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Trade, Labor Reallocation Across Firms and Wage Inequality

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Abstract

This paper develops a framework for studying the effects of higher trade openness on the wage distribution that emphasizes within-industry labor reallocation across firms, strong skill-productivity complementarities in production and heterogenous fixed export costs across firms. Assuming no entry in the industry, an autarkic economy that opens to trade experiences a pervasive rise in wage inequality; a trade liberalization in a trading economy increases inequality at the lower end of the distribution, but may reduce it elsewhere. Assuming free entry, opening to trade could result in pervasively higher inequality or wage polarization. The analysis highlights the importance of new exporters (extensive margin) in shaping the aggregate relative demand for skills, a channel controlled by the distribution of fixed export costs in the model.

Keywords: Trade, firms, workers, supermodularity, wage inequality.
JEL codes: F10, F12, F16.

*All the views expressed in this paper are mine and do not necessarily represent those of the Federal Reserve System.

1 Introduction

In this paper, I develop a general equilibrium trade model with a large number of skill-groups that emphasizes within-industry labor reallocation across firms as the mechanism through which trade affects the wage distribution. The impact of international trade on the income distribution has featured prominently in economic analysis since at least Ohlin (1933). This topic has received renewed attention, as economists have tried to elucidate the causes of the significant rise in wage inequality in many countries since the late 70s. Three implications following from the empirical research on this phenomenon motivate the analysis. First, as discussed in Goldberg and Pavcnik (2007) and Helpman (2016), the evidence provides little support for trade as an important driver of the rise in inequality *through* the channels emphasized by the traditional factor-proportions trade theory.[1] Second, firms may be an important part of the story behind the changes in the wage distribution. For example, Krishna, Poole, and Senses (2014) find substantial within-industry labor reallocation across firms following a trade liberalization that cannot be explained by a random assignment of workers to firms.[2] Third, divergent trends in inequality in different parts of the wage distribution (Autor, Katz, and Kearney 2008) and a rise in within-group (residual) wage inequality (Acemoglu 2002; Attanasio, Goldberg, and Pavcnik 2004), suggests that grouping workers into a few large skill-groups (as typically done in the literature) does not provide enough detail to understand the full distributional consequences of international trade.

The framework extends a standard heterogeneous-firms trade model to include heterogenous workers, strong complementarities in production between worker skill and firm productivity and heterogenous fixed export costs. As in Melitz (2003), labor is the only factor of production, the labor market is perfectly competitive, and final goods are produced by monopolistically competitive firms that differ in their productivity. In addition, the presence of fixed production and export costs leads to selection into activity and into exporting, i.e. only some firms find it optimal to produce and only a subset of them export. Departing from Melitz (2003), the labor force comprises heterogeneous workers of a continuum of skill-types, so firms must choose not only the total number of *production* workers to hire but also the mix of skill-types to employ. With strong production complementarities between worker skill and firm productivity, such a choice implies that more productive firms have workforces of higher average ability in equilibrium. In this setting, a decline in trade costs induces a reallocation of workers across these heterogenous firms that changes the relative demand for skills in the economy. I use the model to study the channels through which this labor reallocation affects the wage distribution, including the entry and exit of firms into the market, the increased demand of incumbent exporters, and the demand of new exporters.

The core of the framework lies in the production and export technology of firms. The output of a firm depends linearly on the number of *production* workers of each skill-type that it employs. The

[1] This evidence includes a rise in the skill-premium in developed and developing countries, driven in both cases by higher demand for skilled workers in all industries (Autor et al. 1998; Goldberg and Pavcnik 2007). In addition, several studies find little labor reallocation across industries in response to trade liberalizations in developing countries.

[2] In addition, as discussed in Card et al. (2016), numerous studies find similar trends in the aggregate dispersion of wages and firms' productivity.

productivity of a production worker at a given firm is a strictly log supermodular function of the worker's skill and the firm's productivity, giving more able workers a comparative advantage in production at more productive firms. As in Costinot and Vogel (2010), these assumptions permit to transform the market equilibrium analysis into the analysis of a matching problem. In particular, the equilibrium allocation of *production* workers among *active* firms is characterized by a strictly increasing and continuous matching function that maps the skill-types of the former to the productivity-types of the latter. Moreover, this matching function is a sufficient statistic for the dispersion wages in this setting, facilitating the analysis of comparative static predictions about wage inequality.

Fixed export costs also play an important role as they determine the shape of the set exporters and their collective demand for skills, a crucial element in the model affecting the distributional consequences of trade. Therefore, I consider a relatively flexible specification of fixed export costs, so that the analysis can incorporate salient empirical features about this set.[3] Specifically, I posit that fixed export costs vary across firms, and model their firm-specific sizes as independent realizations of a nonnegative random variable with an absolutely continuous and increasing CDF. As a result, exporters are, on average, more productive than nonexporters in equilibrium, but high productivity non-exporters coexists with low productivity exporters. Finally, all fixed costs are paid in terms of a "skill bundle" that comprises *non-production* workers of all skill levels, an assumption that allows me to isolate the impact on the wage distribution of the endogenous assignment of production workers to firms.

The cross-section of the model captures several features of the data identified by the trade and labor literatures. The dispersion of wages in the model reflects between-firms wage differences (rather than within-firm differences), a channel that represents around 60% of the wage dispersion in the United States (Davis and Haltiwanger 1991). In addition, more productive firms tend to be larger (in terms of output), have workforces of higher average ability, and pay higher average wages (Card et al. 2016). Per the stochastic representation of fixed export costs, the model features an imperfect positive correlation between size, firm wages and export status (Bernard and Jensen 1995), and between the latter and firm productivity, leading to overlapping productivity distributions for exporters and non-exports (Bernard, Eaton, Jensen, and Kortum 2003). Finally, if workers are classified in large skill-groups, possibly reflecting the imperfect observability of workers ability by the econometrician, then the model features wage heterogeneity within each of these skill groups (Acemoglu 2002; Attanasio et al. 2004).

I carry out the analysis of the effects of trade on the wage distribution under two widely-used assumptions about entry, no free entry a-lá Chaney (2008) and free entry a-lá Melitz (2003). These alternative entry assumptions lead to the no-free-entry and free-entry models analyzed in the paper, whose predictions can be interpreted, respectively, as the short- and long-term effects of trade.[4] These models differ only in the equilibrium condition that pins down the *activity cutoff*, the productivity value below which

[3] The assumption of common fixed export costs, which has been standard in the literature since Melitz (2003), leads to the counterfactual prediction of non-ovelapping productivity distributions for exporters and non-exporters. This unrealistic assumption is not innocuous in this setting.

[4] Exploring the implications of these two alternative entry assumptions also serves a pedagogical purpose. By delivering sharper results, the no-free-entry model facilitates the analysis of the main forces at play, which in turn simplifies the subsequent discussion of the more nuanced implications of the free-entry model.

firms do not find it profitable to produce. Conditional on the activity cutoff, the two models are identical, so they share the cross-sectional implications discussed above.

To study the impact of higher trade openness on the wage distribution, I decompose the associated reallocation of production and employment across firms into three channels. The first channel, the *selection-into-activity* channel, captures the reallocation of resources driven by changes in the set of active firms, i.e. by changes in the activity cutoff. The second channel is the *intensive margin* of trade, and reflects the changes in the production and employment decisions of incumbent exporters that continue serving the foreign market after the decline in trade frictions. Finally, the third channel is the *extensive margin* of trade, which captures the reallocation of employment associated with changes in the set of exporters. These last two channels are largely determined by the distribution of fixed exports costs in the economy, which highlights the importance of properly modeling this element of the model. This decomposition not only highlights the key elements driving the results in the current setting, but also it facilitates the comparison with the implications of other frameworks in the trade literature exploring the connection between international trade, firms and wages.

In the no-free-entry model, an initially autarkic economy that opens to trade always experiences an increase in the activity cutoff and a pervasive rise in wage inequality, in the sense that for any pair of workers, the relative wage of the more skilled one rises. In terms of the three channels discussed above, the selection-into-activity channel induces a pervasive rise in wage inequality, as the exit of the least productive (low-skill-intensive) firms leads to a decline in the relative demand of less skilled workers. With no exporters in the initial autarkic equilibrium, the intensive margin channel is not operational in this counterfactual. Finally, the extensive margin channel also leads to a pervasive rise in wage inequality; the (new) exporters in the open economy are, on average, more productive than non-exporters, so their collective labor demand is biased toward more skilled workers. The importance of this channel, which depends on how fast the fraction of exporting firms increases with productivity, is determined by the CDF of fixed export costs. Regarding the effect on the level of wages, trade always rises the average real wage, but the least skilled workers in the economy could see their real wage decline.

A trade liberalization can lead to additional outcomes.[5] Although a decline in variable trade costs in the no-free-entry model always leads to an increase in the activity cutoff and a rise in wage inequality at the lower end of the wage distribution, little can be said about its impact elsewhere in the distribution. As the activity cutoff rises, the selection-into-activity channel leads to a pervasive rise in wage inequality. The intensive-margin channel also leads to a pervasive rise in wage inequality, reflecting a rise in the more skill-intensive labor demand of incumbent exporters as they expand their production to satisfy a higher foreign demand. In contrast to the previous two channels, the impact of the extensive-margin channel on the wage distribution is ambiguous. Without additional restrictions on the CDF of exports costs, new exporters can be (on average) more or less productive than incumbent firms, so their collective demand may be biased toward more or less skilled workers. Moreover, the ambiguity about the effects of this third channel extends to the overall impact of a trade liberalization on the wage distribution. This result

[5] A trade liberalization is defined as a decline in the variable trade costs faced by an economy that already participates in international trade.

highlights the importance of paying close attention to the modeling of the extensive-margin channel in any study emphasizing the role of heterogenous firms in the distributional consequences of higher trade openness. I present sufficient conditions on the CDF of export costs under which wage inequality rises pervasively after a trade liberalization. Finally, the average real wage always increases following a trade liberalization, but the real wage of the least skilled workers in the economy could decline.

Moving to the predictions of the free-entry model, opening to trade has an ambiguous effect on the wage distribution, as the impact on the activity cutoff cannot be determined without imposing additional restrictions on primitives. However, despite the generality of the assumptions, trade can have only two broad effects on the wage distribution. If the activity cutoff increases, then inequality is pervasively higher in the open economy for the same reasons discussed in the case of the no-free-entry model. If the cutoff decreases, then trade leads to wage polarization, i.e. wage inequality decreases among the least skilled workers, but increases among the most skilled ones. In this case, the selection-into-activity and extensive-margin channels lead to a pervasive decline and a pervasive rise in wage inequality, respectively, with the former channel dominating at lower end of the wage distribution and the latter at the upper end. As in the no-free-entry model, the average real wage is always higher after the economy opens to trade. Perhaps surprisingly, trade raises the real wage of all workers in the economy if and only if it leads to pervasively higher inequality. When trade leads to wage polarization, the real wage of the least skilled workers in the economy necessarily declines.

The case of a trade liberalization in the free-entry model combines all the sources of ambiguity discussed above. If the activity cutoff increases, wage inequality necessarily increases at the lower end of the distribution but may decline elsewhere, a possibility that is eliminated by the same sufficient conditions presented for the no-free-entry model. If the activity cutoff decreases, then wage inequality necessarily decreases at the lower end of the distribution and increases somewhere else in the distribution, but additional outcomes beyond a wage polarization are possible. That said, the analysis shows that higher trade openness never leads to a pervasive decline in wage inequality. The average real wage in the economy always increases with a trade liberalization, while the real wages of the least skilled workers increase if and only if the activity cutoff increases.

Finally, in a methodological contribution, I establish the existence and uniqueness of the equilibrium in this setting, a prerequisite for a theoretical analysis of comparative statics. Conditional on the activity cutoff, the market equilibrium is characterized by a system of nonlinear differential equations involving the matching, price and revenue functions, which together with a set of boundary conditions, defines a nonlinear two-point boundary value problem (BVP). In contrast to the cases of initial value problems (IVP) and linear BVPs, where the standard mathematical theory can handle a wide array of problems, establishing existence and uniqueness of solutions is not trivial in the case of non-linear BVPs. Because of the complexity of the subject, the mathematical literature has typically focused on particular cases of the problem, leading to a multitude of theoretical approaches tailored to each of these cases.[6] In addition, most results in the literature are based on restrictive and not-easily-verifiable assumptions, while those

[6] Bernfeld and Lakshmikantham (1974) survey some of the most common theoretical approaches used in the literature. See Kiguradze (1988) for some results for the general, first-order, two-point BVP.

based on less restrictive assumptions (resembling those used in the standard theory of IVPs) have a local flavor.[7] Despite these difficulties, several studies in the trade literature that use assignment models leading to similar BVPs simply assume or state without proof the existence and uniqueness of the solution. In this paper, I fill this gap in the trade literature by presenting existence and uniqueness results for a nonlinear two-point boundary BVP that encompasses those in this paper and others in the literature.[8] In addition, I derive a set of results which characterize the dependence of the solution to this BVP on its parameters.

This paper is related to a now large literature proposing heterogeneous-firms models in which international trade can affect wage inequality through within-industry mechanisms. Motivated by developments in within-group wage inequality, one line of research introduces labor market frictions so that ex-ante identical workers can earn different wages at different firms. Davis and Harrigan (2011) develop a model of efficiency wages in which the wage that induces worker effort varies across firms due to differences in monitoring technology. Egger and Kreickemeier (2009, 2012) and Amiti and Davis (2012) develop fair-wage models in which the perceived fair wage at which workers supply effort depends on firms' revenue. Finally, bargaining over production surplus resulting from search and matching frictions in labor markets induce wages to vary across firms in Helpman, Itskhoki, and Redding (2010) and Helpman et al. (2016). Another strand of the literature uses models with competitive labor markets and ex-ante heterogeneous workers (from the perspective of firms) to study the effect of trade on wage inequality through its impact on firms' technological choices or on the endogenous assignment of workers to firms; see, for example, Yeaple (2005), Bustos (2011), Monte (2011), Sampson (2014) and Somale (2015).

This paper is also related to a growing number of studies using assignment models to study the distributional consequences of international trade and offshoring, such as Grossman and Maggi (2000), Ohnsorge and Trefler (2007), Costinot (2009), Antràs, Garicano, and Rossi-Hansberg (2006) and Nocke and Yeaple (2008). Methodologically, this paper is closer to a branch of this literature that, building on Costinot (2009), develops two-sided heterogeneity models by embedding in different general equilibrium frameworks a production technology similar to one consider in this paper, giving rise to similar assignment problems. In models with neoclassical roots, Costinot and Vogel (2010) study the assignment of workers to tasks while Grossman, Helpman, and Kircher (2017) study the matching of managers and workers and their sorting into different industries. In monopollistically competitive settings, Sampson (2014) and Somale (2015) extend standard frameworks in the trade literature to study the matching of workers to firms. Sampson (2014) presents a general equilibrium analysis of a model of endogenous technology choice that extends Yeaple (2005) to a continuum of skill-types, and presents some partial equilibrium results for a model that extends Melitz (2003) to allow for worker heterogeneity. Somale (2015) presents a general equilibrium analysis of the effects of opening to trade on wage inequality by extending Chaney (2008). Relative to the last two studies, this paper present a general equilibrium analysis of the effects of opening to trade and of a trade liberalization in models extending both Chaney (2008) and Melitz (2003). In addition, this paper allows fixed export costs to vary across firms, a generalization that makes the model

[7] The existence of a solution is guaranteed only over sufficiently small intervals.

[8] The general BVP considered in this paper encompasses those in Costinot and Vogel (2010), Sampson (2014), Somale (2015), Grossman, Helpman, and Kircher (2017).

better suited to study the effects of trade openness on wage inequality as discussed above. Moreover, this generalization actually simplifies some parts of the analysis, as the matching function does not exhibit kinks as in Sampson (2014) and Somale (2015).

The rest of the paper is organized as follows. Section 2 describes the basic setup of the framework. Sections 3 and 4 characterize the equilibrium in the no-free-entry model and present existence and uniqueness results. Section 5 studies the effects of higher trade openness on wage inequality in the no-free-entry model. Finally, section 6 extends the analysis to the free-entry model.

2 The Model

2.1 Demand

The preferences of the representative consumer are given by a C.E.S utility function over a continuum of goods indexed by ω :

$$U = \left[\int_{\omega \in \Omega} u(\omega)^{\frac{\sigma-1}{\sigma}} d\omega \right]^{\frac{\sigma}{\sigma-1}},$$

where $u(\omega)$ is the quantity consumed of good ω, the measure of the set Ω represents the mass of available goods and $\sigma > 1$ is the elasticity of substitution between goods. The demand and expenditure for individual varieties generated by this utility function are

$$u(\omega) = EP^{\sigma-1} p(\omega)^{-\sigma}, \qquad E(\omega) = EP^{\sigma-1} p(\omega)^{1-\sigma}, \tag{1}$$

where P is the aggregate price level and E is aggregate expenditure,

$$P = \left[\int_{\omega \in \Omega} p(\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}}, \qquad E = \int_{\omega \in \Omega} E(\omega) d\omega. \tag{2}$$

2.2 Production

There is a continuum of *active*, monopolistically competitive firms in the market, each producing a different variety ω.[9] As in Melitz (2003), firms differ in their productivity level ϕ, which they obtain as an independent draw from a distribution $G(\phi)$ with density function $g(\phi)$. I assume that the support of G, $\Phi \equiv \{\phi : g(\phi) > 0\}$, is equal to some bounded interval of non-negative real numbers, $\left[\underline{\phi}, \overline{\phi}\right] \subseteq \mathbb{R}_+$. In contrast to Melitz (2003), the labor force is heterogenous, consisting of a continuum of workers of mass L that differ in their skill level s. The distribution of worker's skills is represented by a nonnegative density $V(s)$, so $LV(s) \geq 0$ represents the inelastic supply of workers with skill s. I only consider skill distributions such that the support of V, denoted by S, is equal to some bounded interval of non-negative real numbers, i.e. $S \equiv \{s : V(s) > 0\} = [\underline{s}, \overline{s}] \subseteq \mathbb{R}_+$. In addition, I assume that the density $V(s)$ is continuously differentiable on S.

[9] A firm is active in the market if it produces positive output.

The production technology of firms is represented by a cost function that exhibits constant marginal cost and fixed overhead costs. After paying the fixed costs described below, a firm must decide the mix of workers to use in production. The total output of a firm with productivity ϕ, $q(\phi)$, is given by

$$q(\phi) = \int_{s \in S} A(s, \phi)\, l(s, \phi)\, ds, \tag{3}$$

where $A(s, \phi)$ is the marginal productivity of a worker of skill s, and $l(s, \phi)$ is the total number of *production* workers of that skill level employed by the firm.[10] More skilled workers are more productive than less skilled workers, regardless of the productivity of the firm that employs them. Also, more productive firms have lower labor input requirements than less productive firms no matter the type of worker considered. In terms of the production function (3), I formally assume that the productivity function $A(s, \phi)$ is strictly positive, strictly increasing and continuously differentiable, i.e. $A(s, \phi) > 0$, $A_s(s, \phi) > 0$ and$A_\phi(s, \phi) > 0$.[11]

In addition to the absolute productivity advantage described above, more skilled workers have a comparative advantage in production at more productive firms. Specifically, I follow Costinot and Vogel (2010) and assume that the function $A(.,.)$ is strictly log-supermodular, i.e.

$$A(s', \phi')\, A(s, \phi) > A(s', \phi)\, A(s, \phi') \text{ for all } s' > s \text{ and } \phi' > \phi. \tag{4}$$

Since $A(s, \phi) > 0$, condition (4) can be rearranged as $A(s', \phi')/A(s', \phi) > A(s, \phi')/A(s, \phi)$, showing that the productivity gains from switching to a more productive firm are higher for more skilled workers. Alternatively, the gains from hiring a more skilled worker are higher for more productive firms.

Following a standard practice in the international trade literature, I assume that fixed costs are paid in terms of labor. Specifically, I assume that firms pay a fixed cost of $fV(s)$ units of each skill $s \in S$, implying that the total fixed cost of a firm is

$$f \int_{\underline{s}}^{\overline{s}} w(s)\, V(s)\, ds = f\overline{w},$$

where $w(s)$ is the wage of a worker with skill level s, and $\overline{w}$ is the average wage in the economy and the numeraire, $\overline{w} = 1$. This specification of fixed costs guarantees that the distribution of skills in the economy is still given by $V(s)$ after all fixed costs have been paid, implying that the demand of labor induced by fixed-costs requirements has no effect on the wage schedule $\{w(s)\}$. As I discuss later, the wage schedule is completely determined by the interactions between the exogenous relative supply of skills, captured by the distribution $V(s)$, and the endogenous relative demand of skills derived from the firm's demand of *production* workers.

[10] In addition to production workers, a firm employs non-production workers to satisfy the fixed costs requirements.

[11] For any function $F(x_1, ..., x_n)$, F_{x_i} denotes the partial derivative of F with respect to variable x_i.

2.3 Variable Costs and Prices

Per the linear production technology (3), workers are perfect substitutes in production. Accordingly, firms employ only those worker-types that entail the lowest cost per unit of output, implying that the marginal cost of a firm with productivity ϕ, $c(\phi)$, is given by

$$c(\phi) = \min_{s \in S} \left\{ \frac{w(s)}{A(s,\phi)} \right\}. \tag{5}$$

For any wage schedule, the marginal cost $c(\phi)$ is strictly decreasing in the productivity level ϕ, as a firm can always hire the same type of workers employed by a less productive competitor and obtain a strictly lower marginal cost due its absolute productivity advantage, $A_\phi(s,\phi) > 0$, i.e.

$$\phi' > \phi \Leftrightarrow c(\phi') < c(\phi). \tag{6}$$

Faced with the iso-elastic demands in (1), firms optimally set their price equal to a constant markup over their marginal costs, $p(\phi) = \frac{\sigma}{\sigma-1} c(\phi)$. This pricing rule and the cost minimization condition (5) imply

$$p(\phi) \leq \frac{\sigma}{\sigma-1} \frac{w(s)}{A(s,\phi)} \text{ for all } s \in S; \qquad p(\phi) = \frac{\sigma}{\sigma-1} \frac{w(s)}{A(s,\phi)} \text{ if } l(s,\phi) > 0. \tag{7}$$

2.4 Entry

I carry out the analysis under two widely-used assumptions regarding entry; no free entry a-lá Chaney (2008) and free entry a-lá Melitz (2003). In the first case, there is a fixed mass of firms in the industry. In the second case, there is unbounded pool of prospective firms that must pay a fixed entry-cost to develop a new product variety and enter the industry. The results obtained under the no-free-entry assumption can be interpreted as the short-term consequences of trade, before investment in the development of new varieties leads new firms to enter the industry. In contrast, the results obtained under the free-entry assumption can be viewed as the long-term effects of trade.

3 No-Free-Entry Equilibrium in the Closed Economy

As in Chaney (2008), there is a fixed mass $\overline{M}$ of firms in the industry. A firm is *active* in the market if and only if it finds it profitable to produce. The pricing rule (7), the consumer's demand and expenditure functions in (1), and the goods-market clearing condition ($u(\omega) = q(\omega)$), imply that a firm's output, revenue and profit from serving the *domestic* market are given by[12]

$$q^d(\phi) = EP^{\sigma-1} \left[\frac{\sigma}{\sigma-1} c(\phi) \right]^{-\sigma}; \; r^d(\phi) = EP^{\sigma-1} \left[\frac{\sigma}{\sigma-1} c(\phi) \right]^{1-\sigma}; \; \pi^d(\phi) = \frac{r^d(\phi)}{\sigma} - f, \tag{8}$$

[12] I make explicit reference to the domestic market here and use the superscript d to denote relevant domestic variables because it facilitates the comparison with the open-economy case analyzed in the next section.

where aggregate expenditure, E, equals aggregate income.[13] The last expression, together with a decreasing marginal cost function $c(\phi)$, implies that a firm's profit is an increasing function of the firm's productivity.

There are combinations of parameters such that all firms are active in equilibrium, $\pi^d(\underline{\phi}) \geq 0$. However, since this case is not theoretically interesting nor empirically relevant, I focus on the conditions that characterize an equilibrium featuring selection into activity, i.e. the least productive firms find it unprofitable to produce and remain inactive, $\pi^d(\underline{\phi}) < 0$.[14] In such an equilibrium, there is a cutoff productivity value $\phi^* \in (\underline{\phi}, \overline{\phi})$ such that only firms with productivity above this value are active in the market. The value of this activity cutoff corresponds to the level of productivity at which firms make zero profits,[15]

$$\pi^d(\phi^*) = 0. \tag{9}$$

In turn, the activity cutoff ϕ^* determines the total mass of active firms in the industry,

$$M = [1 - G(\phi^*)]\,\overline{M}. \tag{10}$$

Finally, the labor market of each type of worker must clear,

$$LV(s) = \int_{\phi^*}^{\overline{\phi}} l^d(s,\phi)\,\frac{g(\phi)}{[1 - G(\phi^*)]}\,d\phi M + MfV(s) \;\text{ for all } s \in S. \tag{11}$$

The left- and right-hand sides of the last expression capture, respectively, the total supply and demand of workers of skill s, with the total demand comprising the demand of *production* workers (first term), and the demand of *non-production* workers derived form the presence of fixed costs of production (second term). Having described all the components of the economy, I state the formal definition of the equilibrium.

Definition 1 *A no-free-entry equilibrium of the closed economy is a mass of active firms* $M > 0$, *a productivity activity-cutoff ,* $\phi^* \in (\underline{\phi}, \overline{\phi})$, *an output function* $q^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, *a labor allocation function* $l^d : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, *a price function* $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ *and a wage schedule* $w : S \to \mathbb{R}_+$ *such that the following conditions hold,*[16]
(i) consumers behave optimally, equations (1) and (2);
(ii) firms behave optimally given their technology, equations (3), (7), (9) and (10);
(iii) goods and labor markets clear, equations (8) and (11), respectively;
(iv) the numeraire assumption holds, $\overline{w} = 1$.

[13] Aggregate income is given by the sum of total labor income and firms' profits.

[14] Intuitively, if the size of the market, captured by the mass of workers L, is sufficiently large relative to the mass of potential firms, then all firms find it profitable to produce. The conditions on primitives that rule out this possibility, so that the equilibrium features selection into activity, are formally stated in proposition 1.

[15] In an equilibrium in which all firms are active in the market, we have $\phi^* = \underline{\phi}$. In addition, in such an equilibrium it is posible that even the least productive firms make strictly positive profits, so condition (9) does not need to hold.

[16] Technically, this definition corresponds to an equilibrium featuring selection into activity. However, since I only consider equilibria of this type, there is no risk of confusion.

3.1 Characterization of the Equilibrium

The log-supermodularity of the productivity function, A, implies that the equilibrium labor allocation is characterized by positive assortative matching, i.e. more productive firms employ *production* workers of higher ability. Specifically, there exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$ such that, all firms of productivity $N(s)$ employ production workers of skill s, and all production workers of skill s are employed at firms with the productivity $N(s)$.[17] Behind this result, formally stated in lemma 1, lies a simple intuition. The cost-minimization condition (5) implies that a firm of productivity ϕ' employing a worker of skill s' cannot reduce its marginal cost of production by employing a worker of a different skill, i.e. $w(s')/A(s', \phi') \leq w(s)/A(s, \phi')$ for all $s \in S$. This observation and the strict log-supermodularity of A imply that, for any skill level $s > s'$ and any productivity level $\phi < \phi'$, the following inequalities hold, $\frac{A(s,\phi)}{A(s',\phi)} < \frac{A(s,\phi')}{A(s',\phi')} \leq \frac{w(s)}{w(s')}$. Accordingly, a firm with productivity $\phi < \phi'$ does not employ workers of skill $s > s'$, as it can obtain a strictly lower marginal cost by hiring a worker of skill s'. Although this argument only proves that the matching function is weakly increasing, it highlights the connection between the log-supermodularity of A and positive assortative matching in equilibrium.

Armed with the previous result, the equilibrium can be characterized in terms of the matching function N, revealing a tight connection between the latter and wage inequality in the current framework. A worker of skill s is matched to a firm with productivity $N(s)$ in equilibrium if and only if the skill level s solves the cost minimization problem (5) for any firm with productivity $\phi = N(s)$. The first order condition for an interior solution of this problem yields the following equilibrium condition,[18]

$$\frac{d \ln w(s)}{ds} = \frac{\partial \ln A(s, N(s))}{\partial s}. \tag{12}$$

The last expression is central in the analysis of wage inequality. It implies that the matching function N is a sufficient statistic for the dispersion of wages in the economy, as it is the only endogenous variable affecting the slope of the wage schedule. The connection between N and wage inequality can be seen more clearly by integrating (12) between s' and $s'' > s'$ to get $w(s'')/w(s') = \exp\{\int_{s'}^{s''} \frac{\partial \ln A(t,N(t))}{\partial s} dt\}$. The last expression, together with the strict log-supermodularity of A, implies that the ratio $w(s'')/w(s')$ is increasing in the values that the matching function takes on the interval $[s', s'']$. Then, any change in the environment leading to an upward shift of the matching function on a given interval also leads to higher relative wages for more skilled workers in that interval. Moreover, the new distribution of wages in the interval is second-order stochastically dominated by the old one, i.e. inequality is pervasively higher after the change.[19]

Letting $H : [\phi^*, \overline{\phi}] \to S$ denote the inverse function of the matching function N, the optimal pricing rule (7) and the expression for revenues in (8) can be used to express firm's prices and revenues as

[17] In somewhat different settings, Costinot and Vogel (2010), Sampson (2014) and Grossman, Helpman, and Kircher (2017) also obtain positive assortative matching in equilibrium as a result of assuming stric log-supermodularity of relevant functions.

[18] The assumptions we made on the primitives of the model imply that all the endgogenous functions considered in this section are differentiable, a result that is formally stated in lemma 1 and proved in the appendix.

[19] In appendix A.1.2, I show that the new distribution is Lorenz dominated by the previous one. The equivalence between Lorenz dominance and normalized second-order stochastic dominance was first shown in Atkinson (1970).

functions of the productivity level ϕ and the value of the function H at that productivity level. Totally differentiating these functions with respect to ϕ and using equation (12) in the resulting expressions yields

$$p_{\phi}(\phi) = -p(\phi)\frac{\partial \ln A(H(\phi),\phi)}{\partial \phi}, \tag{13}$$

$$r^{d}_{\phi}(\phi) = (\sigma-1)\, r^{d}(\phi)\frac{\partial \ln A(H(\phi),\phi)}{\partial \phi}. \tag{14}$$

The last two equations imply that the equilibrium matching of workers and firms is also a sufficient statistic for the dispersion of firms' prices and revenues. In particular, integrating equation (14) between ϕ' and $\phi'' > \phi'$ yields $r^{d}(\phi'')/r^{d}(\phi') = \exp\{(\sigma-1)\int_{\phi'}^{\phi''}\frac{\partial \ln A(H(t),t)}{\partial \phi}dt\}$, so the ratio of the revenues of any two firms depends only on the productivity levels of these firms and the values of the function H between these two levels. Note that the log-supermodularity of A implies that the ratio of revenues $r^{d}(\phi'')/r^{d}(\phi')$ is increasing in the values that the *inverse* of the matching function takes on $[\phi', \phi'']$, so a shift in the matching function will have opposite effects on the dispersion of wages and revenues.

The equilibrium labor allocation implied by the matching function must be consistent with market clearing in the labor and goods markets, i.e. N (or H) must be consistent with conditions (1), (3), (8) and (11). This consistency requirement yields the following equilibrium condition,

$$H_{\phi}(\phi) = \frac{r^{d}(\phi)\, g(\phi)\, \overline{M}}{A(H(\phi),\phi)\left[L - f\left[1-G(\phi^{*})\right]\overline{M}\right] V(H(\phi))\, p(\phi)}, \tag{15}$$

which, after some re-arrangement, states that consumers' expenditure accruing to firms with productivity ϕ, $r^{d}(\phi)\, g(\phi)\, \overline{M}$, must equal the total value of the output that those firms can produce with the workers they employ.[20] Note that the last expression implies that the output of a firm depends positively on the slope of the function H. Intuitively, firms on the interval $[\phi - d\phi, \phi + d\phi]$ employ workers on the interval $[H(\phi) - H_{\phi}(\phi)d\phi, H(\phi) + H_{\phi}(\phi)d\phi]$ in equilibrium. Then, for a given value of $H(\phi)$, a higher value of $H_{\phi}(\phi)$ implies that the same firms employ more workers, so their output is higher.

Given the equilibrium activity cutoff, ϕ^{*}, equations (13)-(15) form a system of nonlinear differential equations that the price function, p, the revenue function, r^{d}, and the inverse of the matching function, H, must satisfy in equilibrium. As is well-known, there is an uncountable family of functions that satisfy a system like (13)-(15), so a set of *boundary conditions* is needed to pin down a particular solution. Two of these boundary conditions are provided by the labor market clearing condition, as all workers must be assigned to some firm in equilibrium, $H(\phi^{*}) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$. A third boundary condition is provided by the zero-profit condition for firms with productivity ϕ^{*}, $r^{d}(\phi^{*}) = \sigma f$. Finally, the activity cutoff ϕ^{*} can be determined from the the following equilibrium condition,

$$\tfrac{\sigma-1}{\sigma}\int_{\phi^{*}}^{\overline{\phi}} r^{d}(\phi) g(\phi)\, d\phi \overline{M} + f\left[1-G(\phi^{*})\right]\overline{M} = L, \tag{16}$$

which states that the total wages paid by firms to production and non-production workers (left) equals

[20] The total value of the output of firms with productivity ϕ is $A(H(\phi),\phi)\left[L - f\left[1-G(\phi^{*})\right]\overline{M}\right] V(H(\phi))\, p(\phi)\, H_{\phi}(\phi)$.

total labor income in the economy, where the expression for the latter uses the numeraire assumption.

The conditions derived above are not only necessary but also sufficient for an equilibrium. In particular, in the appendix I show that if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$ and a triplet of functions $\{p, r^d, H\}$ satisfy those conditions, then they can be used to construct a wage schedule $w(s)$, an output function $q^d(\phi)$, and a labor allocation function $l^d(s, \phi)$ such that all the conditions in the definition of equilibrium are satisfied. I summarize the results in this section in the following lemma that I formally prove in the appendix.

Lemma 1 *In a no-free-entry equilibrium of the closed economy there exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$ (with inverse function H) such that (a) $l^d(s, \phi) > 0$ if and only if $N(s) = \phi$, (b) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$. In addition, the following conditions hold*
(i) The wage schedule w is continuously differentiable and satisfies (12).
(ii) The price, revenue and matching functions, $\{p, r^d, N(\text{and } H)\}$, are continuously differentiable. Given ϕ^*, the triplet $\{p, r^d, H\}$ solves the boundary value problem (BVP) comprising the system of differential equations (13)-(15) and the boundary conditions $r^d(\phi^*) = \sigma f$, $H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$.
(iii) The activity cutoff ϕ^* and the revenue function r^d satisfy (16).
Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$, and functions $p, r^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $H : [\phi^*, \overline{\phi}] \to S$ satisfy conditions (ii)-(iii), then they are, respectively, the productivity activity-cutoff, the price function, the revenue function, and the inverse of the matching function of a no-free-entry equilibrium of the closed economy.

As discussed earlier, one of the contributions of this paper is to formally show the existence and uniqueness of the equilibrium characterized above. However, I defer the formal discussion of this issue to section 4.2, where a more general setting is considered.

4 No-Free-Entry Equilibrium in the Open Economy

Balanced trade takes place between $n+1$ symmetric (identical) economies of the type described above, so the description presented in section 2, including equations (1)-(7), holds for each of these economies. Given that the symmetry assumption ensures that all countries share the same equilibrium variables, I restrict the analysis to the home country. Firms face fixed and variable trade costs. Per-unit trade costs are common to all firms and are modeled in the standard iceberg formulation, whereby $\tau > 1$ units of a good must be shipped in order for 1 unit to arrive in a foreign destination. In contrast, fixed export costs vary across firms. A firm that wishes to export to country i must incur an idiosyncratic fixed cost of y units of a "bundle of skills" comprising $f^x V(s)$ workers of each skill $s \in S$. Setting the average wage as the numeraire, $\overline{w} = 1$, the unit-cost of this bundle of skills is f^x, so the total fixed export cost of the firm is $f^x y$ per foreign market.[21] I model the firm-specific *size* of fixed export costs, y, as the realization of a nonnegative random variable Y with distribution F, which I assume is independent of the productivity distribution, absolutely continuous, and satisfies $F(y) = 0$ for $y \leq \underline{y}$, $dF(y) > 0$ for $y \geq \underline{y}$, where $\underline{y}$ is the

[21] The unit-cost of the bundle of skills is $f^x \int_{\underline{s}}^{\overline{s}} w(s) V(s) \, ds = f^x \overline{w} = f^x$.

lower bound of the support of Y. In addition, I assume that $f_x \underline{y} \tau^{\sigma-1} > f$, which guarantees that a firm's profit in the domestic market is always higher than in any individual foreign market.[22]

These assumptions about fixed export costs have three important implications. First, formulating export costs in terms of the bundle of skills described above guarantees that the demand of labor induced by fixed-export-costs requirements does not affect the wage schedule.[23] Second, in the presence of heterogeneous fixed export costs, a highly productive firm may not find it profitable to export if it faces high fixed export costs, while a less productive competitor may choose to serve the foreign market if its fixed export costs are sufficiently low. As a result, the productivity distributions of exporters and non-exporters overlap in equilibrium, consistent with the evidence in Bernard, Eaton, Jensen, and Kortum (2003). Third, an implication of the restriction $f_x \underline{y} \tau^{\sigma-1} > f$ is that, as in Melitz (2003), the activity status of a firm in the open economy continues to be determined by its domestic profit. Although not essential for the qualitative results in the paper, this implication simplifies the exposition as the condition determining the activity cutoff is unchanged relative to the closed economy.[24]

The determination of the set of active firms and their operations in the domestic market are little changed relative to the closed economy. There is a fixed mass $\overline{M}$ of *potential* firms in the industry. A firm is *active* if and only if it makes non-negative profits in the domestic market. The pricing rule (7) and the expenditure functions in (1) imply that the *potential* domestic output, q^d, revenue, r^d, and profit, π^d, of a firm with productivity ϕ are still given by (8). As before, domestic profits are strictly increasing in ϕ, so the equilibrium is characterized by a cutoff productivity level, $\phi^* \in (\underline{\phi}, \overline{\phi})$, such that a firm is active in the market if and only if its productivity is above this level.[25] Firms with productivity ϕ^* make zero domestic profit, condition (9), while the mass of active firms, M, is given by (10).

The equilibrium in the open economy features selection into trade, i.e. only a subset of active firms export. An active firm serves a foreign market if and only if it can make non-negative profits there. In the presence of variable trade costs, consumers in each country face higher prices for imported goods, $p^x(\phi) = \tau p(\phi)$, so conditions (7) and (1) and the symmetry assumption imply that the *potential* export output, revenue and profit of a firm with productivity ϕ and fixed export costs $f^x y$ are given by

$$q^x(\phi) = \tau^{1-\sigma} q^d(\phi), \; r^x(\phi) = \tau^{1-\sigma} r^d(\phi), \; \pi^x(\phi) = \frac{\tau^{1-\sigma} r^d(\phi)}{\sigma} - f^x y. \tag{17}$$

Then, such a firm exports if and only if $y \leq \tau^{1-\sigma} r_d(\phi) / \sigma f^x$, which, together with the assumptions about y, implies that only a fraction $F\left(\tau^{1-\sigma} r_d(\phi) / \sigma f_x\right)$ of firms with productivity $\phi \geq \phi^*$ export. Note that this fraction is a continuous and increasing function of the productivity level ϕ, so exporters are, on average, more productive than non-exporters.[26] These observations imply that the mass of exporters

[22] A similar relationship between domestic and foreing profits is featured in Melitz (2003).

[23] See the discussion following the specification of fixed production costs.

[24] Alternatively, I could have just assumed that a firm is active if and only if it makes positive profits in the domestic market, regardless of its potential export profits.

[25] As before, we focus on the conditions that characterize an equilibrium featuring selection into activity, i.e. $\pi^d(\underline{\phi}) < 0$.

[26] The productivity distribution of exporters first-order stochastically dominates the distribution of non-exporters.

with productivity ϕ is

$$M^x(\phi) = g(\phi) F\left(\tau^{1-\sigma} r^d(\phi) / \sigma f^x\right) \overline{M}. \tag{18}$$

Finally, the labor market of each type of worker must clear,

$$LV(s) = \int_{\phi^*}^{\overline{\phi}} [l^d(s,\phi) g(\phi) \overline{M} + l^x(s,\phi) M^x(\phi)] d\phi + fMV(s) + \int_{\phi^*}^{\overline{\phi}} nf^x \int_0^{\frac{\tau^{1-\sigma} r^d(\phi)}{\sigma f_x}} ydF(y) g(\phi) \overline{M} d\phi V(s). \tag{19}$$

The left- and right-hand sides of the last expression capture, respectively, the total supply and demand for workers of skill s. Total demand comprises the demand of *production* workers to supply the domestic and foreign markets, first term, and the demand of *non-production* workers derived form the presence of fixed costs of production and fixed export costs, the second and third terms. Conditions (1)-(3), (7)-(10), (17)-(19) and the numeraire assumption completely describe the equilibrium, prompting the formal definition of equilibrium in the appendix, analogous to that for the closed economy.

4.1 Characterization of the Equilibrium

The equilibrium of the open economy shares several features with its closed-economy counterpart. Cost minimization by firms and the strict log-supermodularity of A imply that the equilibrium labor allocation in the open economy is characterized by a strictly increasing matching function, N, that maps the set of skills, S, to the set of productivity levels of active firms, $[\phi^*, \overline{\phi}]$. In addition, equation (12), connecting the wage schedule to the matching function, and equations (13) and (14), connecting the price and domestic-revenue functions to the *inverse* of the matching function, H, continue to hold, as the arguments used in their derivation do not depend on the trade regime of the economy. As before, these equilibrium conditions imply that the labor allocation of workers to firms—as captured by the matching function, N, and its inverse, H—is a sufficient statistic for the dispersion of wages, prices and domestic revenues.

The equilibrium labor allocation must be consistent with labor and goods markets clearing, i.e. N (or H) must be consistent with conditions (3), (8), (17) and (19). This observation and the expression for the mass of exporters, equation (18), yield the following equilibrium condition,

$$H_\phi(\phi) = \frac{r^d(\phi)\left[1 + F\left(\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}\right) n\tau^{1-\sigma}\right] g(\phi)\overline{M}}{A(H(\phi),\phi) V(H(\phi)) p(\phi) \left[L - fM - \int_{\phi^*}^{\overline{\phi}} nf^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF(y) g(\phi')\overline{M} d\phi'\right]}. \tag{20}$$

After some re-arrangement, the last expression states that the total revenue that firms with productivity ϕ make from their sales in the domestic and foreign markets, the numerator on the right-hand side of (20), must equal the total value of the output that those firms can produce with the workers they employ.

Given the equilibrium activity cutoff, ϕ^*, equations (13), (14) and (20) form a system of nonlinear differential equations that the price function, p, the domestic revenue function, r^d, and the inverse of the matching function, H, must satisfy in equilibrium. Two boundary conditions for this system are provided by the labor market clearing condition, as all workers must be assigned to some firm in equilibrium,

$H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$. A third boundary condition is provided by the zero-domestic-profit condition for firms with productivity ϕ^*, $r^d(\phi^*) = \sigma f$. Finally, the open-economy counterpart of equation (16) can be used to determine the activity cutoff ϕ^*,

$$\begin{array}{c} \frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} r^d(\phi)[1+F\left(\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}\right)n\tau^{1-\sigma}]g(\phi)\,d\phi\overline{M}+ \\ fM+\int_{\phi^*}^{\overline{\phi}} nf^x\int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF(y)\,g(\phi')\,\overline{M}d\phi' \end{array} = L, \tag{21}$$

which states that the total value of wages paid by firms to production and non-production workers (left) equals total labor income in the economy, where the expression for the latter uses the numeraire assumption. As in the closed economy case, the conditions derived in this section are not only necessary, but also sufficient for an equilibrium. This characterization of the equilibrium is summarized in lemma 3 in the appendix, which can be easily proved adapting the arguments in the proof of lemma 1.

I conclude this section with a summary of the qualitative properties of the equilibrium in the open economy. In equilibrium more productive firms employ *production* workers of higher ability and pay them higher wages. The stochastic specification of fixed export costs, though, imply an imperfect positive correlation between firms' productivity, average workforce ability, size and export status, which is consistent with the empirical evidence documented in Bernard and Jensen (1995) and Bernard, Eaton, Jensen, and Kortum (2003). In addition, as each firm employs workers of all types (fixed costs as skill-bundles), the dispersion of wages in the model has a between- and within-firm component.

4.2 Existence and Uniqueness of the Equilibrium

I start this section by studying the existence and uniqueness of solutions to the nonlinear, two-point BVPs characterizing the equilibrium in the closed and open economies. In contrast to the cases of initial value problems (IVPs) and linear BVPs, for which there is a standard theory that provides fairly general results under relatively mild restrictions on the *data* of the problem, such a study is not trivial in the case of nonlinear BVPs for two reasons.[27] First, there is no unified theory that can be applied to study these issues for an arbitrary problem. Because of the complexity of the subject, the mathematical literature has typically focus on particular cases of the problem, leading to a multitude of theoretical approaches tailored to these cases.[28] Second, most results in the literature are based on restrictive and not-easily-verifiable assumptions, while those results based on less restrictive assumptions, resembling those used in the standard theory of IVPs, have a local flavor.[29] Despite these difficulties, several studies in the trade

[27] Most textbools on ordinary differential equations cover the standard existence and uniqueness theory for IVPs. Some examples include Coddington and Levinson (1987) and Agarwal and O'Regan (2008a), with the latter also covering basic resutls for linear BVPs. For a more comprehensive treatment of linear BVPs see Stakgold (1998) and Agarwal and O'Regan (2008b).

[28] Bernfeld and Lakshmikantham (1974) present a survey of some of the most common theoretical approaches used in the literature, together with the particular problems to which they have been applied. See Kiguradze (1988) for some results for the general, first-order, two-point BVP.

[29] Following a widely-used approach in the theory of IVPs, Bailey, Shampine, and Waltman (1968) present several existence and uniqueness results for nonlinear BVPs using Piccard's Iteration method (Banach fixed point theorem) when the functions involved satisfy certain Lipschitzian conditions. In all cases, the resuls are local in nature, i.e. the interval over which the

literature that use assignment models and arrive to characterizations of the equilibrium involving a BVP similar to those above, simply assume or state without proof the existence and uniqueness of the solution. In this section I fill this gap in the trade literature by presenting existence and uniqueness results for a nonlinear BVP that encompasses the two BVPs considered above and others in the literature.[30]

For any $\phi_0, \phi_1 \in [\underline{\phi}, \overline{\phi}]$ and $s_0, s_1 \in [\underline{s}, \overline{s}]$, with $\phi_0 < \phi_1$ and $s_0 < s_1$, I consider the nonlinear, two-point BVP (22), comprising the system of differential equations (22a)-(22c) and the boundary conditions (22d),

$$z_\phi(\phi) = -z(\phi)\frac{\partial \ln A(\Gamma(\phi), \phi)}{\partial \phi}, \tag{22a}$$

$$x_\phi(\phi) = (\sigma - 1)\, x(\phi)\frac{\partial \ln A(\Gamma(\phi), \phi)}{\partial \phi}, \tag{22b}$$

$$\Gamma_\phi(\phi) = \frac{x(\phi)\left[1 + F(K_0 x(\phi)) K_1\right]\alpha(\phi) g(\phi)}{A(\Gamma(\phi), \phi)\, V(\Gamma(\phi))\, z(\phi)}, \tag{22c}$$

$$x(\phi) = 1,\ \Gamma(\phi_0) = s_0,\ \Gamma(\phi_1) = s_1, \tag{22d}$$

where $\alpha(\phi)$ is a strictly positive continuous function, $\alpha : [\underline{\phi}, \overline{\phi}] \to \mathbb{R}_{++}$, K_0 and K_1 are nonnegative constants and $\{A, g, V, F\}$ are the functions defined earlier.

The general BVP defined above nests the BVPs corresponding to the closed and open economies, as the latter can be obtained as particular parametrizations of the former. If we set $K_0 = (f/f_x)\tau^{1-\sigma}$, $K_1 = n\tau^{1-\sigma}$, $\phi_0 = \phi^*$, $\phi_1 = \overline{\phi}$ and $\alpha(\phi) = 1$ for all $\phi \in [\underline{\phi}, \overline{\phi}]$, the resulting BVP is equivalent to the BVP of the open economy, in the sense that any solution to one of these two BVPs can be used to construct a solution to the other. To see this, let $\{z, x, \Gamma\}$ be a solution to the BVP (22) parametrized as above. If we define $r^d(\phi) \equiv \sigma f x(\phi)$, $p(\phi) \equiv z(\phi)\sigma f \overline{M}/[L - fM - \int_{\phi^*}^{\overline{\phi}} n f_x \int_0^{f x(\phi')\tau^{1-\sigma}/f_x} y dF(y)\, g(\phi')\, \overline{M} d\phi']$ and $H = \Gamma$, then $\{p, r^d, H\}$ is a solution to the BVP of the open economy. A similar argument shows that any solution to the BVP of the open economy can be used to construct a solution to this particular parametrization of BVP (22). Finally, if we set $K_1 = 0$ in the parametrization above, the resulting BVP is equivalent to the BVP of the closed economy defined in lemma 1.ii.

Lemma 2 states the main results about the general BVP considered in this section.

Lemma 2 *Under the assumptions on the data of the problem, $\{A, g, V, F, \alpha, K_0, K_1\}$, there is a unique continuously differentiable solution to the BVP (22) for any $\phi_0, \phi_1 \in [\underline{\phi}, \overline{\phi}]$ and $s_0, s_1 \in [\underline{s}, \overline{s}]$, with $\phi_0 < \phi_1$ and $s_0 < s_1$. As a function of (ϕ_0, s_0), the solution to the BVP, $\{z(.;\phi_0, s_0), x(.;\phi_0, s_0), \Gamma(.;\phi_0, s_0)\}$, satisfies the following conditions,*
(i) (no crossing) If $K_1 = 0$ and Γ^{-1} denotes the inverse of Γ, then $s_0^a < s_0^b$ implies $\Gamma(\phi;\phi_0, s_0^a) < \Gamma(\phi;\phi_0, s_0^b)$ on $[\phi_0, \phi_1)$, while $\phi_0^a > \phi_0^b$ implies $\Gamma^{-1}(s;\phi_0^a, s_0) > \Gamma^{-1}(s;\phi_0^b, s_0)$ on $[s_0, s_1)$.
(ii) $\phi_0^a > \phi_0^b$ implies $x(\phi;\phi_0^a, s_0) < x(\phi;\phi_0^b, s_0)$ on $[\phi_0^a, \phi_1]$.

The formal proof of the last lemma can be found in the appendix, so here I present a brief outline of

solution is defined has to be sufficiently small.

[30] The general BVP considered in this section encompasses those in Costinot and Vogel (2010), Sampson (2014), Somale (2015), Grossman, Helpman, and Kircher (2017).

the argument. To prove existence, I follow O'Regan (2013) and recast the BVP as a fixed point problem. In particular, I show that a function Γ is part of a solution, $\{z, x, \Gamma\}$, to the BVP (22) if and only if it is a fixed point of some compact functional, Ψ, defined over a convex and closed set K, $\Psi(\Gamma) = \Gamma$. Then, a direct application of the Schauder fixed point theorem yields the existence result. The uniqueness of the solution is established as a consequence of the particular structure of the problem and the strict log-supermodularity of A. The claim in (i) is obtained as a corollary of the uniqueness result. For the case $K_1 = 0$ (closed economy), the claim in (ii) immediately follows from the no-crossing result in (i), (22b) and the log-supermodularity of A. However, this argument cannot be extended to the case $K_1 > 0$ (open economy), as the no-crossing property no longer holds. In the appendix, I present a slightly longer argument that is valid for the general case $K_1 \geq 0$, which also establishes the result as a consequence of the strict log-supermodularity of A.

An important corollary of the discussion so far is that, for a given activity cutoff ϕ^*, the functions r^d and H that solve the BVPs of the closed and open economies do not depend on the mass of firms, $\overline{M}$, nor the mass of production workers.[31] This feature of the solution follows from the uniqueness result in theorem 7, equation (22c) and the correspondence between said BVPs and BVP (22) described above. In fact, the mass of firms and the mass of production workers affect only the level of the solution function p. This result will prove useful in the analysis of the free-entry model in section 6.

Armed with lemma 2, the existence and uniqueness of the equilibrium in the open economy can be easily derived. The structure of the model, together with the fact that the BVP of the open economy has a unique solution, implies that there is exists a unique equilibrium if and only if there is a unique value of the activity cutoff, ϕ^*, that solves equation (21). In turn, this last result can be establish by analyzing the properties of the left- and right-hand sides of said equation as functions of ϕ^*. As discussed above, if r^d is part of the solution to the open-economy BVP, then $r^d(\phi) = \sigma f x(\phi)$, where x is part of the solution to a particular parametrization of the BVP (22). Then, lemma 2.ii implies that $r^d(\phi)$ is strictly decreasing in the activity cutoff ϕ^*, making the left-hand side of (21) strictly decreasing in the value of ϕ^*. In addition, it is readily seen that the right-hand side of (21) does not depend on the value of ϕ^*, so there is a unique solution to (21) if the size of the market, as captured by L, is not too large. The intuition behind this restriction on the market size is simple. The definition and characterization of the equilibrium in the open economy (and the closed economy) correspond to an equilibrium featuring selection into activity. However, such an equilibrium does not exist if the size of the market, as captured by L, is sufficiently large relative to the mass of firms, $\overline{M}$, as in this case even the least productive firms find it profitable to produce. The same argument can be used to show that there exists a unique equilibrium in the closed economy. I summarize this discussion in the next proposition, which also establishes the (constrained) efficiency of the equilibrium in the closed and open economies.

Proposition 1 *Let $\{\underline{p}, \underline{r}^d, \underline{H}\}$ and $\{\underline{p}^a, \underline{r}^{d,a}, \underline{H}^a\}$ be, respectively, the solution to the BVPs characterizing the open and closed economies with $\phi^* = \underline{\phi}$. In addition, let $\beta(r^d, \phi^*)$ and $\beta^a(r^d, \phi^*)$ denote the functions defined by left-hand sides of equations (21) and (16), respectively, in terms of ϕ^* and r^d. Then,*

[31] The mass of production workers in the closed and open economies are given by the term in brackets in the denominator of the right-hand side of equations (15) and (20), respectively.

(i) If $\beta(\underline{r}^d, \underline{\phi}) > L$, then there is a unique no-free-entry equilibrium of the open economy.
(ii) If $\beta^a(\underline{r}^d, \underline{\phi}) > L$, then there is a unique no-free-entry equilibrium of the closed economy.
In addition, the equilibrium of the closed economy is efficient, while that of the open economy is efficient when $f \leq f_x \tau^{1-\sigma}$, and constrained efficient when $f > f_x \tau^{1-\sigma}$.

Being a sufficient statistic for the dispersion of wages in the model, the matching function takes center stage in the subsequent analysis, as any result about wage inequality in this model is essentially a statement about the impact on the matching function of the shock under consideration. Lemma 4 in the appendix collects several results related to the BVP (22) that are instrumental to the analysis, which characterize the dependence of the solution function Γ (and some functionals of Γ) on the parameters of the problem.

5 No-Free-Entry, Trade and Wage Inequality

In this section, I study the effects of higher trade openness on wage inequality in the no-free-entry model described above. In the model, a decline in trade frictions induces a reallocation of production and employment across firms with heterogenous skill demand, affecting the aggregate relative demand for skills and the relative wages in the economy. In the analysis, I distinguish three channels through which trade induces this reallocation, and study the impact of each of these channels on wage dispersion. The first channel is the *intensive margin* of trade, and reflects the changes in the production and employment decisions of those firms that were exporters before the shock and that remain exporters afterwards. The second channel is the *extensive margin* of trade, which captures the reallocation of employment associated with changes in the set of exporters. Finally, the third channel is the *selection-into-activity* effect of trade, capturing the reallocation of resources driven by changes in the set of active firms.

5.1 Autarky vs. Trade

The first instance of higher trade openness that I consider is the case of an initially autarkic economy that opens up to trade. I start this section with one of the main results of the paper, Proposition 2, which states that opening to trade leads to a pervasive increase in wage inequality.

Proposition 2 *Let $\{\phi_a^*, N^a\}$ and $\{\phi_\tau^*, N^\tau\}$ be the activity cutoff and matching function corresponding to the no-free-entry equilibrium of the closed and open economies, respectively. Then the following conditions hold:*
(i) $\phi_\tau^* > \phi_a^*$ and $N^\tau(s) > N^a(s)$ for all $s \in [\underline{s}, \overline{s})$, so inequality is pervasively higher in the open economy.
(ii) The selection-into-activity and extensive-margin channels lead to pervasively higher inequality (intensive-margin channel not operational).

The first result in the last proposition, $\phi_\tau^* > \phi_a^*$, states that the selection-into-activity effects of trade highlighted in Melitz (2003) always hold in the no-free-entry model of this paper, i.e. trade induces the least productive firms to exit the market. Although somewhat trivial in homogenous-workers models a-lá

Melitz/Channey, this result is not immediate in the current framework.[32] For example, in an homogenous-workers version of the no-free-entry model above, assuming that firms with productivity ϕ_a^* are still active after the economy starts trading results in unchanged domestic revenues and labor costs. With aggregate labor costs pinned down by an equilibrium condition, this observation, together with positive export labor costs, implies that a higher activity cutoff is required in the open economy. In contrast, making the same assumption in the heterogeneous-worker framework above leads to lower domestic revenues and labor costs, so establishing the result requires proving that the decline in the latter is more than offset by the new labor costs of exporting (variable and fixed). I do so in the appendix by showing that total wages paid to *production* workers necessarily increase if the activity cutoff remains unchanged, which together with the presence of fixed export labor costs, leads to a rise in the the total wages paid by firms. With total wages pinned down by the numeraire assumption, condition (21), a higher activity cutoff is required in the open economy.[33]

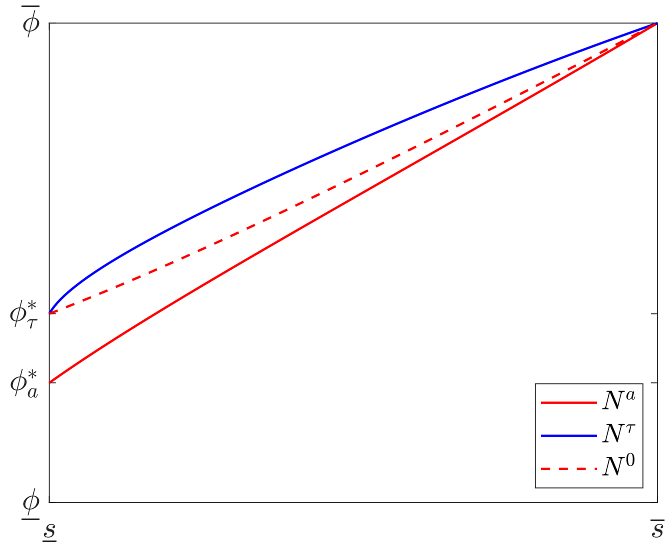
To gain more insight into the effect of trade on wage inequality, I decompose the overall effect into the three channels mentioned above. First of all, note that the intensive-margin channel is not operational in this case, as there were no exporters before the economy started to trade. The selection-into-activity channel captures the impact on wage inequality of the trade-induced increase in the activity cutoff, excluding the impact of changes in the set of exporters. To isolate the effect of this channel, I contrast the matching function of the closed economy with that of an ancillary autarkic economy that differs from the former only in that its activity cutoff is given by that of the open economy. That is, the equilibria of the closed and ancillary economies are characterized by the BVP in lemma 1.ii with $\phi^* = \phi_a^*$ and $\phi^* = \phi_\tau^*$, respectively. The typical situation is depicted in figure 1, where the solid and dashed red lines are, respectively, the matching functions of the closed (N^a) and ancillary (N^0) economies. The no-crossing result in lemma 2.i. implies that the latter lies strictly above the former on $[\underline{s}, \overline{s})$ as shown in the figure. Intuitively, as the firms with productivity in the range $[\phi_a^*, \phi_\tau^*)$ become inactive, the aggregate demand for workers with skills in the range $[\underline{s}, N^a(\phi_\tau^*))$ drops to zero barring any change in the wage schedule. Per the labor market clearing condition, these workers must be reallocated among the firms that remain active, requiring an increase in the relative wages of more skilled workers.

The extensive-margin channel reflects the impact on wage inequality of the increased labor demand by new exporters as they expand their production to serve the foreign market, excluding the effects of changes in the activity cutoff. Put another way, this channel captures the effects of replacing $[1 + F\left(r^d(\phi)\,\tau^{1-\sigma}/\sigma f^x\right) n\tau^{1-\sigma}]$ with 1 in the BVP of the open economy, precisely what the difference between the matching functions of the ancillary (N^0) and open (N^τ) economies in figure 1 captures, with the latter shown in blue. To see why N^τ necessarily lies above N^0 as depicted in the figure, suppose for a moment that the wages of the ancillary economy also prevail in the open economy. In this case, firms of a given productivity level demand the same skill-type of workers in both economies, with exporters in the open economy demanding more labor due to the foreign demand they face. With a constant fraction of exporters across productivity levels, this additional export-driven labor demand would affect all skill

[32] Chaney (2008) develops a parametrized version of Melitz (2003) featuring no free entry.

[33] As explained earlier, the left-hand side of (21) is strictly decreasing in the activity cutoff.

Figure 1: The Effects of Trade on the Matching Function



Note: The solid red and blue lines represent, respectively, the matching functions of the closed (N^a) and open (N^τ) economies. The dashed red line depicts the matching function of an ancillary autarkic economy (N^0) which is obtained by solving the BVP in lemma 1.ii with $\phi^* = \phi^*_\tau$. The difference between N^a and N^0 captures the impact of the selection-into-activity channel, while the difference between N^0 and N^τ captures the impact of the extensive-margin channel.

levels proportionally, leaving unchanged the overall relative demand for skills in the economy. However, as the fraction of exporters in the model increases with firms' productivity, this additional export-driven labor demand is tilted towards more able workers. The resulting rise in the overall relative demand for more skilled workers is inconsistent with labor market clearing, so the relative wages of these workers must be higher in the open economy.[34]

I conclude this section with a discussion of the impact of trade on the *level* of real wages. Although trade always rises the average real wage, the least skilled workers in the economy may see their real wage decline. The pricing rule (7) and the zero profit condition (9) imply that the aggregate price indices of the closed (P^a) and open (P^τ) economies satisfy

$$\left(P^i\right)^\sigma = \frac{\sigma f}{U^i}\left[\frac{\sigma}{(\sigma-1)}\frac{w^i\left(\underline{s}\right)}{A\left(\underline{s},\phi^*_i\right)}\right]^{\sigma-1} \quad \text{for } i = a,\tau, \tag{23}$$

where U^i is the aggregate real expenditure/income in the economy. Per the efficiency result in proposition 1, real income is higher in the open economy, $U^\tau > U^a$.[35] In addition, proposition 2.i, together with the numeraire assumption ($\overline{w}^i = 1$), implies that the open economy exhibits a higher activity cutoff, $\phi^*_\tau > \phi^*_a$, and a lower wage for the least able workers, $w^\tau\left(\underline{s}\right) < w^a\left(\underline{s}\right)$. Accordingly, $P^\tau < P^a$, so the average real

[34] Formally, in the appendix I show that the BVPs of the ancillary and open economies can be conceived as particular parameterizations of the general BVP (22) with $K_1 = 0$ that differ only in the parameter function $\alpha(\phi)$, which is constant in the former and increasing in the latter. The result then follows from a direct application of lemma 4.i in the appendix.

[35] Note that the closed economy allocation is available to the planner of the open economy, so a simple revealed-preference argument yields $U^\tau > U^a$. See the proof of proposition 1 in the appendix for more details.

wage, $\overline{w}/P$, is higher in the open economy.

Finally, recalling that $U^i = E^i/P^i$, equation (23) can be rearranged to get the an expression for the real wage of workers with the lowest skill level, $w^i(\underline{s})/P^i = \frac{(\sigma-1)}{\sigma} A(\underline{s}, \phi_i^*) \left[E^i/\sigma f\right]^{\frac{1}{\sigma-1}}$. This expression implies that trade improves the real wage of even the least able workers in the economy when it induces a rise in aggregate expenditure/income. However, in some parameterizations of the model, trade can induce a decline in the real wage of these workers, as the drop in aggregate expenditure more than offsets the boost coming from a higher activity cutoff.

5.2 Trade Liberalization

Although the preceding analysis sheds light into the effects of higher trade openness on wage inequality, very few, if any, of the countries in the world operate in autarky. For this reason, in this section I study the effects on wage inequality of a trade liberalization, defined as a decline in the variable trade costs faced by an economy that participates in international trade. I find that these effects may differ from those described in the previous section. In particular, although a trade liberalization necessarily rises wage inequality among the least skilled workers in the economy, wage inequality may decline elsewhere in the wage the distribution. Proposition 3 presents the main results of this section.

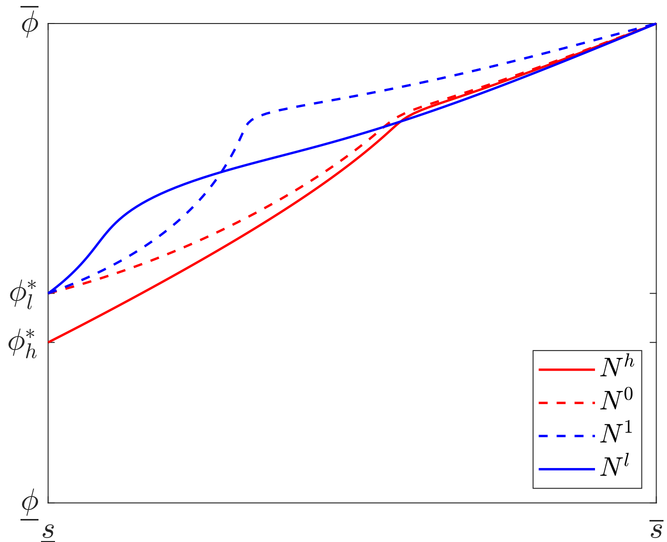
Proposition 3 *Consider a trade liberalization consisting in a decline in variable trade costs from τ_h to τ_l, and let $\{\phi_h^*, N^h\}$ and $\{\phi_l^*, N^l\}$ represent, respectively, the pre- and post-liberalization activity cutoffs and matching functions. Then, the following conditions hold:*
(i) $\phi_l^* > \phi_h^*$, so a trade liberalization raises wage inequality among the least skilled workers in the economy.
(ii) The selection-into-activity and intensive-margin channels lead to pervasively higher inequality, while the effect of the extensive-margin channel is ambiguous.
(iii) If the functions $\eta_0^F(t,\lambda) \equiv \frac{F_y(t\lambda)\lambda}{[1+F(t\lambda)k]}$ and $\eta_1^F(t,\lambda) \equiv \frac{F_y(t\lambda)\lambda^2}{[1+F(t\lambda)k]}$ are, respectively, strictly decreasing and strictly increasing in λ for all $t, k \in \mathbb{R}_{++}$, then a trade liberalization rises wage inequality pervasively.

The first result of the proposition states that, as in the Melitz/Channey models, a trade liberalization always leads to the exit of the least productive of firms from the market, $\phi_l^* > \phi_h^*$. The general line of argument used in the proof of proposition 2.i. can be applied here as well. If the activity cutoff remains unchanged after the decline in trade costs, then total wages paid to production and nonproduction workers necessarily increase. With total wages pinned down by condition (21), the activity cutoff must be higher after the liberalization. This result and the continuity of the matching functions imply that $N^l(s) > N^h(s)$ on some interval of the form $[\underline{s}, s')$, which is equivalent to the second part of the claim in proposition 3.i.[36]

As before, the overall impact of a trade liberalization on wage inequality can be decomposed into the three channels mentioned above. The selection-into-activity channel captures the changes in wage dispersion associated with the rise in the activity cutoff, excluding the impact of changes in the labor

[36] Of note, establishing the consequences of an unchanged activity cutoff is more complicated in the case of a trade liberalization, as multiple crossings of relevant matching functions cannot be ruled out. In this case, the formal argument is based on the results in lemma 4.iv-v.

Figure 2: The Effects of a Trade Liberalization on the Matching Function



Note: The solid red and blue lines represent, respectively, the pre- (N^h) and post-liberalization (N^l) matching functions described in Proposition 3. Starting at N^h, increasing the activity cutoff to $\phi_l^* > \phi_h^*$, while keeping the set of exporters and the level of variable trade costs unchanged yields the matching function depicted by the dashed red line, N^0. Allowing variables trade costs to increase to their post-liberalization level while keeping the set of exporters unchanged yields N^1, the dashed blue line. Adjusting the set of exporters to their post-liberalization composition yields, N^l. Accordingly, the effects of the selection-into-activity, intensive-margin, and extensive-margin channels on the matching function are captured, respectively, by the differences between the pairs $\{N^h, N^0\}$, $\{N^0, N^1\}$, and $\{N^1, N^l\}$.

demand of incumbent exporters and of changes in the set of exporters. To isolate the effect of this channel, I contrast the matching function of the open economy before the liberalization, N^h, with that of an ancillary open economy, N^0, that differs from the former only in that its activity cutoff is given by that prevailing after the liberalization, ϕ_l^*. That is, as I explain in more detail in the appendix, the BVPs associated with N^h and N^0 can be conceived as parameterization of the general BVP (22), with $K_1 = 0$ and $\alpha^h(\phi) \equiv [1 + F\left(r^{d,h}(\phi)\,\tau_h^{1-\sigma}/\sigma f^x\right) n\tau_h^{1-\sigma}]$, that differ only in their boundary conditions.[37] Accordingly, the no crossing result in lemma 2.i. implies that N^0 lies strictly above N^h on $[\underline{s}, \overline{s})$ as depicted by the dashed and solid red lines in figure 2. The intuition for the effects of this channel are the same as before, i.e. the exit of the least productive firms from the market reduces the relative demand of less skilled workers, pushing down their relative wages.

The intensive-margin channel captures the impact on wage inequality of the liberalization-induced rise in the labor demand of incumbent exporters. I isolate this channel by contrasting the matching function N^0 with that of a second ancillary open economy, N^1, with the same set of exporters and active firms, but with variable trade costs given by τ_l. That is, N^1 is obtained by replacing the parameter function $\alpha^h(\phi)$ with $\alpha^1(\phi) \equiv [1 + F\left(r^{d,h}(\phi)\,\tau_h^{1-\sigma}/\sigma f^x\right) n\tau_l^{1-\sigma}]$ in the BVP associated with N^0. As shown by the dashed blue and red lines in figure 2, N^1 necessarily lies above N^0 on $(\underline{s}, \overline{s})$ for the same reasons laid out in the discussion of the extensive-margin channel in proposition 2. Suppose that these ancillary economies

[37] $r^{d,h}$ is the domestic revenue function of the open economy with variable trade costs τ_h.

share the same wage schedule. Then, firms of a given productivity level demand the same skill-type of workers in both economies, with exporters in the N^1-economy (lower trade costs) demanding more labor due to the larger foreign demand they face. As the (common) fraction of exporters in these economies is increasing in firms's productivity, this additional export-driven labor demand in the N^1-economy results in a higher relative demand for more skilled workers, which is inconsistent with labor market clearing. Accordingly, the wages of these workers must be higher in the N^1-economy.[38]

The extensive-margin channel captures the impact on relative wages of allowing the fraction of exporters to adjust, i.e. the effects on wages of replacing $\alpha^1(\phi)$ with $[1 + F\left(r^{d,l}(\phi)\,\tau_l^{1-\sigma}/\sigma f^x\right) n\tau_l^{1-\sigma}]$ in the BVP associated with N^1. Little can be said about these effects at this level of generality. In figure 2, which illustrates only one of the many possibilities, the impact of this channel is given by the difference between N^1 and N^l, the dashed and solid blue lines, respectively. In this example, the weight of some middle-productivity firms among exporters in the post-liberalization economy is larger than in the ancillary N^1-economy. Then, the change in the set of exporters drives up the relative demand for some middle-skill workers, pushing up their wages relative to those of workers with lower and higher skill levels. That said, the impact of this channel could take other forms depending on the distribution function of fixed export costs, F, including a pervasive rise and a pervasive decline in wage inequality. Moreover, the effects of this channel can be strong enough to offset the impact of the other two channels in some parts of the wage distribution, as shown by the crossing of N^h and N^l in figure 2.

Proposition 3.iii presents a set of sufficient conditions on the distribution of fixed exports costs, F, that guarantee that a trade liberalization always leads to a pervasive rise in wage inequality. When the condition on the function η_1^F is satisfied, reducing variable trade costs while keeping the activity cutoff unchanged in the BVP of the open economy (that allows the set of exporters to change) always leads to pervasively higher wage inequality. In addition, when the condition on η_0^F is satisfied, increasing the activity cutoff while keeping variable trade costs constant in said BVP also leads to a pervasive rise in wage dispersion. Accordingly, when both conditions are met, wage inequality increases pervasively following a liberalization, as the effect on relative wages of changes in the set of exporters (extensive-margin channel) never offsets the combined impact of the selection-into-activity and intensive-margin channels. Although these restrictions on F may appear very restrictive to some readers, one should bear in mind that they are sufficient conditions under all paramerizations of the model.[39]

Regarding the impact of a trade liberalization on the *level of wages*, the analysis and conclusions of the previous section also apply to this case. A liberalization increases real income and average real wages, but the least productive workers in the economy could see their real wage decline in some parameterizations of the model.

[38] Formally, the result follows from a direct application of lemma 4.i in the appendix, with α^1 taking the role of α^a in the lemma.

[39] For a Pareto distribution, the condition on η_0^F is always satisfied, while that on η_1^F is satisfied when the shape parameter is small enough. Moroever, a sufficiently small shape parameter typically precludes the crossing of the matching function even when the condition on η_1^F is not satisfied.

5.3 Trade Openness and Wage Dispersion in Alternative Frameworks

The three-channel decomposition of the effects of higher trade openness on wage inequality described above can be a useful tool to analyze differences in the implications of alternative frameworks in the literature. For illustration purposes, I compare the effects of trade on wage inequality in the no-free-entry model in this paper with those in Helpman, Itskhoki, and Redding (2010), henceforth HIR. In the HIR model, firms screen workers to improve the composition of their labor forces as worker ability is not directly observable. As larger firms have higher returns from screening, they do so more intensively and have workforces of higher average ability than smaller firms. This mechanism generates a wage-size premium, implying that both productivity and exporting positively affect the average wages paid by a firm. In this setting, HIR show that wage inequality increases after an economy opens to trade only when there is selection into exporting (only some firms export), but is unchanged when all firms become exporters. Through the lenses of the decomposition analysis above, the intensive-margin channel of trade does not affect wage dispersion in the HIR model, as changes in the activity cutoff do not modify the relative size of firms. In addition, trade affects wage inequality through the extensive-margin channel only when it changes the relative size of firms in the economy, i.e. only when some but not all firms export. In contrast, trade always leads to higher wage inequality in the no-free-entry model of this section. Although trade does not affect wage inequality through the extensive-margin channel when all firms export (as in HIR), it always drives up wage dispersion through the intensive-margin channel.

6 The Free-Entry Model

In the model outlined above, the mass of firms in the industry is fixed at an exogenous level. Although this assumption may be a good approximation to the firm-entry dynamics in the short-run, it does not capture the change in the number of firms through endogenous entry and exit over time. In this section, I relax this assumption by allowing firms to enter the industry for a cost, making the mass of firms in the industry, $\overline{M}$, an additional endogenous variable. Specifically, I assume that there is an unbounded pool of prospective firms that can enter the industry by incurring a fixed entry cost of $f^e V(s)$ units of each skill $s \in S$. Accordingly, the aggregate expenditure on entry costs is $\overline{M} f^e$ when a mass $\overline{M}$ of firms enters the industry.[40] Upon entry, firms obtain their productivity as independent draws from the distribution G, as explained in section 2.2. All the other primitives of the model remain unchanged. Below, I briefly describe the open economy equilibrium in the free-entry model, relegating to the appendix a more detailed exposition.

The new assumptions above do not affect the basic structure of the model described in section 2, so equations (1)-(7) continue to hold. Conditional on the mass of firms, $\overline{M}$, the equilibrium analysis in section 4 applies almost unchanged to the free-entry model, with the caveat that equilibrium conditions now reflect the labor demand derived from the presence of fixed entry costs, i.e. L must be replaced with $L - f^e \overline{M}$ throughout the analysis. The new free-entry assumption implies that, in equilibrium, prospective

[40] The numeraire assumption, $\overline{w} = 1$, yileds $\overline{M} \int_{\underline{s}}^{\overline{s}} f^e w(s) V(s) \, ds = \overline{M} f^e$.

entrants must be indifferent between entering and not entering the industry. Accordingly, expected profits from entering the industry must equal the cost of entry, $[1 - G(\phi^*)]\left[\overline{\pi}^d + \overline{\pi}^x\right] = f^e$, where $\overline{\pi}^d$ and $\overline{\pi}^x$ are, respectively, the average domestic and export profits of active firms.[41] Per the optimal pricing rule, this *free-entry* condition can be written as follows,

$$\int_{\phi^*}^{\overline{\phi}} \left[\frac{r^d(\phi)}{\sigma} - f\right] g(\phi)\, d\phi + \int_{\phi^*}^{\overline{\phi}} \int_0^{\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}} n \left[\frac{r^d(\phi)\,\tau^{1-\sigma}}{\sigma} - f^x y\right] dF(y)\, g(\phi)\, d\phi = f^e. \tag{24}$$

The last equation completes the description of the open-economy equilibrium in the free-entry model, prompting the formal definition in appendix A.5.2.

Lemma 6 of the appendix provides a characterization of the free-entry equilibrium of the open economy that is analogous to the one given in section 4.1 for the no-free-entry model. In particular, given the activity cutoff, ϕ^*, the price, domestic-revenue and inverse-matching functions, $\{p, r^d, H\}$, solve a BVP that differs from that of the no-free-entry model in lemma 3.iii. only in that L is replaced by $L - f^e\overline{M}$ in the equation defining the slope of the inverse-matching function. Moreover, the discussion in section 4.2 implies that conditional on ϕ^*, the BVPs of the no-free-entry and free-entry models have the same parametrization in terms of the general BVP (22), so they share the same solution functions r^d and H. The equilibrium value for ϕ^* in free-entry model is pinned down by the free entry condition (24).[42]

The observations above have important implications. First, all the conclusions reached in section 4.2 about the dependence of $\{r^d, H\}$ on the activity cutoff ϕ^* continue to hold in the free-entry model. Accordingly, many results, such as the existence and uniqueness of the equilibrium in the free-entry model, can be derived in a similar way.[43] Second, the only relevant difference between the no-free-entry and free-entry models regarding the determination of the equilibrium matching function is given by the equation that pins down the activity cutoff in these models, equations (21) and (24), respectively. In the remainder of this section I explore how this difference affects the impact of increased trade openness on wage inequality.

6.1 Autarky vs. Trade in the Free-entry Model

In contrast to the case of the no-free entry model, trade may lead to a rise or a fall in the activity cutoff in the free-entry model, with ambiguous effects on wage inequality through the selection-into-activity channel. Despite this ambiguity at this level of generality, the overall effects of trade on wage dispersion can lead to only two situations, a pervasive increase in wage inequality or wage polarization, as formally stated in proposition 4.

[41] Note that $\overline{\pi}^x$ is not the average export profits among exporters, but among all active firms.

[42] This is the case because ϕ^* and r^d are the only endogenous variables appearing in equation (24). Note that using the analog of equation (21) for the free-entry model to determine the activity cutoff ϕ^* would only give us ϕ^* as a function of the endogenous mass of firms $\overline{M}$.

[43] As $r^d(\phi)$ depends negatively on the activity cutoff, the left-hand side of equation (24) is strictly decreasing in ϕ^*, implying that there is unique free-entry equilibrium if entry costs are not too high.

Proposition 4 *Let* $\{\phi_a^*, N^a\}$ *and* $\{\phi_\tau^*, N^\tau\}$ *be the activity cutoffs and matching functions corresponding to the free-entry equilibrium of the closed and open economies, respectively. Then* ϕ_τ^* *could be lower or higher than* ϕ_a^* *depending on the model's parameters.*
(i) If $\phi_\tau^* \geq \phi_a^*$*, then* $N^\tau(s) > N^a(s)$ *on* $s \in (\underline{s}, \overline{s})$*, so opening to trade leads to pervasively higher wage inequality. The selection-into-activity channel leads to a pervasive rise (no change) in wage inequality if* $\phi_\tau^* > (=)\phi_a^*$*. The extensive-margin channel always leads to a pervasive rise in wage inequality.*
(ii) If $\phi_\tau^* < \phi_a^*$*, then* $N^\tau(s)$ *and* $N^a(s)$ *intersect exactly once on* $(\underline{s}, \overline{s})$*, so opening to trade leads to wage polarization. The selection-into-activity and extensive-margin channels lead, respectively, to pervasively lower and pervasively higher wage inequality.*

A trade-induced decline in the activity cutoff is a theoretical possibility that has important implications for the effect of trade on wage inequality. As this possibility is not present in the no-free-entry model in this paper or even in standard free-entry models with homogeneous workers, such as Melitz (2003), I start the analysis by discussing the elements of the free-entry model above that allow for such an occurrence.

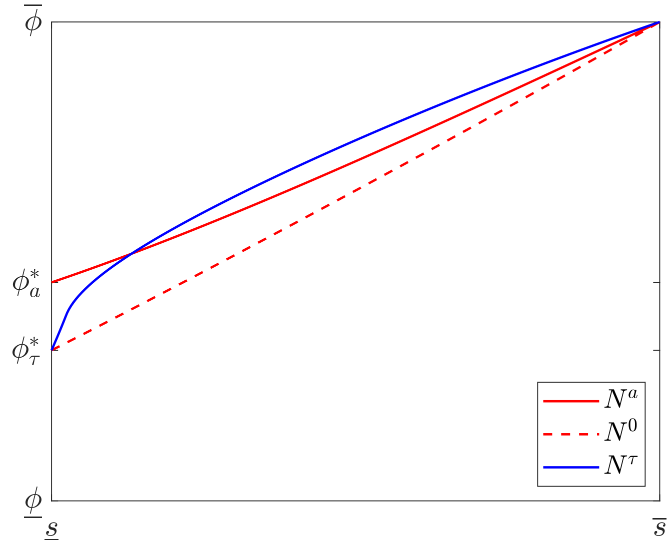
The different equilibrium conditions that determine the activity cutoff in the free-entry and no-free-entry models in this paper imply that trade can lead to a decline in said cutoff in the former but not in the latter. These differences are better understood by comparing the impact that trade has on these equilibrium conditions when the set of active firms and the revenue of the least productive ones are assumed to remain unchanged, $r^d(\phi_a^*) = \sigma f$. As discussed in section 5, in this scenario, trade leads to a rise in the implied total wages paid to production and non-production workers, as total firms' revenue and fixed export costs increase. Accordingly, equation (21) implies that a higher activity cutoff is required in the open economy of the no-free-entry model. In contrast, in the free-entry model, total firms' revenue and fixed export costs enter with opposite signs on the left-hand side of the free-entry condition (24), with an ambiguous net effect, so a lower activity cutoff may be required in the open economy.

Relative to standard free-entry models with homogeneous workers, a trade-induced decline in the activity cutoff is possible in the free-entry model because of the endogenous changes in the matching of heterogeneous workers to firms.[44] As before, it is instructive to compare the impact that trade has on the free-entry condition in these models under the same assumptions described in the previous paragraph. In such a scenario, trade increases export profits from zero (in autarky) to some strictly positive number in both models. With domestic profits remaining unchanged in the homogeneous-workers model (before adjusting the activity cutoff), average/expected profits necessarily increase, so the free-entry condition requires a higher activity cutoff in the open economy. In contrast, in the free-entry model of this paper, trade may lead to a decline in aggregate profits due to changes in the matching function. Specifically, as the matching function N shifts up (H shifts down) in the scenario considered, domestic revenues and profits decline for firms with productivity above ϕ_a^*.[45] For some parameter values, the decline in aggregate domestic profits more than offsets the rise in export profits, so the free-entry condition (24) requires a lower activity cutoff in the open economy.

[44] The stochastic modeling of fixed costs is another difference between the free-entry model in this paper and standard Melitz-type models. However, said difference alone cannot produce a trade-induced declined in the activity cutoff.

[45] See discussion associated to equation (14).

Figure 3: The Effects of Trade on the Matching Function in Free-entry Model



Note: The solid red and blue lines represent, respectively, the matching functions of the closed (N^a) and open (N^τ) economies. The dashed red line depicts the matching function of an ancillary autarkic economy (N^0) which is obtained by solving the BVP in lemma 1.ii with $\phi^* = \phi_\tau^*$. The difference between N^a and N^0 captures the impact of the selection-into-activity channel, while the difference between N^0 and N^τ captures the impact of the extensive-margin channel. The figure depicts the case in which trade induces a decline in the activity cutoff.

Per proposition 4, conditional on its impact on the activity cutoff, trade has a unique qualitative effect on the dispersion of wages, with an unambiguous effect through the selection-into-activity and extensive-margin channels. The case in proposition 4.i, $\phi_\tau^* \geq \phi_a^*$, is essentially the same situation considered in section 5.1 for the no-free-entry model. If $\phi_\tau^* > \phi_a^*$, then the situation is identical to that depicted in figure 1, so the corresponding analysis applies here as well. When $\phi_\tau^* = \phi_a^*$, the only difference is that the selection-into-activity channel has no effect on wage dispersion.

The case in proposition 4.ii, $\phi_\tau^* < \phi_a^*$, requires some additional explanation. As I discuss in the appendix, the matching function of the open economy, N^τ, cannot remain completely below that of the closed economy, N^a, on $[\underline{s}, \overline{s})$. Otherwise, per lemma 2.ii, expected domestic profits in the open economy would be strictly higher than in autarky, implying a violation of the free-entry condition (24). Then, N^τ and N^a must intersect at least once on $(\underline{s}, \overline{s})$. Moreover, adapting the analysis of the extensive-margin channel in section 5.1 to assess the relative position of N^a and N^τ to the right of the first intersection, it can be shown that N^a must remain below N^τ there, so the matching functions must intersect exactly once on $(\underline{s}, \overline{s})$.[46] The situation is depicted in figure 3, where the solid red and blue lines represent N^a and N^τ, respectively. As before, the dashed red line is the matching function of an ancillary autarkic economy, N^0, that is obtained by changing the activity cutoff in the BVP corresponding to N^a from ϕ_a^* to ϕ_τ^*. As

[46] Formally, to the right of the first intersection point, the matching functions of the closed and open economies can be conceived as solutions to particular parameterizations of the general BVP (22) with $K_1 = 0$ that differ only in the parameter function $\alpha(\phi)$, which is constant in the former and increasing in the latter. The result then follows from a direct application of lemma 4.i in the appendix.

discussed in section 5.1, the effects of trade on wage inequality through the selection-into-activity and extensive-margin channels are captured, respectively, by the difference between the pairs $\{N^a, N^0\}$ and $\{N^0, N^\tau\}$. While the selection-into-activity channel pervasively reduces wage inequality, the extensive-margin channel pervasively increases it, with the former channel dominating to the left of the interior intersection point of N^a and N^τ, and the latter dominating to the right. As a result, workers with skill level corresponding to this (interior) intersection point see their wages decline relative to those of all other workers, i.e. trade leads to wage polarization.

Turning to the effects of trade on the *level of real wages*, the results obtained for the no-free-entry model generally go through in the free-entry model. First, the average real wage is always higher in the open economy. As before, the result follows from the (constrained) efficiency of the equilibrium. Second, trade may induce a decline in the real wage of the least skilled workers in the economy, although in the free-entry model this possibility is fully determined by the impact of trade on the activity cutoff. As the free-entry condition implies that the economy's total income and expenditure is given by total labor income, $E = \overline{w}L$, rearranging equation (23) yields $w^i(\underline{s})/P^i = \frac{(\sigma-1)}{\sigma} A(\underline{s}, \phi_i^*)[L/\sigma f]^{\frac{1}{\sigma-1}}$ for $i = a, \tau$, i.e. trade rises the real wage of even the least skilled workers in the economy if and only if it rises the activity cutoff. Note that this observation, together with proposition 4, implies that no worker loses from trade only if wage inequality increases pervasively.

6.2 Trade Liberalization in the Free-entry Model

The effects of a trade liberalization on the wage distribution in the free-entry-model can derived by resorting to the results in propositions 2 to 4, as they largely cover the range of possible outcomes in this case. For the same reasons behind the corresponding result in proposition 4, a trade liberalization could lead to a rise or a fall in the activity cutoff. If the activity cutoff increases, then the situation is identical to that considered in proposition 3 so all the results go through. If the activity cutoff declines, then the pre- and post-liberalization matching functions must intersect at least once on $(\underline{s}, \overline{s})$ to avoid a violation of the free entry condition as discussed in the case of proposition 4.ii. However, in the case of a trade liberalization, more than one crossing on $(\underline{s}, \overline{s})$ cannot be ruled out even when the conditions on the functions $\eta_0^F(t, \lambda)$ and $\eta_1^F(t, \lambda)$ in proposition 3 are satisfied.

7 Conclusion

In this paper, I develop a general equilibrium trade model with a large number of skill-groups that emphasizes the within-industry reallocation of workers across heterogenous firms as the mechanism through which international trade affects the wage distribution. Strong complementarities in production between worker skill and firm productivity lead to positive assortative matching in equilibrium, while heterogeneous fixed export costs imply that the productivity distributions of exporters and non-exporters overlap. As a result, the cross-sectional structure of model captures several features of the data identified by the trade and labor literatures. More productive firms tend to be larger, have workforces of higher average ability and pay higher average wages, and there is an imperfect correlation between firm size, wages and export status. I consider two versions of the model corresponding to two alternative assumptions about firm entry, no free entry a-lá Chaney (2008) and free entry a-lá Melitz (2003).

I use the model to study the theoretical effects of higher trade openness on the wage distribution. In the no-free-entry model, opening to trade always leads to pervasively higher wage inequality. By contrast, a trade liberalization necessarily increases inequality at the lower end of the wage distribution, but may reduce it elsewhere. In the free-entry model, opening to trade leads to pervasively higher inequality (wage polarization) if low-productivity firms exit (enter) the market. In the case of a trade liberalization, all the previous possibilities could arise without additional restrictions on primitives. In all cases, higher trade openness never leads to a pervasive decline in wage inequality. In addition, to gain more insight into the elements driving of these results, I decompose the overall impact of trade on the wage distribution into those associated with the selection-into-activity, the intensive-margin and extensive-margin channels of trade. The analysis highlights the importance of new exporters (extensive margin) in shaping the aggregate relative demand for skills and relative wages, a channel that is controlled by the distribution of fixed export costs in the model.

Finally, I also contribute methodologically to the analysis of assignment problems. In addition to presenting existence and uniqueness results for a general BVP that encompasses those in this paper and others in the literature, I derive general results about the dependence of the solution to this BVP on parameters. These results can be used to analyze comparative statics exercises beyond those considered in this paper.

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A Theoretical Appendix

A.1 Section 3

A.1.1 Proof of Lemma 1

Existence of a matching function N. I start by defining some notation. Let $S(\phi) \equiv \{s \in S : l(s,\phi) > 0\}$ and let $\Phi(s) = \left\{\phi \in \left[\phi^*, \overline{\phi}\right] : l(s,\phi) > 0\right\}$. To clarify the exposition of this part of the proof, I will proceed in a series of steps.

STEP 1: $\Phi(s) \neq \emptyset$ for all $s \in S$ and $S(\phi) \neq \emptyset$ for all $\phi \in \left[\phi^*, \overline{\phi}\right]$.

The full employment condition (11) and $V(s) > 0$ directly imply $\Phi(s) \neq \emptyset$ for all $s \in S$. Now suppose that we have an equilibrium in which there is $\phi \in \left[\phi^*, \overline{\phi}\right]$ such that $S(\phi) = \emptyset$. Then from (3) we have $q(\phi) = 0$ and this is incompatible with the demand given in (1), since for any $p(\phi) \in \mathbb{R}_+$ we have $q(\phi) > 0$. Then in any equilibrium we must have $S(\phi) \neq \emptyset$.

STEP 2: $S(.)$ and $\Phi(.)$ satisfy the following properties: (i) if $s \in S(\phi)$, $s' \in S\left(\phi'\right)$ and $\phi' > \phi$, then $s' \geq s$; and (ii) if $\phi \in \Phi(s)$, $\phi' \in \Phi(s')$ and $s' > s$,then $\phi' \geq \phi$.

(i) Suppose that this is not true and so let $s' < s$. Notice that (7) implies that $s \in S(\phi)$ if and only if $s \in \arg\min_z w(z)/A(z,\phi)$. Then $w(s)/A(s,\phi) \leq w(s')/A(s',\phi)$. In a similar way, $s' \in S\left(\phi'\right)$ implies $w(s')/A\left(s',\phi'\right) \leq w(s)/A\left(s,\phi'\right)$. Combining both inequalities we get $A\left(s,\phi'\right)A(s',\phi) \leq A(s,\phi)A\left(s',\phi'\right)$, but this contradicts the log-supermodularity of A (remember that $\phi' > \phi$ and $s > s'$). Then we must have $s' \geq s$.

(ii) Suppose that this is not true and so let $\phi' < \phi$. Then $\phi \in \Phi(s) \Rightarrow s \in S(\phi)$ and $\phi' \in \Phi(s') \Rightarrow s' \in S\left(\phi'\right)$. Then we have $\phi' < \phi$, $s \in S(\phi)$, $s' \in S\left(\phi'\right)$ and by STEP 2.i this implies $s \geq s'$, which is a contradiction. Then we must have $\phi' \geq \phi$.

STEP 3: (i) $S(\phi)$ is an interval for all $\left[\phi^*, \overline{\phi}\right]$ and $|S(\phi) \cap S\left(\phi'\right)| \leq 1$ for any two different $\phi, \phi' \in \left[\phi^*, \overline{\phi}\right]$; (ii) $\Phi(s)$ is an interval for all $s \in S$ and $|\Phi(s) \cap \Phi(s')| \leq 1$ for any two different $s, s' \in S$.

(i) I will prove the first part by contradiction. Suppose there is $\phi \in \left[\phi^*, \overline{\phi}\right]$ such that $S(\phi)$ is not an interval. Then there we can find $s, s' \in S(\phi)$, with $s < s'$, and some $s'' \in (s, s')$ such that $s'' \notin S(\phi)$. From STEP 1 we know that $\Phi(s'')$ is nonempty and so there must be a $\phi'' \in \left[\phi^*, \overline{\phi}\right]$ such that $s'' \in S\left(\phi''\right)$. We have only two possibilities: $\phi'' > \phi$ and $\phi'' < \phi$. If $\phi'' > \phi$, then STEP 2.i implies $s'' \geq s'$ which is a contradiction. If $\phi'' < \phi$, then STEP 2.i implies $s \geq s''$ which is also a contradiction. Then $S(\phi)$ is an interval for all $\left[\phi^*, \overline{\phi}\right]$.

Let us now show that $S(\phi) \cap S\left(\phi'\right)$ is at most a singleton and as before I will proceed by contradiction. Suppose that the claim is not true. Then there must be $\phi, \phi' \in \left[\phi^*, \overline{\phi}\right]$ such that $s, s' \in S(\phi) \cap S\left(\phi'\right)$ with $s \neq s'$. Without loss of generality assume $\phi' > \phi$ and $s' > s$. Then we have $\phi' > \phi$, $s' \in S(\phi)$, $s \in S\left(\phi'\right)$ and so STEP 2.i implies $s \geq s'$ which is a contradiction. This concludes part i.

(ii) I prove this by contradiction. Suppose there is $s \in S$ such that $\Phi(s)$ is not an interval. Then there we can find $\phi, \phi' \in \Phi(s)$, with $\phi < \phi'$, and some $\phi'' \in \left(\phi, \phi'\right)$ such that $\phi'' \notin \Phi(s)$. From STEP 1 we know that $S\left(\phi''\right)$ is nonempty and so there must be a $s'' \in S$ such that $\phi'' \in \Phi(s'')$. We have only two possibilities: $s'' > s$ and $s'' < s$. If $s'' > s$, then STEP 2.ii implies $\phi'' \geq \phi'$ which is a contradiction.

If $s'' < s$, then STEP 2.ii implies $\phi \geq \phi''$ which is also a contradiction. Then $\Phi(s)$ is an interval for all $s \in S$.

Let us now show that $\Phi(s) \cap \Phi(s')$ is at most a singleton and as before I will proceed by contradiction. Suppose that the claim is not true. Then there must be $s, s' \in S$ such that $\phi, \phi' \in \Phi(s) \cap \Phi(s')$ with $\phi \neq \phi'$. Without loss of generality assume $\phi' > \phi$ and $s' > s$. Then we have $s' > s$, $\phi' \in \Phi(s)$, $\phi \in \Phi(s')$ and so STEP 2.ii implies $\phi \geq \phi'$ which is a contradiction. This concludes part ii.

STEP 4: $S(\phi)$ is a singleton for all but a countable subset of $\left[\phi^*, \overline{\phi}\right]$.

I show this by contradiction. Let $\Phi_0 = \left\{\phi \in \left[\phi^*, \overline{\phi}\right] : |S(\phi)| > 1\right\}$ and suppose Φ_0 is uncountable. Notice that STEP 3.i implies that $S(\phi)$ is a nondegenerate interval for all $\phi \in \Phi_0$. Then for each $\phi \in \Phi_0$ we can pick a rational skill $r(\phi) \in intS(\phi)$ and given that $|S(\phi) \cap S(\phi')| \leq 1$ for any two different ϕ, ϕ' we must have $r(\phi) \neq r(\phi')$ when $\phi \neq \phi'$. Then the function $r : \Phi_0 \rightarrow \mathbb{Q} \cap S$ defined before is injective and so it is a contradiction since Φ_0 is uncountable.

STEP 5: $\Phi(s)$ is a singleton for all but a countable subset of S.

This follows from the same arguments as in STEP 4.

STEP 6: $S(\phi)$ is a singleton for all $\phi \in \left[\phi^*, \overline{\phi}\right]$.

I proceed by contradiction. Suppose there is $\phi \in \left[\phi^*, \overline{\phi}\right]$ such that $S(\phi)$ is not a singleton. Then STEP 3.i implies that $S(\phi)$ is an interval. By STEP 5 $\Phi(s) = \{\phi\}$ for all but a countable subset of $S(\phi)$. Then

$$l(s, \phi) = V(s)\, \delta \left[1 - I_{S(\phi)}\right] \text{ for almost all } s \in S(\phi)$$

where δ is the Dirac delta function. But then $q(\phi) = \int_{s \in S(\phi)} A(s, \phi)\, l(s, \phi)\, ds = \infty$, and this is incompatible with an equilibrium (as defined above). In other words, if $S(\phi)$ is not a singleton, then we would have a positive mass of workers producing in a single type of productivity firms which are of mass zero, and this cannot happen in equilibrium.

STEP 7: $\Phi(s)$ is a singleton for all $s \in S$.

I proceed by contradiction. Suppose there is an $s \in S$ such that $\Phi(s)$ is not a singleton. Then STEP 3.ii implies that $\Phi(s)$ is an interval. By STEP 6 $S(\phi) = \{s\}$ for all $\phi \in \Phi(s)$. Now let $\Phi_0 \subseteq \Phi(s)$ be the set of productivity levels that are assign a strictly positive *conditional*[47] mass of s-skill workers. I will show that Φ_0 is at most countable. The total conditional mass of s-skill workers allocated to productivities in Φ_0 can be expressed as

$$\int_{\Phi_0} l(s, \phi)\, d\phi = \int_{\phi^*}^{\overline{\phi}} k(\phi)\, \delta[1 - I_{\Phi_0}] d\phi$$

where δ is the Dirac delta function and $k(\phi)$ is the conditional mass of worker at productivity $\phi \in \Phi_0$. Notice that $\Phi_0 = \cup_{n=1}^{\infty} \{\phi \in \Phi_0 : k(\phi) \geq 1/n\}$ and because of the full employment condition each

[47] Remeber that the mass of workers of a particular skill s is zero. However, conditional on the skill, we can think of $l(s, \phi)$ as the density that represents the distribution of workers with skill s among the firms indexed by the productivity level. Then conditional on skill s, all s-skill workers have a total mass $V(s) > 0$. Then I say that a set $A \subseteq \left[\phi^*, \overline{\phi}\right]$ has possitive *conditional* mass if

$$\int_{\phi \in A} l(s, \phi)\, d\phi > 0.$$

$\{\phi \in \Phi_0 : k(\phi) \geq 1/n\}$ must be finite. Then Φ_0 is at most countable. This means a zero conditional mass of s-skill workers are allocated to almost all $\phi \in \Phi(s)$, which in turn means that $q(\phi) = 0$ for almost all $\phi \in \Phi(s)$. However this is incompatible with equilibrium since for any $p(\phi) \in \mathbb{R}_+$, the demand of variety ϕ (according to (1)) is strictly positive.

Steps 1,6,7 imply that there is a bijection $N : S \rightarrow [\phi^*, \overline{\phi}]$ such that $l(s, \phi) > 0$ if and only if $\phi = N(s)$ and by STEP 2 it must be strictly increasing.

Conditions i-iii. Consider a no-free-entry equilibrium of the closed economy with activity cutoff ϕ^*, wage schedule $w(s)$, price function $p(\phi)$, domestic revenue function $r^d(\phi)$ and matching function $N(s)$. The cost minimization condition (5) and the existence of the matching function N imply that $s = \arg\min_z w(z)/A(z, N(s))$, so $\frac{w(s)}{A(s,N(s))} \leq \frac{w(s+ds)}{A(s+ds,N(s))}$ and $\frac{w(s+ds)}{A(s+ds,N(s+ds))} \leq \frac{w(s)}{A(s,N(s+ds))}$. Combining these inequalities yields

$$\frac{A(s+ds, N(s))}{A(s, N(s))} \leq \frac{w(s+ds)}{w(s)} \leq \frac{A(s+ds, N(s+ds))}{A(s, N(s+ds))},$$

from which we can obtain the differentiability of $w(s)$ and equation (12), after taking logs, dividing by ds and taking limits as $ds \rightarrow 0$.[48] This proves **condition i**.

The pricing rule (7) and the existence of H imply $\phi = \arg\max_\gamma p(\gamma) A(H(\phi), \gamma)$, so

$$\begin{aligned} p(\phi) A(H(\phi), \phi) &\geq p(\phi + d\phi) A(H(\phi), \phi + d\phi), \\ p(\phi + d\phi) A(H(\phi + d\phi), \phi + d\phi) &\geq p(\phi) A(H(\phi + d\phi), \phi), \end{aligned}$$

Combining both inequalities yields

$$\frac{A(H(\phi), \phi + d\phi)}{A(H(\phi), \phi)} \leq \frac{p(\phi)}{p(\phi + d\phi)} \leq \frac{A(H(\phi + d\phi), \phi + d\phi)}{A(H(\phi + d\phi), \phi)}.$$

The differentiability of $p(\phi)$ and condition (13) are obtained taking logs, dividing by ds and taking limits as $ds \rightarrow 0$ in the last expression. Having established the differentiability of $p(\phi)$, the differentiability of $r^d(\phi)$ and condition (14) follow from the definition of $r^d(\phi)$ in (8).

The pricing rule (7) implies that the variable production cost of a firm equals a fraction $(\sigma - 1)/\sigma$ of its revenue. Then, the total wages paid to production workers employed at firms with productivity weakly lower than ϕ must be equal to a fraction $(\sigma - 1)/\sigma$ of the total revenue generated by those firms,

$$\int_{\underline{s}}^{H(\phi)} w(s) V(s) [L - fM] ds = \tfrac{(\sigma-1)}{\sigma} \int_{\underline{\phi}}^{\phi} r^d(\phi') g(\phi') d\phi' \overline{M} \text{ for all } \phi \in [\phi^*, \overline{\phi}]. \tag{25}$$

Due to the continuity of the revenue function $r^d(\phi)$, the right hand side of (25) is a differentiable function of the limit of integration ϕ. Then, the left hand side must also be a differentiable function of ϕ, which together with the continuity of $V(s)$ and $w(s)$, implies that $H(\phi)$ is differentiable. Differentiating (25)

[48] The limits are well defined since all the functions involved are continuous.

with respect to ϕ and using the pricing rule (7) to substitute for the wage $w(s)$ yields condition (15). Concluding the proof of **condition ii**, the boundary conditions on H follow from the definition of the matching function, while the initial condition on $r^d(\phi)$ is just the the zero-profit condition for firms firms with productivity ϕ^*. Finally, **condition iii** follows from equation (25), evaluated at $\phi = \overline{\phi}$, and the numeraire assumption, $\int_{\underline{s}}^{\overline{s}} w(s) V(s) ds = 1$.

Let us turn to the **sufficient conditions for an equilibrium** stated in the last part of the lemma. Suppose that $\{\phi^*, p, r^d, H\}$ satisfy conditions (ii)-(iii) and define $N \equiv H^{-1}$, $M \equiv [1 - G(\phi^*)]\overline{M}$, $w(s) \equiv \frac{\sigma-1}{\sigma} A(s, N(s)) p(N(s))$, $q(\phi) \equiv \frac{r(\phi)}{p(\phi)}$, and $l(s, \phi) \equiv V(s)[L - fM]\delta(\phi - N(s))$, where $\delta(x)$ is the Dirac-delta function. In what follows I show that $\{M, \phi^*, w, p, q, l\}$ is a no-free-entry equilibrium of the closed economy.

The definitions of $w(s)$, M, and $l(s, \phi)$ above immediately imply that the pricing rule (7), condition (10) and the labor market clearing condition (11) are satisfied. The definition of $q(\phi)$ and equation (15) yield an expression for $q(\phi)$ in terms of H and primitives of the model. The same expression is obtained computing the right hand side of (3) using the labor allocation $l(s, \phi)$ constructed here, so condition (3) is satisfied. The initial condition on the function $r^d(\phi)$ in point ii of the lemma implies that the zero-profit condition (9) holds. Using the definition of w above to substitute for p in equation (15), we arrive at (25) after rearranging and integrating on both sides. Evaluating (25) at $\phi = \overline{\phi}$ and using condition iii of the lemma yields $\int_{\underline{s}}^{\overline{s}} w(s) V(s) ds = 1$, so the numeraire condition holds. Finally, the construction of $q(\phi)$ implies that the consumer's budget constraint is satisfied and, together with conditions (13) and (14), implies $q(\phi')/q(\phi) = [p(\phi')/p(\phi)]^{-\sigma}$, so conditions (1) and (2) hold. This concludes the proof of the lemma.

A.1.2 Matching function and Lorenz dominance

Consider two economies A and B with matching functions N^A, N^B and suppose that $N^B(s) > N^A(s)$ for all $s \in [s_0, s_1] \subseteq [\underline{s}, \overline{s}]$. As discussed in the main text, the strict log-supermodularity implies $w^A(s')/w^A(s) < w^B(s')/w^B(s)$, for all $s' > s$ in $[s_0, s_1]$.

In this context, the poorest ρ fraction of workers in the interval $[s_0, s_1]$ is associated with a skill $s(\rho)$ given by

$$\rho = \int_{s_0}^{s(\rho)} V(s) ds \Big/ \int_{s_0}^{s_1} V(s) ds.$$

The Lorenz Curve is then

$$L(\rho) \equiv \int_{s_0}^{s(\rho)} w(s) V(s) ds \Big/ \int_{s_0}^{s_1} w(s) V(s) ds = \frac{\int_{s_0}^{s(\rho)} \frac{w(s)}{w(s(\rho))} V(s) ds}{\int_{s_0}^{s(\rho)} \frac{w(s)}{w(s(\rho))} V(s) ds + \int_{s(\rho)}^{s_1} \frac{w(s)}{w(s(\rho))} V(s) ds}$$

It is readily seen that this implies that $L^A(\rho) > L^B(\rho)$ for all $\rho \in (0,1)$. Finally, from Atkinson (1970) we know that Lorenz dominance is equivalent to *Normalized* Second-Order Stochastic Dominance.

A.2 Section 4

A.2.1 Definition of Equilibrium

Definition 2 *A no-free-entry equilibrium of the open economy is an activity cutoff* ϕ^**, a mass of active firms* $M > 0$*, a mass of exporters* $M^x(\phi) > 0$ *for each productivity level* $\phi \geq \phi^*$*, output functions* $q^d, q^x : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$*, labor allocations functions* $l^d, l^x : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$*, a price function* $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ *and a wage schedule* $w : S \to \mathbb{R}_+$ *such that the following conditions hold,*
(i) consumers behave optimally, equations (1) and (2);
(ii) firms behave optimally given their technology, equations (3), (7), (9), (10) and (18);
(iii) goods and labor markets clear, equations (8), (17) and (19);
(iv) the numeraire assumption holds, $\overline{w} = 1$*.*

A.2.2 Characterization of Equilibrium

Lemma 3 *In a no-free-entry equilibrium of the open economy with activity cutoff* $\phi^* \in (\underline{\phi}, \overline{\phi})$ *the following conditions hold.*
(i) There exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$*, (with inverse function* H*) such that (i)* $l^d(s,\phi) + l^x(s,\phi) > 0$ *if and only if* $N(s) = \phi$*, (ii)* $N(\underline{s}) = \phi^*$*, and* $N(\overline{s}) = \overline{\phi}$*.*
(ii) The wage schedule w *is continuously differentiable and satisfies (12)*
(iii) The price, domestic revenue and matching functions, $\{p, r^d, N\}$*, are continuously differentiable. Given* ϕ^**, the triplet* $\{p, r^d, H\}$ *solves the BVP comprising the system of differential equations {(13), (14), (20)} and the boundary conditions* $r^d(\phi^*) = \sigma f$*,* $H(\phi^*) = \underline{s}$*,* $H(\overline{\phi}) = \overline{s}$*.*
(iv) The activity cutoff ϕ^* *and the revenue function* r^d *satisfy (21).*
Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$*, and functions* $p, r^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ *and* $H : [\phi^*, \overline{\phi}] \to S$ *satisfy the conditions (iii)-(iv), then they are, respectively, the activity cutoff, the price function, the domestic revenue function, and the inverse of the matching function of a no-free-entry equilibrium of the open economy.*

Proof. Adapt arguments in the proof of lemma 1. ■

A.2.3 Proof of Lemma 2

Existence. My approach to prove the existence of a solution to the BVP (22) relies on fixed-point methods. The first step in such an approach is to recast the BVP under consideration as a fixed point problem of some functional operator. To that end, I define the functional Ψ, mapping the space of continuous functions into itself, as follows

$$\Psi(y)(\phi) \equiv s_0 + [s_1 - s_0] \frac{\int_{\phi_0}^{\phi} h(t, y(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} dt}\right) K_1\right] dt}{\int_{\phi_0}^{\phi_1} h(t, y(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} dt}\right) K_1\right] dt}, \tag{26}$$

where

$$h(t, y(t)) \equiv \frac{A(s_0, \phi_0)}{A(y(t), t)} \frac{V(s_0)}{V(y(t))} \frac{g(t)}{g(\phi_0)} \frac{\alpha(t)}{\alpha(\phi_0)}. \tag{27}$$

The following lemma states that the problem of finding a solution to the BVP (22) is equivalent to the problem of finding a fixed point of the functional Ψ.

Claim 1 *A function Γ belongs to a triplet $\{z, x, \Gamma\}$ solving BVP (22) if and only if it is a fixed point of the functional $\Psi : \mathbf{C}[\phi_0, \phi_1] \to \mathbf{C}[\phi_0, \phi_1]$ defined in (26)-(27).*

Proof. Let us start with the "only if" part of the lemma. Let $\{z, x, \Gamma\}$ be a solution to the BVP (22). It can be shown that each of the functions in the solution triplet must be strictly positive, that x and Γ must be strictly increasing, and that z must be strictly decreasing. Then, equation (22c) implies that for any $t \in (\phi_0, \phi_1]$ we can write

$$\begin{aligned}
\Gamma_\phi(t) &= \Gamma_\phi(\phi_0)\, h(t, \Gamma(t)) \frac{x(t)\, z(\phi_0)\left[1 + F(K_0 x(t)) K_1\right]}{x(\phi_0)\, z(t)\left[1 + F(K_0 x(\phi_0)) K_1\right]} \\
&= \Gamma_\phi(\phi_0) \frac{h(t, \Gamma(t))}{\left[1 + F(K_0) K_1\right]} e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du} \left[1 + F(K_0 x(t)) K_1\right],
\end{aligned}$$

where the second line is obtained using equations (22a)-(22b) and $x(\phi_0) = 1$. Integrating $\Gamma_\phi(t)$ between ϕ_0 and ϕ yields

$$\Gamma(\phi) = \Gamma(\phi_0) + \Gamma_\phi(\phi_0) \int_{\phi_0}^{\phi} \frac{h(t, \Gamma(t))}{\left[1 + F(K_0) K_1\right]} e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du} \left[1 + F(K_0 x(t)) K_1\right] dt.$$

Evaluating the last expression at $\phi = \phi_1$, using the boundary conditions on Γ and solving for $\Gamma_\phi(\phi_0)$ we get

$$\Gamma_\phi(\phi_0) = \frac{\left[s_1 - s_0\right]}{\int_{\phi_0}^{\phi_1} \frac{h(t, \Gamma(t))}{\left[1 + F(K_0) K_1\right]} e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du} \left[1 + F(K_0 x(t)) K_1\right] dt}.$$

The last two expressions, $x(t) = e^{(\sigma - 1) \int_{\phi_0}^{t} \frac{\partial \ln A(H(t), t)}{\partial \phi} dt}$, and the definition of Ψ in (26) yield $\Gamma = \Psi(\Gamma)$, i.e. Γ is a fixed point of Ψ.

Let us turn to the "if" part of the lemma. Let Γ be a fixed point of Ψ. If we define $x(\phi) = e^{(\sigma - 1) \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du}$ and $z(\phi) = \frac{[1 + F(K_0) K_1] \alpha(\phi_0) g(\phi_0) \overline{M}}{A(s_0, \phi_0) V(s_0) \Gamma_\phi(\phi_0)} e^{-\int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du}$, then it is easy to verify that $\{z, x, \Gamma\}$ is a solution to BVP (22). ■

Having recasted the BVP (22) as the problem of finding a fixed point of the functional Ψ defined in (26)-(27), the next step is to establish certain properties of this functional that permit the application of some fixed point theorem in the literature. I do so in the next lemma, in which I state that Ψ is a compact self-map on some closed and convex subset of Banach space.

Claim 2 *Let K be the convex and closed subset of $\mathbf{C}[\phi_0, \phi_1]$ given by*

$$K \equiv \{y \in \mathbf{C}[\phi_0, \phi_1] : s_0 \leq y(\phi) \leq s_1 \text{ for all } \phi \in [\phi_0, \phi_1]\}, \tag{28}$$

and let Ψ be the functional defined in (26)-(27). If $\{V, g, \alpha\}$ are continuous and A is continuously differentiable, then Ψ is a compact self-map on K.

Proof. By definition of Ψ, $\Psi(y)(\phi)$ is a strictly increasing function with $\Psi(y)(\phi_0) = s_0$ and $\Psi(y)(\phi_1) = s_1$, so $\Psi(y) \in K$, i.e. Ψ is a self-map on K. To show that Ψ is compact we have to show that $\Psi(K)$ is relatively compact. Per the Arzela-Ascoli theorem, it enough to show that $\Psi(K)$ is bounded and equicontinuous.

Let us start by showing that $\Psi(K)$ is bounded. To simplify notation, let's define the following constants:

$$\begin{aligned} \overline{h} &\equiv \max_{\phi, y \in [\phi_0, \phi_1] \times [s_0, s_1]} h(\phi, y); \qquad \underline{h} \equiv \min_{\phi, y \in [\phi_0, \phi_1] \times [s_0, s_1]} h(\phi, y); \\ \overline{r} &\equiv \max_{\phi, y \in [\phi_0, \phi_1] \times [s_0, s_1]} \tfrac{\partial \ln A(y,\phi)}{\partial \phi}; \qquad \underline{r} \equiv \min_{\phi, y \in [\phi_0, \phi_1] \times [s_0, s_1]} \tfrac{\partial \ln A(y,\phi)}{\partial \phi} \end{aligned}$$

Since $\{A, V, g, \alpha\}$ are continuous and strictly positive on $\Phi \times S \supseteq [\phi_0, \phi_1] \times [s_0, s_1]$, then the constants $\overline{h}$ and $\underline{h}$ are well-defined and are bounded away from zero. Similarly, the assumptions on A implies that $\frac{\partial \ln A(y,\phi)}{\partial \phi}$ is strictly positive and continuous on $\Phi \times S$, so $\overline{r}$ and $\underline{r}$ are also well-defined and bounded away from zero. Then for any $y \in K$ we have

$$|\Psi(y)(\phi)| \leq s_0 + \frac{[s_1 - s_0]}{(\phi_1 - \phi_0)} \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1 - \phi_0)} (1 + K_1)(\phi - \phi_0) \leq s_0 + [s_1 - s_0] \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1 - \phi_0)} (1 + K_1).$$

The last result implies $\|\Psi(y)\|_\infty \leq s_0 + [s_1 - s_0] \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1 - \phi_0)} (1 + K_1)$, and given that the selection of $y \in K$ was arbitrary, we conclude that $\Psi(K)$ is bounded.

Let us now show that $\Psi(K)$ is equicontinuous. For any $y \in K$ and $\phi' > \phi$ we have

$$\begin{aligned} \left|\Psi(y)(\phi') - \Psi(y)(\phi)\right| &\leq [s_1 - s_0] \frac{\int_{\phi}^{\phi'} h(t, \Gamma(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} dt}\right) K_1\right] dt}{\int_{\phi_0}^{\phi_1} h(t, H(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} dt}\right) K_1\right] dt} \\ &\leq \frac{[s_1 - s_0]}{(\phi_1 - \phi_0)} \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1 - \phi_0)} (1 + K_1) \left|\phi' - \phi\right|. \end{aligned}$$

Given that the selection of $y \in K$ was arbitrary, the last inequality implies that $\Psi(K)$ is equicontinuous on $[\phi_0, \phi_1]$. ■

Per the last two claims, the existence of a solution to the BVP (22) can be obtained as a direct application of the Schauder fixed point theorem (SFPT).[49] A function Γ belongs to a triplet $\{z, x, \Gamma\}$ solving BVP (22) if and only if Γ is a fixed point of the functional Ψ defined in (26)-(27). In addition, this functional is a compact self-map on the closed and convex set K defined in (28), so the SFPT implies that Ψ has a fixed point on K. Then, this fixed point is part of a solution to the BVP (22). Finally, the

[49] For a statement of the SFPT see O'Reagan (1997), OK (20xx), Granas, Gunther and Lee (1985).

continuity of $\{A, V, g, \alpha, F\}$ and (22c) implies that Γ is continuously differentiable.

Uniqueness. I start by proving an intermediate result that is used later. The continuous differentiability of $\{V, g, \alpha, F\}$ and the twice continuous differentiability of A imply that the right-hand side of equations (22a)-(22c) are locally Lipschitz continuous with respect to $\{z, x, \Gamma\}$, as the relevant partial derivatives are bounded on bounded sets. Then, the initial value problem (IVP) given by the differential equations (22a)-(22c) and the initial conditions $x(\phi_0) = 1$, $\Gamma(\phi_0) = s_0$, $z(\phi_0) = z_0$, has at most one solution.

Let us turn to the uniqueness of the solution to the BVP (22). I proceed by contradiction. Suppose that there are two different solutions $\{z', x', \Gamma'\}$ and $\{z, x, \Gamma\}$ to the BVP (22). Then, the uniqueness result in the previous paragraph implies that $z'(\phi_0) \neq z(\phi_0)$, which, together with equation (22c), implies $\Gamma'_\phi(\phi_0) \neq \Gamma_\phi(\phi_0)$. Without loss of generality suppose $\Gamma'_\phi(\phi_0) < \Gamma_\phi(\phi_0)$, i.e. $\Gamma(\phi) > \Gamma'(\phi)$ in some neighborhood (ϕ_0, c), with $c > \phi_0$. By assumption, we know that the functions Γ' and Γ must intersect at least once again on $(\phi_0, \phi_1]$, since $\Gamma(\phi_1) = \Gamma'(\phi_1)$. Let ϕ^+ be the first value to the right of ϕ_0 at which the functions Γ' and Γ intersect, i.e. $\phi^+ \equiv \inf\{\phi \in (\phi_0, \phi_1] : \Gamma'(\phi) = \Gamma(\phi)\}$, and notice that ϕ^+ is well-defined since Γ' and Γ are continuous. Given our assumptions, $\Gamma(\phi) > \Gamma'(\phi)$ for $\phi \in \left(\phi_0, \phi^+\right)$, which together with $\Gamma\left(\phi^+\right) = \Gamma'\left(\phi^+\right)$, implies that $\Gamma'_\phi\left(\phi^+\right) \geq \Gamma_\phi\left(\phi^+\right)$. This and $\Gamma'_\phi(\phi_0) < \Gamma_\phi(\phi_0)$ imply

$$\frac{\Gamma'_\phi\left(\phi^+\right)/\Gamma'_\phi(\phi_0)}{\Gamma_\phi\left(\phi^+\right)/\Gamma_\phi(\phi_0)} > 1. \tag{29}$$

As discussed above, Γ' and Γ are fixed points of the functional Ψ defined in (26), $Z(\phi) = \Psi(Z)(\phi)$, for $Z = \Gamma', \Gamma$, so $Z_\phi(\phi)$ can be obtained differentiating the right-hand side of (26). Doing so yields,

$$Z_\phi\left(\phi^+\right)/Z_\phi(\phi_0) = h\left(\phi^+, Z\left(\phi^+\right)\right) e^{\sigma \int_{\phi_0}^{\phi^+} \frac{\partial \ln A(Z(u),u)}{\partial \phi} du} \frac{\left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{\phi^+} \frac{\partial \ln A(Z(u),u)}{\partial \phi} du}\right) K_1\right]}{\left[1 + F(K_0) K_1\right]},$$

for $Z = \Gamma', \Gamma$. Combining the last expression for both functions yields

$$\frac{\Gamma'_\phi\left(\phi^+\right)/\Gamma'_\phi(\phi_0)}{\Gamma_\phi\left(\phi^+\right)/\Gamma_\phi(\phi_0)} = e^{\sigma \int_{\phi_0}^{\phi^+} \left[\frac{\partial \ln A(\Gamma'(u),u)}{\partial \phi} - \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi}\right] du} \frac{\left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{\phi^+} \frac{\partial \ln A(\Gamma'(u),u)}{\partial \phi} du}\right) K_1\right]}{\left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{\phi^+} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} du}\right) K_1\right]} < 1, \tag{30}$$

where in the last expression I used the fact that $\Gamma'\left(\phi^+\right) = \Gamma\left(\phi^+\right)$, so $h\left(\phi^+, \Gamma'\left(\phi^+\right)\right) = h\left(\phi^+, \Gamma\left(\phi^+\right)\right)$. The log-supermodularity of A, $\Gamma(\phi) > \Gamma'(\phi)$ for $\phi \in \left(\phi_0, \phi^+\right)$ and the fact that F strictly increasing imply that each of the terms on the right-hand side of the last expression is strictly less than 1. However, note that equation (30) contradicts equation (29), so it must be the case that there is only one solution to the BVP (22).

Condition i. Let $\left\{z^i, x^i, \Gamma^i\right\}$ be the unique solution to BVP (22) with $K_1 = 0$ and $s_0 = s_0^i$, for $i = a, b$ and $s_0^a > s_0^b$. To prove the result we show that if Γ^a and Γ^b intersect at some point $\phi^+ \in (\phi_0, \phi_1)$, then there are functions y^i and w^i for $i = a, b$, such that $\{w^a, y^a, \Gamma^a\}$ and $\left\{w^b, y^b, \Gamma^b\right\}$ solve the same IVP on $[\phi_0, \phi_1]$ given by the system (22a)-(22c) and the same initial value at any $\phi \in \left(\phi_+, \phi_1\right)$. Then, the uniqueness result proved at the beginning of the previous section implies that $\{y^a, w^a, \Gamma^a\} = \left\{y^b, w^b, \Gamma^b\right\}$ on $[\phi_0, \phi_1]$, contradicting the initial initial assumption $s_0^a > s_0^b$.

Suppose that there is a $\phi^+ \in (\phi_0, \phi_1)$ and $\Gamma^a\left(\phi^+\right) = \Gamma^b\left(\phi^+\right) \equiv s^+$. If we define the functions $y^i, w^i : [\phi_0, \phi_1] \to \mathbb{R}_+$ as $y^i(\phi) = z^i(\phi)/x^i\left(\phi^+\right)$, $w^i = x^i(\phi)/x^i\left(\phi^+\right)$, it is readily seen that on $\left[\phi^+, \phi_1\right]$ and for $i = a, b$, $\left\{y^i, w^i, \Gamma^i\right\}$ is a solution to the BVP given by the system of differential equations (22a)-(22c) and boundary conditions $w(\phi^+) = 1$, $\Gamma(\phi^+) = s^+$, $\Gamma(\phi_1) = s_1$. As this BVP is just a particular case of BVP (22), it has a unique solution, implying that $\{y^a, w^a, \Gamma^a\} = \left\{y^b, w^b, \Gamma^b\right\}$ on $\left[\phi^+, \phi_1\right]$. Moreover, this result implies that $\{w^a, y^a, \Gamma^a\}$ and $\left\{w^b, y^b, \Gamma^b\right\}$ solve the same IVP on $[\phi_0, \phi_1]$ given by the system (22a)-(22c) and the same initial conditions at any $\phi \in \left(\phi_+, \phi_1\right)$, which is the desired result. The no-crossing result related to the inverses of Γ^i can be establish in a similar way.

Condition ii. Let $\phi_0^a > \phi_0^b$ and suppose that $x^a(\phi) \equiv x(\phi; \phi_0^a) \geq x\left(\phi; \phi_0^b\right) \equiv x^b(\phi)$ for some ϕ on $[\phi_0^a, \phi_1]$. From their definitions, it is clear that $x^a(\phi_0^a) < x^b(\phi_0^a)$, so let ϕ' be the first productivity level such that $x^a(\phi) = x^b(\phi)$. Notice that ϕ' is well defined due to the continuity of the functions involved and due to our initial assumption. By definition of ϕ', we have $x^a\left(\phi'\right) = x^b\left(\phi'\right)$ and $x^a(\phi) < x^b(\phi)$ for $\phi < \phi'$. This means that $x^a(\phi)$ is catching up to $x^b(\phi)$, so this and the log-supermodularity of A imply that there is a $\phi'' < \phi'$, such that $\Gamma^a(\phi) > \Gamma^b(\phi)$ on $\left(\phi'', \phi'\right)$, i.e. Γ^a and Γ^b must intersect at least once strictly to left of ϕ'. Let ϕ_- denote the productivity level corresponding to the first intersection of Γ^a and Γ^b that is **strictly** to the left of ϕ'. Notice that ϕ_- is well defined —due to the continuity of the functions Γ^i and the fact that $\Gamma^a(\phi_0^a) < \Gamma^b(\phi_0^a)$— and that $\phi_- < \phi'$. Similarly, let ϕ_+ denote the productivity level corresponding to the first intersection of Γ^a and Γ^b that is **weakly** to the right of ϕ'. Notice that ϕ_+ is also well defined and that $\phi_+ \geq \phi'$.

From the definitions above we have $\Gamma^a\left(\phi_-\right) = \Gamma^b\left(\phi_-\right)$, $\Gamma^a(\phi) > \Gamma^b(\phi)$ on $\left(\phi_-, \phi_+\right)$ and $\Gamma^a\left(\phi_+\right) = \Gamma^b\left(\phi_+\right)$. Then $\Gamma^a_\phi\left(\phi_-\right) \geq \Gamma^b_\phi\left(\phi_-\right)$ and $\Gamma^a_\phi\left(\phi_+\right) \leq \Gamma^b_\phi\left(\phi_+\right)$, so

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} \leq 1. \tag{31}$$

As discussed above, we can differentiate the right-hand side of (26) to get

$$\frac{\Gamma^i_\phi\left(\phi_+\right)}{\Gamma^i_\phi\left(\phi_-\right)} = h^i\left(\phi_-, \phi_+\right) e^{\sigma \int_{\phi_-}^{\phi_+} \frac{\partial \ln A\left(\Gamma^i(t), t\right)}{\partial \phi} dt} \frac{\left[1 + F\left(K_0 x^i\left(\phi_+\right)\right) K_1\right]}{\left[1 + F\left(K_0 x^i\left(\phi_-\right)\right) K_1\right]} \text{ for } i = a, b,$$

where h is defined in (27). This implies

$$\begin{aligned}\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} &= e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^a(t),t\right)}{\partial\phi}-\frac{\partial \ln A\left(\Gamma^b(t),t\right)}{\partial\phi}\right]dt}\frac{\left[1+F\left(K_0x^a\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^a\left(\phi_-\right)\right)K_1\right]}{\left[1+F\left(K_0x^b\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^b\left(\phi_-\right)\right)K_1\right]}\\ &> \frac{x^a\left(\phi_+\right)/x^a\left(\phi_-\right)}{x^b\left(\phi_+\right)/x^b\left(\phi_-\right)}\frac{\left[1+F\left(K_0x^a\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^a\left(\phi_-\right)\right)K_1\right]}{\left[1+F\left(K_0x^b\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^b\left(\phi_-\right)\right)K_1\right]}, \qquad (32)\end{aligned}$$

where the second line is obtained multiplying the right-hand side by $\exp\int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^b(t),t\right)}{\partial\phi}-\frac{\partial \ln A\left(\Gamma^a(t),t\right)}{\partial\phi}\right]<1$. Per our definitions we have $x^a\left(\phi'\right)=x^b(\phi')$, $x^a\left(\phi_+\right)\geq x^b(\phi_+)$ ($\Gamma^a\left(\phi\right)\geq\Gamma^b\left(\phi\right)$ on $\left[\phi',\phi_+\right]$), and $x^a\left(\phi_-\right)<x^b(\phi_-)$, which together with (32), imply

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)}>1,$$

contradicting (31). Then, it must be the case that $x^a\left(\phi\right)<x^b\left(\phi\right)$ for all $\phi\in[\phi^a_0,\phi_1]$, which is the desired result.

A.2.4 Proof of Proposition 1

The proof of the existence and uniqueness of the equilibrium in the closed and open economies was laid out in the text. Here, I prove the (constrained) efficiency of the equilibrium, starting with the simpler closed-economy case.

Efficiency of the Equilibrium of the Closed Economy

Below I show that an allocation is an equilibrium of the closed economy if and only if it is a solution to the planner's problem

$$\begin{gathered}\max_{\phi^*,\widetilde{q}(\phi),\widetilde{H}(\phi)}\int_{\phi^*}^{\overline{\phi}}\widetilde{q}\left(\phi\right)^{\frac{\sigma-1}{\sigma}}g\left(\phi\right)\overline{M}d\phi\\ \text{subject to}\\ \int_{\phi^*}^{\phi}\frac{\widetilde{q}(\phi')}{A(\widetilde{H}(\phi'),\phi')}g\left(\phi'\right)d\phi'\overline{M}=\int_{\underline{s}}^{\widetilde{H}(\phi)}V(s)ds\left[L-f\left[1-G\left(\phi^*\right)\right]\overline{M}\right]\ \text{ for all }\phi\in\left[\phi^*,\overline{\phi}\right],\\ \widetilde{H}(\phi^*)=\underline{s};\ \widetilde{H}(\overline{\phi})=\overline{s},\end{gathered}\qquad (33)$$

where the left- and right-hand sides of the integral equation represent, respectively, the total mass of workers required to produce $\widetilde{q}\left(\phi'\right)$ units of each variety with productivity below ϕ, and the total mass of workers employed in the production of said varieties. Differentiating both sides of the integral equation above with respect to ϕ yields the the following ordinary differential equation (ODE) for all $\phi\in\left[\phi^*,\overline{\phi}\right]$,

$$\widetilde{H}_\phi\left(\phi\right)=\frac{\widetilde{q}\left(\phi\right)g\left(\phi\right)\overline{M}}{A(\widetilde{H}\left(\phi\right),\phi)V(\widetilde{H}\left(\phi\right))\left[L-f\left[1-G\left(\phi^*\right)\right]\overline{M}\right]}\equiv h^H(\phi^*,\widetilde{q}\left(\phi\right),\widetilde{H}\left(\phi\right),\phi). \qquad (34)$$

Moreover, if (34) is satisfied for all $\phi \in \left[\phi^*, \overline{\phi}\right]$, then we can recover the integral equation above by moving $V(\widetilde{H}(\phi))\left[L - f\left[1 - G(\phi^*)\right]\right]$ to the left-hand side before integrating both sides of the resulting expression between $\left[\phi^*, \phi'\right]$ for each ϕ'. That is, the integral equation is equivalent to the ODE in (34), with the latter being the version of the constraint I consider below.

Following chapter 9 of Luenberger (1969), if $\{\phi^*, \widetilde{q}, \widetilde{H}\}$ solves problem (33), then there is a function of bounded variation, λ^H, and a real number, μ^H, such that the Lagrangian,

$$L(\phi^*, \widetilde{q}, \widetilde{H}) = \int_{\phi^*}^{\overline{\phi}} \widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\, d\phi \overline{M} + \int_{\phi^*}^{\overline{\phi}} \left[\widetilde{H}(\phi) - \underline{s} - \int_{\phi^*}^{\phi} h^H(\phi^*, \widetilde{q}(\phi'), \widetilde{H}(\phi'), \phi') d\phi'\right] d\lambda^H(\phi) + \mu^H \left[\widetilde{H}(\overline{\phi}) - \overline{s}\right]$$

is stationary at $\{\phi^*, \widetilde{q}, \widetilde{H}\}$. Integrating by parts the term involving a double integral and using the fact that λ^H is differentiable, the Lagrangian can be expressed as[50]

$$L(\phi^*, \widetilde{q}, \widetilde{H}) = \left\{ \begin{array}{c} \int_{\phi^*}^{\overline{\phi}} \widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\, d\phi \overline{M} + \int_{\phi^*}^{\overline{\phi}} \widetilde{H}(\phi) \lambda^H_\phi(\phi)\, d\phi + \lambda^H(\phi^*)\underline{s} - \widetilde{H}(\overline{\phi}) \lambda^H(\overline{\phi}) + \cdots \\ \cdots \int_{\phi^*}^{\overline{\phi}} h^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi) d\phi + \mu^H \left[H(\overline{\phi}) - \overline{s}\right] \end{array} \right.$$

The stationarity condition, together with the constraints of the problem, yields the following first order necessary conditions for an optimum

$$\begin{gathered} \widetilde{H}_\phi(\phi) = h^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi) \\ h^H_H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi) + \lambda^H_\phi(\phi) = 0 \\ \tfrac{\sigma-1}{\sigma} \widetilde{q}(\phi)^{-\frac{1}{\sigma}} g(\phi)\overline{M} + h^H_q(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi) = 0 \\ \left[\mu^H - \lambda^H(\overline{\phi})\right] = 0 \\ \widetilde{H}(\phi^*) = \underline{s}, \quad \widetilde{H}(\overline{\phi}) = \overline{s} \\ \int_{\phi^*}^{\overline{\phi}} h^H_{\phi^*}(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi) d\phi = \widetilde{q}(\phi^*)^{\frac{\sigma-1}{\sigma}} g(\phi^*)\overline{M} + h^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi^*). \end{gathered} \tag{35}$$

The first five lines in (35) are the standard necessary conditions of optimal control theory and reflect the constraints of the problem and the implications of stationarity of the Lagrangian with respect to $\{\widetilde{H}, \widetilde{q}\}$. The last line in (35) follows from the stationarity with respect to ϕ^*. Below I show that if $\{\phi^*, \widetilde{q}, \widetilde{H}, \lambda^H\}$ satisfies (35), then we can define functions $\{\widetilde{p}(\phi), \widetilde{r}(\phi)\}$ such that $\{\phi^*, \widetilde{p}(\phi), \widetilde{r}(\phi), \widetilde{H}\}$ satisfy the conditions of lemma 1, proving that a solution to the planner's problem is an equilibrium of the closed economy.

Let $\{\phi^*, \widetilde{q}, \widetilde{H}, \lambda^H\}$ satisfy the conditions in (35). For some (still undefined) positive constant p_0, define

$$\widetilde{p}(\phi) \equiv p_0 \tfrac{\sigma}{\sigma-1} \frac{-\lambda^H(\phi)}{A(\widetilde{H}(\phi), \phi)V(\widetilde{H}(\phi))}, \tag{36}$$

[50] See section 9.5 of Luenberger (1969) for a derivation of the differentiability of λ^H.

which, together with (35), implies

$$\widetilde{p}_{\phi}(\phi) = -\widetilde{p}(\phi) \frac{\partial \ln A(\widetilde{H}(\phi), \phi)}{\partial \phi} \tag{37}$$

Using (36) in the third line of (35) yields

$\widetilde{q}(\phi) = p_0^{\sigma} \left[L - f\left[1 - G(\phi^*)\right] \overline{M}\right]^{\sigma} \widetilde{p}(\phi)^{-\sigma}$,

so defining

$$\widetilde{r}(\phi) \equiv \widetilde{q}(\phi) \widetilde{p}(\phi) \tag{38}$$

implies

$$\begin{gathered} \widetilde{r}(\phi) = p_0^{\sigma} \left[L - f\left[1 - G(\phi^*)\right] \overline{M}\right]^{\sigma} \widetilde{p}(\phi)^{1-\sigma}, \\ \widetilde{r}(\phi) = p_0 \left[L - f\left[1 - G(\phi^*)\right] \overline{M}\right] \widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}} \end{gathered} \tag{39}$$

and

$$\widetilde{r}_{\phi}(\phi) = (\sigma - 1) \widetilde{r}_{\phi}(\phi) \frac{\partial \ln A(\widetilde{H}(\phi), \phi)}{\partial \phi} \tag{40}$$

With these definitions, the the first line of (35) can be expressed as

$$\widetilde{H}_{\phi}(\phi) = \frac{\widetilde{r}(\phi) g(\phi) \overline{M}}{A(\widetilde{H}(\phi), \phi) V(\widetilde{H}(\phi)) \widetilde{p}(\phi) \left[L - f\overline{M}\left[1 - G(\phi^*)\right]\right]}. \tag{41}$$

Finally, noting that the third line in (35) implies $\frac{\sigma-1}{\sigma} \widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi) \overline{M} = -\widetilde{H}(\phi) \lambda^H(\phi)$, the last line in (35) can be expressed as

$\int_{\phi^*}^{\overline{\phi}} -\widetilde{H}(\phi) \lambda^H(\phi) d\phi \frac{f\overline{M} g(\phi^*)}{\left[L - f\left[1 - G(\phi^*)\right] \overline{M}\right]} = \widetilde{q}(\phi^*)^{\frac{\sigma-1}{\sigma}} g(\phi^*) \overline{M} + \widetilde{H}(\phi^*) \lambda^H(\phi^*)$,

$\int_{\phi^*}^{\overline{\phi}} \left[\frac{\widetilde{q}(\phi)}{\widetilde{q}(\phi^*)}\right]^{\frac{\sigma-1}{\sigma}} g(\phi) d\phi \sigma f \overline{M} = \frac{\sigma}{\sigma-1} \left[L - f\left[1 - G(\phi^*)\right] \overline{M}\right]$,

$$\sigma f \int_{\phi^*}^{\overline{\phi}} \frac{\widetilde{r}(\phi)}{\widetilde{r}(\phi^*)} g(\phi) d\phi \overline{M} = \frac{\sigma}{\sigma - 1} \left[L - f\left[1 - G(\phi^*)\right] \overline{M}\right], \tag{42}$$

where the derivation uses (39). If we choose the constant p_0 in (36) such that $\widetilde{r}(\phi^*) = \sigma f$, then the last equation can be expressed as

$$\int_{\phi^*}^{\overline{\phi}} \widetilde{r}(\phi) g(\phi) d\phi \overline{M} = \frac{\sigma}{\sigma - 1} \left[L - f\left[1 - G(\phi^*)\right]\right] \tag{43}$$

Note that conditions $\{(37),(40),(41),(43)\}$, $\{\widetilde{H}(\phi^*) = \underline{s}, \widetilde{H}(\overline{\phi}) = \overline{s}\}$, and $\widetilde{r}(\phi^*) = \sigma f$ are identical to those in lemma 1, so $\{\widetilde{p}, \widetilde{r}, \widetilde{H}\}$ are the price, revenue and inverse matching functions corresponding to the closed economy equilibrium.

On the other direction, let $\left\{\phi_a^*, p, r^d, H\right\}$ be the activity cutoff, price, revenue and inverse matching functions of the closed economy equilibrium, with output function $q^d(\phi) = r^d(\phi) / p(\phi)$. As $\left\{p, r^d, H\right\}$ satisfy the ODE (15), then $\left\{q^d, H\right\}$ satisfy the first condition in (35). Define $\lambda \equiv -\lambda_0 \frac{\sigma-1}{\sigma} p(\phi) A(H(\phi), \phi) V(H(\phi))$

for some positive constant λ_0. Log-differentiating $-\lambda$, together with equilibrium condition (13), yields the second line in (35). Using these definitions in the third condition of (35) yields

$$q^d\left(\phi_a^*\right)^{\frac{\sigma-1}{\sigma}}\left[\frac{q^d\left(\phi\right)}{q^d\left(\phi_a^*\right)}\right]^{\frac{\sigma-1}{\sigma}}-\frac{r^d\left(\phi\right)}{\left[L-f\left[1-G\left(\phi_a^*\right)\right]\overline{M}\right]}\lambda_0=0.$$

Recalling that the CES demand system implies $r^d\left(\phi\right)=Bq^d\left(\phi\right)^{\frac{\sigma-1}{\sigma}}$ for some constant B, the last expression holds for

$$\lambda_0=\frac{\left[L-f\left[1-G\left(\phi_a^*\right)\right]\overline{M}\right]q^d\left(\phi^*\right)^{\frac{\sigma-1}{\sigma}}}{r^d\left(\phi_a^*\right)}.$$

Finally, the derivations above imply that $\left\{q^d,H,\lambda,\phi_a^*\right\}$ satisfy the last line in (35) if and only if they satisfy equation (42), a fact that follows from the zero profit condition $r^d\left(\phi_a^*\right)=\sigma f$ and the numeraire condition (16). Accordingly, $\left\{q^d,H,\lambda,\phi_a^*\right\}$ solves the planner's problem.

Efficiency of the Equilibrium of the Open Economy

In this section, I show that an allocation is an equilibrium of the open economy if and only if it is a solution to the planner's problem

$$\max_{\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}}\int_{\phi^*}^{\overline{\phi}}\widetilde{q}^d\left(\phi\right)^{\frac{\sigma-1}{\sigma}}g\left(\phi\right)\overline{M}d\phi+n\int_{\phi^*}^{\overline{\phi}}\widetilde{q}^x\left(\phi\right)^{\frac{\sigma-1}{\sigma}}F\left(\widetilde{y}\left(\phi\right)\right)g\left(\phi\right)\overline{M}d\phi$$

subject to

$$\begin{gathered}\int_{\phi^*}^{\phi}\frac{\widetilde{q}^d(\phi')}{A(\widetilde{H}(\phi'),\phi')}g\left(\phi'\right)d\phi'\overline{M}+nf_x\int_{\phi^*}^{\phi}\frac{\widetilde{q}^x(\phi')\tau}{A(\widetilde{H}(\phi'),\phi')}F\left(\widetilde{y}\left(\phi'\right)\right)g\left(\phi'\right)\overline{M}d\phi'=\cdots\\ \cdots\int_{\underline{s}}^{\widetilde{H}(\phi)}V(s)dsL^{pw}(\phi^*,\widetilde{y})\text{ for all }\phi\in\left[\phi^*,\overline{\phi}\right],\\ \widetilde{H}(\phi^*)=\underline{s};\ \widetilde{H}(\overline{\phi})=\overline{s}.\end{gathered}\tag{44}$$

where $L^{pw}(\phi^*,\widetilde{y})$ represents the mass of production workers,

$$L^{pw}(\phi^*,\widetilde{y})=\left[L-f\left[1-G\left(\phi^*\right)\right]\overline{M}-nf_x\int_{\phi^*}^{\overline{\phi}}\int_{y}^{\widetilde{y}(\phi')}ydF(y)g\left(\phi'\right)\overline{M}d\phi'\right]$$

As explained in the case of the planner's problem for the closed economy, the integral constraint is equivalent to the following ODE,

$$\begin{gathered}\widetilde{H}_\phi\left(\phi\right)=h^H(\phi^*,\widetilde{q}^d\left(\phi\right),\widetilde{q}^x\left(\phi\right),\widetilde{H}\left(\phi\right),\widetilde{y}\left(\phi\right),\phi),\\ h^H(\cdots,\phi)\equiv\frac{\left[\widetilde{q}^d\left(\phi\right)+\widetilde{q}^x\left(\phi\right)F\left(\widetilde{y}\left(\phi\right)\right)\tau n\right]g\left(\phi\right)\overline{M}}{A(\widetilde{H}\left(\phi\right),\phi)V(\widetilde{H}\left(\phi\right))L^{pw}(\phi^*,\widetilde{y})}.\end{gathered}\tag{45}$$

Following chapter 9 of Luenberger (1969), if $\{\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}\}$ solves problem (44), then there is a function of bounded variation, λ^H, and a real number, μ^H, such that the Lagrangian,

$$L(\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}) = \int_{\phi^*}^{\overline{\phi}} \widetilde{q}^d(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\overline{M}d\phi + n\int_{\phi^*}^{\overline{\phi}} \widetilde{q}^x(\phi)^{\frac{\sigma-1}{\sigma}} F(\widetilde{y}(\phi))\, g(\phi)\overline{M}d\phi + \cdots$$
$$\cdots \int_{\phi^*}^{\overline{\phi}} \left[\widetilde{H}(\phi) - \underline{s} - \int_{\phi^*}^{\phi} h^H(\cdots,\phi')d\phi'\right] d\lambda^H(\phi) + \mu^H\left[\widetilde{H}\left(\overline{\phi}\right) - \overline{s}\right]$$

is stationary at $\{\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H}(\phi),\widetilde{y}\}$. Integrating by parts the term involving a double integral and using the fact that λ^H is differentiable, the Lagrangian can be expressed as

$$L(\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}) = \begin{cases} \int_{\phi^*}^{\overline{\phi}} \widetilde{q}^d(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\overline{M}d\phi + n\int_{\phi^*}^{\overline{\phi}} \widetilde{q}^x(\phi)^{\frac{\sigma-1}{\sigma}} F(\widetilde{y}(\phi))\, g(\phi)\overline{M}d\phi + \cdots \\ \cdots \int_{\phi^*}^{\overline{\phi}} \widetilde{H}(\phi)\lambda_\phi^H(\phi)\,d\phi + \lambda^H(\phi^*)\underline{s} - \widetilde{H}\left(\overline{\phi}\right)\lambda^H(\overline{\phi}) + \cdots \\ \cdots \int_{\phi^*}^{\overline{\phi}} h^H(\cdots,\phi)\lambda^H(\phi)d\phi + \mu^H\left[H\left(\overline{\phi}\right) - \overline{s}\right] \end{cases}$$

The stationarity condition and the constraints of the problem yield the following first order necessary conditions for an optimum

$$\begin{gathered} \widetilde{H}_\phi(\phi) = h^H(\cdots,\phi) \\ h_H^H(\cdots,\phi)\lambda^H(\phi) + \lambda_\phi^H(\phi) = 0 \\ \tfrac{\sigma-1}{\sigma}\widetilde{q}^d(\phi)^{-\frac{1}{\sigma}} g(\phi)\overline{M} + h_{q^d}^H(\cdots,\phi)\lambda^H(\phi) = 0 \\ \tfrac{\sigma-1}{\sigma}\widetilde{q}^x(\phi)^{-\frac{1}{\sigma}} \tau n F(\widetilde{y}(\phi))\, g(\phi)\overline{M} + h_{q^x}^H(\cdots,\phi)\lambda^H(\phi) = 0 \\ n\widetilde{q}^x(\phi)^{\frac{\sigma-1}{\sigma}} F_y(\widetilde{y}(\phi))\, g(\phi)\overline{M}d\phi + \frac{F_y(\widetilde{y}(\phi))\,\tau^{1-\sigma}n}{[1+F(\widetilde{y}(\phi))\,\tau^{1-\sigma}n]} h^H(\cdots,\phi)\lambda^H(\phi) + \cdots \\ \cdots \tfrac{\int_{\phi^*}^{\overline{\phi}} h^H(\cdots,\phi)\lambda^H(\phi)d\phi}{L^{pw}(\phi^*,\widetilde{y})} n f_x \widetilde{y}(\phi)\, F_y(\widetilde{y}(\phi))\, g(\phi)\overline{M} = 0 \\ \left[\mu^H - \lambda^H\left(\overline{\phi}\right)\right] = 0 \\ \widetilde{H}(\phi^*) = \underline{s}, \quad \widetilde{H}(\overline{\phi}) = \overline{s} \\ \int_{\phi^*}^{\overline{\phi}} h_{\phi^*}^H(\cdots,\phi)\lambda^H(\phi)d\phi = \left[\widetilde{q}^d(\phi^*)^{\frac{\sigma-1}{\sigma}} + n\widetilde{q}^x(\phi^*)^{\frac{\sigma-1}{\sigma}} F(\widetilde{y}(\phi^*))\right] g(\phi^*)\overline{M} + h^H(\cdots,\phi^*)\lambda^H(\phi^*). \end{gathered} \tag{46}$$

The first seven lines in (46) are the standard necessary conditions of optimal control theory and reflect the constraints of the problem and the implications of stationarity of the Lagrangian with respect to $\{\widetilde{H},\widetilde{q}^d,\widetilde{q}^x,\widetilde{y}\}$. The last line in (46) follows from the stationarity with respect to ϕ^*. Below, I show that if $\{\phi^*,\widetilde{H},\widetilde{q}^d,\widetilde{q}^x,\widetilde{y},\lambda^H\}$ satisfies (46), then we can define functions $\{\widetilde{p}(\phi),\widetilde{r}(\phi)\}$ such that $\{\phi^*,\widetilde{p}(\phi),\widetilde{r}(\phi),\widetilde{H}\}$ satisfy the conditions of lemma 3 in the appendix, proving that a solution to the planner's problem is an equilibrium of the open economy.

Let $\{\phi^*,\widetilde{H},\widetilde{q}^d,\widetilde{q}^x,\widetilde{y},\lambda^H\}$ satisfy the conditions in (46). The third and fourth lines in (46) yield $\widetilde{q}^x(\phi) = \widetilde{q}^d(\phi)\,\tau^{-\sigma}$. For some (still undefined) positive constant p_0, define

$$\widetilde{p}(\phi) \equiv p_0 \tfrac{\sigma}{\sigma-1} \frac{-\lambda^H(\phi)}{A(\widetilde{H}(\phi),\phi)V(\widetilde{H}(\phi))}, \tag{47}$$

which, together with the second line in (46), implies

$$\widetilde{p}_{\phi}\left(\phi\right)=-\widetilde{p}\left(\phi\right)\frac{\partial\ln A(\widetilde{H}\left(\phi\right),\phi)}{\partial\phi} \tag{48}$$

Using (47) in the third condition in (46) yields

$\widetilde{q}\left(\phi\right)=p_0^{\sigma}L^{pw}(\phi^{*},\widetilde{y})^{\sigma}\widetilde{p}\left(\phi\right)^{-\sigma}$.

Accordingly, defining

$$\widetilde{r}^{d}\left(\phi\right)\equiv\widetilde{q}^{d}\left(\phi\right)\widetilde{p}\left(\phi\right) \tag{49}$$

we get

$$\begin{gathered}\widetilde{r}^{d}\left(\phi\right)=p_0^{\sigma}\left(L^{pw}(\phi^{*},\widetilde{y})\right)^{\sigma}\widetilde{p}\left(\phi\right)^{1-\sigma},\\ \widetilde{r}^{d}\left(\phi\right)=p_0L^{pw}(\phi^{*},\widetilde{y})\widetilde{q}^{d}\left(\phi\right)^{\frac{\sigma-1}{\sigma}}\end{gathered} \tag{50}$$

and

$$\widetilde{r}_{\phi}\left(\phi\right)=\left(\sigma-1\right)\widetilde{r}_{\phi}\left(\phi\right)\frac{\partial\ln A(\widetilde{H}\left(\phi\right),\phi)}{\partial\phi} \tag{51}$$

Noting that the third condition in (46) yields

$\frac{\sigma-1}{\sigma}\widetilde{q}^{d}\left(\phi\right)^{\frac{\sigma-1}{\sigma}}\left[1+F\left(\widetilde{y}\left(\phi\right)\right)\tau^{1-\sigma}n\right]g\left(\phi\right)\overline{M}=-h^{H}(\cdots,\phi)\lambda^{H}\left(\phi\right),$

the fifth line in (46) implies

$$\begin{gathered}\widetilde{y}\left(\phi\right)=\frac{\widetilde{r}^{d}\left(\phi\right)\tau^{1-\sigma}}{\sigma f_xC_0},\\ C_0\equiv\frac{\sigma-1}{\sigma}\frac{\int_{\phi^{*}}^{\overline{\phi}}\widetilde{r}^{d}(\phi)\left[1+F(\widetilde{y}(\phi))\tau^{1-\sigma}n\right]g(\phi)\overline{M}d\phi}{L^{pw}(\phi^{*},\widetilde{y})}.\end{gathered} \tag{52}$$

With the derivations above in mind, the first condition in (46) becomes

$$\widetilde{H}_{\phi}\left(\phi\right)=\frac{\widetilde{r}^{d}\left(\phi\right)\left[1+F\left(\frac{\widetilde{r}^{d}(\phi)\tau^{1-\sigma}}{\sigma f_xC_0}\right)\tau^{1-\sigma}n\right]g\left(\phi\right)\overline{M}}{A(\widetilde{H}\left(\phi\right),\phi)V(\widetilde{H}\left(\phi\right))\widetilde{p}\left(\phi\right)L^{pw}(\phi^{*},\frac{\widetilde{r}^{d}(\phi)\tau^{1-\sigma}}{\sigma f_xC_0})} \tag{53}$$

Finally, using the previous observations and

$\int_{\phi^{*}}^{\overline{\phi}}h_{\phi^{*}}^{H}(\cdots,\phi)\lambda^{H}(\phi)d\phi=-\int_{\phi^{*}}^{\overline{\phi}}h^{H}(\cdots,\phi)\lambda^{H}(\phi)\frac{fg(\phi^{*})+nf_x\int_{y}^{\widetilde{y}(\phi^{*})}ydF(y)g(\phi^{*})\overline{M}}{L^{pw}(\phi^{*},\frac{\widetilde{r}^{d}(\phi)\tau^{1-\sigma}}{\sigma f_xC_0})}d\phi$

in the last condition of (46) yields

$\frac{\left[\sigma f+n\sigma f_x\int_{y}^{\frac{\widetilde{r}^{d}(\phi^{*})\tau^{1-\sigma}}{\sigma f_xC_0}}ydF(y)\right]}{[1+F(\widetilde{y}(\phi^{*}))\tau^{1-\sigma}n]}\int_{\phi^{*}}^{\overline{\phi}}\frac{\widetilde{r}^{d}(\phi)}{\widetilde{r}^{d}(\phi^{*})}\left[1+F\left(\frac{\widetilde{r}^{d}(\phi)\tau^{1-\sigma}}{\sigma f_xC_0}\right)\tau^{1-\sigma}n\right]g\left(\phi\right)\overline{M}d\phi=\frac{\sigma}{\sigma-1}L^{pw}(\phi^{*},\frac{\widetilde{r}^{d}(\phi)\tau^{1-\sigma}}{\sigma f_xC_0}).$

If we choose the constant p_0 in (50) such that

$$\widetilde{r}^{d}\left(\phi^{*}\right)=\frac{\left[\sigma f+n\sigma f_{x}\int_{y}^{\frac{\widetilde{r}^{d}(\phi^{*})\tau^{1-\sigma}}{\sigma f_{x}C_{0}}}ydF(y)\right]}{\left[1+F\left(\frac{\widetilde{r}^{d}(\phi^{*})\tau^{1-\sigma}}{\sigma f_{x}C_{0}}\right)\tau^{1-\sigma}n\right]},\tag{54}$$

then the previous condition becomes

$$\int_{\phi^{*}}^{\overline{\phi}}\widetilde{r}^{d}\left(\phi\right)\left[1+F\left(\frac{\widetilde{r}^{d}(\phi)\tau^{1-\sigma}}{\sigma f_{x}}\right)\tau^{1-\sigma}n\right]g\left(\phi\right)\overline{M}d\phi=\frac{\sigma}{\sigma-1}L^{pw}(\phi^{*},\frac{\widetilde{r}^{d}(\phi)\tau^{1-\sigma}}{\sigma f_{x}})\tag{55}$$

as (52) yields $C_0 = 1$. Note that conditions (48), (51), (53),(55), $C_0 = 1$ and $\{\widetilde{H}(\phi^*) = \underline{s}, \widetilde{H}(\overline{\phi}) = \overline{s}\}$, imply that $\{\phi^*, \widetilde{p}, \widetilde{r}^d, \widetilde{H}\}$ satisfy all the conditions in lemma 3 with the exception of $\widetilde{r}^d(\phi^*) = \sigma f$. Accordingly, per condition (54), $\{\widetilde{p}, \widetilde{r}^d, \widetilde{H}\}$ is an equilibrium of the open economy only if $F\left(\frac{\widetilde{r}^{d}(\phi^{*})\tau^{1-\sigma}}{\sigma f_{x}}\right) = \int_{y}^{\frac{\widetilde{r}^{d}(\phi^{*})\tau^{1-\sigma}}{\sigma f_{x}C_{0}}} ydF(y) = 0$, a condition that is satisfied when the restriction on parameters assumed in the paper holds, $f\tau^{1-\sigma} \leq f_x$. As in the case of the closed economy, we can walk back on this derivations to show that given a triplet $\{p, r^d, H\}$ corresponding to an equilibrium of the open economy, then $\{q^d \equiv \frac{r^d}{p}, H\}$ is a solution to the planner's problem above when $f\tau^{1-\sigma} \leq f_x$.

When the restriction on parameters $f\tau^{1-\sigma} \leq f_x$ is not satisfied, the equivalence between equilibria of the open economy and solutions to problem (44) no longer holds. Intuitively, if $f\tau^{1-\sigma} > f_x$, then the planner is willing to accept some "negative domestic profits", $\widetilde{r}^d(\phi^*) < \sigma f$, because they are more than offset by positive export profits, $\widetilde{r}^d(\phi^*) F\left(\frac{\widetilde{r}^{d}(\phi^{*})\tau^{1-\sigma}}{\sigma f_{x}}\right) n\tau^{1-\sigma} > \sigma f_x \int_{y}^{\frac{\widetilde{r}^{d}(\phi^{*})\tau^{1-\sigma}}{\sigma f_{x}}} ydF(y)$. However, by changing slightly the arguments above, it can be shown that when $f\tau^{1-\sigma} > f_x$, the equilibria of the open economy are equivalent to solutions to constrained planner's problems that feature the following additional constraint

$$\sigma f\int_{\phi^{*}}^{\overline{\phi}}\left[\frac{\widetilde{q}^{d}\left(\phi\right)}{\widetilde{q}^{d}\left(\phi^{*}\right)}\right]^{\frac{\sigma-1}{\sigma}}\left[1+F\left(\widetilde{y}\left(\phi\right)\right)\tau^{1-\sigma}n\right]g\left(\phi\right)\overline{M}d\phi=\frac{\sigma}{\sigma-1}L^{pw}(\phi^{*},\widetilde{y}\left(\phi\right)).$$

Accordingly, $\pi$the equilibrium is constrained efficient in this case.

A.3 Additional Results related to BVP (22)

In this section, I present some results related to BVP (22) that are used in the text and in the proof of other results.

Lemma 4 *For $i = a, b$, let $\{z^i, x^i, \Gamma^i\}$ be the unique solution to the BVP (22) with parameters $\{\alpha^i(\phi), K_0^i, K_1^i\}$ and boundary conditions $x^i(\phi_0) = 1$, $\Gamma^i(\phi_0) = s_0$ and $\Gamma^i(\phi_1) = s_1$.*

(i) Suppose that $K_1^i = 0$, $\frac{\alpha^a(\phi')}{\alpha^a(\phi)} \geq \frac{\alpha^b(\phi')}{\alpha^b(\phi)}$ for all $\phi' > \phi \in [\phi_0, \phi_1]$, and $\frac{\alpha^a(\phi')}{\alpha^a(\phi)} \geq \frac{\alpha^b(\phi')}{\alpha^b(\phi)}$ for all $\phi' > \phi$ on some subinterval $[\phi_l, \phi_h] \subseteq [\phi_0, \phi_1]$. Then $\Gamma^a(\phi) < \Gamma^b(\phi)$ for all $\phi \in (\phi_0, \phi_1)$ and $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$ and

$\Gamma_\phi^a(\phi_1) > \Gamma_\phi^b(\phi_1)$.

(ii) Suppose that $K_0^i = K_0$, $\alpha^i(\phi) = \alpha(\phi)$ *and* $K_1^b < K_1^a$. *Then* $\Gamma_\phi^a(\phi_0) < \Gamma_\phi^b(\phi_0)$, *so there is a* $\phi^+ \in (\phi_0, \phi_1]$ *such that* $\Gamma^a(\phi^+) = \Gamma^b(\phi^+)$ *and* $\Gamma^a(\phi) < \Gamma^b(\phi)$ *for all* $\phi \in (\phi_0, \phi^+)$.

(iii) Let $\Phi^i \equiv \int_{\phi_0}^{\phi_1} x^i(\phi) \frac{\left[1+F\left(K_0^i x^i(\phi)\right)K_1^i\right]}{\left[1+F\left(K_0^i\right)K_1^i\right]} \frac{\alpha^i(\phi)}{\alpha^i(\phi_0)} g(\phi)\, d\phi$. *If* $\Gamma^a(\phi) < \Gamma^b(\phi)$ *for* $\phi \in (\phi_0, \phi_1)$, *then* $\Phi^a > \Phi^b$.

(iv) If $\alpha^i(\phi) = \alpha(\phi)$, $K_0^b = \lambda K_0^a$ *and* $K_1^b = \lambda K_1^a$ *for* $\lambda > 1$, *then* $x^b(\phi)\lambda > x^a(\phi)$ *for all for all* $\phi \in [\phi_0, \phi_1]$.

(v) Let $\delta^i(\phi) \equiv \left[1 + F\left(K_0^i x^i(\phi)\right) K_1^i\right] \alpha^i(\phi)$. *If* $\Gamma^a \neq \Gamma^b$ *and,* $\delta^a(\phi) < \delta^b(\phi)$ *for all* $\phi \in [\phi_0, \phi_1]$, *then*

$$\int_{\phi_0}^{\phi_1} x^a(\phi)\,\delta^a(\phi)\,g(\phi)\,d\phi < \int_{\phi_0}^{\phi_1} x^b(\phi)\,\delta^b(\phi)\,g(\phi)\,d\phi. \tag{56}$$

(vi) Suppose that $\left\{\alpha^i(\phi), K_1^i\right\} = \left\{\alpha(\phi), K_1\right\}$, $K_0^i, K_1 \in \mathbb{R}_{++}$ *and* $K_0^a > K_0^b$. *If the function* $\eta_0(t, \lambda) \equiv \frac{F_y(t\lambda)\lambda K_1}{[1+F(t\lambda)K_1]}$ *is strictly decreasing (increasing) in* λ *for all* $t \in [K_0^b, K_0^b x^b(\phi_1)]$, *then* $\Gamma^a(\phi) > (<)\Gamma^b(\phi)$ *on* (ϕ_0, ϕ_1), *with* $\Gamma_\phi^a(\phi_0) > (<)\Gamma_\phi^b(\phi)$.

(vii) Suppose that $\alpha^i(\phi) = \alpha(\phi)$, $K_0^i, K_1^i \in \mathbb{R}_{++}$ *and* $K_i^a = \lambda K_i^b$ *for* $\lambda > 1$. *If the function* $\eta_1(t, \lambda) \equiv \frac{F_y(t\lambda)\lambda^2 K_1^b}{\left[1+F(t\lambda)\lambda K_1^b\right]}$ *is strictly increasing (decreasing) in* λ *for all* $t \in [K_0^b, K_0^b x^b(\phi_1)]$, *then* $\Gamma^a(\phi) < (>)\Gamma^b(\phi)$ *on* (ϕ_0, ϕ_1) *with* $\Gamma_\phi^a(\phi_0) < (>)\Gamma_\phi^b(\phi_0)$.

Proof. Lemma 4.i. I proceed in steps.

STEP 1: *Under the assumptions of the lemma,* $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ *for all* $\phi \in (\phi_0, \phi_1)$.

Suppose to the contrary that there is a $\phi' \in (\phi_0, \phi_1)$ such that $\Gamma^a(\phi') > \Gamma^b(\phi')$. Let ϕ_- be the first time the functions Γ^a and Γ^b intersect to the left of ϕ' and let ϕ_+ be the first time they intersect to the right of ϕ', i.e. $\phi_- \equiv \max\left\{\phi \leq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\right\}$ and $\phi_+ = \inf\left\{\phi \geq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\right\}$. Note that ϕ_- and ϕ_+ are well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the left and to the right of ϕ' (at ϕ_0 and at ϕ_1). Also note that $\Gamma^a(\phi) > \Gamma^b(\phi)$ for $\phi \in (\phi_-, \phi_+)$. The continuity of Γ_ϕ^a and Γ_ϕ^b, implies $\Gamma_\phi^a(\phi_-) \geq \Gamma_\phi^b(\phi_-)$ and $\Gamma_\phi^a(\phi_+) \leq \Gamma_\phi^b(\phi_+)$, i.e.

$$\frac{\Gamma_\phi^a(\phi_+)/\Gamma_\phi^a(\phi_-)}{\Gamma_\phi^b(\phi_+)/\Gamma_\phi^b(\phi_-)} \leq 1. \tag{57}$$

Differentiating the right-hand side of (26) yields

$$\frac{\Gamma_\phi^i(\phi_+)}{\Gamma_\phi^i(\phi_-)} = h^i(\phi_-, \phi_+)\, e^{\sigma \int_{\phi_-}^{\phi_+} \frac{\partial \ln A\left(\Gamma^i(u), u\right)}{\partial \phi} du}, \tag{58}$$

where $h^i(\phi_-, \phi_+)$ is given by (27) with $\alpha = \alpha^i$. By assumption, we have $\Gamma^a(\phi_-) = \Gamma^b(\phi_-)$ and $\Gamma^a(\phi_+) = \Gamma^b(\phi_+)$, which together with the definition of h^i, imply $\frac{h^a(\phi_-,\phi_+)}{h^b(\phi_-,\phi_+)} = \frac{\alpha^a(\phi_+)/\alpha^a(\phi_-)}{\alpha^b(\phi_+)/\alpha^b(\phi_-)}$. Combining this result and (58) yields

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} = e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A(\Gamma^a(u),u)}{\partial\phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial\phi}\right]du} \frac{\alpha^a(\phi_+)/\alpha^a(\phi_-)}{\alpha^b(\phi_+)/\alpha^b(\phi_-)}. \tag{59}$$

The strict log-supermodularity of A and the fact that $\Gamma^a(\phi) > \Gamma^b(\phi)$ for $\phi \in (\phi_-, \phi_+)$ imply that the first term of the last expression is strictly greater than 1. In addition, the assumption about relative values of α^a and α^b on $[\phi_0, \phi_1]$ implies that the second term is weakly greater than one, i.e. $\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} > 1$. This result contradicts (57), so it must be that $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ for $\phi \in [\phi_0, \phi_1]$.

STEP 2: *Under the assumptions in the lemma, $\Gamma^a(\phi)$ and $\Gamma^b(\phi)$ cannot satisfy $\Gamma^a(\phi) = \Gamma^b(\phi)$ on any non-degenerate interval $I \subseteq [\phi_l, \phi_h]$.*

Suppose to the contrary that $\Gamma^a(\phi) = \Gamma^b(\phi)$ for some non-degenerate interval $I \subseteq [\phi_l, \phi_h]$ and let $\phi_- < \phi_+$ be two interior points of I. Notice that $\Gamma^a(\phi) = \Gamma^b(\phi)$ on I implies that $\Gamma^a_\phi(\phi) = \Gamma^b_\phi(\phi)$ on the interior of I, so

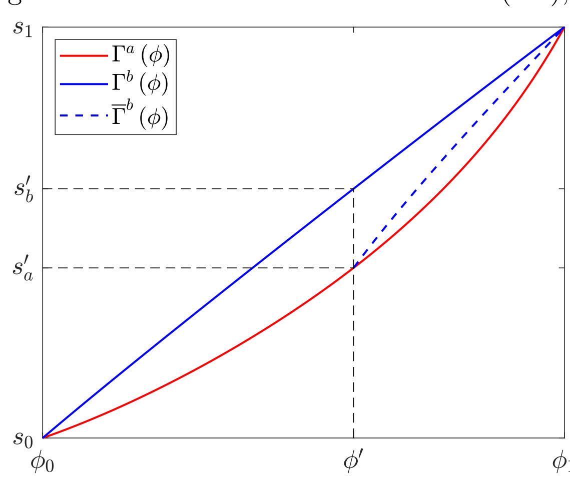
$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} = 1. \tag{60}$$

In addition, equation (59) must also hold in this case, which under the current assumptions yields

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} = \frac{\alpha^a(\phi_+)/\alpha^a(\phi_-)}{\alpha^b(\phi_+)/\alpha^b(\phi_-)} > 1,$$

where the strict inequality follows from $\phi_-, \phi_+ \in [\phi_l, \phi_h]$ and the assumption about relative values of α^a and α^b on this interval. The last expression contradicts (60). Then it must be the case that Γ^a and Γ^b cannot be equal on any non-degenerate interval $I \subseteq [\phi_l, \phi_h]$.

Figure 4: Solutions to the General BVP (22), Γ



Note: The figure depicts solutions to alternative parametrizations of the general BVP (22). The BVPs corresponding to Γ^a and Γ^b differ only in the parameter function $\alpha(\phi)$ as indicated in lemma 4.i Restricted to $[\phi', \phi_1]$,the BVPs corresponding to Γ^b and $\overline{\Gamma}^b$ differ only in their initial conditions.

STEP 3: *Under the assumptions in the lemma,* $\Gamma^a(\phi) < \Gamma^b(\phi)$ for all $\phi \in (\phi_0, \phi_1)$

Steps 1 and 2 imply that there is a $\phi' \in (\phi_l, \phi_h) \subseteq [\phi_0, \phi_1]$ such that $\Gamma^a(\phi') < \Gamma^b(\phi')$. The situation is depicted in figure 4. Now, I prove that $\Gamma^a(\phi) < \Gamma^b(\phi)$ on $[\phi', \phi_1)$. To establish this result I show that there exists a function $\overline{\Gamma}^b : [\phi', \phi_1] \to S$ (dashed blue line), such that $\Gamma^a(\phi) \leq \overline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ for all $\phi \in [\phi', \phi_1)$. Letting $s'_i \equiv \Gamma^i(\phi')$ for $i = a, b$, if we define on $[\phi', \phi_1]$, $w^b(\phi) \equiv x^b(\phi)/x^b(\phi')$ and $y^b(\phi) \equiv z^b(\phi)/x^b(\phi')$, then $\{y^b, w^b, \Gamma^b\}$ is the unique solution to the BVP (22) on $[\phi', \phi_1]$ with parameters $\{\alpha^b(\phi), K_0^b, K_1^b\}$ and boundary conditions $w(\phi') = 1$, $\Gamma^b(\phi') = s'_b$ and $\Gamma^b(\phi_1) = s_1$.[51] Now, let $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ be the unique solution to the BVP (22) on $[\phi', \phi_1]$ with the same parameters and boundary conditions $\overline{x}^b(\phi') = 1$, $\overline{\Gamma}^b(\phi') = s'_a < s'_b$ and $\overline{\Gamma}^b(\phi_1) = s_1$. It is readily seen that $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ and $\{y^b, w^b, \Gamma^b\}$ satisfy the conditions of the no-crossing result in lemma 2.ii with $\overline{\Gamma}^b(\phi') < \Gamma^b(\phi')$, so $\overline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ on $[\phi', \phi_1)$. Defining w^a and y^a on $[\phi', \phi_1]$ from x^a and z^a as I did above implies that $\{y^a, w^a, \Gamma^a\}$ is the unique solution to the BVP (22) on $[\phi', \phi_1]$ with parameters $\{\alpha^a(\phi), K_0^a, K_1^a\}$ and boundary conditions $w^a(\phi') = 1$, $\Gamma^a(\phi') = s'_a$ and $\Gamma^a(\phi_1) = s_1$. Then, $\{w^a, y^a, \Gamma^a\}$ and $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ satisfy the conditions of step 1 above, so $\Gamma^a(\phi) \leq \overline{\Gamma}^b(\phi)$ on $[\phi', \phi_1]$ as depicted in the figure.

The argument in the last paragraph can be easily adapted to show that there is a function $\underline{\Gamma}^b$: $[\phi_0, \phi'] \to S$, such that $\Gamma^a(\phi) \leq \underline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ for all $\phi \in (\phi_0, \phi']$, completing the proof of step 3. Of note, this part of the argument requires slightly different version of the no-crossing in theorem 2.ii. Specifically, in the notation of theorem 2, it can be shown that if we consider the solution to BVP (22) as a function of (s_0, s_1), then $\Gamma(\phi; s_0, s_1^a) < \Gamma(\phi; s_0, s_1^b)$ on $(\phi_0, \phi_1]$ if $s_1^a < s_1^b$.

STEP 4: *Under the same assumptions made in step 3,* $\Gamma_\phi^a(\phi_0) < \Gamma_\phi^b(\phi_0)$ *and* $\Gamma_\phi^a(\phi_1) > \Gamma_\phi^b(\phi_1)$.

Let $\phi' \in (\phi_0, \phi_1)$ and the triplets of functions $\{y^a, w^a, \Gamma^a\}$, $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ and $\{y^b, w^b, \Gamma^b\}$ on $[\phi', \phi_1]$ be defined as in step 3. Given that $\Gamma^a(\phi) \leq \overline{\Gamma}^b(\phi)$ on $[\phi', \phi_1]$, then it must be the case that $\Gamma_\phi^a(\phi_1) \geq \overline{\Gamma}_\phi^b(\phi_1)$, otherwise $\Gamma^a(\phi) > \overline{\Gamma}^b(\phi)$ on some neighborhood of ϕ_1. In a similar way, $\overline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ on on $[\phi', \phi_1)$ implies $\overline{\Gamma}_\phi^b(\phi) \geq \Gamma_\phi^b(\phi)$. Moreover, if $\overline{\Gamma}_\phi^b(\phi) = \Gamma_\phi^b(\phi)$, then$\{\overline{y}^b, \overline{w}^b, \Gamma^b\}$—with $\overline{y}^b(\phi) = \frac{y^b(\phi)}{w^b(\phi_1)}\overline{x}^b(\phi_1)$ and $\overline{w}^b(\phi) = \frac{w^b(\phi)}{w^b(\phi_1)}\overline{x}^b(\phi_1)$—and $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ satisfy the same IVP with initial condition at ϕ_1, so $\overline{\Gamma}^b = \Gamma^b$ on $[\phi', \phi_1]$, contradicting our earlier results. Then it must be the case that $\overline{\Gamma}_\phi^b(\phi_1) > \Gamma_\phi^b(\phi_1)$. Putting together these results we get $\Gamma_\phi^a(\phi_1) \geq \overline{\Gamma}_\phi^b(\phi_1) > \Gamma_\phi^b(\phi_1)$. The other part of the claim can be proved making only minor adjustments to this argument.

Lemma 4.ii. I proceed in steps.

STEP 1: *Under the assumptions of the lemma, there is no* $\phi' \in (\phi_0, \phi_1]$ *such that* $\Gamma^a(\phi) \geq \Gamma^b(\phi)$ *for all* $\phi \in (\phi_0, \phi']$.

Suppose to the contrary that there is such a value $\phi' \in (\phi_0, \phi_1]$. Let ϕ_+ be the first time the functions Γ^a and Γ^b intersect to the right of ϕ', i.e. $\phi_+ = \inf\{\phi \geq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\}$. Note ϕ_+ is well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the right of ϕ' (at ϕ_1). Also note that $\Gamma^a(\phi) \geq \Gamma^b(\phi)$ for $\phi \in (\phi_0, \phi_+)$. The continuity of Γ_ϕ^a and Γ_ϕ^b,

[51] Note that we are using the same notation to denote the restriction of a function to a subset of its domain.

implies $\Gamma^a_\phi(\phi_0) \geq \Gamma^b_\phi(\phi_0)$ and $\Gamma^a_\phi(\phi_+) \leq \Gamma^b_\phi(\phi_+)$, i.e.

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} \leq 1. \tag{61}$$

Differentiating the right-hand side of (26) yields

$$\frac{\Gamma^i_\phi\left(\phi_+\right)}{\Gamma^i_\phi\left(\phi_0\right)} = h^i\left(\phi_0,\phi_+\right) e^{\sigma\int_{\phi_-}^{\phi_+}\frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial\phi}du} \frac{\left[1+F\left(K_0 x^i\left(\phi\right)\right)K_1^i\right]}{\left[1+F\left(K_0\right)K_1^i\right]}, \tag{62}$$

where $h^i\left(\phi_0,\phi_+\right)$ is given by (27). By assumption, we have $\Gamma^a\left(\phi_0\right) = \Gamma^b\left(\phi_0\right)$ and $\Gamma^a\left(\phi_+\right) = \Gamma^b\left(\phi_+\right)$, which together with the definition of h^i, imply $h^a\left(\phi_0,\phi_+\right) = h^b\left(\phi_0,\phi_+\right)$. Combining this result with (62) for $i = a, b$ yields

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} = e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^a(u),u\right)}{\partial\phi}-\frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial\phi}\right]du} \frac{\left[1+F\left(K_0 x^a\left(\phi_+\right)\right)K_1^a\right]/\left[1+F\left(K_0\right)K_1^a\right]}{\left[1+F\left(K_0 x^b\left(\phi_+\right)\right)K_1^b\right]/\left[1+F\left(K_0\right)K_1^b\right]}. \tag{63}$$

The strict log-supermodularity of A and the fact that $\Gamma^a\left(\phi\right) \geq \Gamma^b\left(\phi\right)$ for $\phi \in \left(\phi_0,\phi_+\right)$ imply that the first term of the right-hand side of the last expression is weakly greater than 1. In addition, note that we can write $\frac{\left[1+F\left(K_0 x^i(\phi)\right)K_1^i\right]}{\left[1+F(K_0)K_1^i\right]} = \frac{1}{\left[1+F(K_0)K_1^i\right]} + \frac{F(K_0)K_1^i}{\left[1+F(K_0)K_1^i\right]}\frac{F\left(K_0 x^i(\phi)\right)}{F(K_0)}$,
so $x^a\left(\phi\right) \geq x^b\left(\phi\right)$ for $\phi \in \left(\phi_0,\phi_+\right)$ ($\Gamma^a\left(\phi\right) \geq \Gamma^b\left(\phi\right)$) and $K_1^a > K_1^b$ imply that the second term of the right-hand side of (63) is strictly higher than one. Accordingly, $\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} > 1$, contradicting (61).

STEP 2: *Under the assumptions of the lemma, $\Gamma^a_\phi\left(\phi_0\right) < \Gamma^b_\phi\left(\phi_0\right)$, immediately proving the lemma.*

The result of step 1 immediately yields that $\Gamma^a_\phi\left(\phi_0\right) \leq \Gamma^b_\phi\left(\phi_0\right)$. Otherwise, $\Gamma^a_\phi\left(\phi_0\right) > \Gamma^b_\phi\left(\phi_0\right)$ implies that there is a $\phi' \in (\phi_0,\phi_1]$ such that $\Gamma^a\left(\phi\right) > \Gamma^b\left(\phi\right)$ on $(\phi_0,\phi']$, contradicting the result in step 1. Suppose then that $\Gamma^a_\phi\left(\phi_0\right) = \Gamma^b_\phi\left(\phi_0\right) = \gamma_0$. Note that the (same) boundary conditions of the BVPs under consideration imply $\Gamma^i\left(\phi_0\right) = s_0$, $x^i\left(\phi_0\right) = 1$. In turn, these observations and equations (22a)-(22b) imply $x^i_\phi\left(\phi\right) = \frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}$ and $\frac{z^i_\phi(\phi_0)}{z^i(\phi_0)} = -\frac{\partial\ln A(s_0,\phi_0)}{\partial\phi}$. Log-differentiating both sides of equation (22c) and evaluating at ϕ_0 yields

$\frac{\Gamma^i_{\phi\phi}(\phi_0)}{\Gamma^i_\phi(\phi_0)} = \frac{x^i_\phi(\phi_0)}{x^i(\phi_0)} + \frac{F_y\left(K_0 x^i(\phi_0)\right)K_1^i K_0 x^i_\phi(\phi_0)}{\left[1+F(K_0 x^i(\phi_0))K_1^i\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - [\frac{\partial\ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial s}\Gamma^i_\phi\left(\phi_0\right) + \frac{\partial\ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial\phi} + \frac{V_s\left(\Gamma^i(\phi_0)\right)}{V\left(\Gamma^i(\phi_0)\right)}\Gamma^i_\phi\left(\phi_0\right) + \frac{z^i_\phi(\phi_0)}{z^i(\phi_0)}]$,

$\frac{\Gamma^i_{\phi\phi}(\phi)}{\gamma_0} = \frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi} + \frac{F_y(K_0)K_1^i K_0\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}}{\left[1+F(K_0)K_1^i\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - [\frac{\partial\ln A(s_0,\phi_0)}{\partial s}\gamma_0 + \frac{\partial\ln A(s_0,\phi_0)}{\partial\phi} + \frac{V_s(s_0)}{V(s_0)}\gamma_0 - \frac{\partial\ln A(s_0,\phi_0)}{\partial\phi}]$,

i.e.,

$$\Gamma^a_{\phi\phi}\left(\phi_0\right) - \Gamma^b_{\phi\phi}\left(\phi_0\right) = \frac{F_y(K_0)K_0}{F(K_0)}\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}\gamma_0\left\{\frac{F(K_0)K_1^a}{\left[1+F(K_0)K_1^a\right]} - \frac{F(K_0)K_1^b}{\left[1+F(K_0)K_1^b\right]}\right\} > 0,$$

where the inequality follows from $K_1^a > K_1^b$. The last expression implies that there is some $\phi' \in (\phi_0,\phi_1]$ such that $\Gamma^a_\phi\left(\phi\right) > \Gamma^b_\phi\left(\phi\right)$ on $(\phi_0,\phi']$, which yields a contradiction of step 1. Accordingly, we must have

$\Gamma_{\phi}^{a}(\phi_0) < \Gamma_{\phi}^{b}(\phi_0)$.

Finally, $\Gamma_{\phi}^{a}(\phi_0) < \Gamma_{\phi}^{b}(\phi_0)$ implies $\Gamma^{a}(\phi) < \Gamma^{b}(\phi)$ on some (small enough) interval (ϕ_0, ϕ''), so ϕ^{+} described in the lemma is the first time Γ^a and Γ^b intersect to the right of ϕ''.

Lemma 4.iii.

The idea of the proof is to show that Γ^a and Γ^b can be thought of as the inverse of the matching functions of two artificial economies, and then use this additional information to prove the result. Let $\{z^i, x^i, \Gamma^i\}$ be the solution to the BVP in the statement of the lemma and consider the following artificial economy. In this economy there are no fixed costs of production and no fixed costs to export but the set of active firms and the set of exporters are fixed. In particular, the set of active firms are those with productivity in the range $[\phi_0, \phi_1]$, while the fraction of firms that export at each productivity level is given by $F\left(K_0^i x^i(\phi)\right)$. The set of available workers are those with skills in the range $[s_0, s_1]$. The distribution of skills is given by the restriction of V to $[s_0, s_1]$ and the mass of workers is $\int_{s_0}^{s_1} V(s)\, ds L$. The total mass of firms with productivity ϕ is given by $g(\phi)\alpha^i(\phi)\overline{M}$, so the total mass of firms is $\int_{\phi_0}^{\phi_1} g(\phi)\alpha^i(\phi)\overline{M}$. Finally, τ_i is set to satisfy $K_1^i \equiv \tau_i^{1-\sigma}$.

Now I show that if p^i, $r^{d,i}$ and H^i denote the price, domestic revenue and inverse-matching functions of the economy described above, then $H^i = \Gamma^i$. An argument similar to the one in section 4 implies that $\{p^i, r^{d,i}, H^i\}$ satisfy the differential equations (13), (14) and

$$H_{\phi}^{i}(\phi) = \frac{r^{d,i}(\phi)\left[1 + F\left(K_0^i x^i(\phi)\right) K_1^i\right] g(\phi)\, \alpha^i(\phi)\, \overline{M}}{A\left(H^i(\phi), \phi\right) V\left(H^i(\phi)\right) p^i(\phi)\, L}, \tag{64}$$

with boundary conditions $H^i(\phi_0) = s_0$ and $H^i(\phi_1) = s_1$. Note that we don't have a boundary condition on the domestic revenue function $r^{d,i}$, as the zero-profit condition for firms with productivity ϕ_0 is no longer an equilibrium condition (no fixed costs of production). As a result, the levels of the functions $r^{d,i}$ and p^i cannot be determined without an additional condition (provided below). However, these conditions are enough to pin down H^i. To see this, let $\{p^i, r^{d,i}, H^i\}$ be any triplet of functions satisfying the equilibrium conditions described above, and define $\delta^i(\phi) \equiv \left[1 + F\left(K_0^i x^i(\phi)\right) K_1^i\right] \alpha^i(\phi)$, $v^i(\phi) \equiv r^{d,i}(\phi) / r^{d,i}(\phi_0)$ and $y^i(\phi) \equiv p^i(\phi)\, L / r^{d,i}(\phi_0)\, \overline{M}$. Then, it is readily seen that $\{y^i, v^i, H^i\}$ is the unique solution to the BVP (22) with parameter $K_1 = 0$ and $\alpha = \delta^i$.[52] However, note that, by construction, $\{z^i, x^i, \Gamma^i\}$ is also a solution to this parametrization of the BVP (22), so it must be the case that $H^i = \Gamma^i$.

Let us now derive an additional condition to pin down the revenue function of this artificial economy. In equilibrium, the total revenue of firms with productivity less or equal than ϕ' must equal a constant fraction of the total wages paid to workers employed at those firms,

$$r^{d,i}(\phi_0)\, \alpha^i(\phi_0)\left[1 + F\left(K_0^i\right) K_1^i\right] \int_{\phi_0}^{\phi'} x^i(\phi) \frac{\left[1 + F\left(K_0^i x^i(\phi)\right) K_1^i\right]}{\left[1 + F\left(K_0^i\right) K_1^i\right]} \frac{\alpha^i(\phi)}{\alpha^i(\phi_0)} g(\phi)\, \overline{M} d\phi = \tag{65}$$

$$\cdots \frac{\sigma}{\sigma - 1} L \int_{s_0}^{H^i(\phi')} w^i\left(H^i(\phi)\right) V\left(H^i(\phi)\right) ds, \text{ for } i = a, b.$$

[52] With $K_1 = 0$, the value of K_0 is irrelevant.

Differentiating the left- and right hand sides of the last expression with respect to ϕ', and evaluating the resulting expressions at $\phi' = \phi_0$ yields

$$r^{d,i}\left(\phi_0\right)\alpha^i(\phi_0)\left[1+F\left(K_0^i\right)K_1^i\right]g(\phi_0)\overline{M} = \frac{\sigma}{\sigma-1}Lw^i\left(s_0\right)V\left(s_0\right)H_\phi^i\left(\phi_0\right) \text{ for } i=a,b. \tag{66}$$

The last expression, together with the numeraire assumption, $\int_{s_0}^{s_1} w^i(s)\,V(s)\,ds = 1$, and the inverse matching function H^i, can be used to pin down the value of $r_d^i(\phi_0)$. To see this, note that H^i determines the growth rate of wages along the skill dimension (condition 12), while the numeraire assumption pins down their levels, i.e. the wage schedule is fully determined. Then, equation (66) can be used to pin down $r_d^i(\phi_0)$, the only remaining endogenous variable.

With previous results we are ready to prove the lemma. As $H^a(\phi) < H^b(\phi)$ for $\phi \in [\phi_0, \phi_1]$ by assumption, wages grow faster along the skill dimension in economy a than in economy b, so the numeraire assumption implies $w^a(s_0) < w^b(s_0)$. In addition, $H^a(\phi) < H^b(\phi)$ for $\phi \in [\phi_0, \phi_1]$ also implies that $H_\phi^a(\phi_0) \leq H_\phi^b(\phi_0)$. These observations and (66) imply $r^{d,a}(\phi_0)\,\alpha^a(\phi_0)\left[1+F\left(K_0^a\right)K_1^a\right] < r^{d,b}(\phi_0)\,\alpha^b(\phi_0)\left[1+F\left(K_0^b\right)K_1^b\right]$. Finally, the last inequality, expression (65) evaluated at $\phi' = \phi_1$ for $i=a,b$, and the numeraire assumption yield the desired result.

Lemma 4.iv.

As Γ^i is a fixed point of the functional Ψ^i defined in (26) with parameters $\left\{\alpha^i(\phi), K_0^i, K_1^i\right\}$, $\Gamma^i(\phi) = \Psi^i\left(\Gamma^i\right)(\phi)$, $\Gamma_\phi^i(\phi)$ can be obtained differentiating the right-hand side of (26). Doing so yields,

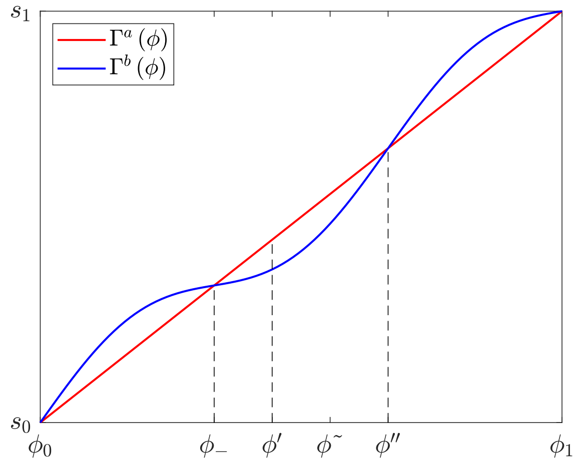
$$\Gamma_\phi^i(\phi) = \left[s_1 - s_0\right]\frac{h^i\left(\phi,\Gamma^i(\phi)\right)x^i(\phi)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^i x^i(\phi)\right)K_1^i\right]}{\int_{\phi_0}^{\phi_1} h^i\left(t,\Gamma^i(t)\right)x^i(t)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^i x^i(t)\right)K_1^i\right]dt}, \tag{67}$$

where in the last expression I used the fact that $x^i(\phi) = e^{(\sigma-1)\int_{\phi_0}^{\phi}\frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial\phi}du}$. The last expression plays a central role in the proof. Specifically, I show that if the claim of the lemma is not satisfied, then it is possible to derive contradicting implications regarding the values of the denominators on the right-hand side of (67), Dem^i for $i=a,b$. Throughout the proof, I denote the numerator on the right hand side of (67) by $Num^i(\phi)$.

Suppose the claim of the lemma is not true and $x^b(\phi)\,\lambda \leq x^a(\phi)$ for some $\phi \in [\phi_0, \phi_1]$. Noting that $x^a(\phi_0) < \lambda x^b(\phi_0)$, let $\phi^{\sim} > \phi_0$ be the lowest productivity value at which $x^b(\phi)\,\lambda = x^a(\phi)$.[53] Clearly, $x^a(\phi)$ must be catching up to $x^b(\phi)\,\lambda$ to the left of $\phi^{\sim}$, so equation (22b) implies that $\Gamma^b(\phi) < \Gamma^a(\phi)$ on some interval (ϕ', ϕ''), with $\phi' < \phi^{\sim} \leq \phi''$ and $\Gamma^b\left(\phi''\right) = \Gamma^a\left(\phi''\right)$. This situation is depicted in figure 5.

[53] Note that $\phi^{\sim}$ is well defined due to the continuity of the functions x^a and x^b.

Figure 5: Hypotetical Solutions to the General BVP (22), Γ



Note: The figure depicts hypothetical solutions to the general BVP (22) with the features implied by the assumption $x^b(\phi)\lambda \leq x^a(\phi)$ given the conditions in lemma 4.iv. as described in the proof. Of note, said assumption implies $\phi^{\sim} \in (\phi', \phi'']$, with the figure showing one of many possibilities.

By construction, $\Gamma^b_\phi(\phi'') \geq \Gamma^a_\phi(\phi'')$ and $x^b(\phi'')\lambda \leq x^a(\phi'')$, where the latter inequality implies

$$\begin{aligned} x^b(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^b x^b(\phi'')\right)K_1^b\right] &= x^b(\phi'')^{\frac{1}{\sigma-1}}\left[x^b(\phi'')+F\left(K_0^a\lambda x^b(\phi'')\right)x^b(\phi'')\lambda K_1^a\right] \\ &< x^a(\phi'')^{\frac{1}{\sigma-1}}\left[x^a(\phi'')+F\left(K_0^a x^a(\phi'')\right)x^a(\phi'')K_1^a\right] \\ &= x^a(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^a x^a(\phi'')\right)K_1^a\right]. \end{aligned} \tag{68}$$

In addition, by definition of h^i we have $h^a(\phi'', \Gamma^a(\phi'')) = h^b(\phi'', \Gamma^b(\phi''))$, which, together with the last expression, implies that $Num^b(\phi'') < Num^a(\phi'')$. This last result, $\Gamma^b_\phi(\phi'') \geq \Gamma^a_\phi(\phi'')$ and expression (67) yield $Dem^b < Dem^a$.

Expression (67) implies

$$\frac{\Gamma^b_\phi(\phi'')/\Gamma^b_\phi(\phi_0)}{\Gamma^a_\phi(\phi'')/\Gamma^a_\phi(\phi_0)} = \frac{x^b(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^b x^b(\phi'')\right)K_1^b\right]}{x^a(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^a x^a(\phi'')\right)K_1^a\right]}\frac{\left[1+F\left(K_0^a\right)K_1^a\right]}{\left[1+F\left(\lambda K_0^a\right)\lambda K_1^a\right]} < 1,$$

where the inequality follows from (68) and $\lambda > 1$. The last result and $\Gamma^b_\phi(\phi'') \geq \Gamma^a_\phi(\phi'')$ imply $\Gamma^b_\phi(\phi_0) > \Gamma^a_\phi(\phi_0)$, i.e. $\Gamma^b(\phi) > \Gamma^a(\phi)$ on some neighborhood of ϕ_0 (excluding ϕ_0). Let ϕ_- be the lowest productivity value to the right of ϕ_0 such that $\Gamma^b(\phi^-) = \Gamma^a(\phi^-)$. As $\Gamma^b(\phi) > \Gamma^a(\phi)$ on (ϕ_0, ϕ_-), we have $\Gamma^b_\phi(\phi_-) \leq \Gamma^a_\phi(\phi_-)$ and $x^b(\phi_-) > x^a(\phi_-)$. Using these results and (67) yields

$$\frac{Dem^b}{Dem^a} = \frac{\Gamma^a_\phi(\phi^-)}{\Gamma^b_\phi(\phi^-)}\frac{x^b(\phi_-)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(\lambda K_0^a x^b(\phi_-)\right)\lambda K_1^a\right]}{x^a(\phi^-)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^a x^a(\phi_-)\right)K_1^a\right]} > 1,$$

contradicting our previous finding, $Dem^b < Dem^a$. Then it must be the case that $x^b(\phi)\lambda > x^a(\phi)$ for all for all $\phi \in [\phi_0, \phi_1]$, which is the desired result.

Lemma 4.v.

As in the case of lemma 4.iii, the idea of the proof is to show that Γ^a and Γ^b can be thought of as the inverse matching functions of two artificial economies, and then use this additional information to prove the result. Moreover, I define these artificial economies here in the same way I did in proof of lemma 4.iii. Let $\{z^i, x^i, \Gamma^i\}$ be the solution to the BVP in the statement of the lemma and consider the following artificial economy. In this economy there are no fixed costs of production and no fixed costs to export but the set of active firms and the set of exporters are fixed. In particular, the set of active firms are those with productivity in the range $[\phi_0, \phi_1]$, while the fraction of firms that export of each productivity level is given by $F\left(K_0^i x^i(\phi)\right)$. The set of available workers are those with skills in the range $[s_0, s_1]$. The distribution of skills is given by the restriction of V to $[s_0, s_1]$ and the mass of workers is $\int_{s_0}^{s_1} V(s)\, dsL$. The total mass of firms with productivity ϕ is given by $g(\phi)\alpha^i(\phi)\overline{M}$, so the total mass of firms is $\int_{\phi_0}^{\phi_1} g(\phi)\alpha^i(\phi)\overline{M}$. Finally, I set τ_i such that $K_1^i \equiv \tau_i^{1-\sigma}$.

The same argument used in the proof of lemma 4.iii implies that if p^i, $r^{d,i}$ and H^i are the price, domestic revenue and inverse-matching functions of the economy described above, then $H^i = \Gamma^i$. In addition, equation (65) also holds in this economy, which can be differentiated with respect to the limit of integration to get

$$r^{d,i}(\phi)\,\delta^i(\phi)\,g(\phi)\overline{M} = \frac{\sigma}{\sigma-1} L w^i\left(H^i(\phi)\right) V\left(H^i(\phi)\right) H_\phi^i(\phi) \text{ for } i = a, b, \tag{69}$$

where $\delta^i(\phi)$ was defined in the statement of the lemma. As discussed in the proof of lemma 4.iii, the last expression and the numeraire assumption, $\int_{s_0}^{s_1} w^i(s)\, V(s)\, ds = 1$, can be used to pin down the level of the domestic revenue function $r^{d,i}$. For this reason, the last expression is central in the proof of this lemma, as the main result is an immediate implication of the values recovered for $r^{d,i}(\phi_0)$ and equation (65).

STEP 1: *Let Φ^* be the set of productivity levels given by*

$$\Phi^* = \left\{\phi \in [\phi_0, \phi_1] : H^b(\phi) = H^a(\phi), H_\phi^b(\phi) \leq H_\phi^a(\phi)\right\},$$

and let S^* denote the set of corresponding skill levels, $S^* \equiv \left\{s \in [s_0, s_1] : s = H^i(\phi) \text{ for some } \phi \in \Phi^*\right\}$. Then, $w^b(s) < w^a(s)$ for some $s \in S^*$.

Suppose that this is not the case and $w^b(s) \geq w^a(s)$ for all $s \in S^*$ and let N^i be the matching function of the artificial economy described above, i.e. N^i is the inverse function of H^i. For any $s \in [s_0, s_1] \backslash S^*$, there are three possibilities, (i) $N^a(s) = N^b(s)$, (ii) $N^a(s) < N^b(s)$, and (iii) $N^a(s) > N^b(s)$. I show that $w^b(s) > w^a(s)$ in all cases.

Let us start with case (i). As $s \notin S^*$, then $H_\phi^b(\phi) > H_\phi^a(\phi)$ for $\phi = N^i(s)$, implying $H^b(\phi') < H^a(\phi')$ on some neighborhood to the left of ϕ. Let ϕ^- be the first time H^a and H^b intersect to the left of ϕ, and let $s^- \equiv H^i(\phi^-)$. By construction, we have $\phi^- \in \Phi^*$ ($s^- \in S^*$) and $H^b(\phi') < H^a(\phi')$ for all $\phi' \in (\phi^-, \phi)$

$(N^b(s') > N^a(s')$ for all $s' \in (s^-, s))$, so

$$w^b(s) = w^b\left(s^-\right) e^{\int_{s^-}^{s} \frac{\partial \ln A\left(t, N^b(t)\right)}{\partial s} dt} > w^a\left(s^-\right) e^{\int_{s_0}^{s} \frac{\partial \ln A\left(t, N^a(t)\right)}{\partial s} dt} = w^a(s), \tag{70}$$

where the last inequality in a consequence of the log-supermodularity of A and $w^b(s^-) \geq w^a(s^-)$.

Turning to case (ii), let s^- and s^+ be the first time N^a and N^b intersect to the left and right of s respectively. These skill levels are well defined due to the continuity of the functions involved and the fact that N^a and N^b intersect at least once to the left and right of s (at s_0 and s_1). Letting $\phi^k \equiv N^i\left(s^k\right)$ for $k = -, +$, by construction we have $N^b(s') > N^a(s')$ for all $s' \in (s^-, s^+)$, so $N_s^b(s^-) \geq N_s^a(s^-)$ $(H_\phi^b\left(\phi^-\right) \leq H_\phi^a\left(\phi^-\right))$, i.e. $s^- \in S^*$. Then inequality (70) also holds in this case.

Let us now turn to case (iii). Let s^- and s^+ be the first time N^a and N^b intersect to the left and right of s respectively. As before, these skill levels are well defined. Letting $\phi^k \equiv N^i\left(s^k\right)$ for $k = -, +$, by construction we have $N^b(s') < N^a(s')$ for all $s' \in (s^-, s^+)$, so $N_s^b(s^+) \geq N_s^a(s^+)$ $(H_\phi^b\left(\phi^+\right) \leq H_\phi^a\left(\phi^+\right))$, i.e. $s^+ \in S^*$. This and the log supermodularity of A imply

$$\frac{w^b(s^+)}{w^b(s)} = e^{\int_{s}^{s^+} \frac{\partial \ln A\left(t, N^b(t)\right)}{\partial s} dt} < e^{\int_{s}^{s^+} \frac{\partial \ln A\left(t, N^a(t)\right)}{\partial s} dt} = \frac{w^a(s^+)}{w^a(s)}.$$

Per our initial assumption and $s^+ \in S^*$ we have $w^b(s^+) \geq w^a(s^+)$, which together with the last expression, yields $w^b(s) > w^a(s)$.

Given that the selection of $s \in [s_0, s_1] \backslash S^*$ was arbitrary, we conclude that $w^b(s) > w^a(s)$ for all $s \in [s_0, s_1] \backslash S^*$. However, notice that $w^b(s) \geq w^a(s)$ on $[s_0, s_1]$ and $w^b(s) > w^a(s)$ on $[s_0, s_1] \backslash S^*$ imply $\overline{w}^b > \overline{w}^a$, which contradicts our numeraire selection. Then it must be the case that $w^b(s) < w^a(s)$ for some $s \in S^*$.

STEP 2: *Let S^* be defined as before, let $s^+ \in S^*$ such that $w^b(s^+) < w^a(s^+)$ and let $\phi^+ = N^i(s^+)$. If $x^b\left(\phi^+\right) \geq x^a\left(\phi^+\right)$, then $r^{d,b}(\phi_0) < r^{d,a}(\phi_0)$.*

By assumption we have $w^b(s^+) < w^a(s^+)$, $H^a\left(\phi^+\right) = H^b\left(\phi^+\right)$ and $H_\phi^b\left(\phi^+\right) \leq H_\phi^b\left(\phi^+\right)$, which, together with equation (69) evaluated at ϕ^+, imply

$$r^{d,b}(\phi_0)\, x^b\left(\phi^+\right) \delta^b\left(\phi^+\right) < r^{d,a}(\phi_0)\, x^a\left(\phi^+\right) \delta^a\left(\phi^+\right).$$

The last expression, $x^b\left(\phi^+\right) \geq x^a\left(\phi^+\right)$, and the assumption in the of the lemma $(\delta^b(\phi) > \delta^a(\phi))$ imply $r^{d,b}(\phi_0) < r^{d,a}(\phi_0)$.

STEP 3: *Let S^*, s^+ and ϕ^+ be defined as in step 2. If $x^b\left(\phi^+\right) < x^a\left(\phi^+\right)$, then $r^{d,b}(\phi_0) < r^{d,a}(\phi_0)$.*

The continuity of H^i and of H^i_ϕ imply that Φ^* and S^* are closed sets, so $s^- \equiv \inf S^* \in S^*$. As H^a and H^b intersect at ϕ_0 and at ϕ^+, the following equality holds for $i = a, b$,

$$\begin{aligned} \ln \frac{A\left(s^+, \phi^+\right)}{A\left(s_0, \phi_0\right)} &= \int_{\phi_0}^{\phi^+} \frac{\partial \ln A\left(H^i(t), t\right)}{\partial s} H^i_\phi(t) dt + \int_{\phi_0}^{\phi^+} \frac{\partial \ln A\left(H^i(t), t\right)}{\partial \phi} dt \\ &= \int_{s_0}^{s^+} \frac{\partial \ln A\left(u, N^i(u)\right)}{\partial s} du + \int_{\phi_0}^{\phi^+} \frac{\partial \ln A\left(H^i(t), t\right)}{\partial \phi} dt. \end{aligned}$$

Coupling the last expression with assumption $x^b\left(\phi^+\right) < x^a\left(\phi^+\right)$ yields $\int_{s_0}^{s^+} \frac{\partial \ln A\left(u, N^b(u)\right)}{\partial s} du > \int_{s_0}^{s^+} \frac{\partial \ln A\left(u, N^a(u)\right)}{\partial s} du$, which, together with condition (12), implies

$$\frac{w^b\left(s^-\right)}{w^b\left(s_0\right)} \frac{w^b\left(s^+\right)}{w^b\left(s^-\right)} = \int_{s_0}^{s^+} \frac{\partial \ln A\left(u, N^b(u)\right)}{\partial s} du > \int_{s_0}^{s^+} \frac{\partial \ln A(u, N^a(u))}{\partial s} du = \frac{w^a\left(s^-\right)}{w^a\left(s_0\right)} \frac{w^a\left(s^+\right)}{w^a\left(s^-\right)}. \tag{71}$$

Now I show that $w^b\left(s^-\right)/w^b\left(s_0\right) \leq w^a\left(s^-\right)/w^a\left(s_0\right)$. If $s^- = s_0$ there is nothing to prove, so let's assume that $s^- > s_0$. First, notice that $N^b\left(s\right) \leq N^a\left(s\right)$ for $s \in [s_0, s^-]$. To see this, suppose to the contrary that $N^b\left(s\right) > N^a\left(s\right)$ for some $s \in (s_0, s^-)$, and let s' be the first time N^b and N^a intersect to the left of s. Then we have $s' < s^-$, $N^b\left(s'\right) = N^a\left(s'\right)$ and $N^b_s\left(s'\right) \geq N^a_s\left(s'\right)$, i.e. $s' \in s^*$ with $s' < s^-$. However, this contradicts the definition of s^-, so it must be the case that $N^b\left(s\right) \leq N^a\left(s\right)$ for $s \in [s_0, s^-]$. This result and the log supermodularity of A implies

$$\frac{w^b\left(s^-\right)}{w^b\left(s_0\right)} = \int_{s_0}^{s^-} \frac{\partial \ln A\left(u, N^b(u)\right)}{\partial s} du \leq \int_{s_0}^{s^-} \frac{\partial \ln A(u, N^a(u))}{\partial s} du = \frac{w^a\left(s^-\right)}{w^a\left(s_0\right)}. \tag{72}$$

The inequalities (71)-(72) and our assumption $w^b\left(s^+\right) < w^a\left(s^+\right)$ imply $w^b\left(s^-\right) < w^a\left(s^-\right)$. Using this result, $H^b_\phi(\phi^-) \leq H^a_\phi(\phi^-)$ and the assumption in the lemma about $\delta^i\left(\phi\right)$ in expression (69) (evaluated at ϕ^-) yields $r^{d,b}\left(\phi^-\right) < r^{d,a}\left(\phi^-\right)$. If $\phi^- = \phi_0$, we are done, so let us assume $\phi^- > \phi_0$. As discussed above, $N^b\left(s\right) \leq N^a\left(s\right)$ for $s \in [s_0, s^-]$ ($H^b\left(\phi\right) \geq H^a\left(\phi\right)$ for $\phi \in \left[\phi_0, \phi^-\right]$), implying

$$\frac{r^{d,b}\left(\phi^-\right)}{r^{d,b}\left(\phi_0\right)} = e^{(\sigma-1)\int_{\phi_0}^{\phi^-} \frac{\partial \ln A(H^b(t), t)}{\partial \phi} dt} \geq e^{(\sigma-1)\int_{\phi_0}^{\phi^-} \frac{\partial \ln A(H^a(t), t)}{\partial \phi} dt} = \frac{r^{d,a}\left(\phi^-\right)}{r^{d,a}\left(\phi_0\right)}.$$

The last expression and $r^{d,b}\left(\phi^-\right) < r^{d,a}\left(\phi^-\right)$ imply $r^{d,b}\left(\phi_0\right) < r^{d,a}\left(\phi_0\right)$, which is the desired result.

STEP 4: *Under the assumptions of the Lemma, inequality (56) holds.*

Steps 2 and 3 together imply that $r^{d,b}\left(\phi_0\right) < r^{d,a}\left(\phi_0\right)$, holds for these two artificial economies. This result, the numeraire assumption for these economies and equation (65) evaluated at $\phi' = \phi_1$ imply that inequality (56) holds.

Lemma 4.vi. I prove the statement for the case in which $\eta_0\left(t, \lambda\right)$ is strictly decreasing in λ.

STEP 1: *Under the assumptions of the lemma, there is no $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a\left(\phi\right) \leq \Gamma^b\left(\phi\right)$ for all $\phi \in (\phi_0, \phi']$.*

Suppose to the contrary that there is such a value $\phi' \in (\phi_0, \phi_1]$. Let ϕ_+ be the first time the functions Γ^a and Γ^b intersect to the right of ϕ', i.e. $\phi_+ = \inf\left\{\phi \geq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\right\}$. Note ϕ_+ is well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the right of ϕ' (at ϕ_1). Also note that $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ for $\phi \in \left(\phi_0, \phi_+\right)$. The continuity of Γ^a_ϕ and Γ^b_ϕ, implies $\Gamma^a_\phi(\phi_0) \leq \Gamma^b_\phi(\phi_0)$ and $\Gamma^a_\phi(\phi_+) \geq \Gamma^b_\phi(\phi_+)$, i.e.

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} \geq 1. \tag{73}$$

Differentiating the right-hand side of (26) yields

$$\frac{\Gamma^i_\phi\left(\phi_+\right)}{\Gamma^i_\phi\left(\phi_0\right)} = h^i\left(\phi_0, \phi_+\right) e^{\sigma \int_{\phi_-}^{\phi_+} \frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial \phi} du} \frac{\left[1 + F\left(K^i_0 x^i(\phi)\right) K_1\right]}{\left[1 + F\left(K^i_0\right) K_1\right]}, \tag{74}$$

where $h^i\left(\phi_0, \phi_+\right)$ is given by (27). By assumption, we have $\Gamma^a(\phi_0) = \Gamma^b(\phi_0)$ and $\Gamma^a\left(\phi_+\right) = \Gamma^b\left(\phi_+\right)$, which together with the definition of h^i, imply $h^a\left(\phi_0, \phi_+\right) = h^b\left(\phi_0, \phi_+\right)$. Combining this result with (74) for $i = a, b$ yields

$$\begin{aligned}
\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} &= e^{\sigma \int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^a(u),u\right)}{\partial \phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial \phi}\right] du} \frac{\left[1+F\left(K^a_0 x^a\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K^a_0\right)K_1\right]}{\left[1+F\left(K^b_0 x^b\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K^b_0\right)K_1\right]}, \\
\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} &\leq e^{\sigma \int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^a(u),u\right)}{\partial \phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial \phi}\right] du} \frac{\left[1+F\left(K^b_0 \lambda x^b\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K^b_0 \lambda\right)K_1\right]}{\left[1+F\left(K^b_0 x^b\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K^b_0\right)K_1\right]},
\end{aligned} \tag{75}$$

where the second line uses $\lambda \equiv K^a_0/K^b_0 > 1$ and $x^b\left(\phi_+\right) \geq x^a\left(\phi_+\right)$, with the latter being a consequence of the strict log-supermodularity of A and the fact that $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ for $\phi \in \left(\phi_0, \phi_+\right)$. Another implication of this the last two observation is that the first term of the right-hand side of the last expression is weakly lower than 1. Focusing on the second term, note that

$$\begin{aligned}
\frac{\left[1+F\left(K^b_0 \lambda x^b\left(\phi_+\right)\right)K_1\right]}{\left[1+F\left(K^b_0 \lambda\right)K_1\right]} &= \exp\left\{\int_{\phi_0}^{\phi_+} \frac{F_y\left(K^b_0 \lambda x^b(\phi)\right)K^b_0 K_1 \lambda x^b_\phi(\phi)}{\left[1+F\left(K^b_0 \lambda x^b(\phi)\right)K_1\right]} d\phi\right\} = \exp\left\{\int_{\phi_0}^{\phi_+} \eta^0\left(K^b_0 x^b(\phi), \lambda\right) K^b_0 x^b_\phi(\phi)\, d\phi\right\} \\
\frac{\left[1+F\left(K^b_0 x^b\left(\phi_+\right)\right)K_1\right]}{\left[1+F\left(K^b_0\right)K_1\right]} &= \exp\left\{\int_{\phi_0}^{\phi_+} \frac{F_y\left(K^b_0 x^b(\phi)\right)K_1 K^b_0 x^b_\phi(\phi)}{\left[1+F\left(K^b_0 x^b(\phi)\right)K_1\right]} d\phi\right\} = \exp\left\{\int_{\phi_0}^{\phi_+} \eta^0\left(K^b_0 x^b(\phi), 1\right) K^b_0 x^b_\phi(\phi)\, d\phi\right\}
\end{aligned} \tag{76}$$

As η^0 is strictly decreasing in λ, the second line in (76) is strictly greater than the first, so the second term on the right-hand side of the second line of (75) is strictly lower than 1, contradicting (73). Accordingly, the statement in step 1 must be true.

STEP 2: *Under the assumptions of the lemma, $\Gamma^a_\phi(\phi_0) > \Gamma^b_\phi(\phi_0)$, so there is a $\phi_+ \in (\phi_0, \phi_1)$ such that $\Gamma^a\left(\phi_+\right) = \Gamma^b\left(\phi_+\right)$ and $\Gamma^a(\phi) > \Gamma^b(\phi)$ on (ϕ_0, ϕ_+).*

The result of step 1 immediately yields that $\Gamma^a_\phi(\phi_0) \geq \Gamma^b_\phi(\phi_0)$. Otherwise, $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$ implies that there is a $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a(\phi) < \Gamma^b(\phi)$ on $(\phi_0, \phi']$, contradicting the result in step 1. Suppose then that $\Gamma^a_\phi(\phi_0) = \Gamma^b_\phi(\phi_0) = \gamma_0$. Note that the (same) boundary conditions of the BVPs under consideration imply $\Gamma^i(\phi_0) = s_0$, $x^i(\phi_0) = 1$. In turn, these observations and equations (22a)-(22b) imply

$x^i_\phi(\phi) = \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial\phi}$ and $\frac{z^i_\phi(\phi_0)}{z^i(\phi_0)} = -\frac{\partial \ln A(s_0,\phi_0)}{\partial\phi}$. Log-differentiating both sides of equation (22c) and evaluating at ϕ_0 yields

$\frac{\Gamma^i_{\phi\phi}(\phi_0)}{\Gamma^i_\phi(\phi_0)} = \frac{x^i_\phi(\phi_0)}{x^i(\phi_0)} + \frac{F_y\left(K^i_0 x^i(\phi_0)\right)K_1 K^i_0 x^i_\phi(\phi_0)}{\left[1+F\left(K^i_0 x^i(\phi_0)\right)K_1\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - \left[\frac{\partial \ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial s}\Gamma^i_\phi(\phi_0) + \frac{\partial \ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial\phi} + \frac{V_s\left(\Gamma^i(\phi_0)\right)}{V(\Gamma^i(\phi_0))}\Gamma^i_\phi(\phi_0) + \frac{z^i_\phi(\phi_0)}{z^i(\phi_0)}\right],$

$\frac{\Gamma^i_{\phi\phi}(\phi)}{\gamma_0} = \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial\phi} + \frac{F_y\left(K^i_0\right)K_1 K^i_0 \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial\phi}}{\left[1+F\left(K^i_0\right)K_1\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - \left[\frac{\partial \ln A(s_0,\phi_0)}{\partial s}\gamma_0 + \frac{\partial \ln A(s_0,\phi_0)}{\partial\phi} + \frac{V_s(s_0)}{V(s_0)}\gamma_0 - \frac{\partial \ln A(s_0,\phi_0)}{\partial\phi}\right],$

i.e.,

$$\Gamma^a_{\phi\phi}(\phi_0) - \Gamma^b_{\phi\phi}(\phi_0) = K^b_0 \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial\phi}\gamma_0 \left\{\frac{F_y\left(K^b_0\lambda\right)\lambda K_1}{\left[1+F\left(K^b_0\lambda\right)K_1\right]} - \frac{F_y\left(K^b_0\right)K_1}{\left[1+F\left(K^b_0\right)K_1\right]}\right\} < 0,$$

where the inequality follows from $\eta^0\left(K^b_0,\lambda\right) < \eta^0\left(K^b_0,1\right)$. The last expression implies that there is some $\phi' \in (\phi_0,\phi_1]$ such that $\Gamma^a_\phi(\phi) < \Gamma^b_\phi(\phi)$ on $(\phi_0,\phi']$, which yields a contradiction of step 1. Accordingly, we must have $\Gamma^a_\phi(\phi_0) > \Gamma^b_\phi(\phi_0)$.

Finally, $\Gamma^a_\phi(\phi_0) > \Gamma^b_\phi(\phi_0)$ implies $\Gamma^a(\phi) > \Gamma^b(\phi)$ on some (small enough) interval $\left(\phi_0,\phi''\right)$, so ϕ^+ described in the statement of the step the first time Γ^a and Γ^b intersect to the right of ϕ''.

STEP 3: *Under the assumptions of the lemma, $\Gamma^a(\phi) > \Gamma^b(\phi)$ on (ϕ_0,ϕ_1).*

I show that $\phi_+ = \phi_0$, where ϕ_+ was defined in step 2. Suppose for a moment that $\phi_+ < \phi_0$. If we define on $[\phi_+,\phi_0]$, $w^i(\phi) \equiv x^i(\phi)/x^i\left(\phi_+\right)$ and $y^i(\phi) = z^i(\phi)/x^i\left(\phi_+\right)$, then it is readily seen that $\left\{y^i, w^i(\phi), \Gamma^i\right\}$ solve BVP (22) in said interval, with $\left\{\alpha^i(\phi), K^i_1\right\} = \{\alpha(\phi), K_1\}$ and parameter $\overline{K}^i_0 = K^i_0 x^i\left(\phi_+\right)$. Per step 2 we have $x^a\left(\phi_+\right) > x^b\left(\phi_+\right)$, so $\overline{K}^a_0 > \overline{K}^b_0$. Then, the BVPs associated to $\left\{y^i, w^i(\phi), \Gamma^i\right\}$ satisfy the conditions of lemma 4.vi, so step 2 implies $\Gamma^a_\phi(\phi_0) > \Gamma^b_\phi(\phi_0)$. However, $\Gamma^a(\phi) > \Gamma^b(\phi)$ on $\left(\phi_0,\phi_+\right)$ implies $\Gamma^a_\phi\left(\phi_+\right) \le \Gamma^b_\phi\left(\phi_+\right)$, so assuming $\phi_+ < \phi_0$ yields a contradiction.

Lemma 4.vii. I prove the statement for the case in which $\eta_1(t,\lambda)$ is strictly increasing in λ.

STEP 1: *Under the assumptions of the lemma, there is no $\phi' \in (\phi_0,\phi_1]$ such that $\Gamma^a(\phi) \ge \Gamma^b(\phi)$ for all $\phi \in (\phi_0,\phi']$.*

Suppose to the contrary that there is such a value $\phi' \in (\phi_0,\phi_1]$. Let ϕ_+ be the first time the functions Γ^a and Γ^b intersect to the right of ϕ', i.e. $\phi_+ = \inf\left\{\phi \ge \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\right\}$. Note ϕ_+ is well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the right of ϕ' (at ϕ_1). Also note that $\Gamma^a(\phi) \ge \Gamma^b(\phi)$ for $\phi \in \left(\phi_0,\phi_+\right)$. The continuity of Γ^a_ϕ and Γ^b_ϕ, implies $\Gamma^a_\phi(\phi_0) \ge \Gamma^b_\phi(\phi_0)$ and $\Gamma^a_\phi(\phi_+) \le \Gamma^b_\phi(\phi_+)$, i.e.

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} \le 1. \tag{77}$$

Differentiating the right-hand side of (26) yields

$$\frac{\Gamma^i_\phi\left(\phi_+\right)}{\Gamma^i_\phi(\phi_0)} = h^i\left(\phi_0,\phi_+\right) e^{\sigma\int_{\phi_-}^{\phi_+}\frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial\phi}du}\,\frac{\left[1+F\left(K^i_0 x^i(\phi)\right)K^i_1\right]}{\left[1+F\left(K^i_0\right)K^i_1\right]}, \tag{78}$$

where $h^i(\phi_0, \phi_+)$ is given by (27). By assumption, we have $\Gamma^a(\phi_0) = \Gamma^b(\phi_0)$ and $\Gamma^a(\phi_+) = \Gamma^b(\phi_+)$, which together with the definition of h^i, imply $h^a(\phi_0, \phi_+) = h^b(\phi_0, \phi_+)$. Combining this result with (78) for $i = a, b$ yields

$$\begin{aligned} \frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} &= e^{\sigma \int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^a(u),u\right)}{\partial \phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial \phi}\right]du} \frac{\left[1+F\left(K_0^a x^a(\phi_+)\right)K_1^a\right]/\left[1+F\left(K_0^a\right)K_1^a\right]}{\left[1+F\left(K_0^b x^b(\phi_+)\right)K_1^b\right]/\left[1+F\left(K_0^b\right)K_1^b\right]}, \\ \frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} &\geq e^{\sigma \int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^a(u),u\right)}{\partial \phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial \phi}\right]du} \frac{\left[1+F\left(K_0^b \lambda x^b(\phi_+)\right)K_1^b\lambda\right]/\left[1+F\left(K_0^b\lambda\right)K_1^b\lambda\right]}{\left[1+F\left(K_0^b x^b(\phi_+)\right)K_1^b\right]/\left[1+F\left(K_0^b\right)K_1^b\right]}, \end{aligned} \tag{79}$$

where the second line uses $\lambda \equiv K_i^a/K_i^b > 1$ and $x^a(\phi_+) \geq x^b(\phi_+)$, with the latter being a consequence of the strict log-supermodularity of A and the fact that $\Gamma^a(\phi) \geq \Gamma^b(\phi)$ for $\phi \in (\phi_0, \phi_+)$. Another implication of the last two observations is that the first term of the right-hand side of the last expression is weakly greater than 1. Focusing on the second term, note that

$$\begin{aligned} \frac{\left[1+F\left(K_0^b \lambda x^b(\phi_+)\right)K_1^b\lambda\right]}{\left[1+F\left(K_0^b\lambda\right)K_1^b\lambda\right]} &= \exp\{\int_{\phi_0}^{\phi_+} \frac{F_y\left(K_0^b\lambda x^b(\phi)\right)K_0^b K_1^b \lambda^2 x^b_\phi(\phi)}{\left[1+F\left(K_0^b\lambda x^b(\phi)\right)K_1^b\lambda\right]} d\phi\} = \exp\{\int_{\phi_0}^{\phi_+} \eta^1\left(K_0^b x^b(\phi), \lambda\right) K_0^b x^b_\phi(\phi)\, d\phi\} \\ \frac{\left[1+F\left(K_0^b x^b(\phi_+)\right)K_1^b\right]}{\left[1+F\left(K_0^b\right)K_1^b\right]} &= \exp\{\int_{\phi_0}^{\phi_+} \frac{F_y\left(K_0^b x^b(\phi)\right)K_0^b K_1^b x^b_\phi(\phi)}{\left[1+F\left(K_0^b x^b(\phi)\right)K_1^b\right]} d\phi\} = \exp\{\int_{\phi_0}^{\phi_+} \eta^1\left(K_0^b x^b(\phi), 1\right) K_0^b x^b_\phi(\phi)\, d\phi\} \end{aligned} \tag{80}$$

As η^1 is strictly increasing in λ, the second line in (80) is strictly lower than the first, so the second term on the right-hand side of the second line of (79) is strictly greater than 1, contradicting (77). Accordingly, the statement in step 1 must be true.

STEP 2: *Under the assumptions of the lemma,* $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$*, so there is a* $\phi_+ \in (\phi_0, \phi_1)$ *such that* $\Gamma^a(\phi_+) = \Gamma^b(\phi_+)$ *and* $\Gamma^a(\phi) < \Gamma^b(\phi)$ *on* (ϕ_0, ϕ_+).

The result of step 1 immediately yields that $\Gamma^a_\phi(\phi_0) \leq \Gamma^b_\phi(\phi_0)$. Otherwise, $\Gamma^a_\phi(\phi_0) > \Gamma^b_\phi(\phi_0)$ implies that there is a $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a(\phi) > \Gamma^b(\phi)$ on $(\phi_0, \phi']$, contradicting the result in step 1. Suppose then that $\Gamma^a_\phi(\phi_0) = \Gamma^b_\phi(\phi_0) = \gamma_0$. Note that the (same) boundary conditions of the BVPs under consideration imply $\Gamma^i(\phi_0) = s_0$, $x^i(\phi_0) = 1$. In turn, these observations and equations (22a)-(22b) imply $x^i_\phi(\phi) = \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi}$ and $\frac{z^i_\phi(\phi_0)}{z^i(\phi_0)} = -\frac{\partial \ln A(s_0,\phi_0)}{\partial \phi}$. Log-differentiating both sides of equation (22c) and evaluating at ϕ_0 yields

$$\frac{\Gamma^i_{\phi\phi}(\phi_0)}{\Gamma^i_\phi(\phi_0)} = \frac{x^i_\phi(\phi_0)}{x^i(\phi_0)} + \frac{F_y\left(K_0^i x^i(\phi_0)\right)K_1^i K_0^i x^i_\phi(\phi_0)}{\left[1+F\left(K_0^i x^i(\phi_0)\right)K_1^i\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - \left[\frac{\partial \ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial s}\Gamma^i_\phi(\phi_0) + \frac{\partial \ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial \phi} + \frac{V_s\left(\Gamma^i(\phi_0)\right)}{V\left(\Gamma^i(\phi_0)\right)}\Gamma^i_\phi(\phi_0) + \frac{z^i_\phi(\phi_0)}{z^i(\phi_0)}\right],$$

$$\frac{\Gamma^i_{\phi\phi}(\phi)}{\gamma_0} = \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi} + \frac{F_y\left(K_0^i\right)K_1^i K_0^i \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi}}{\left[1+F\left(K_0^i\right)K_1^i\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - \left[\frac{\partial \ln A(s_0,\phi_0)}{\partial s}\gamma_0 + \frac{\partial \ln A(s_0,\phi_0)}{\partial \phi} + \frac{V_s(s_0)}{V(s_0)}\gamma_0 - \frac{\partial \ln A(s_0,\phi_0)}{\partial \phi}\right],$$

i.e.,

$$\Gamma^a_{\phi\phi}(\phi_0) - \Gamma^b_{\phi\phi}(\phi_0) = K_0^b \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi}\gamma_0 \left\{\frac{F_y\left(K_0^b\lambda\right)K_1^b\lambda^2}{\left[1+F\left(K_0^b\lambda\right)\lambda K_1^b\right]} - \frac{F_y\left(K_0^b\right)K_1^b}{\left[1+F\left(K_0^b\right)K_1^b\right]}\right\} > 0,$$

where the inequality follows from $\eta^1\left(K_0^b, \lambda\right) > \eta^1\left(K_0^b, 1\right)$. The last expression implies that there is some $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a_\phi(\phi) > \Gamma^b_\phi(\phi)$ on $(\phi_0, \phi']$, which yields a contradiction of step 1. Accordingly, we must have $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$.

Finally, $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$ implies $\Gamma^a(\phi) < \Gamma^b(\phi)$ on some (small enough) interval (ϕ_0, ϕ''), so ϕ^+ described in the statement of the step the first time Γ^a and Γ^b intersect to the right of ϕ''.

STEP 3: *Under the assumptions of the lemma,* $\Gamma^a(\phi) < \Gamma^b(\phi)$ *on* (ϕ_0, ϕ_1).

I show that $\phi_+ = \phi_0$, where ϕ_+ was defined in step 2. Suppose for a moment that $\phi_+ < \phi_0$. If we define on $[\phi_+, \phi_1]$, $w^i(\phi) \equiv x^i(\phi)/x^i(\phi_+)$ and $y^i(\phi) = z^i(\phi)/x^i(\phi_+)$, then it is readily seen that $\{y^i, w^i(\phi), \Gamma^i\}$ solve BVP (22) in said interval, with $\alpha^i(\phi) = \alpha(\phi)$ and parameter $\overline{K}^i_0 = K^i_0 x^i(\phi_+)$. That is, $\overline{K}^a_0 = \lambda_1 \overline{K}^b_0$, where $\lambda_1 \equiv \frac{\lambda x^a(\phi_+)}{x^b(\phi_+)}$. As the BVPs associated with Γ^i satisfy the conditions of lemma 4.iv on $[\phi_0, \phi_+]$, $\lambda_1 > 1$. In addition, step 2 implies $x^a(\phi_+) < x^b(\phi_+)$, so $\lambda_1 < \lambda$.

The previous discussion implies that the BVPs that $\{y^i, w^i(\phi), \Gamma^i\}$ for $i = a, b$ solve on $[\phi_+, \phi_1]$ differ only in the parameters $\{\overline{K}^i_0, K^i_1\}$, with $\overline{K}^a_0 = \lambda_1 \overline{K}^b_0$ and $K^a_1 = \lambda K^b_1$. To understand the implication of this difference, it is convenient to consider a third BVP on $[\phi_+, \phi_1]$ differing from the previous two only in the parameters $\{\overline{K}^c_0, K^c_1\}$, with $\overline{K}^c_0 = \overline{K}^a_0 = \lambda_1 \overline{K}^b_0$ and $K^c_1 = \lambda_1 K^b_1$. Given these definitions, note that the BVPs associated with $\{y^b, w^b(\phi), \Gamma^b\}$ and $\{y^c, w^c(\phi), \Gamma^c\}$ satisfy the conditions in lemma 4.vii, so $\Gamma^c_\phi(\phi_+) < \Gamma^b_\phi(\phi_+)$. In addition, the BVPs associated to $\{y^a, w^a(\phi), \Gamma^a\}$ and $\{y^c, w^c(\phi), \Gamma^c\}$ satisfy the assumptions of 4.ii with $K^a_1 > K^c_1$, so $\Gamma^a_\phi(\phi_+) < \Gamma^c_\phi(\phi_+)$. These inequalities yield $\Gamma^a_\phi(\phi_+) < \Gamma^b_\phi(\phi_+)$. However, step 2 implies that $\Gamma^a_\phi(\phi_+) \geq \Gamma^b_\phi(\phi_+)$, which is a contradiction. Then it must be the case that $\phi_+ = \phi_1$.

This concludes the proof of lemma 4. ■

A.4 Section 5

A.4.1 Proof of Proposition 2

Let us start with the proof of $\phi^*_a < \phi^*_\tau$. For any $\phi^* \in [\underline{\phi}, \overline{\phi}]$, let $\{\overline{p}(.;\phi^*), \overline{r}^d(.;\phi^*), \overline{H}(;\phi^*)\}$ denote the solution to the BVP of the open economy described in lemma 3.iii, and let $\{p(.;\phi^*), r^d(.;\phi^*), H(.;\phi^*)\}$ be the solution to the BVP of the closed economy described in lemma 1.ii where the notation emphasizes the dependence of the solution on the parameter ϕ^*. Note that this notation implies $\{p^a, r^{d,a}, H^a\} = \{p(.;\phi^*_a), r^d(.;\phi^*_a), H(;\phi^*_a)\}$ and $\{p^\tau, r^{d,\tau}, H^\tau\} = \{\overline{p}(.;\phi^*_\tau), \overline{r}^d(.;\phi^*_\tau), \overline{H}(;\phi^*_\tau)\}$, where the superscripts a and τ denote, respectively, the variables corresponding the autarky and trade equilibria of the economy under consideration. Per the discussion leading to proposition 1, the left-hand side of equation (21), which pins down the activity cutoff in the open economy, is strictly decreasing in the value of the parameter ϕ^*. Then, the result is proved if we show that the left-hand side of (21) is strictly greater than the right-hand side at $\phi^* = \phi^*_a$, i.e. if we show $\beta(\overline{r}^d(.;\phi^*_a), \phi^*_a) > \beta^a(r^d(.;\phi^*_a), \phi^*_a) = L$.[54]

First, I show that lemma 4.i implies that when the BVPs of the open and closed economy share the same boundary conditions, then the inverse matching function (matching function) corresponding to the former lies completely below (above) that of the latter. In particular, for any $\phi^* \in [\underline{\phi}, \overline{\phi}]$, $\overline{H}(\phi;\phi^*) < H(\phi;\phi^*)$ for all $\phi \in (\phi^*, \overline{\phi})$. Define $x(\phi;\phi^*) \equiv r^d(\phi;\phi^*)/\sigma f$ and $z(\phi;\phi^*) \equiv \frac{p(\phi;\phi^*)}{\sigma f}\left[L - f\left[1 - G(\phi^*)\right]\overline{M}\right]$. Then,

[54] $\beta^a(.,.)$ and $\beta(.,.)$ are defined in proposition 1 as the functions of r^d and ϕ^* defined by left-hand side of equations (16) and (21), respectively.

$\{z(.;\phi^*), x(.;\phi^*), H(.;\phi^*)\}$ is the unique solution to BVP (22) with parameters $K_1 = 0$, $\alpha(\phi;\phi^*) = 1$ and boundary conditions $x(\phi^*) = 1$, $H(\phi^*) = \underline{s}$ and $H(\overline{\phi}) = \overline{s}$. Similarly, if we define $\overline{x}(\phi;\phi^*) \equiv \overline{r}^d(\phi;\phi^*)/\sigma f$ and $\overline{z}(\phi;\phi^*) \equiv \frac{\overline{p}(\phi)}{\sigma f}[L - fM - \int_{\phi^*}^{\overline{\phi}} f_x \int_0^{f\overline{x}(\phi';\phi^*)\tau^{1-\sigma}/f_x} ydF(y)\, g(\phi')\, \overline{M} d\phi']$, then we can think of the solution to the open economy BVP $\{\overline{z}(.;\phi^*), \overline{x}(.;\phi^*), \overline{H}(.;\phi^*)\}$ as the unique solution to BVP (22) with parameters $\overline{K}_1 = 0$, $\overline{\alpha}(\phi;\phi^*) = \left[1 + F\left(\frac{f\tau^{1-\sigma}}{f_x}\overline{x}(\phi;\phi^*)\right)\tau^{1-\sigma}\right]$ and boundary conditions $\overline{x}(\phi^*) = 1$, $\overline{H}(\phi^*) = \underline{s}$ and $\overline{H}(\overline{\phi}) = \overline{s}$.[55] Given these definitions, it is readily seen that $\{z(.;\phi^*), x(.;\phi^*), H(.;\phi^*)\}$ and $\{\overline{z}(.;\phi^*), \overline{x}(.;\phi^*), \overline{H}(.;\phi^*)\}$ satisfy the conditions of lemma 4.i, with $\{\overline{\alpha}, 1\}$ taking the roles of $\{\alpha^a, \alpha^b\}$, respectively. Then, $\overline{H}(\phi;\phi^*) < H(\phi;\phi^*)$ for all $\phi \in (\phi^*, \overline{\phi})$.

I now show $\beta(\overline{r}(.;\phi_a^*), \phi_a^*) > \beta^a(r^d(.;\phi_a^*), \phi_a^*) = L$. The result in the last paragraph implies that $\{z(.;\phi^*), x(.;\phi^*), H(.;\phi^*)\}$ and $\{\overline{z}(.;\phi^*), \overline{x}(.;\phi^*), \overline{H}(.;\phi^*)\}$ satisfy the conditions of lemma 4.iii, so

$\int_{\phi^*}^{\overline{\phi}} \overline{x}(\phi;\phi^*)\frac{\overline{\alpha}(\phi;\phi^*)}{\overline{\alpha}(\phi^*;\phi^*)} g(\phi)\, d\phi > \int_{\phi^*}^{\overline{\phi}} x(\phi;\phi^*) g(\phi)\, d\phi.$

An implication of this result and $\overline{\alpha}(\phi^*;\phi^*) \geq 1$ is that total wages paid to production workers are higher in the open economy if it shares the activity cutoff with the closed economy,

$$\frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} \overline{r}^d(\phi;\phi^*)\,\overline{\alpha}(\phi;\phi^*)\, g(\phi)\, d\phi\overline{M} = (\sigma-1) f\overline{\alpha}(\phi^*;\phi^*)\int_{\phi^*}^{\overline{\phi}} \overline{x}(\phi;\phi^*)\frac{\overline{\alpha}(\phi;\phi^*)}{\overline{\alpha}(\phi^*;\phi^*)} g(\phi)\, d\phi\overline{M} > \cdots$$
$$\cdots (\sigma-1) f\int_{\phi^*}^{\overline{\phi}} x(\phi;\phi^*) g(\phi)\, d\phi\overline{M} = \frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} r^d(\phi;\phi^*) g(\phi)\, d\phi\overline{M}$$

In addition, per definition we have,

$$\beta^a\left(r^d(\phi;\phi^*), \phi^*\right) = \frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} r^d(\phi;\phi^*) g(\phi)\, d\phi\overline{M} + f\left[1 - G(\phi^*)\right]\overline{M}$$

$$\beta\left(\overline{r}^d(\phi;\phi^*), \phi^*\right) = \left\{\begin{array}{r} \frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} \overline{r}^d(\phi;\phi^*)\,\overline{\alpha}(\phi;\phi^*)\, g(\phi)\, d\phi\overline{M} + \cdots \\ f\left[1 - G(\phi^*)\right]\overline{M} + \int_{\phi^*}^{\overline{\phi}} f_x \int_0^{\frac{\overline{r}^d(\phi;\phi^*)\tau^{1-\sigma}}{\sigma f_x}} ydF(y)\, g(\phi')\,\overline{M} d\phi', \end{array}\right.$$

For $\phi^* = \phi_a^*$, these observations imply

$$\beta\left(\overline{r}^d(.;\phi_a^*), \phi_a^*\right) > \beta^a\left(r^d(\phi;\phi_a^*), \phi_a^*\right) = L,$$

which is the desired result.[56]

Let us now prove the other results in the proposition, i.e. $N^\tau(s) > N^a(s)$ for all $s \in [\underline{s}, \overline{s})$ and proposition 2.ii. Let $N(s;\phi^*)$ be the inverse function of $H(\phi;\phi^*)$. Following the discussion above, these results can be easily proved by decomposing the total effect on the matching function into that of the increase in the exit cutoff (intensive-margin channel) and that of having an increasing share of exporters at each productivity level in the open economy (extensive-margin channel). Starting with the former, the no-crossing result in lemma 2.i and $\phi_a^* < \phi_\tau^*$ imply $N^a(s) = N(s;\phi_a^*) < N(s;\phi_\tau^*)$ on $[\underline{s}, \overline{s})$.[57] Bringing

[55] Note that we are considering $\{\overline{z}(.;\phi^*), \overline{x}(.;\phi^*), \overline{H}(.;\phi^*)\}$ as the solution to a different parametrization of the BVP (22) than the one considered in section 4.2.

[56] In this derivation we used $\sigma f x(\phi;\phi_a^*) = r^{d,a}(\phi)$ and the fact that equation (16) holds in autarky.

[57] As $N(.;\phi^*)$ solves the BVP of the closed economy with activity cutoff ϕ^*, note that $N(s;\phi_\tau^*)$ is the matching function of the ancillary autarkic economy described in the paper.

the effects of exporters into the picture, lemma 4.i implies that $H(\phi;\phi_\tau^*) > \overline{H}(\phi;\phi_\tau^*) = H^\tau(\phi)$ on $(\phi_\tau^*, \overline{\phi})$, i.e. $N(s;\phi_\tau^*) < \overline{N}(s;\phi_\tau^*) = N^\tau(s)$ on $(\underline{s}, \overline{s})$. Combining these observations yield the desired result.

A.4.2 Proof of Proposition 3

Proposition 3.i

Let us start with the proof of $\phi_h^* < \phi_l^*$. For any $\phi^* \in [\underline{\phi}, \overline{\phi}]$ and $i = l, h$, let $\left\{\overline{p}^i(.;\phi^*), \overline{r}^{d,i}(.;\phi^*), \overline{H}^i(;\phi^*)\right\}$ denote the solution to the BVP of the open economy described in lemma 3.iii with variable trade costs τ_i and productivity exit cutoff ϕ^*(the notation emphasizes the dependence of the solution on τ_i and ϕ^*.) With this notation we have $\{p^i, r^{d,i}, H^i\} = \left\{\overline{p}^i(.;\phi_i^*), \overline{r}^{d,i}(.;\phi_i^*), \overline{H}^i(;\phi_i^*)\right\}$, where $\{p^i, r^{d,i}, H^i\}$ are the equilibrium price, revenue and inverse-matching functions of an open economy with variable trade costs τ_i. Let $\beta^i(r^d, \phi^*)$ be the function defined by the left-hand side of equation (21) in terms of r^d and ϕ^* when variable trade costs are given by τ_i.[58] Per the discussion leading to proposition 1, $\beta^i(\overline{r}^{d,i}(.;\phi^*), \phi^*)$ is strictly decreasing in the value of the parameter ϕ^*. Then, to prove the result it is enough to show $\beta^l(\overline{r}^{d,l}(.;\phi_h^*), \phi_h^*) > \beta^h(\overline{r}^{d,h}(.;\phi_h^*), \phi_h^*) = L$.

As a first step, I show that for any $\phi^* \in [\underline{\phi}, \overline{\phi})$, $\overline{r}^{d,l}(\phi;\phi^*)\tau_l^{1-\sigma} > \overline{r}^{d,h}(\phi;\phi^*)\tau_h^{1-\sigma}$ for all $\phi \in [\phi^*, \overline{\phi}]$. Letting

$$\overline{x}^i(\phi;\phi^*) \equiv \overline{r}^{d,i}(\phi;\phi^*)/\sigma f,$$
$$\overline{z}^i(\phi;\phi^*) \equiv \tfrac{\overline{p}^i(\phi)}{\sigma f}[L - fM - \textstyle\int_{\phi^*}^{\overline{\phi}} f_x \int_0^{f\overline{x}^i(\phi';\phi^*)\tau_i^{1-\sigma}/f_x} ydF(y)\, g(\phi')\,\overline{M}d\phi'],$$

then $\{z^i(.;\phi^*), \overline{x}^i(.;\phi^*), \overline{H}^i(.;\phi^*)\}$ is the unique solution to BVP (22) with parameters $K_0^i = \frac{f}{fx}\tau_i^{1-\sigma}$, $K_1^i = \tau_i^{1-\sigma}$, $\alpha^i(\phi;\phi^*) = 1$ and boundary conditions $\overline{x}^i(\phi^*) = 1$, $\overline{H}^i(\phi^*) = \underline{s}$ and $\overline{H}^i(\overline{\phi}) = \overline{s}$. Noting that $K_0^l = \lambda K_0^h$ and $K_1^l = \lambda K_1^h$ with $\lambda = (\tau_l/\tau_h)^{1-\sigma} > 1$, it is readily seen that $\left\{\overline{z}^i(.;\phi^*), \overline{x}^i(.;\phi^*), \overline{H}^i(.;\phi^*)\right\}$ for $i = l, k$, satisfy the conditions of lemma 4.iv, so $\overline{r}^{d,l}(\phi;\phi^*)\tau_l^{1-\sigma} > \overline{r}^{d,h}(\phi;\phi^*)\tau_h^{1-\sigma}$ for all $\phi \in [\phi^*, \overline{\phi}]$.

Let us now show $\beta^l(\overline{r}^{d,l}(.;\phi_h^*), \phi_h^*) > \beta^h(\overline{r}^{d,h}(.;\phi_h^*), \phi_h^*) = L$. To economize on space, I define the following notation

$$\begin{aligned}
\delta^i(\phi) &= \left[1 + F\left(K_0^i \overline{x}^i(\phi;\phi^*)\right) K_1^i\right] \\
R^i(\phi^*) &\equiv \int_{\phi^*}^{\overline{\phi}} \overline{r}^{d,i}(\phi;\phi^*)\left[1 + F\left(\tfrac{\overline{r}^{d,i}(\phi;\phi^*)}{\sigma f_x}\tau_i^{1-\sigma}\right)\tau_i^{1-\sigma}\right] g(\phi)\,d\phi\overline{M}, \\
FF^d(\phi^*) &\equiv f\left[1 - G(\phi^*)\right]\overline{M}, \text{ and } FF^{x,i}(\phi^*) \equiv \int_{\phi^*}^{\overline{\phi}} f_x \int_0^{\frac{\overline{r}^{d,i}(\phi;\phi^*)\tau^{1-\sigma}}{\sigma f_x}} ydF(y)\, g(\phi')\,\overline{M}d\phi',
\end{aligned}$$

where $\{K_0^i, K_1^i, \overline{x}^i\}$ were defined above. These definitions and the result in the previous paragraph imply $\delta^l(\phi) > \delta^h(\phi)$ for all $\phi \in [\phi^*, \overline{\phi}]$, i.e $\{\overline{z}^i(.;\phi^*), \overline{x}^i(.;\phi^*), \overline{H}^i(.;\phi^*)\}$ satisfy the conditions of lemma 4.v, so $R^l(\phi^*) > R^h(\phi^*)$. In addition, the result in the last paragraph also implies $FF^{x,l}(\phi^*) > FF^{x,h}(\phi^*)$.

[58] Note that $\beta^i(.,.)$ is just the function $\beta(.,.)$ defined in proposition 1, where the superscript i in the current notation emphasizes the dependence of this function on τ_i.

These inequalities and the definition of β^i yield

$$\begin{aligned}\beta^l\left(\overline{r}^{d,l}\left(\phi;\phi_h^*\right),\phi_h^*\right) &= \tfrac{\sigma-1}{\sigma}R^l\left(\phi_h^*\right)+FF^d(\phi_h^*)+FF^{x,l}\left(\phi_h^*\right)\\ &> \tfrac{\sigma-1}{\sigma}R^h\left(\phi_h^*\right)+FF^d(\phi_h^*)+FF^{x,h}\left(\phi_h^*\right)\\ &= \beta^h\left(\overline{r}^{d,h}\left(\phi;\phi_h^*\right),\phi_h^*\right)=L.\end{aligned}$$

As discussed above, this result implies $\phi_h^* < \phi_l^*$.

Finally, the continuity of the matching functions and $\phi_h^* < \phi_l^*$ imply that there is a skill level $s' \in (\underline{s},\overline{s}]$ such that $N^l(s) > N^h(s)$ on $[\underline{s},s')$, i.e. inequality necessarily increases among the least skilled workers of the economy after a trade liberalization.

Proposition 3.ii

Here I formally derive the impact on relative wages of the selection-into-activity and the intensive-margin channels discussed in the text. I start by defining some notation. In the sequel, $\{z(\phi;\phi^*,\alpha), x(\phi;\phi^*,\alpha), H(\phi;\phi^*,\alpha)\}$ denotes the unique solution to BVP (22) with constant $K_1=0$, parameter function α, and boundary conditions $\{\overline{x}(\phi^*)=1, H(\phi^*)=\underline{s}, H\left(\overline{\phi}\right)=\overline{s}\}$, where the notation emphasizes the dependence of the solution on $\{\phi^*,\alpha\}$. In addition, I will use $N(\phi;\phi^*,\alpha)$ to denote the inverse of $H(\phi;\phi^*,\alpha)$. For $i=l,h$, let $\left\{\phi_i^*,p^i,r^{d,i},H^i\right\}$ be the activity cutoff, price, domestic revenue and inverse-matching functions of the two open economies in the statement of the proposition (these economies differ only in the variable trade costs they face, with $\tau_l<\tau_h$). Defining the parameter functions $\alpha^i(\phi) \equiv \left[1+F\left(\frac{\tau_i^{1-\sigma}}{\sigma f_x}r^{d,i}(\phi)\right)\tau_i^{1-\sigma}\right]$ for $i=l,h$, we can think of the BVPs associated with each H^i as particular parameterizations of BVP (22) with $K_1=0$ and $\alpha=\alpha^i$.[59] In the notation defined here,

$$\begin{aligned}x\left(\phi;\phi_i^*,\alpha^i\right) &= r^{d,i}\left(\phi;\phi_i^*\right)/\sigma f\\ z\left(\phi;\phi_i^*,\alpha^i\right) &= \frac{p^i(\phi)}{\sigma f}[L-fM-\int_{\phi^*}^{\overline{\phi}} f_x\int_0^{\frac{r^{d,i}(\phi;\phi_i^*)\tau^{1-\sigma}}{\sigma f_x}} ydF(y)\,g\left(\phi'\right)\overline{M}d\phi']\\ H\left(\phi;\phi_i^*,\alpha^i\right) &= H^i\end{aligned}$$

After these preliminaries we are ready to prove the claim.

Let us start with the **selection-into-activity channel**. As discussed in the text, the matching functions N_0 and N^h in figure 2 differ only in their activity cutoffs, i.e. $N_0=N\left(\phi;\phi_l^*,\alpha^h\right)$ and $N^h=N\left(\phi;\phi_h^*,\alpha^h\right)$. Accordingly, the no-crossing result in lemma 2.i implies $N_0(s)>N^h(s)$ on $[\underline{s},\overline{s})$. Note that by sharing the same parameter function α^h, the economies associated with N_0 and N^h have the same fraction of exporters at each productivity (among active firms) and face the same variable costs. Accordingly, their difference captures the effects of the selection-into-activity channel on relative wages.

Let us now turn to the **intensive-margin channel**. Define $\alpha^1(\phi) \equiv \left[1+F\left(\frac{\tau_h^{1-\sigma}}{\sigma f_x}r^{d,h}(\phi)\right)\tau_l^{1-\sigma}\right]$, so $\alpha^1(\phi)$ differs from $\alpha^h(\phi)$ only in the value of the variable trade cost outside the function F. In

[59] See the proof of proposition 2.

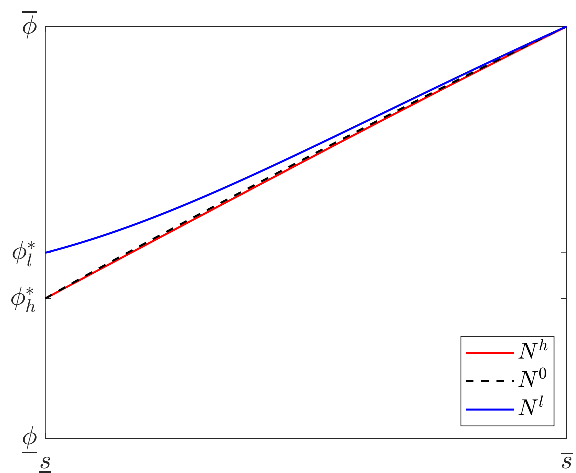
addition, note that for any pair $\phi'', \phi' \in [\phi^*, \overline{\phi}]$ such that $\phi'' > \phi'$ and $F\left(\tau_h^{1-\sigma} r^{d,h}(\phi'')/\sigma f_x\right) > 0$, we have $\alpha^1(\phi'')/\alpha^1(\phi') > \alpha^h(\phi'')/\alpha^h(\phi')$. Finally, as discussed in the text, the matching functions N_0 and N_1 in figure 2 differ only in their parameter function α, i.e. $N_0 = N\left(\phi; \phi_l^*, \alpha^h\right)$ and $N_1 = N\left(\phi; \phi_l^*, \alpha^1\right)$. Accordingly, the BVPs associated with N_0 and N_1 satisfy the conditions of lemma 4.i, so $N_1(s) > N_0(s)$ on $[\underline{s}, \overline{s})$.

Proposition 3.iii

To prove the result it is convenient to break the changes in the BVP of the open economy introduced by the liberalization in two parts, the change associated to the decline in variable trade costs and the change associated to the rise in the activity cutoff (allowing the set of exporters to adjust in each case). Starting with the former, let N_0 be the matching function resulting from reducing τ_h to τ_l in the BVP of the open economy before the liberalization, keeping the activity cutoff unchanged. If the assumption on η_1^F is satisfied, then it is readily seen that F and the open-economy BVPs associated with N^h and N^0 satisfy the conditions in lemma 4.vii with $K_0^h = f\tau_h^{1-\sigma}/f_x$, $K_1^h = n\tau_h^{1-\sigma}$, $K_i^0 = \lambda K_i^h$, and $\lambda = (\tau_l/\tau_h)^{1-\sigma} > 1$. Accordingly, $N^0(s) > N^h(s)$ on $(\underline{s}, \overline{s})$ as shown in figure 6.

Proof. ■

Figure 6: Trade Liberalization under Sufficient Conditions



Note: The solid red and blue lines represent, respectively, the pre- (N^h) and post-liberalization (N^l) matching functions. The dashed black line (N^0) represents the solution to the BVP of the open economy with $\tau = \tau_l$ and $\phi^* = \phi_h^*$. When η_1^F satisfies the sufficient conditions in proposition 3.iii, N^0 lies above N^h. When η_0^F satisfies the sufficient conditions in said proposition, N^l lies above N^0.

Now consider the change in the matching function associated with the rise in the activity cutoff, i.e. the difference between N_0 and N^l in figure 6. Suppose that N_0 and N^l intersect on $(\underline{s}, \overline{s})$ with the first intersection occurring at s', namely $N_0(s') = N^l(s') = \phi'$. If we define on $\left[\phi', \overline{\phi}\right]$, $w^i(\phi) \equiv \frac{r^{d,i}(\phi)}{r^{d,0}(\phi')}$ and $y^i(\phi) \equiv \frac{p^i(\phi)}{r^{d,i}(\phi')\overline{M}}[L - fM^i - \int_{\phi_i^*}^{\overline{\phi}} f_x \int_{\underline{y}}^{\frac{r^{d,i}(\phi)\tau_l^{1-\sigma}}{\sigma f_x}} ydF(y)\, g(\phi)\, \overline{M} d\phi]$, then $\left\{w^i, y^i, H^i\right\}$ is the unique solution to BVP (22) with parameters $\alpha^i(\phi) = 1$, $K_0^i = \frac{r^{d,i}(\phi')\tau_l^{1-\sigma}}{\sigma f}$, $K_1^i = n\tau_l^{1-\sigma}$ and boundary conditions

$w^i(\phi') = 1$, $H^i(\phi') = s'$ and $H^i(\overline{\phi}) = \overline{s}$. In addition, note that the log-supermodularity of A and $H^l(\phi) < H_0(\phi)$ on $[\phi_l^*, \phi')$ implies $r^{d,0}(\phi') > r^{d,l}(\phi')$, so $K_0^0 > K_0^l$. Accordingly, if the assumption on η_0^F is satisfied, then its is readily seen that F and the open-economy BVPs associated with N^l and N^0 satisfy the conditions of lemma 4.vi on $[\phi', \overline{\phi}]$, so $H_\phi^l(\phi') < H_\phi^0(\phi')$. However, $H^l(\phi) < H^0(\phi)$ on $[\phi_l^*, \phi')$ implies $H_\phi^l(\phi') \geq H_\phi^0(\phi')$, which is a contradiction. Then it must be the case that N^l and N^0 do not intersect on $(\underline{s}, \overline{s})$, so N^l lies strictly above N^0 on $[\underline{s}, \overline{s})$ as shown in the picture.

Combining the last two results we get $N^l(s) > N^h(s)$ on $[\underline{s}, \overline{s})$, so inequality is pervasively higher after the liberalization. This concludes the proof of the proposition.

A.5 Section 6

A.5.1 Free-Entry Equilibrium in the Closed Economy

In the free-entry model the mass of firms in the industry, $\overline{M}$, is an additional endogenous variable. As described in the main text, there is an unbounded pool of prospective firms that can enter the industry by incurring a fixed entry-cost of $f^e V(s)$ units of each skill $s \in S$. Upon entry, firms obtain their productivity as independent draws from the distribution G, as explained in section 2.2. Note that the new free-entry assumption does not affect the basic structure of the model described in section 2, so equations (1)-(7) continue to hold.

The analysis of the closed-economy equilibrium in section 3 is valid for any mass of firms, $\overline{M}$, so it applies almost unchanged to the free-entry model once $\overline{M}$ has been determined. In fact, conditional on $\overline{M}$, the analysis needs to be modified only to account for the presence of fixed entry-costs, i.e. L must be replaced with $L - f^e\overline{M}$ throughout the analysis. A *free-entry* condition provides the additional equilibrium condition to pin down the mass of firms.

In the free-entry model, the labor market clearing condition is given by

$$LV(s) = \int_{\phi^*}^{\overline{\phi}} l^d(s, \phi) \frac{g(\phi)}{[1 - G(\phi^*)]} d\phi M + MfV(s) + \overline{M} f^e \text{ for all } s \in S. \tag{81}$$

With unrestricted entry, prospective entrants must be indifferent between entering and not entering the industry, i.e. expected profits from entering must equal the cost of entry, $[1 - G(\phi^*)]\,\overline{\pi}^d = f^e$, where $\overline{\pi}^d$ is the average domestic profit of active firms. Per the optimal pricing rule , this *free-entry* condition can be written as follows,

$$\int_{\phi^*}^{\overline{\phi}} \left[\frac{r^d(\phi)}{\sigma} - f\right] g(\phi)\, d\phi = f^e. \tag{82}$$

Definition 3 *A free-entry equilibrium of the closed economy is a mass of firms $\overline{M} > 0$, a mass of active firms $M > 0$, a productivity activity-cutoff , $\phi^* \in (\underline{\phi}, \overline{\phi})$, an output function $q^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a labor allocation function $l^d : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a price function $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and a wage schedule $w : S \to \mathbb{R}_+$ such that the following conditions hold,*
(i) consumers behave optimally, equations (1) and (2);

(ii) firms behave optimally given their technology, equations (3), (7), (9), and (10);
(iii) goods and labor markets clear, equations (8) and (81), respectively;
(iv) the numeraire assumption holds, $\overline{w} = 1$;
(v) the free-entry condition holds, equation (82).

Given the equilibrium activity cutoff, ϕ^*, the price, domestic-revenue and inverse-matching functions, $\{p, r^d, H\}$, solve a BVP that is almost identical to the one defined in lemma 1.ii for the no-free-entry model. The only difference lies in the slope of the inverse-matching function, which is now given by

$$H_\phi(\phi) = \frac{r^d(\phi)\, g(\phi)\, \overline{M}}{A(H(\phi), \phi)\left[L - fM - f^e\overline{M}\right] V(H(\phi))\, p(\phi)}. \tag{83}$$

The discussion in section 4.2 implies that, for a given activity cutoff ϕ^*, the functions r^d and H that solve this BVP do not depend on the mass of firms nor the mass of production workers. Noting that equations (15) and (83) may differ only in these parameters, the last observation implies that, for a given ϕ^*, the closed-economy BVPs of the no-free-entry and free-entry models share the same solution functions r^d and H.[60]

As the revenue function r^d depends only on ϕ^*, the free-entry condition (82) can be used to determine the equilibrium activity cutoff, ϕ^*. Finally, combining the equilibrium relationship $L = \overline{M}f^e + Mf + \frac{\sigma-1}{\sigma}M\overline{r}^d$ (the counterpart of condition (16) in the no-free-entry model), and free-entry condition we can express the mass of firms as a function of exogenous variables and the activity cutoff ϕ^*,

$$\overline{M} = \frac{L}{\sigma f^e + \sigma f\left[1 - G(\phi^*)\right]}. \tag{84}$$

I summarize this discussion in the following lemma.

Lemma 5 *In a free-entry equilibrium of the closed economy with activity cutoff $\phi^* \in (\underline{\phi}, \overline{\phi})$ the following conditions hold.*
(i) There exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$, (with inverse function H) such that (i) $l^d(s,\phi) > 0$ if and only if $N(s) = \phi$, (ii) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$.
(ii) The wage schedule w is continuously differentiable and satisfies (12).
(iii) The price, revenue and matching functions, $\{p, r^d, N(H)\}$, are continuously differentiable. Given ϕ^*, the triplet $\{p, r^d, H\}$ solves the BVP comprising the differential equations {(13), (14), (83)} and the boundary conditions $r^d(\phi^*) = \sigma f$, $H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$.
(iv) The activity cutoff ϕ^* and the revenue function r^d satisfy the free-entry condition (82).
(v) The mass of firms in the industry, $\overline{M}$, is given by (84).
Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$, and functions $p, r^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $H : [\phi^*, \overline{\phi}] \to S$ satisfy conditions (ii)-(iv), then they are, respectively, the productivity activity-cutoff, the price function, the revenue function, and the inverse of the matching function of a free-entry equilibrium of the closed economy.

[60] These two BVPs are equivalent to the same parametrization of BVP (22).

The discussion preceding proposition 1 implies that $r^d(\phi)$ decreases with ϕ^*, making the left-hand side of (82) strictly decreasing in ϕ^*. If the fixed entry costs are not too high, then there is a unique activity cutoff ϕ^* that solves (82). In turn, this result implies that there is a unique free-entry equilibrium of the open economy.

A.5.2 Free-Entry Equilibrium in the Open Economy

The similarities between the analyses of the closed-economy equilibrium in the no-free-entry and free-entry models extend to the open-economy case. In particular, replacing L with $L - f^e\overline{M}$ throughout the analysis in section 4 yields the characterization of the open-economy equilibrium in the free-entry model, conditional on the mass of firms $\overline{M}$. The free-entry condition provides the additional equilibrium condition to determine $\overline{M}$.

The labor market clearing condition in the open economy is given by

$$LV(s) = \begin{array}{c} \int_{\phi^*}^{\overline{\phi}}[l^d(s,\phi)\,g(\phi)\,\overline{M} + l^x(s,\phi)\,M^x(\phi)]d\phi + \cdots \\ fMV(s) + nf^x\int_0^{\frac{\tau^{1-\sigma}r^d(\phi)}{\sigma f x}} ydF(y)\,M^x(\phi)\,V(s) + f^e\overline{M}V(s) \end{array} \quad \text{for all } s \in S. \tag{85}$$

As before, unrestricted entry implies that expected profits from entering the industry must equal the cost of entry, $[1 - G(\phi^*)]\left[\overline{\pi}^d + \overline{\pi}^x\right] = f^e$, where $\overline{\pi}^d$ and $\overline{\pi}^x$ are, respectively, the average domestic and export profit of active firms.[61] Per the optimal pricing rule , this *free-entry* condition can be written as shown in equation (24) in the main text.

Definition 4 *A free-entry equilibrium of the open economy is a mass of firms $\overline{M}$, an activity cutoff ϕ^*, a mass of active firms $M > 0$, a mass of exporters $M^x(\phi) > 0$ for each productivity level $\phi \geq \phi^*$, output functions $q^d, q^x : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, labor allocations functions $l^d, l^x : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a price function $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and a wage schedule $w : S \to \mathbb{R}_+$ such that the following conditions hold,*
(i) consumers behave optimally, equations (1) and (2);
(ii) firms behave optimally given their technology, equations (3), (7), (9), (10) and (18);
(iii) goods and labor markets clear, equations (8), (17) and (24);
(iv) the numeraire assumption holds, $\overline{w} = 1$;
(v) the free-entry condition holds, equation (24).

Given the equilibrium activity cutoff, ϕ^*, the price, domestic-revenue and inverse-matching functions, $\{p, r^d, H\}$, solve a BVP that is almost identical to the one defined in lemma 3.iii for the no-free-entry model. The only difference lies in the slope of the inverse-matching function, which is now given by

$$H_\phi(\phi) = \frac{r^d(\phi)\left[1 + F\left(\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}\right)n\tau^{1-\sigma}\right]g(\phi)\overline{M}}{A(H(\phi),\phi)V(H(\phi))p(\phi)\left[L - fM - f^e\overline{M} - \int_{\phi^*}^{\overline{\phi}} nf^x\int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF(y)g(\phi')\overline{M}d\phi'\right]}. \tag{86}$$

[61] Note that $\overline{\pi}^x$ is not the average export profits among exporters, but among all active firms.

Noting that equations (20) and (86) may differ only in the mass of firms or the mass of production workers, the discussion in the preceding section implies that, for a given ϕ^*, the open-economy BVPs of the no-free-entry and free-entry models share the same solution functions r^d and H.

As before, the free-entry condition (24) can be used to determine the equilibrium activity cutoff, ϕ^*. Finally, the equilibrium relationship, $L = \overline{M} f^e + M f + \int_{\phi^*}^{\overline{\phi}} n f^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} y dF(y)\, g(\phi')\, \overline{M} d\phi' + \frac{\sigma-1}{\sigma} M \overline{r}^d + \frac{\sigma-1}{\sigma} M \overline{r}^x$, can be combined with the free-entry condition to express the mass of firms in the industry as a function of exogenous parameters, the activity cutoff ϕ^* and the revenue function r^d,

$$\overline{M} = \frac{L}{\sigma \left[f^e + f\left[1 - G(\phi^*)\right] + \int_{\phi^*}^{\overline{\phi}} n f^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} y dF(y)\, g(\phi')\, d\phi' \right]}. \tag{87}$$

I summarize this discussion in the following lemma.

Lemma 6 *In a free-entry equilibrium of the open economy with activity cutoff $\phi^* \in (\underline{\phi}, \overline{\phi})$ the following conditions hold.*
(i) There exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$, (with inverse function H) such that (i) $l^d(s,\phi) + l^x(s,\phi) > 0$ if and only if $N(s) = \phi$, (ii) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$.
(ii) The wage schedule w is continuously differentiable and satisfies (12)
(iii) The price, domestic revenue and matching functions, $\{p, r^d, N\}$, are continuously differentiable. Given ϕ^*, the triplet $\{p, r^d, H\}$ solves the BVP comprising the system of differential equations {(13), (14), (86)} and the boundary conditions $r^d(\phi^*) = \sigma f$, $H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$.
(iv) The activity cutoff ϕ^* and the revenue function r^d satisfy the free-entry condition (24).
(v) The mass of firms in the industry, $\overline{M}$, is given by (87)
Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$, and functions $p, r^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $H : [\phi^*, \overline{\phi}] \to S$ satisfy the conditions (iii)-(iv), then they are, respectively, the activity cutoff, the price function, the domestic revenue function, and the inverse-matching function of a free-entry equilibrium of the open economy.

A.5.3 Proof of Proposition 4

In the free-entry model the activity cutoff may increase or decrease when the economy starts trading. The reasons behind this ambiguity are discussed in the text. In addition, as stated in the text, proposition 4.i considers essentially the same case as proposition 2, so the arguments in the proof of the latter also applies to the former. Here I focus on Proposition 4.ii.

Proposition 4.ii

Let $\phi_\tau^* < \phi_a^*$. If $N^\tau(s) < N^a(s)$ for all $s \in [\underline{s}, \overline{s})$, then lemma 2.ii implies that $r^{d,\tau}(\phi) > r^{d,a}(\phi)$ for all $\phi \geq \phi_a^*$, so domestic profits in the open economy are necessarily higher than in autarky. With strictly positive export profits, this observation implies that total average profits must be higher in the open economy, violating the free entry condition (24). Accordingly, $N^{\tau}(s)$ must lie above $N^{a}(s)$ for some values of s, implying that $N^{\tau}(s)$ and $N^{a}(s)$ must intersect at least once on $(\underline{s}, \overline{s})$.

Next, I show that $N^{\tau}(s)$ and $N^{a}(s)$ intersect exactly once on $(\underline{s}, \overline{s})$. The argument is more easily stated in terms of the inverse functions H^{τ} and H^{a}. Let ϕ_0 be the first time that H^{τ} and H^{a} intersect on $(\phi_a^*, \overline{\phi})$.[62] Note that H^{τ} and H^{a} are part of the unique solutions to parameterizations of BVP (22) that differ only in the parameter function α^i, with $K_1^i = 0$ for $i = \tau, a$, $\alpha^{\tau}(\phi) = 1 + F\left(\frac{r^{d,\tau}\phi}{\sigma f_x}\tau^{1-\sigma}\right)$ and $\alpha^{a}(\phi) = 1$.[63] Then, an immediate application of lemma 4.i yields $H^{\tau}(\phi) < H^{a}(\phi)$ on $(\phi_0, \overline{\phi})$, so H^{τ} and H^{a} ($N^{\tau}(s)$ and $N^{a}(s)$) intersect exactly once on $(\phi_a^*, \overline{\phi})$ ($(\underline{s}, \overline{s})$) at ϕ_0 ($s_0 = H^{i}(\phi_0)$).

The last result implies that, in the open economy, inequality is lower among workers with skill levels below s_0, but higher among workers with skill level above s_0. Put another way, opening to trade leads to wage polarization. The effects of the intensive- and extensive-margin channels can be proved by adapting the arguments in proposition 2.ii.

[62] ϕ_0 is well-defined due to the continuity of H^i, $i = \tau, a$.
[63] See the proof of theorem X (9 I think) for more details.