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Asset Price Learning And Optimal Monetary Policy

Colin Caines* & Fabian Winkler† §

Abstract: We characterize optimal monetary policy when agents are learning about endogenous asset prices. Boundedly rational expectations induce inefficient equilibrium asset price fluctuations which translate into inefficient aggregate demand fluctuations. We find that the optimal policy raises interest rates when expected capital gains, and the level of current asset prices, is high. The optimal policy does not eliminate deviations of asset prices from their fundamental value. When monetary policymakers are information-constrained, optimal policy can be reasonably approximated by simple interest rate rules that respond to capital gains. Our results are robust to a wide range of belief specifications as well as to the inclusion of an investment channel.

Keywords: Optimal Monetary Policy, Natural Real Interest Rate, Learning, Asset Price Volatility, Leaning Against The Wind

JEL classifications: E44, E52

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1 Introduction

How, if at all, should monetary policy react to asset prices? This question has been debated in the macroeconomic literature at least since the seminal article of [Bernanke and Gertler \(2001\)](#). Some economists argue that monetary policy should not pay attention at all to asset prices, or at most in order to improve forecasts of inflation and economic activity. Others argue that asset price misalignments can pose significant risks to macroeconomic and financial stability, and that monetary policy should at least in some circumstances raise interest rates in response to asset price increases.

It emerges relatively quickly that any answer to this question depends on what one thinks about the sources of asset price fluctuations. Are assets always priced at their fundamental value, and if not, what is the nature of price misalignments? Standard macroeconomic models, including workhorse New-Keynesian models used for monetary policy analysis, rule out asset price misalignments by design, and are therefore of limited use in such an exercise. Recently, [Gali \(2014, 2017\)](#) has added rational bubbles to these models. He cautions against a positive reaction of interest rates to bubbles because bubbles grow faster when interest rates are raised. Consistent with that intuition, [Dong *et al.* \(2017\)](#) find that a negative reaction of interest rates to asset prices is optimal in a model with liquidity bubbles in the presence of collateral constraints. Yet, rational bubbles are not the only way through which asset prices can deviate from their “fundamental value”. An alternative and indeed prominent narrative of price misalignments holds that investors have boundedly rational expectations and suffer from bouts of over- and underconfidence which affect prices. This narrative has been formalized using learning models, which are a plausible explanation for many well-known asset price characteristics ([Fuster *et al.* 2012](#); [Collin-Dufresne *et al.* 2013](#); [Adam *et al.* 2015](#); [Barberis *et al.* 2015](#) for stock prices; [Adam *et al.* \(2012\)](#); [Caines \(2016\)](#); [Glaeser and Nathanson \(2017\)](#) for house prices), and are consistent with survey measures of return expectations that are at odds with rational expectations models ([Greenwood and Shleifer, 2014](#)). However, very little work exists on how boundedly rational asset price expectations affect optimal monetary policy.

In this paper, we consider a model in which agents hold subjective, boundedly rational beliefs about asset prices and update these beliefs through a learning process. The model is a simple New-Keynesian model, to which we add a durable asset in fixed supply and learning about the equilibrium asset price. The learning process implies extrapolative expectations, which can lead to endogenous boom-bust cycles in equilibrium price dynamics. We isolate the effects of asset price learning from other expectational channels by restricting beliefs to be model-consistent conditional on subjective asset price expectations

(Winkler, 2016), implying that expectations are consistent with all equilibrium conditions other than asset market clearing.¹ This restriction greatly reduces the degrees of freedom in boundedly rational expectations and also renders the analysis of the learning model particularly transparent. In particular, it implies that agents fully understand the policy strategy followed by the central bank. We then solve analytically for optimal monetary policy under learning.

We show that the optimal policy requires the policy rate to respond positively to asset prices, separately from its reaction to fundamental shocks. The reason is that the natural real rate of interest under learning is no longer a function of just technology and preferences, but depends on subjective asset price beliefs. If agents expect larger capital gains on their assets, then the real interest rate must rise also for the bond market to clear, even if expectations aren't rational. When prices are rigid, setting the interest rate below this "perceived natural rate" inefficiently raises aggregate demand and inflation. The central bank therefore needs to raise interest rates when asset prices are high in order to stabilize inflation. While the optimal interest rate policy is different under learning, the optimal target criterion is not: Because price rigidities are the only distortion in our baseline model, flexible inflation targeting remains optimal. We also numerically evaluate simple Taylor-type interest rate rules which include a reaction to asset prices. We find that this reaction generally mitigates the distortions from non-rational beliefs and stabilizes asset price fluctuations. This latter finding is in direct contrast to Gali (2014) and Dong *et al.* (2017): Higher interest rates increase the size of rational bubbles, but decrease subjective price expectations under learning.

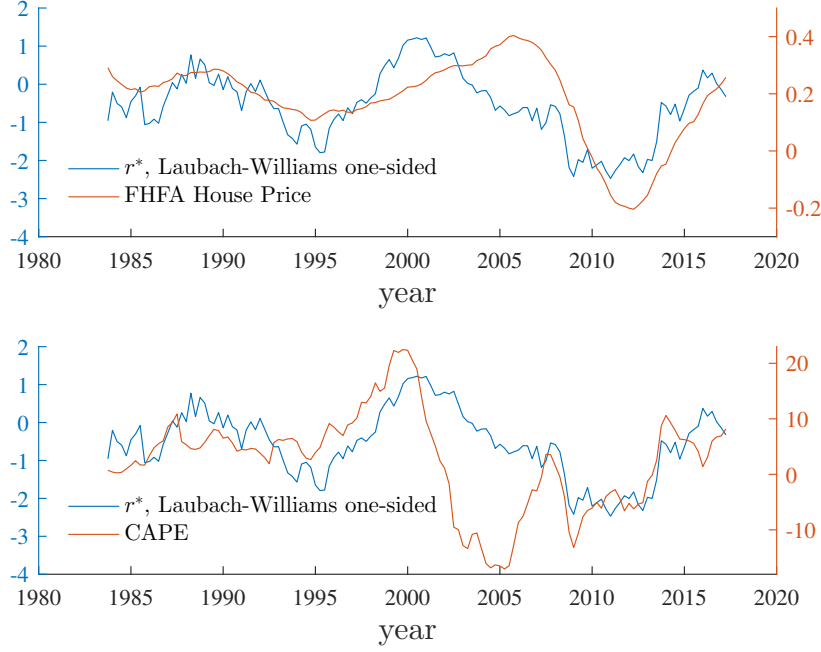
The model predicts positive comovement between the natural real interest rate and realized asset prices. Such comovement seems to be present in the data as well, as Figure 1 shows. The figure plots the 20-quarter change in the well-known Laubach and Williams (2016) natural rate estimate against the 20-quarter change in both the FHFA house price index for the United States as well as the Shiller CAPE ratio². House prices and the natural real rate are positively correlated in both cases.

In our baseline version of the model, non-rational asset price expectations cause distortions because they affect perceived household wealth and therefore aggregate demand. There are of course other, and potentially more important channels through which asset prices can affect the real economy, e.g. credit frictions (Bean, 2004). We see the wealth effect channel in this paper as a stand-in for these more complex transmission channels of asset prices. The advantage of this simplification is that we are able to obtain closed form solutions for optimal monetary policy in the presence of learning. But

¹The concept has also been applied in Caines (2016) and Gandré (2017).

²We thank Kevin Lansing for pointing out this relationship to us.

Figure 1: 20-Quarter Change in the Natural Rate of Interest & Asset Prices.



we also consider an extension of our model in which asset supply is elastic. In this case, subjective belief distortions also cause inefficient fluctuations in real investment. Numerical simulations indicate that interest rate rules still need to respond to asset prices in this case; and moreover, that even the optimal target criterion features “leaning against the wind”, in the sense that the central bank tolerates a downward departure of inflation from the target at times when asset prices are high.

Our baseline model also assumes a particular process of expectation formation in which agents think that asset prices are a random walk with a small time-varying drift. This simple process has been shown to lead to good asset pricing properties in equilibrium (Adam *et al.*, 2017), but our results are not confined to the use of this process. We show that our results carry through to more general class of subjective beliefs, which include for example “natural expectations” (Fuster *et al.*, 2012) and “diagnostic expectations” (Bordalo *et al.*, 2018).

The remainder of this paper is structured as follows. After discussing the related literature in Section 2, we begin by describing the model in Section 3, and our notion of a learning equilibrium in Section 4. We characterize the linearized equilibrium under rational expectations and learning in Section 5. Optimal policy is analyzed in Sections 6, while Section 7 discusses how well certain simple interest rate rules approximate the optimal policy. Sections 8 and 9 discuss extensions to more general beliefs and an investment channel, respectively. Section 10 concludes.

2 Related literature

Our paper shares a number of features with [Adam and Woodford \(2018\)](#), who study so-called robustly optimal policy with house prices. Their model is very similar to ours, but instead of learning they study belief distortions to the fundamentals of the economy and let the policymaker choose a policy that maximizes welfare for the worst possible combination of these distortions.³ The appeal of such an analysis is that the policymaker cannot systematically exploit distorted expectations to its advantage, because the class of possible distortions is large and the policymaker does not know which of these is realized. In our setup, the policymaker could in principle achieve outcomes that are far better than those under rational expectations with a suitable, highly non-linear policy. Because we judge such sophisticated manipulation of beliefs to be unrealistic, we restrict the set of admissible policies to simple linear targeting and instrument rules. Moreover, our results are derived at the fully efficient steady state. By contrast, [Adam and Woodford](#) find a tightening reaction of monetary policy to rising asset prices to be beneficial only when the steady state is distorted in a particular direction.

Our analysis also shares some features with [Christiano *et al.* \(2010\)](#), who study the optimal policy reaction to news shocks about future productivity. In their model, news shocks cause asset prices to rise, and also increase the natural real rate of interest, so that monetary policy should optimally respond by raising interest rates. In this paper, the natural real rate fluctuates in response to endogenous changes in subjective expectations that can be entirely independent of productivity.

[Dupor \(2005\)](#) and [Mertens \(2011\)](#), among others, have argued that monetary policy should react to asset prices in environments that depart from rational expectations. In these papers, distortions to beliefs about asset prices have real effects through investment channels. Monetary policy that reacts to asset prices can then counteract these effects. In this paper we present an argument for leaning against the wind even in the absence of distortions of this kind. Our results are only strengthened if we include an investment channel.

Our paper focuses on learning about asset prices but keeps expectations conditionally model-consistent otherwise. This is not to say that learning or other belief distortions in expectations of other variables doesn't matter; rather, it allows us to isolate the effects of asset prices learning from other such distortions. In so doing, we complement a small but growing number of papers studying monetary policy prescriptions in models with learning about inflation. Fully optimal policy has recently

³One important difference between the different concepts of belief distortions is that our asset prices can deviate from the discounted sum of expected fundamentals, whereas in [Adam and Woodford \(2018\)](#) only expectations of fundamentals are distorted.

been studied in a two-equation model with learning by [Molnar and Santoro \(2014\)](#) and [Eusepi and Preston \(2016\)](#). [Airaudo \(2016\)](#) augments the standard New Keynesian model with a stock market and infinite-horizon learning (as in [Preston \(2006\)](#)) to study conditions under which the rational expectations equilibrium is learnable, but stops short of characterizing optimal policy.⁴ [Eusepi et al. \(2015\)](#) introduce drift in long-run expectations to a New Keynesian model and show that such beliefs introduce an intertemporal trade-off in policy between stabilizing current inflation and anchoring long-horizon beliefs.

3 Model description

Our model is a standard New-Keynesian model in which the representative household also holds a stock of an asset that yields utility. The supply of the asset in the economy is fixed. One can think of the asset as a stock of housing, but we will refer to it as a generic durable asset. Because the asset is in fixed supply, the dynamics of inflation and output are unaffected by its presence under rational expectations. Under learning, however, we will get non-trivial effects of asset prices on allocations.

We first describe the model for a general description of expectations. A representative household provides labor and owns firms. It can also hold nominal bonds promising a nominal return i_t . In addition, the household owns the durable asset. The household's problem is

$$\mathbb{E}^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\phi}}{1+\phi} + \chi \frac{H_t^{1-\theta}}{1-\theta} \right)$$

$$\text{s.t. } C_t = W_t N_t + \Pi_t + T_t - Q_t (H_t - H_{t-1}) + B_t - \frac{1+i_{t-1}}{1+\pi_t} B_{t-1}.$$

Here, C_t is the household's utility from consuming final consumption goods, N_t is the household's labor supply, and T_t are lump-sum taxes. Π_t are the profits received from firms. The quantity of the asset owned by the household is denoted H_t and trades at the price Q_t . B_t are government bonds which are in zero net supply. The price level is P_t and $\pi_t = P_t/P_{t-1} - 1$ is the inflation rate. The expectational operator $\mathbb{E}^{\mathcal{P}}$ has a superscript indicating that agents' expectations are evaluated under a subjective probability measure \mathcal{P} that need not coincide with rational expectations.

⁴Outside of the learning literature, [Gabaix \(2016\)](#) develops a particular form of myopia and studies optimal policy under commitment. In his model, agents have attenuation bias and underestimate the persistence of economic fluctuations. In this paper, agents instead overestimate their persistence, at least along the dimension of asset prices. As a result, the optimal policy needs to react aggressively to the extrapolative bias, while [Gabaix](#) finds that policy can afford to be less aggressive in the presence of attenuation bias.

The first order conditions are:

$$W_t = C_t^\gamma N_t^\phi \quad (1)$$

$$1 = \beta \mathbb{E}_t^{\mathcal{P}} \left(\frac{C_t}{C_{t+1}} \right)^\gamma \frac{1 + i_t}{1 + \pi_{t+1}} \quad (2)$$

$$Q_t = \chi \frac{C_t^\gamma}{H_t^\theta} + \beta \mathbb{E}_t^{\mathcal{P}} \left(\frac{C_t}{C_{t+1}} \right)^\gamma Q_{t+1}. \quad (3)$$

On the production side, a representative intermediate goods producer transforms household labor into intermediate goods using the decreasing returns to scale technology

$$\tilde{Y}_t = A_t N_t^\alpha. \quad (4)$$

It has to hire labor at the real wage rate w_t and sells its goods at the real price M_t . Its first-order condition is

$$W_t = \alpha M_t A_t N_t^{\alpha-1}. \quad (5)$$

There is a continuum of wholesalers indexed by $i \in [0, 1]$ who transform the undifferentiated intermediate good into differentiated goods using a one-for-one technology. They face a standard Dixit-Stiglitz demand function and a Calvo price setting friction. When producer i is able to set a price P_{it} for its output Y_{it} , it solves:

$$\begin{aligned} \max_{P_{it}} \mathbb{E}_t^{\mathcal{P}} \sum_{s=0}^{\infty} \left(\prod_{\tau=1}^s \xi \Lambda_{t,t+\tau} \right) ((1 + \tau_t) P_{it} - M_{t+s} P_{t+s}) Y_{it+s} \\ \text{s.t. } Y_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\sigma} Y_t, \end{aligned}$$

where σ is the demand elasticity of substitution between varieties, $\Lambda_{t,t+\tau} = \beta^\tau C_t^\gamma C_{t+\tau}^{-\gamma}$ is the household discount factor between times t and $t + \tau$, ξ is the probability of not being able to adjust the price in the future. Any profits are distributed to households. The first-order conditions are standard and give rise to the New-Keynesian Phillips curve.

The term τ_t is a time-varying government subsidy to revenue. Shocks to the subsidy will act as cost-push shocks. The steady-state value of the subsidy is set such that it eliminates mark-up distortions. Our steady state is therefore fully efficient.

A representative retailer buys differentiated goods from wholesalers at the price P_{it} and transforms them back into a homogenous final consumption good. The final good sells at price P_t and is produced

according to the technology

$$Y_t = \left(\int_0^1 (Y_{it})^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}. \quad (6)$$

The first order condition gives rise to a constant elasticity of substitution σ between varieties. The price level can be expressed as $P_t = \int_0^1 P_{it} Y_{it} / Y_t$.

The government transfers a lump sum real amount to households

$$T_t = \tau_t \int_0^1 P_{it} Y_{it} di \quad (7)$$

to finance the subsidies to final good producers and offset the tax on stock holdings. Profits and government transfers sum up to $\Pi_t + T_t = Y_t - W_t N_t$. Finally, the central bank sets the nominal interest rate.

We allow for productivity and cost-push shocks which follows first-order autoregressive processes:

$$\log A_t = (1 - \rho_a) \log \bar{A} + \rho_a \log A_{t-1} + \varepsilon_{at} \quad (8)$$

$$\tau_t = (1 - \rho_\tau) \bar{\tau}_t + \rho_\tau \tau_{t-1} + \varepsilon_{\tau t} \quad (9)$$

The innovations are independent white noise with variances σ_A^2 and σ_τ^2 .

Market clearing in the final goods market requires $Y_t = C_t$. Bonds are in zero net supply and the market clearing condition is therefore $B_t = 0$. Finally, the supply of the durable is fixed at unity, so that asset market clearing requires $H_t = 1$.

4 Equilibrium

The equilibrium under rational expectations is standard—it is the textbook New Keynesian model. The asset price Q_t is redundant because the durable asset is in fixed supply. Agents with rational expectations never expect to buy or sell the asset, not because they expect to be unable to do so—they are price takers in a competitive market—but because they expect the price in every state of the world to be such that they will not ever want to change their asset holdings.

Let's recall the formal definition of a rational expectations equilibrium. Let $y_t \in \mathbb{R}^N$ denote the collection of all endogenous model variables—including prices, allocations, and strategies—and by $u_t \in \mathbb{R}^M$ the collection of all exogenous model variables at time t , which I will call “fundamentals”. Stochastic processes for y_t and u_t are defined on the spaces $\Omega_y = \Pi_{t=0}^\infty \mathbb{R}^N$ and $\Omega_u = \Pi_{t=0}^\infty \mathbb{R}^M$, respectively. Further,

denote by $\Omega_u^{(t)}$ the set of all possible histories of exogenous variables up to period t , and its elements by $u^{(t)} \in \Omega_u^{(t)}$. Finally, let \mathbb{P}_u denote the true probability measure for the exogenous variables defined on $(\Omega_u, \mathcal{S}(\Omega_u))$, where $\mathcal{S}(\cdot)$ is the Borel sigma-algebra on a metric space. The topological support of \mathbb{P}_u is denoted by $\text{supp}(\mathbb{P}_u)$.

Definition 1. A *rational expectations equilibrium* is a sequence of mappings $g_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \dots$ such that, for all t and $u^{(t)} \in \text{supp}(\mathbb{P}_u)$:

1. the choices contained in y_t solve the time- t decision problem of each agent in the economy, conditional on decision-relevant⁵ past and current outcomes contained in $u^{(t)}$ and $y^{(t)} = (g_0(u^{(0)}), \dots, g_t(u^{(t)}))$, and evaluating the probability of future external decision-relevant outcomes under the probability measure \mathbb{P} implied by \mathbb{P}_u and the mappings $(g_t)_{t=0}^\infty$;
2. the allocations contained in $y_t = g_t(u^{(t)})$ clear all markets.

Under learning, we assume that agents are not endowed with the knowledge of the equilibrium asset price process. Intuitively, they do not know whether only the representative household is investing in the asset, or whether other investors exist who trade in unknown ways, causing seemingly random price fluctuations. Faced with this lack of knowledge, agents forecast prices using a subjective belief system. As we show in Section 8, this subjective belief system can be made quite general, but here we confine ourselves to our preferred specification which follows [Adam et al. \(2017\)](#). Agents think that the asset price is a simple random walk model with a time-varying drift:

$$\Delta \log Q_t = \mu_{t-1} + \varepsilon_t \quad (10)$$

$$\mu_t = \rho_\mu \mu_{t-1} + \nu_t. \quad (11)$$

where the shocks $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ and $\nu_t \sim \mathcal{N}(0, \sigma_\nu^2)$ are iid white noise, independent of the rest of the economy. Since these two shocks are not observable, agents have to use the Kalman filter to form a belief about the hidden state μ_t . The asset price can equivalently be written just in terms of observables and the filtered state:

$$\Delta \log Q_t = \hat{\mu}_{t-1} + z_t \quad (12)$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t \quad (13)$$

Here, $\hat{\mu}_t$ is the belief about μ_t ; g is the weight agents place on new data when updating their beliefs,

⁵A variable is *decision-relevant* if it enters the agents' decision problem, and a decision-relevant variable is *external* if its value is taken as given by the agent, while it is *internal* if the variable is part of the solution of the agents' decision problem. For example, wholesalers need to get information on current and future aggregate demand Y_t (decision-relevant and external) to set prices P_{it} (decision-relevant and internal), while they do not need to forecast wages since their only production input is the intermediate good.

which is a function of the perceived variances of ε_t and ν_t ; and $z_t \sim \mathcal{N}(0, \sigma_z^2)$ is the forecasting error in the filtering problem. This forecasting error is exogenous normally distributed white noise under agents' subjective expectation, the variance of which is decreasing in the signal-to-noise ratio $\sigma_\nu^2/\sigma_\varepsilon^2$. In order to avoid complications arising from simultaneity in the determination of outcomes and beliefs, we follow Adam *et al.* (2012) and Caines (2016) and assume that in period t agents make choices conditional on $\hat{\mu}_{t-1}$, and update their beliefs according to (12) at the end period.

In order to determine expectations about the remaining variables of the model, including inflation, we follow Winkler (2016) in assuming that agents have so-called “conditionally model-consistent expectations”. This is a restriction on expectations that effectively allows us to isolate the effects of asset price learning from other potential sources of learning in the economy. Conditionally model-consistent expectations are consistent with all equilibrium conditions of the model, except those that would convey knowledge of the price that clears the asset market.

Formally, let $(\Omega_z, \mathcal{S}(\Omega_z), \mathcal{P}_z)$ be the probability space that defines the subjective beliefs for z_t (i.e., the z_t are iid normally distributed with mean zero and variance σ_z^2). Agents' subjective beliefs depend on this perceived stochastic forecast error even though in equilibrium, model outcomes are a function only of fundamentals u_t . The subjective probability measure \mathcal{P} is defined by a mapping from fundamentals u_t and the subjective forecast error z_t to model outcomes y_t .

Definition 2. *Conditionally model-consistent expectations* (CMCE) are a sequence of mappings $h_t : \Omega_u^{(t)} \times \Omega_z^{(t)} \ni (u^{(t)}, z^{(t)}) \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \dots$ such that, for all t and $(u^{(t)}, z^{(t)}) \in \text{supp}(\mathcal{P}_{u,z})$:

1. the choices contained in y_t solve the time- t decision problem of each agent in the economy, conditional on decision-relevant past and current outcomes contained in $u^{(t)}$ and $y^{(t)} = (h_0(u^{(0)}, z^{(0)}), \dots, h_t(u^{(t)}, z^{(t)}))$, and evaluating the probability of future decision-relevant outcomes under the probability measure \mathcal{P} implied by $\mathbb{P}_u \otimes \mathcal{P}_z$ and the mappings $(h_t)_{t=0}^\infty$;
2. the allocations contained in $y_t = h_t(u^{(t)}, z^{(t)})$ clear all markets except the markets for assets and final consumption goods;
3. asset prices under \mathcal{P} follow the law of motion given by (12)–(13).

The definition of the mappings h_t defining expectations is almost identical to the definition of a rational expectations equilibrium, except that asset market equilibrium is not part of the conditions, and instead the price Q_t evolves according to subjective beliefs. Conditional model consistency restricts the subjective belief \mathcal{P} to have the maximum degree of consistency with the model given agents' misspecified belief about asset prices. In analogy to the adaptive learning literature, we call the mappings h_t defining beliefs the *perceived law of motion* (PLM).

Computing the learning equilibrium is an easy two-step procedure: First, compute the PLM

$h_t(u^{(t)}, z^{(t)})$; second, compute the ALM $g_t(u^{(t)})$. Both steps are no more complicated than solving the rational expectations equilibrium.

While under \mathcal{P} , demand for the durable asset does not have to be equal to supply, in equilibrium the market still has to clear:

Definition 3. An *equilibrium* with conditionally model-consistent expectations is a sequence of mappings $r_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto z_t \in \mathbb{R}$ and $g_t : \Omega_u^{(t)} \ni u^{(t)} \mapsto y_t \in \mathbb{R}^N$, $t = 0, 1, 2, \dots$ such that, for all t and $u^{(t)} \in \text{supp}(\mathbb{P}_u)$:

1. $g(u^{(t)}) = h(u^{(t)}, (r_0(u^{(0)}), \dots, r_t(u^{(t)})))$;
2. the allocations contained in $y_t = g_t(u^{(t)})$ clear the asset market.

Market clearing is brought about by finding the right value of the price Q_t that clears the housing market. We call the resulting equilibrium mappings g_t the *actual law of motion (ALM)*. This mapping implies a particular path for z_t , the subjective house price forecast error. In equilibrium, z_t will be a function of the states and the shocks of the model, while under \mathcal{P} it is perceived as an unforecastable exogenous disturbance. It is precisely in this way that \mathcal{P} violates the rational expectations hypothesis.

Agents endowed with conditionally model-consistent expectations may not know the equilibrium pricing function, but they make the smallest possible expectational errors consistent with their subjective view about the evolution of stock prices. This way of setting up expectations is very tractable and can be readily applied in a variety of models, as we have shown in previous papers (Caines, 2016; Winkler, 2016). It also allows us to transparently solve the linearized version of the model.

5 Linearized equilibrium

The analysis in this paper will focus entirely on the linearized version of the model. The conditionally model-consistent expectations under learning imply that the learning equilibrium can be linearized in much the same way as its rational expectations counterpart.

5.1 Rational expectations equilibrium

Under rational expectations, we can linearize around the non-stochastic steady state to obtain:

$$y_t = a_t + \alpha n_t \quad (14)$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} (w_t - a_t + (1-\alpha)n_t) + \eta_t \quad (15)$$

$$w_t = \gamma y_t + \phi n_t \quad (16)$$

$$i_t = \gamma (\mathbb{E}_t y_{t+1} - y_t) + \mathbb{E}_t \pi_{t+1} \quad (17)$$

$$q_t = \gamma y_t - \beta \gamma \mathbb{E}_t y_{t+1} + \beta \mathbb{E}_t q_{t+1}. \quad (18)$$

Here, lower-case variables denote log-linearizations around the zero-inflation steady state, except for i_t which is the difference of the nominal interest rate from its steady-state level, and $\eta_t = \frac{(1-\xi)(1-\beta\xi)}{\xi} (\bar{\tau} - \tau_t)$ is the cost-push shock process. The model is simply the textbook New-Keynesian model with an extra equation for the price of housing. Note that it still has to be closed with an equation describing the conduct of monetary policy, such as an interest rate rule. Note that asset holdings h_t are known to be constant in equilibrium. As a result, h_t does not even enter the rational expectations equilibrium conditions.

An important special case of the model obtains when prices are fully flexible and there are no cost-push shocks ($\xi = 0$ and $\eta_t = 0$). In this case, the allocation in the rational expectations equilibrium is first-best efficient everywhere. Output and the real interest rate are independent of the conduct of monetary policy and are given by:

$$y_t^{n,RE} = \frac{\phi + 1}{\phi + 1 - \alpha(1-\gamma)} a_t \quad (19)$$

$$r_t^{n,RE} = -\frac{\gamma(\phi + 1)}{\phi + 1 - \alpha(1-\gamma)} (1 - \rho_a) a_t. \quad (20)$$

These quantities are also called the natural level of output and the natural real rate, respectively.

As is well known, the equilibrium with sticky prices can be expressed in terms of the deviation from this efficient equilibrium. To this end, denote the output gap by $\hat{y}_t = y_t - y_t^{n,RE}$. The sticky price equilibrium can be summarized with the standard two equations:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{y}_t + \eta_t \quad (21)$$

$$\mathbb{E}_t \hat{y}_{t+1} - \hat{y}_t = \frac{1}{\gamma} \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^{n,RE} \right). \quad (22)$$

where $\kappa = (1 - \xi)(1 - \beta\xi)(1 + \phi - \alpha + \alpha\gamma)/\xi\alpha$. These equations are the standard New-Keynesian Phillips curve and the IS equation.

5.2 Learning equilibrium

In order to solve the learning equilibrium, we have to proceed in two steps. The first is to solve for the agents' policy functions given their beliefs \mathcal{P} ; the second is to impose market clearing in the asset market to back out the equilibrium asset price.

It is easy to verify that the equilibrium under learning has the same non-stochastic steady state as the rational expectations equilibrium, and we take this as our linearization point. The expectations of agents as well as their optimal choices under the subjective measure \mathcal{P} are expressed as the solution to a “perceived law of motion” (PLM) that, in its linearized form, consists of the following equations:

$$y_t = a_t + \alpha n_t \quad (23)$$

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{(1 - \xi)(1 - \beta\xi)}{\xi} (w_t - a_t + (1 - \alpha) n_t) + \eta_t \quad (24)$$

$$w_t = \gamma c_t + \phi n_t \quad (25)$$

$$i_t = \gamma (\mathbb{E}_t^{\mathcal{P}} c_{t+1} - c_t) + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} \quad (26)$$

$$q_t = \gamma c_t - (1 - \beta) \theta h_t - \beta \gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \beta \mathbb{E}_t^{\mathcal{P}} q_{t+1} \quad (27)$$

$$c_t = y_t - \frac{\bar{Q}\bar{H}}{\bar{Y}} (h_t - h_{t-1}) \quad (28)$$

$$q_t = q_{t-1} + \hat{\mu}_{t-1} + z_t \quad (29)$$

$$\hat{\mu}_t = \rho_\mu \hat{\mu}_{t-1} + g z_t \quad (30)$$

This system can be solved as if it were a rational expectations model. However, under \mathcal{P} , there is an additional shock, the asset price forecast error z_t , that is absent under rational expectations. This shock will be predictable in equilibrium, but under \mathcal{P} agents believe it to be unforecastable. The first stage of the solution has to take this into account. Just as under rational expectations, one still needs to add an equation describing the conduct of monetary policy, such as an interest rate rule, for the above system to be fully determined.

Having solved for expectations and optimal choices given \mathcal{P} , the equilibrium under learning (also called “actual law of motion” or ALM) is then found by imposing market clearing in the market for housing. This condition simply reads

$$h_t = 0. \quad (31)$$

This equation implicitly defines the equilibrium realizations of z_t . Contrary to agents' beliefs, this forecast error is not an exogenous shock in equilibrium, but an endogenous variable. It is precisely in this sense that expectations are not rational in this model.

5.2.1 Flexible price PLM

As before, we first describe the flexible price equilibrium ($\xi = 0$ and $\eta_t = 0$). We find the flex-price PLM by solving (23)–(30) under subjective beliefs \mathcal{P} . The learning model has two additional state variables compared to its rational expectations counterpart, q_t and $\hat{\mu}_{t-1}$. We guess and verify that the asset demand function has the following form:

$$h_t^{n,PLM} = k_a a_t + k_h h_{t-1}^{n,PLM} - k_q q_t + k_\mu \hat{\mu}_{t-1} \quad (32)$$

where the coefficients satisfy $k_h \in (0, 1)$, $k_a, k_q, k_\mu > 0$. Exact expressions can be found in the appendix. Asset demand under learning is increasing in productivity, decreasing in the asset price, and increasing in expectations of future asset price growth.

We can also solve for the values of output and the real interest rate under flexible prices. Under subjective expectations, these are functions of the fundamental, last period's asset holdings, the asset price which is perceived as exogenous, and price growth expectations. Below, we write output and the real rate in deviation from their rational expectations counterpart:⁶

$$y_t^{n,PLM} = y_t^{n,RE} + \frac{\alpha\gamma\kappa_1}{1+\phi-\alpha} \left(k_a a_t - (1-k_h) h_{t-1}^{n,PLM} - k_q q_t + k_\mu \hat{\mu}_{t-1} \right) \quad (33)$$

$$\begin{aligned} r_t^{n,PLM} = r_t^{n,RE} + \gamma\kappa_1 \left(k_a (2 - \rho_a - k_h) a_t - (1 - \kappa_h)^2 h_{t-1}^{n,PLM} - k_q (1 - k_h) q_t \right) \\ + \gamma\kappa_1 ((2 - \rho_\mu - k_h) k_\mu + k_q) \hat{\mu}_{t-1}. \end{aligned} \quad (34)$$

where the constant κ_1 is

$$\kappa_1 = \frac{1 + \phi - \alpha}{1 + \phi - \alpha(1 - \gamma)} \frac{\bar{Q}\bar{H}}{\bar{Y}} > 0.$$

This natural rate is increasing in the price growth belief $\hat{\mu}_{t-1}$. Agents' subjective expectations about output under flexible prices are affected by the choice of asset holdings (which are not constant in agents' minds). An increase in expected asset price growth will increase asset demand, and households

⁶Since the PLM includes asset holdings h_t as an endogenous state variable, there are two possible definitions of a natural real rate and natural level of output (Neiss and Nelson, 2003; Woodford, 2003). One can either define them as conditional on the actual level of asset holdings h_{t-1} under sticky prices, or as conditional on the level of asset prices $h_{t-1}^{n,PLM}$ that would obtain had prices been flexible in the past as well, given the history of exogenous shocks. Here, we opt for the latter definition.

will increase their labor supply in order to finance their purchase of the asset, thereby increasing the level of output.

The natural real rate under subjective expectations can be understood by the arbitrage relationship between the return on the durable asset and the return on bonds. Combining the two asset pricing equations (26) and (27), we obtain:

$$r_t^{n,PLM} = \frac{1-\beta}{\beta} (\gamma c_t - \theta h_t - q_t) + \mathbb{E}_t^{\mathcal{P}} q_{t+1} - q_t$$

The expected return on the two assets has to be equal up to first order. An increase in expected durable asset price growth $\hat{\mu}_{t-1} = \mathbb{E}_t^{\mathcal{P}} q_{t+1} - q_t$ increases the expected return to the durable asset, and the real interest rate on bonds therefore has to rise as well.

5.2.2 Flexible price ALM

To find the flex-price equilibrium under learning, i.e. the actual law of motion, one has to impose $h_t = 0$. From the housing demand function (32), one can then immediately solve for the equilibrium asset price and the realization of the subjective forecast error:

$$\begin{aligned} 0 &= k_a a_t - k_q q_t + k_\mu \hat{\mu}_{t-1} \\ \Leftrightarrow q_t &= \frac{k_a a_t + k_\mu \hat{\mu}_{t-1}}{k_q}. \end{aligned} \tag{35}$$

That is, the equilibrium asset price is increasing in both productivity and house price growth expectations. This is intuitive. The demand function (32) is downward-sloping, and so an increase in demand due to either higher productivity (i.e. higher income) or higher expected capital gains has to be met with an increase in the price to bring about equilibrium in the asset market.

It becomes clear that the forecast error z_t is every thing but unforecastable:

$$z_t = \frac{1}{k_q} k_a a_t - q_{t-1} + \left(\frac{k_\mu}{k_q} - 1 \right) \hat{\mu}_{t-1}.$$

This is precisely the way in which rational expectations break in this model. If agents had the correct belief about z_t , then owing to their conditionally-model consistent expectations, their beliefs would be correct and their expectations would be rational.

Substituting the equilibrium price (35) into Equations (33) and (34), we obtain the realized level of

output and the real rate under flexible prices:

$$y_t^{n,ALM} = y_t^{n,RE} \quad (36)$$

$$r_t^{n,ALM} = r_t^{n,RE} + \gamma \kappa_1 ((1 - \rho_a) k_a a_t + ((1 - \rho_\mu) k_\mu + k_q) \hat{\mu}_{t-1}). \quad (37)$$

Under learning and flexible prices, the equilibrium level of output is the same as under rational expectations. This coincidence arises because, under flexible prices, output is determined entirely by intratemporal conditions that are independent of expectations. Nonetheless, the real interest rate does depend on expectations, and its natural level under learning is therefore different from rational expectations. In particular, it is increasing in subjectively expected house price growth.

5.2.3 Sticky prices

Just as under rational expectations, the sticky price equilibrium under learning can be expressed in deviation from the flexible price allocation, which greatly helps our analysis. One only has to be careful in keeping apart the flexible price allocations under the PLM and the ALM. We will use tildes for the former and hats for the latter: That is, $\tilde{h}_t = h_t - h_t^{n,PLM}$ denotes the difference of asset holdings from their flex-price level under the PLM, while the difference with the ALM flex price level is denoted $\hat{h}_t = h_t - h_t^{n,ALM}$. The same notation applies to consumption and output. Then the sticky price equilibrium in the PLM can be summarized with three equations:

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \kappa \left(\tilde{c}_t + \kappa_1 \Delta \tilde{h}_t \right) + \eta_t \quad (38)$$

$$\tilde{c}_t = \mathbb{E}_t^{\mathcal{P}} \tilde{c}_{t+1} - \frac{1}{\gamma} \left(i_t - \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} - r_t^{n,PLM} \right) \quad (39)$$

$$\tilde{h}_t = \frac{\gamma}{\theta(1 - \beta)} (\tilde{c}_t - \beta \mathbb{E}_t^{\mathcal{P}} \tilde{c}_{t+1}) \quad (40)$$

The first equation is the familiar Phillips curve, but where the output gap is replaced by the consumption gap, augmented by asset purchases $\Delta \tilde{h}_t$.⁷ The intuition is that asset purchases are financed by an increase in labor supply, which drives down firms' marginal cost of production and therefore reduces inflation. The second equation is the familiar IS equation, where again the output gap is replaced by the consumption gap since the two are not equal under agents' subjective expectations. The third equation is the Euler equation for housing demand, rewritten in gap form.

Notice that the asset price q_t itself does not appear in the Euler equation for asset holdings in gap

⁷Even though in equilibrium (in the ALM) there can be no housing purchases as the supply of housing is fixed, one cannot simply set $\Delta h_t = 0$ to compute the equilibrium, since agents are not aware of this restriction.

form. Agents perceive it to be an exogenous process, and therefore to be independent of the degree of price stickiness. It therefore drops out of the gap between flexible and sticky prices, and what is left are those variations in asset demand that are due to variations in the household's discount factor. The asset price still implicitly enters equation (40) through the natural rate $r_t^{n,PLM}$.

To find the actual law of motion under sticky prices, one imposes $h_t = h_t^{n,PLM} + \tilde{h}_t = 0$ and solves for q_t . The equilibrium depends crucially on the behavior of the nominal interest rate i_t , which we have not specified yet.

5.3 Numerical illustration

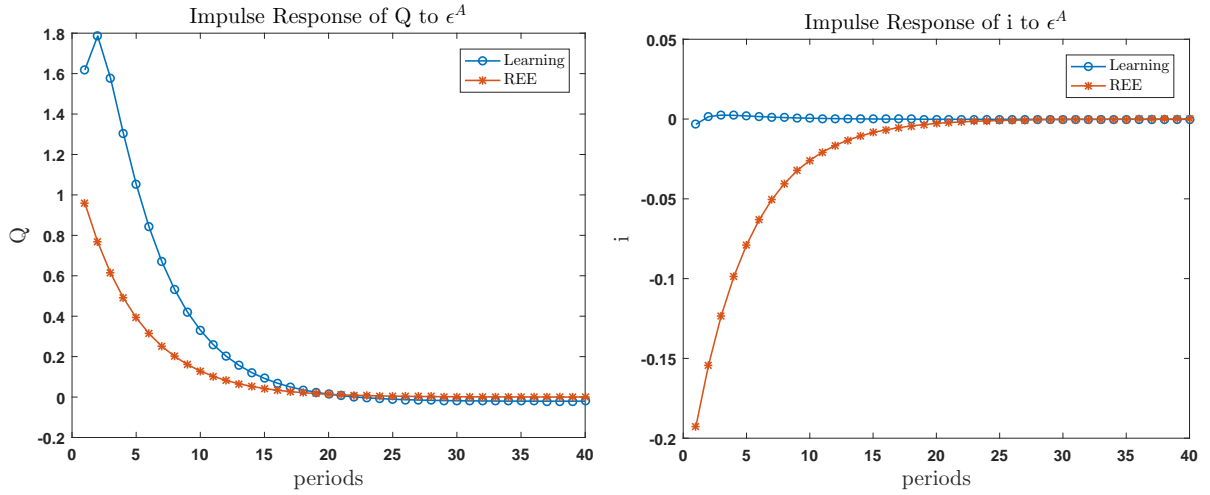
We illustrate the properties of the learning model using a simple calibration in which we interpret the durable asset as housing. We set the labor share in output equal to $\alpha = 0.7$ and the discount factor β equal to 0.995. The coefficient of relative risk aversion is set to $\gamma = 1.39$ (Gandelman and Hernández-Murillo, 2014) and the inverse Frisch elasticity of labor supply is set to $\phi = 0.33$. The utility scaling parameter χ is set to 0.01005 in order to achieve a steady state ratio of asset wealth to output of $QH/Y = 2.01$, which corresponds to the US ratio of real estate holdings over GDP in 2016. The parameter governing the price elasticity of demand for consumption varieties is set to $\sigma = 6$ as in Christiano *et al.* (2010), while price stickiness is set to the standard value $\kappa = 0.75$. We follow Billi (2017) and set the autocorrelation of both the technology and cost-push shocks to 0.8. Finally, we calibrate the remaining four parameters $(\sigma_A, \sigma_\tau, \theta, g)$ to jointly match the volatilities of output growth $\sigma(\Delta Y_t) = 0.64\%$, inflation $\sigma(\pi_t) = 0.82\%$, house price growth $\sigma(\Delta Q_t) = 1.51\%$ and real wage growth $\sigma(\Delta w_t) = 0.10\%$. The resulting parameter values are $\sigma_A = 0.83\%$, $\sigma_p = 0.75\%$, $\theta = 0.0068$ and $g = 0.0041$.

We first document the effect of learning under flexible prices. Here, learning has no effect on equilibrium allocations relative to rational expectations, but manifests itself only in the realized asset price and interest rate process. Figure 2 plots the response of asset prices Q_t and the real interest rate i_t to a technology shock ϵ_A under rational expectations and learning. The effect of learning on Q_t is typical for self-referential asset price learning models. Initially, the asset price Q_t rises on impact because higher wage income raises asset demand, as under rational expectations. But the initial increase now causes a subsequent revision in beliefs $\hat{\mu}_t$ through the learning mechanism. The household believes that the shock has some long-run impact on house price growth and responds by increasing its demand for housing above the rational expectations demand. This response drives a further increase in Q_t in the next period and the shock continues to propagate through belief updating thereafter. At

some point, expected price growth has risen so much that it outstrips realized price growth. At this point, beliefs $\hat{\mu}_t$ decrease, bringing about a reduction in housing demand and therefore in equilibrium house prices, so that the process eventually reverts back to steady state.

The differing response of the interest rate between the learning and rational expectations environments directly shows the effect of expected asset price growth on the natural rate of interest. Even though the impulse response of *realized* consumption is exactly identical under learning and rational expectations, what matters for the interest rate is *expected* consumption growth. Under learning, increases in expected asset price growth in the periods following the shock also increase expected consumption growth, as agents anticipate to sell some of their asset holdings in the future to profit from the capital gains. Higher expected consumption growth implies a higher real rate of interest under the PLM than under rational expectations.

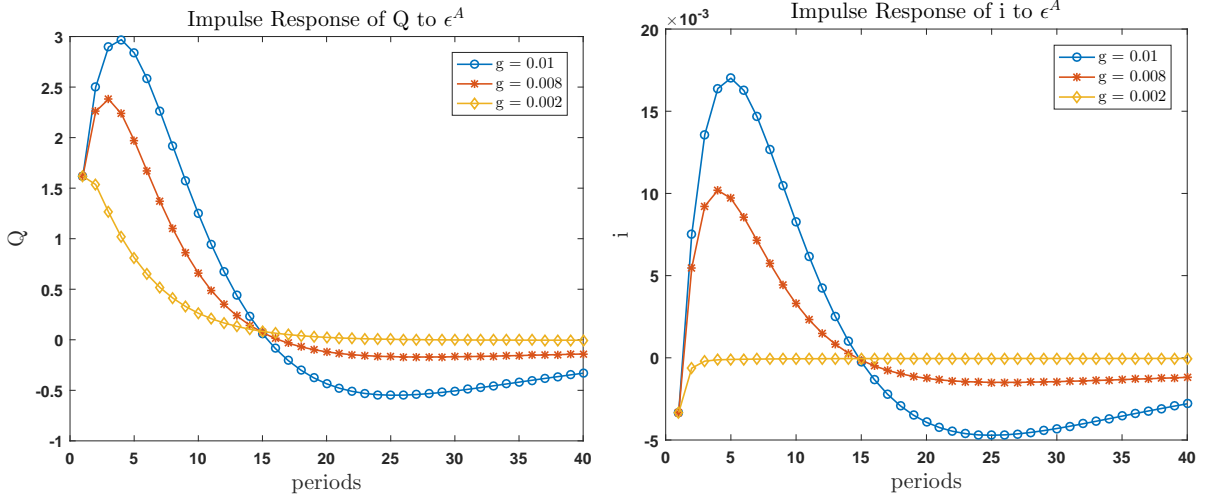
Figure 2: Effect of learning under flexible prices.



Note: Response to a one standard deviation positive technology shock ϵ_{At} . Log percentage points. Flexible prices and zero inflation.

Figure 3 plots the response of Q_t and i_t to a technology shock ϵ_A under learning and flexible prices for different values of the gain parameter, g . A larger learning gain implies that expectations of future asset price growth respond more strongly to shocks, causing larger equilibrium price fluctuations in turn. For a gain close to zero, equilibrium house prices are very close to their rational expectations counterpart. The response of the interest rate, however, remains very different from rational expectations. As the learning gain increases, the asset price as well as the interest rate become more volatile.

Figure 3: Role of the learning gain g , flexible prices.



Note: Response to a one standard deviation positive technology shock ε_{At} . Log percentage points. Flexible prices and zero inflation.

6 Optimal Policy

6.1 Welfare functions

We provide second-order approximations to the expected discounted sum of utility in our model. Under learning, an important distinction has to be made whether welfare is evaluated under the subjective law of motion (in which asset supply is variable and prices are a random walk) or under the actual law of motion (in which asset supply is fixed and prices are functions of the model fundamentals).

If we evaluate welfare under the actual law of motion, then welfare can be approximated by $-\sum_{t=0}^{\infty} \mathbb{E}_0 \mathcal{L}_t$ up to second order, terms independent of policy, and a multiplying positive constant. The period loss function is given by

$$\mathcal{L}_t = \lambda \pi_t^2 + \hat{y}_t^2. \quad (41)$$

where $\lambda = 2\sigma\alpha\xi(1-\xi)^{-1}(1-\beta\xi)^{-1}(1+\phi-\alpha(1-\gamma))^{-1}$ and $\hat{y}_t = y_t - y_t^{n,RE}$ is the deviation of output from its flex-price level. This loss function is identical to that of the standard rational expectations New Keynesian model. It penalizes deviations of inflation from zero as well as deviations of output from the natural rate of output under rational expectations (19). This natural rate of output is first-best efficient under the ALM.

By contrast, the welfare function under the PLM takes a quite different form. Welfare is approxi-

mated by $-\sum_{t=0}^{\infty} \beta^t \mathbb{E}_0^P \mathcal{L}_t^{PLM}$, where the period loss function is given by

$$\begin{aligned} \mathcal{L}_t^{PLM} = & \lambda \pi_t^2 + \tilde{c}_t^2 \\ & + \frac{1 - \alpha + \phi}{1 + \phi - \alpha(1 - \gamma)} \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta \tilde{h}_t \right)^2 + 2\tilde{c}_t \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta \tilde{h}_t \right) \\ & - \frac{(1 - \beta)\theta\alpha}{1 + \phi - \alpha(1 - \gamma)} \frac{\bar{Q}\bar{H}}{\bar{Y}} \tilde{h}_t^2. \end{aligned} \quad (42)$$

where \tilde{c}_t and \tilde{h}_t are the deviations of consumption and asset holdings from their PLM-flexible price levels. Here, the period loss takes the form of deviations from the flexible price allocations under the PLM, and also includes terms for asset holdings. Those terms do not appear in the ALM loss function because asset holdings are constant in equilibrium.

6.2 Optimal policy without cost-push shocks

We now solve for the optimal monetary policy, first under the assumption that there are no cost-push shocks. For exposition, we start by reviewing the optimal policy under rational expectations. As the flexible price equilibrium under rational expectations is first-best efficient, monetary policy is optimal if it manages to replicate the flexible price allocation in the presence of nominal rigidities. This amounts to closing the output gap and completely stabilizing the price level at the same time, as can be seen from the loss function (41). Without cost-push shocks, the “divine coincidence” holds and complete stabilization is achievable. The optimal policy implements

$$\pi_t = 0. \quad (43)$$

From the Phillips curve (21), it immediately follows that $\hat{y}_t = 0$. The optimal policy can be implemented with the following rule:

$$i_t = r_t^{n,RE} + \phi_\pi \pi_t \quad (44)$$

where ϕ_π can be any number satisfying the Taylor principle $\phi_\pi > 1$. The interest rate has to track the natural real rate and react more than one-for-one to inflation, i.e. satisfy the Taylor principle.

Under learning, the question is which welfare criterion to use. Should the central bank aim to maximize agents’ subjectively *expected* discounted utility and minimize the loss function 42 under the PLM? Or should it aim to maximize average *realized* utility and minimize the loss function 41 under the ALM? Fortunately, both welfare criteria prescribe the same optimal outcome here.

Proposition 1. *The optimal monetary policy under learning implements $\pi_t = 0$ and $y_t = y_t^{n,RE}$, regardless of whether welfare is evaluated under the ALM or the PLM. The optimal policy can be implemented with the rule $i_t = r_t^{n,PLM} + \phi_\pi \pi_t$, where $\phi_\pi > 1$.*

Proof. Suppose that the central bank implemented $\pi_t = 0$. The PLM Phillips curve (38) then reduces to the relationship

$$\tilde{y}_t = \frac{\gamma\alpha}{1 + \phi - \alpha(1 - \gamma)} \frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta\tilde{h}_t. \quad (45)$$

Substituting into the housing demand equation (40), we obtain a second-order difference equation of the form

$$(1 - \beta) \frac{\theta}{\gamma} \tilde{h}_t = -\kappa_1 \left(\Delta\tilde{h}_t - \beta \mathbb{E}_t^{\mathcal{P}} \Delta\tilde{h}_{t+1} \right). \quad (46)$$

It is easily verified that the only solution to this equation is $\tilde{h}_t = 0$. But this implies that we implement the flexible price allocation. From the subjective perspective of agents, the flexible price allocation is first-best efficient. Therefore, strict inflation targeting is optimal from the subjective perspective of agents. Moreover, the actual equilibrium in this economy has $\pi_t = 0$ and $y_t = y_t^{n,RE}$, as was shown in the last section. This allocation is also first-best efficient under model-consistent expectations, and therefore optimal under model-consistent expectations as well. \square

It might seem at first that the presence of learning does not alter the prescriptions of optimal policy because the target criterion strict inflation targeting is unchanged. But the implementation of this target requires a different reaction function under learning. The nominal interest rate has to track the natural real interest rate $r_t^{n,PLM}$ as agents perceive it under subjective expectations. This natural rate is very different from the one under rational expectations. Whereas $r_t^{n,RE}$ is a function of productivity a_t only, $r_t^{n,PLM}$ depends additionally on beliefs $\hat{\mu}_t$, prices q_t and the asset holdings h_{t-1} . In particular, the real rate rises when expected asset price growth $\hat{\mu}_t$ increases. In equilibrium, the asset price q_t depend positively on expected price growth, and it is in this sense that the optimal monetary policy leans against the wind: In times of high prices, the interest rate has to be high to track the perceived natural real rate.

The equilibrium realization of the nominal rate under the optimal policy is the expression $r_t^{n,ALM}$ derived in (37). However, an instrument rule that prescribes $i_t = r_t^{n,ALM} + \phi_\pi \pi_t$ would fail to implement the optimal policy. The equilibrium natural rate $r_t^{n,ALM}$ only coincides with the perceived natural rate when $h_t = 0$. While this must be the case in equilibrium, agents under \mathcal{P} contemplate other possible realizations of the house price for which they plan on choosing $h_t \neq 0$. These off-equilibrium states of

the world enter into agents' expectations of future marginal costs. Therefore, the central bank must promise to stabilize inflation even in these off-equilibrium states. Tracking only the equilibrium natural rate is insufficient: It must track the perceived natural rate.

As an illustration, Figure 4 shows impulse responses for the learning model with three interest rate equations:

$$i_t = r_t^{n,PLM} + 1.05\pi_t \quad (47)$$

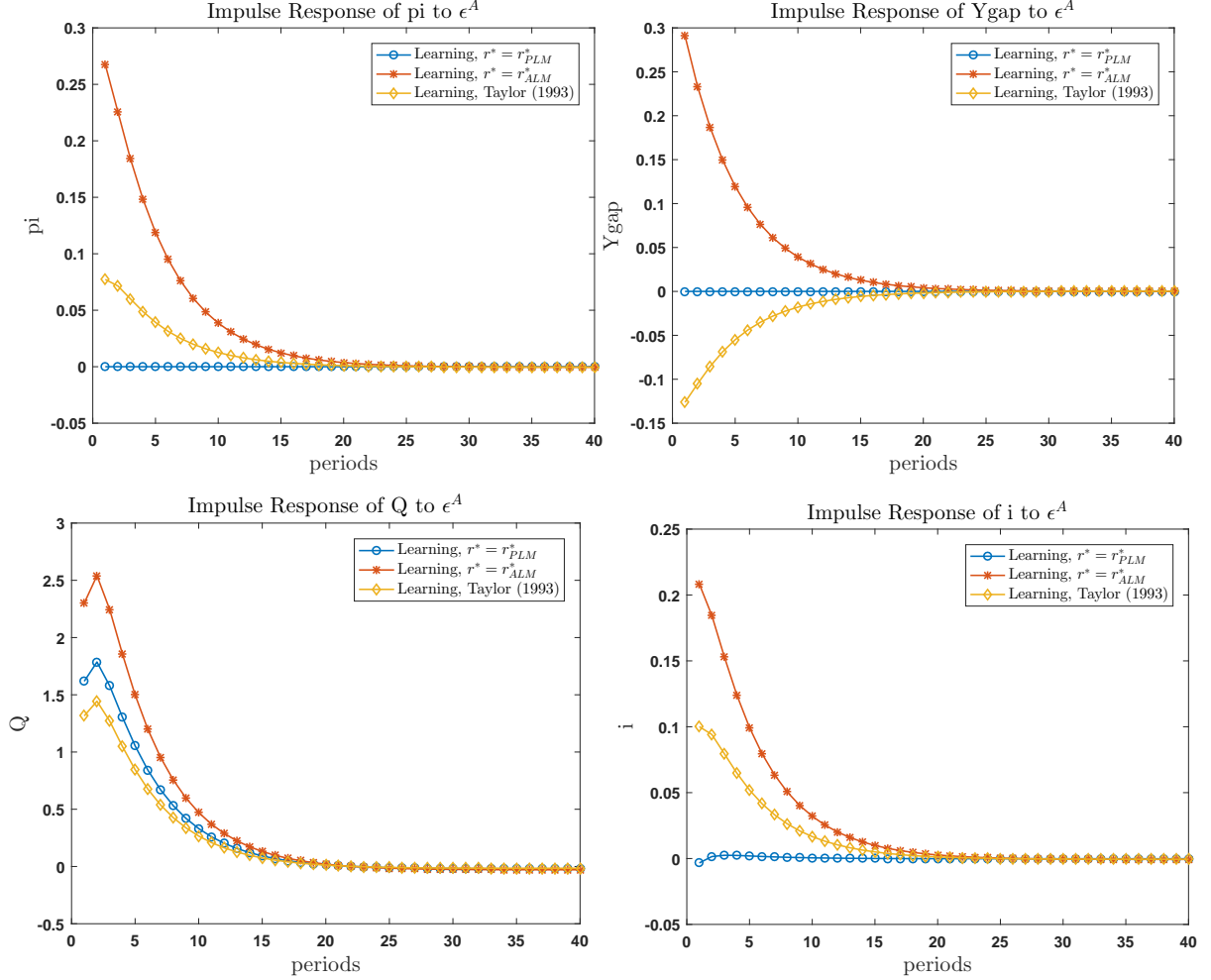
$$i_t = r_t^{n,ALM} + 1.05\pi_t \quad (48)$$

$$i_t = r_{ss} + 1.5\pi_t + 0.125\hat{y}_t. \quad (49)$$

The first equation (47) implements strict inflation targeting as per Proposition 1. The only difference of the second equation (48) is that the monetary authority reacts to the equilibrium process of the natural rate instead of the perceived process. Figure 4 shows how using the ALM natural rate of interest in the the policy rule does not yield a zero inflation outcome. As discussed in the last section, the central bank must promise to stabilize inflation even in those states that are never reached in equilibrium—that is, when the housing market doesn't clear—but contemplated by agents under their subjective expectations. Using the ALM natural rate in the policy rule fails to do so. Due do their beliefs about the process governing Q_t , agents under the PLM do not account for the effect of the technology shock on future asset price growth. Consequently, the initial response of consumption is smaller than under rational expectations. From the standpoint of an agent under the flex price ALM on the other hand, the technology shock has an anticipated positive impact on the path of Q_t due to expected asset demand. As a result, the initial consumption response and subsequent consumption decline will be greater. The ALM natural rate of interest declines more upon the impact of the shock than does the PLM natural rate of interest. When a monetary authority uses the ALM natural rate in its policy rule as in (48), then, the nominal interest does not increase sufficiently to prevent an inflationary response.

Finally, the third equation is a standard Taylor rule. Figure 4 shows that this rule performs somewhat better in terms of outcomes, but is still far from the optimal policy. It is worth notint that the nominal interest rate is more volatile under the Taylor rule than under the optimal rule (47), which reacts to asset prices. The reason is of course that the stabilization benefits of reacting to asset prices make equilibrium nominal rates more stable as well.

Figure 4: Optimal policy and alternatives after a technology shock.



Note: Response to a unit standard deviation positive technology shock ϵ_{At} under sticky prices. Log percentage points. The interest rate rules used are given in Equations (47)–(49).

6.3 Optimal policy with cost-push shocks

The presence of cost-push shocks breaks the so-called “divine coincidence” under rational expectations, so that the first-best allocation is not feasible. Here, we will show first that in principle, a sophisticated policymaker with knowledge of the precise nature of the belief distortions under learning can restore the first-best outcome by twisting private sector expectations to its advantage through a highly non-linear policy. This is clearly the optimal policy, but we do not see it as relevant in practice. Instead, we restrict the set of admissible policies to a known class of linear targeting rules and show that it is possible to replicate the allocations of the RE-optimal policy under discretion and commitment. As in the case without cost-push shocks, implementing these policies requires the nominal interest rate to track the natural rate of interest under the PLM, which is increasing in asset price expectations.

Proposition 2. *With cost-push shocks and learning, it is possible to implement the first-best allocation*

with the targeting rule $\pi_t = -(\beta\rho_\eta)^{-1}\eta_t + b_t z_t$, where b_t is a non-linear state-dependent coefficient.

Proof. See the appendix. \square

Clearly, this is the optimal policy with cost-push shocks, but we see this result as somewhat problematic. First, the policy implies a high degree of belief manipulation by the central bank that is particularly vulnerable to the Lucas critique. This problem was anticipated by [Woodford \(2010\)](#) who wrote that one “might even conclude that the optimal policy under learning achieves an outcome better than any possible rational-expectations equilibrium, by inducing systematic forecasting errors of a kind that happen to serve the central bank’s stabilization objectives”. Moreover, it is difficult to see how such a highly non-linear policy would be credible in the first place.

Here, we will alleviate this problem by restricting the set of admissible policies to those that are familiar from the New-Keynesian literature, and that do not rely on a systematic exploitation of agents’ systematic forecast errors. Under rational expectations, it is well known that the optimal discretionary policy seeking to minimize the loss function (41) satisfies (e.g. [Woodford, 2003](#)):

$$\pi_t = \zeta \eta_t \quad (50)$$

$$\left(y_t - y_t^{n,RE}\right) = -\frac{1 - \zeta(1 - \beta\rho)}{\kappa\zeta} \pi_t \quad (51)$$

where the sensitivity ζ of inflation to the cost-push shock is given by $\zeta_{disc} = (1 - \beta\rho + \lambda\kappa^2)^{-1}$. It is also possible to solve for the weight ζ^* that minimizes the loss function (41) within the class of policies implementing $\pi_t = \zeta\eta_t$. This optimal weight is given by $\zeta^* = (1 - \beta\rho + \lambda\kappa^2 / (1 - \beta\rho))^{-1}$. The interest rate rule that implements this policy is given by

$$i_t = r_t^{n,RE} + \left(\rho + \gamma(1 - \rho) \frac{1 - \zeta(1 - \beta\rho)}{\kappa}\right) \frac{1}{\zeta} \pi_t. \quad (52)$$

Under learning, we obtain the following result:

Proposition 3. *The allocation in (50)–(51) is attainable under learning for any value of ζ in the ALM.*

The nominal interest rate that implements the allocation follows

$$i_t = r_t^{n,PLM} + a \frac{1 + \phi - \alpha}{\kappa_1 \alpha} \tilde{y}_t + \left(\rho + \gamma(1 - \rho) \frac{1 - \zeta(1 - \beta\rho)}{\kappa}\right) \left(\left(1 + \frac{\kappa_1 \gamma(1 - \beta\rho)}{\theta(1 - \beta) + \gamma\beta a}\right)^{-1} + a \frac{1 + \phi - \alpha}{\kappa_1 \alpha \kappa \gamma(1 - \rho)} \right) \frac{1}{\zeta} \pi_t. \quad (53)$$

where the coefficient $a \in (0, 1)$ is defined in the appendix.

In particular, the optimal discretionary policy outcome under RE is attainable under learning with $\zeta = \zeta_{disc}$. Moreover, within the class of policies of the form (50), the weight $\zeta = \zeta^*$ maximizes ALM welfare under learning. The expression for the nominal interest rate shows that the equilibrium interest rate path depends on inflation and the perceived output gap as well as the natural rate. The natural rate is increasing in the level of asset prices as well as the subjective expectation of future asset price growth. Therefore, the nominal interest rate in (88) is effectively reacting to asset prices.

7 Simple rules

Implementing optimal policy in the learning environment requires knowledge of the natural rate of interest under the PLM. In particular, it implies that the monetary authority knows the agents' beliefs about μ_t , which in part determine $r^{n,PLM}$ in equilibrium. An obvious concern is that beliefs that are subjective and privately held are hard to measure. In this section, we show that incorporating a positive reaction to asset prices into a standard interest rate rule can allow a monetary authority who does not observe beliefs to approximate optimal policy under learning. This result is not a natural consequence of the optimal policy analysis, because simple rules can be quite far from the optimal policy. A reaction to asset prices will tend to be beneficial in a rule if periods of elevated asset prices coincide with excess aggregate demand under that particular rule. For our calibrated model and the standard Taylor rule, that turns out to be the case.

We re-compute the model under the assumption that the monetary authority is following a Taylor-type rule of the form:

$$i_t = \rho_i i_{t-1} + (1 - \rho_i) \cdot \left(r_{ss} + \phi_\pi \cdot \pi_t + \phi_y \cdot \hat{y}_t + \phi_q \cdot \sum_{s=0}^{\infty} \omega^s \Delta \log Q_{t-s} \right). \quad (54)$$

The rule depends on inflation and the output gap, and has an additional term for asset prices: a moving average of past price changes, with a weight on past observations that decays at the rate $\tilde{\omega} \in (0, 1)$. In what follows, we keep the coefficient on inflation at $\phi_\pi = 1.5$ and find the tuples $(\rho_i, \phi_y, \phi_q, \tilde{\omega})$ that minimize either (41) under the equilibrium probability measure, or (42) under the subjective probability measure.⁸ We impose the constraint $0 \leq \tilde{\omega} \leq 0.999$. Table 1 shows the optimized rule coefficients and compares them to the outcome of the optimal target criterion from Section 6.2.

⁸If one also optimizes over the coefficient and inflation, then the optimal policy under rational expectations is given by $\phi_\pi \rightarrow \infty$ and $\phi_y/\phi_\pi \rightarrow \zeta > 0$ (Boehm and House, 2014). The outcomes of this limit policy are also attainable under learning with a similar policy that also responds infinitely strongly to inflation and the output gap. In this section, we rule out infinite rule coefficients by keeping the inflation coefficient fixed, and focus only on the tradeoff of reacting to the output gap and asset prices.

Table 1: Performance of optimized simple rules.

Rational Expectations		$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta q_t)$	\mathcal{L}
(1)	$i_t = r_{ss} + 1.5\pi_t + 0.125 \cdot \hat{y}_t$	0.349	0.451	0.801	3.315
(2)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t)$ $\{\rho_i^*, \phi_y^*\} = \{0.844, 0.331\}$	0.249	0.472	0.515	1.888
(3)	$\pi_t = \zeta^* \eta_t$ $\zeta^* = 0.040$	0.050	0.492	1.098	0.577
Learning		$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta q_t)$	\mathcal{L}
(4)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t)$ $\{\rho_i^*, \phi_y^*\} = \{0, 0.041\}$	0.051	0.476	1.741	0.559
(5)	$\pi_t = \zeta^* \eta_t$ $\zeta^* = 0.040$	0.050	0.492	1.879	0.577
(6)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t)$ $\rho_i^* = 0$	0.014	0.541	1.911	0.656
(7)	BG (1999) w/ asset price $i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_q^* \cdot \log q_{t-1})$ $\{\rho_i^*, \phi_q^*\} = \{0, 3.6 \times 10^{-5}\}$	0.013	0.541	1.912	0.656
(8)	BG (2001) w/ asset price $i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_q^* \cdot \log q_t)$ $\{\rho_i^*, \phi_q^*\} = \{0, 7.5 \times 10^{-4}\}$	0.012	0.541	1.911	0.655
(9)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_q^* \cdot \sum_{s=0}^{\infty} \omega^{*s} \Delta \log Q_{t-s})$ $\{\rho_i^*, \phi_q^*, \omega^*\} = \{0, 0.010, 0.268\}$	0.025	0.531	1.792	0.645
(10)	FM (2007) w/ asset price $i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t + \phi_q^* \cdot \log q_t)$ $\{\rho_i^*, \phi_y^*, \phi_q^*\} = \{0.405, 0.064, 0.012\}$	0.045	0.441	1.700	0.476
(11)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t + \phi_q^* \cdot \sum_{s=0}^{\infty} \omega^{*s} \Delta \log Q_{t-s})$ $\{\rho_i^*, \phi_y^*, \phi_q^*, \omega^*\} = \{0.408, 0.064, 0.012, 0.999\}$	0.045	0.442	1.701	0.477

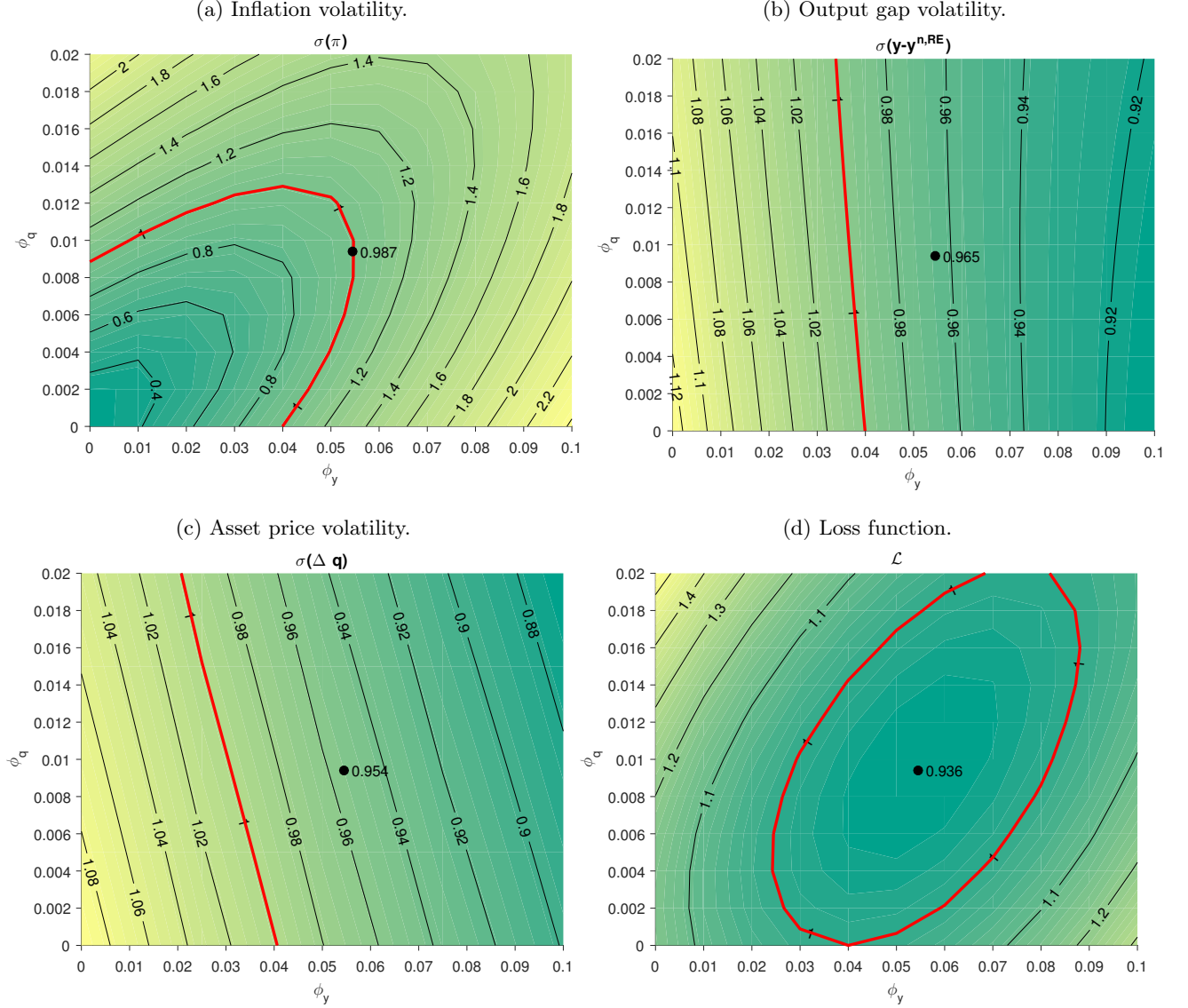
The first three rows show results under rational expectations. Row (1) shows our baseline policy rule while Row (2) shows the optimized values of persistence and the output gap coefficient, holding constant the coefficient on inflation. This rule is more aggressive than the standard Taylor rule and leads to welfare gains from output gap stabilization. Allowing for a non-zero asset price response in the optimization leads to $\phi_q = 0$: There is no benefit from leaning against the wind under rational expectations.⁹ Row (3) also shows the outcomes from the optimal target rule derived in Section 6.2. This target rule dramatically improves welfare, mainly by reducing inflation volatility.

Under learning, the picture is quite different. Row (4) repeats the baseline policy rule, with both interest rate persistence ρ_i and the output gap response ϕ_y set to their optimized value. The resulting rule displays zero interest rate persistence and has a less aggressive output gap response than the standard Taylor rule. Nevertheless, inflation volatility is lower than under rational expectations, thereby improving welfare. Row (5) shows the outcomes from the optimal target rule derived in 6.2. As was shown in Section 6.2, the allocations induced by this rule are identical to those in Row (3). Rows (6) - (8) show the effect of including the asset price level in a rule that does not have an output response as in [Bernanke and Gertler \(1999\)](#) or [Bernanke and Gertler \(2001\)](#). In each case the optimal asset price response is near-zero, with no gain in welfare. The situation is somewhat different when a weighted average of past asset price growth is included instead (Row (9)). The optimal asset price response is positive, with a slight decrease in output gap volatility driving a 1.7 percent gain in welfare. The gains to reacting to asset prices are more pronounced when an output gap response is included in the simple rule. Row (11) shows the results when the model is simulated under the simple rule specified in (54) with optimized coefficient values. Once again reacting to the asset price is optimal. The optimal coefficient on the output gap is positive, and the optimal ω is set very close to one. With this value its dynamics are closer to the subjective belief $\hat{\mu}_t$, which itself is a moving average of past price changes. The reaction to the asset price stabilizes the output gap and lowers the volatility of asset prices, resulting in a near-15 percent welfare gain. The results are similar when the asset price level is included in the rule instead of the weighted average, as in [Faia and Monacelli \(2007\)](#). It is important to highlight that under learning the simple interest rate rules outperform the optimal target criterion when a reaction to both asset prices and the output gap is present. It is also worth noting that the optimized interest rate rules do not reduce asset price volatility to its level under rational expectations. The reason is of course that asset price volatility does not enter the loss function directly, and so the central bank cares only about the effects of asset price movements on inflation and the output gap.

⁹In fact, the optimal coefficient on asset prices would be slightly negative had we not imposed $\phi_q \geq 0$.

To get a better idea of the effects of monetary policy reactions to asset prices and the output gap under learning, we compute loss function values as well as the volatilities of inflation, the output gap and asset prices over a range of parameters. We fix the moving average weight to $\tilde{\omega} = 0.9$, the interest rate persist to $\rho_i = 0$, and vary the magnitude of the response coefficients ϕ_y and ϕ_q on the output gap and inflation.¹⁰ Figure 5 contains the results as surface plots.

Figure 5: Loss values and volatilities for different output gap and asset price coefficients.



Note: Unconditional standard deviation of inflation π_t , house price growth Δq_t and output gap \hat{y}_t under the ALM, and loss function \mathcal{L} , as a function of ϕ_y and ϕ_q , keeping $\tilde{\omega} = 0.9$ throughout. All values are reported relative to the rule in Row (5) of Table 1. Red lines denote contour lines at unity, i.e. the value attained by the optimal coefficient ϕ_y with $\phi_q = 0$. Black dots denote the value attained under the unrestricted optimal coefficients, reported in Row (6) of Table 1.

The effect of changes in the output gap coefficient are as expected: They lower the volatility of the output gap itself, but increase the volatility of inflation. This trade-off arises because the model has

¹⁰Our results are qualitatively robust to changes in the moving average weight ω . In particular, a positive reaction to asset prices $\phi_q > 0$ always reduces asset price volatility.

cost-push shocks in it. A reaction to the output gap also lowers asset price volatility in this model.

But the asset price coefficient also plays an important role. The volatility of asset prices is decreasing in the asset price response ϕ_q . The volatility of the output gap is only little affected by the asset price response, but the volatility of inflation is reduced significantly with $\phi_q > 0$. Therefore, the loss function is minimized at a strictly interior point at which the central bank reacts to both the output gap and asset price growth.

Importantly, a reaction to asset prices always decreases asset price volatility (regardless of the value of $\tilde{\omega}$). This is in stark contrast to the rational bubbles of [Gali \(2014, 2017\)](#). Rational bubbles grow at the rate of interest, and so raising rates when a bubble is growing makes it grow even faster, causing more volatility. By contrast, raising rates in the learning model here has the effect of lowering the house price today: A higher real rate requires a higher expected return on housing. For a given expected capital gain $\hat{\mu}_t$, a higher return needs to be brought about by a lower price today. The reduction in the house price today then reduces optimism about future price growth.

8 Extension: General Asset Price Beliefs

One might wonder whether our results hinge in any way on our assumption that agents' subjective beliefs about asset prices are given by a simple random walk with drift. In this section, we show that this is not the case. All our results so far extend to a very general form of beliefs about asset prices that encompass extrapolative as well as attenuating beliefs relative to rational expectations, “natural expectations” ([Fuster *et al.*, 2012](#)), “diagnostic expectations” ([Bordalo *et al.*, 2018](#)) and other forms of non-rational beliefs. The only assumptions we have to retain are that expectations are conditionally model-consistent in the sense of [Definition 2](#), and that the subjective law of motion for asset prices is independent of policy. While this second assumption is admittedly somewhat limiting, an environment in which agents do think that monetary policy can curb asset price booms probably provides an even stronger rationale for reacting to asset prices than what we discuss here.

We replace the subjective law of motion for asset prices in [\(12\)–\(13\)](#) with a general belief of the form:

$$q_t = A(L) z_t + B(L) u_t. \quad (55)$$

where A and B are arbitrary lag polynomials. Subjective beliefs can depend in an arbitrary way on the fundamental shocks u_t (i.e. productivity and cost-push shocks) as well as a subjective forecast error z_t . The general formula nests rational expectations, our baseline belief system, and a multitude of other

forms of subjective beliefs.

Under flexible prices, we can show that the housing demand function in the PLM (i.e. under \mathcal{P}), which previously was given by (32), is replaced by:

$$h_t^{n,PLM} = k_a a_t + k_h h_{t-1}^{n,PLM} + k_q q_t + \tilde{k}_\mu \sum_{s=0}^{\infty} \tilde{\rho}^s \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+s+1}. \quad (56)$$

The coefficients k_a , k_h and k_q are the same as in the original model, and moreover we have $\tilde{k}_\mu > 0$ and $0 < \tilde{\rho} < \beta$. The natural real rate under the PLM has a somewhat more convoluted form, but importantly, it is still increasing in expectations of asset price growth:

$$\begin{aligned} r_t^{n,PLM} = & r_t^{RE} + \gamma \kappa_1 \left(k_a (2 - \rho_a - k_h) a_t - (1 - \kappa_h)^2 h_{t-1}^{n,PLM} + k_q (1 - k_h) q_t \right) \\ & + \gamma \kappa_1 \left(\frac{\tilde{k}_\mu}{\tilde{\rho}} - k_q \right) \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+1} + \gamma \kappa_1 \tilde{k}_\mu \left(1 - k_h - \frac{1 - \tilde{\rho}}{\tilde{\rho}} \right) \sum_{s=1}^{\infty} \tilde{\rho}^s \mathbb{E}_t^{\mathcal{P}} \Delta q_{t+s+1}. \end{aligned} \quad (57)$$

Moreover, since the asset price q_t is independent of policy under the PLM, it drops out of the equations describing the dynamics of the sticky price equilibrium relative to flexible prices. Equations (38)–(40) continue to hold and the asset price enters only indirectly through the natural real rate $r_t^{n,PLM}$, which itself is independent of policy. As a consequence, all our results from Section 6 continue to hold.

9 Extension: Asset Production

In the model presented thus far, learning causes distortions only through wealth effects affecting aggregate demand. But one of the main concerns about financial stability in macroeconomics is that overoptimism and -pessimism in financial markets might cause inefficient investment fluctuations. Here, we extend the model to allow for the durable asset to be produced instead of being in fixed supply. In this extension, asset price misalignments do distort investment decisions in addition to aggregate demand, and therefore also distort the flexible price allocation. This fundamentally changes the monetary policy tradeoff.

Relative to the baseline model, we now assume that the stock of the durable asset depreciates at the rate δ . The representative household owns firms that can produce I_t durable assets from K_t consumption goods. Their production function has decreasing returns to scale:

$$I_t = A_h K_t^\omega. \quad (58)$$

Production takes place within one period. The profits of the investment firms are:

$$\Pi_t = Q_t I_t - K_t \quad (59)$$

and profit maximization leads to the first order condition:

$$I_t = A_h (\omega Q_t A_h)^{\frac{\omega}{1-\omega}}. \quad (60)$$

The budget constraint of the household becomes

$$C_t + Q_t (H_t - (1 - \delta)H_{t-1}) + \frac{1 + i_{t-1}}{1 + \pi_t} B_{t-1} = W_t N_t + \Pi_t + T_t + B_t. \quad (61)$$

Market clearing in the durable asset market now requires

$$H_t = (1 - \delta) H_{t-1} + I_t. \quad (62)$$

The equilibrium is defined analogously to section 4. Agents do not know the market clearing condition (62), but instead hold subjective beliefs that the asset price follows equations (12)–(13). Beliefs about the hidden state μ_t are updated using the Kalman filter as before, and expectations about the remaining equilibrium objects satisfy conditional model consistency as defined in section 4.

9.1 Linearized equilibrium

We relegate the complete description of the linearized equilibrium to the appendix. Importantly, the natural real rate of interest in the model with asset production continues to be increasing in the asset price belief $\hat{\mu}_t$. As before, we can write the PLM under sticky prices in deviation from the flexible price PLM:

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \kappa \left(\tilde{c}_t + \kappa_1 \left(\tilde{h}_t - (1 - \delta) \tilde{h}_{t-1} \right) \right) + \eta_t \quad (63)$$

$$i_t = \gamma (\mathbb{E}_t \tilde{c}_{t+1} - \tilde{c}_t) + \mathbb{E}_t \pi_{t+1} + r_t^{n, PLM} \quad (64)$$

$$\theta \tilde{h}_t = \gamma \tilde{c}_t - \beta (1 - \delta) \gamma \mathbb{E}_t \tilde{c}_{t+1}. \quad (65)$$

9.2 Welfare functions

We can derive a quadratic approximation of welfare to evaluate different policies. If we evaluate welfare under the actual law of motion, then welfare takes the form $-\sum_{t=0}^{\infty} \mathbb{E}_0 \mathcal{L}_t$, where the period loss function is given by

$$\begin{aligned} \mathcal{L}_t = & 2\sigma \frac{\xi}{1-\xi} \frac{1}{1-\beta\xi} \pi_t^2 + \left(\gamma \frac{\bar{C}}{\bar{Y}} + \frac{1-\alpha+\phi}{\alpha} \frac{\bar{C}^2}{\bar{Y}^2} \right) \hat{c}_t^2 + (1-\beta(1-\delta)) \theta \frac{\bar{Q}\bar{H}}{\bar{Y}} \hat{h}_t^2 \\ & + \left(\frac{1-\alpha+\phi}{\alpha} \frac{\bar{Q}^2 \bar{H}^2}{\bar{Y}^2} + (1-\omega) \frac{\bar{Q}\bar{H}}{\bar{Y}} \right) \left(\hat{h}_t - (1-\delta) \hat{h}_{t-1} \right)^2 \\ & + 2 \frac{1-\alpha+\phi}{\alpha} \frac{\bar{C}}{\bar{Y}} \frac{\bar{Q}\bar{H}}{\bar{Y}} \left(c_t - c_t^{n,RE} \right) \left(\hat{h}_t - (1-\delta) \hat{h}_{t-1} \right). \end{aligned} \quad (66)$$

Compared to the standard New Keynesian model, we have to take into account variation in the asset stock h_t that the household owns, as well as variations in asset investment \tilde{h}_t . As before, the rational expectations equilibrium under flexible prices is first-best efficient, and the loss function can therefore be written in deviations from it.

By contrast, under the PLM, welfare is approximated by $-\sum_{t=0}^{\infty} \mathbb{E}_0^{\mathcal{P}} \mathcal{L}_t^{PLM}$, where the period loss function is given by

$$\begin{aligned} \mathcal{L}_t^{PLM} = & 2\sigma \frac{\xi}{1-\xi} \frac{1}{1-\beta\xi} \pi_t^2 + \left(\gamma \frac{\bar{Y}}{\bar{C}} + \frac{1-\alpha+\phi}{\alpha} \frac{\bar{C}^2}{\bar{Y}^2} \right) \tilde{c}_t^2 - (1-\beta(1-\delta)) \theta \frac{\bar{Q}\bar{H}}{\bar{Y}} \tilde{h}_t^2 \\ & + \frac{1-\alpha+\phi}{\alpha} \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} \right)^2 \left(\tilde{h}_t - (1-\delta) \tilde{h}_{t-1} \right)^2 \\ & + 2 \left(\gamma \frac{\bar{Y}}{\bar{C}} + \frac{1-\alpha+\phi}{\alpha} \right) \frac{\bar{C}}{\bar{Y}} \frac{\bar{Q}\bar{H}}{\bar{Y}} \tilde{c}_t \left(\tilde{h}_t - (1-\delta) \tilde{h}_{t-1} \right). \end{aligned} \quad (67)$$

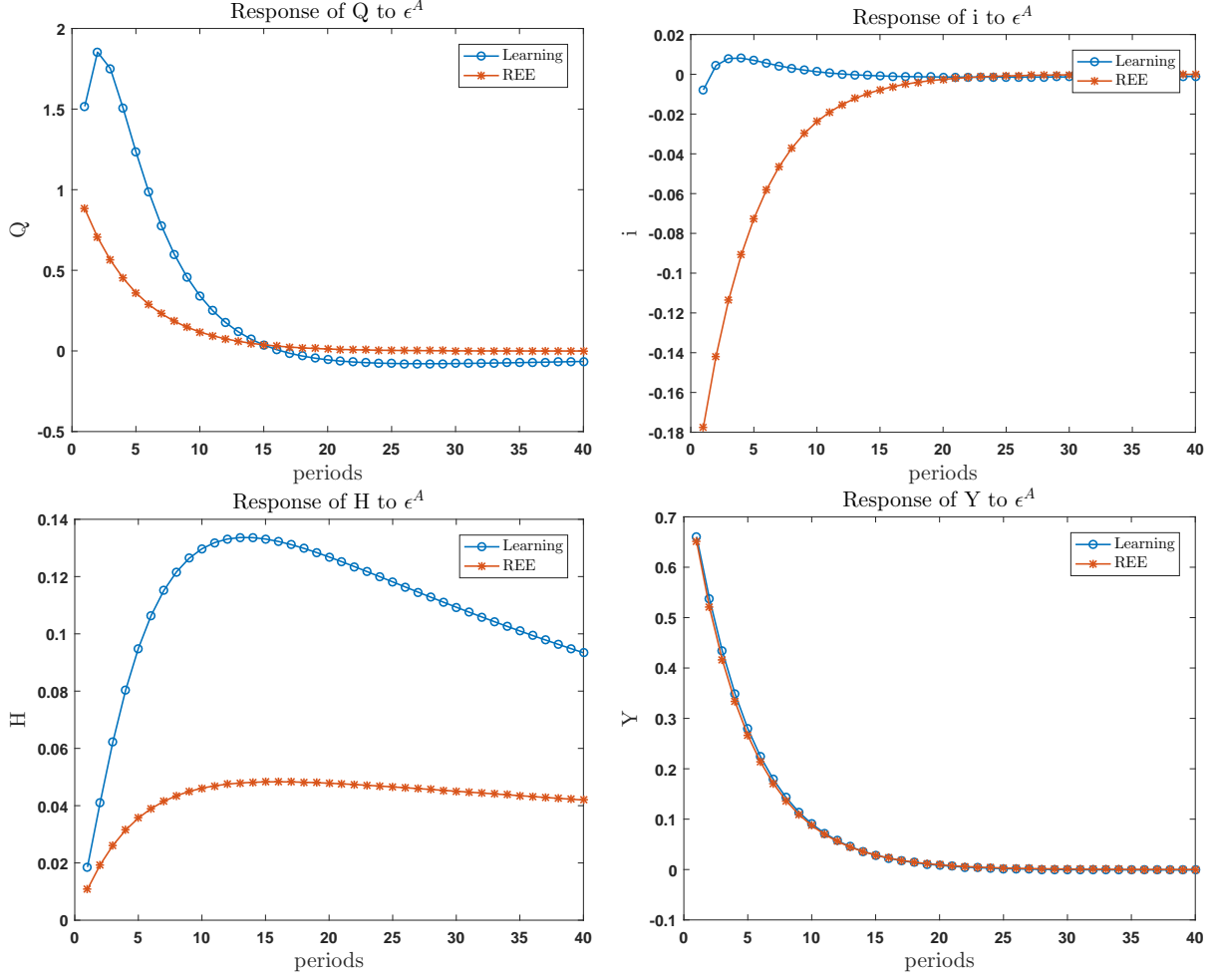
9.3 Numerical results

We illustrate the dynamics of the model with a simple calibration. We take the same parameters as in the baseline model, and in addition set $\omega = 0.636$, corresponding to an elasticity of housing supply of 1.75, as estimated by [Saiz \(2010\)](#). The rate of depreciation of the asset is set to $\delta = 0.007$, corresponding to 2.8% yearly depreciation of housing ([Harding et al., 2007](#)).

Figure (6) illustrates that learning has real effects in this model with production even under flexible prices. It compares the flexible price-responses to a productivity shock under rational expectations and learning. As before, learning increases asset price volatility. But asset production now responds to asset demand, diverting resources away from consumption in an asset price boom.

As was the case with the baseline model, strict inflation targeting requires the policymaker to track

Figure 6: Effect of learning under flexible prices, with asset production.

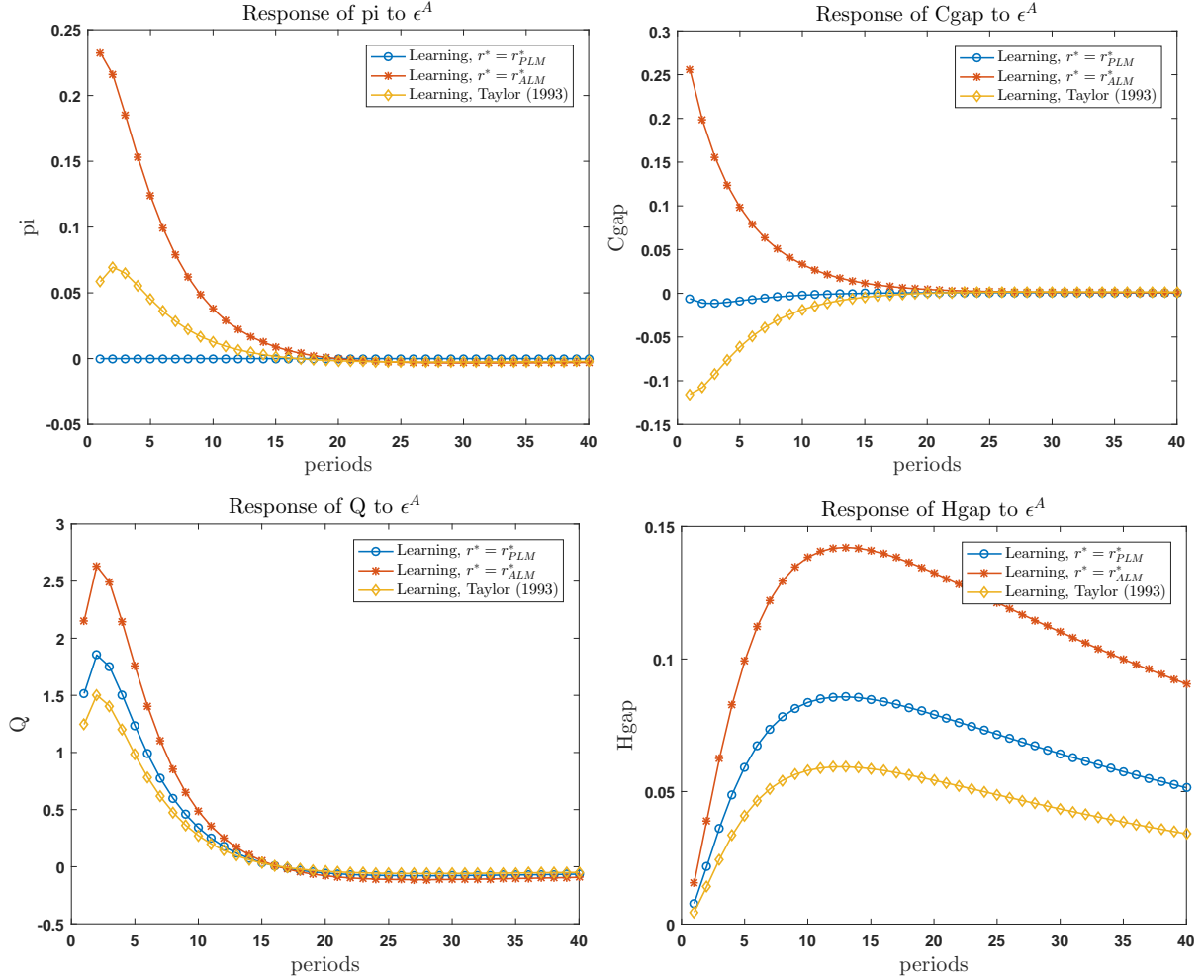


Note: Response to a one standard deviation positive technology shock ϵ_{At} . Log percentage points. Flexible prices and zero inflation, with asset production.

movements in the natural rate of interest in the PLM; but this is no longer optimal. This is illustrated in figure 7, which shows responses to a one standard deviation productivity shock for rules 47–49. The figure shows that when the central bank sets interest rates according to the perceived natural rate of interest $r_t^{n,PLM}$, it achieves strict inflation targeting as before, and therefore implements the flexible price allocation under learning. But this allocation is inefficient because of the investment decisions taken under non-rational expectations and the housing gap \hat{h}_t is not closed. Tracking the equilibrium (ALM) process of the natural rate delivers even worse outcomes. The standard Taylor rule, by contrast, manages to keep the housing gap relatively small following a technology shock, but does not stabilize inflation and also does not close the consumption gap \hat{c}_t as much as rule 47. Therefore, this extension of the model features a non-trivial policy tradeoff even without cost-push shocks.

With an investment channel present, learning does not only change how monetary policy has to be implemented, but also the optimal target criterion. In Table 2, we compare strict inflation targeting in

Figure 7: Inflation targeting vs Taylor rule, with asset production.



Note: Response to a unit standard deviation positive technology shock ϵ_{At} under sticky prices and with asset production. Log percentage points. The interest rate rules used are given in Equations (47) (using the PLM natural rate with asset production) and (49).

the absence of cost-push shocks with a targeting rule that explicitly “leans against the wind”, in that it tolerates inflation below target at times when asset prices are high. This targeting rule would be suboptimal under rational expectations, but turns out to be welfare-increasing under learning.

We also compute optimal simple rules of the form (53) for our calibration of the model with cost-push shocks, as in section 7. The results are tabulated in Table 3. As can be seen in rows (7)–(9), in the absence of an outgap gap response the optimal rule has a negative coefficient on the asset price term. This a result of the allocative inefficiency that learning introduces under asset production. The optimal choice of ϕ_q in this case is driven by the need to control of inefficient under/over investment under learning. As was the case with fixed asset supply, however, the optimal simple rule exhibits a positive response to both asset prices and the output gap, with optimal rule coefficients prescribing a more aggressive reaction to both the output gap and asset prices under learning than in the case with fixed asset supply.

Table 2: “Leaning against the wind” with asset production.

Rational Expectations	$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta \hat{h}_t)$	$\sigma(\Delta q_t)$	\mathcal{L}
$\pi_t = 0$	0.000	0.000	0.000	0.930	0
Learning	$\sigma(\pi_t)$	$\sigma(\hat{y}_t)$	$\sigma(\Delta \hat{h}_t)$	$\sigma(\Delta q_t)$	\mathcal{L}
$\pi_t = 0$	0.000	0.037	4.413	1.661	31.67
$\pi_t = -.311\hat{\mu}_{t-1}$	0.389	0.026	4.369	1.659	31.31

Table 3: Performance of optimized simple rules, with asset production.

Rational Expectations		$\sigma(\pi_t)$	$\sigma(\hat{c}_t)$	$\sigma(\hat{h}_t)$	$\sigma(\Delta q_t)$	\mathcal{L}
(1)	$i_t = r_{ss} + 1.5\pi_t + 0.125 \cdot \hat{y}_t$	0.342	0.470	0.000	0.759	0.779
(2)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t)$ $\{\rho_i^*, \phi_y^*\} = \{0.723, 1.207\}$	0.365	0.271	0.000	0.675	0.475
(3)	$\pi_t = \zeta^* \eta_t$ $\zeta^* = 0.214$	0.299	0.296	0.000	0.968	0.401
Learning		$\sigma(\pi_t)$	$\sigma(\hat{c}_t)$	$\sigma(\hat{h}_t)$	$\sigma(\Delta q_t)$	\mathcal{L}
(4)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t)$ $\{\rho_i^*, \phi_y^*\} = \{0.098, 0.819\}$	1.024	0.512	0.193	1.078	6.207
(5)	$\pi_t = \zeta^* \eta_t$ $\zeta^* = 0.399$	0.566	0.034	0.494	1.662	32.413
(6)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t)$ $\rho_i^* = 0$	0.045	0.602	0.747	1.923	60.771
(7)	BG (1999) w/ asset price $i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_q^* \cdot \log q_{t-1})$ $\{\rho_i^*, \phi_q^*\} = \{0, -0.241\}$	1.735	0.693	0.744	0.778	46.412
(8)	BG (2001) w/ asset price $i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_q^* \cdot \log q_t)$ $\{\rho_i^*, \phi_q^*\} = \{0, -0.072\}$	0.697	0.596	0.742	1.925	59.715
(9)	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_q^* \cdot \sum_{s=0}^{\infty} \omega^{*s} \Delta \log Q_{t-s})$ $\{\rho_i^*, \phi_q^*, \omega^*\} = \{0, -0.064, 0.999\}$	0.618	0.598	0.744	1.927	59.956
(10)	FM (2007) w/ asset price $i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t + \phi_q^* \cdot \log q_t)$ $\{\rho_i^*, \phi_y^*, \phi_q^*\} = \{0.330, 3.548, 1.077\}$	0.638	0.559	0.096	0.927	2.131
(11)*	$i_t = \rho_i^* i_{t-1} + (1 - \rho_i^*) \cdot (r_{ss} + 1.5\pi_t + \phi_y^* \cdot \hat{y}_t + \phi_q^* \cdot \sum_{s=0}^{\infty} \omega^{*s} \Delta \log Q_{t-s})$ $\{\rho_i^*, \phi_y^*, \phi_q^*, \omega^*\} = \{0.329, 3.106, 0.974, 0.999\}$	0.646	0.570	0.095	0.923	2.155

10 Conclusion

In this paper, we have characterized optimal monetary policy in a model in which agents are learning about asset prices. Our model is the standard New-Keynesian model with a durable asset in fixed supply. Agents form expectations about asset price in an extrapolative fashion. However, their expectations remain model-consistent conditional on their beliefs about house prices, which allows us to isolate the effects of learning about asset prices from the many other ways in which distorted beliefs can affect the economy. Learning amplifies asset price fluctuations in the model, and leads to perceived wealth effects that create inefficient fluctuations in aggregate demand.

We gave an analytical solution to the optimal policy in the linearized model with learning, and have showed that inflation targeting remains optimal. However, inflation targeting requires a very different prescription for the nominal interest rate under learning. The interest rate has to increase with asset prices and subjective expectations of future asset price growth. We then showed using numerical simulations that the optimal policy can be reasonably well approximated by simple rules that responds to inflation and a moving average of asset price growth. A reaction to asset prices always reduces asset price volatility, but too much of a reaction can increase inflation volatility. Nonetheless, the optimal weight on asset prices is strictly positive for the rules we considered.

We have also shown that our theoretical results are robust to the specification of subjective beliefs, encompassing a wide range of belief specifications. Another extension of our model showed that the case for reacting to asset prices is strengthened further when investment decisions are affected by asset prices. With an investment channel, it becomes beneficial to include a reaction to asset prices in the targeting rule of monetary policy. This “leaning against the wind” policy effectively curbs inefficient investment fluctuations.

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A Appendix: Details on the derivations

A.1 Asset demand in the model with fixed asset supply

The coefficients in the asset demand function (32) are:

$$k_h = \frac{1}{2\beta} \left(1 + \beta + \frac{\theta}{\gamma\kappa_1} (1 - \beta) - \sqrt{\left(1 + \beta + \frac{\theta}{\gamma\kappa_1} (1 - \beta) \right)^2 - 4\beta} \right) \in (0, 1) \quad (68)$$

$$k_a = \frac{1 + \phi}{1 + \phi - \alpha(1 - \gamma)} \frac{1 - \beta\rho_a}{\kappa_1 + (1 - \beta)\frac{\theta}{\gamma} + \beta\kappa_1(1 - k_h - \rho_a)} > 0 \quad (69)$$

$$k_q = \frac{1}{\gamma} \frac{1 - \beta}{(1 - \beta)\frac{\theta}{\gamma} + \kappa_1(1 - \beta k_h)} > 0 \quad (70)$$

$$k_\mu = \frac{\beta/\gamma}{(1 - \beta)\frac{\theta}{\gamma} + \kappa_1(1 - \beta\rho_\mu + \beta(1 - k_h))} \frac{(1 + \beta)\frac{\theta}{\gamma} + \kappa_1\beta(1 - k_h)}{(1 + \beta)\frac{\theta}{\gamma} + \kappa_1(1 - \beta k_h)} > 0 \quad (71)$$

A.2 Asset demand with extended beliefs

The coefficients \tilde{k}_μ and $\tilde{\rho}$ read as follows:

$$\tilde{k}_\mu = \frac{1}{\gamma} \frac{\beta}{(1 - \beta)\frac{\theta}{\gamma} + \kappa_1(1 + \beta - \beta k_h)} \frac{(1 + \beta)\frac{\theta}{\gamma} + \kappa_1\beta(1 - k_h)}{(1 + \beta)\frac{\theta}{\gamma} + \kappa_1(1 - \beta k_h)} > 0 \quad (72)$$

$$\tilde{\rho} = \frac{\beta}{\beta + (1 - \beta)\frac{\theta}{\gamma\kappa_1} + (1 - \beta k_h)} \in (0, 1). \quad (73)$$

A.3 Attaining the first-best allocation with cost push shocks under learning

Substituting the policy $\pi_t = -(\beta\rho_\eta)^{-1}\eta_t + b_t z_t$ into the Phillips curve (38), we obtain:

$$\pi_t = -(\beta\rho_\eta)^{-1}\eta_t + b_t z_t = \kappa \left(\tilde{c}_t + \kappa_1 \Delta \tilde{h}_t \right). \quad (74)$$

We will find b_t such that $\pi_t = 0$ a.s. under the ALM. In this case, $\kappa \left(\tilde{c}_t + \kappa_1 \Delta \tilde{h}_t \right) = y_t - y_t^{RE} = 0$ under the ALM as well, and the first-best allocation is attained.

In order to solve for b_t , then, we need to compute the equilibrium value of z_t , which amounts to solving for the equilibrium price q_t . To do this, we first need to derive the demand function for the asset under the PLM. Let $x_t = \tilde{c}_t + \kappa_1 \Delta \tilde{h}_t$. In analogy to the computations of the flexible-price equilibrium,

we can compute the asset demand function as the solution to the following system of equations:

$$(1 - \beta) \theta h_t = \gamma c_t - q_t - \beta \mathbb{E}_t^{\mathcal{P}} [\gamma c_{t+1} - q_{t+1}] \quad (75)$$

$$(1 + \phi - \alpha + \alpha \gamma) c_t = \alpha x_t + (1 - \alpha) a_t - (1 + \phi - \alpha) \frac{\bar{Q} \bar{H}}{\bar{Y}} \Delta h_t. \quad (76)$$

With $x_t = (b_t z_t - (\beta \rho_\eta)^{-1} \eta_t) / \kappa$, we can solve:

$$h_t = k_h h_{t-1} + k_a a_t - k_q q_t + k_\mu \hat{\mu}_{t-1} + k_{bz} b_t z_t + k_\eta \eta_t. \quad (77)$$

Here, the coefficients k_h , k_a , k_q and k_μ are the same as in the flexible-price demand function (32). The coefficients k_η and k_{bz} are given by:

$$k_\eta = -\frac{k_h}{\beta \rho_\eta \kappa} \frac{1 - \beta \rho_\eta}{1 - k_h \beta \rho_\eta} \frac{\bar{Y}}{\bar{Q} \bar{H}} \frac{\alpha}{1 + \phi - \alpha} \quad (78)$$

$$k_{z,t} = -\frac{k_h}{\kappa} \frac{\bar{Y}}{\bar{Q} \bar{H}} \frac{\alpha}{1 + \phi - \alpha} \quad (79)$$

Now the ALM is found by imposing $h_t = 0$, and this condition leads to the ALM expression for z_t :

$$z_t = \frac{k_q}{k_q - k_{bz} b_t} \left(\frac{1}{k_q} (k_a a_t + k_\mu \hat{\mu}_{t-1} + k_\eta \eta_t) - q_{t-1} - \hat{\mu}_{t-1} \right) \quad (80)$$

Imposing $\beta \rho_\eta b_t z_t = \eta_t$ then leads to:

$$b_t = \left(\frac{k_{bz}}{k_q} + \frac{\beta \rho_\eta}{k_q} (k_a a_t + k_\mu \hat{\mu}_{t-1} + k_\eta \eta_t) - \beta \rho_\eta (q_{t-1} + \hat{\mu}) \right)^{-1} \eta_t. \quad (81)$$

A.4 Replication of RE-optimal policy with cost push shocks under learning

The solution under learning must be obtained again by first solving the PLM and then imposing equilibrium. We start by imposing (50) as a targeting rule in the PLM. We can make use of the fact that we can represent the equilibrium conditions of the PLM by the three equation (38)–(40). Substituting the targeting rule yields:

$$\tilde{c}_t = -\frac{1 - \zeta (1 - \beta \rho)}{\kappa} \eta_t - \kappa_1 \Delta \tilde{h}_t \quad (82)$$

$$\tilde{h}_t = -\frac{\gamma}{\theta (1 - \beta)} \beta \mathbb{E}_t^{\mathcal{P}} \Delta \tilde{c}_{t+1}. \quad (83)$$

This system has one state variable, the asset holding gap \tilde{h}_{t-1} , and one shock, the cost-push shock η_t . Using the method of undetermined coefficients leads to the following solution:

$$\tilde{c}_t = a\tilde{h}_{t-1} + b\eta_t \quad (84)$$

$$\tilde{h}_t = \gamma \frac{a\tilde{h}_{t-1} + b(1 - \beta\rho)\eta_t}{\theta(1 - \beta) + \gamma\beta a} \quad (85)$$

where

$$a = \frac{1 - \beta}{\beta} \frac{\sqrt{(\theta + \gamma\kappa_1)^2 + 4\frac{\beta}{1-\beta}\gamma\kappa_1\theta} - (\theta + \gamma\kappa_1)}{2\gamma}$$

$$b = -\frac{1 - \zeta(1 - \beta\rho)}{\kappa} \left(1 + \frac{\kappa_1\gamma(1 - \beta\rho)}{\theta(1 - \beta) + \gamma\beta a}\right)^{-1}.$$

We can substitute this solution into the IS equation (39) to solve for the implied nominal interest rate path to obtain:

$$i_t = r_t^{n,PLM} + \rho\eta_t - \gamma(a\Delta\tilde{h}_t + b(1 - \rho)\eta_t). \quad (86)$$

We can equivalently rewrite the purchases of assets in terms of the perceived output gap and inflation by observing that $\eta_t = \pi_t/\zeta$ and

$$\begin{aligned} \kappa_1\Delta\tilde{h}_t &= -\frac{1 - \zeta(1 - \beta\rho)}{\kappa}\eta_t - \tilde{c}_t \\ &= -\frac{1 - \zeta(1 - \beta\rho)}{\kappa}\eta_t - \tilde{y}_t + \frac{\bar{Q}\bar{H}}{\bar{Y}}\Delta\tilde{h}_t \\ &= \frac{1 + \phi - \alpha}{\kappa_1\alpha\gamma} \left(\frac{1 - \zeta(1 - \beta\rho)}{\kappa}\eta_t + \tilde{y}_t \right). \end{aligned} \quad (87)$$

We can then express the interest rate as a function of the perceived natural rate, inflation and the perceived output gap:

$$i_t = r_t^{n,PLM} + a\frac{1 + \phi - \alpha}{\kappa_1\alpha}\tilde{y}_t + \left(\rho - \gamma(1 - \rho)b + a\frac{1 + \phi - \alpha}{\kappa_1\alpha}\frac{1 - \zeta(1 - \beta\rho)}{\kappa}\right)\frac{1}{\zeta}\pi_t. \quad (88)$$

This expression shows that the equilibrium interest rate path depends on inflation and the perceived output gap as well as the natural rate. The natural rate is increasing in the level of asset prices as well as the subjective expectation of future asset price growth. Therefore, the nominal interest rate in (88) is effectively leaning against the wind.

In the second step, we solve for the equilibrium (ALM) allocation and prices. Imposing asset market

clearing requires $h_t = 0$, which can be rewritten as

$$h_t^{n,PLM} = -\tilde{h}_t. \quad (89)$$

Substituting this into the expression for asset demand (32), we can solve for the equilibrium price:

$$\begin{aligned} -\tilde{h}_t &= k_a a_t - k_h \tilde{h}_{t-1} + k_q q_t + k_\mu \hat{\mu}_{t-1} \\ \Rightarrow q_t &= -\frac{1}{k_q} \left(k_a a_t + k_\mu \hat{\mu}_{t-1} + \tilde{h}_t - k_h \tilde{h}_{t-1} \right). \end{aligned} \quad (90)$$

Finally, we solve for the equilibrium output gap. We first rewrite Equation (33) in terms of housing purchases under flexible prices:

$$y_t^{n,PLM} = y_t^{n,RE} + \frac{\alpha\gamma\kappa_1}{1+\phi-\alpha} \left(h_t^{n,PLM} - h_{t-1}^{n,PLM} \right) \quad (91)$$

and use this relationship to arrive at the main result in this section:

$$\begin{aligned} \hat{y}_t &= \tilde{h}_t + y_t^{n,PLM} - y_t^{n,RE} \\ &= \tilde{c}_t + \frac{\bar{Q}\bar{H}}{\bar{Y}} \Delta \tilde{h}_t + y_t^{n,PLM} - y_t^{n,RE} \\ &= \tilde{c}_t + \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} - \frac{\alpha\gamma\kappa_1}{1+\phi-\alpha} \right) \Delta \tilde{h}_t \\ &= \tilde{c}_t + \kappa_1 \Delta \tilde{h}_t \\ &= -\frac{1-\zeta(1-\beta\rho)}{\kappa} \eta_t. \end{aligned} \quad (92)$$

That is, the equilibrium path of the output gap is identical to that under rational expectations.

A.5 Linearized equilibrium conditions and natural rate in the model with asset production

Under rational expectations, the following set of equations describe the linearized equilibrium (up to a monetary policy rule):

$$y_t = a_t + \alpha n_t \quad (93)$$

$$\pi_t = \beta \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} (w_t - a_t + (1-\alpha)n_t) + \eta_t \quad (94)$$

$$w_t = \gamma c_t + \phi n_t \quad (95)$$

$$\bar{Y} y_t = \bar{C} c_t + \bar{Q} \bar{H} (h_t - (1-\delta) h_{t-1}) \quad (96)$$

$$h_t = (1-\delta) h_{t-1} + \frac{\omega \delta}{1-\omega} q_t \quad (97)$$

$$i_t = \gamma (\mathbb{E}_t c_{t+1} - c_t) + \mathbb{E}_t \pi_{t+1} \quad (98)$$

$$q_t = \gamma c_t - (1-\beta(1-\delta)) \theta h_t - \beta(1-\delta) \gamma \mathbb{E}_t c_{t+1} + \beta \delta \mathbb{E}_t q_{t+1}. \quad (99)$$

We can eliminate the asset price, wages, labor and output from this to get a three-equation system:

$$\begin{aligned} \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} \\ &\quad \left(\gamma c_t + \left(\frac{1+\phi-\alpha}{\alpha} \right) \left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{Q} \bar{H}}{\bar{Y}} (h_t - (1-\delta) h_{t-1}) \right) - \frac{1+\phi}{\alpha} a_t \right) + \eta_t \end{aligned} \quad (100)$$

$$i_t = \gamma (\mathbb{E}_t c_{t+1} - c_t) + \mathbb{E}_t \pi_{t+1} \quad (101)$$

$$(1-\tilde{\beta}) \theta h_t = \gamma c_t - \frac{1-\omega}{\omega \delta} (h_t - (1-\delta) h_{t-1}) - \tilde{\beta} \gamma \mathbb{E}_t c_{t+1} + \tilde{\beta} \frac{1-\omega}{\omega \delta} \mathbb{E}_t (h_{t+1} - (1-\delta) h_t) \quad (102)$$

where we introduced $\tilde{\beta} = \beta(1-\delta)$ to ease notation.

Under learning, we can do a similar exercise. We first tackle the PLM. All we do is to replace the market clearing condition with the subjective law of motion for asset prices:

$$y_t = a_t + \alpha n_t \quad (103)$$

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} (w_t - a_t + (1-\alpha)n_t) + \eta_t \quad (104)$$

$$w_t = \gamma c_t + \phi n_t \quad (105)$$

$$\bar{Y} y_t = \bar{C} c_t + \bar{Q} \bar{H} (h_t - (1-\delta) h_{t-1}) \quad (106)$$

$$i_t = \gamma (\mathbb{E}_t^{\mathcal{P}} c_{t+1} - c_t) + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} \quad (107)$$

$$q_t = \gamma c_t - (1-\beta(1-\delta)) \theta h_t - \beta(1-\delta) \gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \beta \delta \mathbb{E}_t^{\mathcal{P}} q_{t+1} \quad (108)$$

$$q_t = q_{t-1} + \hat{\mu}_{t-1} + z_t \quad (109)$$

$$\hat{\mu}_t = \rho_{\mu} \hat{\mu}_{t-1} + g z_t. \quad (110)$$

The three-equation system describing the equilibrium boils down to:

$$\pi_t = \beta \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} \left(\gamma c_t + \left(\frac{1+\phi-\alpha}{\alpha} \right) \left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{Q}\bar{H}}{\bar{Y}} (h_t - (1-\delta)h_{t-1}) \right) - \frac{1+\phi}{\alpha} a_t \right) + \eta_t \quad (111)$$

$$i_t = \gamma (\mathbb{E}_t^{\mathcal{P}} c_{t+1} - c_t) + \mathbb{E}_t^{\mathcal{P}} \pi_{t+1} \quad (112)$$

$$q_t = \gamma c_t - \left(1 - \tilde{\beta} \right) \theta h_t - \tilde{\beta} \gamma \mathbb{E}_t^{\mathcal{P}} c_{t+1} + \tilde{\beta} \mathbb{E}_t^{\mathcal{P}} q_{t+1}. \quad (113)$$

The flexible price equilibrium under rational expectations is characterized as:

$$h_t^{n,RE} = k_a^{RE} a_t + k_h^{RE} h_{t-1}^{RE}, \quad k_a^{RE} > 0, \quad k_h^{RE} \in (0, 1-\delta) \quad (114)$$

$$y_t^{n,RE} = \left(\frac{\bar{C}}{\bar{Y}} \frac{\phi+1}{\bar{C} (1+\phi-\alpha) + \alpha\gamma} + \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} - \frac{\bar{C}}{\bar{Y}} \kappa_1 \right) k_a^{RE} \right) a_t - \left(\frac{\bar{Q}\bar{H}}{\bar{Y}} - \frac{\bar{C}}{\bar{Y}} \kappa_1 \right) (1-\delta - k_h^{RE}) h_{t-1}^{RE} \quad (115)$$

$$c_t^{n,RE} = \left(\frac{\phi+1}{\bar{C} (1+\phi-\alpha) + \alpha\gamma} - \kappa_1 k_a^{RE} \right) a_t + \kappa_1 (1-\delta - k_h^{RE}) h_{t-1}^{RE} \quad (116)$$

$$r_t^{n,RE} = -\gamma \left(\frac{\phi+1}{\bar{C} (1+\phi-\alpha) + \alpha\gamma} - \kappa_1 k_a^{RE} + \kappa_1 (1-\delta - k_h^{RE}) k_a^{RE} \right) (1-\rho_a) a_t - \gamma \kappa_1 (1-\delta - k_h^{RE}) (1 - k_h^{RE}) h_{t-1}^{RE} \quad (117)$$

where the constants κ and κ_1 are now redefined as:

$$\kappa = \frac{1-\xi}{\xi} \frac{1-\beta\xi}{\alpha} \left(\frac{\bar{C}}{\bar{Y}} (1+\phi-\alpha) + \alpha\gamma \right) \quad (118)$$

$$\kappa_1 = \frac{1+\phi-\alpha}{\bar{C} (1+\phi-\alpha) + \alpha\gamma} \frac{\bar{Q}\bar{H}}{\bar{Y}}. \quad (119)$$

Under learning and flexible prices, we can boil things down to these two equations to solve for the PLM:

$$0 = \gamma c_t + \left(\frac{1+\phi-\alpha}{\alpha} \right) \left(\frac{\bar{C}}{\bar{Y}} c_t + \frac{\bar{Q}\bar{H}}{\bar{Y}} \tilde{h}_t \right) - (1+\phi) a_t \quad (120)$$

$$q_t = \gamma c_t - \left(1 - \tilde{\beta} \right) \theta h_t - \tilde{\beta} \gamma \mathbb{E}_t c_{t+1} + \beta \delta \mathbb{E}_t^{\mathcal{P}} q_{t+1}. \quad (121)$$

Guess and verify

$$h_t^{n,PLM} = k_a a_t + k_h h_{t-1}^{n,PLM} + k_q q_t + k_\mu \hat{\mu}_{t-1} \quad (122)$$

where the coefficients are given by:

$$k_h = \frac{1}{2\tilde{\beta}} \left(1 + \tilde{\beta}(1-\delta) + \frac{\theta}{\gamma\kappa_1} (1-\tilde{\beta}) - \sqrt{\left(1 + \tilde{\beta}(1-\delta) + \frac{\theta}{\gamma\kappa_1} (1-\tilde{\beta}) \right)^2 - 4\tilde{\beta}(1-\delta)} \right) \in (0, 1-\delta) \quad (123)$$

$$k_a = \frac{1+\phi}{\frac{\bar{C}}{Y}(1+\phi-\alpha) + \alpha\gamma} \frac{1-\tilde{\beta}\rho_a}{\kappa_1 + \left(1-\tilde{\beta} \right) \frac{\theta}{\gamma} + \tilde{\beta}\kappa_1(1-\delta-k_h-\rho_a)} > 0 \quad (124)$$

$$k_q = -\frac{1}{\gamma} \frac{1-\tilde{\beta}}{\left(1-\tilde{\beta} \right) \frac{\theta}{\gamma} + \kappa_1 \left(1-\tilde{\beta}k_h-\tilde{\beta}\delta \right)} < 0 \quad (125)$$

$$k_\mu = \frac{\tilde{\beta}/\gamma}{\left(1-\tilde{\beta} \right) \frac{\theta}{\gamma} + \kappa_1 \left(1-\tilde{\beta}\rho_\mu + \tilde{\beta}(1-\delta-k_h) \right)} \frac{\left(1-\tilde{\beta} \right) \frac{\theta}{\gamma} + \kappa_1\tilde{\beta}(1-\delta-k_h)}{\left(1-\tilde{\beta} \right) \frac{\theta}{\gamma} + \kappa_1 \left(1-\tilde{\beta}k_h-\tilde{\beta}\delta \right)} > 0 \quad (126)$$

where $\tilde{\beta} = (1-\delta)\beta$. We can characterize the flex-price PLM consumption, output and interest rates:

$$\frac{c_t^{n,PLM} - c_t^{n,RE}}{\kappa_1} = - (k_a - k_a^{RE}) a_t - (k_h - k_h^{RE}) h_{t-1} + (1-\delta-k_h^{RE}) \left(h_{t-1}^{n,PLM} - h_{t-1}^{n,RE} \right) - k_q q_t - k_\mu \hat{\mu}_{t-1} \quad (127)$$

$$\frac{y_t^{n,PLM} - y_t^{n,RE}}{\frac{\bar{Q}\bar{H}}{Y} - \frac{\bar{C}}{Y}\kappa_1} = - \frac{c_t^{n,PLM} - c_t^{n,RE}}{\kappa_1} \quad (128)$$

$$\begin{aligned} - \frac{r_t^{n,PLM} - r_t^{n,RE}}{\gamma\kappa_1} = & - \left((1-k_h)(1-\delta-k_h) - (1-k_h^{RE})(1-\delta-k_h^{RE}) \right) h_{t-1} \\ & - (1-k_h^{RE})(1-\delta-k_h^{RE}) \left(h_{t-1}^{n,PLM} - h_{t-1}^{n,RE} \right) \\ & + (k_a(2-\delta-k_a-\rho_a) - k_a^{RE}(2-\delta-k_a^{RE}-\rho_a)) a_t \\ & + (k_\mu(2-\delta-k_h-\rho_\mu) - k_q) \hat{\mu}_{t-1} + k_q(1-\delta-k_h) q_t \end{aligned} \quad (129)$$

In order to find the ALM under flexible prices, we impose market clearing for the durable asset and obtain:

$$\begin{aligned} (1-\delta) h_{t-1} + \frac{\omega\delta}{1-\omega} q_t &= k_a a_t + k_h h_{t-1} + k_q q_t + k_\mu \hat{\mu}_{t-1} \\ \Leftrightarrow q_t &= \frac{1}{\frac{\omega}{1-\omega}\delta - k_q} (k_a a_t + k_\mu \hat{\mu}_{t-1} - (1-\delta-k_h) h_{t-1}). \end{aligned} \quad (130)$$

The equilibrium price is increasing in productivity, increasing in asset price beliefs, and decreasing in the existing stock of housing.

When the equilibrium asset price q_t from equation (130) is substituted into the expression for the natural rate $r_t^{n,PLM}$, the natural real rate in the ALM $r_t^{n,ALM}$ is increasing in asset price expectations $\hat{\mu}_{t-1}$, just as in the baseline model. The sign of the other coefficients are ambiguous and depend on the parameterization.