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# Derivatives Pricing under Bilateral Counterparty Risk\*

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**Working Paper**

## Abstract

We consider risk-neutral valuation of a contingent claim under bilateral counterparty risk in a reduced-form setting similar to that of Duffie and Huang [1996] and Duffie and Singleton [1999]. The probabilistic valuation formulas derived under this framework cannot be usually used for practical pricing due to their *recursive* path-dependencies. Instead, finite-difference methods are used to solve the quasi-linear partial differential equations that equivalently represent the claim value function. By imposing restrictions on the dynamics of the risk-free rate and the stochastic intensities of the counterparties' default times, we develop path-independent probabilistic valuation formulas that have closed-form solution or can lead to computationally efficient pricing schemes. Our framework incorporates the so-called wrong way risk (WWR) as the two counterparty default intensities can depend on the derivatives values. Inspired by the work of Ghamami and Goldberg [2014] on the impact of WWR on credit value adjustment (CVA), we derive *calibration-implied formulas* that enable us to mathematically compare the derivatives values in the presence and absence of WWR. We illustrate that derivatives values under unilateral WWR need not be less than the derivatives values in the absence of WWR. A sufficient condition under which this inequality holds is that the price process follows a semimartingale with independent increments.

**Keywords:** Reduced-Form Modeling, Counterparty Risk, Wrong Way Risk, Credit Value Adjustment, Basel III

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# 1 Introduction

We consider the problem of valuing a contingent claim under bilateral counterparty risk using the reduced-form approach. Our framework is similar to that of Duffie and Huang [1996] and Duffie and Singleton [1999] in that we make the same recovery modeling assumption which is often referred to as *fractional recovery of market value*.<sup>1</sup> We assume that the contingent claim has a single real-valued promised payoff occurring at a fixed time. Let  $\Pi_T$  denote the payoff at maturity  $T > 0$ , and  $V_t$  denote the risk-neutral value of the claim at time  $t \in [0, T]$  conditional on the survival of both counterparties by time  $t$ . Hereafter, we refer to  $V_t$  as the *survival-contingent* or *pre-default* risk-neutral value of the claim at time  $t$ . Let  $h^A$  and  $h^B$  denote the well-defined risk-neutral intensity processes associated with the default times of counterparty A and counterparty B. We also let  $0 \leq L_t^i \leq 1$  denote the expected fractional loss in market value if counterparty  $i$  were to default at time  $t$  conditional on the information available up to time  $t$ ;  $i = A, B$ .<sup>2</sup> As will be shown in Section 2, the risk-neutral derivatives value at time  $t$  before any counterparty default takes the following probabilistic expression

$$V_t = E_t^{\mathbb{Q}} \left[ \exp \left( - \int_t^T (r_u + s_u^A \mathbf{1}\{V_u < 0\} + s_u^B \mathbf{1}\{V_u \geq 0\}) du \right) \Pi_T \right], \quad (1)$$

where  $r$  is the risk-free rate,  $E_t^{\mathbb{Q}}[\cdot]$  denotes risk-neutral expectation conditional on all information available up to time  $t$ , and  $s_t^i = h_t^i L_t^i$  represents the risk-neutral conditional expected rate of loss of market value at time  $t$  owing to the default of counterparty  $i$ ,  $i = A, B$ . That is,  $s_t^i = h_t^i L_t^i$  is the *risk-neutral mean-loss rate* due to the default of counterparty  $i$ . The probabilistic representation of the survival-contingent risk-neutral derivatives value (1) is path-dependent; it is based on the recursive integral equation containing the value function. This path-dependent *implicit* probabilistic representation is not useful for practical pricing. The value function equivalently solves a quasi-linear partial differential equation (PDE), where the killing rate, in addition to the risk-free rate, also depends on the value function,  $s^A$  and  $s^B$ . This non-linear PDE is often solved by a finite difference method for pricing calculations (Duffie and Huang [1996] and Huge and Lando [1999]). Note that the non-linearity in the pricing PDE or equivalently the recursive path-dependency in the probabilistic representation of the derivatives value need not arise merely due to the presence of the indicator functions of the value process caused by the payoff and so  $V$  being real-valued. This is simply because  $s^A$  or  $s^B$  can also depend on the derivatives value,  $V$ .<sup>3</sup>

We will work under a Markovian framework where the underlying uncertainty is modeled by a multidimensional diffusion. By imposing restrictions on the dynamics of the risk-free rate (short rate),  $s^A$ , and  $s^B$ , we derive path-independent probabilistic valuation formulas that have closed-form solution or can lead to computationally efficient pricing schemes. In the credit literature and credit modeling practice, once a particular type of recovery modeling assumption is made, one

<sup>1</sup>See Chapter 6 of Duffie and Singleton [2003] or Chapter 5 of Lando [2004] for various recovery modeling assumptions and their implications for reduced-form risk-neutral pricing under default risk.

<sup>2</sup>We make the usual assumption that  $A$  and  $B$  cannot default simultaneously.

<sup>3</sup>For instance, consider the derivatives value,  $V_t \geq 0$  for all  $t \in [0, T]$ , under unilateral counterparty risk  $h^A = 0$ . When  $h^B$  is defined as a function of  $V$ , i.e.,  $h^B \equiv h^B(V)$ , the value function (1) remains recursively path-dependent.

often ultimately assumes that the recovery rate, i.e.,  $(1 - L^i)$  in our setting, is constant. So, our restrictions on the dynamics of  $s^A$  and  $s^B$  are to be viewed as restrictions on the dynamics of the counterparty intensities  $h^A$  and  $h^B$ . Starting from an arbitrage-free market model where the money market account growing by short rate is the numeraire, Section 3.1 specifies the short rate dynamics that facilitate a probability measure change under which the survival-contingent value function does not depend on the money market account.<sup>4</sup> Next, to “remove” the recursive dependence of the value function under the *auxiliary* probability measure on counterparty intensities, in Section 3.2 we work with a function of the price process which is required to be a martingale under the *auxiliary* measure. This function of the survival-contingent price process is fully specified by the martingale property and our restrictions on the dynamics of the counterparty intensities. Our path-independent probabilistic valuation formula is then derived by benefiting from the martingale property of the aforementioned function of the price process. This is the first contribution of the paper. Appendix B, which considers derivatives pricing under unilateral counterparty risk, gives an alternative path-independent probabilistic valuation formula using a novel application of well-known change of numeraire techniques.

Since the counterparty default intensities in our framework depend on the derivatives values, our model naturally incorporates *bilateral wrong way risk* (WWR).<sup>5</sup> Recall that the presence of what is often referred to as wrong (right) way risk implies that the counterparty to a derivatives transaction becomes more (less) likely to default when the derivatives value increases. It is natural to consider *bilateral* wrong (right) way risk in derivatives pricing under *bilateral* counterparty risk.<sup>6</sup> For simplicity, hereafter, unless stated otherwise, we avoid explicitly referring to and using the term *right way risk*. Section 4 first outlines the calibration scheme of reduced-form counterparty-defaultable derivatives pricing models under wrong way risk. It then shows that the calibration scheme of our risk-neutral valuation model can be developed similarly, but it will be computationally more involved due to our restrictions on the dynamics of the short rate and counterparty default intensities.

Wrong way risk is often defined and modeled merely in credit value adjustment (CVA) calculations, (see, e.g., Ghamami and Goldberg [2014], Hull and White [2012], Li and Mercurio [2015], and the references therein). It can be easily shown that the very basic definition of CVA as the *market price of counterparty credit risk*, i.e., as “*the counterparty-default-free value of the derivatives minus its counterparty-defaultable value*” need not be equal to the widely-used CVA formulas that take risk-neutral-expected-discounted-loss type forms. Let  $V$  denote the initial value of the claim under bilateral counterparty risk. More generally, it is not difficult to show that the widely-used CVA and debt value adjustment (DVA) formulas appearing as expected-discounted-loss and gain need not lead to the widely-used price decomposition  $V = \hat{V} - \text{CVA} + \text{DVA}$ , where  $\hat{V}$  denotes the risk-neutral initial value of the claim in the absence of counterparty risk. So, it would be less detached from asset

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<sup>4</sup>Appendix A compares our *auxiliary* probability measure change with the well-known change of numeraire techniques.

<sup>5</sup>The term (unilateral) wrong way risk was first introduced by Canabarro and Duffie [2003] in a non-mathematical way.

<sup>6</sup>Bilateral wrong way risk is present, for instance, when a financial institution buys a forward contract on an asset that is highly negatively correlated with the asset values of itself and its counterparty.

pricing theory, and it would be more insightful to consider wrong way risk *directly* in risk-neutral counterparty-defaultable derivatives pricing as opposed to incorporating WWR in CVA calculations using expected-discounted-loss type formulas that do not represent the true market price of counterparty credit risk.<sup>7</sup>

Using the reduced-form approach, Ghamami and Goldberg [2014] show that CVA under wrong way risk, denoted by  $CVA_W$ , need not exceed CVA in the absence of wrong way risk, denoted by  $CVA_I$ . Their result relies on developing *calibration-implied formulas* for  $CVA_I$  that make  $CVA_I$  mathematically comparable to  $CVA_W$  as summarized in Section 5.1. Drawing upon the work of Ghamami and Goldberg [2014], we first consider the unilateral case and develop calibration-implied formulas that enable us to compare derivatives values in the presence and absence of WWR. We show that derivatives values under unilateral WWR need not be less than derivatives values in the absence of WWR. A sufficient condition under which this inequality holds is that the survival-contingent price process follows a semimartingale with independent increments. This is shown by Proposition 1 of Section 5.2. The survival-contingent value function in (1) cannot be used to derive calibration-implied formulas in the bilateral case in the absence of WWR. Proposition 2 of Section 5.2 gives an alternative expression for the survival-contingent price process which facilitates the derivation of the calibration-implied formulas that make the bilateral counterparty-defaultable derivatives values in the absence and presence of WWR mathematically comparable. Similar to our results in the unilateral case, we conclude that no general inequality can be drawn for bilateral counterparty-defaultable derivatives values in the presence and absence of WWR. Our results on the impact of wrong way risk on derivatives values under counterparty risk are the second contribution of the paper. These results have nontrivial implications for finance practitioners, accountants, and bank regulators. Consider, for instance, a broker-dealer that has purchased a derivatives contract from its counterparty. When the contract is viewed as its asset, the dealer records and reports the risk-neutral value of the counterparty-defaultable derivatives as  $\hat{V} - CVA$  on the asset side of its balance sheet at pre-specified points in time, (see, e.g., Ernst and Young [2013]). Under wrong way risk, the dealer, following bank regulators' WWR counterparty risk rules, increases its CVA expected-discounted-loss calculations, (BCBS [2011]).<sup>8</sup> We show that the price decomposition  $\hat{V} - CVA$  loses its validity under wrong way risk in that it will not represent the true risk-neutral counterparty-defaultable derivatives value. We also show that counterparty-defaultable derivatives values need not decrease under WWR.

## 2 Derivatives Pricing under Bilateral Counterparty Risk

Consider a fixed a probability space  $(\Omega, \mathcal{F}, P)$  and a family  $\{\mathcal{F}_t\}_{t \geq 0}$  of sub- $\sigma$ -algebras of  $\mathcal{F}$  satisfying the usual conditions. We suppose that there is a state-variable vector process  $X = (X_1, \dots, X_n)^*$

<sup>7</sup>See, e.g., Chapters 12 and 13 of Gregory [2012] for CVA (DVA) definitions and formulas.

<sup>8</sup>Counterparty risk capital regulations have been premised on the assumption that a financial institution CVA numbers (both model-based and non-model-based) should increase under wrong way risk, (Ghamami and Goldberg [2014]).

that is Markovian under an equivalent martingale measure  $\mathbb{Q}$ .<sup>9</sup> The time-homogeneous diffusions  $X_i$  have stochastic differentials of the form

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^d a_{ij}(t)dW_j(t), \quad (2)$$

where  $W_1, \dots, W_d$  are independent 1-dimensional  $\mathbb{Q}$  standard Brownian motions. For notational simplicity we have suppressed the dependence of  $\mu$  and  $a$  on  $X$ , i.e.,  $\mu(t) \equiv \mu(X(t))$  and  $a(t) \equiv a(X(t))$ . Let the  $n \times d$  matrix  $a = \{a_{ij}\}$  denote the *dispersion matrix* of the above  $n$ -dimensional diffusion process. We make the usual assumption that the symmetric *diffusion matrix*  $b \equiv aa^*$  with elements  $b_{ik} \equiv \sum_{j=1}^d a_{ij}a_{kj}$  is non-negative-definite.<sup>10</sup> We consider a market model driven by the  $n$ -dimensional diffusion  $X$  where there exists a money market account whose balance starts at one and grows at some stochastic short interest rate  $r_t \equiv r(X_t, t) \in \mathbb{R}$ . We make the usual assumption that this economy is arbitrage-free. So, from the first Fundamental Theorem of Asset Pricing, there exists an equivalent martingale measure  $\mathbb{Q}$  associated with the money market account, and under  $\mathbb{Q}$ , a martingale arises whenever the price of any non-dividend paying asset is deflated by the money market account.

Consider a contingent claim which matures at a fixed time  $T > 0$ . The claim's promised payoff is given by a function  $\Pi(x) : \mathbb{R}^n \mapsto \mathbb{R}$ . When the claim expires at  $T$ , it pays off  $\Pi(X_T)$  at  $T$ , assuming neither counterparty has defaulted prior to  $T$ . The claim has no other promised payoffs, either before or after  $T$ . Note that the final promised payoff function  $\Pi(x)$  is not sign-definite, i.e., there is positive probability that this promised final payoff could be either positive or negative. Examples of such claims include forward contracts and risk reversals. Taking counterparty A's perspective, one can assume that counterparty A long the claim and hence receives  $\Pi(X_T)$  at  $T$  assuming no prior default; counterparty B is short the claim and pays  $\Pi(X_T)$  at  $T$  assuming no prior default.

We assume that the default time of counterparty  $i \in \{A, B\}$  denoted by  $\tau^i$  is an  $\mathbb{F}$ -stopping time valued in  $[0, \infty]$ , which accepts a risk-neutral intensity process  $h^i$  such that

$$dH_t^i = (1 - H_t^i)h_t^i dt + dM_t^i, \quad (3)$$

where  $H_t^i = 1\{\tau^i \leq t\}$  is the default indicator of counterparty  $i$  and  $M^i$  is a  $\mathbb{Q}$  martingale.<sup>11</sup> Set  $\tau = \tau^A \wedge \tau^B$ , i.e.,  $\tau$  is the minimum of  $\tau^A$  and  $\tau^B$ . We assume that the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  of the underlying probability space encompasses the filtration generated by the underlying diffusion  $X$  denoted by  $\mathbb{F}^X$  and the filtration generated by the default indicator process  $H_t = 1\{\tau \leq t\}$  denoted

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<sup>9</sup>\* denotes transpose.

<sup>10</sup>See, e.g., Chapter 5 of Karatzas and Shreve [1991], or Chapter 7 of Revuz and Yor [2004]. The *diffusion matrix*  $b \equiv aa^*$  is assumed non-negative-definite because it approximates the rate of change in the covariance matrix of the diffusion vector  $X_t - X_0$  for small values of  $t > 0$ . To see this heuristically, set  $x_i \equiv X_i(0)$  and suppose that  $X_i$ 's have zero drift. For small  $t$  consider the time-discretized approximations to  $X_i$ 's,  $\hat{X}_i(t) = x_i + \sqrt{t} \sum_{j=1}^d a_{ij}(t)Z_j$  with  $Z_1, \dots, Z_d$  being independent standard normal random variables and  $a(t) \equiv a(\hat{X}_t)$ . It is then easy to see that  $E[(\hat{X}_t - x)(\hat{X}_t - x)^*] = tb$  with  $b = \{b_{ij}\}$  and  $b_{ik} \equiv \sum_{j=1}^d a_{ij}a_{kj}$ ,  $i = 1, \dots, n$ . The non-negative-definiteness assumption on the *diffusion matrix*  $b$  is also explicitly used in Section 3.2 when we specify the credit spread dynamics in our framework.

<sup>11</sup>The event  $\tau^i = \infty$  means no default.

by  $\mathbb{H}$ . That is,  $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{H}_t$ . We also assume that  $P(\tau^A = \tau^B) = 0$ . Let  $U$  denote the real-valued value process of the claim from  $A$ 's perspective under bilateral default risk of  $A$  and  $B$ . If a default occurs at time  $t$ , the claim value at time  $t$  is specified as follows

$$U_t^-(\gamma_t^A + \gamma_t^B), \quad (4)$$

where

$$\gamma_t^A = 1\{t = \tau^A\} (1\{U_t^- < 0\}(1 - L_t^A) + 1\{U_t^- \geq 0\}), \quad (5)$$

and

$$\gamma_t^B = 1\{t = \tau^B\} (1\{U_t^- \geq 0\}(1 - L_t^B) + 1\{U_t^- < 0\}). \quad (6)$$

We assume that the fractional loss processes  $L^i$  are bounded by 1 and are predictable. By  $U_t^-$  we mean the price of the claim just before default, i.e.,  $U_t^- \equiv \lim_{s \uparrow t} U_s$ . That is, in words, when the value to counterparty  $A$  is positive, a default by  $B$  causes a sudden downward jump in value from  $U_{t-}$  to  $(1 - L_t^B)U_{t-}$  with  $L_t^B \in [0, 1]$  being the stochastic process describing fractional loss to counterparty  $A$  given default by counterparty  $B$  at time  $t$ . In contrast, when the value to counterparty  $A$  is positive, we assume that a default by  $A$  causes no change in value, since  $B$  still owes  $A$  everything that was owed just prior to  $A$ 's default. Similarly, when the value to counterparty  $A$  is negative, a default by  $B$  causes no change in value, since  $A$  still owes  $B$  everything that was owed just prior to  $B$ 's default. However, when the value to counterparty  $A$  is negative, a default by  $A$  causes a sudden upward jump in value from the negative value  $U_{t-}$  to the less negative value  $(1 - L_t^A)U_{t-}$ , where  $L_t^A \in [0, 1]$  is the stochastic process describing fractional loss to counterparty  $B$  given default by counterparty  $A$  at time  $t$ .

Consider the process  $V$  with the property that  $V_T = \Pi(X_T)$  and  $V_t = U_t$  for  $t < \tau$ . That is,  $V_t$  represents the risk-neutral bilateral counterparty-defaultable derivatives value at time  $t \in [0, T]$  if there has been no default by time  $t$ . We have been referring to  $V$  as the survival-contingent or pre-default price process.

Set  $s_t^i = s^i(V_t, X_t, t) = L_t^i h_t^i$  and  $D_t \equiv \exp(\int_0^t r_u du)$ . Benefiting from the tractability inherent in the *fractional recovery of market value* assumption of (4) - (6), similar to Duffie and Huang [1996] and Duffie and Singleton [1999] probabilistic valuation formulas, the survival-contingent derivatives value at time  $t$  can be expressed as

$$V_t = D_t E^{\mathbb{Q}} \left[ \exp\left(-\int_t^T R_s ds\right) \frac{\Pi(X_T)}{D_T} \middle| X_t \right], \quad (7)$$

where

$$R_t = s_t^A 1\{V_t < 0\} + s_t^B 1\{V_t \geq 0\}. \quad (8)$$

The dependence of the intensities  $h^i$  and so  $s^i$  on the market value  $V$  can capture bilateral wrong way risk. More specifically, wrong (right) way risk can be incorporated in to the valuation framework

when  $h^i$  is defined as an increasing (decreasing) function of  $V$ .<sup>12</sup>

The derivation of the recursive integral equation on the right side of (7) is similar to the method of proof of Theorem 1 of Duffie and Singleton [1999], which is outlined as follows.<sup>13</sup> Set  $H_t = 1\{\tau \leq t\}$ . The discounted gain process  $G$  defined by

$$G_t = \exp\left(-\int_0^t r_s ds\right)V_t(1 - H_t) + \int_0^t \exp\left(-\int_0^s r_u du\right)V_s^-(\gamma_s^A + \gamma_s^B)dH_s,$$

should be a martingale under  $\mathbb{Q}$ . The property that  $G$  is a  $\mathbb{Q}$  martingale and the terminal condition  $V_T = \Pi(X_T)$  gives a complete characterization of arbitrage-free pricing of the contingent claim under bilateral counterparty default risk. Assuming that  $V$  does not jump at  $\tau$  and given the stochastic differential of the default indicators  $H^i$  specified by (3), after using Ito's formula to derive the stochastic differential of the discounted gain process, it can be shown that for  $G$  to be a  $\mathbb{Q}$  martingale, it is necessary and sufficient that

$$V_t = \int_0^t (r_s + R_s)V_s ds + m_t,$$

for some  $\mathbb{Q}$  martingale  $m$ . We know from Lemma 1 of Duffie et al. [1996] that the equality above holds for  $t \leq T$  if and only if

$$V_t = D_t E^{\mathbb{Q}} \left[ \exp\left(-\int_t^T R_s ds\right) \frac{V_T}{D_T} \middle| X_t \right],$$

and so (7) has been derived.

We know from the ‘‘Feynman-Kac’’ formula that, under technical conditions, the survival-contingent value function  $V$  probabilistically represented by (7) equivalently solves the backward Kolmogorov quasi-linear PDE

$$\left( \mathcal{G}_x + \frac{\partial}{\partial t} \right) V(x, t) = (r(x, t) + s^A(V(x, t), x, t)1\{V(x, t) < 0\} + s^B(V(x, t), x, t)1\{V(x, t) \geq 0\}) V(x, t), \quad (9)$$

for  $x \in \mathbb{R}^n, t \in [0, T]$ , where  $\mathcal{G}_x$  is the infinitesimal generator of  $X$ ,

$$\mathcal{G}_x \equiv \sum_{i=1}^n \mu_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} dX_i dX_j, \quad (10)$$

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<sup>12</sup>Recall the definition of CVA. In reduced-form wrong way risk modeling in CVA calculations, Hull and White [2012] and Ghamami and Goldberg [2014] define the counterparty's default intensity as an increasing function of the non-negative part of the derivatives portfolio value (the derivatives portfolio that the financial institution holds with its counterparty).

<sup>13</sup>To derive the probabilistic valuation formula (7), one does not need to impose a Markovian or diffusion dynamics; this can be seen from Duffie and Huang [1996] and Duffie and Singleton [1999]. The derivation of (7) in a non-diffusion setting requires the assumption that  $\Delta V_\tau = 0$  which automatically holds in our diffusion-driven setting.

with the well-known multiplication rule  $(dW_i)(dW_j) = 0$  for  $i \neq j$  and  $(dW_i)^2 = dt$ . The terminal condition restricting the survival-contingent claim value is

$$V(x, T) = \Pi(x), \quad x \in \mathbb{R}^n. \quad (11)$$

Recall that the absence of arbitrage implies that the asset value  $V_t$  deflated by the money market account  $e^{\int_0^t r(X_s, s) ds}$  is a local martingale under  $\mathbb{Q}$ . As a result, the asset value itself must grow in  $\mathbb{Q}$  expectation at the risk-free rate  $r(X_t, t)$ . The reason why the right side of (9) reflects a growth rate different than merely  $r(X_t, t)$  is that the stochastic process actually being described by  $V(x, t)$  is the survival contingent process. The extra proportional drift  $s^A(V(x, t), x, t)1[V(x, t) < 0]$  compensates for a possible jump up by the process  $V$  towards zero that can occur whenever  $V_t < 0$  and  $A$  defaults, while the extra proportional drift  $s^B(V(x, t), x, t)1[V(x, t) \geq 0]$  compensates for a possible jump down by the process  $V$  that can occur whenever  $V_t > 0$  and  $B$  defaults.

**Remark 1** The recursive integral equation on the right side of (7) can not be directly used for calculating the risk-neutral price of the contingent claim. Instead, the “pricing” quasi-linear PDEs, specified by (9) - (11), are usually solved by finite difference methods. For instance, Duffie and Huang [1996] use the Crank-Nicholson method for their numerical results. Or, Huge and Lando [1999] extending Duffie and Huang [1996] model to a rating-based framework use another finite difference method (referred to as Alternating Direction Implicit Finite Difference Method) for their numerical pricing results.

**Remark 2** As stated before, it is more consistent with the asset pricing theory to consider wrong way risk *directly* in risk-neutral counterparty-defaultable derivatives pricing as opposed to considering and incorporating WWR in CVA expected-discounted-loss calculations. Because in the presence of wrong way risk, the CVA expected-discounted-loss type formulas need not coincide with the basic definition of CVA as the *market price of counterparty credit risk*, i.e., *the counterparty-default-free value of the derivatives minus its counterparty-defaultable value*. To see this, consider the unilateral case where  $s^A = 0$ , and assume zero recovery rate for simplicity. Suppose that  $V_t \geq 0$  for all  $t \geq 0$  from counterparty  $A$ 's perspective. Set  $\tau^B \equiv \tau$  and  $s^B = h^B \equiv h$ . Recall (7), the initial value of the counterparty-defaultable derivatives value becomes

$$V_0 = E^{\mathbb{Q}} \left[ e^{-\int_0^T (h_u + r_u) du} \Pi(X_T) \right].$$

Note that

$$\hat{V}_0 = E^{\mathbb{Q}} \left[ e^{-\int_0^T r_u du} \Pi(X_T) \right],$$

denotes the initial value of the claim in the absence of counterparty risk. Now, it is not difficult to see that the so-called *market price of counterparty credit risk*

$$\hat{V}_0 - V_0 = E^{\mathbb{Q}} \left[ e^{-\int_0^T r_u du} \Pi(X_T) (1 - e^{-\int_0^T h_u du}) \right],$$

need not equal to the widely-used CVA expected-loss-formula

$$\text{CVA} \equiv E^{\mathbb{Q}} \left[ e^{-\int_0^\tau r_u du} V_\tau 1\{\tau \leq T\} \right],$$

when  $\tau$ , representing the credit quality of the counterparty, is dependent on  $V$  and  $r$ . Since the expected-discounted-loss CVA formulas do not represent the market price of counterparty risk in the presence of wrong way risk, it would be more insightful to consider any dependence of the counterparties credit quality on the derivatives value *directly* in the reduced-form risk-neutral valuation framework.

### 3 Path-Independent Probabilistic Valuation

We are to derive a path-independent probabilistic representation of the survival-contingent value function  $V(X_t, t)$  solving the quasi-linear PDE (9) subject to the terminal condition (11). The recursive integral equation

$$V(X_t, t) = E^{\mathbb{Q}} \left[ e^{-\int_t^T [r(X_u, u) + s^A(V(X_u, u), X_u, u) 1\{V(X_u, u) < 0\} + s^B(V(X_u, u), X_u, u) 1\{V(X_u, u) \geq 0\}] du} \Pi(X_T) \Big| X_t \right], \quad (12)$$

for  $t \in [0, T]$ , is an implicit probabilistic representation of the solution since the function  $V(X_t, t)$  appears on both sides of (12). If  $s^A(v, x, t) = s^B(v, x, t) = 0$ , then the PDE (9) becomes linear and the probabilistic representation (12) for  $V$  becomes *explicit*, albeit still path-dependent.

To deal with the analytical difficulty arising from the non-linearity in the PDE (9) and equivalently from the recursive path-dependency in (12) we first impose restrictions on the dynamics of the short rate. Next, we will restrict the form of  $s^A$  and  $s^B$  dependence on their three arguments. We will then illustrate how our specification of the short rate,  $s^A$ , and  $s^B$  dynamics lead to path-independent probabilistic formulas for the derivatives values under bilateral counterparty risk. Appendix B, which considers derivatives pricing under unilateral counterparty risk, gives an alternative path-independent probabilistic valuation formula using a novel application of well-known change of numeraire techniques.

#### 3.1 Short Rate Dynamics and the Measure Change

Suppose that the 1-dimensional positive process  $n$  is a  $C^2$  function of  $X_k$  for a given  $1 \leq k \leq n$ . We require that

$$r(x) = \frac{\mathcal{G}_{x_k} n(x)}{n(x)} + \lambda, \quad (13)$$

where  $\lambda$  is a constant and  $\mathcal{G}_{x_k}$  is the generator of  $X_k$ ,

$$\mathcal{G}_{x_k} n(x) = \mu_k \frac{\partial n}{\partial x_k} + \frac{1}{2} \frac{\partial^2 n}{\partial x_k^2} \sum_{j=1}^d a_{kj}^2.$$

It is not difficult to construct simple and realistic examples of short rate processes based on (13). For instance, setting  $n(X_k(t)) = \exp(-X_k^2(t))$  gives

$$r(X_k(t)) = -\mu_k(t)X_k(t) + \frac{1}{2} (X_k^2(t) - 1) \sum_{j=1}^d a_{kj}^2(t) + \lambda.$$

Setting the drift  $\mu_k$  equal to zero and choosing a positive  $\lambda$ ,  $r$  becomes a diffusion that can stay positive almost surely. As will be shown below, the dynamics of the short rate given by (13) enables us to define a  $\mathbb{Q}$ -martingale process denoted by  $N$ . We then make an equivalent measure change going from  $\mathbb{Q}$  to  $\tilde{\mathbb{Q}}$  by using  $N$  as the *Radon-Nikodym* derivative of  $\tilde{\mathbb{Q}}$  with respect to  $\mathbb{Q}$ . As will be seen later, the computational work required for the risk-neutral valuation of the counterparty-defaultable claim can be substantially reduced under the new probability measure  $\tilde{\mathbb{Q}}$  to which we refer hereafter as the auxiliary probability measure. Consider the process

$$N(t) = n(X_k(t)) \exp \left( - \int_0^t \frac{\mathcal{G}_{x_k} n(X_k(u))}{n(X_k(u))} du \right), \quad (14)$$

for any  $0 \leq t \leq T$ . Set  $n' \equiv \frac{\partial n(x)}{\partial x_k}$  and  $n(t) \equiv n(X_k(t))$ . Using Itô's formula, the stochastic differential of  $N$  can be written as

$$dN(t) = N(t) \frac{n'(t)}{n(t)} \sum_{j=1}^d a_{kj}(t) dW_j(t).$$

Given the stochastic differential of  $\log(N_t)$ , we can equivalently write,

$$N(t) = \exp \left\{ \sum_{j=1}^d \int_0^t \frac{n'(u)}{n(u)} a_{kj}(u) dW_j(u) - \frac{1}{2} \int_0^t \left( \frac{n'(u)}{n(u)} \right)^2 \sum_{j=1}^d a_{kj}^2(u) du \right\}, \quad (15)$$

to arrive at the familiar *stochastic exponential* or the *Doleans-Dade exponential* form. Assuming that the Novikov condition

$$E \left[ \exp \left( \frac{1}{2} \int_0^T \left( \frac{n'(u)}{n(u)} \right)^2 \sum_{j=1}^d a_{kj}^2(u) du \right) \right] < \infty,$$

holds,  $\{N_t\}_{t \leq T}$  becomes a true  $\mathbb{Q}$ -martingale. With  $D_t \equiv \exp(\int_0^t r_u du)$  and  $r$  specified by the process  $n$  as in (13), recall that

$$\frac{V_t}{D_t} = E^{\mathbb{Q}} \left[ \exp \left( - \int_t^T R_s ds \right) \frac{\Pi(X_T)}{D_T} \middle| X_t \right],$$

where

$$R_t = s_t^A 1\{V_t < 0\} + s_t^B 1\{V_t \geq 0\},$$

with  $s_t^i \equiv s(V(t), X(t), t)$ , and  $i = A, B$ . Define the new auxiliary probability measure  $\tilde{\mathbb{Q}}$  on  $\mathcal{F}_T$  by

$$d\tilde{\mathbb{Q}} = N_T d\mathbb{Q} \quad \text{on} \quad \mathcal{F}_T.$$

Given the Markov property of  $X$ , using Bayes' Theorem, we can write

$$E^{\tilde{\mathbb{Q}}} \left[ e^{-\int_t^T R_s ds} \frac{\Pi(X_T)}{e^{\lambda T} n_T} \middle| X_t \right] = \frac{E^{\mathbb{Q}} \left[ N_T e^{-\int_t^T R_s ds} \frac{\Pi(X_T)}{e^{\lambda T} n_T} \middle| X_t \right]}{N_t} = \frac{V_t}{e^{\lambda t} n_t} \equiv \tilde{V}_t, \quad (16)$$

where by Girsanov Theorem, under  $\tilde{\mathbb{Q}}$  the process  $X$  evolves with the drift change

$$dX_i(t) = \tilde{\mu}_i(t) dt + \sum_{j=1}^d a_{ij}(t) d\tilde{W}_j(t) \quad (17)$$

where  $\tilde{W}_j$  are  $\tilde{\mathbb{Q}}$  standard Brownian motions and for  $i \neq k$ ,

$$\tilde{\mu}_i = \mu_i + \frac{n'}{n} \sum_{j=1}^d a_{ij} a_{kj}, \quad (18)$$

and

$$\tilde{\mu}_k = \mu_k + \frac{n'}{n} \sum_{j=1}^d a_{kj}^2. \quad (19)$$

The auxiliary survival-contingent value process  $\tilde{V}_t = \frac{V_t}{e^{\lambda t} n_t}$  has the same sign as  $V_t$  for every  $t \in [0, T]$ , and so for  $R$  inside the  $\tilde{\mathbb{Q}}$ -conditional expectation on the left side of (16) we can write  $R_t = s_t^A 1\{\tilde{V}_t < 0\} + s_t^B 1\{\tilde{V}_t \geq 0\}$ . Note that  $X$  dynamics have remained time-homogeneous under  $\tilde{\mathbb{Q}}$ . This has been achieved by our proposed time-homogeneous definition of the short rate in (13) under  $\mathbb{Q}$ . Retaining time-homogeneity aids in our goal of ultimately deriving closed-form or computationally efficient formulas for the value of the contingent claim. Appendix A compares our proposed change of probability measure with the well-known numeraire change techniques.

### 3.2 Default-Intensity Dynamics

The *recursive* path-dependency still exists in the probabilistic representation of the auxiliary survival-contingent value function,

$$\tilde{V}_t = E^{\tilde{\mathbb{Q}}} \left[ e^{-\int_t^T (s_u^A 1\{\tilde{V}_u < 0\} + s_u^B 1\{\tilde{V}_u \geq 0\}) du} \frac{\Pi(X_T)}{e^{\lambda T} n_T} \middle| X_t \right], \quad t \in [0, T], \quad (20)$$

as  $\tilde{V}$  appears on both sides of the above equality. Note that  $s^i$  can also depend on  $\tilde{V}$  in addition to the underlying  $\tilde{\mathbb{Q}}$ -diffusion  $X$  and time,  $t$ . To “remove” the path-dependency on the right side of (20), suppose that we could define the process  $\{f(\tilde{V}_t)\}_{t \leq T}$  to be a  $\tilde{\mathbb{Q}}$ -martingale with  $f$  being a well-defined  $C^2$  function of  $\tilde{V}$ . Then, given that  $\tilde{V}_T = \frac{\Pi(X_T)}{e^{\lambda T} n_T}$ , the martingale property of  $f(\tilde{V})$  would give  $f(\tilde{V}_t) = E^{\tilde{\mathbb{Q}}}[f(\frac{\Pi(X_T)}{e^{\lambda T} n_T})|X_t]$ . So, when  $f$  is invertible, the path-independent probabilistic expression for  $\tilde{V}_t$ , with  $t \in [0, T]$ , becomes

$$\tilde{V}_t = f^{-1} \left( E^{\tilde{\mathbb{Q}}} \left[ f \left( \frac{\Pi(X_T)}{e^{\lambda T} n_T} \right) \middle| X_t \right] \right),$$

where  $f^{-1}$  denotes the inverse of  $f$ . In what follows we illustrate how to specify  $f$  by imposing restrictions on the dynamics of  $s^A$  and  $s^B$  while requiring  $f(\tilde{V})$  to be a  $\tilde{\mathbb{Q}}$ -martingale. Viewing the auxiliary value process  $\tilde{V}$  as a function of the  $n$ -dimensional diffusion  $X$  under  $\tilde{\mathbb{Q}}$ , and using multidimensional Itô’s formula, the stochastic differential of  $\tilde{V}$  can be written as

$$d\tilde{V}(t) = \frac{\partial \tilde{V}}{\partial t} dt + \sum_{i=1}^n \frac{\partial \tilde{V}}{\partial x_i} dX_i(t) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 \tilde{V}}{\partial x_i \partial x_j} dX_i(t) dX_j(t). \quad (21)$$

Given (20), the PDE representation of the auxiliary value process  $\tilde{V}$  via Feynman-Kac becomes

$$\left( \mathcal{G}_x + \frac{\partial}{\partial t} \right) \tilde{V}(x, t) = \left( s^A(\tilde{V}(x, t), x, t) 1\{\tilde{V}(x, t) < 0\} + s^B(\tilde{V}(x, t), x, t) 1\{\tilde{V}(x, t) \geq 0\} \right) \tilde{V}(x, t)$$

with the terminal condition

$$\tilde{V}(x, T) = \frac{\Pi(x)}{e^{\lambda T} n(x)}, \quad x \in \mathbb{R}^n.$$

Note that just as the asset value  $V$  deflated by the money market account is a local martingale under the risk-neutral measure  $\mathbb{Q}$ , the auxiliary asset value  $\tilde{V}$  is a local martingale under the auxiliary measure  $\tilde{\mathbb{Q}}$ . However, if we condition on survival, the  $\tilde{V}$  process drifts down when  $\tilde{V} < 0$  to compensate for the possible jump up upon default by A, and the  $\tilde{V}$  process drifts up when  $\tilde{V} > 0$  to compensate for the possible jump down upon default by B. The magnitude of this compensation for the jumps under  $\tilde{\mathbb{Q}}$  is given by

$$E^{\tilde{\mathbb{Q}}} d\tilde{V}_t = \left( s^A(\tilde{V}(X_t), X_t, t) 1\{\tilde{V}(X_t) < 0\} + s^B(\tilde{V}(X_t), X_t, t) 1\{\tilde{V}(X_t) > 0\} \right) \tilde{V}(X_t),$$

for  $t \in [0, T]$ . Let  $f(\tilde{V})$  be a  $C^2$  function. We use Itô’s formula to derive the stochastic differential of  $f(\tilde{V})$ . We then set its drift equal to zero so that  $f(\tilde{V})$  becomes a local  $\tilde{\mathbb{Q}}$ -martingale. Given the PDE representation of the auxiliary value process  $\tilde{V}$ , setting the drift of the process  $f(\tilde{V})$  equal to zero can be expressed by

$$f'(\tilde{V}_t) \left( s_t^A 1\{\tilde{V}_t < 0\} + s_t^B 1\{\tilde{V}_t \geq 0\} \right) \tilde{V}_t + \frac{1}{2} f''(\tilde{V}_t) \beta = 0, \quad (22)$$

where  $\beta \equiv (d\tilde{V}_t)^2 = (d\tilde{V}_t).(d\tilde{V}_t)$  is computed according to the Itô's multiplication rules and is equal to

$$\beta \equiv \sum_{i=1}^n \left( \frac{\partial \tilde{V}}{\partial x_i} \right)^2 \sum_{j=1}^d a_{ij}^2(t) + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \tilde{V}}{\partial x_i} \frac{\partial \tilde{V}}{\partial x_j} \sum_{k=1}^d a_{ik}(t) a_{jk}(t). \quad (23)$$

Set  $\tilde{V}_x \equiv (\frac{\partial \tilde{V}}{\partial x_1}, \frac{\partial \tilde{V}}{\partial x_1}, \dots, \frac{\partial \tilde{V}}{\partial x_n})^*$ . Note that in matrix notation  $\beta$  takes the quadratic form  $\beta = \tilde{V}_x^* b \tilde{V}_x$ , where  $b = aa^*$  is the non-negative-definite *diffusion matrix* defined in Section 2. That is,  $\beta$  becomes non-negative due to  $b$  being non-negative-definite, and  $\beta$  will be positive when the *diffusion matrix*  $b$  is positive-definite. To derive our path-independent probabilistic representation of the auxiliary value function  $\tilde{V}$ , we now impose structure on  $s^A$  and  $s^B$  by requiring that each non-negative function  $s^i$  take the multiplicative form,

$$s^i(\tilde{v}, x, t) = g^i(\tilde{v}) \beta(x, t), \quad i = A, B, \tilde{v} \in \mathbb{R}, x \in \mathbb{R}^n, t \in [0, T], \quad (24)$$

where  $g^i(\tilde{v}) : \mathbb{R} \mapsto \mathbb{R}^+$ , and  $\beta(x, t) : \mathbb{R}^n \times [0, T] \mapsto \mathbb{R}^+$ . As mentioned before, after any particular type of recovery modeling, the modeler often ultimately assumes that the recovery rate, which is  $(1 - L^i)$  in our setting, is constant. That is, our restrictions on the dynamics of  $s^A$  and  $s^B$  are to be viewed as restrictions on the dynamics of the counterparty default intensities  $h^A$  and  $h^B$ . We require that the positive function  $\beta(x, t)$  be given by (23). So, our intensity dynamics (24) imply that the counterparty default intensities  $h^A$  and  $h^B$  are restricted in their dependence on  $\beta(x, t)$  given in (23) and are *freely* specified by the modeler's choice of  $g^A$  and  $g^B$  as functions of the auxiliary survival-contingent price process  $\tilde{V}$ .

Recall the zero-drift condition (22), given our required intensity dynamics in (24),  $f(\tilde{v})$  solves the following linear ordinary differential equation (ODE),

$$-\frac{1}{2} \frac{f''(\tilde{v})}{f'(\tilde{v})} = g^B(\tilde{v}) \tilde{v}^+ - g^A(\tilde{v}) \tilde{v}^-, \quad \tilde{v} \in \mathbb{R}.$$

whose general solution is

$$f(\tilde{v}) = c_0 + c_1 \int_0^{\tilde{v}} e^{-2 \int_0^y [g^B(z)z^+ - g^A(z)z^-] dz} dy,$$

where  $c_0$  and  $c_1$  are arbitrary real-valued constants. We choose  $c_0 = 0$  and  $c_1 = 1$  for concreteness; so,

$$f(\tilde{v}) = \int_0^{\tilde{v}} e^{-2 \int_0^y [g^B(z)z^+ - g^A(z)z^-] dz} dy. \quad (25)$$

This function is increasing everywhere and hence invertible. Assuming that  $f(\tilde{V}_t)$  is a true  $\tilde{\mathbb{Q}}$  martingale, rather than just a local martingale, the martingale property implies that

$$f(\tilde{V}_t) = E^{\tilde{\mathbb{Q}}}[f(\tilde{V}_T)|X_t] = E^{\tilde{\mathbb{Q}}}\left[f\left(\frac{\Pi(X_T)}{e^{\lambda T} n_T}\right)\middle|X_t\right],$$

where  $n_t \equiv n(X_k(t))$  for  $t \in [0, T]$ .<sup>14</sup> Applying  $f^{-1}$  to both sides of the above equation gives

$$\tilde{V}_t = f^{-1} \left( E^{\tilde{\mathbb{Q}}} \left[ f \left( \frac{\Pi(X_T)}{e^{\lambda T} n_T} \right) \middle| X_t \right] \right). \quad (26)$$

This probabilistic representation for the auxiliary value function  $\tilde{V}$  holds so long as the positive function  $\beta$  is given by (23), where  $\tilde{V}$  is fully specified by both (26) and the  $\tilde{\mathbb{Q}}$  dynamics of  $X$  in (17). Recall that  $V_t = e^{\lambda t} n_t \tilde{V}_t$ . So, the desired path-independent probabilistic representation of the value function  $V$ , which solves the non-linear PDE (9) and is subject to the terminal condition (11), becomes

$$V_t = e^{\lambda t} n_t f^{-1} \left( E_t^{\tilde{\mathbb{Q}}} \left[ f \left( \frac{\Pi(X_T)}{e^{\lambda T} n_T} \right) \right] \right), \quad t \in [0, T], \quad (27)$$

where the subscript  $t$  in  $E_t[\cdot]$  denote conditioning on  $X_t$ . Again, recall that the process  $n_t \equiv n(X_k(t))$  and the constant  $\lambda$  specify the dynamics of the short rate via (13), and the invertible function  $f(\tilde{V})$  is specified by (25). In sum, the path-independent probabilistic representation of the bilateral counterparty-defaultable derivatives value in (27) has been derived due to the special structure imposed on the short rate dynamics in (13) and on the counterparty credit spread dynamics in (24).

**Example** Consider unilateral counterparty-defaultable derivatives pricing where  $s_t^A \equiv 0$  and  $V_t \geq 0$  for all  $t \in [0, T]$ . Suppose that the counterparty's fractional loss  $L^B$  is constant and assume that it is equal to one for simplicity, i.e., assume zero recovery rate. Suppose that the counterparty's intensity  $h^B \equiv h$  is to be defined as a decreasing function of the survival-contingent auxiliary price process  $\tilde{V}$ . For instance, set

$$h(\tilde{V}_t) = \frac{b_t}{\tilde{V}_t} \beta,$$

where  $\beta$  is given in (23), and let  $b$  denote a deterministic function of time to be specified based on the market-implied credit spreads via the model calibration scheme outlined in the next section. The counterparty intensity  $h$  as defined above satisfies our required intensity dynamics in (24) by setting  $g(\tilde{V}_t) = b_t/\tilde{V}_t$ . Recall that our intensity dynamics restrictions lead to the function  $f$  being specified by (25). For this example, we have  $f(\tilde{V}_t) = \frac{1}{2b_t}(1 - \exp(-2b_t\tilde{V}_t))$  and  $f^{-1}(\tilde{V}_t) = -\frac{1}{2b_t} \log(1 - 2b_t\tilde{V}_t)$ . So, given (27), conditional on survival by time  $t$ , the closed-form formula for the risk-neutral value of the derivatives at time  $t \in [0, T]$  becomes

$$V_t = -\frac{1}{2} \frac{e^{\lambda t} n_t}{b_t} \log \left( 1 - \frac{b_t}{b_T} + \frac{b_t}{b_T} E_t^{\tilde{\mathbb{Q}}} \left[ \exp \left( -2b_T \frac{\Pi(X_T)}{e^{\lambda T} n_T} \right) \right] \right),$$

where  $n \equiv n(X_k)$  and the constant  $\lambda$  specifying the dynamics of the short rate are given in (13), and the  $\tilde{\mathbb{Q}}$ -dynamics of  $X$  is specified by (17).

<sup>14</sup>Given the zero-drift condition (22) of the stochastic differential of  $\{f(\tilde{V}_t)\}_{t \leq T}$ , the Novikov's criterion (Theorem 41 of Chapter 2 of Protter [2004]) gives a sufficient condition for  $f(\tilde{V})$  to be a true martingale. Alternatively, after the modeler's choice of  $g^A$  and  $g^B$  in (24), given the specific functional form of  $f$  in (25), one could check whether  $E[f(\tilde{V}_t)] < \infty$  holds for every  $0 \leq t \leq T$  to conclude that  $f(\tilde{V})$  is a true martingale, (Theorem 51 of Chapter 1 of Protter [2004]). For instance, it turns out that since this latter condition holds for our simple unilateral example of Section 3.2, the Novikov's criterion need not assume to be held for that particular example.

## 4 Outline of the Calibration Scheme

Many of the reduced-form models in the credit literature benefit from the the computational convenience of *affine intensity modeling* by assuming that the stochastic intensity of the default time  $\tau$ , denoted by  $h$ , is an affine function of a latent Markov process  $X$ , such that the conditional expectation below representing the survival probabilities can be written as,

$$P(\tau > t | \tau > s) = E_s \left[ e^{-\int_s^t h(X_u) du} \right] = e^{\alpha(s,t) + \beta(s,t)X_s}, \quad (28)$$

where coefficients  $\alpha$  and  $\beta$  depend only on  $s$  and  $t$ ,  $0 < s < t$ .<sup>15</sup> The Markov process  $X$  can be multidimensional; however for simplicity, think of  $X$  as a 1-dimensional process, e.g., a square-root diffusion. The conditional survival probabilities on the left side of Formula (28) are market implied. For instance, they can be approximated from corporate bond spreads or credit default swap spreads. Given the convenient form of the conditional expectation in (28) and given that  $X$  has usually well known distributional properties, statistical estimates of the parameters of  $X$  and  $h$  are often based on (approximate) maximum likelihood estimation methods or the Kalman filter.<sup>16</sup>

Model calibration in derivatives pricing under counterparty risk in the presence of wrong way risk is challenging since the intensity of the counterparty's default time  $h$  - instead of being a function of merely latent Markov processes - is defined as a monotone function of the risk-neutral pre-default price process  $V$ . In what follows we assume that the survival-contingent price process  $V$  is time-homogeneous; we first outline possible calibration schemes of the intensity  $h$  in the absence of our restrictions on the short rate and the counterparty default intensity dynamics. Next, we outline how similar ideas can be applied to our setting that led to the derivation of the path-independent valuation formula (27). For simplicity, the calibration schemes are outlined in the unilateral case.

Consider the time grid  $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$ , and suppose that the counterparty's survival probabilities  $P(\tau > t_i) \equiv p_i$  are approximated from the counterparty's CDS maturity- $t_i$  spreads,  $i = 1, \dots, n$ .<sup>17</sup> That is,  $p_1, p_2, \dots, p_n$  are market-implied. The modeler defines the intensity  $h$  as a function of the pre-default price process  $V$  and an unknown deterministic function of time  $b$ , which is piecewise constant on the time grid. Given,

$$p_i = E \left[ e^{-\int_{t_{i-1}}^{t_i} h(V_u, b_u) du} \right], \quad i = 1, \dots, n, \quad (29)$$

$b$  can be sequentially approximated by replacing the expectation above with the average of market-observed counterparty-defaultable derivatives values when assuming time-homogeneity on the price process. For instance, in the presence of WWR where  $h$  is increasing in  $V$ , set  $h(V_i) = V_i + b_i$  with  $V_i \geq 0$ . Then, the first step of the calibration scheme gives  $b_1 = -t_1^{-1} \log(p_1/\varepsilon_1)$  where  $\varepsilon_1$  is an approximation of  $E[\exp(-\int_0^{t_1} V_u du)]$  obtained from market prices over an interval of length  $t_1$  assuming time-homogeneity on  $V$ . That is, proceeding sequentially, given  $b_i$  specified in the  $i$ th

<sup>15</sup>See, e.g., Duffie and Singleton [2003] and Lando [2004] and the references therein.

<sup>16</sup>See Duffie et al. [2000], Appendix B of Duffie and Singleton [2003], and Lando [2004]. Also, Duffie et al. [2003] and Duffee [1999] are examples of papers using an approximate maximum likelihood estimation method and Kalman filter, respectively.

<sup>17</sup>See, for instance, the calibration scheme of Hull and White [2012].

step,  $p_{i+1}$  approximated from maturity- $t_{i+1}$  CDS spreads, and  $E[\exp(-\int_0^{t_{i+1}} V_u du)]$  approximated by  $\varepsilon_{i+1}$  from the market prices, the calibration scheme uses (29) to approximate  $b_{i+1}$ ,  $i = 1, \dots, n$ .

The calibration scheme of our setting can be outlined similarly. However, it would be computationally more intensive. Recall (24) which specifies and restricts the dynamics of the intensities in our framework. Suppose that the fractional loss process  $L$ , i.e., one minus recovery rate, is constant. Consider the case where the underlying  $\tilde{\mathbb{Q}}$ -diffusion is 1-dimensional,

$$dX_t = \tilde{\mu}(X_t)dt + a(X_t)d\tilde{W}_t, \quad (30)$$

and  $\tilde{\mu}(X_t) = \mu(X_t) + \frac{n'(X_t)}{n(X_t)}a^2(X_t)$ . Set  $L = 1$  for notational simplicity. Then,

$$h_t = g(\tilde{V}_t) \left( a(X_t) \frac{\partial \tilde{V}}{\partial x}(t, X_t) \right)^2, \quad (31)$$

where the auxiliary value process  $\tilde{V}_t = \frac{V_t}{e^{\lambda t} n(X_t)}$  for  $t \in [0, T]$ . Suppose that  $g$  to be chosen by the modeler depends also on an unknown piecewise constant deterministic function of time denoted by  $b$ . For instance, in the presence of wrong way risk, one can define  $g(\tilde{V}_t) = b_t \tilde{V}_t$  with  $\tilde{V}_t \geq 0$ . Set  $\beta_t \equiv (a(X_t) \frac{\partial \tilde{V}}{\partial x}(t, X_t))^2$ . Then, given the intensity dynamics (31) and assuming time-homogeneity of the the price process, our model calibration scheme uses

$$p_i = E \left[ e^{-\int_{t_{i-1}}^{t_i} g(\tilde{V}_u, b_u) \beta_u du} \right], \quad i = 1, \dots, n, \quad (32)$$

by replacing the expectation above by its approximation via market-observed prices to sequentially specify  $b_1, \dots, b_n$ . The calibration scheme of our model that gives the path-independent risk-neutral valuation formula (27) is computationally more involved than the calibration scheme (29) of a model that requires numerically solving the quasi-linear PDEs for pricing. This is so because of the presence of the time-homogeneous process  $n(X_t)$  used to specify our short rate dynamics (13) and also due to the presence of the variance rates  $\beta$  on the right side of (32) that is to be approximated from market-observed derivatives values, realizations of the process  $n$ , and realizations the underlying diffusion  $X$ .

**Remark 3** Note that the expectations in (29) and (32) are under the risk-neutral probability measure  $\mathbb{Q}$  and the auxiliary probability measure  $\tilde{\mathbb{Q}}$ , respectively. That is, we have not specified and used the dynamics of the default intensity process under the physical measure  $\mathbb{P}$  in the calibration scheme as we do not intend to characterize and quantify various types of risk premia in the counterparty credit spreads. For our study it suffices to consider the  $h^{\mathbb{Q}}$  dynamics and develop a calibration scheme that specifies the parameters of  $h^{\mathbb{Q}}$  using the market-implied information, (see, e.g., Eckner [2009]).<sup>18</sup> One can compare our working of merely with  $\mathbb{Q}$ -dynamics to the well-known martingale modeling in the interest rate literature where in the absence of the interest in studying

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<sup>18</sup>See, e.g., Chapter 14 of Singleton [2006], Section 6 of Eckner [2010], Azizpour et al. [2011], and the references therein on using reduced-form models for characterizing the risk premia and empirically studying its structure in the credit markets.

the risk premia in the bond market, the short rate dynamics is specified only under  $\mathbb{Q}$  and the model parameter estimation becomes feasible under the  $\mathbb{Q}$ -dynamics by matching the model-implied term structure to the market-implied (empirical) term structure as closely as possible.<sup>19</sup>

## 5 Wrong Way Risk and Derivatives Pricing

Using the reduced-form approach, Ghamami and Goldberg [2014] show that wrong way CVA,  $CVA_W$ , need not exceed independent CVA,  $CVA_I$ . Their result relies on deriving calibration-implied formulas for  $CVA_I$  that make  $CVA_I$  mathematically comparable to  $CVA_W$  as summarized in Section 5.1. Inspired by the work of Ghamami and Goldberg [2014], in Section 5.2 we derive calibration-implied formulas that enable us to mathematically compare counterparty-defaultable derivatives values in the presence and absence of WWR in our framework. We show that derivatives values under unilateral WWR need not be less than derivatives values in the absence of WWR. A sufficient condition under which this inequality holds is that the survival-contingent price process follows a semimartingale with independent increments. This is shown by Proposition 1 of Section 5.2. Next, considering the bilateral case, Proposition 2 gives an alternative expression for the survival-contingent price process which facilitates the derivation of the calibration-implied formulas that make the counterparty-defaultable derivatives values in the absence and presence of WWR mathematically comparable. Similar to our results in the unilateral case, we conclude that no general inequality can be drawn for bilateral counterparty-defaultable derivatives values in the presence and absence of WWR. Hereafter, to simplify the notation, we do not append the superscript  $\mathbb{Q}$  to  $E$ ; i.e., we set  $E \equiv E^{\mathbb{Q}}$ .

### 5.1 The Impact of WWR on CVA

Recall the widely-used CVA risk-neutral expected-discounted-loss formula at time zero

$$CVA = E[\tilde{D}_\tau V_\tau 1\{\tau \leq T\}],$$

where assuming zero recovery rate,  $V_t$  denotes the non-negative part of the derivatives portfolio value at time  $t$  that a financial institution holds with its counterparty,  $T$  is the longest maturity transaction in the portfolio,  $\tilde{D}_t \equiv \exp(-\int_0^t r_u du)$ , and  $\tau$  is the counterparty's default time, a non-negative random variable with density  $f$ . Suppose that  $\tau$  has a well-defined stochastic intensity  $h$ . Under some technical conditions, it can be shown that

$$CVA_W = \int_0^T E \left[ \tilde{D}_t V_t h_t^w e^{-\int_0^t h_u^w du} \right] dt.$$

Note that under wrong way risk  $h \equiv h^w$  is defined as an increasing function of  $V$ . The calibration-implied formula for  $CVA_I$  is derived as follows

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<sup>19</sup>This is usually referred to as the *inversion of the yield curve* in the interest rate modeling literature, (see, e.g., Chapter 22 of Bjork [2009]).

$$\text{CVA}_I = \int_0^T E[\tilde{D}_t V_t] f_\tau(t) dt = \int_0^T E[\tilde{D}_t V_t] E[h_t^w e^{-\int_0^t h_u^w du}] dt,$$

where the first equality follows due to the independence of  $\tau$ ,  $V$  and  $r$ , and the second equality leading to the calibration-implied formula follows by noting that any model calibration scheme is to approximate the model parameters by matching model-implied survival probabilities  $E[e^{-\int_0^t h_u^w du}]$  to market-implied survival probabilities  $P(\tau > t)$ , for any  $0 \leq t \leq T$ , as closely as possible. Now, given that the wrong way intensity process  $h^w$  appears in the calibration-implied formula, i.e., right side of the  $\text{CVA}_I$  formula above,  $\text{CVA}_W$  and  $\text{CVA}_I$  become mathematically comparable. It then becomes clear to see that one need not exceed the other. Interestingly, Ghamami and Goldberg [2014] give numerical and analytical examples under which  $\text{CVA}_I > \text{CVA}_W$ .

## 5.2 The Impact of WWR on Derivatives Values

We first consider derivatives pricing in our framework under unilateral counterparty default risk. Assume that the fractional loss process  $L^B \equiv L$  is 1 and so  $s^B = h^B \equiv h$ . Recall the survival-contingent valuation formula

$$V_t = E_t \left[ e^{-\int_t^T (r_u + h_u) du} \Pi_T \right].$$

Under wrong way risk  $h$  is defined as an increasing function of  $V$ ,

$$V_t^W = E_t \left[ e^{-\int_t^T (r_u + h_u^w) du} V_T \right],$$

and when  $\tau_B \equiv \tau$ ,  $V$ , and  $r$  are independent we have

$$V_t^I = P(\tau > T | \tau > t) E_t \left[ e^{-\int_t^T r_u du} V_T \right] = E_t \left[ e^{-\int_t^T h_u^w du} \right] E_t \left[ e^{-\int_t^T r_u du} V_T \right], \quad (33)$$

where the right side above is the calibration-implied formula of in our setting. It is derived by noting that any model calibration scheme is to ensure that the model-implied conditional survival probabilities  $E_t \left[ e^{-\int_t^T h_u^w du} \right]$  match the market-implied conditional survival probabilities  $P(\tau > T | \tau > t)$  as closely as possible for any  $t \in [0, T)$ . Since our calibration-implied formula, i.e. right side of (33), is expressed based on the wrong way intensity  $h^w$ , derivatives values in the presence and absence of WWR become mathematically comparable. Consider, for instance, derivatives initial values in the presence and absence of WWR and assume zero (constant) short rate for simplicity,

$$V_0^W = E \left[ e^{-\int_0^T h_u^w du} V_T \right], \quad \text{and} \quad V_0^I = E \left[ e^{-\int_0^T h_u^w du} \right] E[V_T].$$

Knowing that  $h^w$  is an increasing function of  $V$  does not have any implication for the sign of the covariance between  $V_T$  and  $e^{-\int_0^T h_u^w du}$ . For instance, when this covariance is non-negative, the

derivatives initial value under wrong way risk  $V_0^W$  exceed the derivatives initial value in the absence of WWR,  $V_0^I$ . Proposition 1 below identifies sufficient conditions under which the reverse holds, i.e.,  $V_0^W \leq V_0^I$ .

**Proposition 1.** *Consider unilateral counterparty-defaultable derivatives values at time zero,  $V_0^W$ ,  $V_0^I$ , under wrong way risk and in its absence as defined above. Assume that the risk-free rate is constant. Suppose that  $V$  is a semimartingale with independent increments. Let  $h_t^w \equiv h(V_t)$ ,  $0 \leq t \leq T$ , denote the counterparty's stochastic default intensity process under wrong way risk, where  $h$  is a  $\mathcal{C}^2$  function which is increasing in  $V$ . The covariance of two random variables*

$$e^{-\int_0^T h(V_u) du} \quad \text{and} \quad V_T,$$

is non-positive and so  $V_0^W \leq V_0^I$ .

**Proof** To simplify the notation assume zero short rate. Let  $0 \equiv t_0 < t_1 < t_2 < \dots < t_n \equiv T$  denote a time grid on  $[0, T]$  assumed to be equidistant with  $\Delta \equiv t_i - t_{i-1} = T/n$  for notational simplicity. Set  $V_i \equiv V_{t_i}$ ,  $i = 1, \dots, n$ . First consider the covariance of  $\mathcal{E} \equiv \exp(-\Delta \sum_{i=1}^n h(V_i))$  and  $V_T$ , and note that

$$\text{cov}(\mathcal{E}, V_T) = \text{cov}(\mathcal{E}, V_1 + (V_2 - V_1) + \dots + (V_n - V_{n-1})) = \sum_{i=1}^n \text{cov}(\mathcal{E}, V_i - V_{i-1}), \quad (34)$$

with  $V_0 \equiv 0$ . Now, consider the covariance inside the sum on the right side above for a given  $1 \leq k \leq n$  and note that the conditional covariance formula gives,

$$\text{cov}(\mathcal{E}, V_k - V_{k-1}) = \text{cov}(E[\mathcal{E}|V_k - V_{k-1}], V_k - V_{k-1}). \quad (35)$$

To see this, recall the conditional covariance formula,<sup>20</sup>

$$\text{cov}(\mathcal{E}, V_k - V_{k-1}) = E[\text{cov}(\mathcal{E}, V_k - V_{k-1} | V_k - V_{k-1})] + \text{cov}(E[\mathcal{E}|V_k - V_{k-1}], V_k - V_{k-1}),$$

and note that the first term on the right side above is zero.

Given that the random variables  $V_1, (V_2 - V_1), \dots, (V_i - V_{i-1}), \dots, (V_n - V_{n-1})$ , are independent by the independent-increment assumption on  $V$ , and that  $h$  is an increasing function of  $V$ , the conditional expectation on the right side above,

$$E[\mathcal{E}|V_k - V_{k-1}] = E \left[ \exp \left( -\Delta \sum_{i=1}^n h \left( \sum_{j=1}^i (V_j - V_{j-1}) \right) \right) \middle| V_k - V_{k-1} \right] \equiv f(V_k - V_{k-1}),$$

<sup>20</sup>See, for instance, Chapter 3 of Ross [2009].

can be viewed as a non-increasing function of  $(V_k - V_{k-1})$  for any given  $1 \leq k \leq n$ , where we have referred to this function as  $f$  on the right side above. That is, given (35),

$$\text{cov}(\mathcal{E}, V_k - V_{k-1}) = \text{cov}(f(V_k - V_{k-1}), V_k - V_{k-1}) \leq 0, \quad (36)$$

where the inequality above follows from the Chebyshev's algebraic inequality.<sup>21</sup> So, given (34), we conclude,

$$\text{cov}\left(e^{-\Delta \sum_{i=1}^n h(V_i)}, V_T\right) \leq 0. \quad (37)$$

Given that  $V$  is a semimartingale and  $h$  is in  $\mathcal{C}^2$ , we know from the theory of stochastic integration that  $X_n \equiv \Delta \sum_{i=1}^n h(V_i)$  converges to  $X \equiv \int_0^T h(V_u) du$  in  $L^2$  and in probability as  $n \rightarrow \infty$  (equivalently as  $|\Delta|$  converges to zero).<sup>22</sup> Set  $Y \equiv V_T$ . Let  $H$ ,  $F$ , and  $G$  denote the joint cdf, marginal cdf of  $X$ , and marginal cdf of  $Y$ , respectively. Recall Hoeffding [1940] covariance formula,  $\text{cov}(X, Y) = \int_{\mathbb{R}^2} (H(x, y) - F(x)G(y)) dx dy$ . Knowing that convergence in probability implies weak convergence (denoted by  $\Rightarrow$ ), we have  $F_n \Rightarrow F$  and that  $P(X_n \leq x, Y \leq y)$  converges to  $H(x, y) \equiv P(X \leq x, Y \leq y)$  for all  $x$  and  $y$  that are continuity points of  $H$ .<sup>23</sup> So, from bounded convergence and Hoeffding [1940] covariance formula we conclude  $\text{cov}(X_n, Y) \rightarrow \text{cov}(X, Y)$  as  $n$  goes to infinity. That is, (37) implies that

$$\text{cov}\left(e^{-\int_0^T h(V_u) du}, V_T\right) \leq 0,$$

and so  $V_0^W \leq V_0^I$ .

**The Bilateral Case** We now consider risk-neutral valuation under bilateral counterparty risk. We assume zero recovery rate, i.e., we set  $L^A = L^B = 1$ , for simplicity. In what follows we develop calibration-implied formulas for the *independent case* where both counterparty default intensities are independent of derivatives values. Derivatives values then become mathematically comparable in the presence and absence of *bilateral* wrong way risk. Recall the risk-neutral value of the bilateral counterparty-defaultable derivatives value in our framework at time  $t \in [0, T]$  under the zero-recovery assumption,

$$V_t = E_t \left[ e^{-\int_t^T (r_u + h_u^B 1_{\{V_u \geq 0\}} + h_u^A 1_{\{V_u < 0\}}) du} \Pi_T \right].$$

The right side above cannot be used to develop calibration-implied formulas for the independent case. Proposition 2 below provides an alternative expression for the survival-contingent price process. This alternative risk-neutral survival-contingent valuation formula enables us to ultimately derive our desired calibration-implied formulas to mathematically compare the derivatives values in the presence and absence of WWR.

<sup>21</sup>See, for instance, Egozcue et al. [2009].

<sup>22</sup>See, for instance, Section 2 in Chapter 4 of Revuz and Yor [2004], or Section 4 in Chapter 2 of Protter [2004].

<sup>23</sup>See, for instance, Section 2.2. in Chapter 2 of Durrett [2005].

**Proposition 2.** *Assume that both recovery rates are zero. The process  $V_t$*

$$V_t = \hat{D}_t E_t \left[ \hat{D}_T^{-1} V_T + \int_t^T \hat{D}_u^{-1} V_u (1\{V_u \geq 0\} h_u^A + 1\{V_u < 0\} h_u^B) du \right], \quad (38)$$

with

$$\hat{D}_t = \exp \left( \int_0^t (r_u + h_u^A + h_u^B) du \right),$$

can be equivalently expressed as

$$V_t = E_t \left[ e^{-\int_t^T (r_u + h_u^B 1\{V_u \geq 0\} + h_u^A 1\{V_u < 0\}) du} V_T \right], \quad (39)$$

where the subscript  $t$  on the expectation  $E_t$  denotes conditioning on  $\mathcal{F}_t^X$ .

**Proof** Consider (38) and set

$$\hat{R}_t = 1\{V_t \geq 0\} h_t^A + 1\{V_t < 0\} h_t^B.$$

Note that

$$V_t = \hat{D}_t \left( E_t \left[ \hat{D}_T^{-1} V_T + \int_0^T \hat{D}_u^{-1} V_u \hat{R}_u du \right] - \int_0^t \hat{D}_u^{-1} V_u \hat{R}_u du \right). \quad (40)$$

Given (40), the stochastic differential of  $V$  can be written as

$$dV_t = (r_t + h_t^A + h_t^B) V_t dt - V_t \hat{R}_t dt + dm_t = V_t (r_t + h_t^A 1\{V_t < 0\} + h_t^B 1\{V_t \geq 0\}) dt + dm_t, \quad (41)$$

with  $m$  being a  $\mathbb{Q}$  martingale. It is now not difficult to show that the right side of (41) implies (39). Set

$$\tilde{R}_t = \exp \left( - \int_0^t (r_u + h_u^A 1\{V_u < 0\} + h_u^B 1\{V_u \geq 0\}) du \right).$$

Given (41) and the definition of  $\tilde{R}$ , the stochastic differential of  $V_t \tilde{R}_t$  becomes  $d(V_t \tilde{R}_t) = dM_t$  with  $M$  being a  $\mathbb{Q}$  martingale; we now integrate both sides from  $t$  to  $T$  and take conditional expectation with respect to  $\mathcal{F}_t^X$  to recover (39).<sup>24</sup> This completes the proof.

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<sup>24</sup>This last part of the proof is due to Lemma 1 of Duffie et al. [1996]; we have included it in the paper for completeness.

We now compare the risk-neutral derivatives value at time zero in the presence and absence of bilateral wrong way risk. Recall that the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  of the underlying probability space encompasses the filtration generated by the underlying diffusion  $X$ , denoted by  $\mathbb{F}^X$ , and the filtration generated by the default indicator process  $H_t = 1\{\tau \leq t\}$ . Set  $h \equiv h^A + h^B$ . Under the assumption  $P(\tau^A = \tau^B) = 0$ , it well known that,<sup>25</sup>

$$P(\tau > t) = E[e^{-\int_0^t h_u du}] \quad \text{and} \quad P(\tau = \tau^i | \tau = t, \mathcal{F}_T^X) = \frac{h_t^i}{h_t}, \quad i = A, B. \quad (42)$$

Given (42), it is not difficult to show that

$$P(\tau = \tau^i, \tau \leq t) = E \left[ \int_0^t h_u^i e^{-\int_0^u h_s ds} du \right], \quad i = A, B. \quad (43)$$

This is done by noting that  $P(\tau = \tau^i, \tau \leq t) = E[P(\tau = \tau^i, \tau \leq t | \mathcal{F}_T^X)]$  and using (42).

Suppose that both  $h^B \equiv h^{B,w}$  and  $h^A \equiv h^{A,w}$  are monotone increasing in  $V$ . Set  $h^w \equiv h^{A,w} + h^{B,w}$ . Using Proposition 2, the risk-neutral value of the derivatives contract at time zero under bilateral WWR becomes,

$$\begin{aligned} V_0^{WW} &= E \left[ e^{-\int_0^T (r_u + h_u^w) du} V_T \right] + \int_0^T E \left[ V_u^+ h_u^{A,w} e^{-\int_0^u (r_s + h_s^w) ds} \right] du \\ &\quad - \int_0^T E \left[ V_u^- h_u^{B,w} e^{-\int_0^u (r_s + h_s^w) ds} \right] du, \end{aligned} \quad (44)$$

where  $V_t^+ \equiv V_t 1\{V_t \geq 0\}$  and  $V_t^- \equiv -V_t 1\{V_t \geq 0\}$ . Now consider the case where the absence of bilateral wrong way risk implies that counterparty default times  $\tau^A$  and  $\tau^B$  and so their associated default intensities  $h^A$  and  $h^B$  are independent of  $V$ . We also assume that  $h^A$  and  $h^B$  are independent of the short rate  $r$ . Given Proposition (2), the risk-neutral derivatives value at time zero in the independent case is given by

$$\begin{aligned} V_0^I &= P(\tau > T) E \left[ e^{-\int_0^T r_u du} V_T \right] + \int_0^T E \left[ e^{-\int_0^u r_s ds} V_u^+ \right] P(\tau = \tau^A | \tau = u) f_\tau(u) du \\ &\quad - \int_0^T E \left[ e^{-\int_0^u r_s ds} V_u^- \right] P(\tau = \tau^B | \tau = u) f_\tau(u) du, \end{aligned}$$

where  $f_\tau$  denotes the density of  $\tau$ . Note that any calibration scheme is to ensure that model parameters are approximated or statistically estimated such that the model-implied survival probabilities  $E[e^{-\int_0^t h_u^w du}]$  match the market-implied survival probabilities  $P(\tau > t)$  for any  $t \in (0, T]$  as closely as possible. Suppose that, for instance, market-implied survival probabilities are approximated from credit spreads associated with a (fictitious) first-to-default swap referencing only counterparty A

<sup>25</sup>See, for instance, Theorem T15 in Chapter 2 of Bremaud [1981] for the right side of (42) and Chapter 7 of Bielecki and Rutkowski [2004] for the other term.

and counterparty B. Similarly, the model calibration scheme is to ensure that market-implied default probabilities  $P(\tau = \tau^i, \tau \leq t)$  match the model-implied default probabilities  $E \left[ \int_0^t h_u^i e^{-\int_0^u h_s ds} du \right]$ ,  $i = A, B$ , for any  $t \in (0, T]$  as closely as possible. So, the calibration-implied formula for derivatives value at time zero in the independent case becomes

$$\begin{aligned} V_0^I &= E \left[ e^{-\int_0^T r_u du} V_T \right] E \left[ e^{-\int_0^T h_u^w du} \right] + \int_0^T E \left[ e^{-\int_0^u r_s ds} V_u^+ \right] E \left[ h_u^{A,w} e^{-\int_0^u h_s^w ds} \right] du \\ &\quad - \int_0^T E \left[ e^{-\int_0^u r_s ds} V_u^- \right] E \left[ h_u^{B,w} e^{-\int_0^u h_s^w ds} \right] du. \end{aligned} \quad (45)$$

The calibration-implied formula (45) makes the derivatives values under bilateral wrong way risk (44) mathematically comparable with the derivatives values in the independent case. This is because the initial derivatives value in the absence of WWR (45) has been expressed based on wrong way intensities  $h^{A,w}$  and  $h^{B,w}$ . For instance, consider the second terms on the right side of (44) and (45) and assume zero short rate for simplicity. While  $h^{A,w}$  being an increasing function of  $V$  implies  $E[V_t^+ h_t^{A,w}] \geq E[V_t^+] E[h_t^{A,w}]$ , the presence of the exponential terms

$$E[V_t^+ h_t^{A,w} e^{-\int_0^t h_s^w ds}] \quad , \quad E[V_t^+] E[h_t^{A,w} e^{-\int_0^t h_s^w ds}]$$

prevent us from drawing any general inequalities by merely relying on the intensities being monotone functions of the survival-contingent value process. So, similar to our results in the unilateral case, we conclude that no general inequality can be drawn for bilateral counterparty-defaultable derivatives values in the presence and absence of WWR.

## Appendix

### A Comparison with the Numeraire Change

To compare the well-known numeraire change techniques of Geman et al. [1995] to our proposed change of probability measure, consider the following simple example in the absence of default risk. Suppose that the univariate state variable  $X$  has the stochastic differential

$$dX_t = \mu(X_t)dt + a(X_t)dW_t. \quad (46)$$

Consider a contingent claim with maturity  $T > 0$  and payoff  $\Pi(X_T)$ , and a martingale measure  $\mathbb{Q}$  relative to money market account  $D$  as numeraire. That is, the risk-neutral value of the contingent claim at time  $t < T$  is given by

$$\frac{V_t}{D_t} = E^{\mathbb{Q}} \left[ \frac{\Pi(X_T)}{D_T} \middle| X_t \right].$$

Let  $p(t, T)$  denote the time- $t$  value of a default-free zero coupon bond with maturity  $T$ . Using the following change of probability measure

$$d\mathbb{Q}^T = L_T d\mathbb{Q} \text{ on } \mathcal{F}_T \text{ with } L_t = \frac{p(t, T)}{D_t p(0, T)}, \quad 0 \leq t \leq T,$$

the  $T$ -bond becomes the new numeraire. That is, Theorem 1 of Geman et al. [1995] gives

$$\frac{V_t}{p(t, T)} = E^{\mathbb{Q}^T} \left[ \Pi(X_T) \middle| X_t \right].$$

Note that  $p(T, T) = 1$  and that  $p(t, T)$  can be directly observed in the market at time  $t$ . The numeraire change will then ultimately be beneficial when the  $\mathbb{Q}^T$  conditional expectation on the right side above can be computed conveniently. This is often done by assuming deterministic money market account and by imposing structure on the dynamics of the  $T$ -zero coupon bond. For instance, when one assumes deterministic  $D$  and that  $p(t, T)$  evolves according to the following stochastic differential

$$dp(t, T) = r_t p(t, T) dt + \sigma(t, T) p(t, T) dW_t,$$

where coefficient processes are adapted, the process  $L$  takes the familiar stochastic differential

$$dL_t = L_t \sigma(t, T) dW_t,$$

as in Proposition 24.7 of Bjork [2009]. Then, using Girsanov Theorem, the state variable  $X$  evolves under  $\mathbb{Q}^T$  with the drift change,

$$dX_t = [\mu(X_t) + \sigma(t, T) a(X_t)] dt + a(X_t) dW_t^T,$$

with  $W^T$  being a  $\mathbb{Q}^T$  standard Brownian motion. In practical applications of the numeraire change technique, it is often assumed that the numeraire's volatility process and the Girsanov kernel are deterministic, (see, e.g., Section 3.2 of Geman et al. [1995] where the applications of their general option pricing formula are discussed). For instance, in the example of this section, if we further assume that the coefficients  $\mu$ ,  $a$ , and  $\sigma$  are deterministic,  $X$  remains Gaussian under  $\mathbb{Q}^T$  and the integral  $E^{\mathbb{Q}^T} [\Pi(X_T) | X_t]$  can be computed conveniently.

This numeraire change is to be compared with our probability measure change of Section 3. More specifically, in this univariate example, suppose that the constant  $\lambda$  in (13) is zero and the short rate dynamics is given by  $r(x) = \frac{\mathcal{G}_x n(x)}{n(x)}$  with  $\mathcal{G}_x$  being the generator of  $X$ . Then, with  $N_t = \frac{n_t}{D_t}$ , where  $D$  evolves based on our proposed short rate dynamics, the auxiliary probability measure change,

$$d\tilde{\mathbb{Q}} = N_T d\mathbb{Q} \text{ on } \mathcal{F}_T \text{ with } dN_t = N_t \frac{n'(X_t)}{n(X_t)} a(X_t) dW_t, \quad 0 \leq t \leq T,$$

gives,

$$\frac{V_t}{n(X_t)} = E^{\tilde{\mathbb{Q}}} \left[ \frac{\Pi(X_T)}{n(X_T)} \middle| X_t \right],$$

where  $X$  has the stochastic differential

$$dX_t = \left[ \mu(X_t) + \frac{n'(X_t)}{n(X_t)} a^2(X_t) \right] dt + a(X_t) d\tilde{W}_t,$$

under  $\tilde{\mathbb{Q}}$  with  $\tilde{W}$  being a standard  $\tilde{\mathbb{Q}}$  Brownian motion.

In sum, with the numeraire change to the  $T$ -bond, one arrives at the path-independent conditional expectation  $E^{\mathbb{Q}^T}[\Pi(X_T)|X_t]$ . Alternatively, using our proposed auxiliary probability measure, one arrives at the path-independent conditional expectation  $E^{\tilde{\mathbb{Q}}}[\frac{\Pi(X_T)}{n(X_T)}|X_t]$ . To compare the two measure change methods given a specific functional form of the payoff function, in addition to the computational convenience of these conditional expectations under the new probability measures, one should also take in to account and compare the restrictions and assumptions imposed on the dynamics of the underlying processes in each approach, i.e., the restrictions imposed on the dynamics of the  $T$ -bond and the money market account in the more familiar numeraire change technique and the short rate dynamics assumptions of our approach.

## B The Defaultable T-bond as Numeraire

Consider risk-neutral valuation of a contingent claim with maturity  $T > 0$  and sign-definite payoff  $\Pi(X_T) \geq 0$  under unilateral counterparty risk. Recall that our reduced-form framework, which uses the fractional recovery of market value assumption of Duffie and Singleton [1999] and Duffie and Huang [1996], leads to the following survival-contingent value process,

$$V_t = E^{\mathbb{Q}} \left[ e^{-\int_t^T (r_u + s_u) du} \Pi(X_T) \middle| X_t \right], \quad t \in [0, T],$$

where  $s_t = L_t h_t$  with  $L^B \equiv L$  and  $h^B \equiv h$  denoting the fractional loss and hazard rate processes of counterparty B, i.e., a financial institution's counterparty in the derivatives transaction. For simplicity assume that the underlying diffusion is one-dimensional and evolves based on (46). Set  $\check{D}_t \equiv \exp(\int_0^t (r_u + s_u) du)$  and view  $\check{D}$  as a numeraire that makes the normalized survival-contingent price process a  $\mathbb{Q}$ -martingale,

$$\frac{V_t}{\check{D}_t} = E^{\mathbb{Q}} \left[ \frac{\Pi(X_T)}{\check{D}_T} \middle| X_t \right].$$

Let  $\hat{p}(t, T)$  denote the time- $t$  survival-contingent risk-neutral value of counterparty B's defaultable zero coupon bond with maturity  $T$ ;  $\hat{p}(t, T) > 0$  with  $t \in [0, T]$  and  $\hat{p}(T, T) = 1$ . Suppose that  $\hat{p}(t, T)$  is a traded asset. Given that  $\mathbb{Q}$  is a martingale measure for the numeraire  $\check{D}$ , assume that  $\hat{p}(t, T)$  is a positive survival-contingent price process such that  $\hat{p}(t, T)/\check{D}_t$  is a true  $\mathbb{Q}$ -martingale. Now consider the following change of probability measure,

$$d\hat{\mathbb{Q}}^T = \hat{L}_T d\mathbb{Q} \quad \text{on } \mathcal{F}_T \quad \text{with } \hat{L}_t = \frac{\hat{p}(t, T)}{\check{D}_t \hat{p}(0, T)}, \quad 0 \leq t \leq T,$$

through which  $\hat{\mathbb{Q}}^T$  has become a martingale measure for  $\hat{p}(t, T)$  as the (new) numeraire asset. Then, following Theorem 1 of Geman et al. [1995] the survival-contingent price process under  $\hat{\mathbb{Q}}^T$  becomes,

$$V_t = \hat{p}(t, T) E^{\hat{\mathbb{Q}}^T} \left[ \Pi(X_T) \middle| X_t \right]. \quad (47)$$

Assuming that the counterparty's survival-contingent defaultable  $T$ -bond at time  $t$  is market observable, (47) gives the path-independent probabilistic valuation formula under unilateral counterparty risk. It remains to show that how the conditional expectation in (47) can be computed. Suppose that  $\hat{p}(t, T)$  based on the following stochastic differential,

$$d\hat{p}(t, T) = r_t \hat{p}(t, T) dt + \sigma(t, T) \hat{p}(t, T) dW_t, \quad (48)$$

with adapted well-defined coefficient processes. Then, given that

$$\hat{L}_t = \frac{\hat{p}(t, T)}{\tilde{D}_t \hat{p}(0, T)}, \quad 0 \leq t \leq T,$$

is a  $\mathbb{Q}$ -martingale, its stochastic differential becomes  $d\hat{L}_t = \hat{L}_t \sigma(t, T) dW_t$ . Using Girsanov Theorem, the state variable  $X$  evolves under  $\hat{\mathbb{Q}}^T$  with the drift change,

$$dX_t = [\mu(X_t) + \sigma(t, T) a(X_t)] dt + a(X_t) dW_t^T, \quad (49)$$

with  $W^T$  being a  $\hat{\mathbb{Q}}^T$  standard Brownian motion. So, assuming the survival-contingent defaultable  $T$ -bond dynamics (48), the  $\hat{\mathbb{Q}}^T$ -conditional expectation in (47) can be calculated according to the  $\hat{\mathbb{Q}}^T$ -dynamics of the underlying diffusion specified in (49).

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