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When Can Trend-Cycle Decompositions Be Trusted?

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Abstract

In this paper, we examine the results of GDP trend-cycle decompositions from the estimation of bivariate unobserved components models that allow for correlated trend and cycle innovations. Three competing variables are considered in the bivariate setup along with GDP: the unemployment rate, the inflation rate, and gross domestic income. We find that the unemployment rate is the best variable to accompany GDP in the bivariate setup to obtain accurate estimates of its trend-cycle correlation coefficient and the cycle. We show that the key feature of unemployment that allows for precise estimates of the cycle of GDP is that its nonstationary component is "small" relative to its cyclical component. Using quarterly GDP and unemployment rate data from 1948:Q1 to 2015:Q4, we obtain the trend-cycle decomposition of GDP and find evidence of correlated trend and cycle components and an estimated cycle that is about 2 percent below its trend at the end of the sample.

Keywords: Unobserved components model, trend-cycle decomposition, trend-cycle correlation

JEL Classification Numbers: C13, C32, C52

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1 Introduction

When can we trust trend-cycle output decompositions? Univariate studies, such as Morley, Nelson and Zivot (2003) (MNZ hereafter), find a large negative correlation between the innovations of output's trend and cycle as well as small and economically unimportant business cycles. By contrast, studies that include additional variables such as inflation or the unemployment rate typically find estimates of the cyclical component of output that are to a large extent conventional, closely resembling, for example, estimates published by the Congressional Budget Office (CBO). Early examples include Clark (1989), who added the unemployment rate, and Roberts (2001), who incorporated inflation and hours in the analysis. Moreover, both these studies found the output trend-cycle correlation to be small and statistically insignificant. Other bivariate studies, such as Basistha and Nelson (2007) and Basistha (2007), using inflation as an additional variable, find output trend-cycle correlations that are negative and statistically significant, but smaller than in MNZ; they find conventional business cycles estimates.

Basistha (2007) sheds some light on the source of these disparate results using a Monte Carlo study. Basistha shows that a strong estimated trend-cycle correlation can be spurious in a univariate setup. In particular, the correlation can be estimated to be large even when it is zero in the data generating process. By contrast, in a bivariate setup with a variable that resembles inflation, the estimated trend-cycle correlation coefficient is close, on average, to the true correlation of zero.

In this paper, we use Monte Carlo experiments to explore under what conditions an auxiliary variable in a bivariate unobserved components (UC) model of trend-cycle decompositions of output will be helpful for correctly estimating the trend-cycle correlation of output and for identifying more precisely its cyclical component. We start by reviewing the univariate estimation proposed by MNZ, which allows for correlated trend-cycle output innovations. We then assess the conditions under which an auxiliary variable in the UC model will be helpful for estimating accurately the output trend-cycle correlation, for identifying the business cycle, and for testing hypotheses with respect to the trend-cycle correlation coefficient. We examine three specific set-ups, one using an auxiliary variable designed to resemble the unemployment rate, another with a variable resembling inflation, and a third that amounts to using two readings on output, meant to resemble the use of gross domestic income (GDI) as an auxiliary variable.

We find that the univariate model's estimation delivers an output trend-cycle correlation sampling distribution that piles up at -1 and +1 and that the properties of the cycle are not accurately obtained. We also find that there is considerable variation in the ability of the auxiliary variables to distinguish the trend-cycle correlation coefficient and the business cycle. In particular, for some auxiliary variables, the econometrician can obtain spurious estimates of the correlation between trend and cycle, similar but not as bad as those obtained with the univariate setup. For example, if both variables are nonstationary and resemble GDP—as when the bivariate model included both GDP and GDI—spurious correlation results are somewhat likely. We also consider a variable that resembles inflation. As with GDI, we find that spurious correlations can obtain for parametrizations of the model that accord with empirical results in the literature. Thus, simply adding an auxiliary variable does not appear sufficient to allow the proper identification of trend and cycle.

We find, however, that if the auxiliary variable in the bivariate analysis resembles the unemployment rate, the estimation results can be trusted. Based on our experiments, it appears that the key reason the unemployment rate is well-suited to help distinguish the trend and the cycle is that the variance of its unit root component is relatively small compared to the variance of its cyclical component.

Motivated by our Monte Carlo results, we use GDP and unemployment rate data to estimate a bivariate UC model. As in other studies using the unemployment rate, we find a conventional cyclical component of GDP, similar to that published by the CBO. The estimated cycle has a pronounced hump-shaped pattern and complex roots, with a period of 11.1 years. Our empirical results suggest that there is a statistically significant correlation between the output trend and cycle. However, unlike MNZ, we find that the correlation is positive, not negative. That result is suggested by standard statistical tests, and we find in our Monte Carlo work that the size of these tests is approximately correct and that we have sufficient power to reject incorrect parameter values. The resulting business cycle estimates are conventional and similar to those of CBO.

The paper is structured as follows: Section 2 presents a review of the literature on trend-cycle decompositions with UC models. In Section 3, we present the characteristics of the bivariate UC models we will examine. Section 4 presents the results of our Monte Carlo experiments with respect to the estimators of the output trend-cycle correlation and decomposition. The size and power of the LR test of hypothesis on the output trend-cycle correlation coefficient appear in Section 5. Section 6 summarizes the results from the Monte Carlo simulations. In Section 7, we estimate a bivariate UC model including GDP and unemployment data for the U.S. and test for significance of the correlation between trend and cycle components. Section 8 concludes.

2 Contacts with the Literature

Watson (1986) and Clark (1987) were among the first to use a UC model to decompose GDP into independent nonstationary trend and stationary cycle components. The estimates implied that much of the quarterly variability in U.S. economic activity can be attributed to a stationary cyclical component. By contrast, Nelson and Plosser (1982), found that most of the variation in U.S. economic activity can be attributed to a nonstationary trend component. A central assumption of Clark's estimation was the orthogonality between trend and cycle components; the method of Nelson and Plosser (1982) places no restrictions on the correlation between trend and cycle.

In a subsequent paper, Clark (1989) proposed considering GDP and the unemployment rate in a bivariate UC model to decompose GDP into trend and cycle components, allowing a nonzero correlation between trend and cycle innovations. In this case, the trend and cycle disturbances are disentangled by assuming that the cyclical component of output also affects the unemployment rate through an Okun's law relationship. Clark's results provide evidence consistent with the hypothesis that innovations in the trend and cyclical components are independent: the 90 percent confidence interval for the correlation is [-0.4,0.3].

Kuttner (1994) pursued an alternative bivariate approach, adding inflation as an observable and linking inflation and the cycle through a Phillips curve relationship. Kuttner found a business cycle that was similar to Clark (1989). However, Kuttner did not allow for correlation between trend and cycle. Following Kuttner, Roberts (2001) also included a Phillips-curve relationship and further decomposed output into hours and productivity components. Hours and output per hour are each divided into trend and gap components, and the gap affects inflation through a Phillips curve relationship. Both Roberts and Kuttner found that estimates of the trend-cycle decomposition were not much affected by the addition of inflation. In addition, Roberts found that the correlations between trend shocks and the cycle were not statistically significant at conventional levels.

Morley, Nelson and Zivot (2003) carefully explored identification in the univariate UC model. They showed that an unrestricted ARIMA(2,1,2) model implies second moments that can be matched uniquely to the second moments of the UC model. The estimation of the cycle through both the Beveridge-Nelson decomposition of the ARIMA(2,1,2) model and a univariate UC model allowing for correlation between trend and cycle yield estimates in which the cycle is mostly noise and most of the variability in GDP occurs through its trend component, similar to Nelson and Plosser (1982).

Basistha and Nelson (2007) estimate a bivariate UC model with inflation and GDP as observable variables. They introduce the spread between inflation and a survey measure of expectations, motivated by the New Keynesian Phillips curve. They allow for a dense variance-covariance matrix of the shocks and find that the GDP trend and cycle innovations are negatively correlated, as obtained by MNZ, albeit with a smaller absolute magnitude. Their estimated cycle is nonetheless conventional. The authors extend the model to include the unemployment rate as an additional observable via an Okun's law relationship. Results are similar to those when only inflation is included as an additional observable. As noted in the introduction, Basistha (2007) performed a set of Monte Carlo simulations that showed that while a univariate UC specification is not able to identify the correlation coefficient between trend and cycle innovations, a bivariate setup similar to Basistha and Nelson (2007) yields an estimated correlation coefficient that is close on average to the true correlation.

The work of Perron and Wada (2009) is also related to our paper. As in our work, Perron and Wada (2009) consider the estimation of the correlation between trend and cycle when no correlation exists. They, however, look at a model with a deterministic trend whereas our focus is on stochastic trends. The find that when the trend component is deterministic, the estimated correlation between trend and cycle shocks is either -1 or +1, depending on parameter configurations, because the correlation coefficient is not identified when the trend is deterministic. Wada (2012) explores this issue further to offer a more detailed explanation of the lack of identification.

The literature cited above emphasizes a common cycle linking GDP and other variables. Sinclair (2009) pursues an alternative approach in which each variable is allowed to have its own trend and cycle, specifically estimating the trends and cycles for GDP and the unemployment rate in a bivariate UC model. Sinclair finds a statistically significant negative correlation between the trend and cycle components of GDP, as well as in the corresponding innovations of the unemployment rate. The resulting business cycle resembles that of MNZ, with high volatility and relatively small amplitude. In our work, we follow the main body of the literature, which emphasizes a common cycle linking GDP and other variables.

3 UC Models for Trend-cycle Decompositions

We first present the basic structure of the unobserved components model for trend-cycle decomposition. We then examine three specific bivariate extensions and posit a common, nesting framework.

3.1 The basic UC model

$$y_t = \tau_{yt} + c_t \tag{1}$$
$$\tau_{yt} = \mu_y + \tau_{y,t-1} + \eta_{yt} \tag{2}$$
$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \varepsilon_t, \tag{3}$$

In our analysis, $\{y_t\}$ is the log of GDP, $\{\tau_{yt}\}$ is its unobserved trend, assumed to be a random walk with mean growth rate μ_y, and $\{c_t\}$ is the unobserved stationary cycle, assumed to follow an AR(2) process. The roots of $1 - \phi_1 z - \phi_2 z^2 = 0$ are outside the unit circle, and $\{\eta_{yt}\}$ and $\{\varepsilon_t\}$ are potentially correlated disturbances with variance-covariance matrix given by:

$$\begin{bmatrix} \varepsilon_t \\ \eta_{yt} \end{bmatrix} \sim \text{iid } \mathbb{N}\left(\mathbf{0}_{2\times 1}, \begin{bmatrix} \sigma_\varepsilon^2 & \rho_{\eta_y \varepsilon}\sigma_{\eta_y}\sigma_\varepsilon \\ \rho_{\eta_y \varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \sigma_{\eta_y}^2 \end{bmatrix} \right).$$

As noted in the introduction, our main focus is the value of the correlation coefficient $\rho_{\eta_y \varepsilon}$—that is, the correlation between the trend and cycle for output. When it is imposed to be zero—as in Clark (1987) and Watson (1986)—or estimated to be near zero—as in Clark (1989)—the resulting estimates of the cycle are conventional. By contrast, in many univariate studies, such as Nelson and Plosser (1982) and MNZ, it is estimated to be close to -1 and the resulting estimates of the cycle are unconventional, relative, for example, to the output gap estimates of CBO. While the simulation results of Basistha (2007) have shown that these univariate results can be spurious, a key question is when we can trust bivariate results. We therefore consider several bivariate models for trend-cycle decomposition in the presence of nonzero correlation between trend and cycle disturbances and estimation of the correlation coefficient.

3.2 Bivariate UC Model: GDP and the unemployment rate

Clark (1989) proposed extending the univariate UC model for GDP in equations (1)-(3) to include the unemployment rate in the following fashion:

$$u_t = \tau_{ut} + \theta_1 c_t + \theta_2 c_{t-1} \tag{4}$$
$$\tau_{ut} = \tau_{u,t-1} + \eta_{ut}, \tag{5}$$

and variance-covariance matrix:

$$\begin{bmatrix} \varepsilon_t \\ \eta_{yt} \\ \eta_{ut} \end{bmatrix} \sim \text{iid } \mathbb{N}\left(\mathbf{0}_{3\times 1}, \begin{bmatrix} \sigma_\varepsilon^2 & \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \rho_{\eta_u\varepsilon}\sigma_{\eta_u}\sigma_\varepsilon \\ \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \sigma_{\eta_y}^2 & \rho_{\eta_y\eta_u}\sigma_{\eta_y}\sigma_{\eta_u} \\ \rho_{\eta_u\varepsilon}\sigma_{\eta_u}\sigma_\varepsilon & \rho_{\eta_y\eta_u}\sigma_{\eta_y}\sigma_{\eta_u} & \sigma_{\eta_u}^2 \end{bmatrix}\right). \tag{6}$$

In equation (4), $\{u_t\}$ is the unemployment rate, which is decomposed into a trend, $\{\tau_{ut}\}$, assumed to be a random walk with zero drift, and a cyclical component. The cycle of output is allowed to affect the unemployment rate both contemporaneously and with a lag, reflecting the well-known characterization of the unemployment rate as a lagging indicator of the business cycle (see Stock and Watson, 1998). The model allows the correlations between the cycle and trend innovations of GDP, $\rho_{\eta_y\varepsilon}$, and the unemployment rate, $\rho_{\eta_u\varepsilon}$, to be nonzero, as well as the correlation between the two trend shocks, $\rho_{\eta_y\eta_u}$.[1]

There are a number of ways to interpret this model. One is that the system of equations (1)-(3) and (4)-(6) forms a factor model, with $\{c_t\}$ the common factor, normalized so that its effect on $\{y_t\}$ is contemporaneous with a coefficient of one. Another interpretation of equation (4) is Okun's Law, with the unemployment gap related to the output gap contemporaneously and with a lag.

3.3 Bivariate UC Model: GDP and Gross Domestic Income

Gross Domestic Income (GDI) is another plausible cyclical indicator. GDI is an alternative measure of exactly the same concept as GDP, but based on (largely) independent sources of data. Fixler and Nalewaik (2007) and Nalewaik (2010) have shown that GDI is at least as good a measure of aggregate economic activity as GDP; Fleischman and Roberts (2011) find similar results in their multivariate trend-cycle model.

We introduce GDI into the UC model in equations (1)-(3) as follows:

$$z_t = \tau_{zt} + c_t \tag{7}$$

$$\tau_{zt} = \mu_z + \tau_{z,t-1} + \eta_{zt}, \tag{8}$$

and

$$\begin{bmatrix} \varepsilon_t \\ \eta_{yt} \\ \eta_{zt} \end{bmatrix} \sim \text{iid } \mathbb{N}\left(\mathbf{0}_{3\times 1}, \begin{bmatrix} \sigma_\varepsilon^2 & \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \rho_{\eta_z\varepsilon}\sigma_{\eta_z}\sigma_\varepsilon \\ \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \sigma_{\eta_y}^2 & \rho_{\eta_y\eta_z}\sigma_{\eta_y}\sigma_{\eta_z} \\ \rho_{\eta_z\varepsilon}\sigma_{\eta_z}\sigma_\varepsilon & \rho_{\eta_y\eta_z}\sigma_{\eta_y}\sigma_{\eta_z} & \sigma_{\eta_z}^2 \end{bmatrix}\right). \tag{9}$$

In this specification, $\{z_t\}$ is (the log of) GDI, $\{\tau_{zt}\}$ is its unobserved trend, assumed to be a random walk with mean growth rate μ_z, which is the same for GDP, and $\{c_t\}$ is the common unobserved stationary cycle. The correlation coefficient $\rho_{\eta_y\eta_z}$ captures the co-movements between the trends, and the coefficients $\rho_{\eta_y\varepsilon}$ and $\rho_{\eta_z\varepsilon}$ allow for separate trend-cycle correlations for GDP and GDI.

[1] When testing for statistical significance of the trend-cycle correlation coefficient of GDP, $\rho_{\eta_y\varepsilon}$, Clark (1989) assumed that the other two correlation coefficients where zero, implying a less general framework than the one presented here.

3.4 Bivariate UC Model: GDP and the inflation rate

Another alternative introduces inflation in the UC model in equations (1)-(3):

$$\pi_t = \tau_{\pi t} + \theta c_t \tag{10}$$
$$\tau_{\pi t} = \mu_\pi(1-\alpha) + \alpha\pi_{t-1} + \eta_{\pi t}, \tag{11}$$

with $\alpha \in (0,1]$, and

$$\begin{bmatrix} \varepsilon_t \\ \eta_{yt} \\ \eta_{\pi t} \end{bmatrix} \sim \text{iid } \mathbb{N}\left(\mathbf{0}_{3\times 1}, \begin{bmatrix} \sigma_\varepsilon^2 & \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \rho_{\eta_\pi\varepsilon}\sigma_{\eta_\pi}\sigma_\varepsilon \\ \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \sigma_{\eta_y}^2 & \rho_{\eta_y\eta_\pi}\sigma_{\eta_y}\sigma_{\eta_\pi} \\ \rho_{\eta_\pi\varepsilon}\sigma_{\eta_\pi}\sigma_\varepsilon & \rho_{\eta_y\eta_\pi}\sigma_{\eta_y}\sigma_{\eta_\pi} & \sigma_{\eta_\pi}^2 \end{bmatrix}\right). \tag{12}$$

This model incorporates a Phillips curve relationship (equation (10)), where the cyclical component of output helps predict the deviation of inflation, $\{\pi_t\}$, from trend inflation, $\tau_{\pi t}$. The specification for trend inflation nests several alternatives. Kuttner (1994) assumed that inflation followed a unit root process, and hence that $\alpha = 1$. Kuttner also assumed that the correlations between innovations were zero. Roberts (2001) also assumed that $\alpha = 1$ but allowed for correlations between innovations. Basistha (2007, 2009), on the other hand, allowed $\alpha < 1$; he also allowed for correlated trend-cycle innovations.[2]

Stella and Stock (2016) take a different approach and specify the inflation trend as a unit root process, similar to the unemployment rate trend in the GDP-unemployment rate bivariate UC model of Section 3.2, as follows:[3]

$$\tau_{\pi t} = \tau_{\pi,t-1} + \eta_{\pi t}, , \tag{13}$$

with the assumption of orthogonal disturbances. They also introduce measurement errors in the observation equations and stochastic volatilities in all the error terms. The addition of measurement errors makes the model of Stella and Stock (2016), strictly speaking, no longer nested in the class of models we have been considering so far. We nonetheless consider a variant of this specification as one of the alternative models for the sake of completeness; we find that augmenting the model with measurement error does not have an important impact on the results. Similarly, we do not formally consider stochastic volatility but rather examine the implications of the model for the ability to discriminate trend and cycle under different (constant) assumptions about the volatility of trend inflation.

3.5 Nesting the bivariate models

Here, we present a single bivariate UC model specification that encompasses the three models laid out above. In our general specification, x_t is the accompanying variable (the

[2]Another paper that incorporates a Phillips curve relationship in the estimation of the output gap is Basistha and Nelson (2007). These authors included a survey measure of inflation expectations in equation (11) along with lagged inflation.

[3]In fact, the model that Stella and Stock (2016) propose has the unemployment and the inflation rates as observables, as opposed to GDP and the inflation rate. The aim, however, is the same as in the models with GDP as observable: to disentangle the cyclical component of GDP.

unemployment rate, GDI, or the inflation rate) whose trend can be modeled under two alternatives, as shown below:

$$y_t = \tau_{yt} + c_t, \tag{14}$$
$$x_t = \tau_{xt} + \theta c_t, \tag{15}$$
$$\tau_{yt} = \mu_y + \tau_{y,t-1} + \eta_{yt}, \tag{16}$$
$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \varepsilon_t, \tag{17}$$

- Exogenous Auxiliary Trend:

$$\tau_{xt} = \mu_x + \tau_{x,t-1} + \eta_{xt}, \tag{18}$$

- Endogenous Auxiliary Trend:

$$\tau_{xt} = \mu_x(1-\alpha) + \alpha x_{t-1} + \eta_{xt}, \tag{19}$$

  where $\alpha \in (0,1]$,

and

$$\text{var}\left(\begin{bmatrix}\varepsilon_t\\ \eta_{yt}\\ \eta_{xt}\end{bmatrix}\right) = \begin{bmatrix}\sigma_\varepsilon^2 & \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \rho_{\eta_x\varepsilon}\sigma_{\eta_x}\sigma_\varepsilon\\ \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \sigma_{\eta_y}^2 & \rho_{\eta_y\eta_x}\sigma_{\eta_y}\sigma_{\eta_x}\\ \rho_{\eta_x\varepsilon}\sigma_{\eta_x}\sigma_\varepsilon & \rho_{\eta_y\eta_x}\sigma_{\eta_y}\sigma_{\eta_x} & \sigma_{\eta_x}^2\end{bmatrix}.$$

The Exogenous Auxiliary Trend alternative encompasses the bivariate specifications of Sections 3.2 and 3.3, where GDP is accompanied by the unemployment rate and GDI, respectively. It also encompasses the specification of the inflation trend in Stella and Stock (2016). Note that this specification includes only one lag of the cycle entering x_t, whereas the GDP-unemployment rate case outlined in Section 3.2 includes two. To facilitate comparisons across the bivariate models proposed in the literature, we keep one lag only. (In work not presented, we explored using two lags instead of one; the results were very similar.)

When the inflation trend is specified as in equation (11), the Phillips curve cannot be nested with the other models (because the trend is affected by the observation variable). We therefore also consider the Endogenous Auxiliary Trend alternative, which is chosen to encompass the bivariate specification of Section 3.4 where GDP is accompanied by the inflation rate. Depending on the value of the parameter α, inflation can be a stationary process or a process with a unit root.

4 Monte Carlo Exercises

In this section, we assess the properties of the different specifications described above using Monte Carlo exercises. In our Monte Carlo experiments, we repeatedly simulate the various UC models of Section 3 and estimate them by maximum likelihood using the Kalman filter. We will evaluate the specifications along two main dimensions. First, we examine the properties of the estimated trend-cycle correlation for output, $\hat{\rho}_{\eta_y\varepsilon}$, in particular its sampling distribution. We look at the sampling distribution because, as emphasized by Basistha

(2007), standard univariate techniques can find large estimates of the correlation between trend and cycle innovations, even when the true value is zero. The second dimension along which we evaluate the specifications is the properties of the estimated cycle, in particular, its period, which is computed from the estimates of ϕ_1 and ϕ_2, and the estimated proportion of the variance of output that is accounted for by the variance of the cycle. We look at the properties of the cycle because these properties can differ depending on the correlation between trend and cycle. For example, in MNZ's baseline estimation, which allows correlation between trend and cycle, the period of the cycle for the U.S. economy is about 10 quarters. In contrast, the estimated periodicity of the cycle is considerably longer in bivariate specifications that allow for correlation between trend and cycle. In Clark (1989), the period of the cycle is estimated to be about 28 quarters; in Basistha and Nelson (2007), the period of the cycle is infinite in their bivariate setup that includes GDP and inflation only, and about 20 quarters in their trivariate setup that also includes the unemployment rate.

We also look at how each specification characterizes the variance decomposition of output. It is useful to consider the variance decomposition in this framework because it encompasses several dimensions of the estimation that can be affected by the choice of one specification or another, namely the proportion of the variance of output growth attributed to variations in the trend or to variations in the cycle, as well as the impact of the correlation between disturbances. We obtain the variance decomposition of output in the presence of potentially correlated disturbances as follows:

$$\begin{aligned}
\%\text{ of var}\left(\Delta y_t\right)\text{ explained by var}\left(\Delta c_t\right) &= 100 \times \frac{\text{var}\left(\mathbb{E}\left(\Delta y_t | \Delta c_t\right)\right)}{\text{var}\left(\Delta y_t\right)} \\
&= 100 \times \frac{\text{var}\left(\Delta c_t\right)\left(1 + \frac{\rho_{\eta_y \varepsilon}\sigma_{\eta_y}\sigma_\varepsilon}{\text{var}(\Delta c_t)}\right)}{\sigma^2_{\eta_y} + \text{var}\left(\Delta c_t\right) + 2\rho_{\eta_y \varepsilon}\sigma_{\eta_y}\sigma_\varepsilon}, \qquad (20)
\end{aligned}$$

where the estimated parameters replace their theoretical counterparts to compute the estimated variance decomposition. As can be seen, the output trend-cycle correlation influences the variance decomposition.

The central question we aim to answer is whether standard maximum-likelihood techniques can recover the correct value of the correlation between the trend and cycle for output, $\rho_{\eta_y \varepsilon}$, as well as an accurate decomposition of output into trend and cycle. In particular, we are interested in the features of the accompanying variable, x_t, that make the estimation of the correlation coefficient and the properties of the cycle precise. In discriminating these features, it is useful to look at the variance decomposition of x_t, that is, the percent of its variance that is due to the variance of cycle. The variance decomposition for each of the specifications can be written as follows:

- Exogenous Trend:

$$\begin{aligned}
\%\text{ of var}\left(\Delta x_t\right)\text{ explained by var}\left(\Delta c_t\right) &= 100 \times \frac{\text{var}\left(\mathbb{E}\left(\Delta x_t | \Delta c_t\right)\right)}{\text{var}\left(\Delta x_t\right)} \\
&= 100 \times \frac{\text{var}\left(\Delta c_t\right)\left(\theta + \frac{\rho_{\eta_x \varepsilon}\sigma_{\eta_y}\sigma_\varepsilon}{\text{var}(\Delta c_t)}\right)^2}{\sigma^2_{\eta_x} + \theta^2\,\text{var}\left(\Delta c_t\right) + 2\theta\rho_{\eta_x \varepsilon}\sigma_{\eta_x}\sigma_\varepsilon} \qquad (21)
\end{aligned}$$

- Endogenous Trend:
  - Nonstationarity [$\alpha = 1$]:

$$\begin{aligned}\% \text{ of var}(\Delta x_t) \text{ explained by var}(c_t) &= 100 \times \frac{\operatorname{var}(\mathbb{E}(\Delta x_t | c_t))}{\operatorname{var}(\Delta x_t)} \\ &= 100 \times \frac{\operatorname{var}(c_t)\left(\theta + \frac{\rho_{\eta_x \varepsilon}\sigma_{\eta_y}\sigma_\varepsilon}{\operatorname{var}(c_t)}\right)^2}{\sigma^2_{\eta_x} + \theta^2 \operatorname{var}(c_t) + 2\theta\rho_{\eta_x\varepsilon}\sigma_{\eta_x}\sigma_\varepsilon}, \end{aligned} \quad (22)$$

  - Stationarity [$\alpha \in [0, 1)$]:

$$\begin{aligned}\% \text{ of var}(\Delta x_t) \text{ explained by var}(c_t) &= 100 \times \frac{\operatorname{var}(\mathbb{E}(\Delta x_t | c_t))}{\operatorname{var}(\Delta x_t)} \\ &= 100 \times \frac{\operatorname{var}(c_t)\left(\theta + \frac{\rho_{\eta_x \varepsilon}\sigma_{\eta_x}\sigma_\varepsilon}{\operatorname{var}(c_t)} + (\alpha - 1)\frac{\operatorname{cov}(x_{t-1}, c_t)}{\operatorname{var}(c_t)}\right)^2}{\operatorname{var}(\Delta x_t)}, \end{aligned} \quad (23)$$

where

$$\operatorname{var}(\Delta x_t) = \frac{2}{1+\alpha}\left(\sigma^2_{\eta_x} + \theta^2 \operatorname{var}(c_t) + 2\theta\rho_{\eta_x\varepsilon}\sigma_{\eta_x}\sigma_\varepsilon + \theta(\alpha - 1)\operatorname{cov}(x_{t-1}, c_t)\right),$$

$$\operatorname{cov}(x_{t-1}, c_t) = \rho_{\eta_x\varepsilon}\sigma_{\eta_x}\sigma_\varepsilon \sum_{i=0}^{\infty} \alpha^i \psi_{i+1} + \theta\sigma^2_\varepsilon \sum_{i=0}^{\infty}\alpha^i \sum_{j=i+1}^{\infty} \psi_j \psi_{j-i-1},$$

and ψ_j, for $j = 0, 1, 2, ...$, are the coefficients of the Wold representation of the cycle.[4] Notice that as $\alpha \to 1$, the variance decomposition in Equation (23) converges to the variance decomposition in Equation (22).

To explore the features of x_t that makes it a good candidate to find an accurate trend-cycle decomposition of output, we vary two parameters of the processes under the exogenous and endogenous trend alternatives. First, when x_t is nonstationary, we consider several values of the variance of the innovation to the trend, $\sigma^2_{\eta_x}$, which will affect the variance decompositions in equations (21) and (22). Second, when x_t is stationary, and a persistent trend is absent, we vary the persistence coefficient, α, which will change the variance decomposition in equation (23). Given that our benchmark parametrizations assume that $\rho_{\eta_x\varepsilon} = 0$, we can achieve any variance decomposition of x_t by varying $\sigma^2_{\eta_x}$ and keeping all the other coefficients fixed.

The models were estimated using the `fmincon` function in Matlab 2013a. A key constraint on the correlation coefficients to guarantee positive definiteness of the variance-covariance matrix of the innovations ε_t, η_{yt}, and η_{xt} is given by $\rho^2_{\eta_y\varepsilon} + \rho^2_{\eta_x\varepsilon} + \rho^2_{\eta_y\eta_x} + 2\rho_{\eta_y\varepsilon}\rho_{\eta_x\varepsilon}\rho_{\eta_y\eta_x} \leq 1$. We used the interior-point algorithm.

4.1 Monte Carlo Exercises: x_t nonstationary

The parametrizations of the nonstationary models we consider appear in Table 1. Numbers in curly brackets indicate the possible values that the parameters can take over the

[4] $\psi_0 = 1, \psi_1 = \phi_1, \psi_2 = \phi_1\psi_1 + \phi_2, \psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2}$ for $j \geq 3$.

Table 1: Parameter Values - x_t nonstationary

	Parameter Value
μ_y	$\mathbf{0.8}$
ϕ_1	$\mathbf{1.5}$
ϕ_2	$\mathbf{-0.6}$
σ_ε	$\mathbf{0.6}$
σ_{η_y}	$\mathbf{0.7}$
$\rho_{\eta_y \varepsilon}$	$\{-0.9, \mathbf{0}, 0.9\}$
θ	$\mathbf{0.5}$
μ_x	$\mathbf{0}$
σ_{η_x}	$\{0.01, \mathbf{0.1}, 0.2, 0.35, 0.5, 0.7, 1.2, 1.7, 2.5, 3.0\}$
$\rho_{\eta_x \varepsilon}$	$\mathbf{0}$
$\rho_{\eta_y \eta_x}$	$\mathbf{0}$

$T = \{50, 100, \mathbf{200}, 500, 1,000\}$

simulations; bold numbers indicate our benchmark parametrization. The combinations of these values will define the different cases to be explored in the Monte Carlo experiments. This calibration for y_t is similar to the results obtained by MNZ when they assumed no correlation between trend and cycle (labeled UC-0 in MNZ). In particular, the values of ϕ_1 and ϕ_2 in the cycle process, c_t, imply that it will display a hump-shaped pattern; in this case, there are complex roots resulting in a duration of the cycle of about 25 periods. Under the baseline variance decomposition, the variance of the cycle explains about 54 percent of the variations of Δy_t. The mean growth rate, μ_y, is also chosen to match the MNZ results. Notice that the coefficients of the process for y_t, which is the process that represents (the log of) GDP, are the same across experiments, keeping the properties of the cycle fixed across simulations. For convenience, we assume that the cyclical component of y_t enters x_t with a loading coefficient $\theta = 0.5$.[5] We also assume that the drift component of the auxiliary variable is zero—that is, that $\mu_x = 0$.[6]

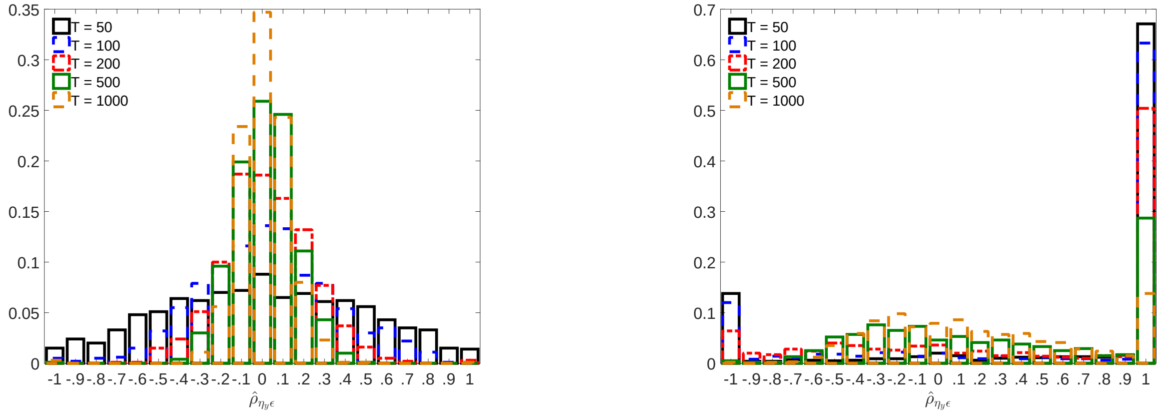
As Basistha (2007) has emphasized, standard univariate techniques can find large estimates of the correlation between trend and cycle innovations even when the true value is zero. We therefore begin by assuming that in the data generating process, $\rho_{\eta_y \varepsilon}$ is zero and then change it to values in its boundary. In addition, because our emphasis is on the correlation between the trend and cycle for output, we will assume throughout the experiments that $\rho_{\eta_y \eta_x}$ and $\rho_{\eta_x \varepsilon}$ are zero.

[5] At first glance, it may appear that this loading factor is only appropriate when x_t is the unemployment rate (up to a sign change). However, an examination of equations (21) and (22) indicates that, for $\rho_{\eta_x \varepsilon} = 0$, we can replicate any variance decomposition of x_t through the appropriate choice of σ_{η_x}. We are therefore able to encompass the parametrizations of inflation or GDI. That is what we do in the simulations, as can be seen from Table 1 where σ_{η_x} takes on a large set of parameter values and θ is fixed.

[6] When x_t corresponds to unemployment or inflation, this assumption is appropriate. In the case of GDI, this coefficient should be equal to μ_y, but the results are not affected by the choice of μ_x.

Figure 1: Frequency Distribution of $\hat{\rho}_{\eta_y \varepsilon}$ under $\rho_{\eta_y \varepsilon} = 0$

(a) Bivariate model (b) Univariate model



4.1.1 Base case: Exogenous auxiliary trend

The first set of Monte Carlo experiments is designed to mimic the GDP-unemployment rate setup of Section 3.2 and the GDP-GDI setup of Section 3.3; it can also be used to interpret the Stella-Stock model of trend inflation discussed in Section 3.4. As stated before, we assume that the trend component of the accompanying variable follows a random walk process.

Figure 1a shows the distribution of the estimated correlation coefficient $\hat{\rho}_{\eta_y \varepsilon}$ obtained from the simulations with benchmark parameter values for sample sizes $T = 50, 100, 200, 500, 1,000$. Under these assumptions, the trend and cycle for output are orthogonal ($\rho_{\eta_y \varepsilon} = 0$) and the auxiliary variable has trend variability similar to that found for the unemployment rate ($\sigma_{\eta_x} = 0.1$) (see Clark, 1989; Fleischman and Roberts, 2011, for example).

The distribution of the maximum likelihood estimator of the correlation coefficient between the trend and cycle innovations of y_t has a conventional shape, with the mass of the distribution concentrated around the true coefficient value as the sample size increases. For typical U.S. quarterly sample sizes of around 200, results are reasonably precise.

The shape of the distribution of the estimator of the trend-cycle correlation coefficient under this bivariate setup contrasts sharply with the distribution of the same correlation coefficient when a univariate estimation is used—as in MNZ. In Figure 1b, results are reported based on data generated with the same benchmark parameter values as in Table 1, but omitting x_t as an observable in the estimation of the UC model. As can be seen, the maximum likelihood estimation of the univariate model implies an estimated trend-cycle correlation coefficient that has a distribution with masses close to -1 and +1. Even with very large sample sizes, the fat tails of the distribution are evident. The results in Figures 1a and 1b suggest that estimation of the bivariate model that includes an x_t process with the baseline features substantially improves the small sample properties of the estimator of the trend-cycle correlation coefficient.

Table 2 reports some key statistics about the estimated cyclical properties—in particular, the median estimated period of the cycle, the percent of simulations that the estimation

Table 2: Features of the Estimated Period and Variance Decomposition under $\rho_{\eta_y \varepsilon} = 0$

	Bivariate model			Univariate model		
Sample Size	Period	% finite	$\frac{\text{simulated } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}{\text{true } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}$	Period	% finite	$\frac{\text{simulated } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}{\text{true } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}$
$T = 50$	17	91	1.10	17	79	1.74
$T = 100$	21	94	1.07	19	67	1.69
$T = 200$	23	92	1.07	21	67	1.60
$T = 500$	25	96	1.05	23	81	1.28
$T = 1,000$	25	98	1.02	24	91	1.12

Note: The true implied period is 24.8 quarters. The true implied variance decomposition $\frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}$ is 54.2%. *% finite* denotes the percentage of times that a finite period was obtained. Results based on 1,000 simulations. Median statistics are reported, except for *% finite*.

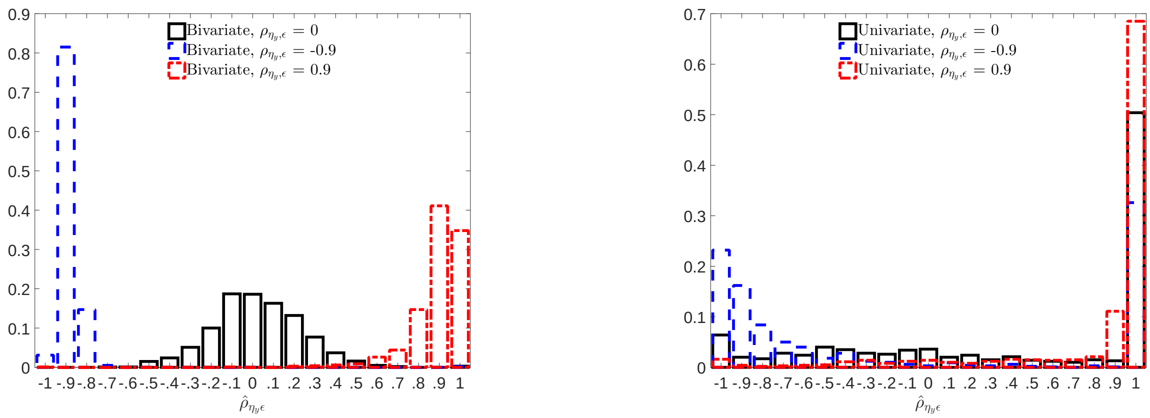
delivers correctly a finite period, and the ratio between the median of the estimated variance decomposition $\frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}$ and its true counterpart for different sample sizes. The results in the second, third and fourth columns show that, in our baseline bivariate setup, the estimated duration of the cycle is close to the implied true duration of about 25 periods with a sample of 200 periods. The percentage of times that the estimation correctly obtains a finite period is above 90 percent. The variance decomposition is reasonably similar to that implied by the true parameters. As the sample size increases, both the period and the variance decomposition approach monotonically their theoretical values, while the percentage of times that a finite period is obtained approaches 100 percent.

The fifth, sixth and seventh columns of Table 2 report the results from the estimation of the univariate setup that includes data on y_t only. The univariate estimation tends to deliver a shorter cyclical period, and the percentage of times that it correctly obtains a finite period is substantially reduced. It also tends to overestimate the fraction of the variation of Δy_t that is explained by the cycle. In particular, with a sample size of 200 observations, the estimated period is 16 percent lower than its theoretical value, compared with 8 percent in the bivariate case. Also, while the median variance decomposition of the bivariate specification is 7 percent above its theoretical counterpart, the univariate model delivers a cyclical contribution that is about 60 percent higher. While the period rises toward its theoretical counterpart as the sample size increases, it does it a slower rate than in the bivariate model, and the estimated variance decomposition still overshoots its theoretical value at the largest sample size considered. Thus, the addition of a nonstationary variable, x_t, with features implied by the benchmark parametrization in Table 1 not only helps reduce the bias in the estimation of the trend-cycle correlation coefficient but also helps reduce the overall bias in the estimation of the period and the amplitude of the cycle.

We also simulate the bivariate and univariate models of this section assuming that there is important correlation between trend and cycle—that is, that the results of MNZ hold. In particular, we simulate the model with the benchmark parametrization of Table 1, except that we also assume that $\rho_{\eta_y \varepsilon} = -0.9$ (MNZ's case) and that $\rho_{\eta_y \varepsilon} = 0.9$. Figure 2 reports the results of the bivariate and univariate models estimations. The plots show that the bivariate specification also does a good job at estimating the trend-cycle correlation coefficient of y_t when the trend and cycle are either negatively or positively correlated, as the peak of the

Figure 2: Frequency Distribution of $\hat{\rho}_{\eta_y\varepsilon}$ under $\rho_{\eta_y\varepsilon} \in \{-0.9, 0, 0.9\}$

(a) Bivariate model (b) Univariate model



distribution occurs close to the true values of -0.9 and 0.9, respectively. The univariate specification, on the other hand, continues to give probability masses that are excessively weighted in the tails. Thus, if the trend and cycle were correlated, the bivariate specification would very likely detect it.

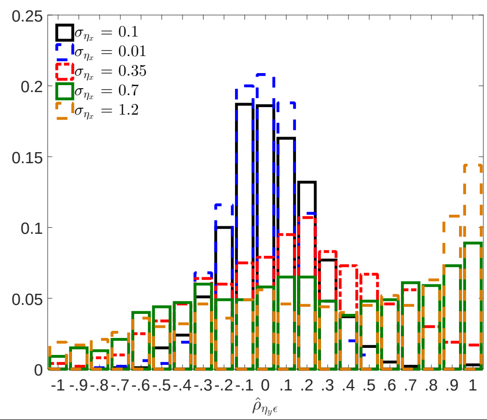
To assess how the estimates are affected by the relative contribution of trend and cycle to the variability of the accompanying variable, we conduct a set of Monte Carlo exercises with the benchmark parametrization of Table 1, except that σ_{η_x} varies in the set $\{0.01, 0.1, 0.2, 0.35, 0.5, 0.7, 1.2\}$. Figure 3 shows the results.[7]

As can be seen, the simulation results imply that increasing the standard deviation of the trend innovation of x_t makes the estimation of $\rho_{\eta_y\varepsilon}$ less accurate. In particular, the distribution of the correlation estimates tends to pile up towards +1. The cyclical properties of the model are also distorted: The cycle period tends to be underestimated; the percentage of times that the estimation correctly delivers a finite period decreases significantly; and the estimated fraction of the variation of Δy_t that is explained by variations in the cycle is further above their theoretical counterparts. A key implication of these results is that adding a variable with large trend variability would not help recover accurately the correlation coefficient or to give precise estimates of the cycle.

With these results, we can also explain why the correlation coefficient and the trend-cycle decomposition would not be accurately estimated using the GDP-GDI bivariate model of Section 3.3. In that setup, $\theta = 1$ and $\mu_x = \mu_y = \mu_z$. However, we know that, since GDI is intended to measure the same concept of overall economic activity as GDP, it should have a variance decomposition very similar to that of GDP. Hence, we can keep the coefficients θ and μ_x at their benchmark values and modify σ_{η_x} to achieve the same variance decomposition of Δy_t. That parametrization corresponds to the case where $\sigma_{\eta_x} = 0.35$ in the table of Figure 3. The results in this case are inferior to those with $\sigma_{\eta_x} = 0.1$, our benchmark case. In particular, the simulated distribution of $\hat{\rho}_{\eta_y\varepsilon}$ has negative skewness and starts to gain mass

[7] In the benchmark parametrization, with $\sigma_{\eta_x} = 0.1$ and $\rho_{\eta_x\varepsilon} = 0$, the variance decomposition implies that the variance of $\theta\Delta c_t$ explains about 94 percent of the variance of Δx_t.

Figure 3: Frequency Distribution of $\hat{\rho}_{\eta_y \varepsilon}$ under $\sigma_{\eta_x} \in \{0.01, 0.1, 0.35, 0.7, 1.2\}$ (Exogenous Trend)



Standard Deviation	true $100 \times \frac{\text{var}(\mathbb{E}(\Delta x_t \mid \Delta c_t))}{\text{var}(\Delta x_t)}$	Period	% finite	$\frac{\text{simulated } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}{\text{true} \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}$
$\sigma_{\eta_x} = 0.01$	99.9	22	96	1.02
$\sigma_{\eta_x} = 0.1$	94.0	23	92	1.07
$\sigma_{\eta_x} = 0.2$	78.4	23	84	1.16
$\sigma_{\eta_x} = 0.35$	54.2	22	71	1.26
$\sigma_{\eta_x} = 0.5$	36.7	21	69	1.37
$\sigma_{\eta_x} = 0.7$	22.9	21	64	1.39
$\sigma_{\eta_x} = 1.2$	9.2	20	67	1.45

Note: The true implied period is 24.8 quarters. The true implied variance decomposition $\frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}$ is 54.2%. *% finite* denotes the percentage of times that a finite period was obtained. Median statistics are reported, except for *% finite*. Results based on 1,000 simulations with sample size $T = 200$.

at positive values of the estimated correlation coefficient. Also, the fourth column of the table shows that the percentage of times that the estimation delivers a finite cycle is 71 percent, compared with 92 percent in the benchmark case. Additionally, the variance decomposition attributes 26 percent more of the variations of Δy_t to the cycle than it should, as can be seen in the last column, compared with only 7 percent in the benchmark parametrization. These results suggest that using the unemployment rate as an auxiliary variable should lead to superior results relative to using GDI. Nonetheless, it is worth noting that the bivariate results with GDI would be more accurate than the univariate results. In particular, the mass of extreme results is significantly smaller than in the univariate case, and the overall performance of the estimates of the periodicity and variance decomposition is better.

We next turn to calibrations in which x_t has properties similar to inflation as in Stella and Stock (2016). As discussed above, Stella and Stock (2016) assume an exogenous inflation trend, so that with appropriate unit transformations, we can use Figure 3 to assess the usefulness of inflation as an auxiliary variable. In particular, their estimated Phillips curve slope coefficient is about -0.37, using the unemployment rate as their cyclical variable. Adjusting for an Okun's law coefficient of -0.5 would imply $\theta = 0.19$ in our setup. In terms of the volatility of the trend component of the auxiliary variable, Stella and Stock (2016) allow the variance of inflation to vary over time. They find that, for the period since 2010—near the end of their sample—the variance of the permanent component of inflation has been about 0.15, implying a standard deviation of a bit less than 0.4 percentage points. By contrast, in the 1970s, the variance of the permanent component of inflation was considerably larger, reaching as high as 1¾ percent in the middle of the decade, implying a standard deviation of the permanent component as high as 1.3 percentage points.

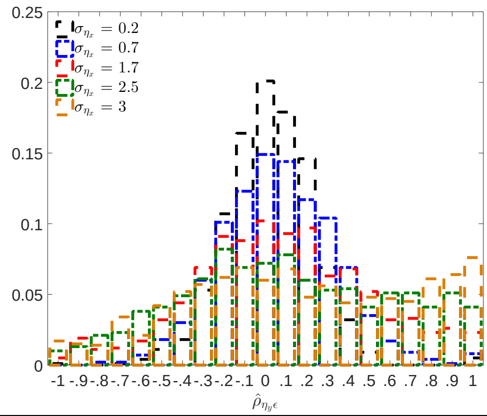
According to equation (21), the cycle would account for around 12 percent of the variation in inflation based on Stella and Stock (2016)'s recent estimate, and 1¼ percent when the inflation trend was more volatile. Those values would put us at the bottom rows (and beyond) of the table of Figure 3, where the variance decomposition of x_t is between 23 and 9 percent. As can be seen, with these values of the standard deviation of the trend of x_t, the correlation between trend and cycle would be poorly estimated, as the sampling distribution shows, and the features of the cycle would imply a lower estimated period and a much higher contribution of the cycle to the variations of Δy_t, compared to their theoretical counterparts. Additionally, the percentage of times that the estimation delivers a finite period is relatively low.[8]

4.1.2 Endogenous auxiliary trend

Our second assumption about the trend in the auxiliary variable accommodates another common model of the trend component of inflation, in which the cyclical component affects the evolution of the trend (see Section 3.4). In this section, we consider the nonstationary variant of the trend component of x_t, assuming that $\alpha = 1$; this specification of the inflation trend has been used by Kuttner (1994) and Roberts (2001). As in Figure 3, we examine the consequences of this particular UC model setup on the estimators of the trend-cycle

[8] As discussed in Section 3.4, Stella and Stock's baseline specification included measurement error in inflation. We explored adding measurement error to the auxiliary variable and found that, qualitatively, the results still hold.

Figure 4: Frequency Distribution of $\hat{\rho}_{\eta_y \varepsilon}$ under $\sigma_{\eta_u} \in \{0.2, 0.7, 1.7, 2.5, 3\}$ (Endogenous Trend)



Standard Deviation	true $100 \times \frac{\text{var}(\mathbb{E}(\Delta x_t \mid c_t))}{\text{var}(\Delta x_t)}$	Period	% finite	$\frac{\text{simulated } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}{\text{true } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}$
$\sigma_{\eta_x} = 0.1$	99.1	22	94	1.02
$\sigma_{\eta_x} = 0.2$	96.7	23	93	1.02
$\sigma_{\eta_x} = 0.5$	82.3	23	91	1.07
$\sigma_{\eta_x} = 0.7$	70.3	23	90	1.08
$\sigma_{\eta_x} = 1.2$	44.6	23	85	1.13
$\sigma_{\eta_x} = 1.7$	28.7	22	83	1.12
$\sigma_{\eta_x} = 2.5$	15.7	22	78	1.23
$\sigma_{\eta_x} = 3.0$	11.4	22	76	1.25

Note: The true implied period is 24.8 quarters. The true implied variance decomposition $\frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}$ is 54.2%. *% finite* denotes the percentage of times that a finite period was obtained. Median statistics are reported, except for *% finite*. Results based on 1,000 simulations with sample size $T = 200$.

correlation and the cycle by varying σ_{η_x} in the set $\{0.1, 0.2, 0.5, 0.7, 1.2, 1.7, 2.5, 3.0\}$ and leaving the other coefficients at their benchmark values. Figure 4 shows the results.

The parametrization of σ_{η_x} in Figure 4 includes higher values than in Figure 3 to capture the small contribution of the cycle to the variability of inflation that is found in the results of the existing literature. For example, rough calculations of the variance decomposition based on the results of Kuttner (1994) and Roberts (2001) imply numbers lower than 15 percent, which correspond to the results reported in the last rows of the table in Figure 4. For those values of σ_{η_x}, the sampling distribution of $\hat{\rho}_{\eta_y \varepsilon}$ is relatively flat and tail events gain in importance, as can be seen in the figure. The fourth column of the table shows that the percentage of times that the estimation delivers a finite period, between 76 and 78 percent, is higher than in the univariate case, but much lower than in our benchmark bivariate setup. The last column shows that the cycle contribution to the variability of y_t is overestimated in the order of around 23 to 25 percent, again better than the univariate case but worse than with a variable that resembles the unemployment rate.

Table 3: Parameter Values - x_t stationary

	Parameter Value
μ_y	$\mathbf{0.8}$
ϕ_1	$\mathbf{1.5}$
ϕ_2	$\mathbf{-0.6}$
σ_ε	$\mathbf{0.6}$
σ_{η_y}	$\mathbf{0.7}$
$\rho_{\eta_y \varepsilon}$	$\mathbf{0}$
θ	$\mathbf{0.5}$
μ_x	$\mathbf{0}$
α	$\{0.1, 0.3, \mathbf{0.5}, 0.7, 0.9, 0.95, 0.99\}$
σ_{η_x}	$\{\mathbf{1}, 3, 7\}$
$\rho_{\eta_x \varepsilon}$	$\mathbf{0}$
$\rho_{\eta_y \eta_x}$	$\mathbf{0}$

$T = 200$

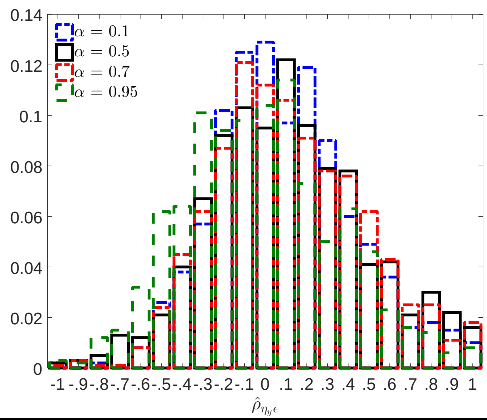
4.2 Monte Carlo Exercises: x_t stationary

In this case, we assume that the inflation trend is given by $\tau_{xt} = \mu_x(1-\alpha) + \alpha x_{t-1} + \eta_{xt}$, as in the Endogenous Trend alternative, except that here, we allow $\alpha \in (0,1)$. We vary the persistence coefficient α inside the unit circle and examine the properties of the trend-cycle correlation and cycle estimates. We also consider three values for the variability of the auxiliary variable, σ_{η_x}. The parametrizations used appear in Table 3. Setting $\sigma_{\eta_x} = 3$ makes the results as $\alpha \to 1$ compatible with the results under the specification of the Endogenous Trend in Section 4.1.2. As discussed in that section, our reading of the literature suggested a relatively high value of σ_{η_x}. To be consistent with the Monte Carlo exercise in Basistha (2007), we also consider a value of $\sigma_{\eta_x} = 1$. Basistha (2007) assumed $\alpha = 0.5$, $\sigma_{\eta_x} = 1$ and that the influence of the cycle on the auxiliary variable, θ, is 0.4.

In Figure 5, $\sigma_{\eta_x} = 1$. With this setting, the results largely confirm the findings in the simulation exercises of Basistha (2007): The distribution of $\hat{\rho}_{\eta_y \varepsilon}$ is well-behaved over a wide range of values for α. As can be seen in Figures 6 and 7, however, as σ_{η_x} increases and a lower proportion of the variability of the auxiliary variable—inflation—is attributed to the cycle, the distribution becomes less well behaved.[9] The figures indicate that the distribution of the estimated trend-cycle correlation coefficient tends to pile up at +1. Three additional features of the estimation can be distinguished as σ_{η_x} increases. First, the estimated period of the cycle tends to decline. Second, the percentage of correctly estimated finite periods also tends to decline. This percentage worsens as the persistence of the auxiliary variable

[9] Notice that the second column of the table in Figure 6 shows that, as the persistence coefficient approaches one, the contribution of the variance of the cycle to the variance of x_t tends to the value when $\alpha = 1$, which can be seen in the second column of the last row of the table of Figure 4. In fact, the contribution of the variance of the cycle to the variance of x_t is 11.35% when $\alpha = 0.999$.

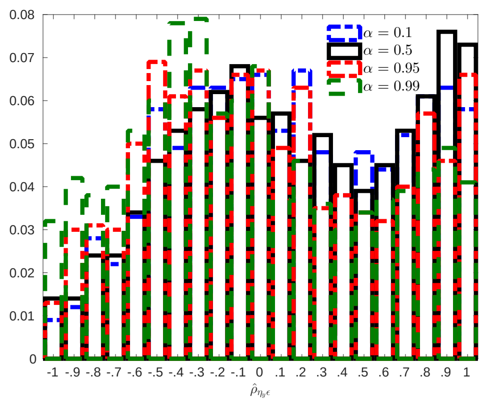
Figure 5: Frequency Distribution of $\hat{\rho}_{\eta_y \varepsilon}$ - x_t stationary, $\sigma_{\eta_x} = 1$



Persistence	true $100 \times \frac{\text{var}(\mathbb{E}(\Delta x_t \mid c_t))}{\text{var}(\Delta x_t)}$	Period	% finite	$\frac{\text{simulated } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}{\text{true} \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}$
$\alpha = 0.1$	0.4	22	88	1.11
$\alpha = 0.3$	1.0	23	87	1.14
$\alpha = 0.5$	2.9	22	87	1.12
$\alpha = 0.7$	9.2	22	89	1.13
$\alpha = 0.9$	30.3	23	83	1.12
$\alpha = 0.95$	40.6	23	81	1.08
$\alpha = 0.99$	50.9	23	75	1.07

Note: The true implied period is 24.8 quarters. The true implied variance decomposition $\frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}$ is 54.2%. *% finite* denotes the percentage of times that a finite period was obtained. Median statistics are reported, except for *% finite*. Results based on 1,000 simulations with sample size $T = 200$.

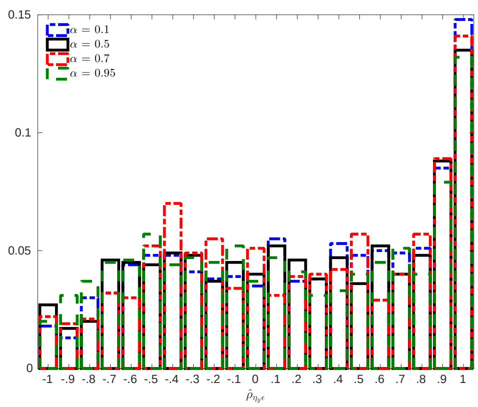
Figure 6: Frequency Distribution of $\hat{\rho}_{\eta_y \varepsilon}$ - x_t stationary, $\sigma_{\eta_x} = 3$



Persistence	true $100 \times \frac{\text{var}(\mathbb{E}(\Delta x_t \mid c_t))}{\text{var}(\Delta x_t)}$	Period	% finite	$\frac{\text{simulated } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}{\text{true} \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}$
$\alpha = 0.1$	0.04	22	77	1.20
$\alpha = 0.3$	0.1	21	76	1.25
$\alpha = 0.5$	0.4	21	77	1.24
$\alpha = 0.7$	1.4	22	77	1.22
$\alpha = 0.9$	5.6	22	72	1.21
$\alpha = 0.95$	8.1	22	71	1.21
$\alpha = 0.99$	10.7	22	68	1.21

Note: The true implied period is 24.8 quarters. The true implied variance decomposition $\frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}$ is 54.2%. *% finite* denotes the percentage of times that a finite period was obtained. Median statistics are reported, except for *% finite*. Results based on 1,000 simulations with sample size $T = 200$.

Figure 7: Frequency Distribution of $\hat{\rho}_{\eta_y \varepsilon}$ - x_t stationary, $\sigma_{\eta_x} = 7$



Persistence	true $100 \times \frac{\text{var}(\mathbb{E}(\Delta x_t \mid c_t))}{\text{var}(\Delta x_t)}$	Period	% finite	$\frac{\text{simulated } \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}{\text{true} \frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}}$
$\alpha = 0.1$	0.01	20	70	1.38
$\alpha = 0.3$	0.02	21	68	1.39
$\alpha = 0.5$	0.07	20	70	1.37
$\alpha = 0.7$	0.3	20	68	1.43
$\alpha = 0.9$	1.1	21	66	1.41
$\alpha = 0.95$	1.6	21	64	1.45
$\alpha = 0.99$	2.2	20	63	1.42

Note: The true implied period is 24.8 quarters. The true implied variance decomposition $\frac{\text{var}(\mathbb{E}(\Delta y_t \mid \Delta c_t))}{\text{var}(\Delta y_t)}$ is 54.2%. *% finite* denotes the percentage of times that a finite period was obtained. Median statistics are reported, except for *% finite*. Results based on 1,000 simulations with sample size $T = 200$.

increases for any value of σ_{η_x}. Finally, the proportion of the variance of output attributed to the cycle tends to be more overestimated.

We believe that the relevant distribution is the one shown in Figure 7. While the empirical estimates in Basistha (2007) of $\alpha = 0.75$ and $\sigma_{\eta_x} = 1.35$ are not far from the values assumed in his Monte Carlo exercise, crucially, he finds an estimate of $\theta = 0.11$. This small value for the influence of the cycle on the auxiliary variable, inflation in this case, dramatically reduces its value as a cyclical indicator relative to his Monte Carlo exercise; the impact of the parameter θ on the variance decomposition can be seen in Equation (23).[10] In particular, Basistha's empirical evidence suggests that the contribution of the cycle to the variance of inflation is on the order of ¼ percent.[11] Our simulations would be able to reproduce that order of the variance decomposition for $\alpha = 0.7$—close to the persistence obtained by Basistha—with a standard deviation of the auxiliary variable's trend, $\sigma_{\eta_x} = 7$. In that case, we would obtain results similar to those in the $\alpha = 0.7$ row of the table in Figure 7. Those results indicate that the distribution of $\rho_{\eta_y \varepsilon}$ would be seriously distorted, the period would be underestimated, only 68 percent of the times a finite period would be obtained, and the cycle explains 43 percent more of the variations in GDP than it should.

It is an open question whether inflation should be modelled as a stationary process, especially over long samples. In Basistha (2007)'s empirical application for the Canadian economy, the results indicate a persistence coefficient of 0.75 with a conventional confidence interval that does not include unity. However, Basistha requires three mean breaks in inflation to obtain that result. Other authors have favored a unit-root specification for inflation. For example, Basistha and Nelson (2007) find that the autoregressive coefficient on U.S. inflation is about 0.88 and the 95% confidence interval includes unity. Along the same lines, Stock and Watson (2007) find that U.S. inflation probably has a unit root.[12] Hence, at least for the case of the United States, inflation is probably nonstationary, in which case the results imply that the trend-cycle correlation and the cycle using inflation as the auxiliary variable would not be accurately estimated.

5 Size and Power of the Likelihood Ratio Test of Hypotheses about $\rho_{\eta_y \varepsilon}$

In this section, we investigate the performance of the Likelihood Ratio (LR) test under nonstationarity of the accompanying variable and the three following assumptions:

- Univariate model and benchmark parametrization.
- Bivariate model under Exogenous Auxiliary Trend and benchmark parametrization.

[10] Basistha's estimation also finds a statistically significant MA(1) term, which, along with the other estimated parameters, implies a lower contribution of the variance of the cycle to the variance of inflation compared with the case in which there is not such a term.

[11] The variance decomposition of the auxiliary variable under the benchmark parametrization in Basistha's Monte Carlo exercise is 5.8%.

[12] Other papers simply assume a unit root in inflation, such as Kuttner (1994), Roberts (2001) or Basistha and Startz (2008)

Under this parametrization, the auxiliary variable has properties similar to the unemployment rate.

- Bivariate model under Endogenous Auxiliary Trend and high trend shock variance of the accompanying variable. Under this parametrization, the accompanying variable resembles inflation.

5.1 Size of the LR Test of Hypotheses about $\rho_{\eta_y \varepsilon}$

We first compute the frequency with which the LR test incorrectly rejects a true hypothesized value—that is, the size of the test. We simulate the bivariate model 1,000 times according to the specification in Table 1, except that we set $\rho_{\eta_y \varepsilon} = \rho^0_{\eta_y \varepsilon}$ for $\rho^0_{\eta_y \varepsilon} \in \{-0.95, -0.9, -0.8, \ldots, -0.1, 0, 0.1, \ldots, 0.8, 0.9, 0.95\}$. We consider the null hypotheses in the univariate and bivariate estimations as $H_0 : \rho_{\eta_y \varepsilon} = \rho^0_{\eta_y \varepsilon}$.

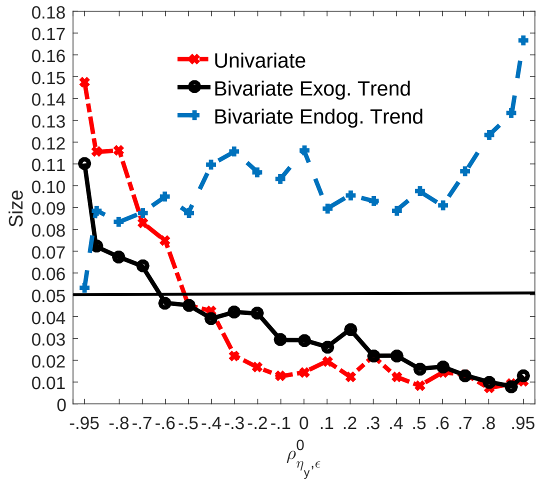
Figure 8 plots the size of the likelihood ratio test for the univariate and bivariate models using a 5 percent significance level. Recall that, ideally, the test should lead to a horizontal line at a size of 0.05—that is, for all values of $\rho_{\eta_y \varepsilon}$, the hypothesis would be rejected 5 percent of the time. As can be seen, none of these models meets that ideal, although of the three, the Bivariate Exogenous Trend alternative comes closest. In particular, the univariate estimation delivers very low size for hypothesized values of the correlation coefficient above -0.3, such that the test will not reject the hypothesized true value often enough. For correlations below -0.5, the size is too large, meaning that the test of the null hypothesis will be rejected too often. Based on their simulations, MNZ argue that the size of the LR test of the hypothesis $H_0 : \rho_{\eta_y \varepsilon} = 0$ is approximately correct. This interpretation is broadly consistent with our findings because we find that at $H_0 : \rho_{\eta_y \varepsilon} = 0$, the null hypothesis will not be rejected often enough, making MNZ's rejection of the hypothesis all the more convincing.

In the case of the Bivariate Endogenous Trend model, the size is uniformly too large, meaning that the LR test under this alternative would reject the null hypothesis too often, in particular the hypothesis $H_0 : \rho_{\eta_y \varepsilon} = 0$. While the size of the test under the Bivariate Exogenous Trend alternative is too large for values of the correlation coefficient lower than -0.7 and too small for values larger than 0.3, it performs almost uniformly better than the other two cases contemplated.

5.2 Power of the LR Test of Hypotheses about $\rho_{\eta_y \varepsilon}$

We now consider the ability of the LR test to reject various false hypothesized values for $\rho_{\eta_y \varepsilon}$—that is, the power of the test. Three exercises are performed in which we simulate the bivariate model 1,000 times. First, we simulate the model according to the benchmark parameter specification in Table 1, that is, assuming that $\rho_{\eta_y \varepsilon} = 0$, and set the null hypotheses in the univariate and bivariate estimations as $H_0 : \rho_{\eta_y \varepsilon} = \rho^0_{\eta_y \varepsilon}$, where $\rho^0_{\eta_y \varepsilon} \in \{-0.95, -0.9, -0.8, -0.7, \ldots, -0.1, 0.1, \ldots, 0.7, 0.8, 0.9, 0.95\}$. In the second exercise, we set $\rho_{\eta_y \varepsilon} = -0.9$ to simulate the model and test the null hypotheses $H_0 : \rho_{\eta_y \varepsilon} = \rho^0_{\eta_y \varepsilon}$, where $\rho^0_{\eta_y \varepsilon} \in \{-0.95, -0.85, -0.8, -0.7, \ldots, -0.1, 0, 0.1, \ldots, 0.7, 0.8, 0.9, 0.95\}$. Finally, we simulate the model setting $\rho_{\eta_y \varepsilon} = 0.9$ and test the null hypotheses $H_0 : \rho_{\eta_y \varepsilon} = \rho^0_{\eta_y \varepsilon}$, where $\rho^0_{\eta_y \varepsilon} \in \{-0.95, -0.90, -0.8, -0.7, \ldots, -0.1, 0, 0.1, \ldots, 0.7, 0.8, 0.85, 0.95\}$

Figure 8: Size of the Likelihood Ratio Test



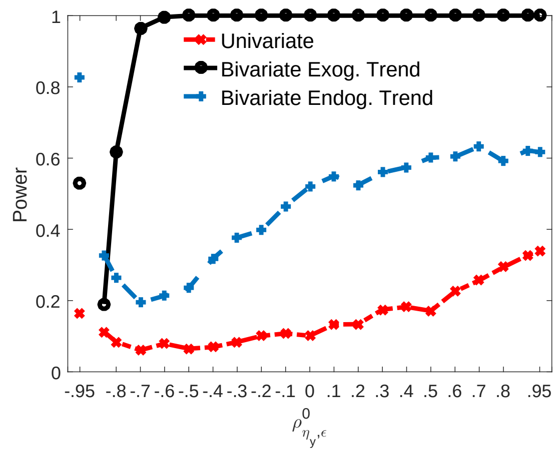
Note: The size corresponds to a significance level $\alpha = 0.05$.

Figure 9 reports the fraction of times the LR test rejects the false hypothesized value, for the univariate and bivariate models, in the three exercises performed. When the true value of $\rho_{\eta_y\varepsilon}$ is zero, Figure 9b shows that the test based on univariate estimation has virtually no power to reject hypothesized values of $\rho_{\eta_y\varepsilon}$ greater than -0.5. The power increases as the value of the hypothesized correlation coefficient approaches the left tail but still reaches only 37 percent for the null hypothesis $H_0 : \rho_{\eta_y\varepsilon} = -0.95$. Hence, the test based on the univariate estimation has very low power to reject the false null hypothesis of negative correlation between trend and cycle. Better performance is obtained under a Bivariate Endogenous Trend. The maximum power of the LR test is about 70 percent, and it is reached when $\rho^0_{\eta_y\varepsilon} = -0.95$. Better still is the Bivariate Exogenous Trend alternative, which has power increasing to 100 percent as the hypothesized value moves away from the true correlation coefficient of zero.

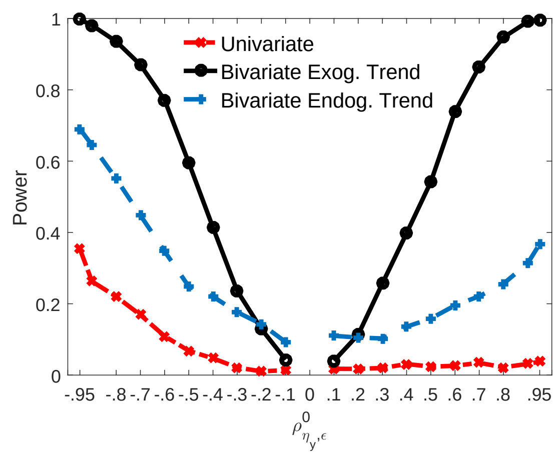
Figure 9a considers the case when the true value of $\rho_{\eta_y\varepsilon}$ is -0.9. In this case, the univariate estimation and Bivariate Endogenous Trend alternatives lack the ability to reject almost any false hypothesized value, whereas the Bivariate Exogenous Trend alternative rapidly approaches a rejection probability of one as the hypothesized values move above the true value of the correlation. In particular, the Bivariate Exogenous Trend model would reject with probability one a hypothesized value of zero for the correlation between trend and cycle innovations if the true correlation were -0.9, compared with a power of around 10 percent in the univariate estimation and 55 percent in the Bivariate Endogenous Trend assumption. A similar performance of the LR test is obtained when the true value of $\rho_{\eta_y\varepsilon}$ is 0.9, as can bee seen in Figure 9c. Once again, the Bivariate Exogenous Trend alternative yields the highest power of the test almost uniformly, and the worst performance is obtained from the univariate specification.

Figure 9: Power of the Likelihood Ratio Test

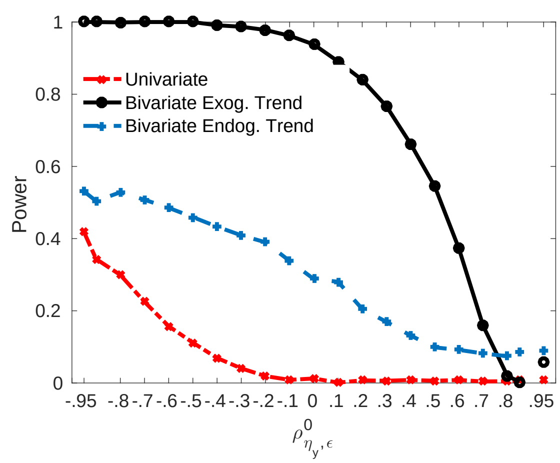
(a) $\rho_{\eta_y \varepsilon} = -0.9$



(b) $\rho_{\eta_y \varepsilon} = 0$



(c) $\rho_{\eta_y \varepsilon} = 0.9$



Note: The power corresponds to a significance level $\alpha = 0.05$.

6 Summary of Monte Carlo Findings

Before proceeding to the empirical application to obtain the trend-cycle decomposition of GDP for the U.S., we summarize the findings of the Monte Carlo exercises:

- Adding a second variable improves the econometrician's ability to distinguish the trend and cycle components of output correctly.
- For example, adding GDI—a second measure of the same concept as GDP—reduces the distortion in the sampling distribution of the correlation between trend and cycle. Nonetheless, in the case of GDI, the distortion in the estimated period remains substantial.
- Performance improves as the variability of the auxiliary variable's trend falls.
- In particular, when the variability of the trend is about the same as that typically found for the unemployment rate, the distortion in the sampling distribution of the correlation between trend and cycle for GDP is practically absent.
- When a stationary accompanying variable is considered with a parametrization that leads to properties similar to inflation—the leading candidate considered in the literature (see Basistha, 2007; Basistha and Nelson, 2007)—the sampling distribution of the correlation coefficient is distorted and the features of the estimated cycle are somewhat far from its theoretical counterpart.
- When we consider treatments that correspond to nonstationary inflation, whether as part of an "accelerationist Phillips curve" as in Kuttner (1994) or Roberts (2001), or as an exogenous trend, as in Stella and Stock (2016), we find that inflation performs poorly as an auxiliary variable. At best—when the variability of trend inflation is relatively low, as Stella and Stock (2016) suggest is the case in relatively recent periods—inflation performs better than GDI. In other settings, such as when the variability of trend inflation is greater, its performance as an auxiliary variable deteriorates.
- The model with the best performance of the LR test of hypothesis on the trend-cycle correlation coefficient in terms of size is the bivariate specification under an exogenous trend and the benchmark parametrization—the specification that resembles the unemployment rate. It is followed by the (realistically calibrated) bivariate specification under an endogenous trend and the univariate model, in that order. The same ranking is obtained when considering the power of the LR test.
- On balance, we find that the best of the three available auxiliary variables we consider is the unemployment rate, with its relatively low trend volatility being the key to its success.

Table 4: Log-likelihood and BIC for Different Restrictions

	Restriction	Log-likelihood	BIC
1	None	-326.52	714.66
2	$\rho_{\eta_y\varepsilon} = 0$	-329.13	714.28
3	$\rho_{\eta_u\varepsilon} = 0$	-326.89	709.80
4	$\rho_{\eta_y\eta_u} = 0$	-328.22	712.46
5	$\rho_{\eta_u\varepsilon} = \rho_{\eta_y\varepsilon} = 0$	-329.56	709.54
6	$\rho_{\eta_y\varepsilon} = \rho_{\eta_y\eta_u} = 0$	-330.64	711.70
7	$\rho_{\eta_u\varepsilon} = \rho_{\eta_y\eta_u} = 0$	-328.37	707.16
8	$\rho_{\eta_u\varepsilon} = \rho_{\eta_y\varepsilon} = \rho_{\eta_y\eta_u} = 0$	-330.71	706.24

Note: BIC is Bayesian Information Criterion

7 Estimation Results of the GDP-Unemployment Bivariate Model

Based on the Monte Carlo exercises, we conclude that, among the three considered, the nonstationary variable that would deliver the most reliable trend-cycle decomposition of GDP is the unemployment rate. Consistent with this finding, we therefore adopt the model of Section 3.2 that includes the unemployment rate, which we reproduce here for convenience.

$$
\begin{aligned}
y_t &= \tau_{yt} + c_t \\
\tau_{yt} &= \mu_y + \tau_{y,t-1} + \eta_{yt} \\
c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + \varepsilon_t, \\
u_t &= \tau_{ut} + \theta_1 c_t + \theta_2 c_{t-1}, \\
\tau_{ut} &= \tau_{u,t-1} + \eta_{ut},
\end{aligned}
$$

$$
\begin{bmatrix} \varepsilon_t \\ \eta_{yt} \\ \eta_{ut} \end{bmatrix} \sim \text{iid } \mathbb{N}\left(\mathbf{0}_{3\times1}, \begin{bmatrix} \sigma_\varepsilon^2 & \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \rho_{\eta_u\varepsilon}\sigma_{\eta_u}\sigma_\varepsilon \\ \rho_{\eta_y\varepsilon}\sigma_{\eta_y}\sigma_\varepsilon & \sigma_{\eta_y}^2 & \rho_{\eta_y\eta_u}\sigma_{\eta_y}\sigma_{\eta_u} \\ \rho_{\eta_u\varepsilon}\sigma_{\eta_u}\sigma_\varepsilon & \rho_{\eta_y\eta_u}\sigma_{\eta_y}\sigma_{\eta_u} & \sigma_{\eta_u}^2 \end{bmatrix}\right).
$$

We use quarterly GDP and unemployment rate data from the St. Louis FRED database as observable variables for the period 1948:1-2015:4 to estimate the unrestricted unobserved components (UC-UR) model above to extract the trend and the cycle of GDP.

In our simulation work, we assumed that the two correlations involving the unemployment-rate trend, $\rho_{\eta_u\varepsilon}$ and $\rho_{\eta_y\eta_u}$, were equal to zero. In Table 4, we assess whether this hypothesis is correct. By comparing lines 1 and 7, we see that the evidence suggests this hypothesis is not rejected: Twice the difference in the log likelihood is 3.7, a difference that is not significant at the 5 percent level for two degrees of freedom, indicating that we are safe in assuming these correlations are zero.

A comparison of lines 7 and 8 provides a test of the restriction $\rho_{\eta_y\varepsilon} = 0$, which is the hypothesis that is explored in detail in the main text. This restriction is strongly rejected. Based on our Monte Carlo analysis, we can be confident that this test is valid. We will

Table 5: Bivariate UC Model Estimates

	$\rho_{\eta_u\varepsilon} = \rho_{\eta_y\eta_u} = 0$				$\rho_{\eta_y\varepsilon} = \rho_{\eta_u\varepsilon} = \rho_{\eta_y\eta_u} = 0$		
	Estimate	Standard Error	Z-statistic		Estimate	Standard Error	Z-statistic
μ_y	0.80	0.04	21.13	μ_y	0.79	0.04	20.96
ϕ_1	1.62	0.06	27.61	ϕ_1	1.56	0.06	26.56
ϕ_2	-0.67	0.06	-11.73	ϕ_2	-0.61	0.06	-10.50
σ_ε	0.41	0.07	5.76	σ_ε	0.58	0.05	12.42
σ_{η_y}	0.60	0.04	14.20	σ_{η_y}	0.60	0.04	16.65
$\rho_{\eta_y\varepsilon}$	0.48	0.18	2.64	$\rho_{\eta_y\varepsilon}$	0	-	-
θ_1	-0.51	0.10	-5.29	θ_1	-0.36	0.04	-8.59
θ_2	-0.17	0.04	-3.85	θ_2	-0.18	0.03	-5.63
σ_{η_u}	0.16	0.02	9.98	σ_{η_u}	0.15	0.02	9.93
$\rho_{\eta_u\varepsilon}$	0	-	-	$\rho_{\eta_u\varepsilon}$	0	-	-
$\rho_{\eta_y\eta_u}$	0	-	-	$\rho_{\eta_y\eta_u}$	0	-	-
	LogL = -328.37, BIC = 707.16				LogL = -330.71, BIC = 706.24		

thus focus on the version of the model that does not impose the $\rho_{\eta_y\varepsilon} = 0$ restriction. We will nonetheless also consider the version of the model with the restriction imposed, in part because it is an important benchmark in the literature. In addition, as noted in the final column, the version of the model with the restriction imposed is preferred according to the Bayesian Information Criterion (BIC). (Although the difference relative to line 7 is "not worth more than a bare mention" (see Kass and Raftery, 1995).)

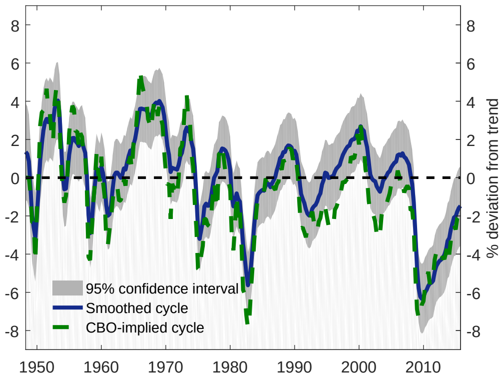
The estimates of the two models appear in Table 5. In the results to the left, the correlation between the output trend and cycle is estimated to be positive, about 0.5. Based on our Monte Carlo results, we are confident that this result is robust. It stands in contrast to the univariate results of MNZ, who found a strongly negative correlation between the output trend and cycle. This result is also different from that of Clark (1989), who found a correlation that was negative but not statistically different from zero at conventional significance levels.

The positive correlation between trend and cycle innovations of GDP means that when the UC model sees a surprising acceleration (deceleration) in GDP, it revises both the trend and the cycle upward (downward). Thus, a positive (negative) perturbation to the trend is likely associated with a positive (negative) perturbation to the cycle. Our model is, of course, purely statistical and there are a number of theoretical interpretations of the positive correlation between trend and cycle. One possibility is hysteresis: Long and pronounced periods of economic slack may affect the trend—for example, as the skills of unemployed workers atrophy (Haltmaier, 2012; Ball, 2014; Reifschneider, Wascher and Wilcox, 2015, see).[13]

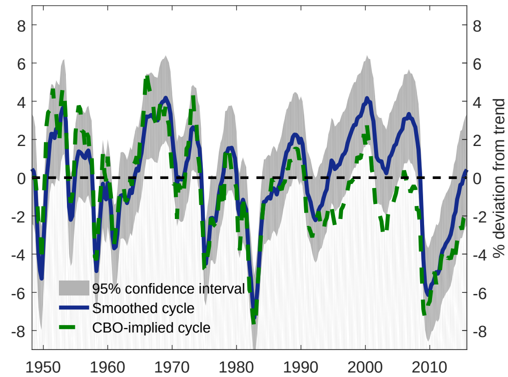
[13] The finding of a positive correlation between trend and cycle appears to be importantly affected by the experience after the Great Recession. In fact, if the estimation were to be conducted with data ending in 2009:4 (or earlier), little statistical evidence would have been found to reject the null hypothesis $\rho_{\eta_y\varepsilon} = 0$.

Figure 10: Smoothed GDP Cycle

(a) $\rho_{\eta_u \varepsilon} = \rho_{\eta_y \eta_u} = 0$



(b) $\rho_{\eta_y \varepsilon} = \rho_{\eta_u \varepsilon} = \rho_{\eta_y \eta_u} = 0$



With respect to the other model parameters, the autoregressive coefficients of the cycle imply a strong hump-shaped pattern of the responses to a business-cycle shock, similar to the results of Fleischman and Roberts (2011) and of MNZ in the estimation of the constrained univariate unobserved components model, the case they label UC-0. The estimated parameters imply complex roots, with a period of 11.1 years. The variance decomposition indicates that about 65 percent of the variation in GDP growth is explained by the cycle. The estimates of the coefficients that relate the cycle to the unemployment rate, θ_1 and θ_2, suggest a conventional Okun's law relationship, with the unemployment rate reacting to cyclical shocks with a lag relative to GDP, and a total effect after two quarters of -0.68, somewhat above in absolute terms from conventional estimates of -0.5 (see Abel, Bernanke and Croushore, 2013).

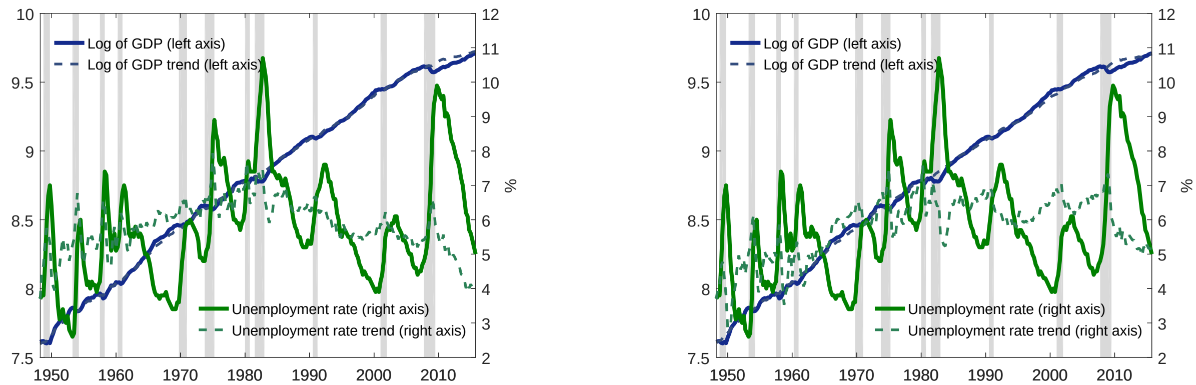
Figure 10a plots the smoothed cycle obtained from the estimation of the model along with the 95 percent confidence intervals. For comparison, we also plot the CBO GDP gap. The CBO and our estimate of the cycle behave similarly, although the cycle obtained from our model shows somewhat less variability than CBO's. At the end of the sample (2015:Q4), our bivariate model predicts that the output gap is still around negative 2 percent, in line with the CBO estimate.

The results in the right-hand part of Table 5 consider the model with no correlations among the trends and cycle innovations. The parameter point estimates are broadly similar to those under the more-restrictive model. The point estimates of ϕ_1 and ϕ_2 indicate that the period of the cycle would be 13.5 years. The variance decomposition would attribute about 60 percent of the variations in GDP growth to variations in the cycle, and the long-run Okun's law coefficient would be about -0.54.

The estimate of the cycle in the absence of trend and cycle correlations appears in Figure 10b. This estimate of the cycle is broadly similar to the estimate when the correlation is freely estimated; their unconditional correlation coefficient is 0.92. there are, however, two key differences. First, the variability of cycle is greater in this version of the model. Second, starting in the mid-1990s, the estimate of the cycle is shifted upwards. We view the smaller variability of the cycle in the first case as a natural consequence of the positive correlation

Figure 11: Observed real GDP and Unemployment Rate and Corresponding Trends

(a) $\rho_{\eta_u \varepsilon} = \rho_{\eta_y \eta_u} = 0$ (b) $\rho_{\eta_y \varepsilon} = \rho_{\eta_u \varepsilon} = \rho_{\eta_y \eta_u} = 0$



between trend and cycle, as a shock that pushes up the cyclical component of output will also tend to raise the trend, softening the cyclical increase. We view the recent higher level of the trend (lower cyclical component) in the model with the unconstrained trend-cycle correlation as being the consequence of the longer recoveries that have been the norm since the 1980s: A long recovery will be accompanied by upward revisions to trend output. Thus, by 2007, the level of trend GDP was about 2 percent higher in the model with correlated trend and cycle. It is interesting to note that in the succeeding recession, this difference in trends was almost entirely eliminated, and in both models, output was about 6 percent below trend at the trough. The gap re-emerged over the next several years, however, and at the end of the sample, the estimated cycle is around zero in the model with no correlation, compared with the 2 percent shortfall in the model with a correlated trend and cycle.

Figure 11 shows the observed (log of) GDP and unemployment rate series along with their estimated trends for the two specifications whose results appear in in Table 5. Both models predict relatively small increases in the trend component of GDP in the latter years of the sample, as is apparent from the dashed blue lines in Figures 11a and 11b. The estimated trend of the model with(out) correlated output innovations rises only at an annualized rate of $1^1/_4$ (1) percent on average from 2010 to 2015, compared with $2^1/_4$ ($2^3/_4$) percent on average from 2007 to 2009 and $2^3/_4$ ($2^1/_2$) percent per year in the four years before that.

Figure 11 also shows the unemployment rate and its trend. The broad movements in the trend unemployment rate are similar across the two specifications (with and without trend-cycle correlation in ouptut): The unemployment rate trend moves up fairly steadily from the beginning of the sample to the mid-1970s, reaching as high as 7 percent in the early 1980s. After a temporary decline in the mid-1980s and a subsequent increase, the trend moves down over the rest of the 1980s and through the 1990s, reaching as low as $5^1/_2$ percent in the model with correlation and 6 percent in the model without correlation. From the mid-1990s to 2007, the unemployment trend stays around $5^1/_2$ percent in the model with correlation and $6^1/_2$ percent percent in the model without correlation. The trend unemployment rate spikes in both models during the financial crisis but then moves down thereafter. As would be expected given Okun's law, at the end of 2015, the model with correlation estimates a trend unemployment rate of around 4 percent and thus an unemployment gap, whereas the model without correlations obtains a trend unemployment rate of around 5 percent.

8 Conclusions

In this paper, we investigated the performance of different bivariate unobserved components models in estimating the trend and cycle of GDP. We found that the best variable to accompany GDP in the bivariate specification is the unemployment rate, which is superior in performance to two alternatives, namely inflation and gross domestic income. Our results suggest that the main reason the unemployment rate is especially helpful is that its unit root component (trend) has a relatively small variance relative to its cyclical component. We estimated the cycle using GDP and unemployment rate data for the U.S. and found that there is evidence of positively correlated trend and cycle innovations and that the cycle has a conventional shape, with a period of about 11 years. Overall, our Monte Carlo experiments suggest that the results of our statistical tests could be trusted when the unemployment rate is included in the model.

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