# Simple Estimators for ARCH Models

## Supplemental Appendix

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**PRELIMINARIES.** This Supplemental Appendix contains detailed proofs of Lemmas 1–12 stated in the Appendix of the main paper. For ease of exposition, each Lemma is also restated here as well. In what follows, $C$ denotes a constant that can assume different values in different places. For matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \geq \mathbf{B}$ means that every element in $\mathbf{A} \geq$ every corresponding element in $\mathbf{B}$. Finally, for a vector $\mathbf{y}$, $\delta_{\mathbf{y}}$ denotes the Dirac measure at $\mathbf{y}$.

**LEMMA 1.** *For ARCH processes that can be cast in terms of the SRE*

$$\sigma_t^2 = \omega_0 + \sigma_{t-1}^2 A_t, \tag{1}$$

*let Assumptions A1 with $k > 3$ and A2 hold. Then Assumption A4 is sufficient for $E\left(\sigma_t^3\right) < \infty$.*

**Proof.**

$$\begin{aligned} \sigma_t^3 &\leq \sigma_t^2 \times \left(\omega_0^{1/2} + \sigma_{t-1} A_t^{1/2}\right). \\ &\leq \left(\omega_0 + \sigma_{t-1}^2 A_t\right) \times \left(\omega_0^{1/2} + \sigma_{t-1} A_t^{1/2}\right) \\ &\leq \omega_0 \sigma_t + \omega_0^{1/2} \sigma_t^2 + \sigma_{t-1}^3 A_t^{3/2}, \end{aligned} \tag{2}$$

where the first inequality follows from the Triangle Inequality, and the third inequality uses $\sigma_t^2 - \omega_0 = \sigma_{t-1}^2 A_t$. Since $\left\{\sigma_t^2\right\}$ is strictly stationary (see; e.g., Mikosch, 1999, Corollary 1.4.38) with a well-defined second moment (see; e.g., Bollerslev, 1986, Theorem 1),

$$\begin{aligned} E\left(\sigma_t^3\right) &\leq C + E\left(\sigma_{t-1}^3\right) E\left(A_t^{3/2}\right) \\ &\leq C\left(1 + E\left(A^{3/2}\right) + E\left(A^{3/2}\right)^2 + \cdots + E\left(A^{3/2}\right)^{k-1}\right) + E\left(\sigma_{t-k}^3\right) E\left(A^{3/2}\right)^k. \end{aligned}$$

As a consequence, $\lim_{k\to\infty} E\left(\sigma_t^3\right) \leq \frac{C}{1-E\left(A^{3/2}\right)} < \infty$ if and only if $E\left(A^{3/2}\right) < 1$. ■

**LEMMA 2.** *For ARCH processes consistent with (1), let Assumptions A1 with $k > 3$, A2 and A4 hold. Consider the following lagged vectors for $h \geq 0$:*

$$\mathbf{Y}_h^{(i)} = \left( |Y_0|^i, \quad \ldots, \quad |Y_h|^i \right), \ i = 1, 2,$$

$$\mathbf{E}_h^{(2)} = \left( \begin{array}{ccccc} \epsilon_0^2, & A_1\epsilon_1^2, & \prod_{j=1}^{2} A_j\epsilon_2^2, & \ldots, & \prod_{j=1}^{h} A_j\epsilon_h^2 \end{array} \right).$$

*If $\sigma$ is RV$(\kappa_0)$, then $\mathbf{Y}_h^{(2)}$ is RV$(\kappa_0/2)$, and $\mathbf{Y}_h^{(1)}$ is RV$(\kappa_0)$.*

**Proof.** That $\sigma$ is RV$(\kappa_0)$; i.e.,

$$P(\sigma > x) \sim c_0 x^{-\kappa_0}, \quad n \to \infty, \tag{3}$$

where $c_0 = c_0(\omega_0, \alpha_{1,0}, \alpha_{2,0})$, the precise value of which is given in Goldie (1991), and $\kappa_0 \in (3,\ p]$ is the unique solution to

$$E(A)^{\kappa_0/2} = 1$$

follows from Mikosch and Stărică (2000, Theorem 2.1). Next,

$$\begin{aligned} \mathbf{Y}_h^{(2)} &= \left( \begin{array}{ccc} \sigma_0^2\epsilon_0^2, & \ldots, & \sigma_h^2\epsilon_h^2 \end{array} \right) \\ &= \left( \begin{array}{cccc} \sigma_0^2\epsilon_0^2, & \sigma_0^2 A_1\epsilon_1^2, & \ldots, & \sigma_{h-1}^2 A_h\epsilon_h^2 \end{array} \right) \\ &\quad + \omega_0 \times \left( \begin{array}{cccc} 0, & \epsilon_1^2, & \ldots, & \epsilon_h^2 \end{array} \right) \\ &= \mathbf{C}_h^{(2)} + \mathbf{R}_h^{(2)}. \end{aligned}$$

Since the tail of $\mathbf{R}_h^{(2)}$ is small relative to the tail of $\mathbf{C}_h^{(2)}$, the tail of $\mathbf{Y}_h^{(2)}$ is determined only by the tail of $\mathbf{C}_h^{(2)}$. Then by induction, the tail of $\mathbf{Y}_h^{(2)}$ is determined by the tail of $\sigma_0^2 \times \mathbf{E}_h^{(2)}$. Given (3) and Mikosch (1999, Proposition 1.5.9), $\sigma_0^2 \times \mathbf{E}_h^2$ is RV$(\kappa_0/2)$ by Mikosch (1999, Proposition 1.3.9(b)). Given $\mathbf{Y}_h^{(2)}$ is RV$(\kappa_0/2)$, $\mathbf{Y}_h^{(1)}$ is RV$(\kappa_0)$ by Mikosch (1999, Proposition 1.5.9). ■

**LEMMA 3.** *For the threshold ARCH(1) model, let Assumptions A1 with $k > 3$, A2 and A4 hold. Consider the following lagged vectors for $h \geq 0$,*

$$\mathbf{Y}_h^i = \left( \begin{array}{ccc} Y_0^i, & \ldots, & Y_h^i \end{array} \right), \quad i = 1, 3,$$

$$\mathbf{E}_h^{(1)} = \left( \begin{array}{ccccc} \epsilon_0, & |\epsilon_0|\,\epsilon_1, & |\epsilon_0|\,|\epsilon_1|\,\epsilon_2, & \ldots, & \prod_{i=0}^{h-1} |\epsilon_i|\,\epsilon_h \end{array} \right).$$

*Then for all $\mathbf{y}_h^1 \in \mathbb{R}^{h+1} \setminus \{\mathbf{0}\}$, $\mathbf{Y}_h^1$ is RV$(\kappa_0)$, and $\mathbf{Y}_h^3$ is RV$(\kappa_0/3)$.*

**Proof.** For the threshold ARCH(1) model,

$$\sigma_t^2(\omega_0, \boldsymbol{\alpha}_0) = \omega_0 + \alpha_{0,t-1} Y_{t-1}^2, \tag{4}$$

where $\boldsymbol{\alpha}_0 = (\alpha_{1,0}, \alpha_{2,0})'$. Define

$$\underline{\alpha} = \min(\alpha_{1,0},\ \alpha_{2,0}) \leq \alpha_{0,t-1}, \quad \overline{\alpha} = \max(\alpha_{1,0},\ \alpha_{2,0}) \geq \alpha_{0,t-1} \quad \forall\ t. \tag{5}$$

Take a first-order Taylor Expansion of $\sigma_h(\omega_0, \boldsymbol{\alpha}_0)$ around $\underline{\omega}$ so that

$$\sigma_h(\omega_0, \boldsymbol{\alpha}_0) = \frac{\alpha_{0,h-1} Y_{h-1}^2}{\sigma_h(\underline{\omega}, \boldsymbol{\alpha}_0)} + \frac{(\omega_0 + \underline{\omega})}{2\sigma_h(\underline{\omega}, \boldsymbol{\alpha}_0)}.$$

Then,

$$\begin{aligned}
\mathbf{Y}_h^1 &= \begin{pmatrix} Y_0, & Y_1, & Y_2, & \ldots, & Y_h \end{pmatrix} \\
&= \sigma_0 \times \left( \epsilon_0, \quad \sigma_0^{-1}\left(\tfrac{\alpha_{0,0} Y_0^2}{\sigma_1(\underline{\omega}, \boldsymbol{\alpha}_0)}\right)\epsilon_1, \quad \sigma_0^{-1}\left(\tfrac{\alpha_{0,1} Y_1^2}{\sigma_2(\underline{\omega}, \boldsymbol{\alpha}_0)}\right)\epsilon_2, \quad \ldots, \quad \sigma_0^{-1}\left(\tfrac{\alpha_{0,h-1} Y_{h-1}^2}{\sigma_h(\underline{\omega}, \boldsymbol{\alpha}_0)}\right)\epsilon_h \right) \\
&\quad + (\omega_0 + \underline{\omega}) \times \left( 0, \quad \tfrac{\epsilon_1}{2\sigma_1(\underline{\omega}, \boldsymbol{\alpha}_0)}, \quad \tfrac{\epsilon_2}{2\sigma_2(\underline{\omega}, \boldsymbol{\alpha}_0)}, \quad \ldots, \quad \tfrac{\epsilon_h}{2\sigma_h(\underline{\omega}, \boldsymbol{\alpha}_0)} \right) \\
&= \mathbf{C}_h^1 + \mathbf{R}_h^1
\end{aligned}$$

Since $\sigma_h^{-1}(\underline{\omega}, \boldsymbol{\alpha}_0)$ is bounded, the tail of $\mathbf{R}_h^1$ is light relative to the tail of $\mathbf{C}_h^1$. As a consequence, the tail of $\mathbf{C}_h^1$ determines the tail of $\mathbf{Y}_h^1$. Let $\mathbf{C}_h^1 = \sigma_0 \times \mathbf{E}_h^{(1)*}$. Since $y_h^1$ is bounded away from zero for all $h$,

$$\frac{\alpha_{0,h-1} Y_{h-1}^2}{\sigma_h(\underline{\omega}, \boldsymbol{\alpha}_0)} \leq \frac{\overline{\alpha} Y_{h-1}^2}{\sigma_h(\underline{\omega}, \boldsymbol{\alpha}_0)} \leq \frac{\overline{\alpha} Y_{h-1}^2}{\underline{\alpha}^{1/2} \left|Y_{h-1}\right|} = \frac{\overline{\alpha}}{\underline{\alpha}^{1/2}} \times \sigma_{h-1} \left|\epsilon_{h-1}\right|,$$

in which case,

$$\mathbf{E}_h^{(1)*} \leq \left( \epsilon_0, \quad \left(\tfrac{\overline{\alpha}}{\underline{\alpha}^{1/2}}\right) \times |\epsilon_0| \, \epsilon_1, \quad \left(\tfrac{\overline{\alpha}}{\underline{\alpha}^{1/2}}\right) \times \left(\tfrac{\sigma_1}{\sigma_0}\right) \times |\epsilon_1| \, \epsilon_2, \quad \ldots, \quad \left(\tfrac{\overline{\alpha}}{\underline{\alpha}^{1/2}}\right) \times \left(\tfrac{\sigma_{h-1}}{\sigma_0}\right) \times \left|\epsilon_{h-1}\right| \epsilon_h \right).$$

Using the Triangle Inequality,

$$\frac{\sigma_1}{\sigma_0} \leq \frac{\omega_0^{1/2} + \alpha_{0,0}^{1/2} |Y_0|}{\sigma_0} \leq \frac{\omega_0^{1/2} + \overline{\alpha}^{1/2} |Y_0|}{\sigma_0} \leq C \times \frac{|Y_0|}{\sigma_0} = C \times |\epsilon_0|,$$

where the final inequality holds because $y_h^1$ is bounded away from zero for all $h$, and

$$\frac{\sigma_2}{\sigma_0} \leq C \times \frac{|Y_1|}{\sigma_0} = C \times \left(\frac{\sigma_1}{\sigma_0}\right) \times |\epsilon_1|.$$

Suppose that

$$\frac{\sigma_{h-2}}{\sigma_0} \leq C \times \prod_{i=0}^{h-3} |\epsilon_i|.$$

Then

$$\frac{\sigma_{h-1}}{\sigma_0} \leq C \times \left(\frac{\sigma_{h-2}}{\sigma_0}\right) \times \left|\epsilon_{h-2}\right| \leq C \times \prod_{i=0}^{h-2} |\epsilon_i|,$$

so that by induction,

$$\mathbf{E}_h^{(1)*} \leq C \times \mathbf{E}_h^{(1)}. \tag{6}$$

Since $E\left(\left|E_h^{(1)}\right|^{\kappa_0+\varepsilon}\right) < \infty$ for all $h$ and some $\varepsilon > 0$, $\sigma_0 \times \mathbf{E}_h^{(1)}$ is RV($\kappa_0$) by Lemma 2 and Basrak, Davis, and Mikosch (2002, Corollary A.2) for $d = 1$, meaning that the tail behavior of $\sigma_0$ determines the tail behavior of the product $\sigma_0 \times \mathbf{E}_h^{(1)}$. Since $\mathbf{C}_h^1 = \sigma_0 \times \mathbf{E}_h^{(1)*}$ is established to determine the tail behavior of $\mathbf{Y}_h^1$, given (6), $\sigma_0$ must also determine the tail behavior of $\mathbf{C}_h^1$. As a consequence, $\mathbf{Y}_h^1$ is RV$(\kappa_0)$; in which case, $\mathbf{Y}_h^3$ is RV$(\kappa_0/3)$ along the same lines as the proof of Resnick (2007, Proposition 7.6), since $\mathbf{Y}_h^{(2)}$ is RV$(\kappa_0/2)$ by Lemma 2. ■

**LEMMA 4.** *Under the same Assumptions as Lemma 3 and for a sequence of constants* $\{a_n\}$,

$$N_n := \sum_{t=1}^{n} \delta_{a_n^{-1}\mathbf{Y}_t} \xrightarrow{d} N := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{P_i \mathbf{Q}_{i,j}}, \tag{7}$$

*where: (i)* $\sum_{i=1}^{\infty} \delta_{P_i}$ *is a Poisson process on* $(0, \infty)$*; (ii) For* $\mathbf{Q}_{i,j} = \left( Q_{ij}^{(0)}, \ \ldots, \ Q_{ij}^{(h)} \right)$, $\sum_{j=1}^{\infty} \delta_{\mathbf{Q}_{i,j}}$, $i \in \mathbb{N}$, *is an i.i.d. sequence of point processes on* $\mathbb{R}_+^{h+1} \setminus \{\mathbf{0}\}$ *with common distribution* $\mathbf{Q}$*; (iii)* $\sum_{i=1}^{\infty} \delta_{P_i}$ *and* $\sum_{j=1}^{\infty} \delta_{\mathbf{Q}_{i,j}}$, $i \in \mathbb{N}$, *are mutually independent.*

**Proof.** (7) is the product of Davis and Mikosch (1998, Theorem 2.8). Conditions for this weak distributional convergence result are (C1) (joint) regular variation of all finite-dimensional distributions of $\mathbf{Y}_t$, (C2) a weak mixing condition for $\{Y_t\}$, and (C3) verification of Davis and Mikosch (1998, Eq. 2.10). Lemmas 2 and 3 establish (C1). $\{Y_t\}$ is strongly mixing by results from Carrasco and Chen (2002); specifically, Corollary 6 for the ARCH(1) and Corollary 10 for the threshold ARCH(1) case, respectively. Lastly, (C3) follows from Davis and Mikosch (1998, Theorem 4.1). ■

**LEMMA 5.** *For the ARCH*(1) *model, let Assumptions A1 with* $k = 6$*, A2 and A4 hold. For* $m = 0, \ldots, h$*, define*

$$\widehat{\gamma}_{(Y,\ Y^2)}(m) = n^{-1} \sum_{t=1}^{n-m} Y_t Y_{t+m}^2, \qquad \gamma_{(Y,\ Y^2)}(m) = E\left(Y_0 Y_m^2\right).$$

*Then for a* $\kappa_0 \in (3,\ 6)$,

$$n a_n^{-3} \left( \widehat{\gamma}_{(Y,\ Y^2)}(m) - \gamma_{(Y,\ Y^2)}(m) \right) \xrightarrow{d} (V_m)_{m=0,\ldots,h}, \qquad h \geq 1, \tag{8}$$

*where* $V_0 := V_0^* - c_3^* \alpha_0^{3/2} \left(1 - c_3 \alpha_0^{3/2}\right)^{-1} V_0^{**}$, $V_m := V_m^* + \alpha_0 V_{m-1}$, *and* $\mathbf{V}_h = \left( V_0, \ \ldots, \ V_h \right)'$ *is jointly* $(\kappa_0/3)-$*stable.*

**Proof.** For an $\varepsilon > 0$, consider

$$\begin{aligned} & a_n^{-3}\sum_t \left(Y_{t+1}^3 - E\left(Y_{t+1}^3\right)\right) \\ = \; & a_n^{-3}\sum_t \sigma_{t+1}^3 \left(\epsilon_{t+1}^3 - c_3^*\right) \times I_{\{|Y_t| \leq a_n\varepsilon\}} \\ & + a_n^{-3}\sum_t \sigma_{t+1}^3 \left(\epsilon_{t+1}^3 - c_3^*\right) \times I_{\{|Y_t| > a_n\varepsilon\}} \\ & + c_3^* a_n^{-3}\sum_t \left(\sigma_{t+1}^3 - E\left(\sigma_{t+1}^3\right)\right) \\ = \; & Ia + IIa + IIIa, \end{aligned} \tag{9}$$

where $\sigma_{t+1}^3 \equiv \sigma_{t+1}^3\left(\omega_0, \alpha_0\right)$. For $Ia$,

$$\begin{aligned} Var\left(Ia\right) & = na_n^{-6} Var\left(\epsilon_1^3\right) \times E\left(\left(\omega_0 + \alpha_0 Y_0^2\right)^3 \times I_{\left\{\left|Y_0\right| \leq a_n\varepsilon\right\}}\right) \\ & \leq Cna_n^{-6} E\left(Y^6 I_{\{|Y| \leq a_n\varepsilon\}}\right) \\ & \sim Ca_n^{-6}\left(a_n\varepsilon\right)^6 nP\left(|Y| > a_n\varepsilon\right) \\ & \longrightarrow C\varepsilon^{6-\kappa_0}, \quad n \to \infty \\ & \longrightarrow 0, \quad \varepsilon \to 0 \end{aligned} \tag{10}$$

where the equality follows because the individual summands are uncorrelated; the (asymptotic) equivalence follows from Mikosch (1999, Proposition 1.3.5) and Karamata's Theorem, and convergence as $n \to \infty$ results because $\{Y_t\}$ is RV($\kappa_0$) and by properties of the Pareto distribution. For $IIa$,

$$IIa = a_n^{-3}\sum_t Y_{t+1}^3 I_{\{|Y_t| > a_n\varepsilon\}} - c_3^* a_n^{-3}\sum_t \sigma_{t+1}^3 I_{\{|Y_t| > a_n\varepsilon\}}.$$

A first-order Taylor Expansion of $\sigma_{t+1}^3$ around $\underline{\omega}$ is (with some simplification),

$$\sigma_{t+1}^3 = C\sigma_{t+1}\left(\underline{\omega}, \alpha_0\right) + \alpha_0 \sigma_{t+1}\left(\underline{\omega}, \alpha_0\right) Y_t^2, \tag{11}$$

so

$$a_n^{-3}\sum_t \sigma_{t+1}^3 I_{\{|Y_t| > a_n\varepsilon\}} = Ca_n^{-3}\sum_t \sigma_{t+1}\left(\underline{\omega}, \alpha_0\right) I_{\{|Y_t| > a_n\varepsilon\}} + \alpha_0 a_n^{-3}\sum_t \sigma_{t+1}\left(\underline{\omega}, \alpha_0\right) Y_t^2 I_{\{|Y_t| > a_n\varepsilon\}}. \tag{12}$$

Next, let $\mathbf{x}_t = \begin{pmatrix} x_t^{(0)}, & \ldots, & x_t^{(h)} \end{pmatrix} \in \mathbb{R}^{h+1} \setminus \{\mathbf{0}\}$, and define for $j \geq 1$,

$$\begin{aligned} T_{j,m,\varepsilon}\left(\sum_{i=1}^{\infty} n_i \delta_{\mathbf{x}_i}\right) & = \sum_{i=1}^{\infty} n_i \left(x_i^{(m)}\right)^j I_{\left\{\left|x_i^{(0)}\right| > \varepsilon\right\}}, \qquad m = 0, 1, \\ T_{j,m,\varepsilon}^{(a)}\left(\sum_{i=1}^{\infty} n_i \delta_{\mathbf{x}_i}\right) & = \sum_{i=1}^{\infty} n_i \left|x_i^{(m)}\right|^j I_{\left\{\left|x_i^{(0)}\right| > \varepsilon\right\}}, \qquad m = 0, 1, \\ T_{m,\varepsilon}^{(1)}\left(\sum_{i=1}^{\infty} n_i \delta_{\mathbf{x}_i}\right) & = \sum_{i=1}^{\infty} n_i x_i^{(0)} \left(x_i^{(m-1)}\right)^2 I_{\left\{\left|x_i^{(0)}\right| > \varepsilon\right\}}, \qquad m \geq 2, \end{aligned}$$

noting that the set $\left\{x \in \mathbb{R}^{h+1} \setminus \{\mathbf{0}\} : \left|x^{(m)}\right| > \varepsilon\right\}$ for any $m \geq 0$ is bounded away from the origin. Then, for the first part of the decomposition in (12),

$$\begin{aligned} 0 &\leq a_n^{-3} \sum_t \sigma_{t+1}(\underline{\omega}, \alpha_0) I_{\{|Y_t| > a_n \varepsilon\}} \\ &\leq \underline{\omega}^{1/2} a_n^{-3} \sum_t I_{\{|Y_t| > a_n \varepsilon\}} + \alpha_0 a_n^{-3} \sum_t |Y_t| I_{\{|Y_t| > a_n \varepsilon\}} \longrightarrow 0, \quad n \to \infty, \end{aligned}$$

since (for sufficiently large $n$),

$$n \left( n^{-1} \sum_t I_{\{|Y_t| > a_n \varepsilon\}} \right) \sim nP(|Y| > a_n \varepsilon) \longrightarrow \epsilon^{-\kappa_0}, \quad n \to \infty, \tag{13}$$

as in (10) and

$$a_n^{-1} \sum_t |Y_t| I_{\{|Y_t| > a_n \varepsilon\}} = T_{1,0,\varepsilon}^{(a)}(N_n) \xrightarrow{d} T_{1,0,\varepsilon}^{(a)}(N), \qquad n \to \infty, \tag{14}$$

by (7), Remark R3 in the Appendix of the main paper, and, given Vaynman and Beare (2014, Lemma A.2), the continuous mapping theorem.[1] For the second part of the decomposition in (12),

$$\begin{aligned} \alpha_0^{3/2} a_n^{-3} \sum_t |Y_t|^3 I_{\{|Y_t| > a_n \varepsilon\}} &\leq \alpha_0 a_n^{-3} \sum_t \sigma_{t+1}(\underline{\omega}, \alpha_0) Y_t^2 I_{\{|Y_t| > a_n \varepsilon\}} \\ &\leq C a_n^{-3} \sum_t Y_t^2 I_{\{|Y_t| > a_n \varepsilon\}} + \alpha_0^{3/2} a_n^{-3} \sum_t |Y_t|^3 I_{\{|Y_t| > a_n \varepsilon\}}, \end{aligned}$$

where the second inequality follows from the Triangle Inequality. Since

$$a_n^{-2} \sum_t Y_t^2 I_{\{|Y_t| > a_n \varepsilon\}} = T_{2,0,\varepsilon}(N_n) \xrightarrow{d} T_{2,0,\varepsilon}(N), \qquad n \to \infty \tag{15}$$

by the same argument that supports (14),

$$a_n^{-3} \sum_t \sigma_{t+1}^3 I_{\{|Y_t| > a_n \varepsilon\}} = \alpha_0^{3/2} a_n^{-3} \sum_t |Y_t|^3 I_{\{|Y_t| > a_n \varepsilon\}} + o_P(1).$$

As a consequence,

$$IIa = T_{3,1,\varepsilon}(N_n) - c_3^* \alpha_0^{3/2} T_{3,0,\varepsilon}^{(a)}(N_n) + o_P(1).$$

Also, given the same argument that supports the simplification of $III$ from Davis and Mikosch (1998, Section 4(B2), p. 2072),

$$\begin{aligned} IIIa &= c_3^* \alpha_0^{3/2} a_n^{-3} \sum_t \left(\omega_0 + \alpha_0 Y_t^2\right)^{3/2} - E\left(\left(\omega_0 + \alpha_0 Y_t^2\right)^{3/2}\right) \\ &= c_3^* \alpha_0^{3/2} a_n^{-3} \sum_t \left(|Y_t|^3 - E|Y_t|^3\right) + o_P(1). \end{aligned} \tag{16}$$

[1] Elsewhere in this Appendix, implicit in applications of the continuous mapping theorem to functions of $N_n$ defined in Lemma 4 is Vaynman and Beare (2014, Lemma A.2).

Next, the same decomposition in (9) is also applicable to

$$a_n^{-3} \sum_t \left( \left| Y_{t+1} \right|^3 - E \left| Y_{t+1} \right|^3 \right) = Ib + IIb + IIIb$$

where $\left| \epsilon_{t+1} \right|^3$ in $Ib$ and $IIb$ is centered around $c_3$. By the same argument that supports (10), $\lim_{n \to \infty} \lim_{\epsilon \to 0} Var\left(Ib\right) = 0$. Reliance on (11), (14), and (15) produces

$$IIb = T_{3,1,\varepsilon}^{(a)}\left(N_n\right) - c_3 \alpha_0^{3/2} T_{3,0,\varepsilon}^{(a)}\left(N_n\right) + o_P\left(1\right)$$

As a consequence,

$$a_n^{-3} \sum_t \left( \left| Y_{t+1} \right|^3 - E \left| Y_{t+1} \right|^3 \right) = \left(1 - c_3 \alpha_0^{3/2}\right)^{-1} \left( T_{3,1,\varepsilon}^{(a)}\left(N_n\right) - c_3 \alpha_0^{3/2} T_{3,0,\varepsilon}^{(a)}\left(N_n\right) \right) + o_P\left(1\right),$$

noting that $IIIa = IIIb$. In addition,

$$\begin{aligned} a_n^{-3} \sum_t \left( Y_{t+1}^3 - E\left(Y_{t+1}^3\right) \right) &= T_{3,1,\varepsilon}\left(N_n\right) \\ &\quad - c_3^* \alpha_0^{3/2} \left(1 - c_3 \alpha_0^{3/2}\right)^{-1} \left( T_{3,1,\varepsilon}^{(a)}\left(N_n\right) - c_3 \alpha_0^{3/2} T_{3,0,\varepsilon}^{(a)}\left(N_n\right) \right) + o_P\left(1\right) \\ &\quad \xrightarrow{d} T_{3,1,\varepsilon}\left(N\right) - c_3^* \alpha_0^{3/2} \left(1 - c_3 \alpha_0^{3/2}\right)^{-1} \left( T_{3,1,\varepsilon}^{(a)}\left(N\right) - c_3 \alpha_0^{3/2} T_{3,0,\varepsilon}^{(a)}\left(N\right) \right) \\ &= S\left(\varepsilon, \infty\right) + c_3^* c_3 \alpha_0^3 \left(1 - c_3 \alpha_0^{3/2}\right)^{-1} S^*\left(\varepsilon, \infty\right) \\ &\quad \xrightarrow{d} V_0^* + c_3^* c_3 \alpha_0^3 \left(1 - c_3 \alpha_0^{3/2}\right)^{-1} V_0^{**}, \end{aligned} \tag{17}$$

where the first "$\xrightarrow{d}$" is with respect to $n \to \infty$ and follows from (7), Remark R3 in the Appendix of the main paper, and the continuous mapping theorem, and the second "$\xrightarrow{d}$" is with respect to $\varepsilon \to 0$ and follows from Davis and Hsing (1995, Proof of Theorem 3.1, pp. 897-898). As a consequence,

$$n a_n^{-3} \left( \widehat{\gamma}_{(Y,\ Y^2)}\left(0\right) - \gamma_{(Y,\ Y^2)}\left(0\right) \right) \xrightarrow{d} V_0^* + c_3^* c_3 \alpha_0^3 \left(1 - c_3 \alpha_0^{3/2}\right)^{-1} V_0^{**} =: V_0$$

Next consider

$$\begin{aligned} & a_n^{-3} \sum_t Y_t Y_{t+1}^2 - E\left(Y_t Y_{t+1}^2\right) \\ = \ & a_n^{-3} \sum_t Y_t \sigma_{t+1}^2 \left(\epsilon_{t+1}^2 - 1\right) \times I_{\{|Y_t| \le a_n \varepsilon\}} \\ & + a_n^{-3} \sum_t Y_t \sigma_{t+1}^2 \left(\epsilon_{t+1}^2 - 1\right) \times I_{\{|Y_t| > a_n \varepsilon\}} \\ & + a_n^{-3} \sum_t Y_t \sigma_{t+1}^2 - E\left(Y_t \sigma_{t+1}^2\right) \\ = \ & Ic + IIc + IIIc \end{aligned} \tag{18}$$

By the same arguments that establish Eq. (10),

$$Var\left(Ic\right) \leq Cna_n^{-6}E\left(Y^6 I_{\{|Y|\leq a_n\varepsilon\}}\right),$$

so that $\lim\limits_{n\to\infty}\lim\limits_{\epsilon\to 0}Var\left(Ic\right)=0$. Since

$$a_n^{-1}\sum_t Y_t I_{\{|Y_t|>a_n\varepsilon\}} = T_{1,0,\varepsilon}\left(N_n\right) \xrightarrow{d} T_{1,0,\varepsilon}\left(N\right), \qquad n\to\infty,$$

given the same arguments that support (14),

$$\begin{aligned} IIc &= a_n^{-3}\sum_t Y_t Y_{t+1}^2 I_{\{|Y_t|>a_n\varepsilon\}} + a_n^{-3}\sum_t Y_t^3 I_{\{|Y_t|>a_n\varepsilon\}} + o_P\left(1\right) \\ &= T_{2,\varepsilon}^{(1)}\left(N_n\right) - \alpha_0 T_{3,0,\varepsilon}\left(N_n\right) + o_P\left(1\right). \end{aligned}$$

Finally, since

$$a_n^{-3}\sum_t Y_t = n^{\frac{\kappa_0-6}{2\kappa_0}}\left(n^{-1/2}\sum_t Y_t\right) \longrightarrow 0, \quad n\to\infty,$$

by Ibragimov and Linnik (1971, Theorem 18.5.3), given that $\{Y_t\}$ is strongly mixing by Carrasco and Chen (2002, Corollary 6),

$$IIIc = \alpha_0 a_n^{-3}\sum_t Y_t^3 - E\left(Y_t^3\right) + o_P\left(1\right)$$

so that

$$\begin{aligned} a_n^{-3}\sum_t Y_t Y_{t+1}^2 - E\left(Y_t Y_{t+1}^2\right) &= T_{2,\varepsilon}^{(1)}\left(N_n\right) - \alpha_0 T_{3,0,\varepsilon}\left(N_n\right) + IIIc + o_P\left(1\right) \\ &\xrightarrow{d} V_1^* + \alpha_0 V_0, \end{aligned}$$

where "$\xrightarrow{d}$" is first with respect to $n\to\infty$ and then $\varepsilon\to 0$, given the same arguments that support (17). As a consequence,

$$na_n^{-3}\left(\gamma_{n,\,(Y,\,Y^2)}\left(1\right) - \gamma_{(Y,\,Y^2)}\left(1\right)\right) \xrightarrow{d} V_1^* + \alpha_0 V_0 =: V_1, \tag{19}$$

and the vector $\mathbf{V}_1$is jointly $(\kappa_0/3)$ −stable. Extending (19) to higher lags (i.e., $m > 1$) is a continuation of the arguments given above. ■

**LEMMA 6.** For the threshold ARCH(1) *model, let Assumptions A1 with* $k = 6$, *A2 and A4 hold. For* $m = 0, \ldots, h$, *define*

$$\widehat{\gamma}_{(Y,\,Y^2)}^{+}\left(m\right) = n^{-1}\sum_{t=1}^{n-m} Y_{t+m}^2 Y_t \times I_{\{Y_t\geq 0\}}, \qquad \gamma_{(Y,\,Y^2)}^{+}\left(m\right) = E\left(Y_m^2 Y_0 \times I_{\{Y_0\geq 0\}}\right),$$

*with* $\widehat{\gamma}^{-}_{(Y,\ Y^2)}(m)$ *and* $\gamma^{-}_{(Y,\ Y^2)}(m)$ *defined analogously using* $I_{\{Y_t<0\}}$*. Then for a* $\kappa_0 \in (3,\ 6)$ and $h > 1$,

$$na_n^{-3}\left(\widehat{\gamma}^{+}_{(Y,\ Y^2)}(m) - \gamma^{+}_{(Y,\ Y^2)}(m)\right) \overset{d}{\longrightarrow} \left(W_m^{+}\right)_{m=0,\ldots,h}, \tag{20}$$

*and*

$$na_n^{-3}\left(\widehat{\gamma}^{-}_{(Y,\ Y^2)}(m) - \gamma^{-}_{(Y,\ Y^2)}(m)\right) \overset{d}{\longrightarrow} \left(W_m^{-}\right)_{m=0,\ldots,h}, \tag{21}$$

*where*

$$W_m^{+} = V_m^{+} + \alpha_{1,0}W_{m-1}^{+}, \qquad W_m^{-} = V_m^{-} + \alpha_{2,0}W_{m-1}^{-},$$

*both* $W_0^{+}$ *and* $W_0^{-}$ *jointly depend on* $V_0^{**}$ *from the proof of Lemma 5, and*

$$\mathbf{W}_h^{(+,-)} = \left(\begin{array}{ccccc} W_0^{+}, & W_0^{-}, & \ldots, & W_h^{+}, & W_h^{-} \end{array}\right)',$$

*which is jointly* $(\kappa_0/3)-$*stable.*

**Proof.** Let $I^{+}(m) \equiv I_{\left\{\epsilon_{t+m}\geq 0\right\}}$ and $I^{-}(m) \equiv I_{\left\{\epsilon_{t+m}<0\right\}}$ for $m = 0, 1$, noting that $I^{+}(m) = I_{\left\{Y_{t+m}\geq 0\right\}}$ and $I^{-}(m) = I_{\left\{Y_{t+m}<0\right\}}$. Then,

$$E\left(Y_{t+1}^3 \times I^{+/-}(1)\right) = E\left(\sigma_{t+1}^3\right)c_3^{+/-},$$

where $c_3^{+/-} = E\left(\epsilon_{t+1}^3 \times I^{+/-}(1)\right)$, and

$$\begin{aligned} & a_n^{-3}\sum_t Y_{t+1}^3 \times I^{+/-}(1) - E\left(Y_{t+1}^3 \times I^{+/-}(1)\right) \\ = \; & a_n^{-3}\sum_t \sigma_{t+1}^3\left(\epsilon_{t+1}^3 \times I^{+/-}(1) - c_3^{+/-}\right) \times I_{\{|Y_t|\leq a_n\varepsilon\}} \\ & + a_n^{-3}\sum_t \sigma_{t+1}^3\left(\epsilon_{t+1}^3 \times I^{+/-}(1) - c_3^{+/-}\right) \times I_{\{|Y_t|> a_n\varepsilon\}} \\ & + c_3^{+/-}a_n^{-3}\sum_t\left(\sigma_{t+1}^3 - E\left(\sigma_{t+1}^3\right)\right) \\ = \; & Ia^{+/-} + IIa^{+/-} + IIIa^{+/-}. \end{aligned}$$

Given the same arguments that support (10), $\lim_{n\to\infty}\lim_{\varepsilon\to 0} Var\left(Ia^{+}\right) = 0$, which, then, also establishes $Var\left(Ia^{-}\right) \longrightarrow 0$, since $Ia = Ia^{+} + Ia^{-}$, and $Var\left(Ia\right) \longrightarrow 0$ from the proof of Lemma 5. Consider next $IIa^{+}$. Given (4),

$$\sigma_{t+1}^3\left(\omega_0, \boldsymbol{\alpha}_0\right) = C\sigma_{t+1}\left(\underline{\omega}, \boldsymbol{\alpha}_0\right) + \alpha_{0,t}\sigma_{t+1}\left(\underline{\omega}, \boldsymbol{\alpha}_0\right)Y_t^2$$

by a first-order Taylor Expansion of $\sigma_{t+1}^3$ around $\underline{\omega}$. Then

$$\begin{aligned} a_n^{-3}\sum_t \sigma_{t+1}^3 \times I_{\{|Y_t|> a_n\varepsilon\}} \;=\; & Ca_n^{-3}\sum_t \sigma_{t+1}\left(\underline{\omega}, \boldsymbol{\alpha}_0\right) \times I_{\{|Y_t|> a_n\varepsilon\}} \\ & + a_n^{-3}\sum_t \alpha_{0,t}\sigma_{t+1}\left(\underline{\omega}, \boldsymbol{\alpha}_0\right)Y_t^2 \times I_{\{|Y_t|> a_n\varepsilon\}}. \end{aligned}$$

Note that

$$\begin{aligned} 0 &\leq a_n^{-3}\sum_t \sigma_{t+1}\left(\underline{\omega}, \boldsymbol{\alpha}_0\right) \times I_{\{|Y_t|>a_n\varepsilon\}} \\ &\leq a_n^{-3}\sum_t \left(\underline{\omega}^{1/2} + \alpha_{0,t}^{1/2}|Y_t|\right) \times I_{\{|Y_t|>a_n\varepsilon\}} \\ &\leq \underline{\omega}^{1/2} a_n^{-3}\sum_t I_{\{|Y_t|>a_n\varepsilon\}} + \overline{\alpha}^{1/2} a_n^{-3}\sum_t |Y_t| \times I_{\{|Y_t|>a_n\varepsilon\}} \longrightarrow 0, \quad n\to\infty \end{aligned}$$

where the second inequality follows from the Triangle Inequality; the third inequality relies on (5), and "$\longrightarrow$" to zero follows from (13) and (14). Also note that, again based on (5),

$$\begin{aligned} a_n^{-3}\sum_t \alpha_{0,t}\sigma_{t+1}\left(\underline{\omega}, \boldsymbol{\alpha}_0\right) Y_t^2 \times I_{\{|Y_t|>a_n\varepsilon\}} &\geq \underline{\alpha} a_n^{-3}\sum_t \left(\underline{\omega} + \underline{\alpha} Y_t^2\right)^{1/2} Y_t^2 \times I_{\{|Y_t|>a_n\varepsilon\}} \\ &\geq \underline{\alpha}^{3/2} a_n^{-3}\sum_t |Y_t|^3 \times I_{\{|Y_t|>a_n\varepsilon\}}, \end{aligned}$$

and

$$\begin{aligned} a_n^{-3}\sum_t \alpha_{0,t}\sigma_{t+1}\left(\underline{\omega}, \boldsymbol{\alpha}_0\right) Y_t^2 \times I_{\{|Y_t|>a_n\varepsilon\}} &\leq \overline{\alpha} a_n^{-3}\sum_t \left(\underline{\omega} + \overline{\alpha} Y_t^2\right)^{1/2} Y_t^2 \times I_{\{|Y_t|>a_n\varepsilon\}} \\ &\leq \overline{\alpha}^{3/2} a_n^{-3}\sum_t |Y_t|^3 \times I_{\{|Y_t|>a_n\varepsilon\}} + \overline{\alpha}\underline{\omega}^{1/2} a_n^{-3}\sum Y_t^2 \times I_{t\,\{|Y_t|>a_n\varepsilon\}} \\ &= \overline{\alpha}^{3/2} a_n^{-3}\sum_t |Y_t|^3 \times I_{\{|Y_t|>a_n\varepsilon\}} + o_P(1), \end{aligned}$$

where the equality follows from (15) so that there exists a constant $C$ for which

$$\begin{aligned} IIa^+ &= a_n^{-3}\sum_t Y_t^3 \times I_{\left\{Y_{t+1}\geq 0\right\}} \times I_{\{|Y_t|>a_n\varepsilon\}} - c_3^+ a_n^{-3}\sum_t \sigma_{t+1}^3 \times I_{\{|Y_t|>a_n\varepsilon\}} \\ &= a_n^{-3}\sum_t Y_t^3 \times I_{\left\{Y_{t+1}\geq 0\right\}} \times I_{\{|Y_t|>a_n\varepsilon\}} - c_3^+ C a_n^{-3}\sum_t |Y_t|^3 \times I_{\{|Y_t|>a_n\varepsilon\}} + o_P(1). \end{aligned}$$

Based on $\mathbf{x}_t$ defined in the proof of Lemma 5 and for the same $j$ and $m$, define

$$T_{j,m,\varepsilon}^+\left(\sum_{i=1}^{\infty} n_i \delta_{\mathbf{x}_i}\right) = \sum_{i=1}^{\infty} n_i \left(x_i^{(m)}\right)^j \times I_{\left\{x_i^{(m)}\geq 0\right\}} \times I_{\left\{\left|x_i^{(0)}\right|>\varepsilon\right\}},$$

and define $T_{j,m,\varepsilon}^-\left(\sum_{i=1}^{\infty} n_i \delta_{\mathbf{x}_i}\right)$ analogously, with $I_{\left\{x_i^{(m)}<0\right\}}$ replacing $I_{\left\{x_i^{(m)}\geq 0\right\}}$. Then

$$IIa^+ = T_{3,1,\varepsilon}^+\left(N_n\right) - c_3^+ C T_{3,0,\varepsilon}^{(a)}\left(N_n\right) + o_P(1).$$

Next, from

$$IIIa^+ = c_3^+\left[a_n^{-3}\sum_t \sigma_{t+1}^3 \times I_{\{Y_t\geq 0\}} - E\left(\sigma_{t+1}^3 \times I_{\{Y_t\geq 0\}}\right) + a_n^{-3}\sum_t \sigma_{t+1}^3 \times I_{\{Y_t<0\}} - E\left(\sigma_{t+1}^3 \times I_{\{Y_t<0\}}\right)\right],$$

where

$$
\begin{aligned}
& a_n^{-3}\sum_t \sigma_{t+1}^3 \times I_{\{Y_t \geq 0\}} - E\left(\sigma_{t+1}^3 \times I_{\{Y_t \geq 0\}}\right) \\
= \; & a_n^{-3}\sum_t \left(\omega_0 + \alpha_{1,0} Y_t^2\right)^{3/2} \times I_{\{Y_t \geq 0\}} - E\left(\left(\omega_0 + \alpha_{1,0} Y_t^2\right)^{3/2} \times I_{\{Y_t \geq 0\}}\right) \\
= \; & \alpha_{1,0}^{3/2} a_n^{-3}\sum_t |Y_t|^3 \times I_{\{Y_t \geq 0\}} - E\left(|Y_t|^3 \times I_{\{Y_t \geq 0\}}\right) + o_P(1),
\end{aligned}
$$

with an analogous decomposition holding for $a_n^{-3}\sum_t \left(\sigma_{t+1}^3 \times I_{\{Y_t<0\}} - E\left(\sigma_{t+1}^3 \times I_{\{Y_t<0\}}\right)\right)$, follows that

$$
\begin{aligned}
IIIa^+ \; = \; & c_3^+ \alpha_{1,0}^{3/2} a_n^{-3}\sum_t |Y_t|^3 \times I_{\{Y_t \geq 0\}} - E\left(|Y_t|^3 \times I_{\{Y_t \geq 0\}}\right) \\
& + c_3^+ \alpha_{2,0}^{3/2} a_n^{-3}\sum_t |Y_t|^3 \times I_{\{Y_t < 0\}} - E\left(|Y_t|^3 \times I_{\{Y_t < 0\}}\right) + o_P(1).
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
& a_n^{-3}\sum_t \left(Y_{t+1}^3 \times I_{\{Y_{t+1} \geq 0\}} - E\left(Y_{t+1}^3 \times I_{\{Y_{t+1} \geq 0\}}\right)\right) \qquad (22) \\
= \; & \left(1 - c_3^+ \alpha_{1,0}^{3/2}\right)^{-1} \left[T_{3,1,\varepsilon}^+(N_n) - c_3^+ C T_{3,0,\varepsilon}^{(a)}(N_n)\right] \\
& + c_3^+ \alpha_{2,0}^{3/2} \left(1 - c_3^+ \alpha_{1,0}^{3/2}\right)^{-1} \left[a_n^{-3}\sum_t |Y_t|^3 \times I_{\{Y_t<0\}} - E\left(|Y_t|^3 \times I_{\{Y_t<0\}}\right)\right] + o_P(1).
\end{aligned}
$$

The same arguments that establish (22) also establish

$$
\begin{aligned}
& a_n^{-3}\sum_t \left(Y_{t+1}^3 \times I_{\{Y_{t+1} < 0\}} - E\left(Y_{t+1}^3 \times I_{\{Y_{t+1} < 0\}}\right)\right) \qquad (23) \\
= \; & \left(1 - c_3^- \alpha_{2,0}^{3/2}\right)^{-1} \left[T_{3,1,\varepsilon}^-(N_n) - c_3^- C T_{3,0,\varepsilon}^{(a)}(N_n)\right] \\
& + c_3^- \alpha_{1,0}^{3/2} \left(1 - c_3^- \alpha_{2,0}^{3/2}\right)^{-1} \left[a_n^{-3}\sum_t \left(Y_{t+1}^3 \times I_{\{Y_t \geq 0\}} - E\left(Y_{t+1}^3 \times I_{\{Y_t \geq 0\}}\right)\right)\right] + o_P(1).
\end{aligned}
$$

From (22) and (23) then follows that

$$
\begin{aligned}
& a_n^{-3}\sum_t \left(Y_{t+1}^3 \times I_{\{Y_{t+1} \geq 0\}} - E\left(Y_{t+1}^3 \times I_{\{Y_{t+1} \geq 0\}}\right)\right) \\
& \xrightarrow{d} \left[1 - \left(c_3^+ \alpha_{1,0}^{3/2} + c_3^- \alpha_{2,0}^{3/2}\right)\right]^{-1} \times \left[\left(1 - c_3^- \alpha_{2,0}^{3/2}\right) T_{3,1,\varepsilon}^+(N) + c_3^+ \alpha_{2,0}^{3/2} T_{3,1,\varepsilon}^-(N) + c_3^+ C T_{3,0,\varepsilon}^{(a)}(N)\right] \\
= \; & \left[1 - \left(c_3^+ \alpha_{1,0}^{3/2} + c_3^- \alpha_{2,0}^{3/2}\right)\right]^{-1} \times \left[S^+(\varepsilon,\infty) + c_3^+ C S^*(\varepsilon,\infty)\right] \\
& \xrightarrow{d} \left[1 - \left(c_3^+ \alpha_{1,0}^{3/2} + c_3^- \alpha_{2,0}^{3/2}\right)\right]^{-1} \times \left[V_0^+ + c_3^+ C V_0^{**}\right]
\end{aligned}
$$

where (as is the case elsewhere) "$\xrightarrow{d}$" is first with respect to $n \to \infty$ and then $\varepsilon \to 0$, with each result following from the same, respective, arguments that support (17). As a consequence,

$$na_n^{-3}\left(\widehat{\gamma}^{+}_{(Y,\ Y^2)}(0)-\gamma^{+}_{(Y,\ Y^2)}(0)\right)\xrightarrow{d}\left[1-\left(c_3^{+}\alpha_{1,0}^{3/2}+c_3^{-}\alpha_{2,0}^{3/2}\right)\right]^{-1}\times\left[V_0^{+}+c_3^{+}CV_0^{**}\right]=:W_0^{+}.$$

Moreover, since following parallel arguments,

$$\begin{aligned}
& a_n^{-3}\sum_t\left(Y_{t+1}^3\times I_{\{Y_{t+1}<0\}}-E\left(Y_{t+1}^3\times I_{\{Y_{t+1}<0\}}\right)\right)\\
& \xrightarrow{d}\left[1-\left(c_3^{+}\alpha_{1,0}^{3/2}+c_3^{-}\alpha_{2,0}^{3/2}\right)\right]^{-1}\times\left[\left(1-c_3^{+}\alpha_{1,0}^{3/2}\right)T_{3,1,\varepsilon}^{-}(N)+c_3^{-}\alpha_{1,0}^{3/2}T_{3,1,\varepsilon}^{+}(N)+c_3^{-}CT_{3,0,\varepsilon}^{(a)}(N)\right]\\
= & \left[1-\left(c_3^{+}\alpha_{1,0}^{3/2}+c_3^{-}\alpha_{2,0}^{3/2}\right)\right]^{-1}\times\left[S^{-}(\varepsilon,\infty)+c_3^{-}CS^{*}(\varepsilon,\infty)\right]\\
& \xrightarrow{d}\left[1-\left(c_3^{+}\alpha_{1,0}^{3/2}+c_3^{-}\alpha_{2,0}^{3/2}\right)\right]^{-1}\times\left[V_0^{-}+c_3^{-}CV_0^{**}\right],
\end{aligned}$$

$$na_n^{-3}\left(\widehat{\gamma}^{-}_{(Y,\ Y^2)}(0)-\gamma^{-}_{(Y,\ Y^2)}(0)\right)\xrightarrow{d}\left[1-\left(c_3^{+}\alpha_{1,0}^{3/2}+c_3^{-}\alpha_{2,0}^{3/2}\right)\right]^{-1}\times\left[V_0^{-}+c_3^{-}CV_0^{**}\right]=:W_0^{-}.$$

Next, define

$$T_{m,\varepsilon}^{+}\left(\sum_{i=1}^{\infty}n_i\delta_{\mathbf{x}_i}\right)=\sum_{i=1}^{\infty}n_ix_i^{(0)}\left(x_i^{(m-1)}\right)^2\times I_{\left\{x_i^{(0)}>\varepsilon\right\}},\qquad m\geq 2,$$

and consider

$$\begin{aligned}
& a_n^{-3}\sum_t Y_{t+1}^2Y_t\times I^{+/-}(0)-E\left(Y_{t+1}^2Y_t\times I^{+/-}(0)\right)\\
= & a_n^{-3}\sum_t\sigma_{t+1}^2Y_t\times I^{+/-}(0)\times\left(\epsilon_{t+1}^2-1\right)\times I_{\{|Y_t|\leq a_n\varepsilon\}}\\
& +a_n^{-3}\sum_t\sigma_{t+1}^2Y_t\times I^{+/-}(0)\times\left(\epsilon_{t+1}^2-1\right)\times I_{\{Y_t>a_n\varepsilon\}}\\
& +a_n^{-3}\sum_t\sigma_{t+1}^2Y_t\times I^{+/-}(0)-E\left(\sigma_{t+1}^2Y_t\times I^{+/-}(0)\right)\\
= & Ib^{+/-}+IIb^{+/-}+IIIb^{+/-}.
\end{aligned}$$

Again following the same arguments that support (10), $\lim_{n\to\infty}\lim_{\varepsilon\to 0}Var\left(Ib^{+}\right)=0$. In addition,

$$\begin{aligned}
IIb^{+} &= a_n^{-3}\sum_t Y_{t+1}^2Y_t\times I_{\{Y_t>a_n\varepsilon\}}-Ca_n^{-3}\sum_t Y_t^3\times I_{\{Y_t>a_n\varepsilon\}}+o_P(1)\\
&= T_{2,\varepsilon}^{+}(N_n)-CT_{3,0,\varepsilon}^{+}(N_n)+o_P(1),
\end{aligned}$$

since

$$\begin{aligned}
\underline{\alpha}a_n^{-3}\sum_t Y_t^3\times I_{\{Y_t>a_n\varepsilon\}}+o_P(1) &\leq a_n^{-3}\sum_t\sigma_{t+1}^2Y_t\times I_{\{Y_t>a_n\varepsilon\}}\\
&\geq \overline{\alpha}a_n^{-3}\sum_t Y_t^3\times I_{\{Y_t>a_n\varepsilon\}}+o_P(1).
\end{aligned}$$

As a consequence,

$$\begin{aligned} & a_n^{-3}\sum_t Y_{t+1}^2 Y_t \times I_{\{Y_t\geq 0\}} - E\left(Y_{t+1}^2 Y_t \times I_{\{Y_t\geq 0\}}\right) \quad (24)\\ = \; & T_{2,\varepsilon}^{+}(N_n) - CT_{3,0,\varepsilon}^{+}(N_n) + \alpha_{1,0}a_n^{-3}\sum_t Y_t^3 \times I_{\{Y_t\geq 0\}} - E\left(Y_t^3 \times I_{\{Y_t\geq 0\}}\right) + o_P(1)\\ & \stackrel{d}{\longrightarrow} V_1^{+} + \alpha_{1,0}W_0^{+}, \end{aligned}$$

where "$\stackrel{d}{\longrightarrow}$" is first with respect to $n\to\infty$ and then $\varepsilon\to 0$ so that

$$na_n^{-3}\left(\widehat{\gamma}_{(Y,\ Y^2)}^{+}(0) - \gamma_{(Y,\ Y^2)}^{+}(0)\right) \stackrel{d}{\longrightarrow} V_1^{+} + \alpha_{1,0}W_0^{+} =: W_1^{+}.$$

Comparable arguments to those establishing (24) then also establish

$$\begin{aligned} & a_n^{-3}\sum_t Y_{t+1}^2 Y_t \times I_{\{Y_t<0\}} - E\left(Y_{t+1}^2 Y_t \times I_{\{Y_t<0\}}\right)\\ & \stackrel{d}{\longrightarrow} V_1^{-} + \alpha_{2,0}W_0^{-} \end{aligned}$$

so that

$$na_n^{-3}\left(\widehat{\gamma}_{(Y,\ Y^2)}^{-}(0) - \gamma_{(Y,\ Y^2)}^{-}(0)\right) \stackrel{d}{\longrightarrow} V_1^{-} + \alpha_{2,0}W_0^{-} =: W_1^{-}. \quad (25)$$

As is the case for Lemma 5, extending (25) to higher lags (i.e., $m>1$) remains a continuation of the arguments given above. ■

**LEMMA 7.** *Let Assumptions A1 with $k=8$, A2 and A6 with $l=2$ hold. For $m=0,1$ define*

$$\widehat{\gamma}_{Y^2}^{+}(m) = n^{-1}\sum_{t=1}^{n-m} Y_{t+m}^2 Y_t^2 \times I_{\{Y_t\geq 0\}}, \qquad \gamma_{Y^2}^{+}(m) = E\left(Y_m^2 Y_0^2 \times I_{\{Y_0\geq 0\}}\right),$$

*with $\widehat{\gamma}_{Y^2}^{-}(m)$ and $\gamma_{Y^2}^{-}(m)$ defined analogously using $I_{\{Y_t<0\}}$. Then for a $\kappa_0\in(4,\ 8)$,*

$$na_n^{-4}\left(\widehat{\gamma}_{Y^2}^{+}(m) - \gamma_{Y^2}^{+}(m)\right) \stackrel{d}{\longrightarrow} \left(Q_m^{+}\right)_{m=0,1},$$

*and*

$$na_n^{-4}\left(\widehat{\gamma}_{Y^2}^{-}(m) - \gamma_{Y^2}^{-}(m)\right) \stackrel{d}{\longrightarrow} \left(Q_m^{-}\right)_{m=0,1},$$

*where*

$$Q_1^{+} = U_1^{+} + \alpha_{1,0}Q_0^{+}, \qquad Q_1^{-} = U_1^{-} + \alpha_{2,0}Q_0^{-},$$

*jointly depend on $U_1$ from Theorem 2, and*

$$\mathbf{Q}_1^{(+,-)} = \begin{pmatrix} Q_1^{+}, & Q_1^{-} \end{pmatrix}'$$

*is jointly $(\kappa_0/4)-$stable.*

**Proof.** Following the notation introduced in the proof to Lemma 6, if $c_4^{+/-} = E\left(\epsilon_{t+1}^4 \times I^{+/-}(1)\right)$, then

$$\begin{aligned}
& a_n^{-4}\sum_t Y_{t+1}^4 \times I^{+/-}(1) - E\left(Y_{t+1}^4 \times I^{+/-}(1)\right) \\
= \; & a_n^{-4}\sum_t \sigma_{t+1}^4 \left(\epsilon_{t+1}^4 \times I^{+/-}(1) - c_4^{+/-}\right) \times I_{\{|Y_t|\le a_n\varepsilon\}} \\
& + a_n^{-4}\sum_t \sigma_{t+1}^4 \left(\epsilon_{t+1}^4 \times I^{+/-}(1) - c_4^{+/-}\right) \times I_{\{|Y_t|> a_n\varepsilon\}} \\
& + c_4^{+/-} a_n^{-4}\sum_t \left(\sigma_{t+1}^4 - E\left(\sigma_{t+1}^4\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
& a_n^{-4}\sum_t Y_{t+1}^2 Y_t^2 \times I^{+/-}(0) - E\left(Y_{t+1}^2 Y_t^2 \times I^{+/-}(0)\right) \\
= \; & a_n^{-4}\sum_t \sigma_{t+1}^2 Y_t^2 \times I^{+/-}(0) \times \left(\epsilon_{t+1}^2 - 1\right) \times I_{\{|Y_t|\le a_n\varepsilon\}} \\
& + a_n^{-4}\sum_t \sigma_{t+1}^2 Y_t^2 \times I^{+/-}(0) \times \left(\epsilon_{t+1}^2 - 1\right) \times I_{\{Y_t> a_n\varepsilon\}} \\
& + a_n^{-4}\sum_t \sigma_{t+1}^2 Y_t^2 \times I^{+/-}(0) - E\left(\sigma_{t+1}^2 Y_t^2 \times I^{+/-}(0)\right).
\end{aligned}$$

Following the same, general, steps provided in the proof to Lemma 6 (while recognizing that $\sigma_{t+1}^4$ has an exact expression and, so, does not require a first-order Taylor approximation), it follows that

$$a_n^{-4}\sum_t Y_{t+1}^4 \times I^{+}(1) - E\left(Y_{t+1}^4 \times I^{+}(1)\right) \xrightarrow{d} \frac{U_0^+ + c_4^+ C U_0^{**}}{1 - \left(c_4^+ \alpha_{1,0}^2 + c_4^- \alpha_{2,0}^2\right)} =: Q_0^+,$$

where $U_0^{**}$ is a component of $U_1$ in Theorem 2 and

$$a_n^{-4}\sum_t Y_{t+1}^2 Y_t^2 \times I^{+}(0) - E\left(Y_{t+1}^2 Y_t^2 \times I^{+}(0)\right) \xrightarrow{d} U_1^+ + \alpha_{1,0} Q_0^+ =: Q_1^+.$$

In addition, following from parallel arguments,

$$a_n^{-4}\sum_t Y_{t+1}^4 \times I^{-}(1) - E\left(Y_{t+1}^4 \times I^{-}(1)\right) \xrightarrow{d} \frac{U_0^- + c_4^- C U_0^{**}}{1 - \left(c_4^+ \alpha_{1,0}^2 + c_4^- \alpha_{2,0}^2\right)} =: Q_0^-$$

and

$$a_n^{-4}\sum_t Y_{t+1}^2 Y_t^2 \times I^{-}(0) - E\left(Y_{t+1}^2 Y_t^2 \times I^{-}(0)\right) \xrightarrow{d} U_1^- + \alpha_{2,0} Q_0^- =: Q_1^-.$$

■

**LEMMA 8.** *For the ARCH(p) model, let Assumptions A1 with $k > 3$ and A2 hold. Then Assumption A8 is sufficient for $E\left(\sigma_t^3\right) < \infty$.*

**Proof.** The proof is by induction.

$$\begin{aligned}\sigma_t^3 &\leq \sigma_t^2 \times \left(\omega_0^{1/2} + \sum_{i=1}^{p} \alpha_{i,0}^{1/2} \left|Y_{t-i}\right|\right) \\ &\leq \omega_0^{3/2} + \omega_0 \sum_{i=1}^{p} \alpha_{i,0}^{1/2} \left|Y_{t-i}\right| + \omega_0^{1/2} \sum_{i=1}^{p} \alpha_{i,0} Y_{t-i}^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_{i,0}\alpha_{j,0}^{1/2} Y_{t-i}^2 \left|Y_{t-j}\right|,\end{aligned}$$

where the first inequality follows from the Triangle Inequality. Then, using Bollerslev (1986, Theorem 1),

$$\begin{aligned}E\left(\sigma_t^3\right) &\leq C + \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_{i,0}\alpha_{j,0}^{1/2} E\left(Y_{t-i}^2 \left|Y_{t-j}\right|\right) \\ &\leq C + \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_{i,0}\alpha_{j,0}^{1/2} E\left|Y_{t-j}\right|^3 \\ &\leq C + c_3 \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_{i,0}\alpha_{j,0}^{1/2} E\left(\sigma_{t-j}^3\right)\end{aligned}$$

From Lemma 1,

$$C + c_3 \alpha_{1,0}^{3/2} E\left(\sigma_{t-1}^3\right) \leq C + c_3 \alpha_{1,0}^{3/2} E\left(\sigma_t^3\right).$$

Suppose

$$C + c_3 \sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \alpha_{i,0}\alpha_{j,0}^{1/2} E\left(\sigma_{t-j}^3\right) \leq C + c_3 \left(\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \alpha_{i,0}\alpha_{j,0}^{1/2}\right) E\left(\sigma_t^3\right).$$

Then

$$\begin{aligned}E\left(\sigma_t^3\right) &\leq C + c_3 \left(\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \alpha_{i,0}\alpha_{j,0}^{1/2}\right) E\left(\sigma_t^3\right) + c_3 \left(\sum_{i=1}^{p} \alpha_{i,0}\alpha_{p,0}^{1/2} + \sum_{j=1}^{p-1} \alpha_{i,0}^{1/2}\alpha_{p,0}\right) E\left(\sigma_{t-p}^3\right) \\ &\leq \overline{C} + DE\left(\sigma_{t-p}^3\right) \\ &\leq \overline{C}\left(1 + D + D^2 + \ldots\right) \\ &\leq \frac{\overline{C}}{1-D} \\ &\leq \frac{C}{1 - c_3 \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_{i,0}\alpha_{j,0}^{1/2}}\end{aligned}$$

■

**LEMMA 9.** *For the ARCH(p) model let Assumptions A1 with $k > 3$, A2 and A8 hold. Consider*

$$X_t = \mathbf{X}_{t-1}' \boldsymbol{\alpha}_0 + W_t \tag{26}$$

*as it is defined in Section 2.3 of the main text and the set of instruments*

$$\mathbf{Z}_{t-1} = \left( \begin{array}{ccc} Y_{t-1}, & \ldots, & Y_{t-h} \end{array} \right)',$$

*where, in this case,* $h = p$*. Given Assumption A3,* $\mathbf{Z}_{t-1}$ *identifies* $\alpha_0$*.*

**Proof.** The proof is by induction. When $p = 1$, $\mathbf{Z}_{t-1}$ identifies $\boldsymbol{\alpha}_0$ (see Section 2.1 in the main paper). From (26),

$$\begin{aligned} X_t &= \sum_{i=1}^{p-1} X_{t-i}\alpha_{i,0} + X_{t-p}\alpha_{p,0} + W_t \\ &= \widetilde{\mathbf{X}}_{t-1}'\widetilde{\boldsymbol{\alpha}}_0 + X_{t-p}\alpha_{p,0} + W_t. \end{aligned}$$

Let

$$\widetilde{\mathbf{Z}}_{t-1} = \left( \begin{array}{ccc} Y_{t-1}, & \ldots, & Y_{t-p+1} \end{array} \right)',$$

and assume that $E\left(\widetilde{\mathbf{Z}}_{t-1}\widetilde{\mathbf{X}}_{t-1}'\right)$ is nonsingular. Then

$$\widetilde{\boldsymbol{\alpha}}_0 = E\left(\widetilde{\mathbf{Z}}_{t-1}\widetilde{\mathbf{X}}_{t-1}'\right)^{-1}\left[E\left(\widetilde{\mathbf{Z}}_{t-1}X_t\right) - E\left(\widetilde{\mathbf{Z}}_{t-1}X_{t-p}\right)\alpha_{p,0}\right]. \tag{27}$$

Further let

$$\begin{aligned} \mathbf{L}_0 &= E\left(Y_{t-p}\widetilde{\mathbf{X}}_{t-1}'\right)E\left(\widetilde{\mathbf{Z}}_{t-1}\widetilde{\mathbf{X}}_{t-1}'\right)^{-1}E\left(\widetilde{\mathbf{Z}}_{t-1}X_t\right), \\ \mathbf{M}_0 &= E\left(Y_{t-p}\widetilde{\mathbf{X}}_{t-1}'\right)E\left(\widetilde{\mathbf{Z}}_{t-1}\widetilde{\mathbf{X}}_{t-1}'\right)^{-1}E\left(\widetilde{\mathbf{Z}}_{t-1}X_{t-p}\right), \end{aligned}$$

noting that $\mathbf{M}_0$ is a scalar. Then given (27),

$$\alpha_{p,0} = \frac{E\left(Y_{t-p}X_t\right) - \mathbf{L}_0}{E\left(Y_{t-p}^3\right) - \mathbf{M}_0},$$

where $E\left(Y_{t-p}^3\right) - \mathbf{M}_0 \neq 0$ given A3 and Guo and Phillips (2001, Lemma 1). ■

**LEMMA 10.** *For the ARCH(p) model, let Assumptions A1 with* $k = 6$*, A2 and A8 hold. Then*

$$a_n^{-3}\sum_t \sigma_{t+1}^3 - E\left(\sigma_{t+1}^3\right) \xrightarrow{d} V_{0,\sigma}$$

*when* $\kappa_0 \in (3,\ 6)$*, where* $V_{0,\sigma}$ *is* $(\kappa_0/3)-$*stable.*

**Proof.**

$$\begin{aligned}
& a_n^{-3}\sum_t \sigma_{t+1}^3 - E\left(\sigma_{t+1}^3\right) \\
= \; & a_n^{-3}\sum_t \left(\sigma_{t+1}^3 - E\left(\sigma_{t+1}^3\right)\right) \times I_{\left\{\sigma_{t+1}\leq a_n\varepsilon\right\}} \\
& + a_n^{-3}\sum_t \left(\sigma_{t+1}^3 - E\left(\sigma_{t+1}^3\right)\right) \times I_{\left\{\sigma_{t+1}> a_n\varepsilon\right\}} \\
= \; & Ia + IIa
\end{aligned}$$

Given Carrasco and Chen (2002, Proposition 12), $\{\sigma_t\}$ is strictly stationary. Then from $Ia$, given Lemma 8,

$$\begin{aligned}
a_n^{-3}\sum_t E\left(\sigma_{t+1}^3\right) \times I_{\left\{\sigma_{t+1}\leq a_n\varepsilon\right\}} &= n^{\frac{\kappa_0-6}{2\kappa_0}} E\left(\sigma^3\right) n^{-1/2}\sum_t I_{\left\{\sigma_{t+1}\leq a_n\varepsilon\right\}} \\
&\longrightarrow 0,
\end{aligned}$$

as $n \to \infty$ by the CLT in Ibragimov and Linnik (1971, Theorem 18.5.3). Also, since

$$Var\left(a_n^{-3}\sum_t \sigma_{t+1}^3 \times I_{\left\{\sigma_{t+1}\leq a_n\varepsilon\right\}}\right) = n a_n^{-6} E\left(\sigma^6 I_{\left\{\sigma\leq a_n\varepsilon\right\}}\right),$$

$\lim_{n\to\infty}\lim_{\epsilon\to 0} Var\left(Ia\right) = 0$, as shown elsewhere in this Appendix. Next,

$$\begin{aligned}
a_n^{-3}\sum_t E\left(\sigma_{t+1}^3\right) \times I_{\left\{\sigma_{t+1}> a_n\varepsilon\right\}} &= n a_n^{-3} E\left(\sigma^3\right) n^{-1}\sum_t I_{\left\{\sigma_{t+1}> a_n\varepsilon\right\}} \\
&\sim a_n^{-3} E\left(\sigma^3\right) n P\left(\sigma_{t+1} > a_n\varepsilon\right) \\
&\longrightarrow 0,
\end{aligned}$$

as $n \to \infty$ as in (10). As a result,

$$\begin{aligned}
IIa &= a_n^{-3}\sum_t \sigma_{t+1}^3 \times I_{\left\{\sigma_{t+1}> a_n\varepsilon\right\}} + o_p\left(1\right) \\
&= T_{3,0,\epsilon}\left(N_n\right) + o_p\left(1\right)
\end{aligned}$$

so that

$$\begin{aligned}
a_n^{-3}\sum_t \sigma_{t+1}^3 - E\left(\sigma_{t+1}^3\right) &= Ia + IIa + o_p\left(1\right) \\
&\xrightarrow{d} T_{3,0,\epsilon}\left(N\right) \\
&\xrightarrow{d} V_{0,\sigma}
\end{aligned}$$

where "$\xrightarrow{d}$" is with respect to $n \to \infty$ first and then $\epsilon \to 0$. The first "$\xrightarrow{d}$" relies on Basrak, Davis, and Mikosch (2002, Corollary 3.5(B) to establish $\left(\left(\begin{array}{cc} Y_t, & \sigma_t \end{array}\right)\right)$ as RV($\kappa_0$) and Basrak, Davis, and Mikosch (2002, Theorem 2.10), which is a generalization of Lemma 4 to $\widetilde{\mathbf{Y}}_t$, since $\{\sigma_t\}$

is also strongly mixing given Carrasco and Chen (2002, Proposition 12). The second "$\xrightarrow{d}$" follows from Davis and Hsing (1995, Proof of Theorem 3.1, pp. 897-898). ■

**LEMMA 11.** *For the ARCH(p) model, let Assumptions A1 with $k=6$, A2 and A8 hold. Then*

$$a_n^{-3}\sum_t Y_t^2 Y_{t+m} \xrightarrow{d} \left(R_{p,m}\right)_{m=1,\ldots,p},$$

*when $\kappa_0 \in (3,\ 6)$, where each $R_{p,m}$ is $(\kappa_0/3)-$stable.*

**Proof.** To begin,

$$E\left(Y_t^2 Y_{t+m}\right) = E\left(Y_t^2 \sigma_{t+m} E\left(\epsilon_{t+m} \mid \digamma_{t-m+1}\right)\right) = 0.$$

Then,

$$\begin{aligned} a_n^{-3}\sum_t Y_t^2 Y_{t+m} &= a_n^{-3}\sum_t Y_t^2 Y_{t+m} \times I_{\{|Y_t| \le a_n\varepsilon\}} + a_n^{-3}\sum_t Y_t^2 Y_{t+m} I_{\{|Y_t| > a_n\varepsilon\}} \\ &= Ib + IIb \end{aligned}$$

Since

$$Var\left(Ib\right) = n a_n^{-6} Var\left(\epsilon_{t+m}\right) E\left(\sigma_{t+m}^2 Y_t^4 \times I_{\{|Y_t| \le a_n\varepsilon\}}\right),$$

and

$$\begin{aligned} E\left(\sigma_{t+m}^2 Y_t^4 \times I_{\{|Y_t| \le a_n\varepsilon\}}\right) &= \omega_0 E\left(Y_t^4 \times I_{\{|Y_t| \le a_n\varepsilon\}}\right) + \sum_{i=1}^{p} \alpha_{i,0} E\left(Y_{t+m-i}^2 Y_t^4 \times I_{\{|Y_t| \le a_n\varepsilon\}}\right) \\ &\le CE\left(Y^6 \times I_{\{|Y| \le a_n\varepsilon\}}\right), \end{aligned}$$

$\lim_{n\to\infty}\lim_{\epsilon\to 0} Var\left(Ia\right) = 0$, as in (10). Next, building off of the definitions introduced in the proof of Lemma 5, consider

$$T_{m,\epsilon}^{(2)}\left(\sum_{i=1}^{\infty} n_i \delta_{\mathbf{x}_i}\right) = \sum_{i=1}^{\infty} n_i \left(x_i^{(0)}\right)^2 x_i^{(m-1)} I_{\left\{\left|x_i^{(0)}\right| > \varepsilon\right\}}, \qquad m \ge 2.$$

Then

$$\begin{aligned} a_n^{-3}\sum_t Y_t^2 Y_{t+m} &= Ib + T_{m,\epsilon}^{(2)}\left(N_n\right) \\ &\xrightarrow{d} T_{m,\epsilon}^{(2)}\left(N\right) \\ &\xrightarrow{d} R_{p,m}, \end{aligned}$$

where "$\xrightarrow{d}$" is with respect to $n \to \infty$ first, and then $\epsilon \to 0$. As for Lemma 10, the first "$\xrightarrow{d}$" relies on Basrak, Davis, and Mikosch (2002, Corollary 3.5(B) and Theorem 2.10) and the continuous mapping theorem. As is true elsewhere in this Appendix, the second "$\xrightarrow{d}$" follows from Davis and Hsing (1995, Proof of Theorem 3.1, pp. 897-898). ■

**LEMMA 12.** *For the ARCH$(p)$ model, let Assumptions A1 with $k=6$, A2 and A8 hold. Then, given the definitions of $\widehat{\gamma}_{(Y,\ Y^2)}(m)$ and $\gamma_{(Y,\ Y^2)}(m)$ in Lemma 5,*

$$na_n^{-3}\left(\widehat{\gamma}_{(Y,\ Y^2)}(m)-\gamma_{(Y,\ Y^2)}(m)\right)\xrightarrow{d}\left(V_{p,m}\right)_{m=0,\ldots,h} \tag{28}$$

*for a $\kappa_0\in(3,\ 6)$, where $V_{p,m}:=V_{p,m}^{*}-\alpha_{1,0}V_{p.m-1}$, $V_{p,0}:=V_{p,0}^{*}+c_3^{*}V_{0,\sigma}$, and the vector*

$$\mathbf{V}_{p,h}=\left(\begin{array}{ccc} V_{p,0}, & \ldots, & V_{p,h}\end{array}\right)'$$

*is jointly $(\kappa_0/3)-$stable.*

**Proof.** Begin by considering the following modification to (9)

$$\begin{aligned} & a_n^{-3}\sum_t\left(Y_{t+1}^3-E\left(Y_{t+1}^3\right)\right)\\ = & a_n^{-3}\sum_t\sigma_{t+1}^3\left(\epsilon_{t+1}^3-c_3^{*}\right)\times I_{\left\{\sigma_{t+1}\le a_n\varepsilon\right\}}\\ & +a_n^{-3}\sum_t\sigma_{t+1}^3\left(\epsilon_{t+1}^3-c_3^{*}\right)\times I_{\left\{\sigma_{t+1}>a_n\varepsilon\right\}}\\ & +c_3^{*}a_n^{-3}\sum_t\left(\sigma_{t+1}^3-E\left(\sigma_{t+1}^3\right)\right)\\ = & Ia+IIa+IIIa\end{aligned}$$

introduced to deal with the complications posed by a multi-lag parameterization of $\sigma_{t+1}^2$. From this decomposition,

$$Var\left(Ia\right)=na_n^{-6}Var\left(\epsilon_{t+1}^3\right)E\left(\sigma^6\times I_{\left\{\sigma\le a_n\varepsilon\right\}}\right)$$

so that $\lim\limits_{n\to\infty}\lim\limits_{\epsilon\to 0}Var\left(Ia\right)=0$, as in (10). Next,

$$\begin{aligned} IIa &= a_n^{-3}\sum_t Y_{t+1}^3\times I_{\left\{\left|Y_{t+1}\right|>a_n\varepsilon\right\}}-c_3^{*}a_n^{-3}\sum_t\sigma_{t+1}^3\times I_{\left\{\sigma_{t+1}>a_n\varepsilon\right\}}+o_P(1)\\ &= T_{3,0,\epsilon}\left(N_n\right)-c_3^{*}T_{3,0,\epsilon}^{*}\left(N_n\right)+o_P(1)\end{aligned}$$

where the first equality follows from Basrak, Davis and Mikosch (2002, proof of Theorem 3.6), and $T_{3,0,\epsilon}^{*}\left(N_n\right)$ denotes that $N_n$ is defined in terms of $\sigma_{t+m}$, while $T_{3,0,\epsilon}\left(N_n\right)$ retains its definition from the proof of Lemma 5, where $N_n$ is a function of $Y_{t+m}$. As a result,

$$\begin{aligned} a_n^{-3}\sum_t\left(Y_{t+1}^3-E\left(Y_{t+1}^3\right)\right) &= T_{3,0,\epsilon}\left(N_n\right)-c_3^{*}T_{3,0,\epsilon}^{*}\left(N_n\right)+IIIa+o_P(1)\\ &\xrightarrow{d} V_{p,0}^{*}+c_3^{*}V_{0,\sigma},\end{aligned} \tag{29}$$

where "$\xrightarrow{d}$" is with respect to $n\to\infty$ first, and then $\epsilon\to 0$. Here, "$\xrightarrow{d}$" follows from Basrak, Davis, and Mikosch (2002, Corollary 3.5(B) and Theorem 2.10), Lemma 10, and Davis and Hsing

(1995, Theorem 3.1, pp. 897-898) and grants that

$$na_n^{-3}\left(\widehat{\gamma}_{(Y,\ Y^2)}(0)-\gamma_{(Y,\ Y^2)}(0)\right)\stackrel{d}{\longrightarrow}V_{p,0}:=V_{p,0}^{*}+c_3^{*}V_{0,\sigma}. \tag{30}$$

Consider next the decomposition in (18). From this decomposition,

$$\begin{aligned} Var(Ic) &= na_n^{-6}Var\left(\epsilon_1^2\right)\times E\left(Y_t^2\sigma_{t+1}^2\times I_{\{|Y_t|\le a_n\varepsilon\}}\right)\\ &\le Cna_n^{-6}E\left(Y^6\times I_{\{|Y|\le a_n\varepsilon\}}\right), \end{aligned}$$

since

$$\begin{aligned} E\left(Y_t^2\sigma_{t+1}^2\times I_{\{|Y_t|\le a_n\varepsilon\}}\right) &= \omega_0^2E\left(Y_t^2\times I_{\{|Y_t|\le a_n\varepsilon\}}\right)+2\omega_0\sum_{i=1}^{p}\alpha_{i,0}E\left(Y_t^2Y_{t+1-i}^2\times I_{\{|Y_t|\le a_n\varepsilon\}}\right)\\ &\quad+\sum_{i=1}^{p}\sum_{j=1}^{p}\alpha_{i,0}\alpha_{j,0}E\left(Y_t^2Y_{t+1-i}^2Y_{t+1-j}^2\times I_{\{|Y_t|\le a_n\varepsilon\}}\right), \end{aligned}$$

making the now familiar $\lim_{n\to\infty}\lim_{\epsilon\to 0}Var(Ic)=0$ apply. Next,

$$\begin{aligned} IIc &= a_n^{-3}\sum_t Y_tY_{t+1}^2\times I_{\{|Y_t|>a_n\varepsilon\}}-\alpha_{1,0}a_n^{-3}\sum_t Y_t^3\times I_{\{|Y_t|>a_n\varepsilon\}}\\ &\quad -a_n^{-3}\sum_t\sum_{i=2}^{p}\alpha_{i,0}Y_t^2Y_{t+1-i}^2\times I_{\{|Y_t|>a_n\varepsilon\}}+o_P(1)\\ &= T_{2,\epsilon}^{(1)}(N_n)-\alpha_{1,0}T_{3,0,\epsilon}(N_n)-\sum_{i=2}^{p}\alpha_{i,0}T_{i,\epsilon}^{(2)}(N_n)+o_P(1). \end{aligned}$$

Finally,

$$IIIc=\alpha_{1,0}a_n^{-3}\sum_t Y_t^3-E\left(Y_t^3\right)+a_n^{-3}\sum_t\sum_{i=2}^{p}\alpha_{i,0}Y_t^2Y_{t+1-i}^2+o_P(1),$$

so that

$$\begin{aligned} a_n^{-3}\sum_t Y_tY_{t+1}^2-E\left(Y_tY_{t+1}^2\right) &= Ic+T_{2,\epsilon}^{(1)}(N_n)-\alpha_{1,0}T_{3,0,\epsilon}(N_n)-\sum_{i=2}^{p}\alpha_{i,0}T_{i,\epsilon}^{(2)}(N_n)\\ &\quad+IIIc+o_P(1)\\ &\stackrel{d}{\longrightarrow}V_{p,1}^{*}+\alpha_{1,0}V_{p,0}, \end{aligned}$$

where "$\stackrel{d}{\longrightarrow}$" is with respect to $n\to\infty$ first (following from the same arguments that support convergence as $n\to\infty$ in (29) and Lemma 11) and $\epsilon\to 0$ second (as established elsewhere in this appendix) so that

$$na_n^{-3}\left(\widehat{\gamma}_{(Y,\ Y^2)}(1)-\gamma_{(Y,\ Y^2)}(1)\right)\stackrel{d}{\longrightarrow}V_{p,1}:=V_{p,1}^{*}+\alpha_{1,0}V_{p,0}. \tag{31}$$

As argued elsewhere in this Supplemental Appendix, extending (31) to higher lags (i.e., $m>1$) is a continuation of the arguments given above. ■

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TABLE 4

| $\lambda$ | skew. | $\kappa$ | est. | $m$ | mean bias | med. bias | sd | dec. rge. | rmse | mae | mdae | Efficiency Ratio rmse | Efficiency Ratio mae | Efficiency Ratio mdae |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\eta = 4.1$ | | | | | | | |
| -0.10 | -0.65 | 3.44 | TSLS | 100 | -0.034 | -0.047 | 0.123 | 0.318 | 0.128 | 0.105 | 0.094 | 2.77 | 3.18 | 3.60 |
| | | | | 50 | -0.025 | -0.039 | 0.133 | 0.347 | 0.135 | 0.111 | 0.100 | 2.93 | 3.36 | 3.86 |
| | | | | 25 | -0.016 | -0.033 | 0.140 | 0.367 | 0.141 | 0.115 | 0.104 | 3.06 | 3.50 | 4.01 |
| | | | OLS | | -0.048 | -0.062 | 0.088 | 0.205 | 0.101 | 0.083 | 0.077 | 2.19 | 2.53 | 2.95 |
| | | | QMLE | | 0.000 | -0.005 | 0.046 | 0.102 | 0.046 | 0.033 | 0.026 | 1.00 | 1.00 | 1.00 |
| -0.20 | -1.27 | 3.35 | TSLS | 100 | -0.034 | -0.046 | 0.106 | 0.261 | 0.111 | 0.090 | 0.078 | 2.29 | 2.57 | 2.80 |
| | | | | 50 | -0.032 | -0.045 | 0.110 | 0.271 | 0.115 | 0.092 | 0.081 | 2.36 | 2.64 | 2.91 |
| | | | | 25 | -0.030 | -0.045 | 0.112 | 0.276 | 0.116 | 0.093 | 0.081 | 2.39 | 2.67 | 2.93 |
| | | | OLS | | -0.051 | -0.066 | 0.089 | 0.210 | 0.103 | 0.086 | 0.079 | 2.11 | 2.46 | 2.86 |
| | | | QMLE | | 0.000 | -0.006 | 0.049 | 0.107 | 0.049 | 0.035 | 0.028 | 1.00 | 1.00 | 1.00 |
| -0.40 | -2.32 | 3.15 | TSLS | 100 | -0.038 | -0.050 | 0.084 | 0.192 | 0.092 | 0.074 | 0.066 | 1.59 | 1.80 | 2.01 |
| | | | | 50 | -0.037 | -0.050 | 0.085 | 0.194 | 0.092 | 0.075 | 0.066 | 1.60 | 1.81 | 2.02 |
| | | | | 25 | -0.037 | -0.049 | 0.085 | 0.193 | 0.092 | 0.075 | 0.066 | 1.60 | 1.81 | 2.01 |
| | | | OLS | | -0.060 | -0.076 | 0.089 | 0.208 | 0.107 | 0.091 | 0.087 | 1.86 | 2.21 | 2.65 |
| | | | QMLE | | 0.000 | -0.007 | 0.058 | 0.125 | 0.058 | 0.041 | 0.033 | 1.00 | 1.00 | 1.00 |
| -0.80 | -3.48 | 2.95 | TSLS | 100 | -0.048 | -0.060 | 0.073 | 0.164 | 0.087 | 0.073 | 0.069 | 1.17 | 1.37 | 1.63 |
| | | | | 50 | -0.048 | -0.059 | 0.073 | 0.165 | 0.087 | 0.073 | 0.069 | 1.18 | 1.38 | 1.63 |
| | | | | 25 | -0.047 | -0.059 | 0.072 | 0.164 | 0.087 | 0.073 | 0.068 | 1.17 | 1.37 | 1.63 |
| | | | OLS | | -0.040 | -0.054 | 0.092 | 0.224 | 0.101 | 0.083 | 0.076 | 1.36 | 1.57 | 1.81 |
| | | | QMLE | | 0.001 | -0.010 | 0.074 | 0.162 | 0.074 | 0.053 | 0.042 | 1.00 | 1.00 | 1.00 |
| | | | | | | | $\eta = 6.1$ | | | | | | | |
| -0.10 | -0.34 | 4.29 | TSLS | 100 | -0.016 | -0.022 | 0.111 | 0.285 | 0.112 | 0.090 | 0.076 | 4.43 | 4.52 | 4.55 |
| | | | | 50 | -0.010 | -0.017 | 0.123 | 0.319 | 0.123 | 0.100 | 0.087 | 4.86 | 4.99 | 5.22 |
| | | | | 25 | -0.002 | -0.011 | 0.131 | 0.341 | 0.131 | 0.106 | 0.092 | 5.17 | 5.29 | 5.46 |
| | | | OLS | | -0.020 | -0.030 | 0.064 | 0.140 | 0.067 | 0.052 | 0.044 | 2.65 | 2.60 | 2.65 |
| | | | QMLE | | 0.000 | -0.001 | 0.025 | 0.064 | 0.025 | 0.020 | 0.017 | 1.00 | 1.00 | 1.00 |
| -0.20 | -0.67 | 4.16 | TSLS | 100 | -0.015 | -0.022 | 0.090 | 0.221 | 0.091 | 0.071 | 0.059 | 3.44 | 3.42 | 3.34 |
| | | | | 50 | -0.014 | -0.021 | 0.094 | 0.228 | 0.095 | 0.074 | 0.062 | 3.57 | 3.55 | 3.52 |
| | | | | 25 | -0.013 | -0.020 | 0.096 | 0.232 | 0.097 | 0.076 | 0.063 | 3.66 | 3.61 | 3.56 |
| | | | OLS | | -0.022 | -0.032 | 0.066 | 0.146 | 0.070 | 0.055 | 0.047 | 2.63 | 2.61 | 2.69 |
| | | | QMLE | | 0.000 | -0.001 | 0.027 | 0.066 | 0.027 | 0.021 | 0.018 | 1.00 | 1.00 | 1.00 |
| -0.40 | -1.23 | 3.85 | TSLS | 100 | -0.015 | -0.023 | 0.068 | 0.152 | 0.070 | 0.053 | 0.044 | 2.28 | 2.23 | 2.18 |
| | | | | 50 | -0.015 | -0.023 | 0.069 | 0.155 | 0.070 | 0.054 | 0.045 | 2.30 | 2.25 | 2.22 |
| | | | | 25 | -0.015 | -0.023 | 0.069 | 0.156 | 0.071 | 0.054 | 0.044 | 2.31 | 2.25 | 2.19 |
| | | | OLS | | -0.029 | -0.040 | 0.071 | 0.156 | 0.076 | 0.061 | 0.054 | 2.50 | 2.55 | 2.68 |
| | | | QMLE | | 0.000 | -0.002 | 0.031 | 0.076 | 0.031 | 0.024 | 0.020 | 1.00 | 1.00 | 1.00 |
| -0.80 | -1.88 | 3.58 | TSLS | 100 | -0.020 | -0.029 | 0.059 | 0.132 | 0.063 | 0.049 | 0.042 | 1.63 | 1.64 | 1.68 |
| | | | | 50 | -0.020 | -0.028 | 0.060 | 0.134 | 0.063 | 0.049 | 0.043 | 1.64 | 1.65 | 1.71 |
| | | | | 25 | -0.020 | -0.028 | 0.059 | 0.132 | 0.063 | 0.049 | 0.042 | 1.63 | 1.64 | 1.68 |
| | | | OLS | | -0.041 | -0.053 | 0.073 | 0.160 | 0.083 | 0.069 | 0.064 | 2.16 | 2.30 | 2.54 |
| | | | QMLE | | 0.000 | -0.002 | 0.039 | 0.095 | 0.039 | 0.030 | 0.025 | 1.00 | 1.00 | 1.00 |
| | | | | | | | $\eta = 8.1$ | | | | | | | |
| -0.10 | -0.27 | 4.82 | TSLS | 100 | -0.011 | -0.014 | 0.105 | 0.271 | 0.105 | 0.084 | 0.072 | 4.96 | 4.98 | 5.04 |
| | | | | 50 | -0.006 | -0.011 | 0.117 | 0.307 | 0.118 | 0.095 | 0.082 | 5.53 | 5.60 | 5.76 |
| | | | | 25 | 0.000 | -0.007 | 0.126 | 0.327 | 0.126 | 0.101 | 0.087 | 5.93 | 5.99 | 6.07 |
| | | | OLS | | -0.012 | -0.019 | 0.052 | 0.112 | 0.053 | 0.040 | 0.033 | 2.49 | 2.36 | 2.33 |
| | | | QMLE | | 0.000 | -0.001 | 0.021 | 0.054 | 0.021 | 0.017 | 0.014 | 1.00 | 1.00 | 1.00 |
| -0.20 | -0.53 | 4.66 | TSLS | 100 | -0.010 | -0.014 | 0.083 | 0.200 | 0.083 | 0.065 | 0.053 | 3.77 | 3.68 | 3.53 |
| | | | | 50 | -0.009 | -0.014 | 0.086 | 0.211 | 0.087 | 0.068 | 0.056 | 3.93 | 3.85 | 3.74 |
| | | | | 25 | -0.008 | -0.013 | 0.089 | 0.214 | 0.089 | 0.069 | 0.057 | 4.04 | 3.93 | 3.79 |
| | | | OLS | | -0.013 | -0.022 | 0.054 | 0.117 | 0.056 | 0.042 | 0.036 | 2.53 | 2.42 | 2.38 |
| | | | QMLE | | 0.000 | -0.001 | 0.022 | 0.056 | 0.022 | 0.018 | 0.015 | 1.00 | 1.00 | 1.00 |
| -0.40 | -0.98 | 4.29 | TSLS | 100 | -0.010 | -0.015 | 0.060 | 0.138 | 0.061 | 0.046 | 0.038 | 2.47 | 2.35 | 2.25 |
| | | | | 50 | -0.009 | -0.016 | 0.061 | 0.140 | 0.061 | 0.047 | 0.038 | 2.48 | 2.38 | 2.27 |
| | | | | 25 | -0.009 | -0.015 | 0.061 | 0.141 | 0.062 | 0.047 | 0.038 | 2.49 | 2.38 | 2.26 |
| | | | OLS | | -0.018 | -0.028 | 0.060 | 0.130 | 0.062 | 0.048 | 0.041 | 2.53 | 2.46 | 2.46 |
| | | | QMLE | | 0.000 | -0.001 | 0.025 | 0.063 | 0.025 | 0.020 | 0.017 | 1.00 | 1.00 | 1.00 |
| -0.80 | -1.52 | 3.97 | TSLS | 100 | -0.013 | -0.020 | 0.053 | 0.118 | 0.054 | 0.041 | 0.035 | 1.79 | 1.73 | 1.71 |
| | | | | 50 | -0.012 | -0.019 | 0.053 | 0.119 | 0.055 | 0.042 | 0.035 | 1.81 | 1.74 | 1.73 |
| | | | | 25 | -0.012 | -0.019 | 0.053 | 0.119 | 0.054 | 0.041 | 0.034 | 1.80 | 1.73 | 1.69 |
| | | | OLS | | -0.028 | -0.038 | 0.064 | 0.141 | 0.070 | 0.056 | 0.050 | 2.31 | 2.34 | 2.49 |
| | | | QMLE | | 0.000 | -0.001 | 0.030 | 0.077 | 0.030 | 0.024 | 0.020 | 1.00 | 1.00 | 1.00 |

## TABLE 5

| $\lambda$ | skew. | $\kappa$ | est. | $m$ | mean bias | med. bias | sd | dec. rge. | rmse | mae | mdae | Efficiency Ratio rmse | Efficiency Ratio mae | Efficiency Ratio mdae |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\eta = 4.1$ | | | | | | | | | | | | | | |
| -0.10 | -0.65 | 3.44 | TSLS | 100 | -0.014 | -0.024 | 0.111 | 0.279 | 0.112 | 0.088 | 0.072 | 6.90 | 7.45 | 7.54 |
| | | | | 50 | -0.011 | -0.024 | 0.114 | 0.285 | 0.115 | 0.089 | 0.073 | 7.06 | 7.57 | 7.59 |
| | | | | 25 | -0.009 | -0.022 | 0.117 | 0.289 | 0.117 | 0.091 | 0.074 | 7.22 | 7.72 | 7.73 |
| | | | OLS | | -0.038 | -0.049 | 0.078 | 0.173 | 0.087 | 0.070 | 0.061 | 5.36 | 5.92 | 6.38 |
| | | | QMLE | | 0.000 | -0.001 | 0.016 | 0.036 | 0.016 | 0.012 | 0.010 | 1.00 | 1.00 | 1.00 |
| -0.20 | -1.27 | 3.35 | TSLS | 100 | -0.017 | -0.027 | 0.081 | 0.179 | 0.083 | 0.062 | 0.050 | 4.63 | 4.90 | 4.96 |
| | | | | 50 | -0.017 | -0.027 | 0.082 | 0.181 | 0.083 | 0.063 | 0.049 | 4.66 | 4.92 | 4.87 |
| | | | | 25 | -0.017 | -0.027 | 0.082 | 0.178 | 0.084 | 0.063 | 0.050 | 4.68 | 4.92 | 4.93 |
| | | | OLS | | -0.041 | -0.052 | 0.080 | 0.175 | 0.089 | 0.072 | 0.064 | 4.99 | 5.70 | 6.36 |
| | | | QMLE | | 0.000 | -0.002 | 0.018 | 0.039 | 0.018 | 0.013 | 0.010 | 1.00 | 1.00 | 1.00 |
| -0.40 | -2.32 | 3.15 | TSLS | 100 | -0.021 | -0.031 | 0.061 | 0.125 | 0.065 | 0.049 | 0.042 | 2.89 | 3.20 | 3.45 |
| | | | | 50 | -0.021 | -0.031 | 0.061 | 0.124 | 0.065 | 0.049 | 0.042 | 2.88 | 3.19 | 3.43 |
| | | | | 25 | -0.021 | -0.031 | 0.061 | 0.124 | 0.065 | 0.049 | 0.041 | 2.87 | 3.18 | 3.42 |
| | | | OLS | | -0.049 | -0.062 | 0.081 | 0.178 | 0.095 | 0.079 | 0.071 | 4.22 | 5.10 | 5.89 |
| | | | QMLE | | 0.000 | -0.002 | 0.023 | 0.047 | 0.023 | 0.015 | 0.012 | 1.00 | 1.00 | 1.00 |
| -0.80 | -3.48 | 2.95 | TSLS | 100 | -0.030 | -0.040 | 0.055 | 0.113 | 0.063 | 0.051 | 0.047 | 2.12 | 2.52 | 2.93 |
| | | | | 50 | -0.030 | -0.040 | 0.055 | 0.112 | 0.062 | 0.050 | 0.046 | 2.11 | 2.50 | 2.90 |
| | | | | 25 | -0.030 | -0.039 | 0.055 | 0.111 | 0.062 | 0.050 | 0.046 | 2.11 | 2.50 | 2.89 |
| | | | OLS | | -0.065 | -0.077 | 0.079 | 0.172 | 0.102 | 0.088 | 0.084 | 3.46 | 4.36 | 5.26 |
| | | | QMLE | | 0.000 | -0.003 | 0.029 | 0.061 | 0.029 | 0.020 | 0.016 | 1.00 | 1.00 | 1.00 |
| $\eta = 6.1$ | | | | | | | | | | | | | | |
| -0.10 | -0.34 | 4.29 | TSLS | 100 | -0.003 | -0.007 | 0.077 | 0.179 | 0.077 | 0.058 | 0.045 | 9.32 | 8.92 | 8.28 |
| | | | | 50 | -0.003 | -0.006 | 0.079 | 0.184 | 0.079 | 0.059 | 0.046 | 9.53 | 9.10 | 8.42 |
| | | | | 25 | -0.002 | -0.007 | 0.080 | 0.187 | 0.080 | 0.060 | 0.047 | 9.71 | 9.26 | 8.55 |
| | | | OLS | | -0.010 | -0.017 | 0.046 | 0.086 | 0.047 | 0.033 | 0.026 | 5.63 | 5.07 | 4.80 |
| | | | QMLE | | 0.000 | 0.000 | 0.008 | 0.021 | 0.008 | 0.006 | 0.005 | 1.00 | 1.00 | 1.00 |
| -0.20 | -0.67 | 4.16 | TSLS | 100 | -0.004 | -0.008 | 0.050 | 0.105 | 0.050 | 0.035 | 0.027 | 5.77 | 5.20 | 4.81 |
| | | | | 50 | -0.003 | -0.008 | 0.050 | 0.106 | 0.050 | 0.036 | 0.027 | 5.78 | 5.22 | 4.82 |
| | | | | 25 | -0.003 | -0.008 | 0.051 | 0.107 | 0.051 | 0.036 | 0.028 | 5.84 | 5.26 | 4.85 |
| | | | OLS | | -0.011 | -0.019 | 0.049 | 0.093 | 0.050 | 0.036 | 0.029 | 5.75 | 5.24 | 5.01 |
| | | | QMLE | | 0.000 | 0.000 | 0.009 | 0.022 | 0.009 | 0.007 | 0.006 | 1.00 | 1.00 | 1.00 |
| -0.40 | -1.23 | 3.85 | TSLS | 100 | -0.005 | -0.010 | 0.038 | 0.075 | 0.038 | 0.026 | 0.021 | 3.80 | 3.36 | 3.16 |
| | | | | 50 | -0.005 | -0.010 | 0.038 | 0.075 | 0.038 | 0.026 | 0.021 | 3.79 | 3.35 | 3.18 |
| | | | | 25 | -0.004 | -0.010 | 0.038 | 0.075 | 0.038 | 0.026 | 0.021 | 3.80 | 3.35 | 3.14 |
| | | | OLS | | -0.017 | -0.026 | 0.054 | 0.108 | 0.057 | 0.042 | 0.035 | 5.67 | 5.40 | 5.34 |
| | | | QMLE | | 0.000 | 0.000 | 0.010 | 0.025 | 0.010 | 0.008 | 0.007 | 1.00 | 1.00 | 1.00 |
| -0.80 | -1.88 | 3.58 | TSLS | 100 | -0.007 | -0.014 | 0.036 | 0.071 | 0.037 | 0.027 | 0.022 | 2.95 | 2.69 | 2.71 |
| | | | | 50 | -0.007 | -0.014 | 0.036 | 0.071 | 0.037 | 0.026 | 0.022 | 2.94 | 2.68 | 2.67 |
| | | | | 25 | -0.007 | -0.014 | 0.036 | 0.072 | 0.037 | 0.026 | 0.022 | 2.93 | 2.67 | 2.65 |
| | | | OLS | | -0.027 | -0.037 | 0.058 | 0.119 | 0.065 | 0.051 | 0.045 | 5.17 | 5.16 | 5.44 |
| | | | QMLE | | 0.000 | -0.001 | 0.012 | 0.031 | 0.012 | 0.010 | 0.008 | 1.00 | 1.00 | 1.00 |
| $\eta = 8.1$ | | | | | | | | | | | | | | |
| -0.10 | -0.27 | 4.82 | TSLS | 100 | -0.002 | -0.003 | 0.062 | 0.148 | 0.062 | 0.047 | 0.037 | 8.91 | 8.60 | 8.06 |
| | | | | 50 | -0.002 | -0.003 | 0.064 | 0.150 | 0.064 | 0.048 | 0.038 | 9.13 | 8.81 | 8.32 |
| | | | | 25 | -0.001 | -0.003 | 0.065 | 0.153 | 0.065 | 0.049 | 0.039 | 9.27 | 8.96 | 8.47 |
| | | | OLS | | -0.004 | -0.009 | 0.031 | 0.059 | 0.031 | 0.021 | 0.016 | 4.50 | 3.88 | 3.59 |
| | | | QMLE | | 0.000 | 0.000 | 0.007 | 0.018 | 0.007 | 0.005 | 0.005 | 1.00 | 1.00 | 1.00 |
| -0.20 | -0.53 | 4.66 | TSLS | 100 | -0.001 | -0.004 | 0.038 | 0.085 | 0.038 | 0.028 | 0.022 | 5.29 | 4.89 | 4.58 |
| | | | | 50 | -0.001 | -0.004 | 0.039 | 0.085 | 0.039 | 0.028 | 0.022 | 5.33 | 4.93 | 4.68 |
| | | | | 25 | -0.001 | -0.004 | 0.039 | 0.086 | 0.039 | 0.028 | 0.022 | 5.38 | 4.97 | 4.68 |
| | | | OLS | | -0.005 | -0.010 | 0.034 | 0.064 | 0.034 | 0.023 | 0.018 | 4.76 | 4.12 | 3.89 |
| | | | QMLE | | 0.000 | 0.000 | 0.007 | 0.018 | 0.007 | 0.006 | 0.005 | 1.00 | 1.00 | 1.00 |
| -0.40 | -0.98 | 4.29 | TSLS | 100 | -0.002 | -0.005 | 0.028 | 0.059 | 0.028 | 0.020 | 0.016 | 3.54 | 3.14 | 2.95 |
| | | | | 50 | -0.002 | -0.005 | 0.028 | 0.059 | 0.028 | 0.020 | 0.016 | 3.54 | 3.14 | 2.95 |
| | | | | 25 | -0.002 | -0.005 | 0.028 | 0.059 | 0.029 | 0.020 | 0.016 | 3.54 | 3.14 | 2.93 |
| | | | OLS | | -0.008 | -0.015 | 0.040 | 0.077 | 0.041 | 0.029 | 0.023 | 5.08 | 4.55 | 4.40 |
| | | | QMLE | | 0.000 | 0.000 | 0.008 | 0.020 | 0.008 | 0.006 | 0.005 | 1.00 | 1.00 | 1.00 |
| -0.80 | -1.52 | 3.97 | TSLS | 100 | -0.003 | -0.008 | 0.028 | 0.056 | 0.028 | 0.020 | 0.016 | 2.90 | 2.54 | 2.42 |
| | | | | 50 | -0.003 | -0.008 | 0.028 | 0.055 | 0.028 | 0.020 | 0.016 | 2.89 | 2.53 | 2.43 |
| | | | | 25 | -0.003 | -0.008 | 0.028 | 0.056 | 0.028 | 0.020 | 0.016 | 2.88 | 2.52 | 2.41 |
| | | | OLS | | -0.015 | -0.023 | 0.046 | 0.091 | 0.049 | 0.037 | 0.031 | 5.02 | 4.72 | 4.73 |
| | | | QMLE | | 0.000 | 0.000 | 0.010 | 0.025 | 0.010 | 0.008 | 0.007 | 1.00 | 1.00 | 1.00 |

**Notes to Tables 4–5**. All simulations are conducted for the ARCH(1) model with $\omega_0 = 0.005$ and $\alpha_0 = 0.25$, each selected to match the empirical features of high frequency financial returns. All simulations are also conducted for $10,000$ trials where, within each trial, the first 200 observations are dropped to avoid initialization effects. The estimators under study are TSLS, OLS, and QMLE. For TSLS, instrument vectors of 100, 50, and 25 lags are considered. Summary statistics are the mean bias and median bias, each measured relative to the true parameter value, the standard deviation, decile range (the difference between the 90th and 10th percentiles), and the root mean squared error, mean absolute error, and median absolute error, also each measured with respect to the true parameter value. The Efficiency Ratio is the root mean squared error, mean absolute error, and median absolute error of the given estimator divided by the corresponding measure for the QMLE. $\{\epsilon_t\}$ is drawn from the student's t density of Hansen (1994) for $\lambda = -\begin{pmatrix} 0.10, & 0.20, & 0.40, & 0.80 \end{pmatrix}$ and $\eta = \begin{pmatrix} 4.1, & 6.1, & 8.1 \end{pmatrix}$, noting that moments up to the $\eta$th are well defined. Skew is the skewness in $\{\epsilon_t\}$ and, hence, $\{Y_t\}$. Values of $\kappa$ for $\{Y_t\}$ are obtained from separate simulations of $10,000$ trials on sample sizes of $10,000$ observations. Lastly, for Tables 4 and 5, the sample size is $10,000$ and $100,000$, respectively.