)} = \frac{\phi(z)}{\sigma^2 x^2}\left(c\,h_N(z) - z - \sigma\right), \qquad z = z(x)\,.$$

The prefactor is strictly positive and $z(x)$ ranges over all of $\mathbb{R}$ as x ranges over $(0, \infty)$, so condition

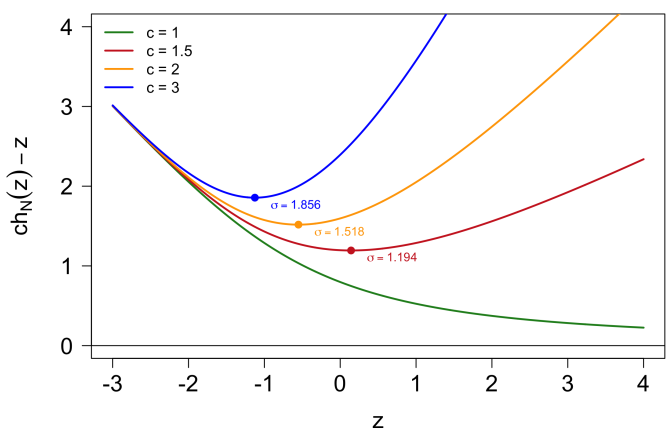
(F.3) holds for every $x > 0$ if and only if $c\,h_N(z) - z \geq \sigma$ for every $z \in \mathbb{R}$, that is, if and only if

$$\sigma \;\leq\; \sigma_c^* \;\equiv\; \inf_{z\in\mathbb{R}} \{c\,h_N(z) - z\} \;. \tag{F.4}$$

Each family member therefore delivers an exact upper bound on the volatility of the log-normal.

By part (*ii*) of Lemma F.2, $h_N'(z) = h_N(z)[h_N(z) - z]$, so the first-order condition for the minimizer is $c\,h_N(z)[h_N(z) - z] = 1$; by part (*iii*), $c\,h_N'' > 0$, so the function we are minimizing is strictly convex and, since $0 < h_N' < 1$, the interior minimizer z_c^* is unique whenever $c > 1$. The first-order condition is transcendental and admits no elementary closed-form solution, so we compute σ_c^* numerically; we plot the objective function and mark the minimizers in Figure F.1. For $c = 1$ the function $h_N(z) - z$ is strictly decreasing with $\inf_z h_N(z) - z = 0$, which is not attained: no positive σ satisfies (F.4), and the log-normal never has an increasing promise hazard rate. For $c = 3/2$, $z^* \approx 0.14$ and $\sigma^* \approx 1.19$; for $c = 2$ (ILHR), $z^* \approx -0.55$ and $\sigma^* \approx 1.52$; and for $c = 3$, $z^* \approx -1.13$ and $\sigma^* \approx 1.86$.

Figure F.1: Functions that Bound σ with Log-Normal Distributions



Notes: The objective function $c\,h(z) - z$ for the values of c used in the paper, where h is the standard normal hazard rate. Markers show the interior minimizers z_c^* and the implied volatility bounds $\sigma_c^* = \inf_z c\,h(z) - z$ of equation (F.4).