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On Default and Uniqueness of Monetary Equilibria*

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Abstract

We examine the role that credit risk in the central bank's monetary operations plays in the determination of the equilibrium price level and allocations. Our model features trade in fiat money, real assets and a monetary authority which injects money into the economy through short-term and long-term loans to agents. Short-term loans are riskless, but long-term loans are collateralized by a portfolio of real assets and are subject to credit risk. The private monetary wealth of individuals is zero, i.e., there is no outside money. When there is no default in equilibrium, there is indeterminacy. Positive default in *every* state of the world on some long-term loan endogenously creates positive liquid wealth that supports positive interest rates and resolves the aforementioned indeterminacy. Hence, a non-Ricardian policy across loan markets can determine the equilibrium allocations while it allows the central bank to earn profits from seigniorage in order to compensate for any losses.

Keywords: Determinacy, Liquid wealth, Default, Collateral, Monetary policy

JEL Classification: D5, E4, E5

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1 Introduction

The analysis of the interaction between the real and monetary sectors of the economy calls for theories of how the price level is determined in equilibrium. In general equilibrium models where all transactions take place in real terms, the general price level is indeterminate and only relative prices can be determined; a consequence of Walras law. The introduction of money or other monetary assets not only does not resolve the issue of indeterminacy, but can also lead to indeterminacy of real equilibrium allocations when asset markets are incomplete (Balasko and Cass, 1989; Geanakoplos and Mas-Colell, 1989). The presence of a monetary sector is essential to provide the additional restrictions to determine the price level. Thus, the dichotomy between the nominal and real sectors of the economy vanishes when policy resolves the real as well as the nominal indeterminacy of monetary equilibria (see Grandmont (1985) for an extensive discussion of the classical dichotomy).

The focus of this paper is on how the monetary authority can determine the price level rather on how it chooses policy optimally to achieve certain economic outcomes. Thus, we suggest and examine a sufficient condition such that monetary equilibria are determinate, which has not been studied in the existing literature. Many authors have suggested sufficient conditions to pin down the price level. Using a variant of the sell-all model of Lucas and Stokey (1987), Magill and Quinzii (1992) show that the monetary authority can determine the price level if it buys all the endowments of agents in exchange for a fixed supply of money, which subsequently the agents use to buy back goods. Determinacy is achieved because the quantity theory of money holds by construction and the monetary authority can choose the price level by exchanging a fixed amount of money with the total endowments in the economy, i.e., money and prices move in lockstep.

However, the non-trivial quantity theory of money can fail when we move towards a bid-offer model where agents endogenously decided the quantity of goods they buy and sell. Again, the monetary authority can inject a fixed amount of money in the economy, for example in the form of a loan to agents, but agents may choose to spend only part of it to purchase goods and keep the rest as “idle" cash. The hoarding behavior can be discouraged if there is an opportunity cost of holding cash, i.e., if the central bank charges a positive interest rate for the amount of money it injects in the economy. Dubey and Geanakoplos (2006) show that a sufficient condition to guarantee determinacy is the existence of private monetary endowments that can support positive interest rates in equilibrium (Dubey and Geanakoplos, 1992; Dubey and Geanakoplos, 2003*a*; Dubey and Geanakoplos, 2003*b*). These monetary endowments (outside money) are free and clear of any offsetting obligation, thus they are different from the money stock (inside money) that the monetary authority injects into the economy and which needs to be repaid in the end.[1] A key distinction between outside and inside money is that inside money is an asset of the central bank and a liability of agents, while outside money is an asset that belongs to the balance sheet of only one agent (see Gurley and Shaw (1960) for more on the distinction between inside and outside money).

Our purpose is to establish an alternative sufficient condition to guarantee the determinacy of monetary equilibria in the absence of outside money. For that purpose, we consider a two period monetary model introduced by Dubey and Geanakoplos (2006), which features both short-term and long-term nominal loans issued by the central bank, and extend it in four aspects. First, we introduce real assets, akin to “Lucas trees" that promise uncertain dividends in the second period. Second, we require that long-term loans are collateralized by a portfolio of these Lucas trees. The central bank has the authority to choose the collateral requirements. Third, we allow the central bank to offer a menu of long-term loans parametrized by the different set of collateral requirements and different promised interest rates or loan supplies. Finally, we allow agents to endogenously default on a long-term

[1]Numerous papers use this framework: Tsomocos (2008) applies the argument to show the determinacy of international monetary equilibria, while Giraud and Tsomocos (2010) prove uniqueness under the limit-price exchange process. Espinoza, Goodhart and Tsomocos (2008) and Espinoza and Tsomocos (2014) use a similar framework to connect the supply of liquidity by the central bank to the term structure of interest rates and provide an explanation for the term premium. Goodhart, Sunirand and Tsomocos (2005), Goodhart, Sunirand and Tsomocos (2006), Goodhart, Peiris, Tsomocos and Vardoulakis (2010), Tsenova (2014) and Tsomocos (2003) use the same model of money to analyze the financial stability in a monetary economy.

loan, if the value of the collateral is lower that the promised loan repayment.

Our result can be summarized as follows: When the central bank undertakes some credit risk in its intertemporal monetary operations, i.e., if agents default on some loan in every state of the world in the second period, then it can *endogenously* create private liquid wealth for the agents in the economy which can be used to support positive interest rates in all states and periods (see also Tsomocos (1996)). In the initial period agents borrow both in the short-term and long-term money markets. At the end the period, agents return to the central bank the principal from short-term money operations and can pay the interest by using part of the money they borrow long-term, while they transfer the rest as deposits to the second period. Agents are able to act this way, because they do not need to repay the long-term loans in full, but can default on them. The extra private liquid wealth, which constitutes a loss on long-term money market operations by the central bank, can be used to repay positive interest rates on short-term loans in every state and period.

We show that the central bank can choose interest rates and collateral requirements for a menu of long-term loans such that there is default on some loan in every state in the second period and additionally positive short-term interest rates can be supported within each state and period. In turn, this guarantees that agents will not hold “idle" cash, hence the quantity theory of money obtains. In other words, the monetary authority can determine monetary equilibria by appropriately choosing the level of short-term and long-term interest rates and additionally the level of collateral requirements on long-term loans (see section 6). The conditions to obtain determinacy when the monetary authority targets the money supply instead of the interest rates level are more restrictive and are discussed in section 7.

We should note that the fact that the central bank assumes credit risk and eventually losses in its long-term monetary operations does not imply that its budget constraint at the end of time is violated. On the contrary, the losses in the long-term loans are compensated by seigniorage revenues through the short-term monetary operations. This introduces the notion of non-Ricardian policy not only across time, but also across loan markets. Put differently, the accounting inflows and outflows need not be balanced either across time or within each loan market. This is sufficient to create positive liquid wealth and guarantee determinacy. The presence of outside money in Dubey and Geanakoplos (2006) makes their model non-Ricardian as well.[2]

Recapitulating, indeterminacy obtains whenever the government satisfies its budget constraint period by period, i.e., when it follows a Ricardian policy. Such a policy can be achieved either by zero interest rates or by the redistribution of seigniorage revenues as dividends to agents (Drèze and Polemarchakis, 2000; Bloise, Drèze and Polemarchakis, 2005). On the contrary, a non-Ricardian policy can be achieved by violating the period by period government budget constraint. Note that in Dubey and Geanakoplos (2006) determinacy is assured by not distributing the interest rate payments as dividends in the same period, while in our argument determinacy relies on the fact that the government/central bank does not tax *in advance* the agents so as to offset the realized losses accruing from future default.

As an aside note, the monetary authority in our model can choose collateral requirements to create a full set of state-contingent bonds if the payoff matrix of the real assets has full rank. McMahon, Peiris and Polemarchakis (2014) and Peiris and Polemarchakis (2014) show that the equilibrium cannot be determined even in the presence of outside money, unless the monetary authority specifies the composition of the long-term loans, or uses a term-structure or forward guidance rule as in Adão, Correia and Teles (2014), Magill and Quinzii (2014*a*) and Magill and Quinzii (2014*b*). The need for "comprehensive monetary policy" has been pointed out in Drèze and Polemarchakis (2000). Monetary policy in our model is comprehensive, since the monetary authority chooses long-term interest rates (or money supplies) and collateral requirements for each long-term contract independently, and in addition sets the short-term interest rates (or money supplies).

The rest of the paper proceeds as follows. In section 2 we describe the structure of the economy and the market interactions, while in sections 3 and 4 we present agents'

[2] See Buiter (2002), Cochrane (2001), Sims (1994) and Woodford (1994) for the importance of non-Ricardian policy in determining the equilibrium price level.

optimization problems and market clearing conditions. In section 5 we prove a number of lemmas that we use in sections 6 and 7 to prove the determinacy of monetary equilibria under interest rate and money supply targeting by the central bank, respectively. Finally, section 8 concludes.

2 The Economy

We consider a two-period economy, $t \in T = \{0,1\}$, with the set of states of nature as $S^* = \{0,1,...,S\}$. State 0 occurs in period 0, and nature selects one of the states $s \in S = \{1,...,S\}$ which occur in period 1. We also denote s^* as one state in S^*.

There are $L = \{1,...,L\}$ perishable commodities. The commodity space may be viewed as $\mathbb{R}_+^{S^*L}$. The set of durable, real assets is $K = \{1,...,K\}$, which, without loss of generality, pays out in terms of good $\ell = 1$. Denote the real payoffs of an asset k in state s as X_s^k. The asset payoffs' space may be viewed as $\mathbb{X}_+^{SK}$, which is a $S \times K$ matrix.

Let the price for commodity ℓ in state $s^* \in S^*$ be $p_{s^*}^{\ell}$, and the price for asset k at $t = 0$ be p^k. Thus, the nominal payoff of asset k in state s is $X_s^k \cdot p_s^1$.

The set of agents is $H = \{1,...,H\}$. The agents are endowed with both the commodities and real assets. Let the initial endowment of commodity ℓ in state $s^* \in S^*$ by agent h be $e_{s^*}^{\ell}(h)$. We assume that no agent has the null endowment of commodities in any state, i.e., for $s^* \in S^*$ and $h \in H$:

$$e_{s^*}(h) = (e_{s^*}^1(h),...,e_{s^*}^L(h)) \neq 0.$$

Moreover, each named commodity is actually present in the aggregate, i.e.,

$$\sum_{h \in H} e_{s^*}^{\ell}(h) >> 0, \; \forall \ell \in L.$$

Let $e^k(h)$ denote agent h's endowment of asset k in period 0. Agent h has utility of consumption $u^h : \mathbb{R}_+^{S^*L} \to \mathbb{R}$. We also assume that each u^h is concave and smooth (i.e., second partial derivatives exist and are continuous), and strictly monotonic.

Agents trade on both commodities $\ell \in L$ and real assets $k \in K$. In each state $s^* \in S^*$, agent h spends $b^{\ell}_{s^*}(h)$ amount of money to purchase commodity ℓ or sells $q^{\ell}_{s^*}(h)$ amount of commodity ℓ. Trades on the real asset k occurs only at $t = 0$. Agent $h \in H$ will purchase the real asset by spending $b^k(h)$ amount of money or sell $q^k(h)$ amount of assets.

We introduce the demand for money through cash-in-advance constraints. All commodities and assets are traded exclusively for money, which is fiat, and thus does not provide any direct utility to agents. Money is not only the stipulated means of exchange, but it is also a store of value, i.e., it is perfectably durable and can be carried forward for future use.

Money can enter the economy either in the form of a loan, which creates a liability for agents to repay a certain amount in the future (inside money), or in the form of monetary endowments, which are free and clear of any offsetting obligation. Dubey and Geanakoplos (2006), McMahon, Peiris and Polemarchakis (2014) and Tsomocos (2008) all consider both types of money, and show that positive outside money is necessary (but not always sufficient) to determine the price level. Our most important departure from the previous literature is that we only consider inside money. Yet, we are still able to prove determinacy of monetary equilibria.

In particular, we consider a central bank which extends loans to private agents. For simplicity, we consider two types of bank loans. A short-term loan is traded at the beginning of each state $s^* \in S^*$ and promises $\mu_{s^*}(h)$ dollars at the end of the period. Let r_{s^*} denote the interest rate for the short-term loan in state $s^* \in S^*$. Long-term loans are traded at the beginning of state 0 and promise $\bar{\mu}^c(h)$ dollars before commodity trade in every future state $s^* \in S$ in period 1. The long-term loans are collateralized. Let $c \in C \equiv \{1, ..., C\}$ index the long-term bank loan that has corresponding collateral requirements given by γ^c_k, $k \in K$, and $\bar{r}_c$ denote the interest rate on bond c.

Each unit of long-term loan c needs to be collateralized by a bundle of assets where γ^c_k

is the value of assets $k \in K$ that an agent should hold if he takes a long-term loan indexed by $c \in C$. Note, that γ_k^c are the collateral requirements per one unit of loan, i.e., a long-term loan of size $\bar{\mu}^c(h)$ requires assets k worth $\bar{\mu}^c(h) \cdot \gamma_k^c$ to be pledged as collateral. In this sense, γ_k^c is akin to (the inverse of) a loan-to-value ratio, and the quantity of asset k pledged as collateral is equal to γ_k^c/p^k. The monetary authority chooses γ_k^c, $\forall k \in K, c \in C$. The collateral constraint should hold at $t = 0$ for each asset individually. Thus,

$$\sum_c \bar{\mu}^c(h)\frac{\gamma_k^c}{p^k} \leq e^k(h) - q^k(h) + \frac{b^k(h)}{p^k} \quad \forall k \in K.^3 \tag{1}$$

Agent h can choose to default at $t = 1$ if the value of her collateral is lower than the loan obligation (Geanakoplos and Zame, 2013), i.e., she delivers

$$\bar{\mu}^c(h) f_s^c \quad \forall c \in C, \tag{2}$$

where $f_s^c = \min\left[1, \sum_k \gamma_k^c X_s^k \frac{p_s^1}{p^k}\right]$.

Define $\mathbb{I}_s^c(h)$ an indicator variable for every $c \in C$, $s \in S$ and $h \in H$, such that $\mathbb{I}_s^c(h) = 1$

[3] The collateral constraint links the monetary value of required collateral to the total value of asset holdings for each asset k. Hence, the amount of collateral is endogenously determined, i.e., the monetary authority neither specifies the quantity of assets to be pledged as collateral nor their nominal price, but only its total value. It should be noted that an alternative collateral requirement that links nominal loans to real asset holdings would resolve the indeterminacy of monetary equilibria if binding, which is typically the case when default obtains in equilibrium. In such an arrangement, the price of collateral would be determined *a priori* as in Shubik and Tsomocos (1992) who allow the monetary authority to set the exchange price of fiat money to gold (the durable asset in our framework). On the contrary, we focus on a collateral constraint that specifies the loan-to-value ratio for long-term loans and emphasize the role of default on some loans in supporting positive interest rates on other non-defaultable loans due to capability of rolling over loans. The argument behind our determinacy result relies crucially on the fact that default endogenously creates “outside" money such that positive interest rates can be supported rather than on the fact that default results in binding collateral constraints. Put differently, it is the positive interest rate and not the binding collateral constraint that pins down prices. However, we hasten to emphasize that our argument requires the overlapping of both defaultable and non-defaultable loans, which are the long-term and short-term loans in our framework respectively, so that to allow for rolling over of debts. Most importantly, the non-defaultable loans need to carry positive interest rates and the demand for them should be positive in equilibrium. Naturally, agents would prefer to first take loans that default, but the borrowing capacity is restricted by the collateral constraint. If the benefits of borrowing at positive interest rates are high enough, i.e., if the "gains-to-trade" hypothesis is satisfied (see Dubey and Geanakoplos (2003*b*)), then there will be demand for non-defaultable loans as well. Nevertheless, our argument may not be robust to a specification whereby these additional intra-period loans do not exist. In principle, our result should also obtain with only defaultable and non-defaultable overlapping intetermporal loans.

when there is no default and $\mathbb{I}_s^c(h) = 0$ when there is default.

The monetary authority can use two alternative ways to implement monetary policy. At one extreme we may suppose that the central bank sets quantity targets and pre-commits to the size of its borrowing or lending, letting interest rates be determined endogenously at equilibrium. At the other extreme we may suppose that the central bank sets interest rate targets, and pre-commits to supply whatever amount of money demanded at those rates. We show that both policies lead to determinacy, but the conditions for the former policy are more restrictive.

3 Agents Optimization Problem

For convenience, we list prices as (r, p) where r is the $(S^* + C)$ -dimentional vector of interest rates for the short-term loans and long-term collateralized loans; and p is the $(S^*L + K)$ -dimentional price vector, including the prices for commodities, $p_{s^*}^{\ell}$ and for real assets, p^k.

The agents take (r, p) as given. Given (r, p), with $r \geq 0$ and $p >> 0$, the budget set $B(r, p)(h)$ available to agent h - specifying the sequence of market actions and consumption choices $\sigma(h) = \{b_{s^*}^{\ell}(h), q_{s^*}^{\ell}(h), b_{s^*}^{k}(h), q_{s^*}^{k}(h), \bar{\mu}^c(h), \mu_{s^*}(h), \bar{d}(h), c_s^{\ell}(h)\}$ that are feasible for her - is depicted in the following table:

Steps	Descriptions
(i)	borrow short-term and long-term, trade in commodities and real assets
(ii)	repay the short-term loan and consume
$(iii)_s$	borrow short-term, repay or default on long-term loans, and trade in commodities
$(iv)_s$	repay short-term loan and consume

We require that the outflow of money at each point in time cannot exceed its stock on hand, letting $\Delta_{s^*}(h)$ and $\bar{\Delta}(h)$ denote the difference between the right-hand-side and the left-hand-side of the corresponding budget constraints below. $\Delta_{s^*}(h)$ represent intratemporal deposits or “idle" money, while $\bar{\Delta}(h)$ are intertemporal deposits or money “carried forward".

Every agent $h \in H$ tries to maximize her expected utility from the consumption of L goods,[4]

$$\max_{\sigma(h)} \mathbb{U}^h = \sum_{\ell} u_0^h \left(c_0^\ell(h) \right) + \sum_s \sum_\ell u_s^h \left(c_s^\ell(h) \right), \tag{3}$$

subject to the following constraints:

$$\sum_k b^k(h) + \sum_\ell b_0^\ell(h) + \Delta_0(h) = \frac{\mu_0(h)}{1+r_0} + \sum_c \frac{\bar{\mu}^c(h)}{1+\bar{r}_c}, \tag{4}$$

(i.e., Money spent on purchase of real assets and commodities in state 0 + money unspent $=$ money borrowed on short and long loans),

$$\mu_0(h) + \bar{\Delta}(h) = \Delta_0(h) + \sum_k p^k q^k(h) + \sum_\ell p_0^\ell q_0^\ell(h), \tag{5}$$

(i.e., Money repaid on short loan + money unspent $=$ money unspent in (4) + money obtained from sales of real assets and commodities),

$$c_0^\ell(h) \equiv e_0^\ell(h) - q_0^\ell(h) + \frac{b_0^\ell(h)}{p_0^\ell} \quad \forall \ell \in L, \tag{6}$$

(i.e., Consumption of commodity $\ell \equiv$ endowment of ℓ - sales of ℓ + purchase of ℓ),

$$\sum_c \bar{\mu}^c(h) \frac{\gamma_k^c}{p^k} \leq e^k(h) - q^k(h) + \frac{b^k(h)}{p^k} \quad \forall k \in K, \tag{7}$$

(i.e., Amount of real asset k required as collateral for long loans $\leq$ endowment k - sales of k + purchase of k),

$$\sum_\ell b_s^\ell(h) + \sum_c \mathbb{I}_s^c(h) \bar{\mu}^c(h) + \Delta_s(h) = \frac{\mu_s(h)}{1+r_s} + \bar{\Delta}(h), \tag{8}$$

[4] To simplify the intricate equilibrium equations that arise with incomplete markets we confine attention to “active” equilibria in which each agent chooses to buy something in every state. See Tsomocos (2008) for a formal treatment of indeterminacy in the presence of inactive markets.

(i.e., Money spent on purchase of commodities in state s + money repaid fully to some long loans + money unspent = money borrowed short-term + money unspent from (5)),

$$\mu_s(h) \leq \Delta_s(h) + \sum_{\ell} p_s^{\ell} q_s^{\ell}(h), \tag{9}$$

(i.e., Money repaid on short loan $\leq$ money unspent at (8) + money obtained from sales of commodities),

$$e_s^k(h) \equiv \left[e^k(h) - q^k(h) + \frac{b^k(h)}{p_k} \right] - \sum_c (1 - \mathbb{I}_s^c(h)) \bar{\mu}^c(h) \frac{\gamma_k^c}{p^k} \quad \forall k \in K, \tag{10}$$

(i.e., real asset k owned at state $s \equiv$ asset owned at state 0 - asset foreclosed),

$$c_s^{\ell}(h) \equiv e_s^{\ell}(h) - q_s^{\ell}(h) + \frac{b_s^{\ell}(h)}{p_s^{\ell}} \quad \forall \ell \in L/\{1\}, \tag{11}$$

(i.e., consumption of commodity ℓ at state $s \equiv$ endowment of commodity ℓ at state s - sales of commodity ℓ at state s + purchase of commodity ℓ at state s),

and

$$c_s^1(h) \equiv \sum_k e_s^k(h) X_s^k + e_s^1(h) - q_s^1(h) + \frac{b_s^1(h)}{p_s^1} \quad \text{for } \ell = 1, \tag{12}$$

(i.e., consumption of commodity 1 in state $s \equiv$ commodity 1 produced by real assets + endowment of commodity 1 - sales of commodity 1 + purchase of commodity 1).

If agent h chooses to default on contract c in state s, then the monetary authority seizes the collateral and puts it up for sale in the market for good 1. Recall that assets deliver in terms of this good in the beginning of period 1. Instead of receiving $\bar{\mu}^c$, the monetary authority receives $\sum_k \gamma_k^c X_s^k p_s^1 / p^k$.

4 Market Clearing and Equilibrium

4.1 Goods markets

Total sales should be equal to total purchases, i.e.,

$$p_{s^*}^{\ell} = \frac{\sum_h b_{s^*}^{\ell}(h)}{\sum_h q_{s^*}^{\ell}(h)} \quad \forall \ell \in L \setminus \{1\}, s^* \in S^*, \tag{13}$$

$$p_0^{\ell} = \frac{\sum_h b_0^{\ell}(h)}{\sum_h q_0^{\ell}(h)} \quad \text{for } \ell = 1, \tag{14}$$

$$p_s^{\ell} = \frac{\sum_h b_s^{\ell}(h)}{\sum_h q_s^{\ell}(h) + q_s^M} \quad \text{for } \ell = 1. \tag{15}$$

where $q_s^M \equiv \sum_h \sum_k \sum_c (1 - \mathbb{I}_s^c(h))\, \bar{\mu}^c(h) \gamma_k^c X_s^k / p^k$, is the amount of collateral that the monetary authority liquidates in the event of default.

4.2 Assets markets

Total sales by agents should be equal to total purchases, i.e.,

$$p^k = \frac{\sum_h b^k(h)}{\sum_h q^k(h)} \quad \forall k \in K. \tag{16}$$

4.3 Short-term money markets

Total short-term loans demand should be equal to money supply, i.e.,

$$1 + r_{s^*} = \frac{\sum_h \mu_{s^*}(h)}{\sum_h M_{s^*}}. \tag{17}$$

4.4 Long-term money markets

Total long-term loans demand for each $c \in C$ should be equal to money supply, i.e.,

$$1 + \bar{r}_c = \frac{\sum_h \bar{\mu}^c(h)}{\bar{M}^c} \quad \forall c \in C. \tag{18}$$

Definition of Equilibrium: *$((r,p),(M_{s^*},\bar{M}^c_{c\in C}),\sigma(h)_{h\in H})$ is a Monetary Collateral Equilibrium (MCE) for the economy $\mathbb{E} = \left(\left(\mathbb{U}^h, e_{s^*}(h), e^k(h)\right)_{h\in H, s^*\in S^*}, \mathbb{X}, (\gamma^c_k)_{k\in K, c\in C}\right)$ if and only if equations (13)-(18) hold, $\sigma(h) \in argmax_{\sigma(h)\in B(r,p)(h)} \mathbb{U}^h, \forall h \in H$. In sum, all markets clear, expectations are rational, i.e. future prices and interest rates are correctly anticipated, and agents optimize given their budget sets.*

5 Equilibrium Analysis

In this section we prove a number of lemmas that we will use to prove the determinacy of monetary equilibria in sections 6 and 7. We, first, establish some restrictions on equilibrium variables. In particular, we show that interest rates cannot be negative (lemma 1), that all money will return to the central bank at the end of the final period (lemma 2), how the term-structure of interest rates is determined (lemma 3), how the intretemporal deposits are determined (lemma 4 and 5), and that there are no wash sales of commodities and assets in equilibrium (lemmas 6 and 7).

Lemma 1: *At any monetary equilibrium, $r_{s^*}, \bar{r}_c \geq 0$, $\forall s^* \in S^*$ and $c \in C$*

Proof. Let $r_{s^*} < 0$, then agents could infinitely arbitrage the central bank. The collateral constraint puts a bound on the demand for long-term loans, hence $\bar{r}_c$ can be less than r_0 in equilibrium when positive, while agents continue to borrow in both markets. However, when there is no default in any state on loan c, the collateral constraint does not bind, and

$\bar{r}_c \gg r_0$. Otherwise, agents would not borrow short-term at $t = 0$ and the market would not clear. □

Lemma 2: *No worthless cash at end.*

Proof. Suppose that agent h keeps worthless cash at the end of period 1, thus constraint (9) is non-binding. Then h can borrow a little more on r_s, use the money to buy more commodities in state s (leaving all his other actions unchanged), without violating constraint (9), i.e., with enough money at hand to repay the extra loan. This improves his utility, a contradiction. □

Lemma 3: *Term-structure of interest rates.*

Proof. Sum equations (4) and (5) over all agents and apply market clearing conditions (13), (14) and (17) to get:

$$r_0 M_0 + \bar{\Delta} = \sum_c \bar{M}^c, \tag{19}$$

where $\bar{\Delta} = \sum_h \bar{\Delta}(h)$. Similarly, sum equations (8) and (9) over all agents (realize that $\mathbb{I}_s^c$ is independent of the identity of the borrowing agent h) and apply market clearing conditions (13), (15), (17) and (18) to get:

$$\begin{aligned} & r_s M_s + \sum_c f_s^c \sum_h \bar{\mu}^c(h) = \bar{\Delta} \\ \Rightarrow & r_s M_s + \sum_c f_s^c \bar{M}^c (1 + \bar{r}_c) = \bar{\Delta} \quad \forall s \in S \end{aligned} \tag{20}$$

Combine equations (19) and (20) to get

$$r_0 M_0 + r_s M_s + \sum_c [f_s^c \bar{M}^c (1 + \bar{r}_c) - \bar{M}^c] = 0 \quad \forall s \in S. \tag{21}$$

□

Lemma 4: *Suppose* $r_0 > 0$ *and* $r_s > 0$, $\forall s \in S$. *Then,* $\Delta_s(h) = 0$, $\forall s \in S$ *and* $h \in H$; $\Delta_0(h) = 0$ *if agent h borrows short-term in state* 0*; however,* $\Delta_0(h)$ *may be positive if agent h does not borrow in period* 0.

Proof. Suppose that agent h hoards money within state s, i.e., $\Delta_s(h) > 0$. Then, she could have reduced her money holdings by ε, borrowed $\varepsilon(1+r_s)$ less from the short-term loan market without violating constraint (8) and reduce the sale of good ℓ by $\frac{\varepsilon r_s}{p_s^\ell}$ without violating constraint (9). This results in a utility gain $\frac{\varepsilon r_s}{p_s^\ell}\nabla_{s\ell}^h$, hence a contradiction with optimality. Similarly, $\Delta_0(h) = 0$ if agent h borrows on the short-term loan in period 0. However, if agent h does not borrow short-term, i.e., $\mu_0(h) = 0$, then Δ_0^h can be positive. □

Lemma 5: *Sales in state s imply short-term borrowing but sales in state* 0 *may not.*

Proof. Agent h would not sell in $s \in S$ if she does not borrow on r_s because the sales revenue for $s \in S$ are too late for anything except repayment of loans. At $t = 0$, an agent might sell in order to deposit long-term. Meanwhile, she may not need to borrow short-term since she may borrow long-term to purchase goods. Note that she may choose to borrow long-term and hold cash across periods at the same time (see lemma 10). □

Lemma 6: *No wash sales of commodities. Suppose* $r >> 0$. *Then* $b_s^\ell(h)q_s^\ell(h) = 0$, $\forall$ $h \in H, s \in S^*$ *and* $\ell \in L$.

Proof. Suppose $b_s^\ell(h)q_s^\ell(h) > 0$ and $\mu_s(h) > 0$ for some $s \in S^*$. Let h borrow s less on the short-term loan in state s (i.e. reduce $\mu_s(h)$ by $(1+r_s)\varepsilon$), spend ε less on the purchase of ℓ, and sell $(1+r_s)\varepsilon/p_s^\ell$ less of ℓ. This would increase her consumption of ℓ by $\varepsilon r_s/p_s^\ell$ and improve her utility without violating the budget constraints.

Note that, since we consider an active economy where all agents buy some goods in every period, they also sell some other goods given that $b_s^\ell(h)q_s^\ell(h) > 0$. Hence, $\mu_s(h) > 0$, because the only reason to sell in $s \in S$ is to pay off the loan.

Suppose $\mu_0 = 0$, but $\bar{\mu}^c(h) > 0$ for some long-term loan c and $b_s^\ell(h)q_s^\ell(h) > 0$. Again, let

h borrow ε less on the loan term loan c, spend ε less on the good ℓ and reduce her sales revenue from ℓ by $(1+\bar{r})\varepsilon$. This would improve her consumption of ℓ by $\varepsilon\bar{r}/p^l_s$, a contradiction.

Finally, if $\mu_0(h) = \bar{\mu}(h) = 0$, then h does not have money to purchase anything, thus $b^\ell_s(h)q^\ell_s(h) = 0$. $\square$

Lemma 7: *No wash sales of real assets. Suppose $r >> 0$. Then $b^k(h)q^k(h) = 0$, for all $h \in H$ and $k \in K$.*

Proof. Similar as lemma 6. $\square$

We now turn to prove how the choice of collateral requirements determines default in equilibrium and what the effect on interest rates is. If the monetary authority sets collateral requirements such that there is no default in equilibrium, then all interest rates are zero (lemma 8). This leads to indeterminacy of monetary equilibria in the absence of outside money which is extensively discussed in Dubey and Geanakoplos (2006). In lemma 9, we show that the monetary authority can choose collateral requirements to achieve any profile of default it wants and in lemma 11 we show that positive default is a necessary and sufficient condition to support positive short-term interest states in every state and period of the world. The underlying reason is that default creates endogenously liquid wealth (equivalent to outside money), which can be distributed between period 0 and period 1 short-term loan markets. This requires that agents can take both long-term loans and deposit intertemporally in equilibrium, which is shown in lemma 10.

Lemma 8: *If $\sum_k \gamma^c_k X^k_s p^1_s / p^k > 1\ \forall c \in C, s \in S$, then $r_0 = r_s = \bar{r}_c = 0$.*

Proof. From equation (2), this implies that there is no default on any contract c and state s. Thus, $\mathbb{I}^c_s = 1$ for all $c \in C$ and $s \in S$ and equation (21) becomes $r_0 M_0 + r_s M_s + \sum_c \bar{r}_c \bar{M}^c = 0$. Hence, $r_0 = r_s = \bar{r}_c = 0$ for all states s and contracts c. $\square$

Lemma 9: $\exists\, \gamma_k^c$*,* $k \in K$*,* $c \in C$ *such that there is default on some long-term loan c in every state s.*

Proof. The monetary authority needs to choose collateral requirements γ_k^c for every asset k such that $\sum_k \gamma_k^c X_s^k p_s^1 / p^k < 1$ for some c in any state s. Also, the collateral requirement for at least one k needs to be strictly positive for every contract c, such that demand from long-term loans is bounded by the collateral constraint (1). Given that prices are bounded, the monetary authority can always choose $\gamma_k^c > 0$, such that default occurs in every state for some contract. To illustrate this, suppose that $K = 1$. Then, the monetary authority could choose $0 < \gamma_k < \dfrac{1}{max_s \left[X_s^k sup(p_s^1)/inf(p^k)\right]}$, which yields default in every state, while demand is bounded by (1). Obviously, with complete markets the monetary authority can choose collateral requirements such that every contract defaults in exactly one state. $\square$

Let $\nabla_{s^*\ell}^h = \partial \mathbb{U}^h / \partial c_{s^*}^\ell(h)$, $\forall s^* \in S^*, \ell \in L, h \in H$. Given lemma 6, we can distinguish between goods that agent h buys and goods that she sells. $K(+)_{s^*}(h)$ and $K(-)_{s^*}(h)$ are defined similarly.

Denote by $L(+)_{s^*}(h) = \left\{\ell \in L : b_{s^*}^\ell(h) > 0 \text{ and } q_{s^*}^\ell(h) = 0\right\}$ and $L(-)_{s^*}(h) = \left\{\ell \in L : b_{s^*}^\ell(h) = 0 \text{ and } q_{s^*}^\ell(h) > 0\right\}$. Moreover, denote by $\ell_{s^*}^+(h)$ one element of $L(+)_{s^*}(h)$ and by $\ell_{s^*}^-(h)$ one element of $L(-)_{s^*}(h)$.

Lemma 10: *Agent h may hold cash across periods and borrow in the long term collateralized loan market simultaneously, i.e.,* $\bar{\Delta}(h) > 0$ *and* $\bar{\mu}^c(h) > 0$ *for some* $c \in C$ *can coexist.*

Proof. If agent h takes a long-term loan c and does not default on it later, she would not carry cash across period. Otherwise, if $\bar{\Delta}(h) > 0$, since $\bar{r}^c >> r_0, \forall c \in C$, h would be better off reducing the amount of inventory by ε, borrowing $\dfrac{\varepsilon}{1+r_0}$ amount more on the short-term loan and borrowing $\dfrac{\varepsilon}{1+r_0}$ amount less on the long term loan. In the next period, she will receive ε amount less of inventory, while repay $\dfrac{\varepsilon(1+\bar{r}^c)}{(1+r_0)}$ less on the long-term loan.

Since $\frac{\varepsilon(1+\bar{r}^c)}{(1+r_0)} >> \varepsilon$, agent h will be better off.

However, if h does default on the long-term loan c, she may choose to carry cash across periods at the same time. To see this,consider the first-order condition for $\bar{\Delta}(h)$:

$$-\frac{\nabla^h_{0\ell_0^+(h)}}{p_0^{\ell_0^+(h)}} + (1+r_0)\sum_s \frac{\nabla^h_{s\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} = 0 \quad \text{for } \bar{\Delta}(h) > 0$$

or, if an agent does not borrow short-term at t=0,

$$-\frac{\nabla^h_{0\ell_0^+(h)}}{p_0^{\ell_0^+(h)}} + \sum_s \frac{\nabla^h_{s\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} = 0 \quad \text{for } \bar{\Delta}(h) > 0.$$

The first-order conditions for long-term loans require:

$$\frac{\nabla^h_{0\ell_0^+(h)}}{p_0^{\ell_0^+(h)}} - (1+\bar{r}_c)\sum_s \frac{\nabla^h_{s\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} f_s^c - \sum_k \bar{\lambda}^k \frac{\Upsilon_k^c}{p^k} = 0 \quad \forall c \in C$$

To have the agent borrowing long term and inventory cash simultaneously requires:

$$(1+r_0)\sum_s \frac{\nabla^h_{s\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} = (1+\bar{r}^c)\sum_s \frac{\nabla^h_{s\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} f_s^c + \sum_k \bar{\lambda}^k \frac{\Upsilon_k^c}{p^k}$$

or

$$\sum_s \frac{\nabla^h_{s\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} = (1+\bar{r}^c)\sum_s \frac{\nabla^h_{s\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} f_s^c + \sum_k \bar{\lambda}^k \frac{\Upsilon_k^c}{p^k}$$

where $\bar{\lambda}^k$ is the multiplier related to the binding collateral constraints k. Since f_s^c is smaller than one for defaulted loans, the above equation can be satisfied. □

Lemma 11: *$\exists$ interest rates $\bar{r}_c > 0$ and Υ_k^c, $k \in K$, $c \in C$ such that all r_0 and r_s are strictly positive when $0 < \bar{\Delta} < \sum_c \bar{M}^c$.*

Proof. From equation (19), $r_0 > 0$ when $\bar{\Delta} < \sum_c \bar{M}^c$. $\bar{\Delta}$ is the amount of money that can be collected as seigniorage at $t = 1$ on short-term loan and long-term loans. By appropriately setting the collateral requirement γ_k^c for each contract c, the monetary authority can choose which long-term loans that will be repaid in full, i.e., $\mathbb{I}_s^c = 1$, such that equation (20) is satisfied for some $r_s, \bar{r}^c > 0$ as long as $\bar{\Delta} > 0$, which can happen in equilibrium (lemma 10). □

6 Determinacy with interest rate targets

The monetary authority can set collateral requirements such that there is active default in every state of the world in the last period (lemma 9). In turn, this is sufficient to support positive interest rates in every period and state (lemma 11), which the central bank chooses as well. Thus, the quantity theory of money holds and agents do no hoard money within each period while borrowing short-term (lemma 4). This resolve the indeterminacy of monetary equilbria as we prove in the rest of the section.

The strategy of our proof is to represent a monetary equilibrium with default as the solution to a system of simultaneous equations with the same number of unknowns, and then to apply the transversality theorem to prove that “generically" the solution to this system is a zero-dimensional manifold.

The monetary authority targets interest rates $r = \left((\bar{r}_c)_{c\in C}, (r_{s^*})_{s^*\in S^*}\right) \gg 0$ and chooses the collateral requirements $\gamma = (\gamma_k^c)_{k\in K, c\in C}$.

Our exogenous variables are $(u) = \left(\left(u_{s^*}^{\ell}\right)_{s^*\in 0\cup S, \ell\in L, h\in H}\right)$, as we hold $\left(r, \mathbb{X}, \gamma, \left(e_{s^*}^{\ell}(h), e^k(h)\right)_{s^*\in S^*, \ell\in L, k\in K, h\in H}\right)$ fixed.

Our endogenous variables are $\Bigg((M_{s^*})_{s^*\in S^*}, (\bar{M}^c)_{c\in C}, \left(p_{s^*}^{\ell}\right)_{s^*\in S^*, \ell\in L}, \left(p^k\right)_{k\in K},$ $\left(c_{s^*}^{\ell}(h), b_{s^*}^{\ell}(h), b_{s^*}^{k}(h), q_{s^*}^{\ell}(h), q^k(h), \mu_{s^*}(h), \bar{\mu}^c(h), \bar{\lambda}^k(h)\right)_{s^*\in S^*, \ell\in L, k\in K, c\in C, h\in H}\Bigg).$

Given lemma 2, the endogenous variables $\left(\left(p_{s^*}^{\ell}\right)_{s^*\in S^*, \ell\in L}, \left(p^k\right)_{k\in K},\right.$

$(M_{s^*})_{s^*\in S^*}, (\bar{M}^c)_{c\in C}, \left(\left(c^\ell_{s^*}(h)\right)_{s^*\in S^*,\ell\in L,h\in H}\right)$ will be forced by equations (13)-(15), (16), (17), (18), (6), (11), (12), and will be functions of the remaining free endogenous variables. Denote the set of the free endogenous variables by σ with domain $D(\sigma)$.

The free variables will be determined by the following first order conditions and the remaining budget constraints (4), (5), (8) and (9). The first-order conditions for commodities require:

$$\frac{\nabla^h_{s^*\ell}}{p^\ell_{s^*}} - \frac{\nabla^h_{s^*\ell^+_{s^*}(h)}}{p^{\ell^+_{s^*}(h)}_{s^*}} = 0 \quad \text{for } \ell \in L(+)_{s^*}(h) \setminus \left\{\ell^+_{s^*}(h)\right\}, \tag{22}$$

$$\frac{\nabla^h_{s^*\ell}}{p^\ell_{s^*}} - \frac{\nabla^h_{s^*\ell^-_{s^*}(h)}}{p^{\ell^-_{s^*}(h)}_{s^*}} = 0 \quad \text{for } \ell \in L(-)_{s^*}(h) \setminus \left\{\ell^-_{s^*}(h)\right\}, \tag{23}$$

$$\frac{\nabla^h_{s^*\ell^+_{s^*}(h)}}{p^{\ell^+_{s^*}(h)}_{s^*}} - (1+r_{s^*})\frac{\nabla^h_{s^*\ell^-_{s^*}(h)}}{p^{\ell^-_{s^*}(h)}_{s^*}} = 0$$

or, if an agent does not borrow short-term,

$$\frac{\nabla^h_{s^*\ell^+_{s^*}(h)}}{p^{\ell^+_{s^*}(h)}_{s^*}} - \frac{\nabla^h_{s^*\ell^-_{s^*}(h)}}{p^{\ell^-_{s^*}(h)}_{s^*}} = 0. \tag{24}$$

The first-order condition for $\bar{\Delta}(h)$ requires:

$$-\frac{\nabla^h_{s^*\ell^+_0(h)}}{p^{\ell^+_0(h)}_0} + (1+r_0)\sum_s \frac{\nabla^h_{s^*\ell^+_s(h)}}{p^{\ell^+_s(h)}_s} = 0 \quad \text{for } \bar{\Delta}(h) > 0$$

or, if an agent does not borrow short-term at $t = 0$,

$$-\frac{\nabla^h_{s^*\ell^+_0(h)}}{p^{\ell^+_0(h)}_0} + \sum_s \frac{\nabla^h_{s^*\ell^+_s(h)}}{p^{\ell^+_s(h)}_s} = 0 \quad \text{for } \bar{\Delta}(h) > 0. \tag{25}$$

The first-order conditions for long-term loans require:

$$\frac{\nabla^h_{s^*\ell_0^+(h)}}{p_0^{\ell_0^+(h)}} - (1+\bar{r}^c)\sum_s \frac{\nabla^h_{s^*\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} f_s^c - \sum_k \bar{\lambda}^k \frac{\gamma_k^c}{p^k} = 0 \quad \forall c \in C. \tag{26}$$

The first order conditions for asset purchases and sales, respectively, at $t=0$ require:

$$-\frac{p^k}{p_0^{\ell_0^+(h)}} \nabla^h_{s^*\ell_0^+(h)} + \sum_s \frac{\nabla^h_{s^*\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} X_s^k p_s^1 + \bar{\lambda}^k = 0 \quad \text{for } k \in K(+)_{s^*}(h), \tag{27}$$

$$\frac{p^k}{p_0^{\ell_0^+(h)}} \nabla^h_{s^*\ell_0^+(h)} - (1+r_0)\sum_s \frac{\nabla^h_{s^*\ell_s^+(h)}}{p_s^{\ell_s^+(h)}} X_s^k p_s^1 - \bar{\lambda}^k = 0 \quad \text{for } k \in K(-)_{s^*}(h). \tag{28}$$

Finally, the complementarity slackness conditions require:

$$\bar{\lambda}^k \left[e^k(h) - q^k(h) + \frac{b^k(h)}{p^k} - \sum_c \bar{\mu}^c(h) \frac{\gamma_k^c}{p^k} \right] = 0 \quad \forall k \in K. \tag{29}$$

The following table matches free variables with equations:

Free variable	Equation
$b^\ell_{s^*}, q^\ell_{s^*}\ \ell \notin \ell^+_{s^*}(h), \ell^-_{s^*}(h)$	Equations (22),(23)
$q^\ell_{s^*}\ \ell \in \ell^-_{s^*}(h)$	Equations (24)
$b^\ell_{s^*}\ \ell \in \ell^+_{s^*}(h)$	Equations (4), (8)
q^k_0, b^k_0	Equations (27), (28)
μ_{s^*}	Equations (5), (9) or $\mu_0(h) = 0$
$\bar{\mu}^c$	Equations (26)
$\bar{\lambda}^k, k \in K$	Equation (29)
$\bar{\Delta}$	Equations (25) or $\bar{\Delta} = 0$
Δ_{s^*}	Lemma 4 or equation (5)

The space $\mathbb{U}^h$ of utilities of agent $h \in H$ consists of all linear perturbations of some fixed utility $\bar{u}^h : \mathbb{R}^{TL}_+ \to \mathbb{R}$,. i.e., $u^h(c) = \bar{u}^h(c) + \delta \cdot c$, where $\delta \in \mathbb{R}^{TL}_+$. Let $\mathbb{U} = \times_{h\in H}\mathbb{U}^h$.

Consider a matching of the free variables in σ to equations, as in table 6. Consider the map $\psi : \mathbb{U} \times D \to \mathbb{R}^d$ given by $\psi(u,\sigma) = LHS$ of the d equations in the matching. If $\sigma \in D$ is an active equilibrium of $u \in \mathbb{U}$, then $\psi(u,\sigma) = 0$. i.e., $\sigma \in \psi_u^{-1}(0)$ where

$\psi_u \equiv \psi(u,\sigma)$. Since dimension $D = d$, $\psi_u^{-1}(0)$ will be a zero-dimensional manifold provided that $\psi_u : D \to \mathbb{R}^d$ is transverse to 0. This follows from the transversality theorem for almost all $u \in \mathbb{U}$ if each map ψ is transverse to 0. Theorem 1 proves that this is the case when the monetary authority sets strictly positive interest rates and chooses collateral requirements such that there is default on some long-term loan in every state in period 1.

Theorem 1: *Assume that for every agent h, and for the collateralized loans and real assets traded by the agent (denoted by $C(h)$ and $K(h)$), the real payoff of a riskless bond $1/p$, the real payoff of the long-term collateralized loans $f^c/p, c \in C(h)$ and the real payoff of the asset $X^k p_s^1/p, k \in K(h)$ are linearlly independent. The set of equilibrium outcomes is determinate for generic u in U when the monetary authority targets positive interest rates $r = \left((\bar{r}^c)_{c\in C}, (r_{s^*})_{s^*\in S}\right) \gg 0$ and chooses collateral requirements γ_k^c $c \in C$ such that there is default in at least one contract c in every state s.*

Proof. To perturb (22) or (23), adjust $\nabla^h_{s^*\ell}$ $\ell \notin \ell^+_{s^*}(h), \ell^-_{s^*}(h)$, and leave all other equations undisturbed.

To perturb (24), adjust $\nabla^h_{s^*\ell}$ $\ell \in \ell^-_{s^*}(h)$, which disturbs (23). To restore (23), adjust $\nabla^h_{s^*\ell}$ $\ell \in L(-)_{s^*}(h) \setminus \left\{\ell^-_{s^*}(h)\right\}$, which leaves all other equations undisturbed.

Next consider the set of first-order conditions for inventory, the long loan, and the assets. Note, following lemma (10), (25) and (26) can either invoked together or not. Recall that the vectors $\dfrac{1}{p}$, $f^c/p, c \in C(h)$ and $X^k p_s^1/p, k \in K(h)$ are linearlly independent. Therefore, we can adjust $\nabla^h_{s^*\ell_s^+(h)}/p_s^{\ell_s^+(h)}$ in a direction perpendicular to all but one of these vectors, and thereby unilaterally perturb any one of the equations from this set. In the process (22) or (24) will be disturbed. We restore these by adjusting $\nabla^h_{s^*\ell}$, for $\ell \in L(+)_{s^*}(h) \setminus \left\{\ell^+_{s^*}(h)\right\}$ and $\nabla^h_{s^*\ell_s^-(h)}$. The latter further disturb (23), which is restored via $\nabla^h_{s^*\ell}$, for $\ell \in L(-)_{s^*}(h) \setminus \left\{\ell^-_{s^*}(h)\right\}$

To perturb (7), adjust γ_k^c without affecting $\mathbb{I}_s^c$. This disturbs (22) and (22). Adjust as described above.

To perturb (4), increase $\mu_0(h)$ by $\varepsilon > 0$, thus h spends $\dfrac{\varepsilon}{1+r_0}$ more on his good. Then, (5) is distorted for some other agents $h \in H$, who receive ε^h more money. Assume that one agent take it all, thus $\varepsilon^h = \dfrac{\varepsilon}{1+r_0}$. In the following, we proceed by considering two cases:

Case 1: assume that the agent h increases his borrowing, which means that she can increase her purchases by $\dfrac{\varepsilon}{(1+r_0)^2}$. Continuing with this logic and assuming that every subsequent agent borrows at r_0, the total increase in borrowing is $\sum_t^{\infty} \dfrac{\varepsilon}{(1+r_0)^t} = \varepsilon \dfrac{1+r_0}{r_0}$. Thus, total expenditures on goods go up by $\dfrac{\varepsilon}{r_0}$, which is finite.

Case 2: alternatively, assume that the agent h does not borrow at r_0. Then, she would carry the amount ε^h into the next period and will increase her purchases of goods by ε^h. Then, (9) would be distorted for some agents, who receive ε^h more money. Assume that one agent h' again take it all, thus $\varepsilon^{h'} = \dfrac{\varepsilon}{1+r_s}$. Agent h' will increase her borrowing in the same way as described in case 1. Note that agent h' as well as all subsequent agents will borrow, following lemma (5).

It follows that the forced variables $p_s^\ell, c_s^\ell(h)$ also change infinitely small for all $\ell \in L, h \in H$. Perturbing utilities as above, we can restore the old ratios $\dfrac{\nabla^h_{s^*\ell}}{p^\ell_{s^*}}$.

The proofs for (5), (8) and (9) are on the same line.

To perturb (29), adjust $\bar{\mu}^c$ for some c which perturbs (4), and (8) or (22) or (23) (through (11) and (12)). Adjust the way described above.

Thus we see that the map ψ is indeed transverse to 0, proving the determinacy of active monetary equilibrium outcomes. □

7 Determinacy with money supply targets

Instead of targeting interest rates, the monetary authority could target the money supplies $\left((M_{s^*})_{s^* \in S^*}, (\bar{M}^c)_{c \in C}\right)$ and let the interest rates be endogenously determined from the market clearing condition (17) and (18). The remaining variables and optimality condition are the same as in interest rate targeting. If short-term interest rates r_{s^*} are positive in equilibrium in every state, then the analysis in Theorem 1 goes through and the equilibrium outcomes are determinate. However, we also need to account for the possibility that short-term interest rates are zero, i.e., intratemporal money holding $\Delta_{s^*}(h)$ cannot be determined from lemma 4.

We show in lemma 12 below that collateral requirements exist such that short-term interest rates in all states $s \in S$ at $t = 1$ are positive. Hence, $\Delta_s(h) \cdot \mu_s(h) = 0$ and the only indeterminacy is due to the initial price level, i.e., $\Delta_0(h)$ cannot be determined when $r_0 = 0$. We, then, discuss a way to determine the initial price level.

Lemma 12: $\exists$ *interest rates* $\bar{r}_c \geq 0$ *and* γ_k^c, $k \in K$, $c \in C$ *such that all* r_s *are strictly positive when* $r_0 = 0$.

Proof. From equation (19), $r_0 = 0$ requires that $\bar{\Delta} = \sum_c \bar{M}^c$. Then, (21) implies that:

$$r_s M_s + \sum_c f_s^c \bar{M}^c (1 + \bar{r}_c) = \sum_c \bar{M}^c$$
$$\sum_s \lambda_s r_s M_s + \sum_c \sum_s \lambda_s f_s^c \bar{M}^c (1 + \bar{r}_c) = \sum_c \sum_s \lambda_s \bar{M}^c, \tag{30}$$

where $\lambda_s = \frac{\nabla^h_{s^* \ell_s^+(h)}}{p_s^{\ell_s^+(h)}}$. Combining the first order condition (25) and (26) for $\bar{\Delta}(h)$ and $\bar{\mu}^c(h)$ yields:

$$\sum_s \lambda_s f_s^c (1 + \bar{r}_c) + \sum_k \bar{\lambda}^k \gamma_k^c (1 + \bar{r}_c)/p^k = \sum_s \lambda_s$$
$$\sum_c \sum_s \lambda_s f_s^c \bar{M}^c (1 + \bar{r}_c) + \sum_c \sum_k \bar{\lambda}^k \gamma_k^c \bar{M}^c (1 + \bar{r}_c)/p^k = \sum_c \sum_s \lambda_s \bar{M}^c. \tag{31}$$

Combining (30) and (31), we get that

$$\sum_s \lambda_s r_s M_s = \sum_c \sum_k \bar{\lambda}^k \gamma_k^c \bar{M}^c (1 + \bar{r}_c)/p^k > 0. \tag{32}$$

If $r_s = 0$ for some state s, then the monetary authority can choose a different collateral requirement to reduce the interest rate $\bar{r}_c$ that an agent is willing to pay for some constract c that delivers fully in that state s. In the extreme, the monetary authority can choose collateral requirements such that all contracts $c \in C$ default in that state, thus the shortfall in the repayment of long-term debt accrues to the short-term money market. □

It remains to pin down the price level at $t = 0$. In particular, we need a way to unilaterally perturb the period 0 budget constraints of agents that choose to hoard cash within the period. The presence of private monetary endowments (outside money) plays this role. The period 0 budget constraints can be perturb by varying the level of outside money for each agents that hoards liquidity, while all other equations are not disturbed. We refer the reader to Dubey and Geanakoplos (2006) for a detailed analysis given that our focus has been to show the determinacy of monetary equilibria in the absence of outside money. Note, however, that we require outside money only in the initial period, while Dubey and Geanakoplos (2006) also need outside money in every future state $s1$ when the monetary authority targets the money supply.

8 Conclusions

We examine how the credit risk that the monetary authority undertakes in its operations relates to the determinacy of monetary equilibria. Our model features trade in fiat money, real assets and a monetary authority that either injects money into the economy through short-term and long-term loans to agents, or sets interest rates. Short-term loans are safe, but long-term loans are collateralized by a portfolio of real assets and are subject to credit risk, i.e., agents can choose to default if the value of the collateral is less than the promised loan repayment.

If the monetary authority chooses collateral requirements such that there is no default in equilibrium all interest rates are zero. As a result, the amount of money used by agents to purchase goods and assets cannot be pinned down by monetary policy and the equilibrium outcomes manifest indeterminacy. Alternatively, the monetary authority can set collateral requirements on long-term loans such that there is default on some contract in every future state of the world. The shortfall for the central bank endogenously creates private liquid wealth for agents, which can be used for trade. Eventually, all money will end up with the central bank, thus the presence of private liquid wealth can support positive short-term interest rates. Consequently, agents do not hold “idle" cash in equilibrium, a non-trivial quantity theory of money obtains and the equilibrium is determinate.

The central bank makes losses on its long-term operations, but makes equal profits from seigniorage on its short-term ones. Nevertheless, its policy is non-Ricardian not only over time, but also across loan markets. The latter is a novel feature of our analysis, which is a consequence of default. Absent default, non-Ricardian policy requires that the central bank injects outside money in the economy in the form of private monetary endowments, which are free and clear of any offsetting obligation.

Our analysis is focused on how the monetary authority can determine the equilibrium price level. Such a policy will also affect the real equilibrium allocations unless the indeterminacy was purely nominal to begin with and the economy was populated by a representative agent. The classical dichotomy between the real and monetary sectors breaks with the resolution of real and nominal indeterminacy. However, we have not addressed whether the undertaking of credit risk in the central banks monetary operations is an optimal policy. Goodhart, Tsomocos and Vardoulakis (2010), Lin, Tsomocos and Vardoulakis (2014) and Peiris and Vardoulakis (2014) present monetary models with collateralized debt where the monetary policy interacts with default and affects equilibrium allocation and, thus, welfare.

References

Adão, Bernardino, Isabel Correia and Perdo Teles (2014), 'Short and long interest rate targets', *Journal of Monetary Economics* **66**, 95–107.

Balasko, Yves and David Cass (1989), 'The structure of financial equilibrium with exogenous yields: the case of incomplete markets', *Econometrica* **57**(1), 135–162.

Bloise, Gaetano, Jacques H. Drèze and Herakles Polemarchakis (2005), 'Monetary equilibria over an infinite horizon', *Economic Theory* **25**, 51–74.

Buiter, Willem H. (2002), 'The fallacy of the fiscal theory of the price level: A critique', *Economic Journal* **112**(481), 459–480.

Cochrane, John H. (2001), 'Long-term debt and optimal policy in the fiscal theory of the price level', *Econometrica* **69**(1), 69–116.

Drèze, Jacques H. and Herakles Polemarchakis (2000), Monetary equilibria, *in* G.Debreu, W.Neufeind and W.Trockel, eds, 'Economic Essays: A Festschrift for Werner Hildenbrand', Springer, pp. 83–108.

Dubey, Pradeep and John Geanakoplos (1992), 'The value of money in a finite-horizon economy: A role for banks', *in P. Dasgupta, D. Gale, D. Hart, and E. Maskin (eds.), Economic Analysis of Markets and Games, Essays in Honor of Frank Hahn. Cambridge, MA: MIT Press* pp. 407–444.

Dubey, Pradeep and John Geanakoplos (2003*a*), 'Inside and outside money, gains to trade, and is-lm', *Economic Theory* **21**(2), 347–397.

Dubey, Pradeep and John Geanakoplos (2003*b*), 'Monetary equilibrium with missing markets', *Journal of Mathematical Economics* **39**(5-6), 585–618.

Dubey, Pradeep and John Geanakoplos (2006), 'Determinacy with nominal assets and outside money', *Economic Theory* **27**(1), 79–106.

Espinoza, Raphael A., Charles A.E. Goodhart and Dimitrios P. Tsomocos (2008), 'State prices, liquidity, and default', *Economic Theory* **39**(2), 177–194.

Espinoza, Raphael A. and Dimitrios P. Tsomocos (2014), 'Monetary transaction costs and the term premium', *Economic Theory, forthcoming* .

Geanakoplos, John and Andreu Mas-Colell (1989), 'Real indeterminacy with financial assets', *Journal of Economic Theory* **47**(1), 22–38.

Geanakoplos, John and William R. Zame (2013), 'Collateral equilibrium, I: a basic framework', *Economic Theory* **56**(3), 443–492.

Giraud, Gaël and Dimitrios P. Tsomocos (2010), 'Nominal uniqueness and money non-neutrality in the limit-price exchange process', *Economic Theory* **45**(1-2), 303–348.

Goodhart, Charles A.E., Dimitrios P. Tsomocos and Alexandros P. Vardoulakis (2010), 'Modelling a housing and mortgage crisis', *in Financial Stability, Monetary Policy, and Central Banking, edited by Rodrigo A. Alfaro. Central Bank of Chile* .

Goodhart, Charles A.E, Pojanart Sunirand and Dimitrios P. Tsomocos (2005), 'A risk assessment model for banks', *Annals of Finance* **1**(2), 197–224.

Goodhart, Charles A.E, Pojanart Sunirand and Dimitrios P. Tsomocos (2006), 'A model to analyse financial fragility', *Economic Theory* **27**(1), 107–142.

Goodhart, Charles A.E, Udara Peiris, Dimitrios P. Tsomocos and Alexandros P. Vardoulakis (2010), 'On dividend restrictions and the collapse of the interbank market', *Annals of Finance* **6**(4), 455–473.

Grandmont, Jean-Michel (1985), *Money and Value*, Cambridge University Press.

Gurley, John G. and Edward S. Shaw (1960), 'Money in a Theory of Finance', *Brookings Institution, Washington, D.C.* .

Lin, Li, Dimitrios P. Tsomocos and Alexandros P. Vardoulakis (2014), 'Debt deflation effects of monetary policy', *Finance and Economics Discussion Series 2014-37. Board of Governors of the Federal Reserve System* .

Lucas, Robert E. and Nancy L. Stokey (1987), 'Money and rates of interest in a cash-in-advance economy', *Econometrica* **55**(3), 491–513.

Magill, Michael and Martine Quinzii (1992), 'Real effects of money in general equilibrium', *Journal of Mathematical Economics* **21**(4), 301–342.

Magill, Michael and Martine Quinzii (2014*a*), 'Anchoring expectations of inflation', *Journal of Mathematical Economics* **50**, 86–105.

Magill, Michael and Martine Quinzii (2014*b*), 'Term structure and forward guidance as instruments of monetary policy', *Economic Theory* **56**(1), 1–32.

McMahon, Michael, Udara Peiris and Herakles Polemarchakis (2014), 'Perils of quantitative easing', *Warwick WP* .

Peiris, Udara and Alexandros P. Vardoulakis (2014), 'Collateral and the efficiency of monetary policy', *working paper* .

Peiris, Udara and Herakles Polemarchakis (2014), 'Monetary policy and quantitative easing in an open economy: Prices, exchange rates and risk premia', *Warwick WP* .

Shubik, Martin and Dimitrios P. Tsomocos (1992), 'A strategic market game with a mutual bank with fractional reserves and redemption in gold', *Journal of Economics* **55**(2), 123–150.

Sims, Christopher A. (1994), 'A simple model for the study of the determination of the price level and the interaction of monetary and fiscal policy', *Economic Theory* **4**(3), 381–399.

Tsenova, Tsvetomira (2014), 'International monetary transmission with bank heterogeneity and default risk', *Annals of Finance* **10**(2), 217–241.

Tsomocos, Dimitrios P. (1996), 'Essays on Monetary Theory, Banking and General Economic Equilibrium', *Ph.D Thesis, Yale University* .

Tsomocos, Dimitrios P. (2003), 'Equilibrium analysis, banking and financial instability', *Journal of Mathematical Economics* **39**(5-6), 619–655.

Tsomocos, Dimitrios P. (2008), 'Generic determinacy and money non-neutrality of international monetary equilibria', *Journal of Mathematical Economics* **44**(7-8), 866–887.

Woodford, Michael (1994), 'Monetary policy and price level determinacy in a cash-in-advance economy', *Economic Theory* **4**(3), 345–380.