1 Standardizing the Fernández-Steel t distribution

Recall the density of the Fernández-Steel t (FS-t) distribution:

\[
    f_{FS}(x; \gamma, \zeta) = \frac{2\gamma}{\sigma(1 + \gamma^2)} \begin{cases} 
        f_t \left( \frac{x-\mu}{\sigma}; \zeta \right) & x \leq \mu \\
        f_t \left( \frac{x-\mu}{\gamma \sigma}; \zeta \right) & x > \mu
    \end{cases}
\]  

for location parameter \( \mu \) and scale parameter \( \sigma \) and where \( f_t \) is the density of the t distribution. In this appendix, we show how these parameters are specified to obtained a standardized distribution of mean zero and unit variable.

Let

\[
    m_1(\zeta) = \frac{2\Gamma\left(\frac{\zeta+1}{2}\right)\zeta}{\sqrt{\pi}\Gamma\left(\frac{\zeta}{2}\right)(\zeta-1)}
\]

\[
    m_2(\zeta) = \frac{\zeta}{\zeta-2}.
\]

These are the first and (non-central) second moments of the absolute value of a Student t distributed random variable with \( \zeta \) degrees of freedom. In the case of a normal distribution
they simplify to $m_1(\infty) = \sqrt{2/\pi}$ and $m_2(\infty) = 1$. The required scaling and location parameters in (OS.1) to give a distribution with mean 0 and variance 1 are

$$\sigma(\zeta, \gamma) = \frac{(m_2(\zeta) - m_1(\zeta)^2) (\gamma^2 + \gamma^{-2}) + 2m_1(\zeta)^2 - m_2(\zeta)}{2}$$
$$\mu(\zeta, \gamma) = -\sigma(\zeta, \gamma)m_1(\zeta) (\gamma - \gamma^{-1}).$$

### 2 Special cases of the hypergeometric $3F_2(1)$ function

We seek solutions to the second moments and cross-moments of the transformed PIT values when the standardized kernel densities take the form

$$\tilde{g}_\nu(u) = u^a - 1 (1 - u)^b - 1$$

for parameters ($a > 0, b > -1/2$). As discussed in Appendix C of the main text, this involves the integral

$$M(a_1, b_1, a_2, b_2) = \int_0^1 (1 - u)\tilde{g}_1(u)\tilde{G}_2(u)du$$

(OS.2)

where $\tilde{G}_i$ denotes the transform function for kernel $i$ and has parameters $(a_i, b_i)$. The general solution to $M$ can be written in two ways:

$$M(a_1, b_1, a_2, b_2) = \frac{B(a_1 + a_2, 1 + b_1)}{a_2} 3F_2(a_2, a_1 + a_2, b_1 + b_2; 1 + a_2, 1 + a_1 + a_2 + b_1, 1)$$

(OS.3)

$$= \frac{B(a_1 + a_2, 1 + b_1 + b_2)}{a_2} 3F_2(1, a_1 + a_2, a_2 + b_2; 1 + a_2, 1 + a_1 + a_2 + b_1 + b_2, 1)$$

(OS.4)

In this appendix we document special cases for which the $3F_2(1)$ terms have known closed-form solution.

#### 2.1 Case $b_1 = 0$

When $b_1 = 0$ and $b_2 \neq 0$, we apply [Wolfram Research](https://www.wolframalpha.com/input/?i=3F2%28a%2C+b%29) to (C.3) to obtain

$$M(a_1, 0, a_2, b_2) = \frac{1}{a_1} (B(a_2, b_2) - B(a_1 + a_2, b_2))$$

(OS.5)

When $b_1 = b_2 = 0$, we instead apply [Wolfram Research](https://www.wolframalpha.com/input/?i=3F2%28a%2C+b%29) to (OS.4):

$$M(a_1, 0, a_2, 0) = \frac{1}{a_1} (\psi(a_1 + a_2) - \psi(a_2))$$

(OS.6)

where $\psi$ is the digamma function. One can derive this by noting that [Wolfram Research](https://www.wolframalpha.com/input/?i=Limit%28%281%2Fb%29-B%28a%2Cb%29%2C+b%3D0%29) together imply the limit

$$\lim_{b \to 0} \left( \frac{1}{b} - B(a, b) \right) = \psi(a) + \gamma$$

where $\gamma$ is the Euler-Mascheroni constant.
2.2 Case $b_1 = n \in \mathbb{N}$

Binomial expansion of the term $(1 - u)^{b_1}$ in the integral form (C.5) of $M$ leads to

$$M(a_1, n, a_2, b_2) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} M(a_1 + k, 0, a_2, b_2)$$

(OS.7)

This approach would be practical for small values of $n$.

2.3 Case $a_2 = n \in \mathbb{N}$

For the moment, assume $b_2 \neq 0$. We have from Wolfram Research (2023, 06.19.03.0003.01):

$$B(u; n, b) = B(n, b) \left(1 - (1 - u)^b \sum_{k=0}^{n-1} \frac{(b)_k u^k}{k!}\right)$$

(OS.8)

where $(b)_k$ is Pochhammer’s notation defined recursively by $(b)_0 = 1$, $(b)_{m+1} = (b+m) \cdot (b)_m$.

This leads to

$$M(a_1, b_1, n, b_2) = \frac{(n-1)!}{(b_2)_n} \left(B(a_1, 1 + b_1) - \sum_{k=0}^{n-1} \frac{(b_2)_k}{k!} B(a_1 + k, 1 + b_1 + b_2)\right).$$

(OS.9)

This can be computed more effectively in recursive form:

$$M(a_1, b_1, 1, b_2) = \frac{1}{b_2} (B(a_1, 1 + b_1) - B(a_1, 1 + b_1 + b_2))$$

(OS.10)

$$M(a_1, b_1, m + 1, b_2) = \frac{1}{b_2 + m} (m M(a_1, b_1, m, b_2) - B(a_1 + m, 1 + b_1 + b_2))$$

(OS.11)

To take the limit of (OS.10) as $b_2 \to 0$, we have by Wolfram Research (2023, 06.18.06.0013.01) that

$$B(a, b + z) = B(a, b)(1 + z(\psi(b + 1) - \psi(a + b + 1))) + \mathcal{O}(z^2)$$

which implies

$$M(a_1, b_1, 1, 0) = B(a_1, 1 + b_1) (\psi(1 + a_1 + b_1) - \psi(1 + b_1))$$

(OS.12)

Observe that (OS.11) is well-behaved at $b_2 = 0$.

2.4 Case $b_2 = n \in \mathbb{N}$

We have from Wolfram Research (2023, 06.19.03.0001.01):

$$B(u; a, 1 + m) = B(a, 1 + m) u^a \sum_{k=0}^{m} \frac{(a)_k (1 - u)^k}{k!}$$

(OS.13)

Substituting into the integral for $M$:

$$M(a_1, b_1, a_2, n) = \frac{(n-1)!}{(a_2)_n} \sum_{k=0}^{n-1} \frac{(a_2)_k}{k!} B(a_1 + a_2, b_1 + k + 1).$$

(OS.14)
This could also be expressed compactly in recursive form. An especially simple subcase is

\[ M(a_1, b_1, a_2, 1) = \frac{1}{a_2} B(a_1 + a_2, b_1 + 1). \]  

(OS.15)

3 Examples of TLSF score tests

Test based on normal distribution: We have \( R = \Phi, \rho = \phi, \lambda_\rho(x) = x, \) and \( \lambda_\rho'(x) = 1. \) Consequently, integrals of type \( B_{\rho,k} \) have solution \( B_{\rho,0}(\alpha) = \alpha, \ B_{\rho,1}(\alpha) = A_{\rho,0}(\alpha), \) and \( B_{\rho,2}(\alpha) = A_{\rho,1}(\alpha). \)

Test based on logistic distribution: When \( R(x) \) is the logistic function \( S(x), \) we have \( \rho(x) = S(x)S(-x), \lambda_\rho(x) = S(x) - S(-x) \) and \( \lambda_\rho'(x) = 2\rho(x). \) The inverse cdf is \( R^{-1}(p) = \ln(p/(1-p)), \) which leads to the convenient expressions \( \rho(R^{-1}(\alpha)) = \alpha(1-\alpha), C_1(R^{-1}(\alpha)) = 1-\alpha, C_2(R^{-1}(\alpha)) = -\alpha, C_2(R^{-1}(\alpha)) = -\alpha(1-\alpha). \)

Integrals of type \( B_{\rho,k} \) have solutions

\[ B_{\rho,0}(\alpha) = \frac{1}{3} \alpha^2 (3 - 2\alpha) \]  

(OS.16)

\[ B_{\rho,1}(\alpha) = \begin{cases} 
\frac{1}{3} (\alpha(1-\alpha) + \ln(1-\alpha)) + B_{\rho,0}(\alpha) R^{-1}(\alpha) & \alpha < 1 \\
0 & \alpha = 1
\end{cases} \]  

(OS.17)

\[ B_{\rho,2}(\alpha) = \begin{cases} 
\frac{2}{3} \left( \alpha + \text{Li}_2 \left( \frac{-\alpha}{1-\alpha} \right) \right) + 2B_{\rho,1}(\alpha) R^{-1}(\alpha) - B_{\rho,0}(\alpha) R^{-1}(\alpha)^2 & \alpha < 1 \\
\frac{2}{3} (\frac{-\alpha^2}{1-\alpha}) & \alpha = 1
\end{cases} \]  

(OS.18)

The dilogarithm function \( \text{Li}_2 \) is available in the GSL package.

For the lower bound on permissible \( \alpha_1, \) \([14]\) simplifies to the condition \( xR(x) = 1, \) which has solution \( x_1 = W_0(1/e) + 1, \) where \( W_0 \) here denotes the Lambert \( W- \) function. This leads to the bound \( \alpha_1 \geq 1/(1 + W_0(1/e)) \approx 0.782. \)

Tests based on Gumbel distribution: For the standard Gumbel, we have \( R(x) = \exp(-\exp(-x)) \) and \( \rho(x) = \exp(-x)R(x), \) so \( \lambda_\rho(x) = 1 - \exp(-x) = 1 - C_1(x). \) The inverse cdf is \( R^{-1}(p) = -\ln(-\ln(p)), \) which leads to \( \rho(R^{-1}(\alpha)) = -\alpha \ln(\alpha) \) and \( \lambda_\rho(R^{-1}(\alpha)) = 1 + \ln(\alpha). \) For \( B_{\rho,0}(\alpha) \) and \( B_{\rho,1}(\alpha) \) we have

\[ B_{\rho,0}(\alpha) = \begin{cases} 
0 & \alpha = 0 \\
\alpha(1 - \ln(\alpha)) & \alpha > 0
\end{cases} \]

\[ B_{\rho,1}(\alpha) = \begin{cases} 
\text{li}(\alpha) - \alpha + B_{\rho,0}(\alpha) R^{-1}(\alpha) & \alpha < 1 \\
\gamma - 1 & \alpha = 1
\end{cases} \]

where \( \text{li} \) is the logarithmic integral function and \( \gamma \) is the Euler-Mascheroni constant. \( B_{\rho,2}(\alpha) \) is best solved numerically.

For the complementary Gumbel, we apply Remark \([3.6]\) to obtain \( R_c, \lambda_c, \) etc., and equations \([D.16] - [D.17]\) to provide solutions to \( A_{\rho,k}^c \) and \( B_{\rho,k}^c. \)
Tests based on logistic-beta distribution:  The logistic-beta distribution has density
\[ ρ(x; a, b) = \frac{S(x)^a S(-x)^b}{B(a, b)} \]
for parameters \( a > 0, b > 0 \). \( S(x) = 1/(1 + \exp(-x)) \) is the logistic function and \( B(a, b) \) is the beta function.

The logistic function satisfies \( S(-x) = 1 - S(x) \) and \( S'(x) = S(x)S(-x) \) from which it follows that
\[ \lambda_ρ(x) = bS(x) - aS(-x) \]
\[ \lambda'_ρ(x) = (a + b)S(x)S(-x) > 0 \]
The cumulants of the distribution are
\[ κ_m(a, b) = ψ^{(m-1)}(a) + (-1)^m ψ^{(m-1)}(b) \]
where \( ψ^{(j)} \) is the \( j \)th derivative of the digamma function. An implication is that the skew of the distribution has the same sign as \( a - b \), since \( ψ^{(2)}(x) \) is strictly increasing for \( x > 0 \).

The cdf is a composition of the beta cdf \( I(z; a, b) \) and the logistic function, i.e., \( R(x) = I(S(x); a, b) \). The inverse cdf is therefore
\[ R^{-1}(p) = \text{logit}(I^{-1}(p; a, b)) \]
where \( \text{logit}(u) = \ln(u/(1-u)) \). The well-known symmetry for the beta distribution implies a symmetry of the same form for the inverse cdf:
\[ 1 - I^{-1}(p; a, b) = I^{-1}(1 - p; b, a) \]
Consequently, we can write
\[ R^{-1}(p) = \ln \left( I^{-1}(p; a, b) \right) - \ln \left( I^{-1}(1 - p; b, a) \right) \]
(OS. 19)

Observe that
\[ λ'_ρ(x; a, b)ρ(x; a, b) = \frac{(a + b)}{B(a, b)} S(x)^{a+1} S(-x)^{b+1} \]
\[ = \frac{(a + b)B(a + 1, b + 1)}{B(a, b)} \rho(x; a + 1, b + 1) = \frac{ab}{1 + a + b} \rho(x; a + 1, b + 1) \]
which implies that
\[ B_{ρ,0}(1) = \frac{ab}{1 + a + b} \]
\[ B_{ρ,1}(1) = B_{ρ,0}(1)κ_1(a + 1, b + 1) \]
\[ B_{ρ,2}(1) = B_{ρ,0}(1)(κ_2(a + 1, b + 1) + κ_1(a + 1, b + 1)^2) \]

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For any $\alpha$, we have
\[ B_{\rho,0}(\alpha) = B_{\rho,0}(1) R (R^{-1}(\alpha; a, b); a + 1, b + 1). \]

For $k = 1, 2$, we can obtain solutions to $B_{\rho,k}(\alpha)$ in terms of $\,_3F_2$ and $\,_4F_3$ hypergeometric functions. In practice, it is easiest to pre-calculate interpolation tables for these functions for select values of $(a, b)$. The numeric integrals appear to be straightforward and knowing the end-value of $B_{\rho,k}(1)$ ensures a high degree of accuracy.

One special case of possible special interest is the skew sech distribution. The density is usually written
\[ \rho^{ss}(x; \theta) = \frac{\cos(\theta)}{2} \exp(\theta x) \text{sech} \left( \frac{\pi}{2} x \right) \]
where the skew parameter $|\theta| < \pi/2$ and $\theta = 0$ corresponds to the standard sech distribution. We can easily show that if $X$ is distributed skew sech with parameter $\theta$ then $\pi X$ is distributed logistic-beta with parameters $(a, 1 - a)$ for $a = 1/2 + \theta/\pi \in (0, 1)$.

References