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Optimal Government Spending at the Zero Lower Bound: A Non-Ricardian Analysis*

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Abstract

This paper analyzes the implications of distortionary taxation and debt financing for optimal government spending policy in a sticky-price economy where the nominal interest rate is subject to the zero lower bound constraint. Regardless of the type of tax available and the initial debt level, optimal government spending policy in a recession is characterized by an initial increase followed by a reduction below, and an eventual return to, the steady state. The magnitude of variations in the government spending as well as their welfare implications depend importantly on the available tax instrument and the initial debt level.

JEL: E32, E52, E61, E62, E63

Keywords: Commitment, Distortionary Taxation, Government Spending, Liquidity Trap, Nominal Debt, Optimal Policy, Zero Lower Bound.

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1 Introduction

Recent theoretical studies on fiscal policy have emphasized the role of the zero lower bound (ZLB) constraint on the nominal interest rate in determining its effects on the economy. Some have studied the effect of an exogenous increase in government spending and have found that the government spending multiplier on output is larger when the ZLB constraint binds than when it does not (Christiano, Eichenbaum, and Rebelo (2011); Eggertsson (2011); Woodford (2011)). Others have considered normative questions and have shown that it is optimal to increase government spending when the nominal interest rate is at the ZLB (Werning (2012); Nakata (2013a); Schmidt (2013)). While many have analyzed the government spending multipliers in settings with a distortionary tax and debt, normative analyses of government spending have focused on the environment in which a lump-sum tax is available. As most countries raise their revenues through distortionary taxes and debt issuance, it would be useful to reassess optimal government spending policy at the ZLB in the model with a distortionary tax and debt.

Accordingly, this paper studies optimal government spending and monetary policy in economies where the nominal interest rate is subject to the ZLB constraint and the government spending needs to be financed by either a distortionary tax or debt issuance. The analysis is conducted in a standard New Keynesian model. I consider two models with a distortionary tax—one with a labor income tax and nominal debt and the other with a consumption tax and nominal debt—and contrast them to the model with lump-sum taxation. Throughout the paper, I assume that the government can commit; the government chooses a sequence of policy instruments at time one and adheres to the announced policy path in the future.

I find that, regardless of the type of tax instrument available, the optimal government spending path is characterized by an initial expansion followed by a reduction below, and an eventual return to, the steady state. In all three models, this pattern reflects the desire of the government to align the marginal utility of government spending with the marginal utilities of consumption and leisure. While the pattern of optimal government spending path is the same across all three models, the magnitude of variation in government spending and its welfare effect are not. In particular, the variation in government spending and the resulting welfare effects are much smaller in the model with a consumption tax and debt than in the other two models.

The variation in government spending and the welfare gain of government spending policy are larger in the economy with a larger initial debt. In the model with a labor income tax and debt, the welfare gain of government spending policy is equivalent to a one-time transfer of 0.2 percent of the steady-state consumption in the economy with initial debt-to-annualized output ratio of 50 percent. However, the number increases to 0.4 and 1.1 percent in the economy with initial debt-to-annualized output ratio of 100 and 200 percent, respectively. Similar results are obtained in the model with a consumption tax and debt. This result arises because, in the model with a distortionary tax and debt, the ZLB constraint is welfare reducing both in the short run and the long run: While the ZLB constraint reduces welfare by restricting the government's ability to stabilize consumption and output in the short run, it also reduces welfare, as it prevents the economy from converging to a steady state with lower debt and less distortion in the long run. The access to government spending policy mitigates the distortions arising from the ZLB on both fronts. When the initial debt level is large, the long-run distortion is large and the government spending policy has more to contribute.

My paper builds on the work of Werning (2012), Nakata (2013a), and Schmidt (2013). They characterize optimal government spending and monetary policy in the model with the ZLB under the assumption of lump-sum taxation. I extend their analysis to the models with distortionary taxation and debts. My analysis complements the work of Burgert and Schmidt (2014) and Mateev (2014), who have also studied the implications of debt financing for optimal government spending policy in the model with the ZLB. They assume that the government does not have an explicit commitment technology. In my model, the government is assumed to be able to commit to future policies. Also, while they focus on the model with a labor income tax, I also analyze the model with a consumption tax.

Some authors have analyzed the model with distortionary taxes at the ZLB. Correia, Farhi, Nicolini, and Teles (2013) analyzed the model in which the government can simultaneously choose all of labor-income tax, consumption tax, and lump-sum tax, and they showed that there is no role for government spending in that environment. Eggertsson and Woodford (2004) studied the optimal mix of a distortionary taxation and debt, assuming that the government spending is not available, and Eggertsson (2011) studied the effects of exogenous changes in distortionary taxes on allocations at the ZLB. In my model, government spending is an endogenous variable chosen optimally by the government.

This paper is related to a large literature on optimal fiscal and monetary policy in sticky-price models under commitment. Early contributions include Benigno and Woodford (2004), Schmitt-Grohe and Uribe (2004), and Siu (2004). While most authors have considered the question of how to use a distortionary tax and debt to best finance an exogenous stream of government spending, some have recently analyzed the model in which the government spending is also an endogenous choice variable of the government, as in my paper. Examples are Adam (2011), Leith and Wren-Lewis (2013), Leith, Moldovan, and Rossi (2015), and Motta and Rossi (2014). All these papers abstract from the ZLB constraint on the policy rate. I build on these earlier contributions and focus on the implications of the ZLB.

The rest of the paper is organized as follows. Section 2 describes the model, defines the private-sector equilibrium, and states the government's problem. Section 3 discusses parametrization and the solution method. Section 5 presents the results, and a final section concludes.

2 Model

The model is a standard New Keynesian economy formulated in discrete time with an infinite horizon. The economy is populated by four types of agents: the representative household, the final-goods producer, a continuum of intermediate-goods producers, and the government. I will consider three versions of the model: The first version allows only for lump-sum taxation, the second version allows for labor-income taxation and debt, and the third version allows for consumption taxation and debt. While I will describe the model with all of these fiscal instruments, it should be understood that a subset of them is set to zero in any version of the model.

2.1 Household

The representative household chooses consumption, labor supply, and the holdings of a one-period risk-free nominal bond to maximize the expected discounted sum of the future period utilities. The household likes consumption and dislikes labor. Following the setup of the previous work on fiscal policy at the ZLB, I assume that the household also values government spending. The period utility is assumed to be separable. The household problem is given by

$$\max_{\{C_t, N_t, B_t\}_{t=1}^{\infty}} E_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[\prod_{s=0}^{t-1} \delta_s\Big] \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n} + \chi_{g,0} \frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}}\Big], \tag{1}$$

subject to

$$(1+\tau_{c,t}) P_t C_t + R_t^{-1} B_t \leq (1-\tau_{n,t}) W_t N_t + B_{t-1} - P_t T_t + \Gamma_t, \tag{2}$$

and δ_1 is given. C_t is the consumption of the final goods, N_t is the labor supply, and G_t is government spending. P_t is the price of consumption good, W_t is nominal wage, and Γ_t is the profit from the intermediate goods producers. B_t is a one-period risk-free bond that pays one unit of money at t+1, and R_t is the return on the bond. T_t, $\tau_{n,t}$, and $\tau_{c,t}$ are lump-sum taxation, the labor income tax rate, and the consumption tax rate, respectively.

The discount rate at time t is given by $\beta\delta_t$. δ_t is the discount factor shock that alters the weight of the future utility at time t+1 relative to the period utility at time t. $\{\delta_t\}_{t=1}^{\infty}$ is exogenously given, and evolves deterministically as follows:

$$\delta_1 = 1 + \epsilon_\delta, \quad \delta_t = 1 + \rho_\delta(\delta_{t-1} - 1) \text{ for } t \geq 2.$$

ϵ_δ is revealed at the beginning of t=1 before the agents make decisions. δ_0 multiplies all period utilities and is normalized to 1.

2.2 Firms

There is a representative final-good producer and a continuum of intermediate-goods producers indexed by $i \in [0,1]$. The representative final-good producer purchases the intermediate goods and combines them into the final good using CES technology. It then sells the final good to the household and the government as well as to the intermediate-goods producers if they change the prices. At each time t, the optimization problem of the final-good producer is given by

$$\max_{\{Y_{i,t}\}_{i\in[0,1]}} \quad P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di, \tag{3}$$

subject to the CES production function, $Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}}$.

Intermediate-goods producers use labor inputs and produce imperfectly substitutable intermediate goods according to a linear production function. They set the nominal prices in a staggered fashion: Each period, with probability α, each firm i sets the price of its own good to maximize the expected discounted sum of future profits

$$\max_{P_t^*(i)} \quad E_t \sum_{j=1}^{\infty} \alpha^j \beta^{j-1} \Big[\prod_{j=0}^{t+j-1} \delta_{t+j} \Big] \lambda_{t+j} \Big[P_t^*(i) - W_{t+j} \Big] Y_{t+j}(i), \tag{4}$$

subject to $Y_{i,t} = \left[\frac{P_{i,t}}{P_t}\right]^{-\theta} Y_t$, and $Y_{i,t} = N_{i,t}$. λ_t is the Lagrangian multiplier on the household budget constraint at time t, and $\left[\prod_{j=0}^{t-1} \delta_j\right] \lambda_t$ measures the marginal value of an additional unit of profits to the household.

2.3 Government's Policy Instruments

The set of potential policy instruments is $[R_t, G_t, T_t, \tau_{n,t}, \tau_{c,t}, B_t]$. The ZLB constraint on the nominal interest rate is given by

$$R_t \geq 1. \tag{5}$$

As mentioned previously, I consider three versions of the model, each associated with a unique set of fiscal instruments. In the first version, the government has access to a lump-sum tax alone, and the government budget constraint is given by

$$P_t G_t = P_t T_t. \tag{6}$$

In the second version, the government has access to a labor income tax and nominal debt. In this case, the government budget constraint is given by

$$B_{t-1} + P_t G_t = \tau_{n,t} W_t N_t + R_t^{-1} B_t. \tag{7}$$

Finally, in the third version, the government has access to a consumption tax and nominal debt. In this case, the government budget constraint is given by

$$B_{t-1} + P_t G_t = \tau_{c,t} P_t C_t + R_t^{-1} B_t. \tag{8}$$

2.4 Market Clearing Conditions

The labor market and good market clearing conditions are given by

$$N_t = \int N_t(i) di \quad \text{and} \tag{9}$$

$$Y_t = C_t + G_t \tag{10}$$

The bond market clearing condition is already embedded in the notation, as I use the same notation, B_t, in the representative household budget constraint and the government budget constraint.

2.5 An Implementable Equilibrium

Given an initial level of debt, B_0, the distribution of initial prices $P_{i,0}$ for all $i \in [0,1]$, and a sequence of discount factor shocks $\{\delta_t\}_{t=1}^{\infty}$, an equilibrium of this economy consists of $\{C_t, N_t, Y_t,\ P_{i,t}, G_t, R_t, T_t, \tau_{n,t}, \tau_{c,t}, B_t\}_{t=1}^{\infty}$ such that (i) $\{C_t, N_t, B_t\}_{t=1}^{\infty}$ solves the household problem, (ii) $\{P_{i,t}\}_{t=1}^{\infty}$ solves the firms' problem, (iii) the government budget constraint is satisfied, and (iv) all markets clear. It is straightforward to show that an equilibrium can be recursively characterized by $\{C_t,\ N_t,\ Y_t,\ \Pi_t,\ p_t^*,\ s_t,\ C_{d,t},\ C_{n,t},\ R_t,\ G_t,\ T_t,\ \tau_{n,t},\ \tau_{c,t},\ b_t\}_{t=1}^{\infty}$ satisfying the following set of equations:

$$\frac{C_t^{-\chi_c}}{(1+\tau_{c,t})R_t} = \beta\delta_t \frac{C_{t+1}^{-\chi_c}\Pi_{t+1}^{-1}}{1+\tau_{c,t+1}}, \tag{11}$$

$$\frac{1-\tau_{n,t}}{1+\tau_{c,t}} w_t = N_t^{\chi_n} C_t^{\chi_c}, \tag{12}$$

$$s_t = (1-\zeta_p)\left[p_t^*\right]^{-\theta} + \zeta_p \Pi_t^{\theta} s_{t-1}, \tag{13}$$

$$1 = (1-\zeta_p)\left[p_t^*\right]^{1-\theta} + \zeta_p \Pi_t^{\theta-1}, \tag{14}$$

$$p_t^* = \frac{\theta}{\theta-1}\frac{C_{n,t}}{C_{d,t}}, \tag{15}$$

$$C_{n,t} = \frac{1}{1+\tau_{c,t}} Y_t w_t C_t^{-\chi_c} + \zeta_p \beta\delta_t \Pi_{t+1}^{\theta} C_{n,t+1}, \tag{16}$$

$$C_{d,t} = \frac{1}{1+\tau_{c,t}} Y_t C_t^{-\chi_c} + \zeta_p \beta\delta_t \Pi_{t+1}^{\theta-1} C_{d,t+1}. \tag{17}$$

$$Y_t s_t = N_t, \tag{18}$$

$$Y_t = C_t + G_t, \tag{19}$$

$$\mathsf{GBC_t}, \quad \text{and} \tag{20}$$

$$R_t \geq 1, \tag{21}$$

where $\Pi_t := \frac{P_t}{P_{t-1}}$, $w_t := \frac{W_t}{P_t}$, and $b_t \equiv \frac{B_t}{P_t}$. p_t^* is the price set by the optimizing firms normalized by the aggregate price, and s_t is cross-sectional price dispersion defined as $s_t := \int_0^1 (\frac{P_t(i)}{P_t})^{-\theta_t} di$. $C_{d,t}$ and $C_{n,t}$ are auxiliary variables introduced to describe the equilibrium conditions recursively. Notice that the equilibrium depends on the initial real debt b_0 and the initial price dispersion s_0. The equation for the government budget constraint, $\mathsf{GBC_t}$, is given by one of the three equations (6-8), depending on which model is being analyzed.

2.6 Government's Problem

The government's problem is to find an allocation that maximizes the household welfare and price/policy variables that decentralize the allocation as an equilibrium. Given b_0 and s_0 , the government optimization problem is given by

$$\max_{\{u_t\}_{t=1}^T} \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n} + \chi_{g,0} \frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}} \Big], \tag{22}$$

subject to the set of equations characterizing the implementable equilibria where

$$u_t := [C_t, N_t, Y_t, w_t, p_t^*, \Pi_t, C_{d,t}, C_{n,t}, R_t, G_t, \text{one of } (T_t, \{\tau_{c,t}, b_t\}, \{\tau_{n,t}, b_t\})]. \tag{23}$$

Let ω_t be the vector of the Lagrangean multipliers on the equations characterizing the implementable equilibria at period t. Given b_0 and s_0, *the Ramsey equilibrium* is defined as a sequence of u_t and ω_t satisfying the first-order necessary conditions of this government problem, and *a Ramsey steady state* is defined as a set of scalars, $u_{ss} := \{C_{ss}$, N_{ss}, Y_{ss}, w_{ss}, p_{ss}^*, Π_{ss}, $C_{d,ss}$, $C_{n,ss}$, R_{ss}, G_{ss}, one of $[T_{ss}$, $(b_{ss}$, $\tau_{n,ss})$, $(\tau_{c,ss}$, $b_{ss})]$, $\omega_{1,ss}$, $\omega_{2,ss}$, ... ,$\omega_{11,ss}\}$ satisfying the first order necessary conditions of the Lagrangean problem for $t \geq 2$.

Modified government's problem: As in Ramsey equilibria in many other contexts, the first order necessary conditions at $t = 1$ differ from those at $t \geq 2$. Therefore, even without any discount factor shocks, the solution exhibits initial dynamics before it converges to a terminal Ramsey steady state.[1] In order to focus on the economy's response to the discount rate shock, I will take a *timeless perspective* and modify the government's problem so as to eliminate the initial dynamics that would prevail even in the absence of any shock. Specifically, following

[1] In this model, if the initial debt level is larger than the one associated with the Ramsey steady state with the highest value, the government will reduce the level of debt at time one by levying a high labor income or consumption tax.

Khan, King, and Wolman (2003), the government's objective function is modified to include penalty terms involving hypothetical time-zero Lagrange multipliers.[2] Given the initial debt, b_0, the modified objective function of the government is given by

$$\sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n} + \chi_{g,0} \frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}} \Big] + p(\omega_{1,ss}, \omega_{6,ss}, \omega_{7,ss} u_1),$$

where the penalty term, $p(\cdot)$, is given by

$$p(u_{ss}, u_1) := -\omega_{1,ss} \frac{C_1^{-\chi_c}}{1+\tau_{c,1}} \Pi_1^{-1} - \omega_{6,ss} \zeta_p \Pi_1^{\theta} C_{n,1} - \omega_{7,ss} \zeta_p \Pi_1^{\theta-1} C_{d,1},$$

where $\omega_{1,ss}$, $\omega_{6,ss}$, $\omega_{7,ss}$ are the Ramsey steady-state values of Lagrange multipliers associated with b_0.

3 Parameter Values and Solution Method

3.1 Parameter Values

Table 1 lists baseline parameter values. The discount rate (β), the risk aversion parameter (χ_c), the substitutability of intermediate goods (θ), and the Calvo price parameter (α) are standard. For the parameters describing the household's preference for government spending, I choose $\chi_{g,0}$, the weight on the utility from government spending relative to the utility from private consumption, to be 0.25 so that the steady-state level of government spending to output ratio is roughly about 20 percent. I set the inverse of the intertemporal elasticity of substitution for government spending, $\chi_{g,1}$, to be unity. I will consider alternative parameter values in the appendix to show the robustness of the results.

Following Levin, López-Salido, Nelson, and Yun (2010), I set the magnitude of the shock, $\epsilon_{\delta,1}$, to be 0.02, which makes the natural rate of interest rate negative five percent at period one, and the persistence of the shock, ρ_δ, to be 0.85. They have demonstrated that a shock of this magnitude and persistence leads to a large output decline even when the government has a commitment technology, and has suggested that fiscal and financial policies may play an important role.

In the model with a distortionary tax and debt, the initial debt level, b_0, affects the dynamic response of the economy to the discount rate shock. In the baseline, I choose b_0 so that the ratio of debt to annualized output in a Ramsey steady state associated with $b_{ss} = b_0$ is 0.5, which is slightly above the ratio of publicly held debt to GDP in the U.S. during

[2] The thought experiment behind this modified government's problem is as follows. Suppose that the government solved the Ramsey problem a long time ago, and that the economy is at its Ramsey steady-state at t=0. Now, if the government were hypothetically given an opportunity to reoptimize at t=1, it would use this opportunity to improve the welfare from that point on by deviating from what it promised at t=0. The terms involving the time-zero Lagrangian multiplier that appear in the modified objective function penalizes such a deviation so that, in the absence of any shocks, the government continues to choose the same allocation and policy as it chose at time zero.

Table 1: Parameterization

Parameter	Description	Parameter Value
β	Discount factor	$\frac{1}{1+0.075}$
χ_c	Inverse intertemporal elasticity of substitution for C_t	$\frac{1}{6}$
χ_n	Inverse labor supply elasticity	1.0
$\chi_{g,0}$	Utility weight on G_t	0.25
$\chi_{g,1}$	Intertemporal elasticity of substitution for G_t	1.0
θ	Elasticity of substitution among intermediate goods	10
α	Calvo parameter	0.75
$b_0/4Y_0$	Initial debt-to-output ratio	0.5
ϵ_δ	The size of the discount factor shock	0.02
ρ_δ	AR(1) coefficient for discount factor shock	0.85

the fiscal year 2008, a period just before the Federal Reserve lowered the policy rate to the effective lower bound.[3] A key exercise of the paper will be to examine the effects of the initial debt level on government spending and nominal interest rate policies.

3.2 Solution Method

I use a variation of the Newton method to solve the model in its original nonlinear form. The method is a further modification of a modified Newton algorithm by Juillard, Laxton, McAdam, and Pioro (1998). They modify a standard Newton algorithm so as to take advantage of the recursive structure common in infinite-horizon structural models. However, their algorithm requires the knowledge of the terminal steady state. In the model with a distortionary tax and debt, the terminal steady-state values are unknown quantities to be solved for. Thus, I embed the Newton algorithm in a shooting algorithm in which the terminal steady state is searched by a bisection method. In each iteration of the shooting algorithm, I guess the terminal steady-state level of debt, solve the model by the Newton method, and check whether or not the endpoint of the solution is consistent with the steady state associated with the guess of the terminal debt level. The details of the solution method are described in the appendix B.

4 Ramsey Steady States

While there is a unique Ramsey steady state in the model with a lump-sum tax, there are infinite Ramsey steady states in the model with debt, each indexed by the level of debt. In the models with a distortionary tax and debt, the terminal steady state is typically different from the initial steady state and, as a result, government spending policy affects allocations not only in the short run, but also in the long run. To prepare ourselves for discussions of

[3]The ratio was 0.394 in the United States (Source: St. Louis Fed's FRED; Gross Federal Debt Held by the Public as Percentage of Gross Domestic Production in 2008).

long-run effects of government spending policy later, it is useful to understand several key properties of the Ramsey steady states in the model with debt at this stage.

Figure 1: Ramsey Steady States in the Model with Labor Income Tax and Debt

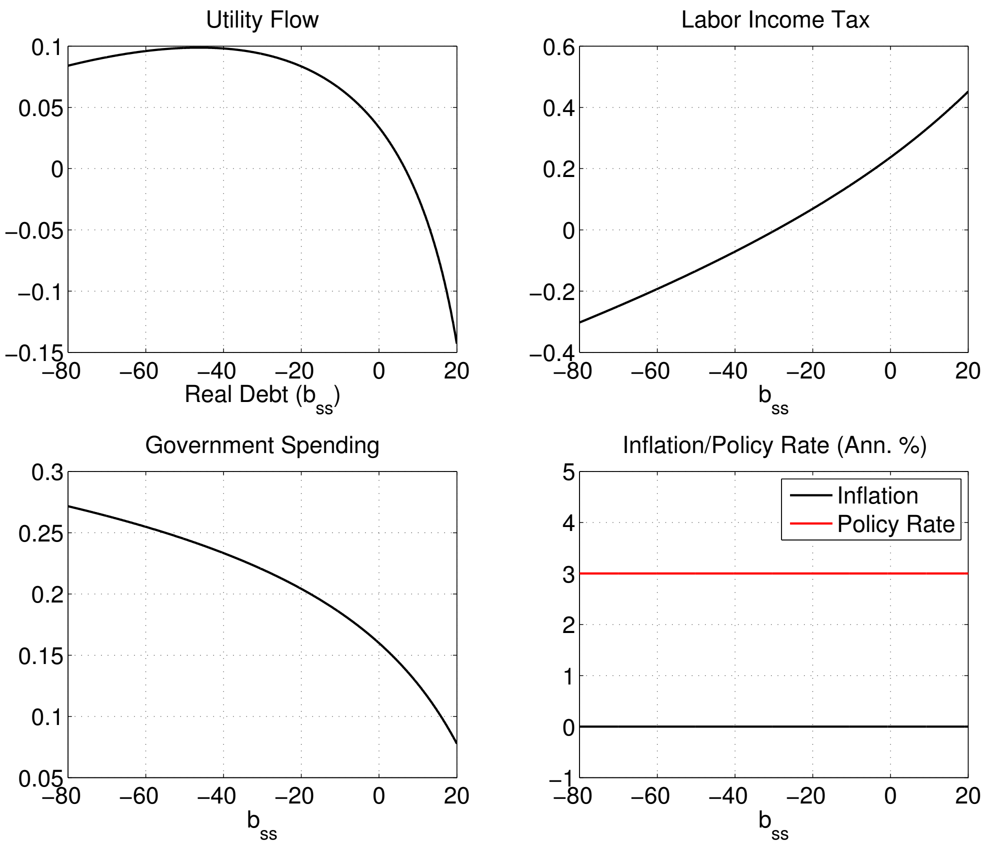
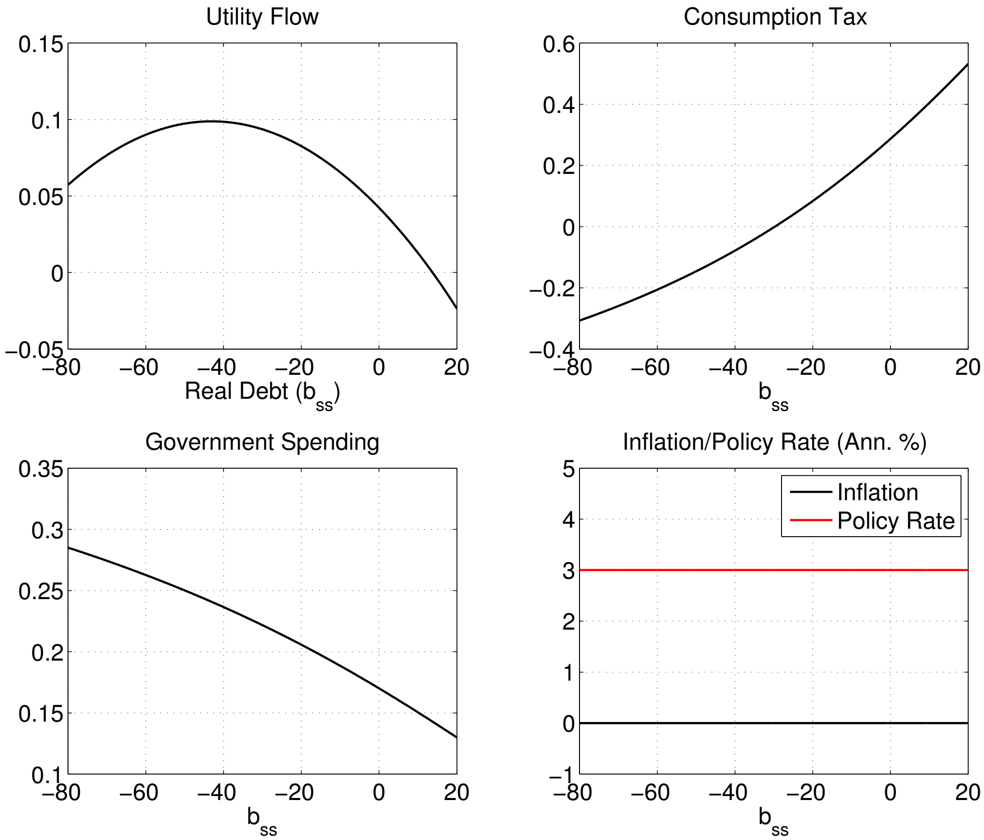


Figure 1 shows utility flows, tax rates, government spending, and inflation at various Ramsey steady states associated with different levels of debt for the model with labor income tax and debt. Figure 2 shows the same for the model with a consumption tax and debt. In both models, the larger debt is associated with a higher tax rate, as the higher tax rate allows the government to cover the higher interest expenses on its debt. The Ramsey steady state with the highest period-flow utility is the one in which the government has large asset holdings—large enough to pay for the labor income subsidy or consumption subsidy that eliminates the distortion arising from imperfect competition in the product market. The government spending is lower in the steady state with larger debt.

Since one Ramsey steady state yields the best utility flow, the existence of alternative steady states may strike one as odd at first. Why wouldn't the government adjust the debt level if the debt level differs from the one associated with the highest utility? The reason is that the government's desire to inflate away the nominal debt, or alternatively the desire to deflate away the asset holdings, to set the real value of the debt at the level consistent with the highest utility flow, is countered by the production inefficiency associated with price dispersion caused by such inflation or deflation. Constrained by the given initial debt level,

Figure 2: Ramsey Steady-States in the Model with Consumption Tax and Debt



the government chooses not to pay these costs in order to move to the Ramsey steady state with the highest utility flow.

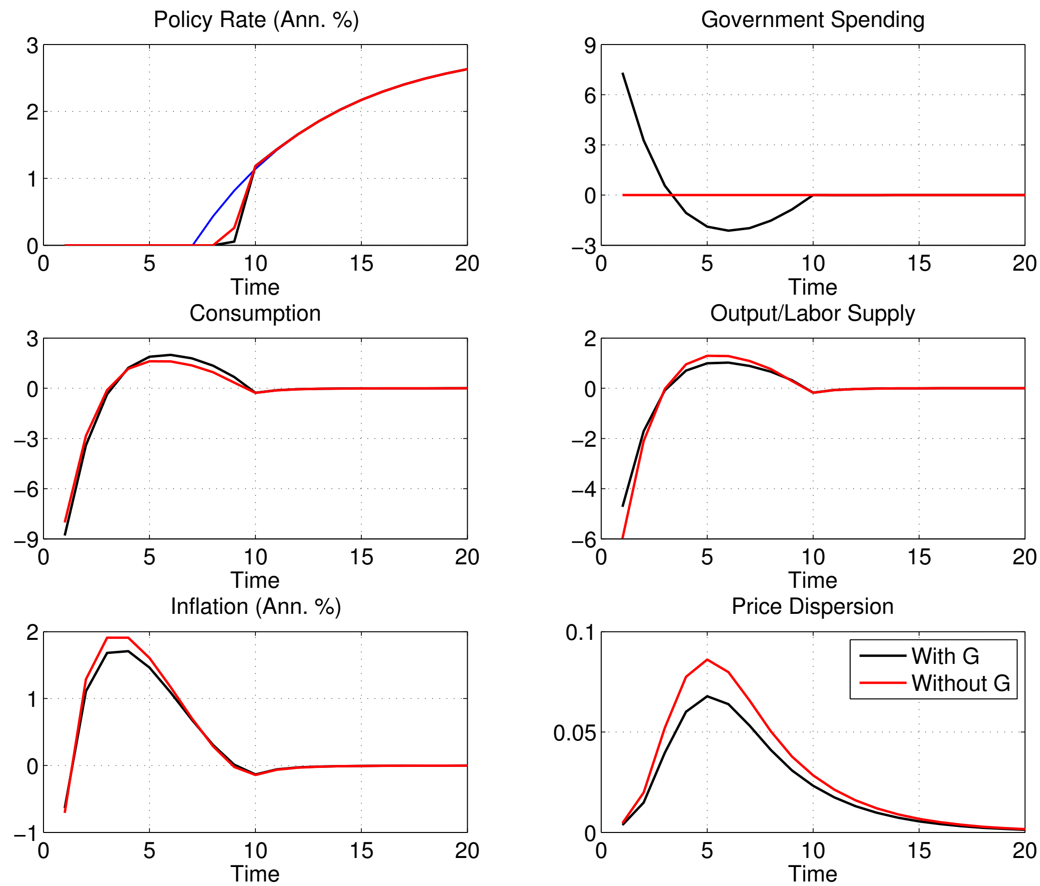
While some variables take different values across different Ramsey steady states, other variables take the same values in any Ramsey steady states. For example, as shown in the black line of the bottom-right panels of figures 1 and 2, inflation is zero in any steady state for both models of a distortionary taxation. Zero inflation is optimal because it leads all firms to set an identical price, which minimizes the misallocation of labor inputs across firms. With zero inflation, the nominal interest rate is $\frac{1}{\beta}$ at any Ramsey steady state, as shown in the red line of the same panel. Accordingly, the optimal reset price (p^*_{ss}) and the price dispersion (s_{ss}) are one in any Ramsey steady state.

5 Optimal Policy

5.1 Model with Lump-Sum Taxation

Figure 3 shows the impulse response functions of the model's key variables in the economy with lump-sum taxation. Black and red lines are for the unconstrained economy where the government can adjust its spending and the constrained economy where the government is not allowed to adjust its spending from the initial steady-state level, respectively.

Figure 3: Optimal Policy in the Model with Lump-Sum Taxation



*All variables are expressed as a percentage deviation from its initial steady-state value unless indicated otherwise.

Regardless of the availability of government spending, optimal interest rate policy is characterized by an extended period of holding the policy rate at zero. The natural rate of interest—the nominal interest rate that would completely neutralize the discount rate shock—is positive after time 7, so it is feasible for the government to stabilize the economy from that point on. However, the government can improve welfare by promising to keep the policy rate at the ZLB longer. Overshooting of consumption, output, and inflation associated with the extended period of zero nominal interest rate mitigates the initial declines in consumption, output, and inflation because the private sector agents are forward looking.

Optimal government spending policy is characterized by a mirror image of the optimal path of consumption and output. As shown in the top-right panel, the government spending increases initially, declines below its steady state, and eventually returns to the steady state. This pattern reflects the desire of the government to balance the marginal utilities of government spending and leisure. When output is below the steady-state level, leisure is high and thus the marginal cost of increasing leisure (i.e., reducing output) is low relative to the marginal benefit of increasing government spending. Thus, the government can improve welfare by increasing output/labor supply and the government spending. The consumption path is slightly more volatile as a result of this government spending policy, but this cost is countered by the benefit of a better mix of government spending and output. These results are consistent with those in Werning (2012), Nakata (2013b), and Schmidt (2013).

Optimal variation in government spending is modest. The initial increase is about 7 percent of its steady-state level, which is about 1.5 percent of the steady-state output. The output multiplier at time 1 is roughly one, and variations in the government spending have no significant effect on consumption. One key difference between the constrained and unconstrained economies is in the response of price dispersion; price dispersion is smaller in the unconstrained economy than in the constrained economy, reflecting a more subdued rise in inflation in the unconstrained economy when the ZLB is binding.

The first row of Table 2 shows the welfare cost of the ZLB with and without government spending policy. The table shows how much consumption goods, as a percentage of the steady-state consumption, the government needs to give to the household in the economy with the ZLB at time one so that she is as well off as the household in the hypothetical economy without the ZLB. The welfare cost of the ZLB is 1.44 percent with government spending policy while it is 1.63 percent without it. The welfare gain of government spending policy—the difference between the welfare costs of the ZLB with and without government spending policy—is about 0.2 percentage points.

Table 2: Welfare Effects of Government Spending Policy

	Welfare Cost of the ZLB		
	All variables chosen optimally	G_t fixed	Welfare Gain of Government Spending Policy
Model with lump-sum tax	1.44	1.63	(0.19)
Model with labor income tax/debt	1.36	1.54	(0.18)
Model with consumption tax/debt	0.06	0.07	(0.01)

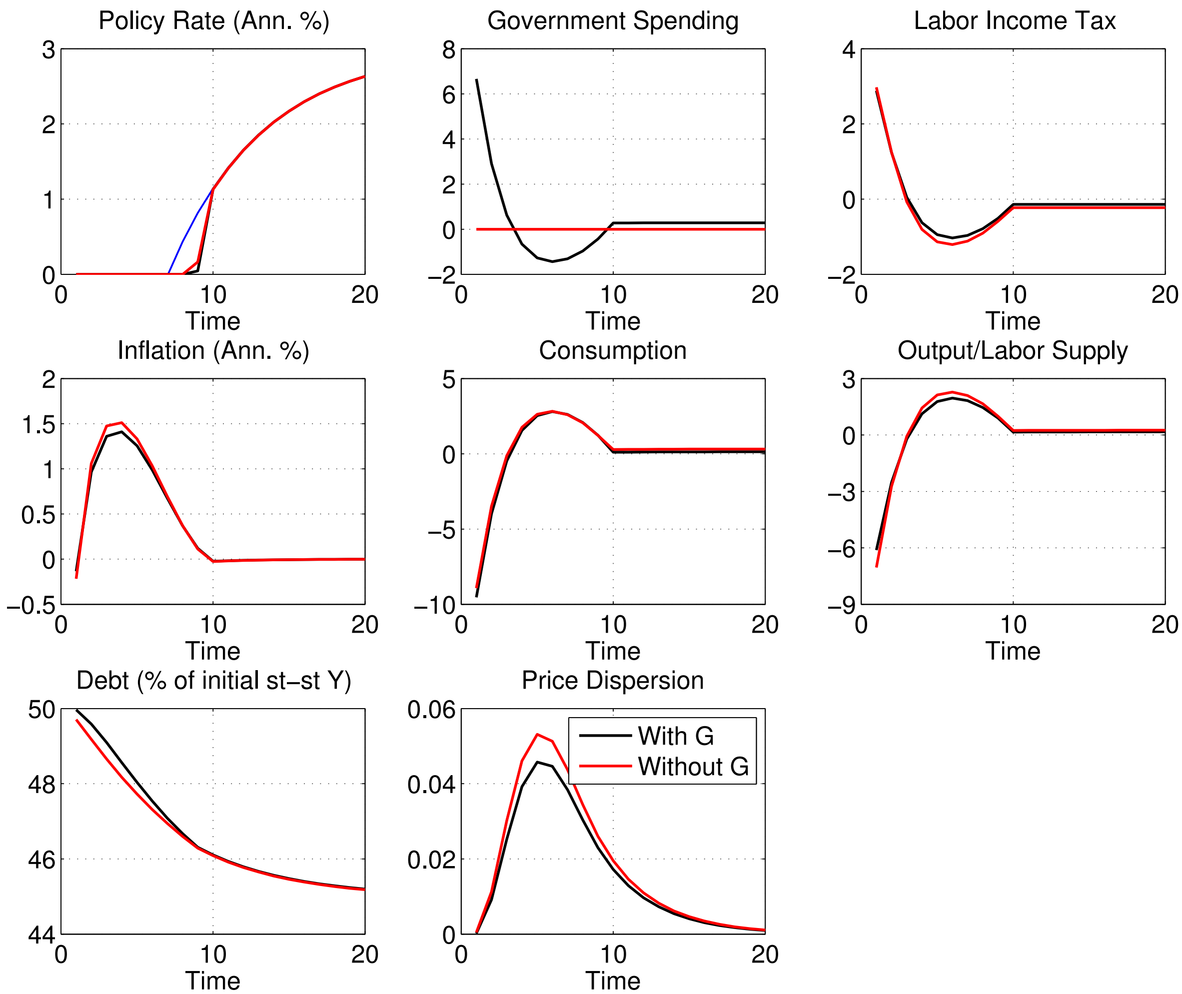
5.2 Model with Labor Income Tax and Debt

Figure 4 shows the impulse response functions of the model's key variables in the economy with labor income taxation and debt. Black and red lines are for the unconstrained and the constrained economies, respectively.

As in the model with lump-sum taxation, optimal interest rate policy is characterized by an extended period of holding the policy rate at the ZLB. Optimal allocations are again characterized by overshooting of consumption, output, and inflation while the policy rate is at the ZLB. These promises of overshooting mitigate their initial declines through expectations.

Optimal government spending policy is characterized by an initial expansion followed by a decline below, and eventual return to, the terminal steady state, as in the model with lump-sum taxation. The optimal path of the labor income tax rate follows the same pattern, rising

Figure 4: Optimal Policy in the Model with Labor Income Taxation and Debt



*All variables are expressed as a percentage deviation from its initial steady-state value unless indicated otherwise.

and falling together with government spending. Similar to the model with lump-sum taxation, an increase in the government spending at the time when output is low increases welfare because it would lead to a better mix of the government spending and output. The initial increase in the labor tax rate improves welfare because an increase in the labor income tax increases the marginal costs of production and mitigates the deflationary pressure at the ZLB. The beneficial effects of increasing labor income tax rate while the policy rate is constrained at the ZLB is consistent with the result in Eggertsson (2011), who has demonstrated the expansionary effects of exogenous changes in the labor income tax rate at the ZLB.

The economy converges to a terminal steady state that is different from the initial steady state. Despite the initial expansion in government spending, debt declines steadily from time one. Debt declines because a prolonged period of below-trend nominal interest rates implies lower interest rate expenses. As a result, debt eventually settles at a new steady state where the debt-to-output ratio is lower than that in the initial steady state. Since the debt level is lower, consumption, output, and government spending are higher, and the labor income tax rate is lower, in this terminal steady state than in the initial steady state. This is consistent with the earlier analysis of the Ramsey steady states in section 4.

The initial increase in government spending is modest, about 7 percent of its initial steady-state level, and is roughly the same as that in the model with lump-sum taxation. As in the model with lump-sum taxation, a rise in inflation is more subdued with government spending policy than without it. As a result, price dispersion rises by a smaller amount. Without access to government spending policy, debt declines by more and converges to a new steady-state level that is slightly lower than the level where it would converge with the government spending.

With the size of variations in government spending being roughly the same as in the model with lump-sum taxation, the welfare implication of government spending policy in this model is similar to that in the model with lump-sum taxation. As shown in the second row of table 2, the welfare cost of the ZLB is 1.36 percent with government spending policy while it is 1.54 percent without it. The welfare gain of government spending policy is about 0.2 percentage points.

5.3 Model with Consumption Tax and Debt

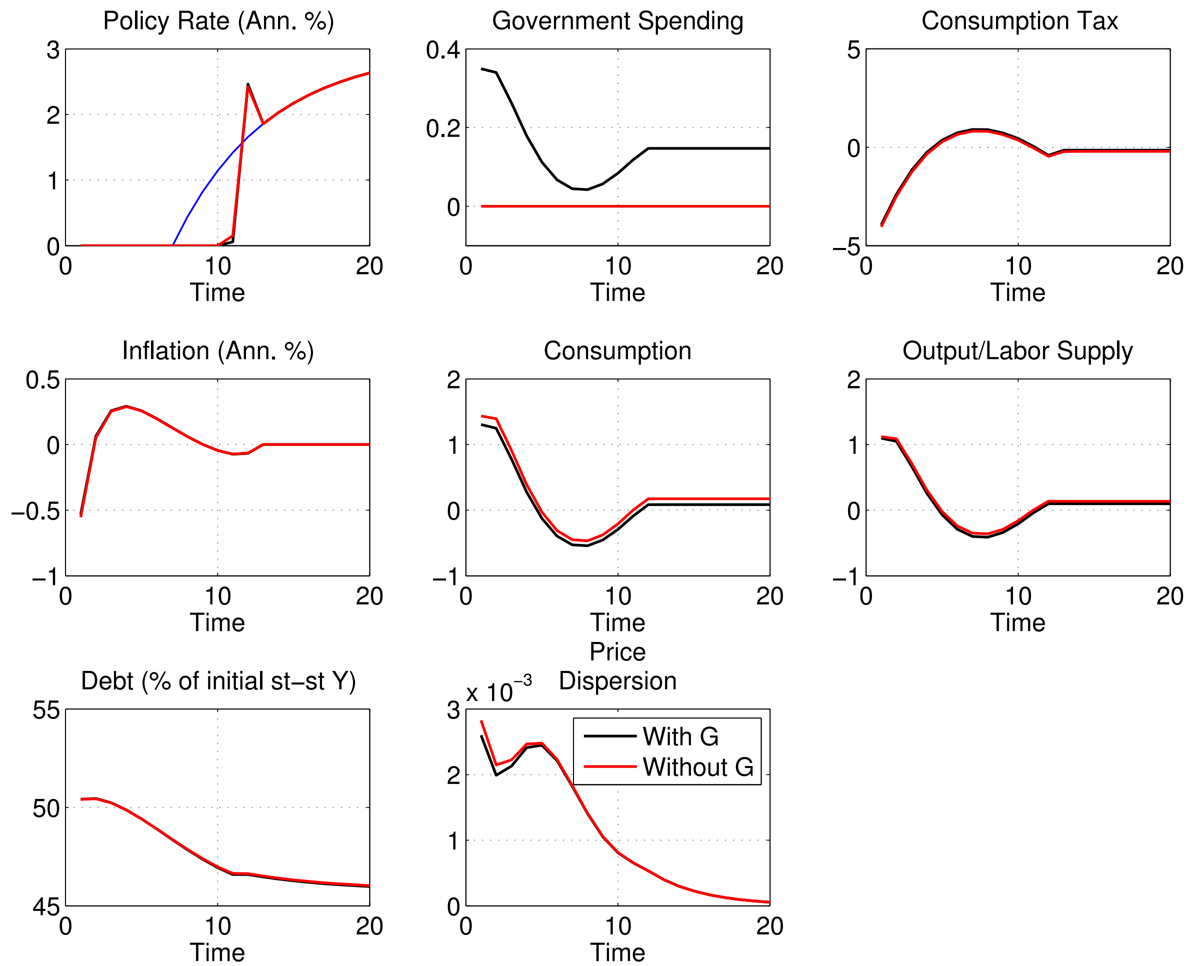
Figure 5 shows the impulse response functions of the model's key variables in the model with a consumption tax and debt. Black and red lines are for the unconstrained and the constrained economies, respectively.

As in the other two models, optimal interest rate policy is characterized by an extended period of keeping the policy rate at the ZLB. Inflation overshoots while at the ZLB, which mitigates the initial decline in inflation.

Optimal path of consumption tax is characterized by an initial decline followed by an overshooting and eventual return to the new steady state. An *expected change* in the consumption tax affects the household's intertemporal decision in the same way as the *level* of the interest rate affects it. From the consumption Euler equation (11), it can be seen that an expected increase in the consumption tax rate induces the household to spend more today and less tomorrow, just as a reduction in the nominal interest does so. As a result, the government can stimulate consumption today by promising to increase the consumption tax rate in the future. In an environment where the household discount rate is high, the government finds it optimal to stimulate consumption by reducing the consumption tax rate initially and raising it afterwards. This consumption tax policy induces the household to even *increase* consumption initially.

Optimal government spending policy follows the same pattern as that in the models with lump-sum taxation and labor income taxation. In this model, consumption is initially higher than the steady-state level, so the marginal cost of reducing consumption is low. Thus, the

Figure 5: Optimal Policy in the Model with Consumption Taxation and Debt



*All variables are expressed as a percentage deviation from its initial steady-state value unless indicated otherwise.

government can increase welfare by increasing government spending and reducing consumption as a result. Increasing government spending has the cost of increasing output—thereby reducing the utility flow of leisure—and the optimal increase in government spending balances this cost against the benefit of a better mix of consumption and government spending.

As in the model with labor income taxation, a prolonged period of below-trend nominal interest rates leads debt to decline, and the debt eventually converges to a level that is lower than the level that prevailed in the initial steady state. Accordingly, consumption, output, and government spending are higher, and the consumption tax is lower, in the terminal steady state than in the initial steady state.

While the pattern of government spending policy is the same as in the other two models, the magnitude of the variation is different: It is much smaller in this model with consumption tax. Since the expected change in the consumption tax rate acts as a substitute for the interest rate in mitigating the adverse effects of the preference shock, the ZLB constraint on the nominal interest rate is less destabilizing. Consumption and output are much more stabilized in this model than the other two models. There is not much left for the government spending policy to contribute.

Reflecting the small variation in government spending in this model, the welfare gain of government spending policy is small. According to the last row of table 2, the welfare cost of the ZLB is 0.06 percent with government spending policy while it is 0.07 percent without it. The welfare gain of optimally choosing government spending is only 0.01 percentage points, substantially lower than those in the models with lump-sum tax and labor income tax/debt.

5.4 Importance of the Initial Debt Level

Thus far, we have analyzed the economies with a distortionary tax and debt, assuming a specific initial debt level that is broadly consistent with the debt level in the U.S. a year before the Great Recession. However, some countries have larger debts than others. I now discuss how optimal policy depends on the initial debt level.

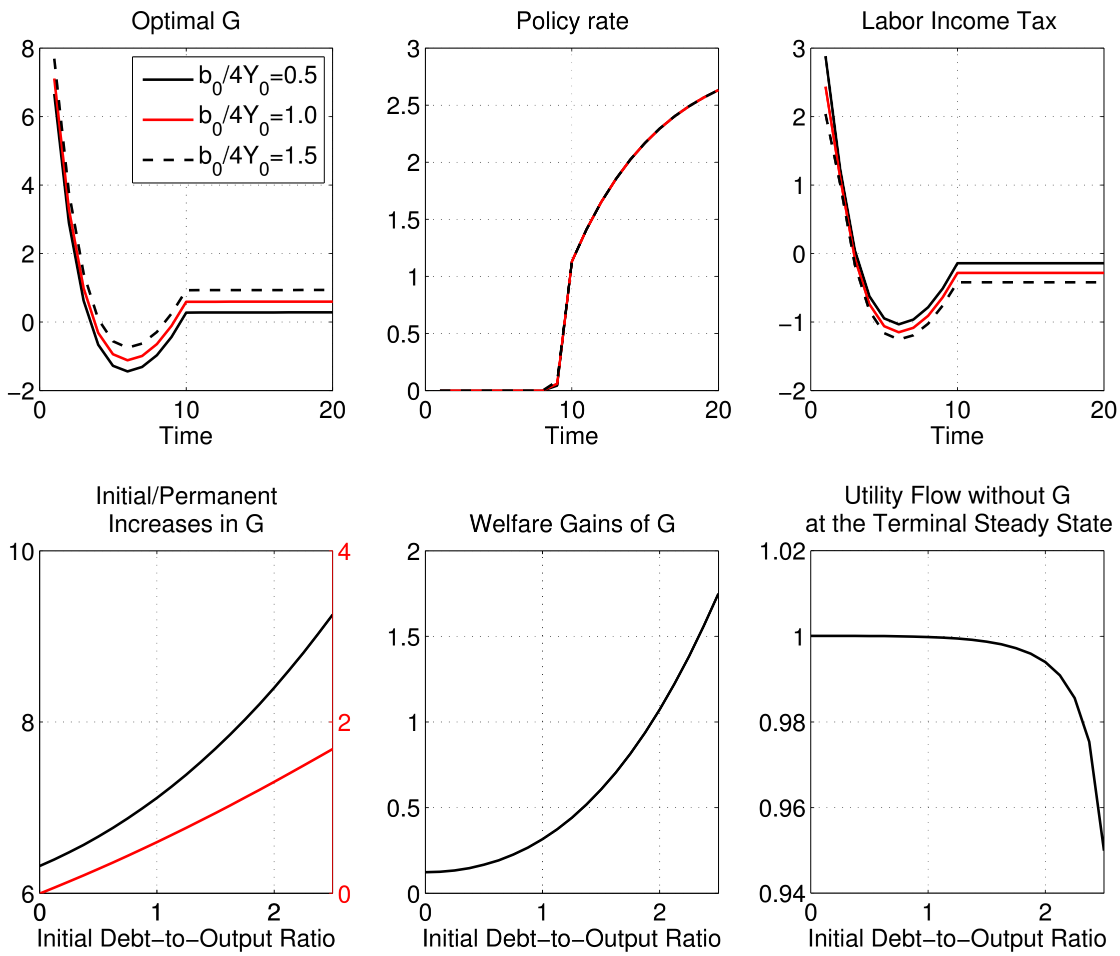
Model with labor income tax and debt: The top panels of figure 6 show the impulse response function of government spending, the nominal interest rate, and the labor income tax rate for the economies with labor income taxation with three alternative initial debt-to-GDP ratios (0.5, 1.0, 1.5).

According to the figure, the patterns of optimal policies do not depend on the initial debt-to-output ratio. Optimal government spending policy is characterized by an initial expansion followed by an undershooting and eventual return to the terminal steady state. Optimal nominal interest rate policy is characterized by an extended period of keeping the policy rate at the ZLB. Optimal labor income tax policy is characterized by an initial increase followed by reduction below, and the eventual return to, the terminal steady state.

While the pattern of optimal government spending policy does not depend on the initial debt level, the magnitude of variation does. In particular, the initial expansion in government spending is larger with a higher debt, and government spending converges to a higher level (relative to the initial steady-state level) when the debt is higher, as shown in the bottom-left panel of figure 6. This result arises because, given that the nominal interest rate policy is invariant to the initial debt level, the reduction in the debt from the initial steady state to the terminal steady state is larger when the initial debt is higher. A reduction in debt means that the economy is closer to the efficient levels: The tax rate is lower, and consumption, output, and government spending are higher. Thus, a larger reduction in debt means that the consumption, output, and government spending increase by more from their initial steady-state levels.

The welfare gain of government spending policy is larger with a higher initial debt, as shown in the bottom-middle panel of figure 6. In the model with debt, the government spending policy not only has short-run stabilization effects, but also the long-run effects on

Figure 6: Optimal Policy with Alternative Initial Debt Levels:
Model with Labor Income Taxation and Debt



*For the bottom-left panel, the black and red lines are for the initial and permanent increases in the govenment spending, respectively.
**For the bottom-right panel, "utility flow without G at the terminal steady state" shows the ratio of the utility flow without government spending policy to the utility flow with government spending policy.

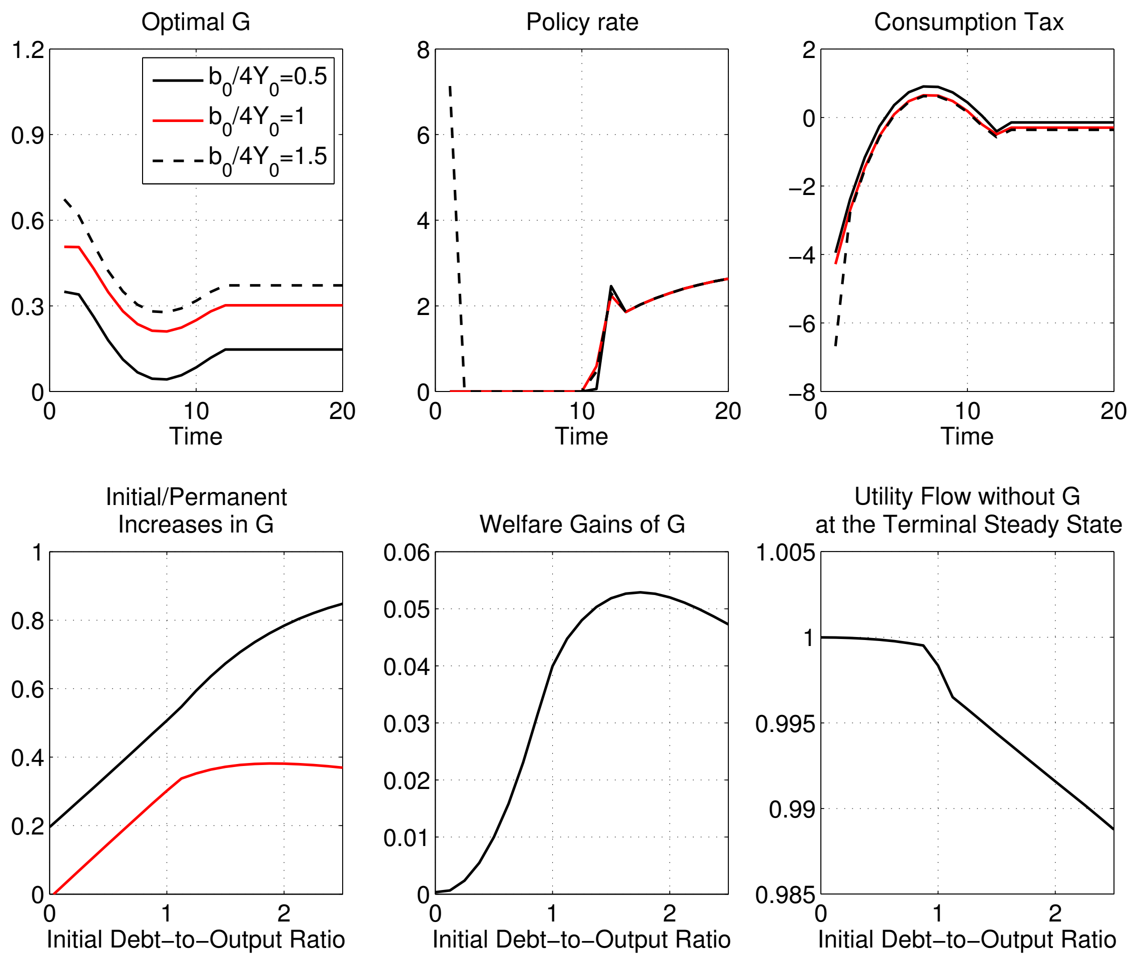
the terminal steady state: It allows the economy to settle in a terminal steady state with a higher utility flow. As we have seen, in the model with debt, the economy will settle in a steady state with lower debt and higher government spending as a consequence of the discount rate shock. If the government is not allowed to adjust its spending from the initial level, then the government spending is suboptimally low in the terminal steady state. This long-run effect is absent in the model with lump-sum taxation.

Since the optimal government spending level in the terminal steady state exceeds that of the initial steady state by more when the initial debt is higher, the lack of access to government spending reduces welfare *at the terminal steady-state* by more when the initial debt is higher, as shown in the bottom-right panel. As a result, the *overall welfare cost* of not being able to adjust government spending is higher when the initial debt is higher.[4]

[4]As discussed shortly, the increase in the terminal utility flows does not *necessarily* mean that the wel-

Model with consumption tax and debt: Figure 7 shows the same set of panels for the model with a consumption tax and debt.

Figure 7: Optimal Policy with Alternative Initial Debt Levels:
Model with Consumption Taxation and Debt



*For the bottom-left panel, the black and red lines are for the initial and permanent increases in govenment spending, respectively.
**For the bottom-right panel, "utility flow without G at the terminal steady state" shows the ratio of the utility flow without government spending policy to the utility flow with government spending policy.

The patterns of optimal government spending and consumption tax policies are invariant to the initial debt level. While optimal government spending policy is characterized by the initial increase followed by the reduction below, and the eventual convergence to, the new steady state, optimal consumption tax policy is characterized by a mirror image of optimal government spending path.

The pattern of the nominal interest rate policy is also invariant to the initial debt level

fare gain, as measured by the time-one transfer of consumption as a percentage of the initial steady-state consumption, is larger, because the initial steady-state consumptions are different across different initial debt levels.

when the initial debt level is below a certain threshold; the government lowers the policy rate to zero at time one and keeps it there for an extended period of time. When the debt level is above that threshold, optimal policy prescribes that the government not lower the policy rate to the ZLB. This result is obtained because the changes in consumption tax rates acts as a substitute for the nominal interest rate policy. When the debt level is sufficiently high, the initial decline in the consumption tax rate is so large and a positive nominal interest rate is needed to restrain consumption. Consistent with this nominal interest rate policy, the Lagrange multiplier on the ZLB constraint at time one becomes zero when the initial debt level is sufficiently high, as shown in figure 8. When the initial debt level is such that optimal policy rate is positive at time one, the policy rate is higher with a higher initial debt.

The welfare gain of government spending policy varies with the initial debt level in a non monotonic way, as shown in the bottom-middle panel of figure 7. It increases with the initial debt level up to a point for the same reason as in the model with a labor income taxation and debt, namely that the access to government spending policy allows the economy to settle in a steady state that is more efficient. However, the increase in the terminal utility flows does not necessarily mean that the welfare gain, as measured by the time-one transfer of consumption as a percentage of the initial steady-state consumption, is larger, because the initial steady-state consumptions are different across different initial debt levels. When the initial debt-to-output ratio is sufficiently high—higher than 1.75 in this parameterization—higher initial debt is associated with a lower welfare gain of government spending. Nevertheless, for the range of initial debt levels considered in this paper, even when the initial debt-to-output ratio is higher than 1.75, the welfare gain of government spending policy remain substantially higher than that in the baseline parameterization with the initial debt-to-output ratio of 0.5.

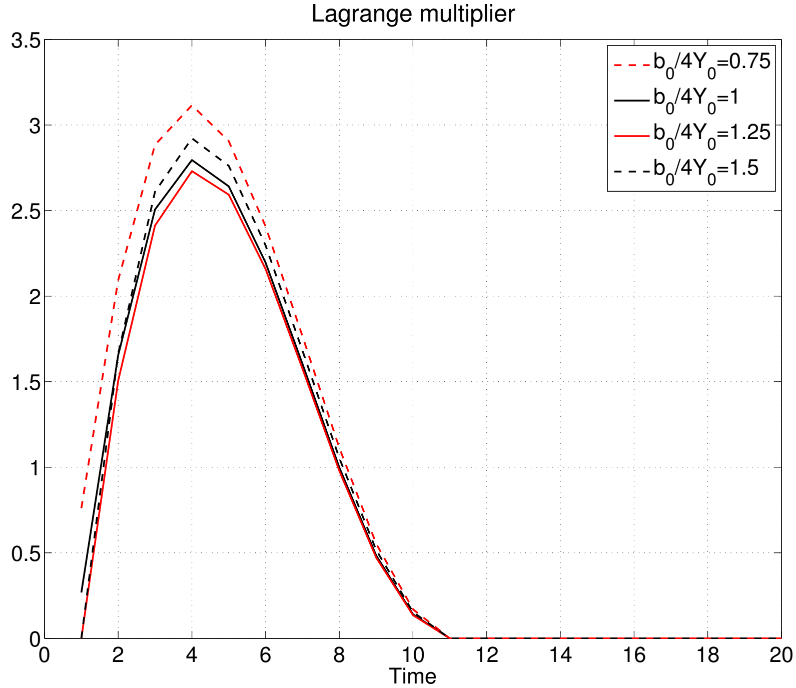
5.5 Sensitivity Analyses

Various results shown thus far are robust to alternative values of structural parameters. For the sake of brevity, the robustness of the results are shown in the appendix.

6 Conclusion

The paper has characterized optimal government spending and monetary policy when the nominal interest rate is subject to the the ZLB constraint in a sticky-price model. I departed from the previous literature by analyzing the economies with distortionary taxation and nominal debt. I have shown that, regardless of the tax instrument, optimal government spending policy is characterized by an initial expansion followed by a sharp decline below, and an eventual return to, the steady state. While the pattern of optimal government spending policy does not depend on the available tax instrument, its magnitude does. In particular, the amount of variation in government spending is much smaller in the model with a consumption tax than in the model with a labor-income or lump-sum tax. I also documented

Figure 8: The Shadow Value of the ZLB Constraint
in the Model with Consumption Taxation and Debt



that the magnitude of optimal variations in the government spending and the welfare gain from government spending policy increase with the initial debt level. If the economy starts with a large initial debt, the welfare gains can be large as the access to government spending policy improves welfare not only in the short run, but also in the long run.

In this paper, I have broken the Ricardian equivalence by abandoning the lump-sum tax assumption and introducing a distortionary tax into the model. There are many other plausible ways in which the Ricardian equivalence may fail in reality, such as the presence of liquidity-constrained households or the presence of frictions in the labor or financial markets. Some have recently studied the implications of these frictions for fiscal multipliers at the ZLB.[5] It would be useful to study their implications for optimal government spending policy in future research.[6]

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[5]For example, Carrillo and Poilly (2013) analyze the government spending multiplier in a model with financial frictions. Roulleau-Pasdeloup (2014) examines the government spending multiplier in a model with search and matching frictions in the labor market.

[6]Bilbiie, Monacelli, and Perotti (2014) study the welfare effects of exogenous changes in government spending in the model where a fraction of households do not have access to the bond market.

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Appendices

Appendix A demonstrates the robustness of key results in the main text to alternative values of key structural parameters. Appendix B explains the solution method. Appendix C presents the Lagrangean problem of the government and its first-order necessary conditions.

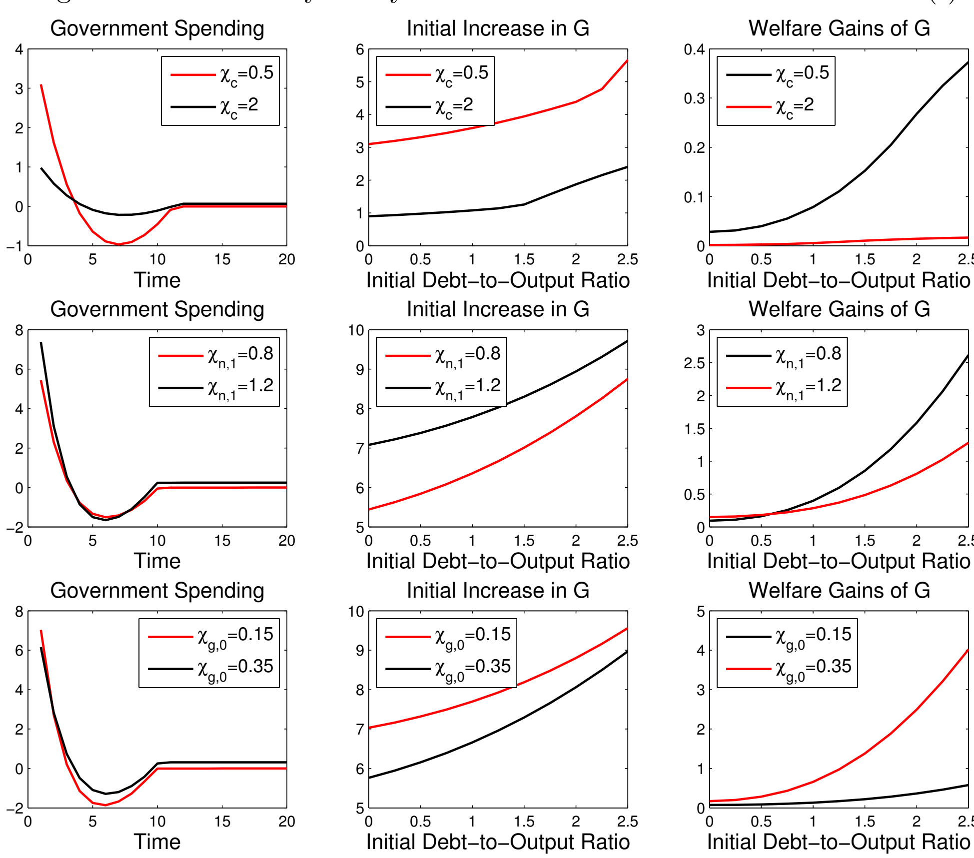
A Robustness

This section documents the robustness of key features of optimal policy in the models with a distortionary tax and debt. The results I focus on are (1) the pattern of optimal government spending policy, (2) how the magnitude of optimal government spending depends on the initial debt level, and (3) how the welfare effects of optimal government spending policy depend on the initial debt level.

A.1 Model with Labor Income Taxation

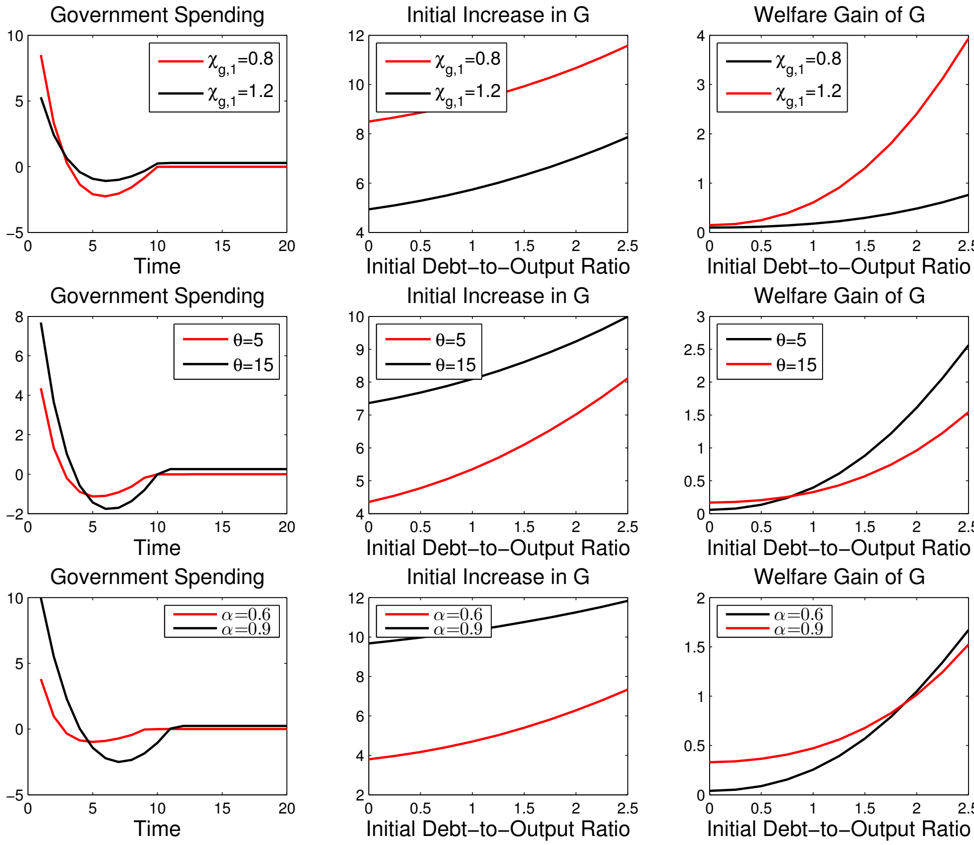
The left column of figure A.1 shows optimal government spending policy paths for alternative values of χ_c, $\chi_{n,1}$, and $\chi_{g,0}$ in the model with a labor income taxation and debt. The middle column of the same figure shows how the magnitude of optimal government spending varies with the initial debt level under the alternative values of χ_c, $\chi_{n,1}$, and $\chi_{g,0}$. The right column shows how the welfare gains from government spending policy depend on the initial debt level, again under the alternative values of χ_c, $\chi_{n,1}$, and $\chi_{g,0}$.

Figure A.1: Sensitivity Analyses: Model with Labor Income Taxation (I)



The left column of figure A.2 shows optimal government spending policy paths for alternative values of $\chi_{g,1}$, θ, and α in the model with a labor income taxation and debt. The middle column of the same figure shows how the magnitude of optimal government spending varies with the initial debt level under the alternative values of $\chi_{g,1}$, θ, and α. The right column shows how the welfare gains from government spending policy depend on the initial debt level under the alternative values of $\chi_{g,1}$, θ, and α.

Figure A.2: Sensitivity Analyses: Model with Labor Income Taxation (II)

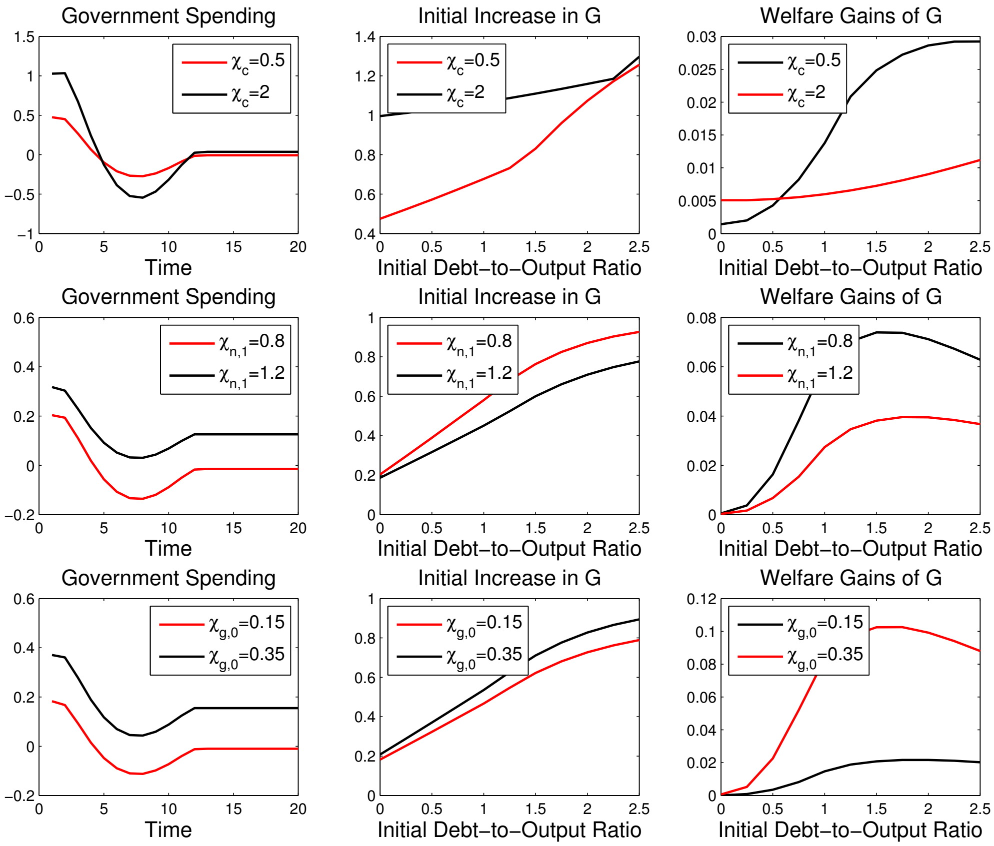


According to the left columns of these two figures, optimal government spending policy is characterized by the initial increase, followed by the reduction below, and the eventual return to, the new steady state, regardless of the parameter values. The middle columns of these two figures demonstrate the robustness of the result that the initial increase in the government spending is larger when the initial debt is larger. Finally, the right columns show that the welfare gain of government spending policy increases with the initial debt level, regardless of the parameter values.

A.2 Model with Consumption Taxation

The left column of figure A.4 shows optimal government spending policy paths for alternative values of χ_c, $\chi_{n,1}$, and $\chi_{g,0}$ in the model with a consumption taxation and debt. The middle column of the same figure shows how the magnitude of optimal government spending varies with the initial debt level under the alternative values of χ_c, $\chi_{n,1}$, and $\chi_{g,0}$. The right column shows how the welfare gains from government spending policy depend on the initial debt level, again under the alternative values of χ_c, $\chi_{n,1}$, and $\chi_{g,0}$.

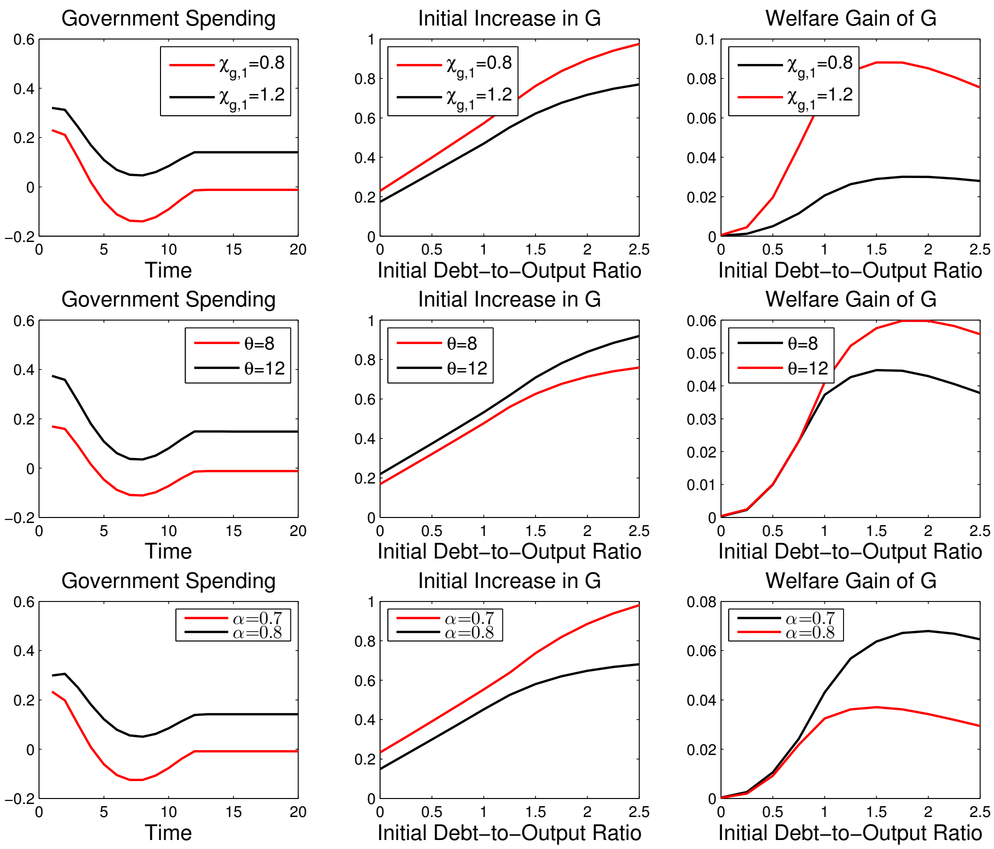
Figure A.3: Sensitivity Analyses: Model with Consumption Taxation (I)



The left column of figure A.3 shows optimal government spending policy paths for alternative values of $\chi_{g,1}$, θ, and α in the model with a consumption taxation and debt. The middle column of the same figure shows how the magnitude of optimal government spending varies with the initial debt level under the alternative values of $\chi_{g,1}$, θ, and α. The right column shows how the welfare gains from government spending policy depend on the initial debt level under the alternative values of $\chi_{g,1}$, θ, and α.

According to the left columns of these two figures, optimal government spending policy is characterized by the initial increase, followed by the reduction below, and the eventual return to, the new steady state, regardless of the parameter values. The middle columns of these two figures demonstrate the robustness of the result that the initial increase in the government spending is larger when the initial debt is larger. Finally, the right columns show that the welfare gain of government spending policy increases with the initial debt level unless the initial debt is sufficiently large, regardless of the parameter values. However, when the initial debt level is sufficiently large, the welfare gains decrease with the initial debt level as in the baseline parameterization of the model with consumption tax.

Figure A.4: Sensitivity Analyses: Model with Consumption Taxation (II)



B Nonlinear solution method

B.1 Problem

Let z_t be the vector of all time-t variables in the model. Given z_0, the problem is to find $\{z_t\}_{t=1}^{\infty}$ and two integers T_f and T_l such that

$$\begin{aligned}
&\ldots\\
f(z_{T_l+3}, z_{T_l+2}, z_{T_l+1}) &= 0 \text{ and } R_{T_l+2} > 1\\
f(z_{T_l+2}, z_{T_l+1}, z_{T_l}) &= 0 \text{ and } R_{T_l+1} > 1\\
g(z_{T_l+1}, z_{T_l}, z_{T_l-1}) &= 0 \text{ and } \omega_{11,T_l} > 0\\
&\ldots\\
g(z_{T_f+2}, z_{T_f+1}, z_{T_f}) &= 0 \text{ and } \omega_{11,T_f+1} > 0\\
g(z_{T_f+1}, z_{T_f}, z_{T_f-1}) &= 0 \text{ and } \omega_{11,T_f} > 0\\
f(z_{T_f}, z_{T_f-1}, z_{T_f-2}) &= 0 \text{ and } R_{T_f-1} > 1\\
&\ldots\\
f(z_3, z_2, z_1) &= 0 \text{ and } R_2 > 1\\
f(z_2, z_1, z_0) &= 0 \text{ and } R_1 > 1,
\end{aligned} \tag{24}$$

where T_f and T_l are the first and last periods for which $R_t = 1$ and $f(\cdot)$ and $g(\cdot)$ are the vectors of *functions* containing the first-order necessary conditions of the Lagrangean problem when $R_t > 1$ and $R_t = 1$, respectively. $f(\cdot)$ and $g(\cdot)$ are identical except for the last element; the last elements of $f(\cdot)$ and $g(\cdot)$ are $\omega_{11,t}$ and $R_t - 1$, respectively.

In order to reduce the problem into a finite problem, I assume that the economy will have converged to a terminal Ramsey steady state, z_{tss}, after t=S for some large integer S. The problem then becomes that of finding $\{z_t\}_{t=1}^{S}$, two integers, T_f and T_l, and the terminal Ramsey steady state, z_{tss}, such that

$$\begin{aligned}
f(z_{tss}, z_{tss}, z_S) &= 0 \text{ and } R_{S+1} > 1 \\
f(z_{tss}, z_S, z_{S-1}) &= 0 \text{ and } R_S > 1 \\
f(z_S, z_{S-1}, z_{S-2}) &= 0 \text{ and } R_{S-1} > 1 \\
&\dots \\
f(z_{T_l+2}, z_{T_l+1}, z_{T_l}) &= 0 \text{ and } R_{T_l+1} > 1 \\
g(z_{T_l+1}, z_{T_l}, z_{T_l-1}) &= 0 \text{ and } \omega_{11,T_l} > 0 \\
&\dots \\
g(z_{T_f+2}, z_{T_f+1}, z_{T_f}) &= 0 \text{ and } \omega_{11,T_f+1} > 0 \\
g(z_{T_f+1}, z_{T_f}, z_{T_f-1}) &= 0 \text{ and } \omega_{11,T_f} > 0 \\
f(z_{T_f}, z_{T_f-1}, z_{T_f-2}) &= 0 \text{ and } R_{T_f-1} > 1 \\
&\dots \\
f(z_3, z_2, z_1) &= 0 \text{ and } R_2 > 1 \\
f(z_2, z_1, z_0) &= 0 \text{ and } R_1 > 1.
\end{aligned} \tag{25}$$

B.2 Solution Method: A Big Picture

I use the "Newton-within-Shooting" algorithm to solve this problem. The algorithm proceeds as follows.

- Step 1: Guess T_f and T_l
  - Step 1.A: Guess b_{tss}, a level of debt at the terminal steady state.
  - Step 1.B: Compute z_{tss}, the terminal Ramsey steady state, associated with b_{tss}.
  - Step 1.C: Given z_{tss}, use the modified Newton algorithm described below to solve for $\{z_t\}_{t=1}^{S}$.
  - Step 1.D: Check the first-order necessary conditions at $t = S + 1$ are satisfied. If not, adjust the debt level at the terminal Ramsey steady state and go back to Step 1.B.
- Step 2: Check if $\omega_{11,t} \geq 0$ and $R_t \geq 1$ for all $1 \leq t \leq S$. If not, adjust T_f and T_l and go back to Step 1.

The Newton algorithm I use in step 1.C is the modified Newton method of Juillard, Laxton, McAdam, and Pioro (1998), which I shall describe shortly. Given T_f, T_l, and z_{tss}, the goal of the modified Newton algorithm is to find $\{z_t\}_{t=1}^S$ satisfying the equilibrium conditions from $t = 1$ to $t = S$. The number of equations is the same as the number of variables.[7]

B.3 Modified Newton Algorithm

By stacking z_t into a vector $Y = [z_1', z_2', ..., z_{S-1}', z_S']'$, we can express the problem as that of finding Y such that

$$F(Y) = 0, \tag{26}$$

where $F(\cdot)$ is a vector of functions stacking $f(\cdot)$ and $g(\cdot)$ from $t = 1$ to $t = S$. As in any Newton algorithm, we start by an initial guess of Y, $Y^{(1)}$. At a k-th iteration, given a previous guess $Y^{(k)}$, I compute an adjustment factor ΔY by solving the following system of linear equations:

$$\left[\frac{\partial F(Y)}{\partial Y}\right]_{Y^{(k)}} \Delta Y = -F(Y^{(k)}). \tag{27}$$

If $||\Delta Y|| < \epsilon_{tol}$, the algorithm ends. Otherwise, I set our next guess of Y as

$$Y^{(k+1)} = Y^{(k)} + \mu \Delta Y \tag{28}$$

I set $\mu = 0.5$ and $\epsilon_{tol} = 10e - 15$.

When the system of equations is small, we can find the ΔY that satisfies equation (27) by inverting $\left[\frac{\partial F(Y)}{\partial Y}\right]_{Y^{(k)}}$. However, when the system is large, inverting this matrix is computationally costly. Juillard, Laxton, McAdam, and Pioro (1998) proposed to solve for ΔY in equation (27) by making use of the sparseness of the matrix, which comes from the recursive structure of the problem.

I now turn to the details of their algorithm. For the clarity of exposition, I focus on the case where the the zero lower bound binds initially (i.e., $T_f = 0$) in what follows. Equation (27) can be written as

[7] While I have not been able to prove the uniqueness of the solution, this Newton algorithm returns the unique solution regardless of the starting values used to initiate the algorithm.

$$
\begin{bmatrix}
\bar{J}_S^{(k)} & M_{S-1}^{(k)} & 0 & 0 & 0 & ... & ... & & 0 & 0 \\
L_S^{(k)} & \bar{J}_{S-1}^{(k)} & M_{S-2}^{(k)} & 0 & 0 & ... & ... & & 0 & 0 \\
 & & & & & & & & & 0 \\
 & & & & & & & & & 0 \\
0 & 0 & 0 & L_{T+1}^{(k)} & J_T^{(k)} & M_{T-1}^{(k)} & ... & & ... & 0 \\
0 & 0 & 0 & 0 & L_T^{(k)} & J_{T-1}^{(k)} & M_{T-2}^{(k)} & & ... & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & L_3^{(k)} & J_2^{(k)} & M_1^{(k)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & L_2^{(k)} & J_1^{(k)}
\end{bmatrix}
\begin{bmatrix}
\Delta z_S^{(k+1)} \\
\Delta z_{S-1}^{(k+1)} \\
... \\
\Delta z_{T+1}^{(k+1)} \\
\Delta z_T^{(k+1)} \\
\Delta z_{T-1}^{(k+1)} \\
... \\
\Delta z_3^{(k+1)} \\
\Delta z_2^{(k+1)} \\
\Delta z_1^{(k+1)}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-f(z_{tss}, z_S^{(k)}, z_{S-1}^{(k)}) \\
-f(z_S^{(k)}, z_{S-1}^{(k)}, z_{S-2}^{(k)}) \\
... \\
-f(z_{T+1}^{(k)}, z_T^{(k)}, z_{T-1}^{(k)}) \\
-g(z_T^{(k)}, z_{T-1}^{(k)}, z_{T-2}^{(k)}) \\
-g(z_{T-1}^{(k)}, z_{T-2}^{(k)}, z_{T-3}^{(k)}) \\
... \\
-g(z_3^{(k)}, z_2^{(k)}, z_1^{(k)}) \\
-g(z_2^{(k)}, z_1^{(k)}, z_{0,ss})
\end{bmatrix}
=:
\begin{bmatrix}
d_S^{(k)} \\
d_{S-1}^{(k)} \\
... \\
d_T^{(k)} \\
d_{T-1}^{(k)} \\
d_{T-2}^{(k)} \\
... \\
d_2^{(k)} \\
d_1^{(k)}
\end{bmatrix},
\tag{29}
$$

where

$$L_t^{(k)} \equiv \frac{\partial f(z_t^{(k)})}{\partial z_{t+1}} \left(\equiv \frac{\partial g(z_t^{(k)})}{\partial z_{t+1}}\right) \tag{30}$$

$$\bar{J}_t^{(k)} \equiv \frac{\partial f(z_t^{(k)})}{\partial z_t} \tag{31}$$

$$J_t^{(k)} \equiv \frac{\partial g(z_t^{(k)})}{\partial z_t} \tag{32}$$

$$M_t^{(k)} \equiv \frac{\partial f(z_t^{(k)})}{\partial z_{t-1}} \left(\equiv \frac{\partial g(z_t^{(k)})}{\partial z_{t-1}}\right). \tag{33}$$

Notice that the difference between the zero-bound regime and the normal regime shows up only in the derivative matrix with respect to z_t (i.e., J and $\bar{J}$). This system of equations can also be written as

$$
\begin{aligned}
\bar{J}_S^{(k)}\Delta z_S^{(k+1)} + M_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} &= d_S^{(k)} \\
L_S^{(k)}\Delta z_S^{(k+1)} + \bar{J}_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} + M_{S-2}^{(k)}\Delta z_{S-2}^{(k+1)} &= d_{S-1}^{(k)} \\
&\dots \\
L_{T+2}^{(k)}\Delta z_{T+2}^{(k+1)} + \bar{J}_{T+1}^{(k)}\Delta z_{T+1}^{(k+1)} + M_T^{(k)}\Delta z_{T+1}^{(k+1)} &= d_{T+1}^{(k)} \\
L_{T+1}^{(k)}\Delta z_{T+1}^{(k+1)} + J_T^{(k)}\Delta z_T^{(k+1)} + M_{T-1}^{(k)}\Delta z_T^{(k+1)} &= d_T^{(k)} \\
L_T^{(k)}\Delta z_T^{(k+1)} + J_{T-1}^{(k)}\Delta z_{T-1}^{(k+1)} + M_{T-2}^{(k)}\Delta z_{T-2}^{(k+1)} &= d_{T-1}^{(k)} \\
&\dots \\
L_3^{(k)}\Delta z_3^{(k+1)} + J_2^{(k)}\Delta z_2^{(k+1)} + M_1^{(k)}\Delta z_1^{(k+1)} &= d_2^{(k)} \\
L_2^{(k)}\Delta z_2^{(k+1)} + J_1^{(k)}\Delta z_1^{(k+1)} &= d_1^{(k)}.
\end{aligned}
\tag{34}
$$

Notice that finding ΔY in equation (27) is equivalent to finding a sequence of $\{\Delta z_t^{(k+1)}\}_{t=1}^S$ that satisfies this system.

To find $\{\Delta z_t^{(k+1)}\}_{t=1}^S$, notice that

$$
\begin{aligned}
\Delta z_S^{(k+1)} &= \left[\bar{J}_S^{(k)}\right]^{-1} d_S^{(k)} - \left[\bar{J}_S^{(k)}\right]^{-1} M_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} \\
&= \Lambda_S^{(k)} + \Phi_S^{(k)}\Delta z_{S-1}^{(k+1)}.
\end{aligned}
\tag{35}
$$

Plugging this into the equation for $t = S-1$, I obtain

$$
L_S^{(k)}[\Lambda_S^{(k)} + \Phi_S^{(k)}\Delta z_{S-1}^{(k+1)}] + \bar{J}_{S-1}^{(k)}\Delta z_{S-1}^{(k+1)} + M_{S-2}^{(k)}\Delta z_{S-2}^{(k+1)} = d_{S-1}^{(k)} \tag{36}
$$

$$\Rightarrow$$

$$
\begin{aligned}
\Delta z_{S-1}^{(k+1)} &= \left[L_S^{(k)}\Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1}\left[d_{S-1}^{(k)} - L_S^{(k)}\Lambda_S^{(k)}\right] - \left[L_S^{(k)}\Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} M_{S-2}^{(k)}\Delta z_{S-2}^{(k+1)} \\
&= \Lambda_{S-1}^{(k)} + \Phi_{S-1}^{(k)}\Delta z_{S-2}^{(k+1)}.
\end{aligned}
\tag{37}
$$

This process can be repeated until $t = 1$. The outcome of this process is the following law of motion for Δz_t:

$$
\begin{aligned}
\Delta z_S^{(k+1)} &= \Lambda_S^{(k)} + \Phi_S^{(k)}\Delta z_{S-1}^{(k+1)} \\
\Delta z_{S-1}^{(k+1)} &= \Lambda_{S-1}^{(k)} + \Phi_{S-1}^{(k)}\Delta z_{S-2}^{(k+1)} \\
&\dots \\
\Delta z_{T+1}^{(k+1)} &= \Lambda_{T+1}^{(k)} + \Phi_{T+1}^{(k)}\Delta z_T^{(k+1)} \\
\Delta z_T^{(k+1)} &= \Lambda_T^{(k)} + \Phi_T^{(k)}\Delta z_{T-1}^{(k+1)} \\
&\dots \\
\Delta z_2^{(k+1)} &= \Lambda_2^{(k)} + \Phi_2^{(k)}\Delta z_1^{(k+1)} \\
\Delta z_1^{(k+1)} &= \Lambda_1^{(k)},
\end{aligned}
\tag{38}
$$

where the time-varying coefficients are computed recursively as follows:

$$
\begin{aligned}
\Lambda_S^{(k)} &= \left[\bar{J}_S^{(k)}\right]^{-1} d_S^{(k)} \\
\Phi_S^{(k)} &= -\left[\bar{J}_S^{(k)}\right]^{-1} M_{S-1}^{(k)} \\
\Lambda_{S-1}^{(k)} &= \left[L_S^{(k)}\Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} \left[d_{S-1}^{(k)} - L_S^{(k)}\Lambda_S^{(k)}\right] \\
\Phi_{S-1}^{(k)} &= -\left[L_S^{(k)}\Phi_S^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} M_{S-2}^{(k)} \\
&\ldots \\
\Lambda_{T+1}^{(k)} &= \left[L_{T+1}^{(k)}\Phi_{T+2}^{(k)} + \bar{J}_T^{(k)}\right]^{-1} \left[d_T^{(k)} - L_{T+1}^{(k)}\Lambda_{T+2}^{(k)}\right] \\
\Phi_{T+1}^{(k)} &= -\left[L_{T+1}^{(k)}\Phi_{T+2}^{(k)} + \bar{J}_{S-1}^{(k)}\right]^{-1} M_{T-1}^{(k)} \\
\Lambda_T^{(k)} &= \left[L_T^{(k)}\Phi_S^{(k)} + J_{T-1}^{(k)}\right]^{-1} \left[d_{S-1}^{(k)} - L_S^{(k)}\Lambda_{T+1}^{(k)}\right] \\
\Phi_T^{(k)} &= -\left[L_T^{(k)}\Phi_S^{(k)} + J_{T-1}^{(k)}\right]^{-1} M_{S-2}^{(k)} \\
&\ldots \\
\Lambda_2^{(k)} &= \left[L_3^{(k)}\Phi_3^{(k)} + J_2^{(k)}\right]^{-1} \left[d_2^{(k)} - L_3^{(k)}\Lambda_3^{(k)}\right] \\
\Phi_2^{(k)} &= -\left[L_3^{(k)}\Phi_3^{(k)} + J_2^{(k)}\right]^{-1} M_1^{(k)} \\
\Lambda_1^{(k)} &= \left[L_2^{(k)}\Phi_2^{(k)} + J_1^{(k)}\right]^{-1} \left[d_1^{(k)} - L_2^{(k)}\Lambda_2^{(k)}\right].
\end{aligned} \tag{39}
$$

The knowledge of L_t, J_t, $\bar{J}_t$, and M_t is sufficient for us to compute these time-varying coefficients given in the set of equations (39). Once the coefficients are computed, one can use the law of motion given in the set of equations (38) to find the sequence of $\{\Delta z_t^{(k+1)}\}_{t=1}^S$.

C The Lagrangean Problem

The Lagrangean problem of the government is given by

$$
\begin{aligned}
&L(b_0, s_0, \omega_{1,0}, \omega_{6,0}, \omega_{7,0}) = \\
&\quad \min_{\{\omega_t\}_{t=1}^{\infty}} \max_{\{u_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \chi_{n,0} \frac{N_t^{1+\chi_{n,1}}}{1+\chi_{n,1}} + \chi_{g,0} \frac{G_t^{1-\chi_{g,1}}}{1-\chi_{g,1}} \Big] \\
&\qquad -\omega_{1,0} C_1^{-\chi_c} \Pi_1^{-1} - \omega_{6,0} \zeta_p \Pi_1^{\theta} C_{n,1} - \omega_{7,0} \zeta_p \Pi_1^{\theta-1} C_{d,1} \\
&\qquad + \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \Bigg[\omega_{1,t} [\frac{C_t^{-\chi_c}}{(1+\tau_{c,t}) R_t} - \beta \delta_t \frac{C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}}{1+\tau_{c,t+1}}] \\
&\qquad + \omega_{2,t} [\frac{1-\tau_{n,t}}{1+\tau_{c,t}} w_t - \chi_{n,0} N_t^{\chi_{n,1}} C_t^{\chi_c}] \\
&\qquad + \omega_{3,t} [s_t - (1-\zeta_p) \big[p_t^* \big]^{-\theta} - \zeta_p \Pi_t^{\theta} s_{t-1}] \\
&\qquad + \omega_{4,t} [1 - (1-\zeta_p) \big[p_t^* \big]^{1-\theta} - \zeta_p \Pi_t^{\theta-1}] \\
&\qquad + \omega_{5,t} [p_t^* - \frac{\theta}{\theta-1} \frac{C_{n,t}}{C_{d,t}}] \\
&\qquad + \omega_{6,t} \big[C_{n,t} - \frac{1-\nu_t}{1+\tau_{c,t}} Y_t w_t C_t^{-\chi_c} - \zeta_p \beta \delta_t \Pi_{t+1}^{\theta} C_{n,t+1} \big] \\
&\qquad + \omega_{7,t} \big[C_{d,t} - \frac{1}{1+\tau_{c,t}} Y_t C_t^{-\chi_c} - \zeta_p \beta \delta_t \Pi_{t+1}^{\theta-1} C_{d,t+1} \big] \\
&\qquad + \omega_{8,t} [Y_t s_t - N_t] \\
&\qquad + \omega_{9,t} [Y_t - C_t - G_t] \\
&\qquad + \omega_{10,t} GBC_t \\
&\qquad + \omega_{11,t} [R_t - 1] \Bigg],
\end{aligned} \tag{40}
$$

where the problem is indexed by $\omega_{1,0}$, $\omega_{6,0}$, $\omega_{7,0}$ in addition to b_0 and s_0. The first-order necessary conditions of this problem is given by

$$
\begin{aligned}
C_t : & C_t^{-\chi_c} - \omega_{1,t} \chi_c \frac{C_t^{-\chi_c - 1}}{(1+\tau_{c,t}) R_t} + \omega_{1,t-1} \chi_c \frac{C_t^{-\chi_c - 1} \Pi_t^{-1}}{1+\tau_{c,t}} - \omega_{2,t} \chi_c \chi_{n,0} N_t^{\chi_{n,1}} C_t^{\chi_c - 1} \\
& + \omega_{6,t} \chi_c \frac{Y_t w_t C_t^{-\chi_c - 1}}{1+\tau_{c,t}} + \omega_{7,t} \chi_c \frac{Y_t C_t^{-\chi_c - 1}}{1+\tau_{c,t}} - \omega_{9,t} - \omega_{10,t} \tau_{c,t} = 0
\end{aligned} \tag{41}
$$

$$
Y_t : -\omega_{6,t} \frac{w_t C_t^{-\chi_c}}{1+\tau_{c,t}} - \omega_{7,t} \frac{C_t^{-\chi_c}}{1+\tau_{c,t}} + \omega_{8,t} s_t + \omega_{9,t} = 0 \tag{42}
$$

$$
N_t : -\chi_{n,0} N_t^{\chi_{n,1}} - \omega_{2,t} \chi_{n,1} \chi_{n,0} N_t^{\chi_{n,1} - 1} C_t^{\chi_c} - \omega_{8,t} - \omega_{10,t} \tau_{n,t} w_t = 0 \tag{43}
$$

$$
w_t : \omega_{2,t} \frac{1-\tau_{n,t}}{1+\tau_{c,t}} - \omega_{6,t} \frac{1-\nu_t}{1+\tau_{c,t}} Y_t C_t^{-\chi_c} - \omega_{10,t} \tau_{n,t} N_t = 0 \tag{44}
$$

$$\begin{aligned}
\Pi_t &: \omega_{1,t-1} C_t^{-\chi_c} \Pi_t^{-2} - \omega_{3,t} \theta \zeta_p \Pi_t^{\theta-1} s_{t-1} - \omega_{4,t} (\theta - 1) \zeta_p \Pi_t^{\theta-2} \\
&\quad - \omega_{6,t-1} \theta \zeta_p \Pi_t^{\theta-1} C_{n,t} - \omega_{7,t-1} (\theta - 1) \zeta_p \Pi_t^{\theta-2} C_{d,t} = 0 && (45) \\
s_t &: \omega_{3,t} - \omega_{3,t+1} \beta \delta_t \zeta_p \Pi_{t+1}^{\theta} + \omega_{8,t} Y_t - \omega_{10,t} b_{t-1} \Pi_t^{-2} = 0 && (46) \\
p_t^* &: \omega_{3,t} \theta (1 - \zeta_p) \left[p_t^*\right]^{-\theta-1} + \omega_{4,t} (\theta - 1)(1 - \zeta_p) \left[p_t^*\right]^{-\theta} + \omega_{5,t} = 0 && (47) \\
C_{n,t} &: -\omega_{5,t} \frac{\theta}{\theta - 1} \frac{1}{C_{d,t}} + \omega_{6,t} - \omega_{6,t-1} \zeta_p \Pi_t^{\theta} = 0 && (48) \\
C_{d,t} &: \omega_{5,t} \frac{\theta}{\theta - 1} C_{n,t} C_{d,t}^{-2} + \omega_{7,t} - \omega_{7,t-1} \zeta_p \Pi_t^{\theta-1} = 0 && (49) \\
R_t &: -\omega_{1,t} C_t^{-\chi_c} \Pi_t^{-1} R_t^{-2} + \omega_{10,t} R_t^{-2} b_t + \omega_{11,t} = 0 && (50) \\
&\omega_{11,t} = 0 \text{ if } R_t > 1, \omega_{11,t} > 0 \text{ if } R_t = 1 && (51) \\
G_t &: \chi_{g,0} G_t^{-\chi_{g,1}} - \omega_{9,t} + \omega_{10,t} = 0 && (52) \\
\tau_{c,t} &: -\omega_{1,t} (1 + \tau_{c,t})^{-2} C_t^{-\chi_c} + \omega_{1,t-1} (1 + \tau_{c,t})^{-2} C_{t-1}^{-\chi_c} \\
&\quad - \omega_{2,t} (1 + \tau_{c,t})^{-2} w_t + \omega_{6,t} (1 + \tau_{c,t})^{-2} Y_t w_t C_t^{-\chi_c} \\
&\quad + \omega_{7,t} (1 + \tau_{c,t})^{-2} Y_t C_t^{-\chi_c} - \omega_{10,t} C_t = 0 && (53) \\
\tau_{n,t} &: -\omega_{2,t} \frac{1}{1 + \tau_{c,t}} w_t - \omega_{10,t} w_t N_t = 0 && (54) \\
b_t &: -\omega_{10,t} R_t^{-1} + \omega_{10,t+1} \beta \delta_t \Pi_{t+1}^{-1} = 0 && (55) \\
T_t &: -\omega_{10,t} = 0 && (56)
\end{aligned}$$

as well as the equations characterizing the equilibrium of the model shown in section 2.