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Solving asset pricing models with stochastic volatility*

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Abstract

This paper provides a closed-form solution for the price-dividend ratio in a standard asset pricing model with stochastic volatility. The solution is useful in allowing comparisons among numerical methods used to approximate the non-trivial closed-form.

Keywords: Endowment model, Price-dividend ratio, Closed-form solution
JEL classifications: C61, C62, G12

1 Introduction

The purpose of this paper is to obtain an exact expression for the price-dividend ratio for a simple asset pricing model with stochastic volatility. Stochastic volatility has become an important feature of macroeconomic models that seek to jointly explain stylized business cycle and asset pricing facts. Since closed-form solutions elude richer macroeconomic models, various numerical methods have been proposed to provide an approximated solution. The contribution of this paper is to present a simple stochastic volatility model in which an exact solution exists, which may serve as a benchmark from which to compare alternative numerical approximation methods.

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Burnside (1998) provided an exact solution for the Lucas (1978) asset pricing model with Gaussian, autoregressive dividend growth shocks and time separable constant relative risk aversion (CRRA) preferences.[1] Tsionas (2003) extended Burnside's solution to an arbitrary shock distribution while Chen, Cosimano, and Himonas (2008) and Collard, Féve, and Ghattassi (2006) extended it to the case with non-time separable preferences through habits in consumption. In each case, the solutions provide a useful benchmark against which to test numerical solution algorithms. This paper follows in that tradition. It extends the Burnside model by adding stochastic volatility to the dividend growth process.

Since Bansal and Yaron (2004) showed the importance of stochastic volatility to account for stylized asset pricing facts, the use of stochastic volatility has become a widespread addition to macro-finance models. Stochastic volatility is attractive because it generates heteroskedastic aggregate fluctuations, a basic property of many time series (such as consumption) and adds extra flexibility in accounting for asset-pricing patterns. Due to the increasing importance of stochastic volatility, which naturally adds additional non-linearity into the solution of models, a growing literature has been testing how different numerical solution methods that solve equilibrium models with stochastic volatility perform. Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao (2012), for example, compare perturbation methods (of second and third order), Chebyshev polynomials and value function iteration in a real business cycle model with stochastic volatility.

In this paper, I show the exact solution for the price-dividend ratio of a simple asset pricing model as a non-trivial function of the model's two state variables, the current dividend growth rate and the current volatility of the dividend growth process. The solution has the following properties: First, the price-dividend ratio increases when the volatility of dividend growth increases as well as when the volatility of the stochastic volatility process increases. Second, the sensitivity of the price-dividend ratio to a change in the volatility state is increasing in the persistence of the stochastic volatility process. In addition, I derive an expression for the unconditional mean of the price-dividend process that is also increasing in the volatility and persistence of the stochastic volatility process. Finally, I provide parameter conditions under which the price-dividend ratio and its unconditional mean are finite.

The rest of the paper is structured as follows. Section 2 presents the basic asset-pricing model with stochastic volatility and section 3 presents and discusses the results. Section 4 concludes. The appendix provides a detailed derivation of the key results of the paper as well as discussing a variant of the basic model.

[1] An early contribution by Labadie (1989) also provided the solution in a slightly more general context.

2 The asset pricing model

There is a representative agent who maximizes the expected discounted stream of utility

$$\mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \tag{1}$$

subject to the budget constraint

$$c_t + s_{t+1} p_t \leq (d_t + p_t) s_t, \tag{2}$$

where $\mathbf{E}_t$ is mathematical expectations operator conditional on the time t information set, c_t is consumption and s_t denotes units of an asset whose price at date t is p_t with dividends, d_t. The discount factor is $\beta \in (0,1)$ and the coefficient of relative risk aversion is $\gamma > 0$ and $\gamma \neq 1$. The growth rate of dividends, denoted $x_t \equiv \log(d_t/d_{t-1})$, is assumed to follow a Gaussian $AR(1)$ process

$$x_t = x + \rho (x_{t-1} - x) + \sqrt{\eta_t} \varepsilon_t, \tag{3}$$

where x is the steady state growth rate of dividends, $\rho \in [0,1)$ is the persistence parameter and ε_t is a sequence of i.i.d. innovations from the standard normal distribution. The innovations to x_t are scaled by $\sqrt{\eta_t}$. η_t is therefore the conditional variance of dividend growth and is time varying. In particular, it follows an $AR(1)$ process

$$\eta_t = \eta + \rho_\eta (\eta_{t-1} - \eta) + \omega \varepsilon_{\eta,t}, \tag{4}$$

where η is its steady state, $\rho_\eta \in [0,1)$ is the persistence of the stochastic volatility process, ω is a scalar and $\varepsilon_{\eta,t}$ is a sequence of i.i.d. innovations from the standard normal distribution.[2]

The first-order equilibrium condition of the agent's maximization problem, equations (1)-(2), is

$$c_t^{-\gamma} p_t = \mathbf{E}_t \beta c_{t+1}^{-\gamma} (p_{t+1} + d_{t+1}).$$

Market clearing, $s_t = 1$, implies that $c_t = d_t$, and, in defining the price-dividend ratio as

[2] This formulation of the stochastic volatility process ensures a closed-form expression for the price-dividend ratio but could technically cause the standard deviation of dividend growth to become negative. However, under reasonable calibrations of the process, this happens rarely. Bansal and Yaron (2004) use the same process and choose the following parameter values based on a monthly frequency: $\eta = 1.232 \times 10^{-3}$, $\rho_\eta = 0.987$, and $\omega = 0.04658 \times 10^{-3}$. Simulating this process 10^5 times for 840 quarters results in the process turning negative in 0.13% of the simulations. A discussion of the model solution using an appropriately truncated normal distribution is provided in Appendix A.2.

$y_t \equiv p_t/d_t$, the first-order equilibrium condition becomes

$$y_t = \mathbf{E}_t \beta \left(\frac{d_{t+1}}{d_t}\right)^{1-\gamma} (y_{t+1} + 1) \, . \tag{5}$$

Iterating forward and making using of x_t, we are left with

$$y_t = \sum_{i=1}^{\infty} \beta^i \mathbf{E}_t \exp\left((1-\gamma) \sum_{j=1}^{i} x_{t+j}\right). \tag{6}$$

3 The model solution

Equation (6) shows that, in this asset pricing model, the price-dividend ratio at time t is simply a function of expected future dividend growth. Finding an exact solution for y_t means finding a closed-form expression for $\mathbf{E}_t \exp\left((1-\gamma)\sum_{j=1}^{i} x_{t+j}\right)$ for $i = 1, 2, ...$in terms of the current state, x_t and η_t. In the case without stochastic volatility Burnside (1998) derived such a solution. The theorem below shows an exact solution *with* stochastic volatility.

Theorem 1 *The solution to equation (6) is*

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp\left(A_i x + B_i (x_t - x) + C_i \eta + D_i (\eta_t - \eta) + F_i \omega^2\right) \tag{7}$$

where

$$\begin{gathered}
A_i \equiv (1-\gamma)\, i, \qquad B_i \equiv \left(\tfrac{1-\gamma}{1-\rho}\right) \rho \left(1-\rho^i\right) \\
C_i \equiv \tfrac{1}{2} \left(\tfrac{1-\gamma}{1-\rho}\right)^2 \left(i - 2\rho \tfrac{1-\rho^i}{1-\rho} + \rho^2 \tfrac{1-\rho^{2i}}{1-\rho^2}\right), \\
D_i \equiv \tfrac{1}{2} \left(\tfrac{1-\gamma}{1-\rho}\right)^2 \left(\rho_\eta \tfrac{1-\rho_\eta^i}{1-\rho_\eta} - 2\rho_\eta^i \rho \tfrac{1-\left(\rho_\eta^{-1}\rho\right)^i}{1-\rho_\eta^{-1}\rho} + \rho_\eta^i \rho^2 \tfrac{1-\left(\rho_\eta^{-1}\rho^2\right)^i}{1-\rho_\eta^{-1}\rho^2}\right), \\
F_i \equiv \tfrac{1}{4} \left(\tfrac{1-\gamma}{1-\rho}\right)^4 \left(\begin{array}{c}
i\phi_1^2 + \phi_2^2 \frac{1-\rho_\eta^{2i}}{1-\rho_\eta^2} + \phi_3^2 \frac{1-\rho^{2i}}{1-\rho^2} + \phi_4^2 \frac{1-\rho^{4i}}{1-\rho^4} \\
+2\phi_1\phi_2 \frac{1-\rho_\eta^i}{1-\rho_\eta} + 2\phi_1\phi_3 \frac{1-\rho^i}{1-\rho} + 2\phi_1\phi_4 \frac{1-\rho^{2i}}{1-\rho^2} \\
+2\phi_2\phi_3 \frac{1-\left(\rho_\eta \rho\right)^i}{1-\rho_\eta \rho} + 2\phi_2\phi_4 \frac{1-\left(\rho_\eta \rho^2\right)^i}{1-\rho_\eta \rho^2} + 2\phi_3\phi_4 \frac{1-\rho^{3i}}{1-\rho^3}
\end{array}\right),
\end{gathered}$$

and where

$$\phi_1 \equiv \frac{1}{1-\rho_\eta}, \quad \phi_2 \equiv \frac{-\rho_\eta \left(\rho_\eta + \rho\right)(1-\rho)^2}{\left(\rho_\eta - \rho^2\right)\left(1-\rho_\eta\right)\left(\rho_\eta - \rho\right)}, \quad \phi_3 \equiv \frac{2\rho^2}{\rho_\eta - \rho} \rho^{i-1}, \quad \textit{and } \phi_4 \equiv -\left(\frac{\rho^4}{\rho_\eta - \rho^2}\right).$$

Proof. See Appendix A.1. ∎

In Burnside (1998), the solution without stochastic volatility is

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp\left(A_i x + B_i (x_t - x) + C_i \eta\right),$$

therefore, it is the term $D_i(\eta_t - \eta) + F_i\omega^2$ inside the exponential function in equation (7) that is novel. It is straightforward to show (see equation (13) and (15) in Appendix A.1) that both $D_i > 0$ and $F_i > 0$.[3] It follows that $\frac{\partial y_t}{\partial(\eta_t-\eta)} > 0$ and $\frac{\partial y_t}{\partial\omega^2} > 0$: A rise in the volatility of dividend growth unambiguously increases the price-dividend ratio as does a rise in the volatility of the stochastic volatility process itself. Since the agent is risk averse, greater uncertainty reduces the agent's demand for the asset, reducing the price. It also follows that $\frac{\partial|\partial y_t/\partial(x_t-x)|}{\partial(\eta_t-\eta)} > 0$ and $\frac{\partial|\partial y_t/\partial(x_t-x)|}{\partial\omega^2} > 0$: The price-dividend ratio responds more to movements in the dividend growth rate in a high volatility state than in a low volatility state as well as in an environment with greater stochastic volatility. The insight from this result is that the heteroskedasticity (inherent in the exogenous dividend growth process) will be more pronounced in the endogenous price-dividend ratio. Equations (13) and (15) also show clearly that $\frac{\partial D_i}{\partial\rho_\eta}, \frac{\partial F_i}{\partial\rho_\eta} > 0$: A rise in the persistence of the stochastic volatility process increases the sensitivity of the price-dividend ratio to both changes in dividend growth and volatility.

Since the price-dividend ratio is the sum of an infinite sequence, it is not clear from equation (7) whether the price-dividend ratio is finite. The following theorem states the parameter conditions under which the price-dividend ratio *is* finite.

Theorem 2 *The series in equation (7) converges if and only if*

$$\beta \exp\left((1-\gamma)\,x + \frac{1}{2}\left(\frac{1-\gamma}{1-\rho}\right)^2\left(\eta + \frac{1}{2}\left(\frac{1}{1-\rho_\eta}\right)^2\omega^2\right)\right) < 1. \tag{8}$$

Proof. See Appendix A.3. ■

In Burnside (1998), the convergence criterion is

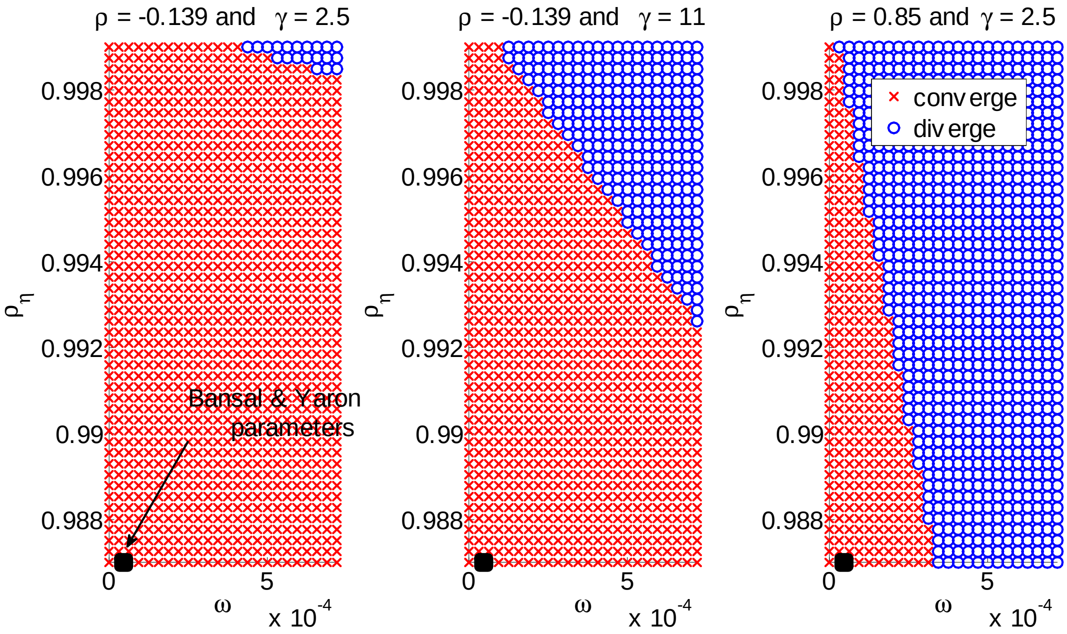
$$\beta \exp\left((1-\gamma)\,x + \frac{1}{2}\left(\frac{1-\gamma}{1-\rho}\right)^2\eta\right) < 1,$$

and thus less demanding that the condition in Theorem 2, conditional on the same parameters for β, γ, x, ρ and η.

To get a better understanding of the restriction the condition in Theorem 2 places on the parameters of the stochastic volatility process, I following Schmitt-Grohé and Uribe (2004) and Bansal and Yaron (2004) in parameterizing the asset pricing model as follows: $\beta = 0.95$, $x = 0.0179$, and $\eta = 6.084 \times 10^{-5}$. In addition, I consider three different parameterizations of the pair (ρ, γ) using $\rho = \{-0.137, 0.9\}$ and $\gamma = \{2.5, 11\}$. I ignore the high persistence, high risk aversion combination since the price-dividend

[3] The exception is logarithmic preferences ($\gamma = 1$) in which case $A_i = B_i = C_i = D_i = F_i = 0$ and the price-dividend ratio becomes constant. With logarithmic preferences $B_i = 0$ because the wealth and subsitution effects of a change in the dividend-growth rate exactly offset eachother. Since the price-dividend ratio remains constant in response to dividend growth movements, it follows that the price-dividend ratio is also invariant to changes in the volatility of those movements.

Figure 1: Regions of convergence in the parameter space



Note: Red crosses mark the parameter space for which the condition in Theorem 2 holds, blue circles the parameter space for which the condition is violated and the price-dividend ratio is no longer finite. The black square denotes parameters values $\rho_\eta = 0.987$, and $\omega = 0.0465 \times 10^{-3}$ used by Bansal and Yaron (2004). Remaining parameters are $\beta = 0.95$, $x = 0.0179$ and $\eta = 6.084 \times 10^{-5}$.

ratio is never finite in this case. Figure 1 shows the (ρ_η, ω) pairs (the two parameters describing the stochastic volatility process) for which the condition for a finite price-dividend ratio (in Theorem 2) holds. The plots show that when both the persistence of the endowment growth process and risk aversion are low (the left panel), then the conditions on the stochastic volatility process to ensure that the price-dividend ratio is finite are relatively weak. Bansal and Yaron (2004) choose parameter values of $\rho_\eta = 0.987$ and $\omega = 0.0465 \times 10^{-3}$ (indicated in the figure), significantly inside the convergent parameter space. However, as either the level of risk aversion (middle panel) or the persistence of the dividend growth process (right panel) increases, the parameter space for the stochastic volatility process consistent with a finite price-dividend process shrinks considerably.

The same condition as in Theorem 2 also ensures that the unconditional mean of the price-dividend ratio is finite, as stated in the next theorem.

Theorem 3 *The mean of the price-dividend ratio is*

$$\mathbf{E}(y_t) = \sum_{i=1}^{\infty} \beta^i \exp\left(\begin{array}{c} A_i x + C_i \eta + F_i \omega^2 \\ +\frac{1}{2}\frac{B_i^2 \eta}{1-\rho^2} + \frac{\omega^2}{2}\left(\frac{\gamma_{i,1}^2}{1-\rho_\eta^2} - \frac{2\gamma_{i,1}\gamma_{i,2}}{1-\rho_\eta \rho^2} + \frac{\gamma_{i,2}^2}{1-\rho^4}\right) \end{array} \right),$$

where
$$\gamma_{i,1} \equiv \left(\frac{B_i^2}{2} \frac{\rho_\eta}{\rho_\eta - \rho^2} + D_i \right) \quad , \quad \gamma_{i,2} \equiv \frac{B_i^2}{2} \frac{\rho^2}{\rho_\eta - \rho^2},$$
and is finite if and only if the condition in Theorem 2 holds.

Proof. See Appendix A.4. ■

The unconditional mean price-dividend ratio is increasing in both the volatility, ω and the persistence ρ_η of the price-dividend ratio (as is made clear by the quadratic expression in (19) in Appendix A.4).

4 Conclusion

This paper provides an exact expression for the price-dividend ratio in an endowment asset pricing model with CRRA preferences, Gaussian autoregressive shocks *and* stochastic volatility. The solution provides a useful benchmark against which to test the performance of alternative numerical solution algorithms which one may wish to use to solve more elaborate macro-finance models with stochastic volatility.

Since the structure of the model with stochastic volatility shares many of the properties of the basic Burnside asset pricing model, it should be possible to derive an exact solution for this stochastic volatility model with the addition of multivariate and higher order autoregressive processes as in Burnside (1998) or with habits in consumption as in Chen, Cosimano, and Himonas (2008) and Collard, Féve, and Ghattassi (2006). This would be a fruitful direction for future research.

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A Appendix

A.1 Solution: Proof of Theorem 1

The ultimate aim is to rewrite the expression

$$\mathbf{E}_t \exp\left((1-\gamma)\sum_{j=1}^{i} x_{t+j}\right) \quad \text{for} \quad i = 1, 2, ... \tag{9}$$

in terms of the time t state variables, x_t and η_t. Iterating forward the dividend growth process, equation (3), so that x_{t+j} is in terms of x_t gives

$$x_{t+j} = x + \rho^j (x_t - x) + \sum_{k=1}^{j} \rho^{j-k} \sqrt{\eta_{t+k}}\varepsilon_{t+k}.$$

Substituting this into (9) gives

$$\mathbf{E}_t \exp\left((1-\gamma)\sum_{j=1}^{i}\left(x + \rho^j (x_t - x) + \sum_{k=1}^{j} \rho^{j-k}\sqrt{\eta_{t+k}}\varepsilon_{t+k}\right)\right).$$

Collecting terms for x, $(x_t - x)$ and each ε_{t+j} gives

$$\mathbf{E}_t \exp\left((1-\gamma)\begin{pmatrix} \sum_{j=1}^{i}\left(x + \rho^j (x_t - x)\right) \\ +\sum_{j=1}^{i}\left(\sum_{k=1}^{i-j+1}\rho^{k-1}\right)\sqrt{\eta_{t+j}}\varepsilon_{t+j} \end{pmatrix}\right).$$

Using the standard results of geometric progressions gives

$$\mathbf{E}_t \exp\begin{pmatrix} (1-\gamma)\, ix + (1-\gamma)\,\rho\frac{1-\rho^i}{1-\rho}(x_t - x) \\ +\frac{(1-\gamma)}{1-\rho}\sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)\sqrt{\eta_{t+j}}\varepsilon_{t+j} \end{pmatrix}.$$

Since the first row in the previous expression is only in terms of x and $(x_t - x)$, the expectations operator can be moved, leaving

$$\exp\left(A_i x + B_i\left(x_t - x\right)\right) \mathbf{E}_t \exp\left(\theta \sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)\sqrt{\eta_{t+j}}\varepsilon_{t+j}\right), \tag{10}$$

where

$$A_i \equiv (1-\gamma)\, i, \quad B_i \equiv \theta\rho\left(1-\rho^i\right) \quad \text{and} \quad \theta \equiv \left(\frac{1-\gamma}{1-\rho}\right).$$

At this stage it is instructive to rewrite the expression with the expectations operator in (10) as an integral of probabilistic outcomes

$$\int_{\varepsilon_{\eta,t+1}} \cdots \int_{\varepsilon_{\eta,t+i}} \int_{\varepsilon_{t+1}} \cdots \int_{\varepsilon_{t+i}} \exp\left(\theta \sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)\sqrt{\eta_{t+j}}\varepsilon_{t+j}\right) \underset{\varepsilon_{t+1}}{dF} \cdots \underset{\varepsilon_{t+i}}{dF}\, \underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG},$$

where F and G are the density functions for the i.i.d. random variables ε and ε_η, respectively. Since the ε innovations are independent, we can rewrite the problem as

$$\int_{\varepsilon_{\eta,t+1}} \cdots \int_{\varepsilon_{\eta,t+i}} \left(\prod_{j=1}^{i} \int_{\varepsilon_{t+j}} \exp\left(\theta\left(1-\rho^{i-j+1}\right)\sqrt{\eta_{t+j}}\varepsilon_{t+j}\right) \underset{\varepsilon_{t+j}}{dF}\right) \underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG},$$

Using a standard result for random variables, namely that if $z \sim N(0,1)$ and k is a scalar, then $\mathbf{E}\left(\exp\left(kz\right)\right) = \exp\left(\frac{k^2}{2}\right)$, we get

$$\int_{\varepsilon_{\eta,t+1}} \cdots \int_{\varepsilon_{\eta,t+i}} \left(\prod_{j=1}^{i} \exp\left(\frac{\theta^2}{2}\left(1-\rho^{i-j+1}\right)^2 \eta_{t+j}\right)\right) \underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG},$$

or

$$\int_{\varepsilon_{\eta,t+1}} \cdots \int_{\varepsilon_{\eta,t+i}} \exp\left(\frac{\theta^2}{2}\sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)^2 \eta_{t+j}\right) \underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG}. \tag{11}$$

If we assumed $\eta_{t+i} = \eta$ for all $i = 1, 2, ...$ the expectations operator would disappear from the above expression and with a little further manipulation we would recover the solution in Burnside (1998). Instead, with stochastic volatility there is more work to do. Iterating forward the stochastic volatility process, equation (4), so that η_{t+j} is in terms of η_t gives

$$\eta_{t+j} = \eta + \rho_\eta^j\left(\eta_t - \eta\right) + \sum_{k=1}^{j} \rho_\eta^{j-k}\omega\varepsilon_{\eta,t+k}.$$

Substituting this expression into (11) gives

$$\int_{\varepsilon_{\eta,t+1}} \cdots \int_{\varepsilon_{\eta,t+i}} \exp\left(\frac{\theta^2}{2}\sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)^2\left(\eta + \rho_\eta^j\left(\eta_t-\eta\right) + \sum_{k=1}^{j}\rho_\eta^{j-k}\omega\varepsilon_{\eta,t+k}\right)\right) \underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG}.$$

Collecting terms for η, $(\eta_t - \eta)$ and each $\varepsilon_{\eta,t+j}$ gives

$$\int\limits_{\varepsilon_{\eta,t+1}} \cdots \int\limits_{\varepsilon_{\eta,t+i}} \exp\left(\frac{\theta^2}{2}\left(\begin{array}{c} \sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)^2\eta \\ +\sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)^2\rho_\eta^j\left(\eta_t-\eta\right) \\ +\omega\sum_{j=1}^{i}\left(\sum_{k=1}^{i-j+1}\left(1-\rho^{i-j+2-k}\right)^2\rho_\eta^{k-1}\right)\varepsilon_{\eta,t+j}\end{array}\right)\right) \underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG}.$$

Since the first two rows in the previous expression are only in terms of η and $(\eta_t - \eta)$, the integral can be moved, leaving

$$\begin{aligned} &\exp\left(C_i\eta + D_i\left(\eta_t - \eta\right)\right) \\ &\times \int\limits_{\varepsilon_{\eta,t+1}} \cdots \int\limits_{\varepsilon_{\eta,t+i}} \exp\left(\frac{\theta^2\omega}{2}\sum_{j=1}^{i}\left(\sum_{k=1}^{i-j+1}\left(1-\rho^{i-j+2-k}\right)^2\rho_\eta^{k-1}\right)\varepsilon_{\eta,t+j}\right)\underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG}, \end{aligned} \tag{12}$$

where

$$C_i \equiv \frac{\theta^2}{2}\sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)^2 \quad \text{and} \quad D_i \equiv \frac{\theta^2}{2}\sum_{j=1}^{i}\left(1-\rho^{i-j+1}\right)^2\rho_\eta^j. \tag{13}$$

Notice that $D_i \geq 0$, $\frac{\partial D_i}{\partial \rho} \leq 0$ and $\frac{\partial D_i}{\partial \rho_\eta} \geq 0$. Expanding the quadratic terms in C_i and D_i gives

$$\begin{aligned} C_i &= \tfrac{\theta^2}{2}\sum_{j=1}^{i}\left(1 - 2\rho^i\rho^{-(j-1)} + \rho^{2i}\rho^{-2(j-1)}\right) \\ D_i &= \tfrac{\theta^2}{2}\sum_{j=1}^{i}\left(\rho_\eta\rho_\eta^{j-1} - 2\rho_\eta\rho^i\left(\rho_\eta\rho^{-1}\right)^{j-1} + \rho_\eta\rho^{2i}\left(\rho_\eta\rho^{-2}\right)^{j-1}\right), \end{aligned}$$

and using the standard results of geometric progressions gives

$$\begin{aligned} C_i &= \tfrac{\theta^2}{2}\left(i - 2\rho\tfrac{1-\rho^i}{1-\rho} + \rho^2\tfrac{1-\rho^{2i}}{1-\rho^2}\right) \\ D_i &= \tfrac{\theta^2}{2}\left(\rho_\eta\tfrac{1-\rho_\eta^i}{1-\rho_\eta} - 2\rho_\eta^i\rho\tfrac{1-\left(\rho_\eta^{-1}\rho\right)^i}{1-\rho_\eta^{-1}\rho} + \rho_\eta^i\rho^2\tfrac{1-\left(\rho_\eta^{-1}\rho^2\right)^i}{1-\rho_\eta^{-1}\rho^2}\right) \end{aligned}$$

The final expression left to evaluate is the integral expression in (12),

$$\int\limits_{\varepsilon_{\eta,t+1}} \cdots \int\limits_{\varepsilon_{\eta,t+i}} \exp\left(\frac{\theta^2\omega}{2}\sum_{j=1}^{i}\left(\sum_{k=1}^{i-j+1}\left(1-\rho^{i-j+2-k}\right)^2\rho_\eta^{k-1}\right)\varepsilon_{\eta,t+j}\right)\underset{\varepsilon_{\eta,t+1}}{dG} \cdots \underset{\varepsilon_{\eta,t+i}}{dG}, \tag{14}$$

which becomes $\exp\left(F_i\omega^2\right)$ where

$$F_i \equiv \frac{\theta^4}{4}\sum_{j=1}^{i}\left(\sum_{k=1}^{i-j+1}\left(1-\rho^{i-j+2-k}\right)^2\rho_\eta^{k-1}\right)^2. \tag{15}$$

Notice that $F_i \geq 0$, $\frac{\partial F_i}{\partial \rho} \leq 0$ and $\frac{\partial F_i}{\partial \rho_\eta} \geq 0$. The above expression is another geometric progression (albeit a more tedious one). Expand to give

$$F_i = \frac{\theta^4}{4}\sum_{j=1}^{i}\left(\sum_{k=1}^{i-j+1}\left(\rho_\eta^{k-1} - 2\rho^{i-j+1}\left(\rho_\eta\rho^{-1}\right)^{k-1} + \rho^{2(i-j+1)}\left(\rho_\eta\rho^{-2}\right)^{k-1}\right)\right)^2.$$

Using (for the penultimate time) the results of geometric progressions gives

$$F_i = \frac{\theta^4}{4} \sum_{j=1}^{i} \left(\frac{1-\rho_\eta^{i-j+1}}{1-\rho_\eta} - 2\rho^{i-j+1} \frac{1-\left(\rho_\eta \rho^{-1}\right)^{i-j+1}}{1-\rho_\eta \rho^{-1}} + \rho^{2(i-j+1)} \frac{1-\left(\rho_\eta \rho^{-2}\right)^{i-j+1}}{1-\rho_\eta \rho^{-2}} \right)^2 .$$

It is useful to reverse the indexation for $j = 1, ..., i$ by rewriting $i + j - 1 = j$, in which case

$$F_i = \frac{\theta^4}{4} \sum_{j=1}^{i} \left(\frac{1-\rho_\eta^{j}}{1-\rho_\eta} - 2\rho^{j} \frac{1-\left(\rho_\eta \rho^{-1}\right)^{j}}{1-\rho_\eta \rho^{-1}} + \rho^{2j} \frac{1-\left(\rho_\eta \rho^{-2}\right)^{j}}{1-\rho_\eta \rho^{-2}} \right)^2 .$$

Further manipulation gives

$$F_i = \frac{\theta^4}{4} \sum_{j=1}^{i} \left(\phi_1 + \phi_2 \rho_\eta^{j-1} + \phi_3 \rho^{j-1} + \phi_4 \rho^{2(j-1)} \right)^2 .$$

where

$$\phi_1 \equiv \frac{1}{1-\rho_\eta}, \quad \phi_2 \equiv \frac{-\rho_\eta \left(\rho_\eta + \rho\right)\left(1-\rho\right)^2}{\left(\rho_\eta - \rho^2\right)\left(1-\rho_\eta\right)\left(\rho_\eta - \rho\right)}, \quad \phi_3 \equiv \frac{2\rho^2}{\rho_\eta - \rho} \rho^{i-1}, \quad \text{and} \quad \phi_4 \equiv -\left(\frac{\rho^4}{\rho_\eta - \rho^2} \right).$$

Multiplying out the quadratic term gives

$$F_i = \frac{\theta^4}{4} \sum_{j=1}^{i} \left(\begin{array}{c} \phi_1^2 + \phi_2^2 \rho_\eta^{2(j-1)} + \phi_3^2 \rho^{2(j-1)} + \phi_4^2 \rho^{4(j-1)} \\ +2\phi_1\phi_2 \rho_\eta^{j-1} + 2\phi_1\phi_3 \rho^{j-1} + 2\phi_1\phi_4 \rho^{2(j-1)} \\ +2\phi_2\phi_3 \left(\rho_\eta \rho\right)^{j-1} + 2\phi_2\phi_4 \left(\rho_\eta \rho^2\right)^{j-1} + 2\phi_3\phi_4 \rho^{3(j-1)} \end{array} \right).$$

Using (for the final time) the results of geometric progressions gives

$$F_i = \frac{\theta^4}{4} \left(\begin{array}{c} i\phi_1^2 + \phi_2^2 \frac{1-\rho_\eta^{2i}}{1-\rho_\eta^2} + \phi_3^2 \frac{1-\rho^{2i}}{1-\rho^2} + \phi_4^2 \frac{1-\rho^{4i}}{1-\rho^4} \\ +2\phi_1\phi_2 \frac{1-\rho_\eta^{i}}{1-\rho_\eta} + 2\phi_1\phi_3 \frac{1-\rho^{i}}{1-\rho} + 2\phi_1\phi_4 \frac{1-\rho^{2i}}{1-\rho^2} \\ +2\phi_2\phi_3 \frac{1-\left(\rho_\eta \rho\right)^i}{1-\rho_\eta \rho} + 2\phi_2\phi_4 \frac{1-\left(\rho_\eta \rho^2\right)^i}{1-\rho_\eta \rho^2} + 2\phi_3\phi_4 \frac{1-\rho^{3i}}{1-\rho^3} \end{array} \right).$$

This completes the proof.■

A.2 Ruling out negative volatility with a truncated normal

Drawing the ε_η innovations from the standard normal distribution creates the technical possibility that we get negative values for η_t. One candidate solution might be to draw from a truncated standard normal distribution which, with appropriate truncation, can guarantee non-negative values for η_t. To find the natural truncation point, we can calculate the value of η_{t+i} (without loss of generality, we set $\eta_t = \eta$) following a sequence of lowest-possible realizations of ε_η, namely $\varepsilon_\eta^{\min}$ to give

$$\eta_{t+i}^{\min} = \eta + \rho_\eta^{i-1}\omega\varepsilon_\eta^{\min} + \cdots + \omega\varepsilon_\eta^{\min}.$$

The non-negativity constraint requires $\lim_{i\to\infty}\eta_{t+i}^{\min} > 0$, in which case

$$\eta + \lim_{i\to\infty}\frac{1-\rho_\eta^i}{1-\rho_\eta}\omega\varepsilon_\eta^{\min} > 0 \quad \text{or} \quad \varepsilon_\eta^{\min} > -\frac{\eta\left(1-\rho_\eta\right)}{\omega}.$$

This expression implies that for a small ω relative to a large η (and low persistence, ρ_η), the probability of η_t becoming negative can be extremely small and of no practical concern. Bansal and Yaron (2004) use the following parameterization for the stochastic volatility process: $\eta = 1.232\times10^{-3}$, $\rho_\eta = 0.987$, and $\omega = 0.04658\times10^{-3}$. In this case $\varepsilon_\eta^{\min} = -0.344$. However, drawing from this distribution would also lower the volatility of the process that Bansal and Yaron targeted since

$$\operatorname{var}\left(\varepsilon_\eta^{trunc}\right) = 1 + \frac{2\varepsilon_\eta^{\min}\phi\left(\varepsilon_\eta^{\min}\right)}{1-2\Phi\left(\varepsilon_\eta^{\min}\right)} < 1,$$

where the *trunc* superscript denotes that it is the truncated random variable and the 1 on the right-hand side of the expression is the variance of the non-truncated standard normal. To consider how the model solution would be altered by the additional truncation, reconsider (14), reproduced here

$$\int\limits_{\varepsilon_{\eta,t+1}}\cdots\int\limits_{\varepsilon_{\eta,t+i}} \exp\left(\frac{\theta^2\omega}{2}\sum_{j=1}^{i}\left(\sum_{k=1}^{i-j+1}\left(1-\rho^{i-j+2-k}\right)^2\rho_\eta^{k-1}\right)\varepsilon_{\eta,t+j}\right)\underset{\varepsilon_{\eta,t+1}}{dG}\cdots\underset{\varepsilon_{\eta,t+i}}{dG},$$

and rewrite it as

$$\prod_{j=1}^{i}\int\limits_{\varepsilon_{\eta,t+j}} \exp\left(\frac{\theta^2\omega}{2}\left(\sum_{k=1}^{i-j+1}\left(1-\rho^{i-j+2-k}\right)^2\rho_\eta^{k-1}\right)\varepsilon_{\eta,t+j}\right)\underset{\varepsilon_{\eta,t+j}}{dG}.$$

In general, the moment generating function of a stochastic variable X with distribution G is

$$M\left(t\right) = \mathbf{E}\exp\left(\tau X\right) = \int_{-\infty}^{\infty}\exp\left(\tau X\right)dG\left(\varepsilon_\eta\right), \quad \tau\in\mathbf{R}.$$

Rewriting (14) using the moment generating function becomes

$$\prod_{j=1}^{i} M\left(\frac{\theta^2\omega}{2}\left(\sum_{k=1}^{i-j+1}\left(1-\rho^{i-j+2-k}\right)^2\rho_\eta^{k-1}\right)\right). \tag{16}$$

Using the results of geometric progressions gives

$$\left(\frac{1}{1-\rho_\eta}-2\frac{\rho^{i-j+2}}{\rho-\rho_\eta}+\frac{\rho^{2(i-j+2)}}{\rho^2-\rho_\eta^2}\right)-\left(\frac{1}{1-\rho_\eta}-2\frac{\rho}{\rho-\rho_\eta}+\frac{\rho^2}{\rho^2-\rho_\eta}\right)\rho_\eta^{i-j+1},$$

for the summation in expression (16). The term $F_i\omega^2$ in equation (7) from the main text is therefore replaced with the following expression:

$$\sum_{j=1}^{i}\log M\left(\frac{\theta^2\omega}{2}\left(\begin{array}{c}\frac{1}{1-\rho_\eta}-2\frac{\rho^{i-j+2}}{\rho-\rho_\eta}+\frac{\rho^{2(i-j+2)}}{\rho^2-\rho_\eta^2}\\ -\left(\frac{1}{1-\rho_\eta}-2\frac{\rho}{\rho-\rho_\eta}+\frac{\rho^2}{\rho^2-\rho_\eta}\right)\rho_\eta^{i-j+1}\end{array}\right)\right).$$

With ε_η drawn from a symmetrically truncated standard normal distribution with $\varepsilon_\eta^{\min}=-\frac{\eta(1-\rho_\eta)}{\omega}$, the moment generating function is given by

$$\exp\left(\frac{\tau^2}{2}\right)\left(\frac{\Phi\left(-\varepsilon_\eta^{\min}-\tau\right)-\Phi\left(\varepsilon_\eta^{\min}-\tau\right)}{1-2\Phi\left(\varepsilon_\eta^{\min}\right)}\right).$$

In the limit, $\frac{\eta(1-\rho_\eta)}{\omega}\to\infty$, the moment generating function would be $\exp\left(\frac{\tau^2}{2}\right)$, recovering the solution in the main text.

A.3 Convergence: Proof of Theorem 2

The aim is to show that the infinite summation

$$\sum_{i=1}^{\infty}\beta^i\exp\left(A_ix+B_i\left(x_t-x\right)+C_i\eta+D_i\left(\eta_t-\eta\right)+F_i\omega^2\right),$$

convergences to a finite number. First, I define

$$z_i=\beta^i\exp\left(A_ix+B_i\left(x_t-x\right)+C_i\eta+D_i\left(\eta_t-\eta\right)+F_i\omega^2\right),$$

so that the price-dividend ratio given by

$$y_t=\sum_{i=1}^{\infty}z_i.$$

To test convergence, it is sufficient to show that $\lim_{i\to\infty}\left|\frac{z_{i+1}}{z_i}\right|<1$. It follows that

$$\left|\frac{z_{i+1}}{z_i}\right|=\beta\exp\left(\begin{array}{c}\left(A_{i+1}-A_i\right)x+\left(B_{i+1}-B_i\right)\left(x_t-x\right)\\ +\left(C_{i+1}-C_i\right)\eta+\left(D_{i+1}-D_i\right)\left(\eta_t-\eta\right)+\left(F_{i+1}-F_i\right)\omega^2\end{array}\right),$$

which, when defining $\widetilde{X_i} \equiv X_{i+1} - X_i$ becomes

$$\left|\frac{z_{i+1}}{z_i}\right| = \beta \exp\left(\widetilde{A}x + \widetilde{B}_i\left(x_t - x\right) + \widetilde{C}_i\eta + \widetilde{D}_i\left(\eta_t - \eta\right) + \widetilde{F}_i\omega^2\right),$$

where

$$\begin{gathered}
\widetilde{A} \equiv 1-\gamma, \qquad \widetilde{B}_i \equiv \left(1-\gamma\right)\rho^{i+1} \\
\widetilde{C}_i \equiv \tfrac{\theta^2}{2}\left(1 - 2\rho^{i+1} + \rho^{2(i+1)}\right), \\
\widetilde{D}_i \equiv \tfrac{\theta^2}{2}\left(\left(\rho_\eta^{i+1} - \tfrac{2\rho_\eta\rho}{\rho_\eta-\rho}\left(\rho^i\left(1-\rho\right) - \rho_\eta^i\left(1-\rho_\eta\right)\right) + \tfrac{\rho_\eta\rho^2}{\rho_\eta-\rho^2}\left(\rho^{2i}\left(1-\rho^2\right) - \rho_\eta^i\left(1-\rho_\eta\right)\right)\right)\right), \\
\text{and } \widetilde{F}_i \equiv \tfrac{\theta^4}{4}\left(\begin{array}{c}
\phi_1^2 + \phi_2^2\rho_\eta^{2i} + \phi_3^2\rho^{2i} + \phi_4^2\rho^{4i} \\
+2\phi_1\phi_2\rho_\eta^i + 2\phi_1\phi_3\rho^i + 2\phi_1\phi_4\rho^{2i} \\
+2\phi_2\phi_3\left(\rho_\eta\rho\right)^i + 2\phi_2\phi_4\left(\rho_\eta\rho^2\right)^i + 2\phi_3\phi_4\rho^{3i}
\end{array}\right).
\end{gathered}$$

Taking the limit of these terms gives

$$\begin{gathered}
\lim_{i\to\infty}\widetilde{A} = 1-\gamma, \qquad \lim_{i\to\infty}\widetilde{B}_i = 0 \\
\lim_{i\to\infty}\widetilde{C}_i = \tfrac{1}{2}\left(\tfrac{1-\gamma}{1-\rho}\right)^2, \\
\lim_{i\to\infty}\widetilde{D}_i = 0, \text{ and } \lim_{i\to\infty}\widetilde{F}_i = \left(\tfrac{\theta^2}{2\left(1-\rho_\eta\right)}\right)^2.
\end{gathered}$$

It then follows that

$$\lim_{i\to\infty}\left|\frac{z_{i+1}}{z_i}\right| = \beta \exp\left(\left(1-\gamma\right)x + \frac{1}{2}\left(\frac{1-\gamma}{1-\rho}\right)^2\eta + \left(\frac{\theta^2}{2\left(1-\rho_\eta\right)}\right)^2\omega^2\right).$$

A.4 Mean price-dividend ratio: Proof of Theorem 3

In order to calculate the unconditional mean, it is necessary to appropriately capture the autocorrelation created by the ε_η innovations in the dividend growth process. Iterating backward the stochastic volatility process, equation (4), so that η_t is in terms of a sequence of past ε_η realizations gives

$$\eta_t - \eta = \rho_\eta^k\left(\eta_{t-k} - \eta\right) + \omega\sum_{s=1}^k \rho_\eta^{s-1}\varepsilon_{\eta,t+1-s}. \tag{17}$$

Taking the limit gives

$$\lim_{k\to\infty}\eta_t - \eta = \omega\sum_{s=1}^\infty \rho_\eta^{s-1}\varepsilon_{\eta,t+1-s},$$

in which case

$$\eta_{t+1-j} - \eta = \omega\sum_{s=1}^\infty \rho_\eta^{s-1}\varepsilon_{\eta,t+2-j-s}.$$

Similarly, x_t can be written as

$$x_t - x = \rho^k \left(x_{t-k} - x\right) + \textstyle\sum_{j=1}^{k} \rho^{j-1} \sqrt{\eta_{t+1-j}} \varepsilon_{t+1-j},$$

and

$$\lim_{k\to\infty} x_t - x = \textstyle\sum_{j=1}^{\infty} \rho^{j-1} \sqrt{\eta_{t+1-j}} \varepsilon_{t+1-j}.$$

Substituting in for equation (17) gives

$$x_t - x = \textstyle\sum_{j=1}^{\infty} \rho^{j-1} \left(\sqrt{\eta + \omega \sum_{s=1}^{\infty} \rho_\eta^{s-1} \varepsilon_{\eta,t+2-j-s}}\right) \varepsilon_{t+1-j}. \tag{18}$$

The unconditional mean of y_t is

$$\mathbf{E}\left(y_t\right) = \textstyle\sum_{i=1}^{\infty} \beta^i \exp\left(A_i x + C_i \eta + F_i \omega^2\right) \mathbf{E} \exp\left(B_i \left(x_t - x\right) + D_i \left(\eta_t - \eta\right)\right),$$

which means we need only evaluate the expectations term

$$\mathbf{E} \exp\left(B_i \left(x_t - x\right) + D_i \left(\eta_t - \eta\right)\right).$$

To do this, first substitute using equation (18), which gives

$$\mathbf{E} \exp\left(B_i \left(\textstyle\sum_{j=1}^{\infty} \rho^{j-1} \left(\sqrt{\eta + \omega \sum_{s=1}^{\infty} \rho_\eta^{s-1} \varepsilon_{\eta,t+2-j-s}}\right) \varepsilon_{t+1-j}\right) + D_i \left(\omega \sum_{j=1}^{\infty} \rho_\eta^{j-1} \varepsilon_{\eta,t+1-j}\right)\right).$$

At this stage it is instructive to rewrite the expectations operator as an integral of probabilistic outcomes

$$\int_{\varepsilon_{\eta,t}} \cdots \int_{\varepsilon_{\eta,t-\infty}} \int_{\varepsilon_t} \cdots \int_{\varepsilon_{t-\infty}} \exp\left(\begin{array}{c} B_i \sum_{j=1}^{\infty} \rho^{j-1} \left(\sqrt{\eta + \omega \sum_{s=1}^{\infty} \rho_\eta^{s-1} \varepsilon_{\eta,t+2-j-s}}\right) \varepsilon_{t+1-j} \\ +D_i \omega \left(\sum_{j=1}^{\infty} \rho_\eta^{j-1} \varepsilon_{\eta,t+1-j}\right) \end{array}\right) \underset{\varepsilon_t}{dF} \cdots \underset{\varepsilon_{t-\infty}}{dF} \underset{\varepsilon_{\eta,t}}{dG} \cdots \underset{\varepsilon_{\eta,t-\infty}}{dG},$$

Rearranging the above expression gives

$$\begin{aligned} &\int_{\varepsilon_{\eta,t}} \cdots \int_{\varepsilon_{\eta,t-\infty}} \left(\prod_{j=1}^{\infty} \int_{\varepsilon_t} \cdots \int_{\varepsilon_{t-\infty}} \exp\left(B_i \rho^{j-1} \left(\sqrt{\eta + \omega \sum_{s=1}^{\infty} \rho_\eta^{s-1} \varepsilon_{\eta,t+2-j-s}}\right) \varepsilon_{t+1-j}\right) \underset{\varepsilon_t}{dF} \cdots \underset{\varepsilon_{t-\infty}}{dF}\right) \\ &\times \exp\left(D_i \omega \left(\sum_{j=1}^{\infty} \rho_\eta^{j-1} \varepsilon_{\eta,t+1-j}\right)\right) \underset{\varepsilon_{\eta,t}}{dG} \cdots \underset{\varepsilon_{\eta,t-\infty}}{dG}. \end{aligned}$$

Using the same standard result as before for Gaussian shocks gives

$$\begin{aligned} &\int_{\varepsilon_{\eta,t}} \cdots \int_{\varepsilon_{\eta,t-\infty}} \prod_{j=1}^{\infty} \exp\left(\frac{B_i^2}{2} \rho^{2(j-1)} \left(\eta + \omega \sum_{s=1}^{\infty} \rho_\eta^{s-1} \varepsilon_{\eta,t+2-j-s}\right)\right) \\ &\times \exp\left(D_i \omega \left(\sum_{j=1}^{\infty} \rho_\eta^{j-1} \varepsilon_{\eta,t+1-j}\right)\right) \underset{\varepsilon_{\eta,t}}{dG} \cdots \underset{\varepsilon_{\eta,t-\infty}}{dG}, \end{aligned}$$

which can be rewritten as

$$\underset{\varepsilon_{\eta,t}}{\int}\cdots\underset{\varepsilon_{\eta,t-\infty}}{\int}\exp\left(\frac{B_i^2}{2}\sum_{j=1}^{\infty}\rho^{2(j-1)}\left(\eta+\omega\sum_{s=1}^{\infty}\rho_\eta^{s-1}\varepsilon_{\eta,t+2-j-s}\right)+D_i\omega\left(\sum_{j=1}^{\infty}\rho_\eta^{j-1}\varepsilon_{\eta,t+1-j}\right)\right)\underset{\varepsilon_{\eta,t}}{dG}\cdots\underset{\varepsilon_{\eta,t-\infty}}{dG},$$

Removing the constants term from the integral gives

$$H_i\underset{\varepsilon_{\eta,t}}{\int}\cdots\underset{\varepsilon_{\eta,t-\infty}}{\int}\exp\left(\frac{B_i^2\omega}{2}\sum_{j=1}^{\infty}\rho^{2(j-1)}\left(\sum_{s=1}^{\infty}\rho_\eta^{s-1}\varepsilon_{\eta,t+2-j-s}\right)+D_i\omega\left(\sum_{j=1}^{\infty}\rho_\eta^{j-1}\varepsilon_{\eta,t+1-j}\right)\right)\underset{\varepsilon_{\eta,t}}{dG}\cdots\underset{\varepsilon_{\eta,t-\infty}}{dG},$$

where

$$H_i\equiv\exp\left(\frac{1}{2}\frac{B_i^2\eta}{1-\rho^2}\right).$$

Focussing on the integral term, the above expression is rearranged in order to bring together ε_η innovations with the same time subscript:

$$\underset{\varepsilon_{\eta,t}}{\int}\cdots\underset{\varepsilon_{\eta,t-\infty}}{\int}\exp\left(\sum_{j=1}^{\infty}\left(\frac{B_i^2\omega}{2}\rho_\eta^{j-1}\left(\sum_{s=1}^{j}\left(\rho_\eta^{-1}\rho^2\right)^{s-1}\right)+D_i\omega\rho_\eta^{j-1}\right)\varepsilon_{\eta,t+1-j}\right)\underset{\varepsilon_{\eta,t}}{dG}\cdots\underset{\varepsilon_{\eta,t-\infty}}{dG}.$$

Again, using the results of standard normals and geometric series gives

$$\exp\left(\frac{\omega^2}{2}\sum_{j=1}^{\infty}\left(\frac{B_i^2}{2}\rho_\eta^{j-1}\left(\frac{1-\left(\rho_\eta^{-1}\rho^2\right)^j}{1-\rho_\eta^{-1}\rho^2}\right)+D_i\rho_\eta^{j-1}\right)^2\right).$$

This can be rewritten as

$$\exp\left(\frac{\omega^2}{2}\sum_{j=1}^{\infty}\left(\gamma_1\rho_\eta^{j-1}-\gamma_2\rho^{2(j-1)}\right)^2\right),\tag{19}$$

where

$$\gamma_{i,1}\equiv\left(\frac{B_i^2}{2}\frac{\rho_\eta}{\rho_\eta-\rho^2}+D_i\right)\text{ and }\gamma_{i,2}\equiv\frac{B_i^2}{2}\frac{\rho^2}{\rho_\eta-\rho^2}.$$

Multiplying out the quadratic term in expression (19) gives

$$\exp\left(\frac{\omega^2}{2}\sum_{j=1}^{\infty}\left(\gamma_{i,1}^2\rho_\eta^{2(j-1)}-2\gamma_{i,1}\gamma_{i,2}\left(\rho_\eta\rho^2\right)^{j-1}+\gamma_2^2\rho^{4(j-1)}\right)\right),$$

And using the standard results of geometric series gives

$$\exp\left(\frac{\omega^2}{2}\left(\frac{\gamma_1^2}{1-\rho_\eta^2}-\frac{2\gamma_1\gamma_2}{1-\rho_\eta\rho^2}+\frac{\gamma_2^2}{1-\rho^4}\right)\right).$$

Thus, the unconditional mean price-dividend ratio is

$$\mathbf{E}\left(y_t\right)=\sum_{i=1}^{\infty}\beta^i\exp\left(\begin{array}{c}A_ix+C_i\eta+F_i\omega^2\\+\frac{1}{2}\frac{B_i^2\eta}{1-\rho^2}+\frac{\omega^2}{2}\left(\frac{\gamma_{i,1}^2}{1-\rho_\eta^2}-\frac{2\gamma_{i,1}\gamma_{i,2}}{1-\rho_\eta\rho^2}+\frac{\gamma_{i,2}^2}{1-\rho^4}\right)\end{array}\right).$$

Next, it is necessary to show that the condition for convergence of the infinite summation in the expression above is the same as the condition stated in Theorem 2. Let

$$z_i = \beta^i \exp\left(\begin{array}{c} A_i x + C_i\eta + F_i\omega^2 \\ +\frac{1}{2}\frac{B_i^2\eta}{1-\rho^2} + \frac{\omega^2}{2}\left(\frac{\gamma_{i,1}^2}{1-\rho_\eta^2} - \frac{2\gamma_{i,1}\gamma_{i,2}}{1-\rho_\eta\rho^2} + \frac{\gamma_{i,2}^2}{1-\rho^4}\right)\end{array}\right),$$

so that $\mathbf{E}\left(y_t\right) = \sum_{i=1}^{\infty} z_i$. Then

$$\left|\frac{z_{i+1}}{z_i}\right| = \beta\exp\left(\begin{array}{c} \widetilde{A}x + \widetilde{C}_i\eta + \widetilde{F}_i\omega^2 + \frac{\eta}{2(1-\rho^2)}\left(B_{i+1}^2 - B_i^2\right) + \frac{\omega^2}{2} \\ \times\left(\left(\frac{1}{1-\rho_\eta^2}\left(\gamma_{i+1,1}^2 - \gamma_{i,1}^2\right) - \frac{2}{1-\rho_\eta\rho^2}\left(\gamma_{i+1,1}\gamma_{i+1,2} - \gamma_{i,1}\gamma_{i,2}\right) + \frac{1}{1-\rho^4}\left(\gamma_{i+1,2}^2 - \gamma_{i,2}^2\right)\right)\right)\end{array}\right).$$

The parameters $\widetilde{A}, \widetilde{C}_i$, and $\widetilde{F}_i$ are the same as in Section A.3. Since Section A.3 also shows that $\lim_{i\to\infty}\widetilde{B}_i = \lim_{i\to\infty}\widetilde{D}_i = 0$, it follows naturally (or after much tedious manipulation[4]) that this result also implies that

$$\lim_{i\to\infty}\left(B_{i+1}^2 - B_i^2\right) = \lim_{i\to\infty}\left(D_{i+1} - D_i\right) = 0,$$

$$\lim_{i\to\infty}\left(\gamma_{i+1,1}^2 - \gamma_{i,1}^2\right) = \lim_{i\to\infty}\left(\gamma_{i+1,1}\gamma_{i+1,2} - \gamma_{i,1}\gamma_{i,2}\right) = \lim_{i\to\infty}\left(\gamma_{i+1,2}^2 - \gamma_{i,2}^2\right) = 0.$$

[4] Available upon request.