Finance and Economics Discussion Series
Divisions of Research & Statistics and Monetary Affairs
Federal Reserve Board, Washington, D.C.

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2016-065

Please cite this paper as:
Dobrev, Dobrislav, Travis D. Nesmith, and Dong Hwan Oh (2016). "Accurate Evaluation of Expected Shortfall for Linear Portfolios with Elliptically Distributed Risk Factors," Finance and Economics Discussion Series 2016-065. Washington: Board of Governors of the Federal Reserve System, http://dx.doi.org/10.17016/FEDS.2016.065.

Accurate Evaluation of Expected Shortfall for Linear Portfolios with Elliptically Distributed Risk Factors

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Abstract: We provide an accurate closed-form expression for the expected shortfall of linear portfolios with elliptically distributed risk factors. Our results aim to correct inaccuracies that originate in [1] and are present also in the recent comprehensive survey [2] on estimation methods for expected shortfall. In particular, we show that the correction we provide in the popular multivariate Student t setting eliminates understatement of expected shortfall by a factor varying from at least 4 to more than 100 across different tail quantiles and degrees of freedom. As such, the resulting economic impact in financial risk management applications could be significant. More generally, our findings point to the extra scrutiny required when deploying new methods for expected shortfall estimation in practice.

Keywords: Expected shortfall; elliptical distributions; multivariate Student t distribution; accurate closed-form expression

MSC: 91G10

JEL: C46,G11

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1. Introduction

Important advantages of expected shortfall (ES) over value at risk (VaR) as a coherent risk measure [see 3] have drawn the attention of financial risk managers, regulators and academics alike. For instance, a key element of a recent proposal by the Basel Committee on Banking Supervision [4] is moving the quantitative risk metrics system in regard to trading book capital requirement policies from 99% VaR to 97.5% ES. The surge in interest in ES estimation methods has been reflected also in the recent survey by Nadarajah *et al.* [2], which emphasizes many new developments and covers over 140 references on the subject. In this context, the case of elliptically distributed risk factors emerges as one of the simplified multivariate settings of choice because elliptical distributions offer a great deal of flexibility and analytical tractability. These benefits, however, require restricting all risk factors to have equally heavy tails. One popular elliptical example is given by the multivariate Student t distribution, which allows setting the tail index directly as a function of the number of degrees of freedom.

Further restricting attention to linear portfolios, the main purpose of this paper is to correct the inaccuracies in analytical expressions for ES both in the general elliptical case and in the multivariate Student t case, as originally presented in Kamdem [1] and subsequently reproduced in the recent survey [2]. In particular, we show that the correction we provide in the popular multivariate Student t setting eliminates understatement of expected shortfall by a factor varying from at least 4 to more than 100 across different tail quantiles and degrees of freedom. As such, the resulting economic impact in financial risk management applications can be fairly significant.

The paper proceeds as follows. Section 2 derives the correct ES expression in the general elliptical case. Section 3 deals with the additional correction that needs to be made in the multivariate Student t case. Section 4 verifies the corrected ES expressions via a mapping to the univariate Student t case. Section 5 conducts an assessment of the resulting economic impact. Section 6 provides a summary and conclusions.

2. Accurate ES in the General Elliptical Case

Following the notations in [1], we consider a linear portfolio with a weight row vector $\delta = (\delta_1, \delta_2, ..., \delta_n)$ in n elliptically distributed risky returns $X = (X_1, ..., X_n)$ with mean μ, scale matrix $\Sigma = AA'$, and pdf of X taking the form

$$f_X(x) = |\Sigma|^{-1/2} g\left((x-\mu)\Sigma^{-1}(x-\mu)'\right)$$

for some non-negative density generator function g.

The expected shortfall associated with the continuous portfolio returns $\Delta\Pi \equiv \delta X' = \delta_1 X_1 + ... + \delta_n X_n$ is then given by[1]

$$\begin{aligned} -ES_\alpha &= E\left(\Delta\Pi | \Delta\Pi \leq -VaR_\alpha\right) \\ &= \frac{1}{\alpha} E\left(\Delta\Pi \cdot 1\left\{\Delta\Pi \leq -VaR_\alpha\right\}\right) \\ &= \frac{1}{\alpha} \int_{\{\delta x' \leq -VaR_\alpha\}} \delta x' f(x)\, dx\,. \end{aligned}$$

After the same change of variables as in [1, section 2], we arrive at

$$\begin{aligned} -ES_\alpha &= \frac{1}{\alpha} \int_{\{|\delta A| z_1 \leq -\delta\mu - VaR_\alpha\}} \left(|\delta A| z_1 + \delta\mu\right) g\left(\|z\|^2\right) dz \\ &= \frac{1}{\alpha} \int_{\{|\delta A| z_1 \leq -\delta\mu - VaR_\alpha\}} |\delta A| z_1 g\left(\|z\|^2\right) dz + \delta\mu\,, \end{aligned}$$

where the norm $\|\cdot\|$ is defined as the Euclidean norm. By writing $\|z\|^2 = z_1^2 + \|z'\|^2$ and introducing spherical coordinates $z' = r\xi, \xi \in S_{n-2}$, the integral on the right hand side above can be expressed as

$$-ES_\alpha = \delta\mu + \frac{|S_{n-2}|}{\alpha} \int_0^\infty r^{n-2} \left[\int_{-\infty}^{-q_\alpha} |\delta A| z_1 g\left(z_1^2 + r^2\right) dz_1\right] dr\,,$$

where $|S_{n-2}| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ is the surface measure of the unit-sphere in R^{n-1} and $q_\alpha = \frac{\delta\mu + VaR_\alpha}{|\delta A|}$. We can then change the variable z_1 to $-z_1$ and the variable r to u as given by $u = z_1^2 + r^2$, which then implies

[1] See [5] for various definitions of ES for continuous or discontinuous portfolio returns

that $du = 2r\,dr\,dr = \frac{du}{2r} = \frac{du}{2\sqrt{u-z_1^2}}$ and $r = \sqrt{u - z_1^2}$. Substituting the change of variables leads to the following equivalent expression for $-ES_\alpha$,

$$\begin{aligned} -ES_\alpha &= \delta\mu - \frac{|S_{n-2}|}{\alpha} \int_{q_\alpha}^{\infty} \int_0^{\infty} r^{n-2}\, |\delta A|\, z_1 g\left(z_1^2 + r^2\right) dr\, dz_1 \\ &= \delta\mu - \frac{|S_{n-2}|\,|\delta A|}{2\alpha} \int_{q_\alpha}^{\infty} \int_{z_1^2}^{\infty} z_1 \left(u - z_1^2\right)^{\frac{n-3}{2}} g\left(u\right) du\, dz_1 \,. \end{aligned}$$

Changing the order of the two integrals further yields

$$\begin{aligned} -ES_\alpha &= \delta\mu - \frac{|S_{n-2}|\,|\delta A|}{2\alpha} \int_{q_\alpha^2}^{\infty} \int_{q_\alpha}^{\sqrt{u}} z_1 \left(u - z_1^2\right)^{\frac{n-3}{2}} g\left(u\right)\, dz_1\, du \\ &= \delta\mu - \frac{|S_{n-2}|\,|\delta A|}{2\alpha} \int_{q_\alpha^2}^{\infty} g\left(u\right) \int_{q_\alpha}^{\sqrt{u}} z_1 \left(u - z_1^2\right)^{\frac{n-3}{2}}\, dz_1\, du \,. \end{aligned}$$

Because the inner integral simplifies to $\frac{1}{n-1}\left(u - q_\alpha^2\right)^{\frac{n-1}{2}}$, and by definition $|\delta A| = |\delta\Sigma\delta'|^{1/2}$, we obtain the following final result,

$$\begin{aligned} ES_\alpha &= -\delta\mu + |\delta A|\, \frac{|S_{n-2}|}{2\alpha} \int_{q_\alpha^2}^{\infty} \frac{1}{n-1} \left(u - q_\alpha^2\right)^{\frac{n-1}{2}} g\left(u\right)\, du \\ &= -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{2\pi^{\frac{n-1}{2}}}{\underbrace{2\alpha\Gamma\left(\frac{n-1}{2}\right)(n-1)}_{=2\Gamma\left(\frac{n+1}{2}\right)}} \int_{q_\alpha^2}^{\infty} \left(u - q_\alpha^2\right)^{\frac{n-1}{2}} g\left(u\right)\, du \\ &= -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{\pi^{\frac{n-1}{2}}}{2\alpha\Gamma(\frac{n+1}{2})} \int_{q_\alpha^2}^{\infty} \left(u - q_\alpha^2\right)^{\frac{n-1}{2}} g\left(u\right)\, du \,. \end{aligned}$$

Thus, we have proved at the following theorem for ES in the general elliptical case:

Theorem 1. *The expected shortfall ES_α at quantile α of a linear portfolio δX in elliptically distributed risk factors X with pdf defined by $f_X(x) = |\Sigma|^{-1/2} g\left((x-\mu)\Sigma^{-1}(x-\mu)'\right)$ is given by*

$$ES_\alpha = -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{\pi^{\frac{n-1}{2}}}{2\alpha\Gamma(\frac{n+1}{2})} \int_{q_\alpha^2}^{\infty} \left(u - q_\alpha^2\right)^{\frac{n-1}{2}} g\left(u\right)\, du\,, \tag{1}$$

where $q_\alpha = \frac{\delta\mu + VaR_\alpha}{|\delta\Sigma\delta'|^{1/2}}$.

Comparing the above result to the corresponding expressions in [1, equation (4.1) in Theorem 4.1] as well as in [2, equation (15) in section 3.20], we conclude that our second term on the right hand side of equation (1) is smaller by a factor of 2. In particular, this corrects a two-fold overstatement of ES in the zero-mean case of typical interest in many short-term financial risk management applications.

3. Accurate ES in the Multivariate Student t Case

An important special case of an elliptical distribution of traditional interest in risk management applications is given by the Multivariate Student t distribution, which has the following pdf,

$$f_X(x) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{|\Sigma|(\nu\pi)^n}} \left(1 + \frac{(x-\mu)\Sigma^{-1}(x-\mu)'}{\nu}\right)^{-\frac{(n+\nu)}{2}} .$$

Thus, substituting $g(u) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\nu\pi)^n}}\left(1+\frac{u}{\nu}\right)^{-\frac{(n+\nu)}{2}}$ in equation (1) above further specializes the obtained general expression for ES to the multivariate Student t case, so that

$$\begin{aligned} ES_\alpha &= -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{\pi^{\frac{n-1}{2}}}{2\alpha\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\nu\pi)^n}} \int_{q_\alpha^2}^{\infty} \left(u - q_\alpha^2\right)^{\frac{n-1}{2}} \left(1+\frac{u}{\nu}\right)^{-\frac{(n+\nu)}{2}} du \\ &= -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{1}{2\alpha\sqrt{(\nu)^n\pi}} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\Gamma(\frac{n+1}{2})} \int_{q_\alpha^2}^{\infty} \left(u - q_\alpha^2\right)^{\frac{n-1}{2}} \left(1+\frac{u}{\nu}\right)^{-\frac{(n+\nu)}{2}} du . \end{aligned}$$

By [1, Lemma 2.1], we can then use the following equality

$$\int_{q_\alpha^2}^{\infty} \left(u - q_\alpha^2\right)^{\frac{n-1}{2}} \left(1+\frac{u}{\nu}\right)^{-\frac{(n+\nu)}{2}} du = \nu^{\frac{n+\nu}{2}} \left(q_\alpha^2 + \nu\right)^{-\left(\frac{\nu-1}{2}\right)} B\left(\frac{\nu-1}{2}, \frac{n+1}{2}\right) .$$

Substituting the equality into the prior equation, along with $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ yields in turn:

$$\begin{aligned} ES_\alpha &= -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{1}{2\alpha\sqrt{(\nu)^n\pi}} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\Gamma(\frac{n+1}{2})} \nu^{\frac{n+\nu}{2}} \left(q_\alpha^2+\nu\right)^{-\left(\frac{\nu-1}{2}\right)} B\left(\frac{\nu-1}{2}, \frac{n+1}{2}\right) \\ &= -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{1}{2\alpha\sqrt{(\nu)^n\pi}} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\Gamma(\frac{n+1}{2})} \nu^{\frac{n+\nu}{2}} \left(q_\alpha^2+\nu\right)^{-\left(\frac{\nu-1}{2}\right)} \frac{\Gamma(\frac{\nu-1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+\nu}{2})} \\ &= -\delta\mu + \left|\delta\Sigma\delta'\right|^{1/2} \frac{\nu^{\frac{\nu}{2}}}{2\alpha\sqrt{\pi}} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})} \left(q_\alpha^2+\nu\right)^{-\left(\frac{\nu-1}{2}\right)} \end{aligned}$$

With this, we obtain the following result for ES in the multivariate Student t case:

Theorem 2. *The expected shortfall ES_α at quantile α of a linear portfolio δX in risk factors X having multivariate Student t distribution with pdf $f_X(x) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{|\Sigma|(\nu\pi)^n}}\left(1+\frac{(x-\mu)\Sigma^{-1}(x-\mu)'}{\nu}\right)^{-\frac{(n+\nu)}{2}}$ is given by*

$$ES_\alpha = -\delta\mu + es_{\alpha,\nu} \cdot \left|\delta\Sigma\delta'\right|^{1/2} , \tag{2}$$

with

$$es_{\alpha,\nu} = \frac{\nu^{\frac{\nu}{2}}}{2\alpha\sqrt{\pi}} \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})} \left(q_{\alpha,\nu}^2 + \nu\right)^{-\left(\frac{\nu-1}{2}\right)} \tag{3}$$

and

$$q_{\alpha,\nu} = \frac{\delta\mu + VaR_\alpha}{\left|\delta\Sigma\delta'\right|^{1/2}} , \tag{4}$$

where $q_{\alpha,\nu}$ is uniquely determined by solving a transcendental equation given by [1, Theorem 2.2] and [2, section 3.20].[2]

A close inspection of our expression for $es_{\alpha,\nu}$ in equation (3) above in comparison to the corresponding equations in [1, Theorem 4.2] and [2, section 3.20] reveals a difference of 1 in the power of the last term $\left(q_{\alpha,\nu}^2+\nu\right)^{-\left(\frac{\nu-1}{2}\right)}$ in addition to the extra scaling factor by 2 inherited from the correction made above in the general elliptical case. In particular, the correct power of that term is $-(\frac{\nu-1}{2})$ rather than $-(\frac{\nu+1}{2})$. As such, these two corrections taken together eliminate an understatement of ES by a factor of $\frac{(q_{\alpha,\nu}^2+\nu)}{2} > 1$ in the zero-mean multivariate Student t setting with $\nu \geq 2$ degrees of freedom of particular interest in many applications. Before conducting a numerical assessment of this combined effect of correcting the two separate inaccuracies in the ES expressions found in the general elliptical case and the multivariate Student t case overlooked in both [1] and [2], we first provide an alternative way to check our results through the univariate Student t case.

4. Comparison to ES in the Univariate Student t Case

The ES expression in Theorem 2 holds for any linear portfolio. In particular, it should hold if only a single asset is held, for example if $\delta = [1, 0, \cdots, 0]$. Consequently, the formula for expected shortfall for the multivariate Student t should reproduce the formula for a univariate Student t.

The results in [6, section 2.2.2] show that the expected shortfall for a zero-mean univariate Student t random variable is given by[3]

$$ES_\alpha = \frac{1}{\alpha\sqrt{\pi}\sqrt{\nu}}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\left(1+\frac{q_\alpha^2}{\nu}\right)^{-\left(\frac{\nu+1}{2}\right)}\left(\frac{\nu+q_\alpha^2}{\nu-1}\right)$$

Collecting terms and using known identities allows us to equivalently express this as

$$\begin{aligned} ES_\alpha &= \frac{1}{\alpha\sqrt{\pi}}\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\left(\nu+q_\alpha^2\right)^{-\left(\frac{\nu+1}{2}\right)}\nu^{\left(\frac{\nu}{2}\right)}\left(\frac{\nu+q_\alpha^2}{\nu-1}\right) \\ &= \frac{1}{\alpha\sqrt{\pi}}\frac{\frac{\nu-1}{2}\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}\left(\nu+q_\alpha^2\right)^{-\left(\frac{\nu+1}{2}\right)}\nu^{\left(\frac{\nu}{2}\right)}\left(\frac{\nu+q_\alpha^2}{\nu-1}\right) \\ &= \frac{1}{\alpha\sqrt{\pi}}\frac{\Gamma(\frac{\nu-1}{2})}{2\Gamma(\frac{\nu}{2})}\left(\nu+q_\alpha^2\right)^{-\left(\frac{\nu-1}{2}\right)}\nu^{\left(\frac{\nu}{2}\right)} \\ &= \frac{\nu^{\frac{\nu}{2}}}{2\alpha\sqrt{\pi}}\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}\left(q_\alpha^2+\nu\right)^{-\left(\frac{\nu-1}{2}\right)}. \end{aligned}$$

This result matches exactly our equation (3) from Theorem 2 on the preceding page with $n = 1$. Conversely, combining this expression from the univariate case with the scaling properties of the multivariate Student t distribution (as an elliptical distribution) with respect to $|\delta\Sigma\delta'|^{1/2}$ can also be used to confirm the validity of our expressions in Theorem 2. The consistency of our result in the univariate case also readily applies to general elliptical distributions. By contrast, the formulas contained in both [1]

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[2] Accurate numerical values of $q_{\alpha,\nu}$ for different α and ν of interest have been tabulated by [1, section 2.2].

[3] Choosing the mean to be zero is done for simplicity; if the univariate Student t has a mean then both formulas are adjusted by adding the mean.

and [2] for elliptical distributions and multivariate Student t distributions are not consistent with the respective univariate formulas.

5. Economic Impact of The Correction

In order to assess the resulting economic impact, we study numerically the combined effect of the above corrections of the inaccuracies in the ES expressions found in [1] and [2]. To this end, we further restrict attention to the zero-mean multivariate Student t setting and tabulate in Table 1 the accurate (panel A) and inaccurate (panel B) values of $es_{\alpha,\nu}$ as well as their ratio (panel C) across different tail quantiles $\alpha = 0.01, 0.025, 0.05$ and degrees of freedom $\nu = 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 200, 250$.

Table 1. Numerical Comparison of the Accurate versus Inaccurate Expression for Expected Shortfall in the Multivariate Student t Case. The table reports the accurate (panel A) and inaccurate (panel B) values of $es_{\alpha,\nu}$ as well as their ratio (panel C) across different tail quantiles $\alpha = 0.01, 0.025, 0.05$ (different rows) and degrees of freedom $\nu = 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 200, 250$ (different columns) of the multivariate Student t distribution governing the individual risky returns in a linear portfolio. The accurate expression in panel A reflects the derivations in this paper (Theorems 1 and 2), while the inaccurate expression in panel B is originally due to Kamdem [1, Theorem 4.2] and is reproduced in Nadarajah *et al.* [2, section 3.20].

ν	2	3	4	5	6	7	8	9	10	100	200	250
Panel A: The accurate $es_{\alpha,\nu} = \frac{\nu^{\frac{\nu}{2}}}{2\alpha\sqrt{\pi}} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(q^2_{\alpha,\nu} + \nu\right)^{-\left(\frac{\nu-1}{2}\right)}$ derived in sections 2 and 3 above												
$es^t_{0.010,\nu}$	14.071	7.004	5.221	4.452	4.033	3.770	3.591	3.462	3.363	2.722	2.717	2.665
$es^t_{0.025,\nu}$	8.832	5.040	3.994	3.522	3.256	3.087	2.970	2.884	2.819	2.379	2.358	2.354
$es^t_{0.050,\nu}$	6.164	3.874	3.203	2.890	2.711	2.595	2.514	2.515	2.891	2.093	2.078	2.075
Panel B: The inaccurate $es_{\alpha,\nu} = \frac{\nu^{\frac{\nu}{2}}}{\alpha\sqrt{\pi}} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(q^2_{\alpha,\nu} + \nu\right)^{-\left(\frac{\nu+1}{2}\right)}$ in [1] and [2]												
$es^t_{0.010,\nu}$	0.557	0.593	0.579	0.546	0.508	0.472	0.438	0.408	0.381	0.052	0.026	0.021
$es^t_{0.025,\nu}$	0.861	0.768	0.682	0.607	0.543	0.490	0.446	0.409	0.377	0.046	0.023	0.019
$es^t_{0.050,\nu}$	1.171	0.908	0.750	0.638	0.555	0.490	0.439	0.409	0.453	0.041	0.021	0.016
Panel C: Ratio $\left(q^2_{\alpha,\nu} + \nu\right)/2$ of the accurate (panel A) to the inaccurate (panel B) values of $es_{\alpha,\nu}$												
$es^t_{0.010,\nu}$	25.253	11.808	9.020	8.161	7.938	7.994	8.195	8.480	8.819	52.795	102.741	127.750
$es^t_{0.025,\nu}$	10.256	6.564	5.854	5.804	5.994	6.296	6.659	7.059	7.482	51.968	101.944	126.939
$es^t_{0.050,\nu}$	5.263	4.269	4.272	4.530	4.888	5.295	5.729	6.143	6.378	51.378	101.365	126.363

It stands out that the ratio $\left(q^2_{\alpha,\nu} + \nu\right)/2$ of the accurate versus inaccurate values reported in panel C of Table 1 is quite large and varies from at least just above 4 (for $\alpha = 0.05$ and $\nu = 3$ or 4) to more than 100 (for $\nu \geq 200$). The later discrepancies occur as the results should be converging to the results for a Gaussian distribution; the results in panel A are converging to the Gaussian unlike the values in panel B.

Importantly, in the case of $\alpha = 0.025$ the minimum ratio of the accurate to inaccurate values of ES is about 6. Given that moving the quantitative risk metrics system in regard to trading book capital requirement policies from 99% VaR to 97.5% ES (i.e. in our notation $\alpha = 0.025$) is a key element of the recent proposal by the [4], the numerical results imply that the correction would eliminate at least a six-fold and potentially much larger understatement of risk in the popular zero-mean multivariate Student t setting. Clearly, the resulting economic impact in financial risk management applications could be significant.

Noting again the magnitude and potential economic impact of our correction, one could look for possible reasons as to why such large inaccuracies in the otherwise quite popular multivariate Student t setting could have gone unnoticed. One important observation to make in this regard is that, as recently noted by [7] among others, the literature on backtesting ES is fairly new, thereby leaving a potential loophole for any such errors in ES computations to go unnoticed for a while in financial industry applications that may not yet perform routine and powerful enough ES backtesting. Another possibility to keep in mind is that instead of using the inaccurate expressions in [1] and [2] one could alternatively take directly the values of $es_{\alpha,\nu}$ tabulated by [1, section 4.1], which happen to be offset by yet another separate mistake by a factor of 10, thereby mechanically, but still inaccurately, shifting the discrepancy in the opposite more conservative direction across much of the range of tabulated different tail quantiles and degrees of freedom. The magnitude of ES underestimation even using these wrongly tabulated values still remains very large for corner choices of ν such as 2 or much larger than 10. However, for most other values of ν there is at least some partial cancellation effect with all other errors as a result of this third inaccuracy in [1]; the result would be to bring most discrepancies down to somewhat more moderate magnitudes in the order of 20% to 50%.

All in all, partial offsetting of different errors and challenges with ES backtesting could have played a role for ES discrepancies of even such large magnitude and potential economic impact as the ones we report in panel C of Table 1 to elude detection for some time. Our findings provide a word of caution about the scrutiny required when deploying any new methods for ES estimation in practice, as may be happening as a result of the proposed new guidelines issued by the Basel Committee on Banking Supervision [4].

6. Conclusions

The case of elliptically distributed risk factors is a popular simplified multivariate setting in financial risk management offering a great deal of flexibility and analytical tractability at the expense of restricting all risk factors to have equally heavy tails. Our accurate closed-form expressions for the expected shortfall of linear portfolios with any elliptically distributed and, more particularly, commonly used multivariate Student t distributed risk factors correct major inaccuracies in the original development by Kamdem [1] and more recently repeated in the comprehensive survey of expected shortfall estimation methods by Nadarajah *et al.* [2]. In terms of magnitude, the resulting correction in the zero-mean multivariate Student t setting eliminates understatement of expected shortfall by a factor varying from at least 4 to more than 100 across different tail quantiles and degrees of freedom. As such, the economic impact from using our accurate expected shortfall expressions in financial risk management applications with elliptically distributed risk factors can be fairly significant.

Another important area of applications of the accurate closed-form results we obtain for expected shortfall with elliptically distributed risk factors is gauging the statistical precision of alternative non-parametric ES estimation methods relying on Monte Carlo simulations in the spirit of the analysis by Yamai and Yoshiba [8]. In particular, the ability to study the performance of alternative non-parametric ES estimators in controlled experiments for multivariate heavy-tailed settings with accurately known analytical results can help provide some useful guidance in the context of the proposal by the Basel Committee on Banking Supervision [4] to move the quantitative risk metrics system in regard to trading book capital requirement policies from 99% VaR to 97.5% ES. More generally, our findings point to the extra scrutiny required when deploying new methods for expected shortfall estimation in practice, especially also in light of the widely acknowledged separate challenges with backtesting expected shortfall.

Author Contributions: The research problem was identified by D.D.; the analytical solution and all numerical results were obtained independently by both D.D. and D.O.; the paper was written jointly by D.D. and D.O. under supervision by T.D.N.

Conflicts of Interest: All authors currently work at the Federal Reserve Board of Governors. The views expressed in this paper are those of the authors and should not be interpreted as reflecting the views of the Federal Reserve Board of Governors or any other person associated with the Federal Reserve System.

Abbreviations

The following abbreviations are used in this manuscript:

ES: Expected shortfall
VaR: Value-at-risk

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