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Monetary Policy, Incomplete Information, and the Zero Lower Bound

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Abstract

In the context of a stylized New Keynesian model, we explore the interaction between imperfect knowledge about the state of the economy and the zero lower bound. We show that optimal policy under discretion near the zero lower bound responds to signals about an increase in the equilibrium real interest rate by less than it would when far from the zero lower bound. In addition, we show that Taylor-type rules that either include a time-varying intercept that moves with perceived changes in the equilibrium real rate or that respond aggressively to deviations of inflation and output from their target levels perform similarly to optimal discretionary policy. Our analysis of first-difference rules highlights that rules with interest rate smoothing terms carry forward current and past misperceptions about the state of the economy and can lead to suboptimal performance.

JEL classification: E32, E52
Key words: imperfect information, zero lower bound, discretionary policy, monetary policy rules.

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1 Introduction

Policymakers routinely have to make decisions based on incomplete knowledge about the state of the economy. Once source of incomplete knowledge is the level of the equilibrium real rate.[1] As former Chairman Bernanke recently wrote:[2]

> If the Fed wants to see full employment of capital and labor resources (which, of course, it does), then its task amounts to using its influence over market interest rates to push those rates toward levels consistent with the equilibrium rate, or–more realistically–its best estimate of the equilibrium rate, which is not directly observable.

Imperfect knowledge about the equilibrium real rate raises the possibility that policymakers might misperceive its value, which has important implications for the design of monetary policy. In this paper, we discuss the ways in which optimal policy (under discretion) should take account of variation in the level of the equilibrium real interest rate in the context of a simple New Keynesian (NK) model in which policymakers have imperfect information about the level of the equilibrium real rate and are constrained by the zero lower bound.

We find that uncertainty about the current state of the economy affects optimal policy in a number of important ways. First, there is an attenuation of the policy response to perceived changes in the equilibrium real rate coming from the filtering problem facing the monetary authority, as in Aoki (2003). Second, the zero lower bound non-linearly alters the distribution of equilibrium prices and quantities for both the private sector and the central bank, as has been recently discussed by Evans et al. (2015). Third, in our setup, the private sector has access to different information than the monetary authority and thus forms different beliefs about both current and future prices and quantities. Because the central bank has to form expectations about those private-sector expectations, they are not *ex-post* equal and the optimal policy response to perceived changes in the equilibrium real rate is further attenuated by the information processing problem. This muted response reflects asymmetric risks faced by the central bank near the zero lower bound. Finally, the private sector understands that the monetary authority may in the future respond to noise in the signals it receives. Near the zero lower bound, the private sector internalizes that

[1] In the context of the New Keynesian model that we use, the equilibrium real rate corresponds to the rate of interest associated with the efficient economy that is undistorted by nominal or real rigidities.

[2] Ben Bernanke's Blog, "Why are interest rates so low?" March 30, 2015, http://www.brookings.edu/blogs/ben-bernanke/posts/2015/03/30-why-interest-rates-so-low

the monetary authority can only respond in one direction, which increases the mean of the private sector's expected future paths for short-term nominal (and real) interest rates. This tightening bias affects prices and quantities in the current period.

Optimal monetary policy takes this tightening bias into account by attenuating the response of the nominal interest rate to positive signals about the equilibrium real rate. In particular, when the policy rate is constrained by the zero lower bound and the policymaker receives a noisy signal that the equilibrium rate has increased, the optimal policy is to make a mistake on the side of keeping the real policy rate below the perceived efficient rate so as to reduce the likelihood that an adverse shock will weaken the economy enough, to require a return to the zero lower bound. That is, policymakers "take out insurance" against situations in which the information about the equilibrium real rate might lead them to mistakenly raise the policy rate.

Optimal policy under discretion in the neighborhood of the zero lower bound prescribes a complicated, nonlinear reaction function for the policy rate that may, in practice, be difficult to communicate and implement. An alternative approach would be to commit to a policy rule for the short-term nominal interest rate. Thus, we also study the potential benefits of committing to a rule-based approach in setting monetary policy within the same incomplete information framework. In particular, we compare the performance of simple Taylor-type rules with a constant intercept term to alternatives in which the intercept term is time-varying and changes based on a central bank's estimate of the equilibrium real rate. We also compare their performance to first-difference rules (e.g., Orphanides and Williams (2002)) in which there is no intercept term and it is thus unnecessary to take a stand on how the equilibrium real rate should enter this rule. The performance of simple Taylor rules, using conventional parameter values, are substantially improved when the intercept in the rule is adjusted for movements in the equilibrium real rate. However, if the intercept of the rule is held constant, we find that simple Taylor rules can perform almost as well as rules with a time-varying equilibrium real rate when they respond much more aggressively to inflation and the output gap. Similarly, we find that first-difference rules with an aggressive response to inflation tend to perform well.

Most of the existing literature studying optimal monetary policy at the zero lower bound focuses on the case in which policymakers have complete information. Thus, one of our contributions is to assume that the monetary authority has less information about the state of the economy than the private sector. A justification for this assumption is that private agents make decisions after seeing shocks that affect them directly. By contrast, policymakers have to learn about the shocks that cause households and firms to make demand, production, and pricing decisions, which we model as the monetary authority receiving noisy signals about the state of the economy. If there were no noise in the signals, the monetary authority would know the underlying structural shocks exactly, which would mean it had perfect information about equilibrium prices and quantities as well. In this sense, the central bank's informational disadvantage relative to the private sector captures that, in reality, central banks need to now-cast prices and quantities when they make their policy decisions.

While we prefer to interpret our setup from the perspective of a now-casting problem faced by the central bank, another way to rationalize our framework is from a sequential structure to the decisions made in the model. In particular, a model economy in which nominal interest rate decisions have to be made before private-sector decisions in the same period requires that the central bank base its decisions on less information than the private sector has available at that time. This type of timing assumption captures the fact that, in reality, policy rate decisions are not made on a daily basis, but rather during meetings that occur infrequently. While policymakers may be able to react to incoming data received during the intermeeting periods, historically policy decisions have only been infrequently made outside of regularly scheduled meetings. Thus, policymakers regularly set short-term nominal interest rates based on information available only at the beginning of the period in which that policy decision will be operative. Moreover, with lags in official data releases and revisions to real-time data, the information available even about the recent past is likely to be affected by noise, further emphasizing the informational problem faced by central banks.

This paper is related to a growing literature in NK models exploring the implications of the zero lower bound for optimal policy. Eggertsson and Woodford (2003) emphasized that optimal commitment policy can be very effective in stabilizing the economy at the zero lower bound. However, as noted in Levin et al. (2010), optimal commitment policies at the zero lower bound suffer from a severe time inconsistency problem. Accordingly, we focus on optimal discretion to abstract from how the central bank can have access to a credible commitment technology.

Adam and Billi (2007) and Nakov (2008) consider optimal discretionary policy at the zero lower bound with full information, while Aoki (2003) studies the optimal discretionary policy without the zero lower bound but when the monetary authority has imperfect knowledge about the state of the economy. We consider both imperfect information and the zero lower bound in a unified framework and thus, unlike Aoki (2003), optimal policy deviates from certainty equivalence. Several other studies have focused on the use of simple feedback rules for conducting monetary policy near the zero lower bound (e.g., Reifschneider and Williams (2000) and Coenen and Wieland (2003)). We also consider simple rules but in the context of a central bank with imperfect knowledge about the state of the economy. By considering optimal simple rules in the context of imperfect information, our paper builds on work by Boehm and House (2014). Our paper is also closely related to a recent contribution by Evans et al. (2015). In their complete information setup, they focus on uncertainty about future values of the equilibrium real rate. By contrast, in our incomplete information setup, the central bank does not observe the current equilibrium real rate, and private-sector beliefs about equilibrium prices and quantities differ from those of the central bank.

The remainder of this paper is organized as follows. Section 2 discusses the basic elements of the model that we use to characterize the determinants of the equilibrium real rate. Section 3 reviews optimal policy under discretion in the presence of the zero lower bound. It also describes the information structure of the model and characterizes the optimal filtering problem. This section presents the main results regarding optimal discretionary policy near the zero lower bound when the central bank has imperfect information about the true state of the economy. Section 4 then discusses the performance of simple rules in this environment. Finally, section 5 offers some concluding remarks.

2 A Simple New Keynesian Model

Our analysis builds on Woodford (2003) and Galí (2008) who use the simple NK model to characterize optimal monetary policy. The model has two key equations: an IS equation that relates output to the real interest rate (that is, the nominal short-term rate adjusted by expected inflation) and expected future output, and a Phillips curve relationship that relates inflation to the output gap and expected future inflation. We leave the details of the model for an appendix and devote this section to defining the natural and the efficient levels of the real interest rates using a log-linear approximation of the model economy.

As discussed in the appendix, the private sector (log-linearized) equilibrium conditions around the non-stochastic steady state can be written as follows:

$$y_t = E_t\{y_{t+1}\} - E_t\{i_t - \pi_{t+1} - \eta_t\} \tag{1}$$

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa[y_t + \mu_t] \tag{2}$$

where E_t represents the expectation conditional on time t information; y_t and π_t refer to the log deviations of output and inflation from their values in nonstochastic steady state; and i_t represents the short-term nominal interest rate in deviations from its non-stochastic steady state. The parameter β represents the household's discount factor. The parameter $\kappa = \frac{\epsilon-1}{\varphi}$ is inversely related to firms' costs of changing prices (φ) and directly related to the elasticity of demand (ϵ). The variables η_t and μ_t represent exogenous shocks to the discount rate and the firm's marginal cost, respectively.[3] A decrease in η_t represents an exogenous factor that induces a temporary rise in households' propensity to save, and reduces current aggregate household demand for goods. An increase in μ_t corresponds to an exogenous cost-push shock that increases inflation for given levels of output and expected future inflation; therefore, it introduces a tradeoff between inflation and output.

The natural rate of output and the natural real interest rate are defined as the levels of output and the real interest rate in the economy without price rigidities. As shown in the appendix, deviations of the natural real rate from its steady state value can be expressed as follows:

$$r_t^n = \eta_t - E_t\{\mu_{t+1} - \mu_t\}$$

Thus, there are two sources of exogenous disturbances to the natural rate of interest: shocks to the discount factor and shocks to the marginal cost. A decrease in the desire to save (an increase in η_t) and expected decrease in marginal costs cause the natural rate of interest to rise.

The efficient equilibrium corresponds to that of the economy with flexible prices and no exogenous variation in marginal costs, as those shocks push equilibrium quantities and prices from their efficient values. Formally, as shown in the appendix, the log-linear approximation around the efficient non-stochastic steady state implies that:[4]

$$r_t^e = \eta_t,$$

where r_t^e denotes the deviation of the efficient real interest rate from its steady state value.

As discussed in Woodford (2003) and Galí (2008), the presence of a gap between the natural and the efficient levels of output due to exogenous changes in real marginal costs implies that the relevant concept for welfare is the difference between the observed level and the efficient level of output. Formally, fluctuations in output relative to its efficient level, x_t, can be decomposed as follows:

$$x_t = y_t - y_t^e = (y_t - y_t^n) + (y_t^n - y_t^e) = (y_t - y_t^n) - \mu_t$$

[3] The relationship of these shocks with the specification of preferences and technology is described in the appendix. Both shocks follow first-order autoregressive processes.

[4] As shown in the Appendix, the non-stochastic efficient level of output is constant and normalized to one, that is $y_t^e = 0$.

Thus, x_t can be decomposed in the sum of deviations of output from the natural rate (due to the presence of nominal frictions) plus deviations of the natural level of output from its efficient level (due to changes in real marginal costs, μ_t, exogenous to monetary policy). Using this definition, the aggregate demand equation can be written in terms of deviation of output from its natural rate as follows:

$$y_t - y_t^n = E_t\{y_{t+1} - y_{t+1}^n\} - E_t\{i_t - \pi_{t+1} - r_t^n\}$$

The inflation equation can also be written in terms of the output gap:

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa[x_t + \mu_t], \tag{3}$$

given that there are not exogenous variation in productivity as described in the appendix $y_t^e = 0$. We can also rewrite equation (1) in terms of the output gap as follows:

$$x_t = E_t\{x_{t+1}\} - E_t\{i_t - \pi_{t+1} - r_t^e\} \tag{4}$$

Because of the presence of the cost-push shocks, μ_t, the "divine coincidence" property discussed in Blanchard and Galí (2007) is not present in our model. Thus, it will also not be optimal for the policymaker to pursue a policy in which the *ex-ante* real rate equals the efficient real interest rate.

3 Optimal Discretionary Policy

We now discuss the optimal monetary policy when the policymaker is unable to credibly commit to future policy actions, and when the nominal interest rate is constrained by the zero lower bound. We consider the cases in which the central bank has perfect information first and later imperfect information regarding the current state of the economy.

3.1 The Optimal Policy Problem

We build on Eggertsson and Woodford (2003) who first introduced the zero lower bound constraint into the linear-quadratic framework originally studied by Rotemberg and Woodford (1997) and Clarida et al. (1999).[5] Following Woodford (2003) we assume that the monetary authority seeks to minimize the following quadratic loss function,

$$\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left\{\pi_t^2 + \lambda x_t^2\right\},$$

[5] Adam and Billi (2007) and Nakov (2008) also offer discussions of the effect of the effective zero lower bound on optimal monetary policy under discretion.

where β is the discount factor, λ is a trade-off parameter and E_t represents the expectation conditional on time t information. The monetary authority is constrained by the private sector equilibrium conditions, equations (4) and (3), as well as by the zero lower bound on the nominal interest rate:

$$i_t \geq \log\left(\frac{\beta}{\overline{\overline{\Pi}}}\right). \tag{5}$$

In this paper we focus on the optimal stabilization policy at the zero lower bound and abstract from the issue on how the zero lower bound will affect the optimal average rate of inflation. Instead, π_t in the loss function represents the deviation of inflation from a two percent target which is consistent with the longer-run objective of the Federal Reserve.[6] This allows us to focus on how the nonlinearity introduced by the zero lower bound limits the optimal real rate from being equal to the efficient real rate. At the zero lower bound, any shock creates a tradeoff between inflation and output stabilization for the policymaker because of the constraint it places on monetary policy actions.

We assume that the monetary authority chooses policy under discretion taking private-sector expectations as given. Under these assumptions and under the assumption of complete information, the first-order conditions characterizing optimal policy are:

$$\lambda x_t = \kappa\theta_t - \phi_t$$
$$\pi_t = -\theta_t$$
$$\gamma_t = \phi_t,$$

where ϕ_t, θ_t, and γ_t denote the Lagrange multipliers on equations (3), (4), and (5), respectively. The complementary slackness condition is:

$$[i_t - \log(\frac{\beta}{\overline{\overline{\Pi}}})] \cdot \gamma_t = 0.$$

In normal circumstances (e.g., far away from the zero lower bound), the previous first-order conditions correspond to those analyzed by Clarida et al. (1999). In this case $\phi_t = \gamma_t = 0$, and hence the optimal discretionary policy gives rise to the following optimality condition or targeting rule:

$$x_t = -\frac{\kappa}{\lambda}\pi_t.$$

Without exogenous shocks to marginal costs and under the assumption that the economy will never be at the zero lower bound, the optimal discretionary policy consists of

[6] See Coibion et al. (2012) and the references therein, for a discussion of how the optimal steady state inflation is affected by the presence of the zero lower bound.

setting the actual real rate equal to the efficient real rate, r_t^e. In these circumstances, this policy implies that inflation is equal to target and the output gap is closed at every date (i.e., $\pi_t = x_t = 0$).

However, the presence of the zero lower bound will change this prescription. At the zero lower bound, the equilibrium under discretion implies that the targeting rule characterizing optimal policy is affected by the Lagrange multiplier associated with that constraint:

$$x_t = -\frac{\kappa}{\lambda}\pi_t - \frac{1}{\lambda}\phi_t. \tag{6}$$

In normal times, the output gap moves proportionately with the inflation gap; however, at the zero lower bound, a tightening in the constraint implies a larger decline in the output gap relative to the decline in the inflation gap, since $\phi_t > 0$.

3.2 Information Structure

Our novel modeling contribution is to describe how optimal policy at the zero lower bound changes because the monetary authority faces imperfect information about the state of the economy. Importantly, we assume that the monetary authority has less knowledge about the state than the private sector. A justification for this assumption in the context of our model is that private agents (households and firms) make decisions after seeing shocks that affect them directly. By contrast, policymakers have to learn about the nature of fluctuations in the economy that cause private agents to make demand, production, and pricing decisions.

Formally, at every time t households and firms know the entire history of exogenous shocks and endogenous prices and quantities up to date t. However, for the monetary authority, information about the history of exogenous shocks and endogenous prices and quantities is available up to date $t-1$, and the monetary authority only receives noisy signals about current values. In particular, at time t, the monetary authority observes:

$$s_t^\eta = \eta_t + e_{\eta t}, \text{ where } e_{\eta t} \sim N(0, \sigma_{e\eta}^2)$$
$$s_t^\mu = \mu_t + e_{\mu t}, \text{ where } e_{\mu t} \sim N(0, \sigma_{e\mu}^2).$$

That is, the monetary authority observes noisy signals about the discount rate and the marginal cost shocks at time t. If there were no noise in the signals, the monetary authority would know η_t and μ_t exactly, which would mean that it had perfect information about π_t and x_t as well. In this sense, uncertainty about η_t and μ_t is a stand-in for uncertainty about the equilibrium values of π_t and x_t. In a very general sense, the values π_t and x_t are determined in equilibrium by a mapping from states and shocks to equilibrium prices and quantities:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = f(\eta_{t-1}, \mu_{t-1}, e_{\eta t}, e_{\mu t}, \varepsilon_{\eta t}, \varepsilon_{\mu t}),$$

where the time-invariant function f represents the equilibrium mapping. When $\sigma^2_{e\eta} = \sigma^2_{e\mu} = 0$, s^η_t and s^μ_t reveal $\varepsilon_{\eta t}$ and $\varepsilon_{\mu t}$, which in turn reveal equilibrium quantities and prices, x_t and π_t. In our setup, the monetary authority has to form beliefs about $e_{\eta t}$, $e_{\mu t}$, $\varepsilon_{\eta t}$ and $\varepsilon_{\mu t}$.

We assume that the central bank has an informational disadvantage relative to the private sector so as to capture that, in reality, the central banks need to now-cast prices and quantities when they make policy decisions. There is a long literature about the now-casting problems that central banks face and their associated policy implications. For example, a string of work initiated by Orphanides and Williams (2002) emphasizes that real-time measurement errors can lead to persistent policy mistakes if monetary policy rules do not confront this now-casting problem. In the context of NK models, the literature on monetary policy with imperfect information (typified by Svensson and Woodford (2003), Svensson and Woodford (2004), and Aoki (2003)) often specifies signals as noisy indicators of endogenous variables. Typically, signals take the form:

$$s^\pi_t = \pi_t + e_{\pi t}$$
$$s^y_t = y_t + e_{yt}.$$

When signals are specified in this way, the zero lower bound on the short-term nominal interest rate complicates the joint signal extraction problem and optimal policy problem dramatically. Our information structure allows us to model central bank beliefs using a Kalman filter (discussed below) to form beliefs, which is optimal because η_t and μ_t evolve independently of endogenous prices and quantities. That is, there is no simultaneity problem in our setup because the conditioning set of the monetary authority does not contain any quantities that are functions of endogenous values that are determined in response to the policy decision. This leads to a separation between the optimal filtering problem and the optimal policy problem, which greatly simplifies the information processing problem. We view this imposition of separation as a compromise that embodies an information friction while keeping the problem tractable in the context of the zero lower bound.

An alternative interpretation of our approach is that the distributions of $e_{\pi t}$ and e_{yt} are not normal and instead are such that the signals s^π_t and s^y_t embody the *same* information as s^η_t and s^μ_t. That is, rather than $e_{\pi t}$ and e_{yt} being independently and normally distributed around the endogenous variables that they simultaneously determine, we assume that they have a complex and unknown distribution that makes our assumed signal structure (and the separation between the filtering problem and the optimal policy problem) exactly correct.

While we prefer to interpret our setup from the perspective of a now-casting problem faced by the central bank, another way to rationalize our framework is through a sequential decision structure. In particular, a model economy in which nominal interest rate decisions for period t have to be made before private agents make decisions in period t requires that the central bank bases them on less information than the private sector has available at time t. If, for example, the central bank makes its policy rate decision at the beginning of period t, that decision will be based on information available at time $t-1$ and potentially, any signals the central bank may have regarding current shocks that affect private sector decisions.

This type of timing assumption captures the fact that policy rate decisions are not made on a daily basis, but rather during meetings that occur infrequently. While policymakers may be able to react to incoming data received during the intermeeting periods, historically policy decisions have only been infrequently made outside of regularly scheduled meetings. Thus, most central banks regularly set short-term nominal interest rates based on information available only at the beginning of the period in which that policy decision will be operative. Moreover, with lags in official data releases and frequent revisions to real-time data, even the information available to the central bank about the recent past is likely to be affected by noise.

3.3 Information Processing

To define the signal-extraction problem of the central bank, it is convenient to denote the vector of current state variables by:

$$\xi_t = \begin{bmatrix} \mu_t \\ \eta_t \end{bmatrix}.$$

Given our assumptions in the model, these exogenous variables evolve according to:

$$\xi_t = F\xi_{t-1} + v_t, \tag{7}$$

where

$$F = \begin{bmatrix} \rho_\mu & 0 \\ 0 & \rho_\eta \end{bmatrix} \text{ and } v_t \tilde{} N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{matrix} \sigma_\mu^2 & 0 \\ 0 & \sigma_\eta^2 \end{matrix} \right) = N\left(0, Q\right).$$

As described above, the monetary authority sees all prices and quantities, as well as all state variables, up through time $t-1$. In addition, the monetary authority has access to noisy signals about the current values of the state variables, which we specified above, and can be written as

$$s_t = \left[\begin{array}{c} s_t^{\mu} \\ s_t^{\eta} \end{array} \right] = \xi_t + w_t, \tag{8}$$

where

$$w_t \tilde{} N(0, R),$$

where R is defined accordingly given the definitions above. In terms of our notation, the central bank information set is

$$\{s_t, \xi_{t-1}, \xi_{t-2,} ...\}.$$

We assume that the private sector has complete information, meaning that its information set includes everything the monetary authority sees, as well as the true values of current-period state variables.

Conditional on this information structure, the problem of the monetary authority is to determine the optimal policy rate, given its expectations about equilibrium prices and quantities. The linearity of equations (7) and (8), the normality of the innovations, and the normality of the noise in the signals implies that the beliefs of the central bank about ξ_t will be normally distributed. In fact, given a signal, s_t, the central bank optimally forms beliefs about ξ_t using the Kalman filter so that

$$E(\xi_t | \{s_t, \xi_{t-1}, \xi_{t-2,} ...\}) = F\xi_{t-1} + Q(Q+R)^{-1}(s_t - F\xi_{t-1})$$
$$Var(\xi_t | \{s_t, \xi_{t-1}, \xi_{t-2,} ...\}) = \left(I - Q(Q+R)^{-1}\right) Q$$

where the Kalman gain is given by $Q(Q+R)^{-1}$. These objects are the mean and the variance of the normal posterior distribution of the central bank. The monetary authority can use this distribution to form beliefs about about inflation and the output gap by integrating over the distribution of ξ_t. The distribution of ξ_t can then be used to solve the optimal policy problem of the central bank. Under incomplete information, the first-order necessary condition associated with minimizing the central bank loss function is:

$$E[\lambda x_t + \pi_t \kappa | \{s_t, \xi_{t-1}, \xi_{t-2,} ...\}] + \phi_t = 0.$$

The expectations operator in this first-order condition reflects policymakers use their best estimate of inflation and the output gap in their optimal choice of the nominal interest rate. This condition when coupled with the expressions for x_t and π_t, the zero lower bound constraint, and the complementary slackness condition defines the equilibrium conditions under imperfect information. To see how this first order condition affects the central bank's choice of i_t, it is useful to substitute for x_t in order to get

$$i_t = \frac{1}{\lambda}\phi_t + E\left[E_t\{x_{t+1}\} + E_t\left\{\pi_{t+1} + r_t^e\right\} + \frac{\kappa}{\lambda}\pi_t | \left\{s_t, \xi_{t-1}, \xi_{t-2}, ...\right\}\right].$$

Written in this way, it is clear that even if the private sector has perfect information at time t, the monetary authority needs to form expectations about private sector expectations in order to solve its optimal policy problem.

Uncertainty about the state of the economy in our setup affects optimal policy in a number of ways. As we will show below, there is an attenuation coming from imperfect information. Additionally, the zero lower bound alters the distribution of equilibrium objects for both the private sector and the central bank. Furthermore, because of its informational disadvantage, the monetary authority has to form beliefs about private sector expectations. As such, the private sector expectations about future inflation and output can be thought of as time t variables about which the central bank is uncertain. In our setup, the private sector has different beliefs about both current and future prices and quantities than the monetary authority. Because the central bank has to form expectations about those private sector expectations, they are not *ex-post* equal. This additional uncertainty about private sector expectations arises because of our informational setup, and affects the optimal policy response near the zero lower bound. Finally, the private sector understands that the monetary authority may in the future respond to noise in the signals it receives. Near the zero lower bound, the private sector internalizes that the monetary authority can only respond in one direction, which shifts the mean of the private sector's expected future paths for the short-term nominal interest rates. This tightening bias affects prices and quantities in the current period, which the central bank internalizes.

3.4 Results

As discussed in the appendix, we solve the model using a global solution method that allows us to characterize how imperfect information near the zero lower bound affects optimal policy. We pick relatively standard values for the parameters of the model (see, e.g., Woodford (2003) and more recently Evans et al. (2015)). The discount factor, β, is chosen to be consistent with a quarterly model and is set equal to 0.995. As a result, the steady state real rate equals 2 percent on an annual basis. The central bank's inflation target, $\bar{\Pi}$, is chosen to be consistent with a 2 percent annual rate. The parameter κ which governs the slope of the Phillips curve, is set equal to 0.01. Following Woodford (2003), we choose $\lambda = 0.0156$. This quarterly weight translates into an annualized weight of $\lambda \times 16 \approx 0.25$. We set the parameters describing the shocks as follows. The discount factor shocks has an autocorrelation, ρ_η, equal to 0.8 with a standard deviation, σ_η, of 0.45. Likewise, the markup shock has an autocorrelation coefficient, ρ_μ, equal to 0.3 with a standard deviation, σ_μ, of 0.1. The standard deviations of the noise in the signals, $\sigma_{e\eta}$ and $\sigma_{e\mu}$, are calibrated to be 0.9 and 0.1, respectively. This implies signal-to-noise ratios of 1.25 and 2, respectively.

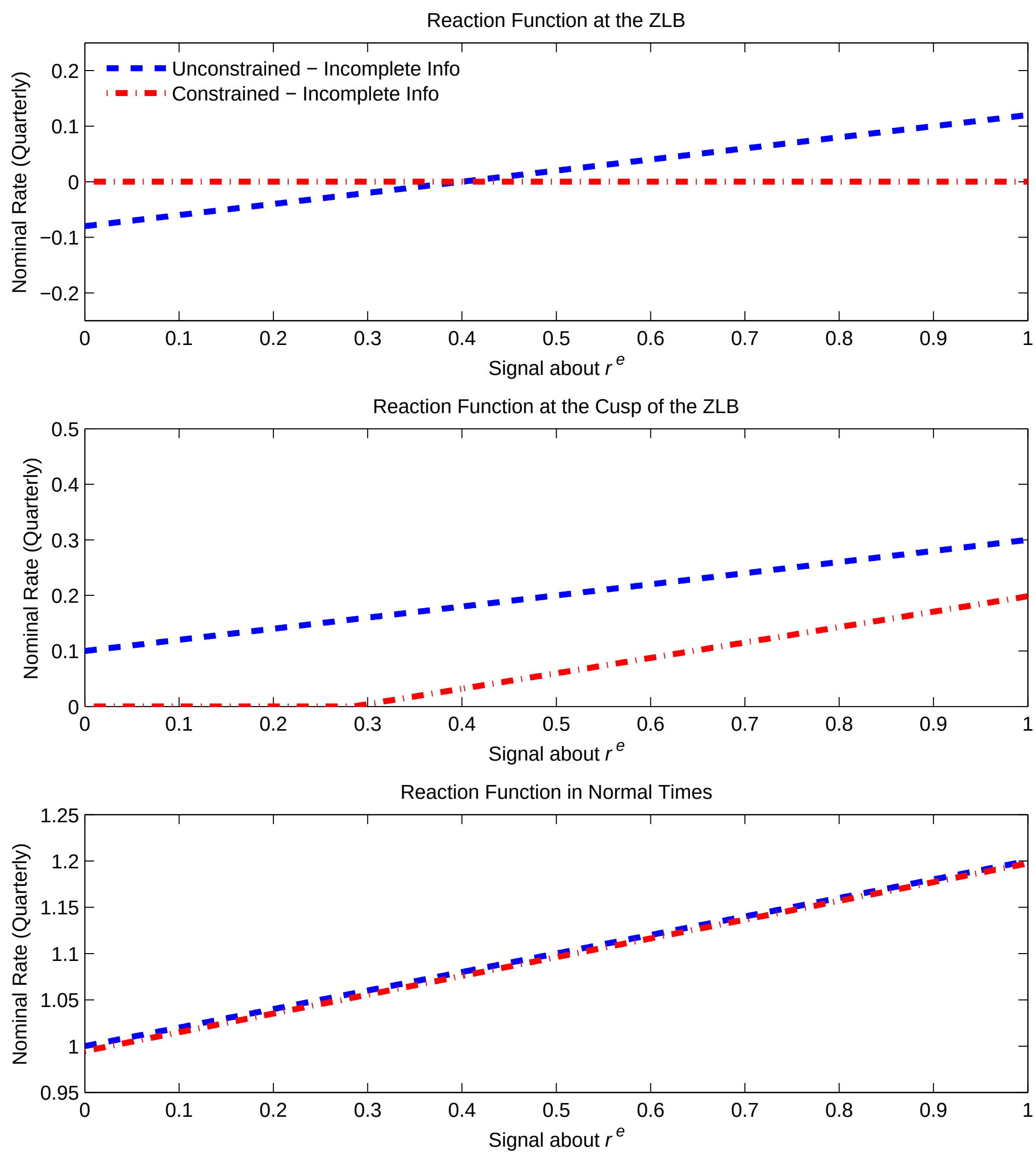
Figure 1 illustrates how incomplete information about the efficient real interest rate interacts with the zero lower bound to affect optimal policy. Each panel in Figure 1 shows the optimal response to a positive signal about the efficient real rate. That is, for each panel, we show how the policymaker optimally sets the nominal short-term interest rate (on the vertical axes, measured in percentage points) in response to an innovation that signals that the natural rate has increased (the horizontal axes show the magnitude of that positive signal, measured in percentage points). The three panels differ in their assumed initial values for the efficient rate.

The top panel assumes that the efficient rate has been well below zero and, as result, the zero lower bound is binding. Even for relatively large positive signals about the efficient rate, the optimal policy leaves the nominal rate unchanged. By contrast, if the lower bound were irrelevant, as shown by the dashed blue line, the optimal policy would set the nominal rate below zero for small, positive signals about the level of the efficient rate. For signals larger than forty basis points the optimal unconstrained policy rate would be positive. Notably, once the zero lower bound is taken into account, the presence of uncertainty keeps the nominal rate at the zero lower bound even in situations where the unconstrained policy would be positive. These lower policy prescriptions reflect that optimal policy opts to insure against particular bad outcomes associated with the zero lower bound.

The middle panel repeats the exercise assuming that the efficient rate has been low enough to put the short-term nominal interest rate at the cusp of the zero lower bound. If the monetary authority acts as if policy is not constrained by the zero lower bound, the nominal rate rises about ten basis point for a positive signal of fifty basis points. When we account for the lower bound constraint, the optimal policy reacts less, rising only five basis points for a signal of fifty basis points, because the policy rate remains at the lower bound until the signal rises above thirty basis points. As the signal grows larger, the reaction of the optimal policy non-linearly approaches the reaction of the unconstrained optimal policy.

The bottom panel shows the effects in normal times corresponding to an initial value of the efficient real rate that is high enough so that the zero lower bound is very unlikely

Figure 1: Optimal Discretionary Policy with Incomplete Information



to bind in the near future. In this case, the policymaker translates any change in beliefs about the natural rate into the nominal interest rate in the same way as a policymaker who is unconstrained by the lower bound.

In Figure 2, we repeat the above experiments for the cases of incomplete and complete information while accounting for the zero lower bound on nominal short-term interest rates. As shown in the bottom panel, policymakers operating in a world of incomplete information downweight the signal they receive about the efficient rate—because they do not know that the signal is correct—whereas policymakers under complete information do not. As can be seen, the optimal policy response in the presence of incomplete information is substantially muted relative to its complete-information counterpart in which the monetary authority translates one-for-one any change in beliefs about the natural rate into the short-term nominal interest rates. Thus, as discussed in Aoki (2003), the optimal policy is muted under incomplete information relative to complete information.

The top panel shows that when the zero lower bound is binding, incomplete information can keep optimal policy at the zero lower bound even when optimal policy with complete information would increase the short-term nominal rate. In sum, the effects of imperfect information are magnified at the zero lower bound. This can be seen by comparing the distance between the lines in the top panel and the distance between the lines in the bottom panel at the far right of each chart.

More generally, the combination of the zero lower bound and imperfect information makes the optimal policy response a complicated, non-linear function of signals about the state of the economy. For small revisions to the equilibrium real rate, the policy rate remains at the zero lower bound regardless of whether information is complete or incomplete. But if the estimate of efficient rate is known to be imprecise, a larger perceived increase is required to move away from the zero lower bound, and the policy responses under these conditions are even more attenuated than they are in normal times. The reason is as follows. When the policy rate is constrained by the zero lower bound and the policymaker receives a noisy signal that the efficient rate has increased, the optimal policy is to make a mistake on the side of keeping the real policy rate below the perceived efficient rate so as to reduce the likelihood that an adverse shock will weaken the economy enough, to require a return to the zero lower bound. That is, policymakers "take out insurance" against situations in which the information about the equilibrium real rate might lead them to mistakenly raise the policy rate.

In contrast to the model studied by Aoki (2003) and Boehm and House (2014), the

Figure 2: Optimal Discretionary Policy under Alternative Information Sets

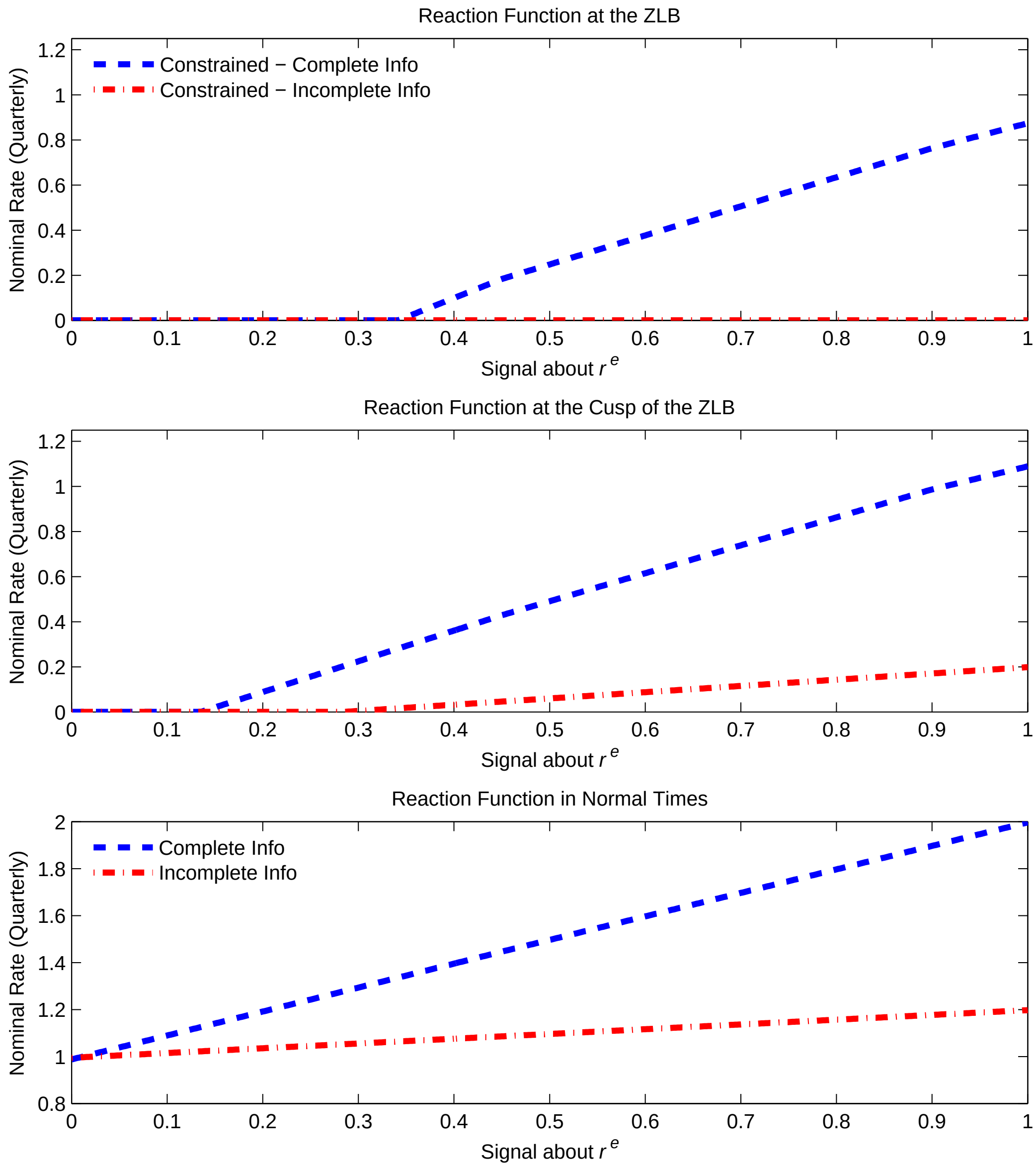
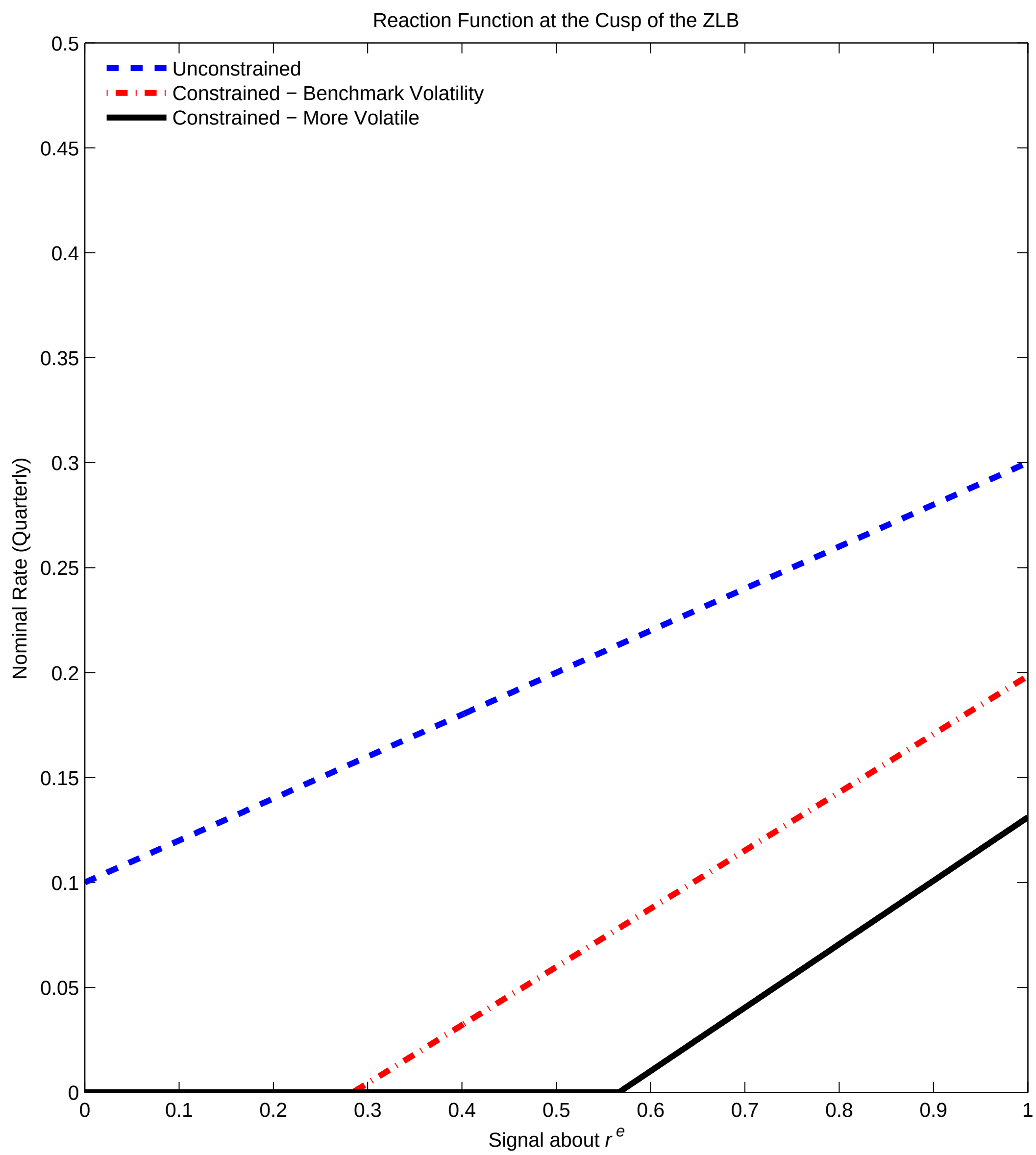


Figure 3: Optimal Discretionary Policy near the ELB: More Volatile Economy



zero lower bound makes the optimal policy deviate from certainty equivalence. Figure 3 shows the implications for optimal policy of increasing the variance of shocks in the economy.[7] In a more uncertain environment, optimal policy becomes even less responsive to positive signals about the state of the economy when the economy is at the cusp of the lower bound.

Figure 4 shows the outcomes for inflation and the output gap when *ex-post* the signal about the change in the efficient real rate is either correct or entirely noise. *Ex-ante* the optimal policy reacts to the signal itself, so the policy rate is identical whether the signal turns out to be correct or not (as shown in the top-right panel). Because the economy is at the cusp of the zero lower bound, with a sufficiently positive signal the central bank increases the short-term interest rate. For the same *ex-ante* policy, *ex-post* if the signal turns out to be entirely noise (the red dot-dashed line), the output gap is relatively small and inflation is near target. If the signal turns out to reflect an actual change in the natural rate, then the output gap and inflation move in the same direction as the natural rate. However, responses of the output gap and inflation are much larger when the efficient rate has *ex-post* declined as opposed to increased. That is, it is more costly to (*ex-ante*) overestimate the natural rate, and to raise the nominal rate without an actual increase in the natural rate, than it is to (*ex-ante*) underestimate the natural rate and leave the nominal rate at the lower bound. This captures the essence of a risk-management approach at the cusp of the lower bound: Optimal policy preserves the option to maintain the zero short-term rate so that if the (*ex-ante*) signal about the efficient real rate is proven to be wrong the costs of misinterpreting the signal (deferring the rate increase) are low.

Another way to grasp the effects of imperfect information about the efficient real rate near the zero lower bound is to look at the distribution of economic outcomes. Figure 5 shows the distribution of the efficient rate, the nominal rate, the output gap, and inflation when we simulate the economy at the cusp of the lower bound (dot-dashed lines) and in normal times (solid lines).[8] As reflected in top-left panel, in normal times the mean of this distribution is higher as compared to when the economy is at the cusp of the lower bound. Accordingly, during normal times the distribution of nominal interest is centered to the right of the distribution of the nominal rates when the economy is at the cusp of the zero lower bound. When the economy is at the cusp of the lower bound there is substantial probability that the nominal interest rate will remain at its lower bound. For the output gap, there is

[7] When we increase the volatility of the underlying shocks, we keep the same signal-to-noise ratio as in the previous exercises.

[8] For the simulations, the innovations (including the noisy ones) for the efficient real rate and marginal cost shocks are simulated according to their exogenous stochastic distributions, while initial values for the marginal cost shocks is set to zero.

Figure 4: The Effect of Actual and Perceived Innovations to the Natural Rate

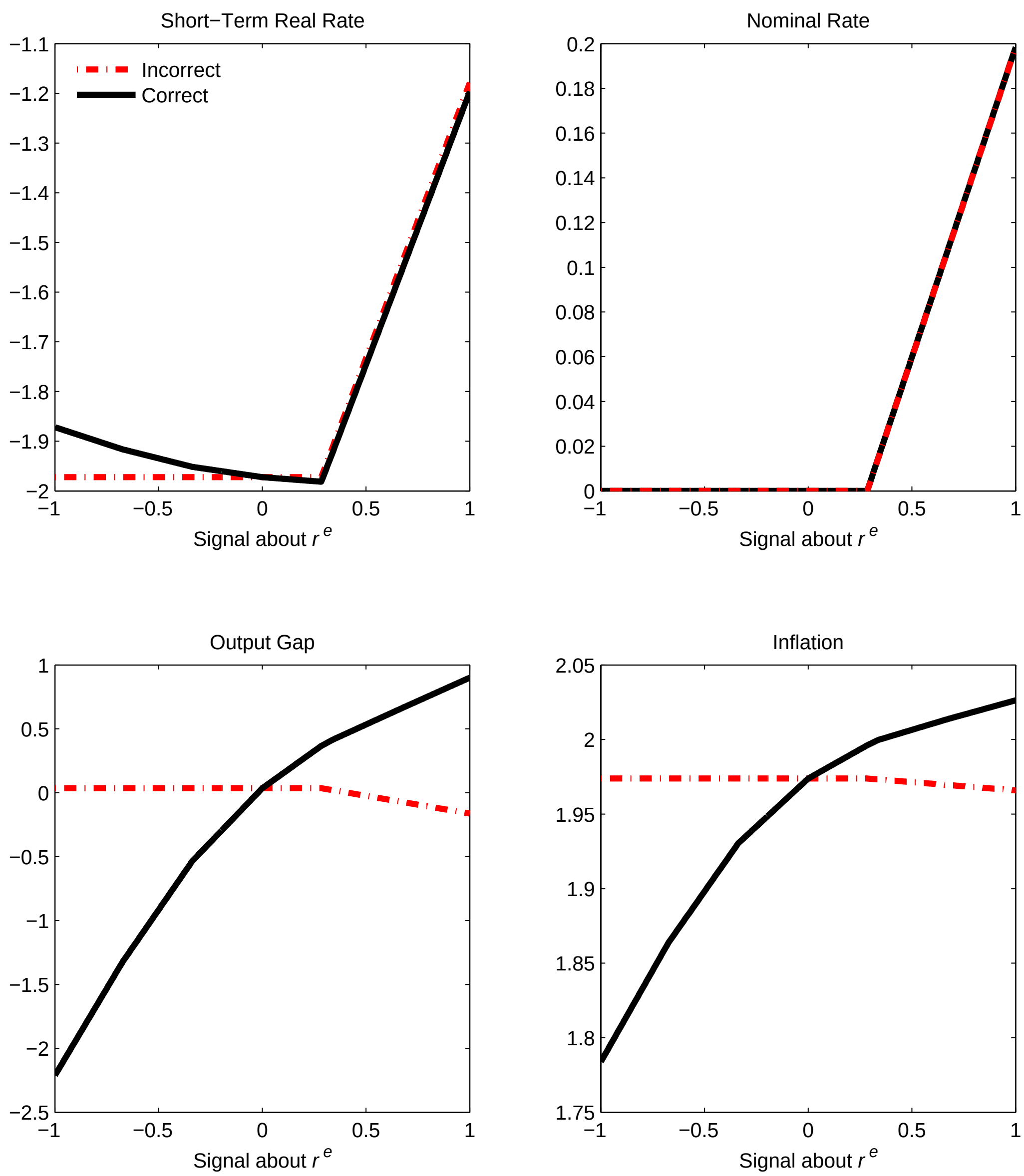
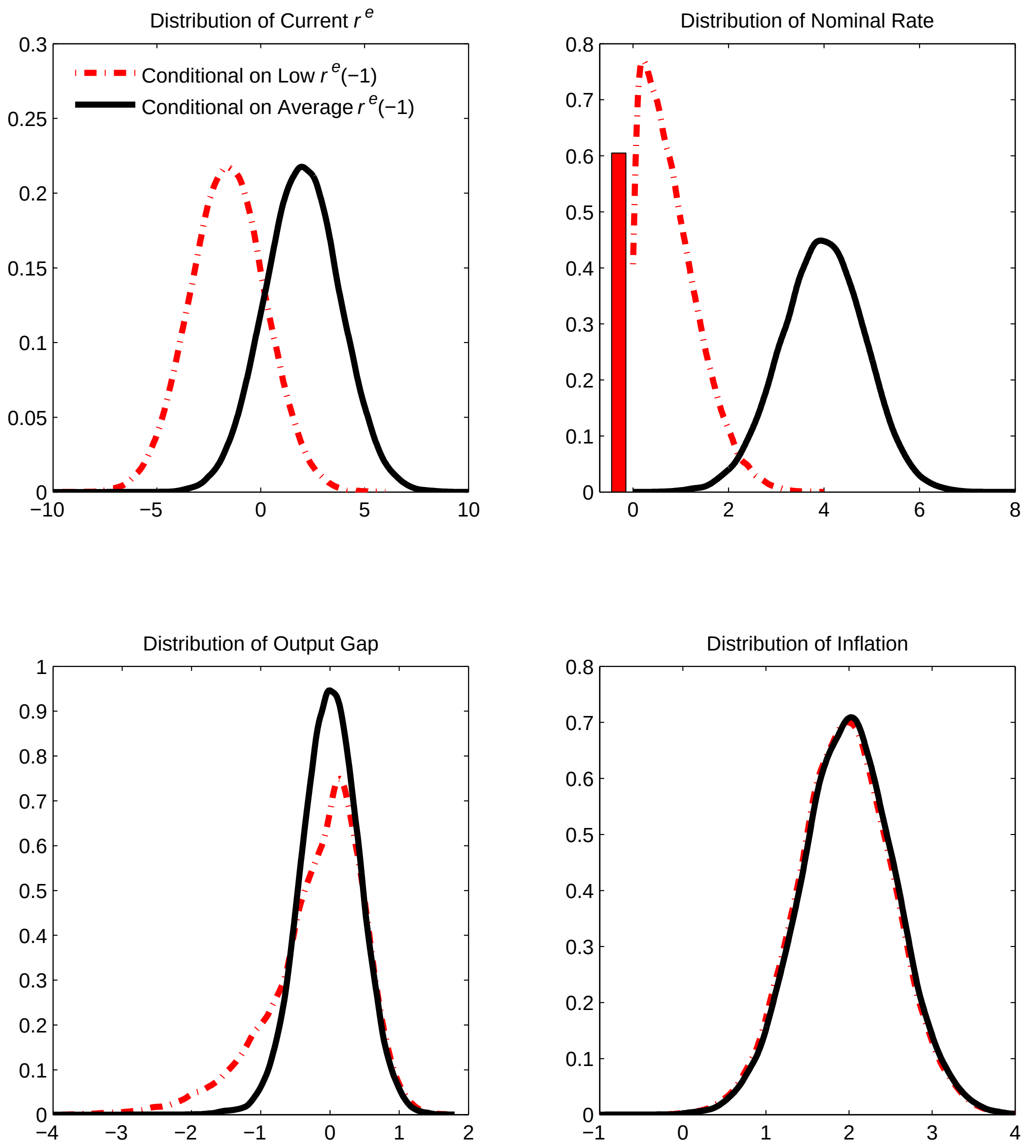


Figure 5: Distributions Conditional on Alternative Natural Real Rates



a notable difference in the two distributions, as the distribution with the economy near the cusp of the lower bound is markedly skewed downward. These adverse outcomes are what leads to the muted response we discussed above. In effect, an optimal policy that accounts for the imperfect knowledge about the efficient rate will provide precautionary stimulus to offset the downward bias in economic outcomes caused by the zero lower bound.

4 Simple Rules

The analysis up to now has focused on optimal discretionary policy, which consists of a strategy such that the policymaker sets the nominal rate in response to incoming data on a period-by-period basis without committing itself to future actions as a way of affecting private sector expectations. The results highlight the interplay of the zero lower bound and imperfect knowledge about the efficient rate in shaping optimal discretion as a basis for pursuing a *risk-management* approach that insures the economy against undesirable outcomes. In this section, we study the effectiveness of committing to a rule-based approach in setting monetary policy within the same economic framework.

4.1 Taylor-type Rules

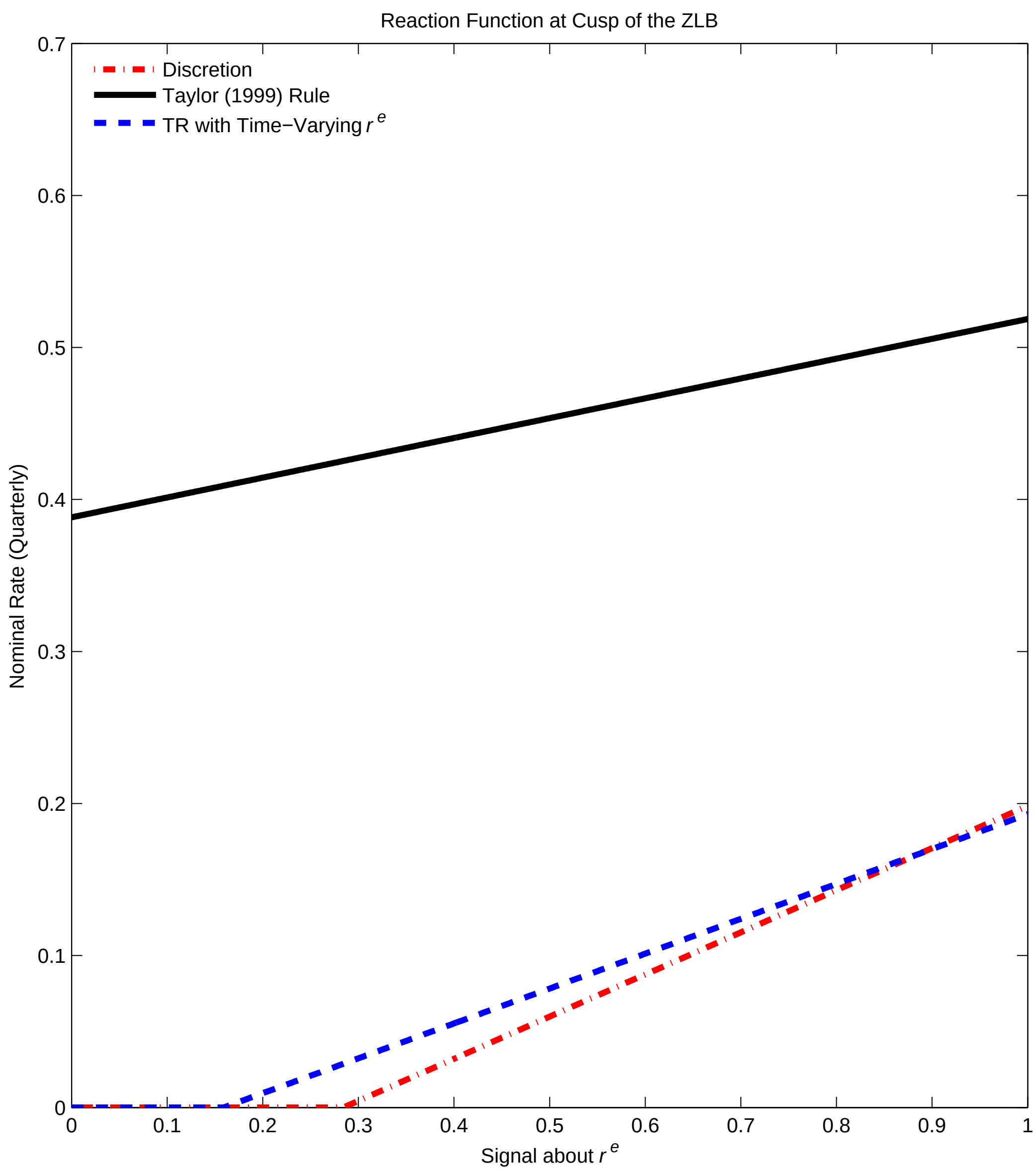
Common simple rules in the literature on monetary policy are Taylor-type rules which take the form of the rules studied in Taylor (1993) and Taylor (1999). We compare the performance of Taylor-type rules to optimal policy. The Taylor-type rule that we consider has the form:

$$i_t = \max\{\log(\beta\overline{\Pi}^{-1}), E\left[\alpha r_t^e + \gamma_\pi \pi_t + \gamma_x x_t \right| \{s_t, \xi_{t-1}\}]\}.$$

The rule studied in Taylor (1999) serves as a baseline case and corresponds to parameter values of $\alpha = 0$, $\gamma_\pi = 1.5$, and $\gamma_x = 1/4$. When $\alpha = 0$, the rule does not includes a time-varying intercept to reflect movements in the efficient real rate (r_t^e). Notably, because policymakers have imperfect information about the exact values of the underlying state of the economy, we assume that the use their best estimate of inflation, the output gap, and the efficient real rate as arguments to the rule. Even if policymakers were to commit to a simple policy rule, operationalizing a rule for the time t interest rate that is written in terms of time t inflation and the time t output gap would require measurement of inflation and output and estimation of the deviation of output from its potential level. Our imperfect information setting is meant to capture that process.

Figure 6 shows the response of the nominal interest rate to a positive signal about

Figure 6: Taylor Rules and Optimal Discretion at the Cusp of the ZLB



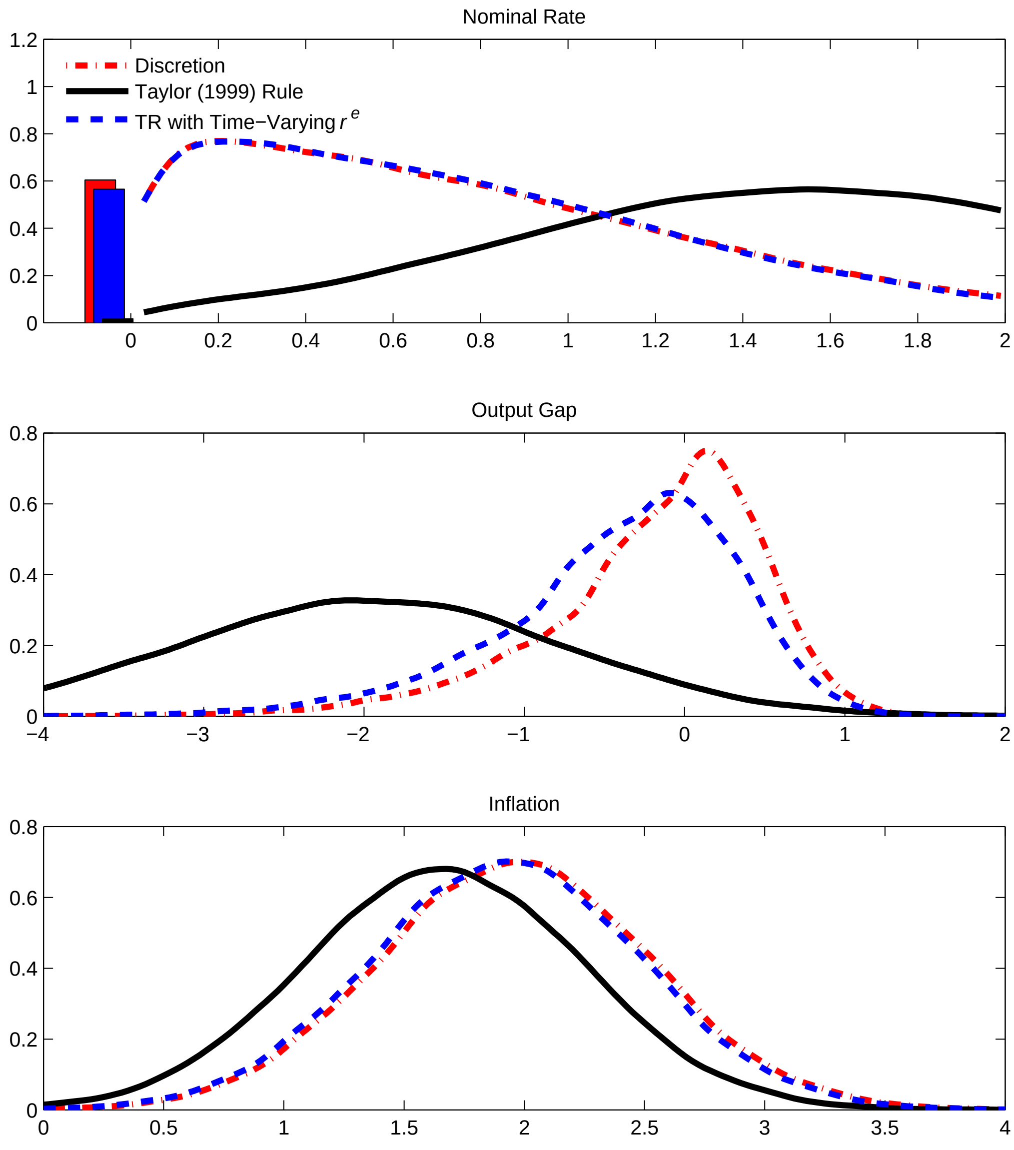
and increase in r_t^e. We show the policy response assuming that the economy is initially in a situation where the equilibrium real rate is low and the zero lower bound is just binding for optimal policy under discretion. The policy response under discretion is shown as the dot-dashed red line. The policy response for the Taylor-type rule where $\alpha = 0$ corresponds to the blue solid line, which is notably higher than the optimal policy under discretion. That the Taylor-type rule prescribes higher interest rates than the optimal policy under discretion is reminiscent of results reported in Boehm and House (2014), who show that the parameter values considered by Taylor (1999) systematically under-react to movements in endogenous variables relative to the optimal policy rule.

The blue dotted line shows the reaction of the Taylor-type rule with $\alpha = 1$, that is, the case when the intercept term of the Taylor rule adjusts with perceived movements in the efficient real rate. The prescriptions of this rule are notably lower than the case in which $\alpha = 0$ because the central bank's estimate of the efficient real rate directly enters the rule. That the prescriptions of this rule are similar to optimal discretionary policy might not be entirely surprising given that optimal policy in a model that ignores the zero lower bound (in the absence of markup shocks) is to set the nominal interest rate equal to the efficient real rate. The fact that the rule does mimic optimal discretion when $\alpha = 1$ reflects the presence of the zero lower bound which makes optimal policy even more accommodative than the policy prescribed by the rule.

Figure 7 shows the distributions of outcomes, conditional on the economy being at the cusp of the zero lower bound (as assumed in the previous figure) for the Taylor-type rules with $\alpha = 0$ and $\alpha = 1$. The top panel shows the distribution of the time t nominal interest rate. Because of our assumed initial conditions for the equilibrium real rate, there is a significant probability that the nominal rate remains at the zero lower bound for optimal policy under discretion as well as under the Taylor rule with a time-varying intercept term that tracks perceived movements in the natural real rate. The Taylor rule with a constant intercept equal to the steady state real rate produces a distribution of higher nominal interest rates, with much less mass at the zero lower bound. This result is a direct effect of the relatively high prescriptions for the policy rate coming from a Taylor rule with a constant intercept (shown in the previous figure).

The relatively high nominal interest rate from the Taylor rule with a constant intercept leads to lower average outcomes for the output gap and inflation, shown in the middle and bottom panels of Figure 7. That this rule underreacts to perceived changes in the equilibrium real rate is reflected in the relatively dispersed distributions for the output gap and inflation

Figure 7: Conditional Distributions (at the Cusp of the ZLB) under Alternative Taylor Rules



when compared to the outcomes under optimal discretionary policy. By contrast, if the intercept of the Taylor rule reflects perceived changes in the equilibrium real rate (the case of $\alpha = 1$), the distributions of the nominal interest rate, the output gap, and inflation closely resemble the outcomes under optimal discretion.

The differences between the two specifications of these Taylor-type rules are also apparent in an unconditional sense. Figure 8 shows the unconditional distributions of the nominal interest rate, the output gap, and inflation under optimal policy and the Taylor-type rules when $\alpha = 0$ and when $\alpha = 1$. The distributions of outcomes when $\alpha = 1$ are very similar to those under optimal policy. Given the close relationship between the two policies when the zero bound is ignored, this is perhaps not surprising. The Taylor-type rule with a constant intercept performs notably worse than the others in that the distributions of inflation and output are much more dispersed than under optimal discretionary policy. These dispersed outcomes are the result of the rule systematically under-responding to movements in the efficient real interest rate, which can be seen by the relatively thin tails of the distribution of the nominal interest rate, shown in the top panel.

In our simulations, the nominal interest rate under optimal policy spends more time, on average, at the zero lower bound than either specification of the Taylor-type rule. This reflects the nonlinearity in the optimal policy response that causes policymakers to ease policy in the vicinity of the zero lower bound as a risk-management approach to policy. That is, deviating from a Taylor-type rule with a constant intercept term by keeping policy lower near the zero lower bound is an optimal response, which can be approximated by adjusting the Taylor-rule intercept to changes in the central bank's best estimate of the efficient real rate.

Figure 8: Unconditional Distributions under Alternative Taylor Rules

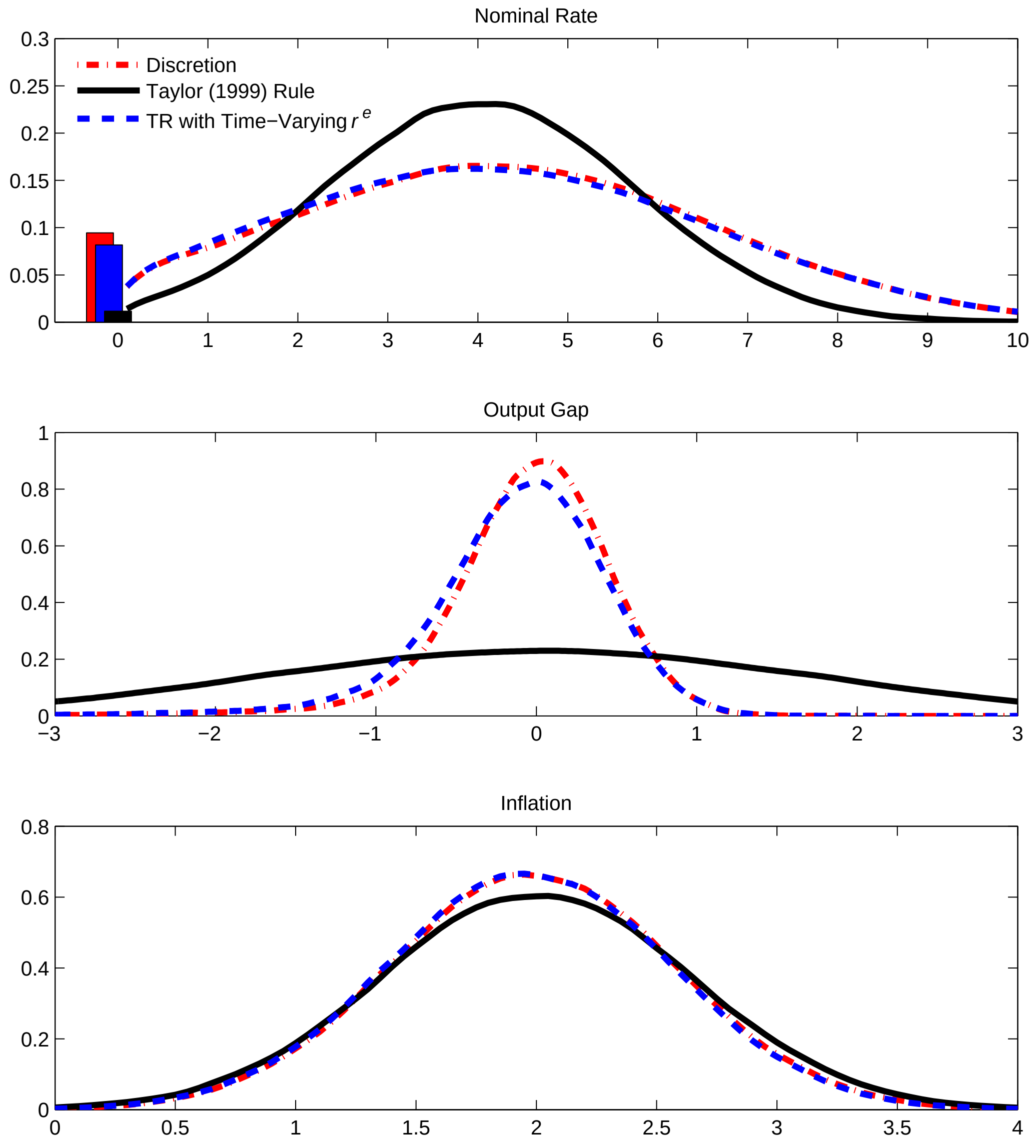


Table 1. Unconditional Welfare Losses for Alternative Monetary Policy Rules

Policy Rules	Welfare Losses	Contributions of Inflation $(\pi - E(\pi))^2$	$E(\pi) - \overline{\pi}$	Output $(x - E(x))^2$	$E(x)$
Optimal	5.375	0.02233	-0.0038	0.2877	-0.051
Taylor-type					
Constant Intercept	14.720	0.0265	0.00017	3.0203	0.0018
Time-Varying Intercept	5.529	0.0221	-0.0077	0.3356	-0.1211
Constant Intercept, Optimal	5.654	0.0224	-0.0050	0.3747	-0.0327
First-Difference, Optimal	19.830	0.0153	-0.0032	5.3746	-0.0228

The unconditional distributions of the inflation and the output gap determine the relative performance of these rules as evaluated by the monetary authority's unconditional loss function. Table 1 shows the relevant moments as well as the unconditional losses across rules. Notably, the zero lower bound shifts the means of the distributions down for almost all of the rules, although by small amounts.[9] Differences in relative performance across rules are almost entirely driven by differences in the dispersion of the distributions of the output gap and inflation. The more dispersed outcomes under the standard Taylor rule ($\alpha = 0$) lead to greater losses than the other specification of the Taylor rule ($\alpha = 1$). Interestingly, in our model neither specification performs better than optimal discretion.

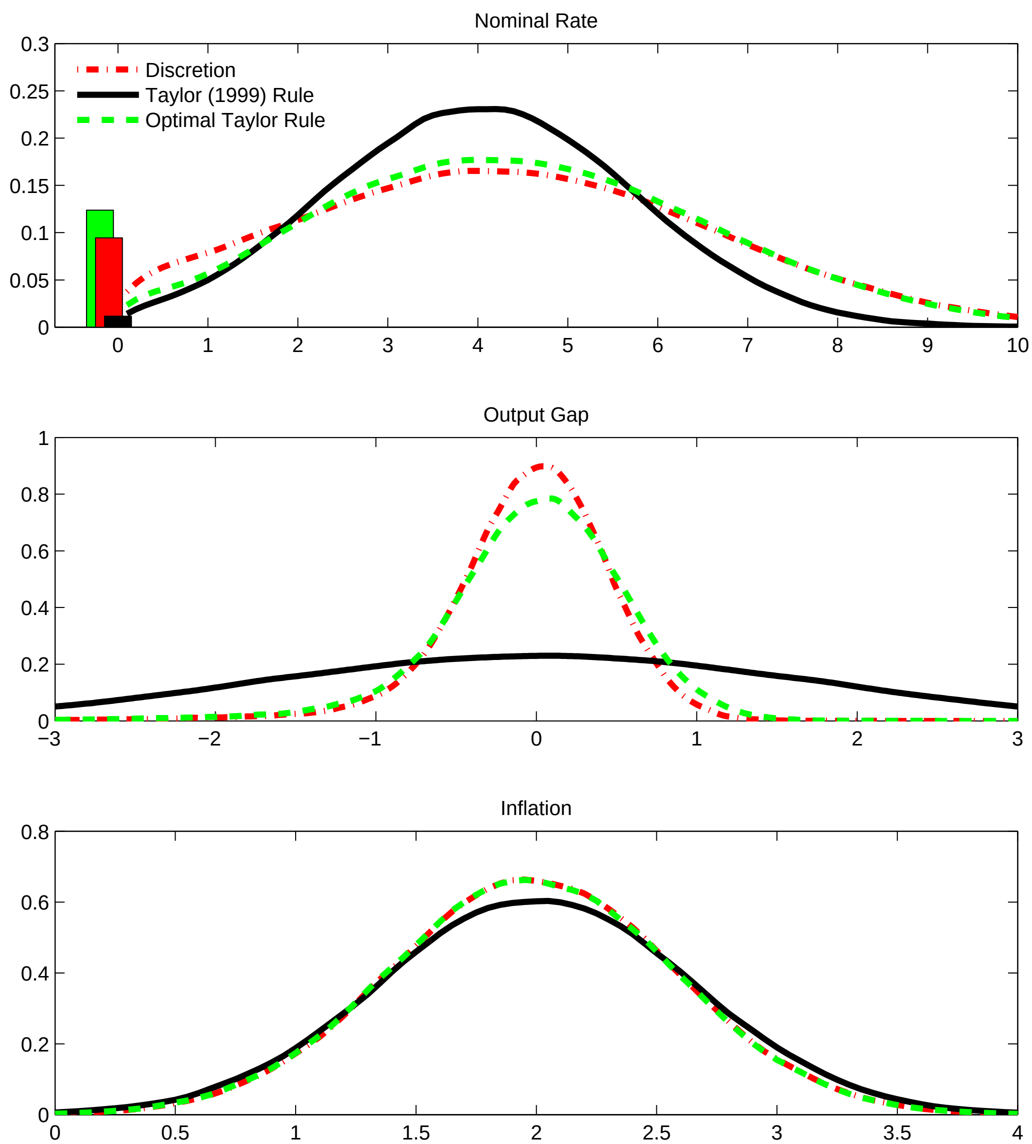
While the Taylor-type rule with a constant intercept performs relatively poorly for the parameter values chosen above, we also consider an optimal version of the rule, where the parameters γ_π and γ_x are chosen to minimize the unconditional loss function of the monetary authority. The optimal parameter values (which are about $\gamma_\pi = 18$ and $\gamma_x = 10$) are much higher than those commonly used in the literature on monetary policy rules.[10] Notably, they imply that the monetary authority should optimally respond more strongly to deviations of inflation and output from their targets than prescribed by more-traditional parameter values.[11] This aggressive response is evidenced by the fatter tails of the unconditional distribution of the nominal interest rate with the optimal Taylor-type rule with a

[9] All of the numbers in Table 1 are expressed in percent.

[10] The welfare loss is relatively flat for relatively high values of these coefficients.

[11] These results are reminiscent of results reported in Boehm and House (2014), who do not consider the interest-rate lower bound.

Figure 9: Unconditional Distributions under Optimal Taylor Rules



constant coefficient in Figure 9, relative to the Taylor-type rule with a constant intercept and our baseline parameters.

Interestingly, as shown in Figure 9, the unconditional distribution of the nominal interest rate under the optimal Taylor-type rule is similar to the distribution under optimal policy, however optimal policy has a somewhat fatter left tail, even though it spends less time at the zero lower bound, reflecting the complex nonlinear reaction function of the monetary authority under optimal policy. Notably, the distributions of inflation and output under the optimal Taylor-type rule are similar to the distributions under discretion. That is, the outcomes under optimal policy, in our model, can be well approximated by a Taylor-type rule that either adjusts its intercept to changes in the equilibrium real rate or by a Taylor-type rule that responds aggressively to deviations of inflation and output from their target values. Although the unconditional distributions are similar, the optimal Taylor rule still slightly underperforms optimal discretionary policy as shown in Table 1.

4.2 First-difference Rules

Because of the uncertainty surrounding the equilibrium real rate, some researchers have argued that rules that depend on it are flawed (e.g., Orphanides and Williams (2002)). As an alternative, they have advocated for first-difference rules as a guide for the conduct of policy, as these rules do not require responding to changes in the equilibrium real rate because they are rules for the change in the nominal interest rate, rather than the level. We revisit this debate by evaluating the performance of first-difference rules when there is imperfect information about the equilibrium real rate and compare their performance to the Taylor-type rules discussed above, as well as optimal discretionary policy.

We consider first-difference rules of the form:

$$i_t = \max\{\log(\beta\overline{\Pi}^{-1}), i_{t-1} + E\left[\gamma_\pi \pi_t + \gamma_y (y_t - y_{t-1}) \middle| \{s_t, \xi_{t-1}\}\right]\}.$$

Although this rule eliminates the equilibrium rate from its specification, policymakers still need to form expectations about (now-cast) current inflation and the current level of output. We specify the rule in terms of output growth, rather than the output gap as in the Taylor-type rules above, because Orphanides and Williams (2002) have emphasized that responding to the output gap requires estimating the potential level of output, which is unobservable and potentially difficult to learn about. However, unlike in Orphanides and Williams (2002), we assume that the monetary authority is not perfectly informed about the output and inflation terms that enter the rule at time t.

Figure 10: Unconditional Distributions under Optimal First-Difference Rules

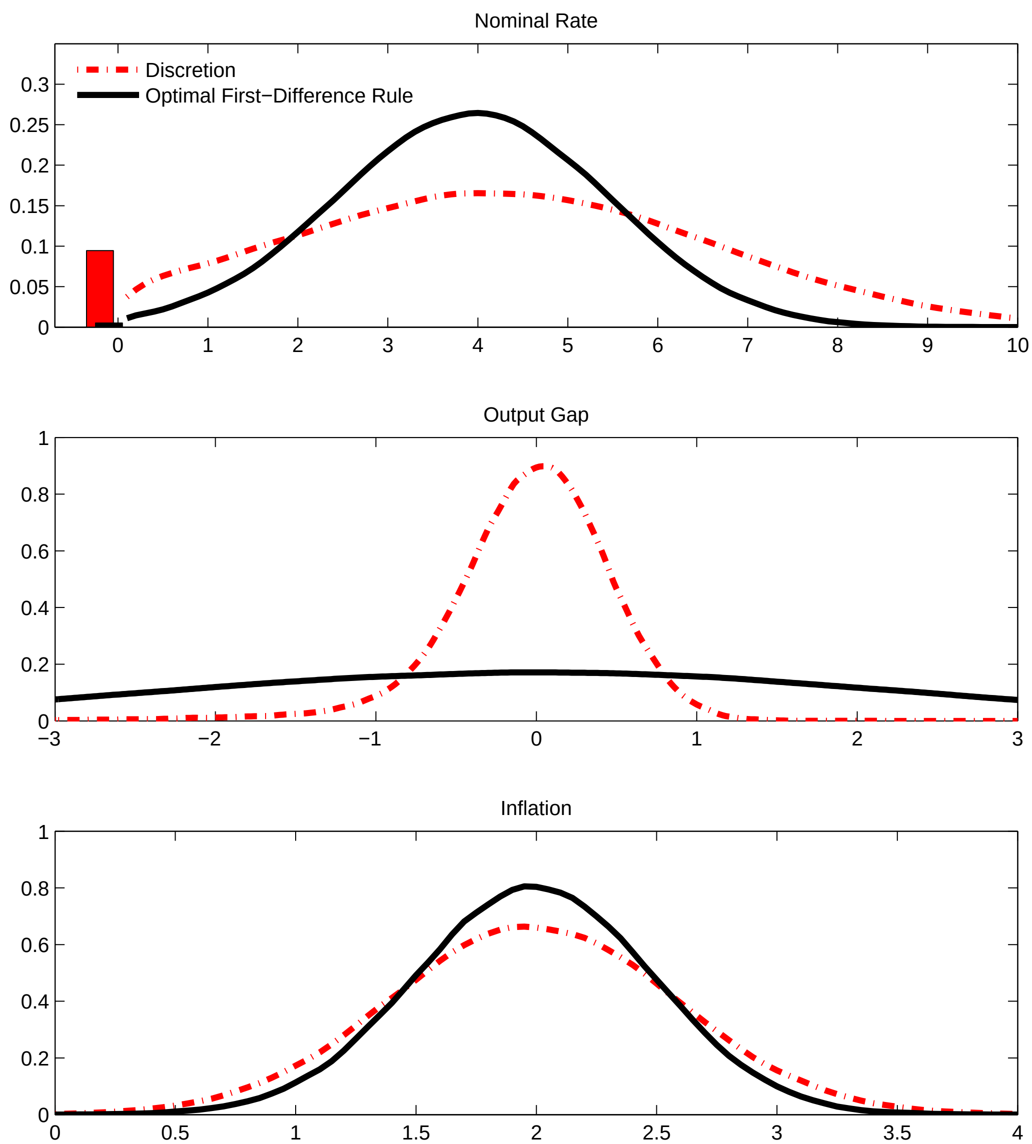


Figure 10 shows the unconditional distributions of the nominal interest rate, the output gap, and inflation under the first-difference rule in which the coefficients have been chosen to minimize the unconditional loss function facing the monetary authority. Even when we optimally choose the parameters, the unconditional distribution of the nominal interest rate from the first-difference rule is less dispersed than under optimal policy. While we find that the optimal first-difference rule responds aggressively to inflation $\gamma_\pi = 2.1$, we find that the optimal coefficient on output growth is almost zero. This muted response to changes in output growth leads to the wide dispersion of the output gap, shown in the middle panel of figure. While the unconditional distribution of inflation under the first-difference rule is less dispersed than the distribution of inflation under optimal policy, this reduction in variance does not lead to better welfare outcomes (as shown in the last row of Table 1) than under discretionary optimal policy because of the wide distribution of the output gap.

The reason that the optimal first-difference rule does not respond more strongly to output or inflation is that doing so would also respond more strongly to incorrect signals. When noise in the monetary authority's information set at time t enters the first-difference rule, the noise is incorporated into the level of the nominal interest rate and is not discarded in the next period. That is, the first-difference rules carries imperfect information forward, magnifying the effects. By contrast, Taylor-type rules (without interest rate smoothing) do not carry forward past misperceptions about the state of the economy directly in the rule.

While we could of course lag the inflation and output terms in the first-difference rule by one period so that there would no longer be any quantities in the rule about which the monetary authority has imperfect information, that solution would be specific to the information structure assumed in our model. If measures of inflation and output will never be free of noise (e.g. measurement error), even after many subsequent periods of measurement, the optimal policy problem will always have to confront the imperfect information problem, and if imperfectly observed quantities enter a first-difference rule, misperceptions will be carried forward.

5 Conclusions

Our analysis has shown that, in a simple NK model, imperfect knowledge about the equilibrium real interest rate interacts with the zero lower bound to cause optimal policy under discretion to respond to signals about an increase in the equilibrium real rate by less than it would during more-normal circumstances. Because optimal policy under discretion in the neighborhood of the zero lower bound prescribes a complicated, nonlinear reaction function for the policy rate that may, in practice, be difficult to communicate and implement, we also study simple policy rules. In our model, we show that Taylor-type rules that either include a time-varying intercept that moves with perceived changes in the equilibrium real rate or that respond aggressively to deviations of inflation and output from their target levels perform similarly to optimal discretionary policy. While we do not explicitly model credibility and communications concerns for the monetary policy authority, these simple rules may be easier to implement and communicate to the public. Our analysis of first-difference rules highlights that rules with smoothing terms carry forward current and past misperceptions about the state of the economy and can lead to suboptimal performance.

Appendix

A Detailed Description of the Model The economy consists of a representative household, a continuum of firms producing differentiated intermediate goods, perfectly competitive final goods firms, and a central bank in charge of monetary policy.

Households There is a representative household that values streams of consumption C_t and hours worked H_t according to preferences given by the utility function:

$$E_t \sum_{t=1}^{\infty} \delta_{t-1} \left(\log \left[C_t \right] - \mu_t H_t \right) \tag{9}$$

where E_t represents the mathematical expectation conditional on all exogenous shocks and endogenous prices and quantities up to time t. The household maximizes expected utility flows subject to a sequence of budget constraints,

$$P_t C_t + R_t^{-1} B_t = W_t H_t + B_{t-1} + T_t. \tag{10}$$

where household nominal expenditures on consumption at date t are given by $P_t C_t$, where P_t denotes the aggregate price level, B_t are nominal bonds sold at the price R_t^{-1}, and $R_t = (1+i_t)$ denotes the gross nominal return on these bonds. The nominal bonds purchased by a household pay one unit of the numéraire in the next period with certainty. A household receives income from any bonds carried over from last period in addition to its labor income, $W_t H_t$, from supplying its labor services to the economy's firms. A household also pays lump-sum taxes and receives dividends from intermediate goods firms, the sum of which is denoted by T_t. The labor supply shock, μ_t, is assumed to follow an AR(1) process in logs, such that:

$$\log(\mu_t) = \rho_\mu \log(\mu_{t-1}) + \varepsilon_{\mu,t} \tag{11}$$

where ρ_μ denotes the persistence of the shock and σ_μ denotes the standard deviation of the innovations, $\varepsilon_{\mu,t}$.

To capture exogenous changes in the household's desire to save, we allow for a shock to the discount rate. The rate of time discounting at time t is given by $\delta_t = \beta^t \left(\prod_{s=0}^{t} \eta_s \right)^{-1}$, where $0 < \beta < 1$. Hence, $\frac{\delta_{t+1}}{\delta_t} = \frac{\beta}{\eta_{t+1}}$, and η_{t+1} is the shock to the discount rate (or the natural rate of interest, as will be clear below). The discount rate shock follows an AR(1) process in logs, such that:

$$\log(\eta_t) = \rho_\eta \log(\eta_{t-1}) + \varepsilon_{\eta,t} \tag{12}$$

where ρ_η denotes the persistence of the shock and σ_η denotes the standard deviation of the innovations, $\varepsilon_{\eta,t}$. A decrease in η_t represents an exogenous factor that induces a temporary rise in a household's propensity to save, and reduces current aggregate household demand for goods.

The first order necessary conditions for optimality from the household's problem can be written as follows:

$$1 = \frac{\beta}{\eta_t} R_t E_t \left\{ \frac{C_t}{C_{t+1}} \Pi_{t+1}^{-1} \right\}, \tag{13}$$

$$\mu_t w_t = C_t \tag{14}$$

where $w_t = \frac{W_t}{P_t}$ denotes the real wage, and $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the inflation rate between t and $t+1$. The linear specification for labor services in equation (9) along with a competitive labor market implies that there is a perfectly elastic supply of labor available to the economy's firms.

Firms There is a continuum of monopolistically competitive firms producing differentiated intermediate goods. The latter are used as inputs by perfectly competitive firms that produce a single final good, Y_t, using a constant returns to scale production technology, $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \, dj \right)^{\frac{\epsilon}{\epsilon-1}}$, where $Y_t(j)$ is the quantity of intermediate good j used as an input, and $\epsilon > 1$ is the elasticity of substitution. Profit maximization and perfect competition yield the set of demand schedules, $Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t$ and aggregate price index, $P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} \, dj \right)^{\frac{1}{1-\epsilon}}$.

The production function for intermediate good j is given by:

$$Y_t(j) = H_t(j). \tag{15}$$

Since intermediate goods are imperfect substitutes, the intermediate goods-producing firms sell their output in a monopolistically competitive market. During period t, the firm sets its nominal price $P_t(j)$, subject to the requirement that it satisfies the demand of the representative final goods producer at that price. As in Rotemberg (1982), the intermediate good producer faces a quadratic cost of adjusting its nominal price between periods, measured in terms of the finished good and given by: $\frac{\varphi}{2} \left(\frac{P_t(j)}{\bar{\Pi} P_{t-1}(j)} - 1 \right)^2 Y_t$, where φ governs the obstacles to price adjustment and $\bar{\Pi}$ denotes the central bank's inflation target. The cost of price adjustment makes the problem of the intermediate good producer dynamic; that is, it chooses $P_t(j)$, taking as given $\bar{\Pi}$, to maximize its present discount value of expected profits:

$$E_0 \sum_{t=1}^{\infty} \frac{\delta_{t-1}}{C_t} \left\{ \left(\frac{P_t(j)}{P_t} \right)^{1-\epsilon} Y_t - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} (1+\tau) \mu_t w_t Y_t - \frac{\varphi}{2} \left[\frac{P_t(j)}{\bar{\Pi} P_{t-1}(j)} - 1 \right]^2 Y_t \right\},$$

where $\frac{\delta_{t-1}}{C_t}$ measures the marginal value—in utility terms—to the representative household of an additional unit of real profits received in the form of dividends during period t, w_t is the firm's real marginal cost. τ is an employment subsidy. The first order condition for this problem is:

$$(1-\epsilon)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} + \epsilon\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon-1}(1+\tau)\mu_t w_t - \varphi\left[\frac{P_t(j)}{\bar{\Pi}P_{t-1}(j)} - 1\right]\frac{P_t(j)}{\bar{\Pi}P_{t-1}(j)} \tag{16}$$
$$+\varphi\frac{\beta}{\eta_t}E_t\left\{\frac{C_t}{C_{t+1}}\left[\frac{P_{t+1}(j)}{\bar{\Pi}P_t(j)} - 1\right]\frac{P_{t+1}(j)}{\bar{\Pi}P_t(j)}\frac{P_t}{P_t(j)}\frac{Y_{t+1}}{Y_t}\right\} = 0.$$

Notice that under flexible prices, i.e. $\varphi = 0$, then expression (16) leads to the well-known expression where firm set prices as a time-varying mark-up, $\frac{\epsilon}{\epsilon-1}$, over marginal cost, i.e. $P_t = \frac{\epsilon}{\epsilon-1}(1+\tau)\mu_t w_t$.

Market Clearing and Symmetric Equilibrium Conditions In a symmetric equilibrium, all intermediate goods producing firms make the same decisions so that aggregate output satisfies $Y_t = Y_t(j)$ and aggregate labor satisfies $H_t = H_t(j)\ \forall j$. It then follows that $Y_t = H_t$ and goods market clearing can be written in aggregate terms as:

$$Y_t = C_t + \frac{\varphi}{2}\left(\frac{\Pi_t}{\bar{\Pi}} - 1\right)^2 Y_t. \tag{17}$$

Households trade the nominal bond among themselves so that bond market clearing implies $B_t = 0$.

The symmetric equilibrium conditions from households and firms as well as the goods market clearing condition are:

$$1 = \frac{\beta R_t}{G\eta_t}E_t\left\{\frac{C_t}{C_{t+1}}\Pi_{t+1}^{-1}\right\}, \tag{18}$$

$$\varphi\left(\frac{\Pi_t}{\bar{\Pi}} - 1\right)\frac{\Pi_t}{\bar{\Pi}} = \frac{\varphi\beta}{\eta_t}E_t\left\{\frac{C_t}{C_{t+1}}\left(\frac{\Pi_{t+1}}{\bar{\Pi}} - 1\right)\frac{\Pi_{t+1}}{\bar{\Pi}}\frac{Y_{t+1}}{Y_t}\right\} + \epsilon C_t(1+\tau)\mu_t + (1-\epsilon) \tag{19}$$

$$Y_t = C_t + \frac{\varphi}{2}\left(\frac{\Pi_t}{\bar{\Pi}} - 1\right)^2 Y_t \tag{20}$$

These conditions along with the monetary policy rule and the stochastic processes for the shocks must be satisfied in equilibrium. Equation (18) is the consumption Euler equation that jointly with the next equation, the aggregate resource constraint, constitute the basis for the aggregate demand relationship. Equation (19) is the nonlinear version of the New Keynesian Phillips curve that relates current inflation to past and expected future inflation and output.

Natural and Efficient Levels The natural rate of output and the real interest rate are defined as the levels in an associated economy without price rigidities. The flexible price equilibrium is characterized by the following two equations:

$$1 = \frac{\beta RR_t^n}{\eta_t} E_t \left\{ \frac{Y_t^n}{Y_{t+1}^n} \right\}$$
$$(1+\tau)\mu_t Y_t^n = \frac{\epsilon - 1}{\epsilon}$$

We assume that, in the non-stochastic steady state, the employment subsidy is set such that it offsets the monopoly distortion. A log-linearized approximation around the natural equilibrium steady state yields:

$$r_t^n = \eta_t - E_t\{\mu_{t+1} - \mu_t\}$$
$$y_t^n = -\mu_t$$

Thus, there are two sources of exogenous disturbances to the natural rate of interest: shocks to the discount factor and shocks to the marginal cost. A decrease in the desired to save (an increase in η_t) and expected decrease in marginal costs cause the natural rate of interest to rise. The natural level of output moves with exogenous changes in the marginal costs.

The efficient equilibrium corresponds to that of flexible prices and no exogenous variation in marginal costs. Formally, the efficient allocations can be described as follows:

$$1 = \frac{\beta RR_t^e}{\eta_t} E_t \left\{ \frac{Y_t^e}{Y_{t+1}^e} \right\}$$
$$Y_t^e = 1$$

The log-linear approximation around the efficient non-stochastic steady state implies that:

$$r_t^e = \eta_t$$

and the efficient level of output is constant and normalized to one. Thus, shocks to the marginal costs reduce output relative to its efficient level.

Solution Method To solve the nonlinear model, under optimal discretionary policy, we use a projection method similar to Gust et al. (2012) and Christiano and Fisher (2000). The solution algorithm involves parameterizing the unknown functions $E_t x_{t+1} = f^x(\xi_t, s_t, \xi_{t-1})$, $E_t \pi_{t+1} = f^\pi(\xi_t, s_t, \xi_{t-1})$, where E_t denotes rational expectations based on private sector information. We also parameterize the function $f^r(s_t, \xi_{t-1})$, which satisfies the central bank's first order condition. In particular, this function can be derived by substituting the private sector equilibrium conditions into the central bank's first order condition and rewriting it as:

$$i_t - \frac{\phi_t}{\lambda + \kappa^2} = f^r(s_t, \xi_{t-1}) + E\{r_t^e + \left(\frac{\kappa^2}{\lambda + \kappa^2}\right) \mu_t | \{s_t, \xi_{t-1}\}\}$$

where i is the steady state value of the nominal interest rate. To solve for the optimal policy we use the fact that $\phi_t = 0$ when i_t is greater than or equal to the lower bound constraint.

To be consistent with the information structure, the function f^r does not take ξ_t as an argument. This ensures that the solution imposes that the monetary authority is unable to use information encoded in the private sector decisions in the current period. To solve for the unknown parameters of these functions, we conjecture a guess and iterate until the parameters satisfy the private sector equilibrium conditions and the central bank's first order condition at a finite number of points. Note that the above expression can be used to determine the short-term interest rate and the Lagrange multiplier ϕ_t as follows. If the right-hand side is positive ($f^r(s_t, \xi_{t-1}) + E\{r_t^e + \left(\frac{\kappa^2}{\lambda+\kappa^2}\right) \mu_t | \{s_t, \xi_{t-1}\}\} > 0$) then $i_t = f^r(s_t, \xi_{t-1}) + E\{r_t^e + \left(\frac{\kappa^2}{\lambda+\kappa^2}\right) \mu_t | \{s_t, \xi_{t-1}\}\}$; otherwise $i_t = 0$ and $\phi_t = -[f^r(s_t, \xi_{t-1}) + E\{r_t^e + \left(\frac{\kappa^2}{\lambda+\kappa^2}\right) \mu_t | \{s_t, \xi_{t-1}\}\}\}]$.

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