Finance and Economics Discussion Series
Divisions of Research & Statistics and Monetary Affairs
Federal Reserve Board, Washington, D.C.

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Caggese and Perez-Orive (2017)

2017-009

Please cite this paper as:
Caggese, Andrea and Ander Perez-Orive (2017). "Capital Misallocation and Secular Stagnation," Finance and Economics Discussion Series 2017-009. Washington: Board of Governors of the Federal Reserve System, https://doi.org/10.17016/FEDS.2017.009.

Capital Misallocation and Secular Stagnation*

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(This Version: January 17, 2017)

Abstract

The widespread emergence of intangible technologies in recent decades may have significantly hurt output growth–even when these technologies replaced considerably less productive tangible technologies–because of structurally low interest rates caused by demographic forces. This insight is obtained in a model in which intangible capital cannot attract external finance, firms are credit constrained, and there is substantial dispersion in productivity. In a tangibles-intense economy with highly leveraged firms, low rates enable more borrowing and faster debt repayment, reduce misallocation, and increase aggregate output. An increase in the share of intangible capital in production reduces the borrowing capacity and increases the cash holdings of the corporate sector, which switches from being a net borrower to a net saver. In this intangibles-intense economy, the ability of firms to purchase intangible capital using retained earnings is impaired by low interest rates, because low rates increase the price of capital and slow down the accumulation of corporate savings.

Keywords: Intangible Capital, Borrowing Constraints, Capital Reallocation, Secular Stagnation
JEL Classification: E22, E43, E44

* A previous version of this paper was entitled "Reallocation of Intangible Capital and Secular Stagnation". We thank Andrew Abel, Fiorella de Fiore (discussant), Wouter Den Haan (discussant), Andrea Eisfeldt, Antonio Falato, Maryam Farboodi (discussant), Simon Gilchrist, Adam Guren, Matteo Iacoviello, Arvind Krishnamurthy, Tim Landvoigt (discussant), Claudio Michelacci (discussant), Guillermo Ordonez, Enrico Perotti, Vincenzo Quadrini, and Stephen Terry, and seminar participants at Boston University, Boston College, the Federal Reserve Board, the Bank of Spain, the CREI macro lunch, the 7th Meeting of the Macro Finance Society (UCLA 2016), the 2016 Barcelona Summer Forum Workshop on Financial Markets and Asset Prices, the 2016 NBER SI Workshop on Macro, Money and Financial Frictions, the Midwest Macro Meetings, the Cleveland Fed Day-Ahead Meeting on Productivity, the BoE Workshop on Finance, Investment and Productivity, and the Bank of Italy Workshop on Macroeconomic Dynamics for very helpful comments. We also thank Christoph Albert for excellent research assistance. Andrea Caggese acknowledges financial support from the Ministry of Economics of Spain and from Resercaixa. The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or of anyone else associated with the Federal Reserve System. All errors are, of course, our own responsibility.

1 Introduction

Real interest rates have decreased in past decades, while economic growth has fallen short of previous trends, developments that have been linked to a process of 'secular stagnation' (Summers (2015), Eichengreen (2015)). At the same time, the developed world has experienced a technological change toward a stronger importance of information technology and of knowledge, human and organizational capital, which has gradually reduced the reliance on physical capital (Corrado and Hulten (2010a)) and has been linked to a significant decrease in corporate net borrowing (Falato, Kadyrzhanova, and Sim (2014), Döttling and Perotti (2015)).[1]

This paper argues that the increased reliance on intangible capital and the low real interest rates interact to hurt capital reallocation and reduce productivity and output growth. Aggregate productivity depends on an efficient reallocation of resources from declining or exiting firms to new entrants or expanding firms. The rise of intangible capital implies a growing importance of the reallocation of intangible assets such as patents, brand equity, and human and organizational capital. These assets cannot be collateralized, and their acquisition has to be financed mostly using retained earnings. As a result, the corporate sector borrows less, holds an increasing amount of cash, and switches from being a net borrower to a net saver. A decrease in interest rates increases the price of these intangible assets and reduces the ability of credit-constrained expanding firms to purchase them. Lower interest rates also decrease the rate at which non-investing firms can accumulate savings to finance future expansions. We show that the rise in intangibles, via these effects, alters the dynamic relationship between interest rates and efficiency in the allocation of capital.

We formalize this intuition by developing a model of an economy in which firms use tangible capital, intangible capital, and labor as complementary factors in the production of consumption goods. A subset of firms have high productivity and suffer from financing constraints that prevent them from issuing equity or from borrowing any amount in excess of the collateral value of their holdings of tangible and intangible capital. We follow Kiyotaki and Moore (2012) in assuming that these high-productivity firms can invest only occasionally. In equilibrium, they save as much as possible in non-investing periods, and invest all of their accumulated net savings plus their maximum available borrowing in investing periods. Any residual capital not absorbed by the high-productivity firms is used by low-productivity firms, which are financially unconstrained.

[1] The decrease in corporate net borrowing has translated into a shift in the net financial position of the nonfinancial corporate sector from a net borrowing position roughly before the year 2000 to a net saving position from 2000 onward (Armenter and Hnatkovska (2016), Quadrini (2016), Chen, Karabarbounis, and Neiman (2016), Shourideh and Zetlin-Jones (2016)).

In our economy, the consumer sector is modeled as overlapping generations of households displaying a realistic life cycle, in a way that enables us to obtain an equilibrium interest rate in the steady state that is not necessarily equal to the household rate of time preference. This specification of the consumer sector allows us to consider some of the main structural forces that multiple studies have identified to have pushed real interest rates lower in recent decades. In particular, we focus on a higher propensity to save in the household sector due to an increase in household longevity and a decrease in the rate of time preference.[2] The increased corporate net savings due to a higher intangibles usage also contributes to the downward pressure on real rates.

We first inspect the analytical solution of a simplified version of the model to describe four channels through which lower interest rates interact with the intensity of intangible capital in firms' production function to affect the steady state equilibrium of our economy. First, a *debt overhang channel* allows net borrowing high-productivity firms to pay down their debt more easily when interest rates are low and enables them to absorb more capital. Second, and conversely, a *savings channel* operates when the firm sector is a net saver: reductions in the interest rate decrease the speed of accumulation of savings and hurt capital reallocation. Third, lower interest rates that increase the price of tangible and intangible assets reduce the amount of capital that high-productivity firms can purchase for a given amount of net worth and borrowing capacity–a *capital purchase price channel*. Fourth, a lower interest rate increases the present value of the collateral pledged next period, and reduces the size of the downpayment necessary to purchase capital, improving capital reallocation through a *borrowing/collateral value channel*. The analytical solution of the simplified model provides a clear illustration of the main theoretical finding of the paper: in an economy with a relatively low collateral value of capital, the *savings* and the *capital purchase price* channels dominate and a drop in the interest rate worsens the allocation of resources and reduces aggregate investment, productivity, and output.

In the remaining sections of the paper, we calibrate and simulate our full general equilibrium model to study how the parallel developments in the household and corporate sectors have interacted to generate aggregate patterns consistent with the secular stagnation hypothesis. In the household sector, as discussed earlier, we model a progressive decrease in individuals' rate of time preference and a progressive increase in their life expectancy, both of which put downward pressure on the equilibrium interest rate. In the corporate sector, we introduce a gradual shift

[2] We interpret our exercise as a shortcut for a collection of different factors, such as population aging, wealth and income inequality, financial deepening, and foreign-sector developments, which have contributed to increase households' demand for savings in the past 40 years.

in the reliance on intangible capital of firms.[3]

We find that while the household sector developments in isolation and the corporate sector developments in isolation are both expansionary, the combination of both developments is contractionary. The drop in the interest rate increases high-productivity firms' ability to borrow and pay down their debt while firms still rely strongly on tangible capital. As firms use increasingly more intangible capital and become net savers, low rates reduce efficient capital allocation by increasing capital prices and by slowing the accumulation of corporate savings. The share of output produced by the high-productivity firms drops significantly. The lower corporate borrowing itself also puts downward pressure on interest rates, which amplifies the misallocation of capital. Despite the fact that the economy is shifting toward a higher reliance on a more productive type of capital, aggregate productivity falls by 6.5%, and even though low rates encourage capital creation, output is 2% lower than in the case in which only household sector or only corporate sector developments occur.

We interpret this comparative static exercise as capturing the developments in the U.S. economy following the rise in the share of intangible capital and the rise in net household and foreign-sector savings in the past 40 years. In this respect, this model is remarkably consistent with a series of well-documented trends during this period: (i) net corporate savings increased as a fraction of gross domestic product (GDP), (ii) household leverage increased as a fraction of GDP, (iii) the real interest rate fell, (iv) intra-industry dispersion in productivity has increased, and (v) output and productivity progressively declined relative to their previous trends.

An important question is whether the trends identified in this paper are likely to persist, as in the secular stagnation hypothesis, or reverse. While the technological shift identified in the paper is likely to be permanent and possibly intensify, the developments that are keeping interest rates low may fade in the future. Some developments pushing down rates, such as the lower retirement age or the increase in the net demand for safe assets, may prove temporary, while others, such as population ageing, the decline in the growth rate of population, or the drop in the relative price of capital, might be more persistent.[4]

Overall, our results suggest that the interaction between low interest rates, intangible technologies, and corporate financing patterns might be an important factor behind secular stagnation.

[3] We set the reliance on intangible capital to match its observed evolution from a pre-1980 value of 20% of aggregate capital to a post-2010 value of 60% of aggregate capital (Corrado and Hulten (2010a), Falato, Kadyrzhanova, and Sim (2014), Döttling and Perotti (2015)). Since we assume that intangible capital is more productive than tangible capital, this gradual shift is consistent with the notion of the transition to intangible capital as a privately optimal choice of firms adopting technologies that are more productive.

[4] For a detailed discussion of the causes of low real interest rates and the likelihood that these causes remain in the future, see Baldwin and Teulings (2014), Summers (2014), and Blanchard, Furceri and Pescatori (2014).

Related Literature

The secular stagnation hypothesis as an explanation of recent economic trends has been proposed by, among others, Summers (2015) and Eichengreen (2015). One prominent example of a formalization of these ideas is Eggertsson and Mehrotra (2014), who show how a persistent tightening of the debt limit facing households can reduce the equilibrium real interest rate and, in the presence of sticky prices and a zero lower bound in nominal interest rates, generate permanent reductions in output.[5] Our paper contributes to this literature by identifying and formalizing a novel misallocation effect of endogenously low real interest rates. Our alternative explanation of the secular stagnation hypothesis can account for a large drop in aggregate output, does not rely on the zero lower bound or sticky prices, and is consistent with a broad set of well-documented trends.

The rising use of intangible capital has been documented by Corrado and Hulten (2010a), and its relation to the decrease in corporate borrowing and the rise in corporate cash holdings has been shown empirically by Bates, Kahle, and Stulz (2009). Giglio and Severo (2012), Falato, Kadyrzhanova, and Sim (2014) and Döttling and Perotti (2015) introduce models that describe how the rise in intangibles can lower the equilibrium interest rate by decreasing firms' net borrowing. We add to this literature by describing a mechanism through which the rise in intangibles can have a negative effect on aggregate capital reallocation and growth.

Our paper is also related to the literature on financial frictions, firm dynamics, and misallocation (Buera, Kaboski, and Shin (2011), Caggese and Cuñat (2013), Moll (2014), Midrigan and Xu (2014), and Buera and Moll (2015)). With respect to these papers, our contribution is to provide novel theoretical insights on the relation between interest rates, the collateralizability of capital, and misallocation.[6]

The rest of the paper is organized as follows. Section 2 introduces the empirical evidence that motivates this paper. We describe a very simple model in Section 3 that conveys the basic intuition of the mechanisms we introduce, and we develop a full-fledged general equilibrium extension in Section 4. The steady state and calibration of the general equilibrium model are described in Section 5 and the simulation results are discussed in Section 7. Section 8 concludes.

[5] Other recent theoretical papers with alternative explanations of secular stagnation are Bacchetta, Benhima, and Kalantzis (2016) and Benigno and Fornaro (2015).

[6] Gopinath et al. (2016) also consider a model with financial frictions and heterogenous firms in which declining interest rates cause an increase in the dispersion in the productivity of capital. However, their mechanism is fundamentallty different from ours. In their model, when the interest rate falls, all firms invest more and expand aggregate capital and ouput. Productivity dispersion increases because larger firms are able to grow more rapidly than smaller and more financially constrained ones. In our model, instead, low rates tighten financial constraints of high-productivity firms that utilize intangible capital, and reduce their investment.

2 Empirical Motivation

In this section, we summarize the key stylized facts that motivate our model.

1 - Developed economies are significantly more reliant on intangible capital now than in the 1980s, and this technological shift has been linked to the simultaneous transition of the corporate sector from net debtor to net saver.

The developed world has experienced a technological change toward a stronger importance of information technology and of knowledge, human, and organizational capital, which has gradually reduced the reliance on physical capital (Brown, Fazzari and Petersen (2009), Corrado and Hulten (2010a), Falato, Kadyrzhanova, and Sim (2014)). In the United States, intangible capital as a share of total capital went from around 0.2 in the 1970s to 0.5 in the 2000s (Falato, Kadyrzhanova, and Sim (2014)). In parallel, there has been a shift in the net financial position of the nonfinancial corporate sector from a net borrowing position roughly before the year 2000 to a net saving position from 2000 onward (Armenter and Hnatkovska (2016), Quadrini (2016), Chen, Karabarbounis, and Neiman (2016), Zetlin-Jones and Shourideh (2016)).

The empirical evidence suggests that these two trends are related. The process of technological change has been linked to a lower availability of collateral for the corporate sector, which has lowered its debt capacity. Brown, Fazzari, and Petersen (2009) document that U.S. firms finance most of their research and development (R&D) expenditures out of retained earnings and equity issues, an observation in line with the conclusion in Hall (2002) that R&D-intensive firms feature much lower leverage, on average, than less R&D-intensive firms. Gatchev, Spindt, and Tarhan (2009) document that, in addition to R&D, marketing expenses and product development are also mostly financed out of retained earnings and equity. In contrast, tangible assets are mostly financed with debt.[7] The process of technological change has also been linked to an increase in the precautionary motives for cash accumulation to avoid future financial shortages (Bates, Kahle, and Stulz (2009), Falato, Kadyrzhanova, and Sim (2014), Falato and Sim (2014), Döttling and Perotti (2015), Begenau and Palazzo (2016)).[8]

Furthermore, firm-level empirical evidence suggests that the observed link between intangible intensity and high cash holdings is driven by financial frictions. Begenau and Palazzo (2016)

[7] Eisfeldt and Rampini (2009) report that a big share of machinery, equipment, buildings and other structures is financed with debt. Inventory investment and other tangible short-term assets attract substantial debt finance in the form of trade credit and bank credit lines (Petersen and Rajan (1997), Sufi (2009)). Finally, investment in commercial real estate is primarily financed with mortgage loans (Benmelech, Garmaise, and Moskowitz (2005)).

[8] Lack of access to debt financing of firms that rely on intangible capital could be compensated by easy access to equity financing. While easy access to equity financing would be consistent with the observed lower leverage of these firms, it would be harder to reconcile with the remarkable accumulation of cash holdings. A large body of evidence shows that external equity financing is significantly costly (Altinkilic and Hansen (2000), Gomes (2001), Belo, Lin and Yang (2016)).

introduce evidence showing that an important determinant of the increase in cash holdings of public firms is the increase in frequency of new firms that are very R&D intensive, and they suggest that these trends are consistent with a model in which cash holdings are driven by financial frictions of the R&D-intensive firms and costly equity financing. Similarly, Falato, Kadyrzhanova, and Sim (2014) show empirically that the relation between reliance on intangible capital and cash holdings is stronger among firms for which financing frictions are more severe.

2 - Productivity dispersion has increased in intangibles sectors during recent decades, while it has remained roughly constant in tangibles sectors.

Kehrig (2015) analyzes establishment-level manufacturing data from the U.S. census and documents a significant increasing trend in the dispersion of productivity across firms within sectors over the past 40 years. Earlier, we provided evidence that the rising intangible capital share is related to an increase in firm-level cash holdings to overcome external finance constraints. If the misallocation of resources caused by financial constraints is a factor contributing to the increase in productivity dispersion, we should expect the latter to be more pronounced in sectors with higher intensity of intangible capital.[9] In order to investigate the relation between the rise in intangibles and productivity dispersion, we use accounting data of 34,900 U.S. corporations obtained from Compustat, covering the period from 1980 to 2015, and containing 379,318 firm-year observations. We define intangible capital as the sum of knowledge capital and organizational capital. We consider two alternative productivity measures: labor productivity (y) and total factor productivity (TFP) (A) (see Appendix A for details). Our measure of misallocation, the productivity dispersion, is computed as the standard deviation of the difference between the logs of the productivity of firm i and the aggregate productivity of the industry s in which firm i operates.

[FIGURE 1 ABOUT HERE]

[FIGURE 2 ABOUT HERE]

Figures 1 and 2 plot the dispersion of labor productivity and TFP, respectively, in 2-digit SIC industries over time (normalized by the value in 1980). In both figures, the left graph shows average dispersion for all sectors, and it replicates the upward-sloping trend already documented

[9] It is important to note that this paper, like Kehrig (2015), analyzes the dynamics of the cross-sectional dispersion of productivity, not the dispersion of business growth rates. Davis et al. (2006) focus on the latter, and using both firm- and establishment-level data document a negative trend instead. These opposite trends are consistent with the findings of our model, in which a decline in the growth rate of expanding firms reduces reallocation of capital and increases steady state productivity differences.

by Kehrig (2015) using establishment-level data. In the right graph, the red dashed line shows the mean of the dispersion measure across industries (weighted by sales) in the top 50%, and the blue line in the bottom 50%, of the distribution of the industry-wide ratio of intangible capital to total capital (averaged across years).[10] Both figures show that the constant rise in the within-industry dispersion of productivity is driven by the sectors with higher average shares of intangible capital. This evidence is consistent with the hypothesis that intangible capital exacerbates misallocation problems caused by financial frictions. Appendix A discusses two additional exercises that provide robustness to these results.

3 Simple and Intuitive Explanation of the Mechanisms

We introduce in this section the simplest possible model that can describe our proposed mechanisms and deliver analytical results. Our main interest is studying how exogenous interest rate variations affect the allocation of capital and aggregate output depending on the degree of tangibility of capital. This framework is extended in Section 4 in a full-fledged general equilibrium setup that can be used for realistic quantitative analysis.

Consider an infinite-horizon, discrete-time model of the final goods producers of an economy. Firms use capital, which is in constant aggregate supply $\overline{K}$, to produce a homogeneous consumption good using a constant-returns-to-scale technology. There are two types of firms, *high-productivity* and *low-productivity.* Efficiency is determined by the share of $\overline{K}$ allocated to high-productivity firms. Here we present the aggregate steady state equilibrium conditions and introduce the details of the derivation of this simple model in Appendix B.

Aggregate output in the steady state is

$$Y = Y^p + Y^u + Y^e = zK + z^u \left(\overline{K} - K\right), \tag{1}$$

where z captures the productivity of high-productivity firms and $z^u < z$ captures the productivity of low-productivity firms.

Aggregate capital holdings K of the high-productivity firms, which are assumed to be finan-

[10] The sectors with high shares of intangible capital are: Chemicals and Allied Products; Industrial and Commercial Machinery and Computer Equipment; Electronic & Other Electrical Equipment & Components; Transportation Equipment; Measuring, Photographic, Medical, & Optical Goods, & Clocks; Miscellaneous Manufacturing Industries; Wholesale Trade - Durable Goods; Home Furniture, Furnishings and Equipment Stores; Miscellaneous Retail Business Services; and Engineering, Accounting, Research, and Management Services.

The sectors with low shares of intangible capital are: Oil and Gas Extraction; Food and Kindred Products; Paper and Allied Products; Rubber and Miscellaneous Plastic Products; Stone, Clay, Glass, and Concrete Products; Primary Metal Industries; Fabricated Metal Products; Wholesale Trade - Nondurable Goods; General Merchandise Stores; Food Stores; Apparel and Accessory Stores; and Eating and Drinking Places.

cially constrained, are

$$K = \frac{A^e(1+r) + Y^e}{q\left(1 - \frac{\theta}{1+r}\right)}, \tag{2}$$

where

$$q = \frac{z^u}{r+\xi} \tag{3}$$

is the price of capital. Low-productivity firms, which are financially unconstrained, have aggregate capital holdings of $\overline{K} - K$, are the marginal buyers of capital, and price it according to their marginal productivity. The parameter ξ captures a pricing wedge (such as a risk premium).[11]

The numerator of (2) captures the total funds available to high-productivity firms to invest and is assumed to be positive in equilibrium. It is equal to the aggregate net savings or liabilities of the high-productivity firms $A^e(1+r)$, including their return r this period, plus output generated this period, Y^e.[12] The denominator of (2) captures the downpayment necessary to purchase one unit of capital. High-productivity firms can borrow using one-period debt up to a fraction θ ($0 \leq \theta \leq 1$) of the value of capital next period and have to pay q per unit.

We capture reliance on intangible capital by two features: positive A^e and low θ. Intangible capital is poor collateral (low θ), so firms that rely on intangible capital do not have a large borrowing capacity and instead accumulate retained earnings and are more likely to be net savers ($A^e > 0$). Tangible capital has a high collateral value (high θ), so firms that rely on tangible capital are able to borrow more and are more likely to be net borrowers ($A^e < 0$). Importantly, this negative relationship between the tangibility of capital θ and the financial wealth of high-productivity firms is consistent with the empirical evidence, which we discussed in the previous section, and it arises endogenously in the full model derived in Section 4.

We now describe the four mechanisms through which interest rates affect the allocation of capital and aggregate output. Inspecting dK/dr, which can be expressed as

$$\frac{dK}{dr} = \frac{A^e}{q\left(1 - \frac{\theta}{1+r}\right)} + \frac{A^e(1+r) + Y^e}{q\left(1 - \frac{\theta}{1+r}\right)} \left[\frac{1}{r+\xi} - \frac{\theta}{(1+r-\theta)(1+r)}\right], \tag{4}$$

we can identify these four channels. If $A^e > 0$, an exogenous increase in r benefits capital allocation by increasing available savings to high-productivity firms to invest. That is the *savings channel*. If $A^e < 0$, an increase in r hurts capital allocation by increasing the debt burden of high-productivity firms and decreasing their available funds. That is the *debt overhang channel*. The first term in (4) is positive if $A^e > 0$, capturing the savings channel, and is negative

[11] In the full general equilibrium model of Section 4, a positive wedge ξ arises endogenously because of capital depreciation and because of decreasing returns to scale in the low-productivity firms' production function.

[12] Equation (2) is derived in Appendix B from the equilibrium of a model in which overlapping generations of firms live for two periods, and receive an endowment of $A^e(1+r) + Y^e$ when they are born.

if $A^e < 0$, capturing the debt overhang channel. The *capital purchase price channel* is the mechanism through which increases in r benefit capital reallocation by decreasing q and making capital cheaper. The first term inside the square brackets in (4) captures this channel and is always positive. Finally, the *collateral value channel* is the channel through which increases in r hurt capital reallocation by decreasing the value of firms' collateral (the term $\theta/(1+r)$) and tightening the borrowing constraint. The second term inside the brackets represents this channel and is always negative.

How do these four channels depend on the intensity of intangible capital? In other words, how does the tangibility of capital matter for the effect of variations in r on the efficiency of this economy? For clarity of exposition, assume that a tangibles-intensive economy is one in which $A^e < 0$ and $\theta > 0$, and that an intangibles-intensive economy is one in which $A^e > 0$ and $\theta = 0$. Then

$$sign\left[\frac{dK}{dr}(\text{tangible})\right] = \underbrace{\frac{A^e}{q\left(1-\frac{\theta}{1+r}\right)}}_{<0} + \frac{A^e(1+r)+Y^e}{q\left(1-\frac{\theta}{1+r}\right)}\left[\underbrace{\frac{1}{r+\xi}}_{>0} - \underbrace{\frac{\theta}{(1+r-\theta)(1+r)}}_{<0}\right] < 0 \quad \text{if (5) is met,}$$

and

$$sign\left[\frac{dK}{dr}(\text{intangible})\right] = \underbrace{\frac{A^e}{q}}_{>0} + \frac{A^e(1+r)+Y^e}{q}\left[\underbrace{\frac{1}{r+\xi}}_{>0}\right] > 0 \quad \text{always.}$$

The derivative $\frac{dK}{dr}$ is positive in an intangibles economy, meaning that a reduction in r is unambiguously contractionary. It is instead most likely expansionary in a tangibles economy, particularly if the responsiveness of q to r is limited (ξ is high) and the borrowing capacity is large (θ is high). More specifically, a decrease in r is unambiguously expansionary in a tangibles economy if the following condition is satisfied:

$$\theta > \frac{1+2r+r^2}{1+2r+\xi}. \tag{5}$$

Taken together, these results suggest that the degree of tangibility of capital in an economy matters importantly for how exogenous variations in the interest rate affect capital allocation and output and describe four important channels through which these effects occur. Crucially, they show that falling interest rates can become contractionary in an economy that relies on intangible capital.

Section 4 provides a full-fledged model in which we endogenize firms' financing constraints, saving and borrowing decisions, and investment, households' consumption and savings, and the interest rate, wages, and the prices of tangible and intangible capital.

The Investment Demand Curve

To provide a deeper understanding of how the features of the equilibrium of this economy change as a result of a transition from an economy reliant on tangible capital to one in which intangible capital acquires a larger importance, we represent the equilibrium in the credit market in Figure 3. The main objective is to provide an empirically relevant assessment of the slope of the investment demand curve for different values of θ. To do so, we calibrate the parameters at the annual frequency to be broadly consistent with observed moments of U.S. data. We postpone a more thorough calibration to the full model developed in Section 4. We study a range of the real interest rate between $r = 6\%$ and $r = 0\%$, consistent with the observed evolution of real rates between the early 1980s and the present. We normalize the productivity of low-productivity firms to $z^u = 1$ and the output endowment to $Y^e = 1$. We consider a tangibles economy to feature a pledgeability parameter of capital θ equal to 0.9 and a net borrowing position equivalent to 20% of output ($A^e = -0.2$). We consider an intangibles economy to feature a pledgeability parameter of capital θ equal to 0.4 and a net saving position equivalent to 20% of output ($A^e = 0.2$). The interest rate wedge ξ is set at 20% and is meant to capture a combination of factors such as risk premia, default premia, and capital depreciation.

[FIGURE 3 ABOUT HERE]

In Figure 3, the upward-sloping savings curve captures the combination of the (unmodeled) net savings of the household sector. Higher interest rates induce households to save more, under the empirically realistic assumption that the substitution effect dominates the income effect for them. The demand for capital by investing firms is equal to the amount borrowed by them plus (minus) the savings (debt) they carry over from the previous period. This curve can be upward or downward sloping depending on the relevance of intangible capital in the production function. In an economy where capital is interpreted to be of a tangible nature ($\theta = 0.9$ and $A^e = -0.2$), an increase in aggregate savings has the effect of lowering interest rates and increasing capital purchases from expanding firms. When there is a shift outward in the savings curve, the economy moves from point A to point B. The collateral value channel and the debt overhang channel dominate. As a result, a larger share of the capital stock is in the hands of high-productivity firms, which improves the allocation of resources and increases aggregate productivity and output. Instead, in an economy where capital is interpreted to be of an intangible nature ($\theta = 0.4$ and $A^e = 0.2$), the demand for capital curve is upward sloping due to the strength of the capital price and savings channels. As interest rates rise, firms demand more capital because they have larger savings and the price of capital is lower. In this case, an outward shift in the savings schedule generates a decrease in the equilibrium capital purchases of high-productivity firms, because the decrease in interest rates the shift in savings generates hurts the reallocation of capital toward high-productivity firms. The economy moves from point A to point C, worsening the allocation of resources and reducing aggregate productivity and output.

4 General Equilibrium Model

We introduce an infinite-horizon, discrete-time economy populated by an intermediate sector that produces capital; by a final good sector in which firms use labor and capital to produce consumption goods; and by households, which provide labor and own both sectors. There are several important extensions to the simple model analyzed in Section 3, and we describe the main ones here. We introduce an intermediate capital producing sector that allows us to endogenize in equilibrium the aggregate stock of capital. In the final good sector, we model explicitly tangible and intangible capital, and we derive endogenously the accumulation of financial and physical assets of firms that live multiple periods. The household sector is modeled as a life-cycle framework, which allows us to endogenize the interest rate and study how it is affected by demographic changes and other demand-side factors.

4.1 The Capital-Producing Sector

A representative firm in this sector chooses investment in tangible and intangible capital, respectively I_t^T and I_t^I, in order to maximize profits:

$$\max_{I^J} q_{J,t} I_t^J - b_t^J \left(\frac{I_t^J}{\varphi} \right)^{\varphi}, \tag{6}$$

where $\varphi > 1, b_t^J > 0$, and $q_{J,t}$ is the price of the type of capital $J \in \{T, I\}$. We allow for b_t^T and b_t^I to be time varying in order to capture trends in the evolution of the relative price of capital. The first order condition yields $I_t^J = \varphi \left(\frac{q_{J,t}}{b_t^J} \right)^{\frac{1}{\varphi-1}}$, and profits are $\pi_t^J = \frac{q_{J,t}^{\frac{\varphi}{\varphi-1}}}{{b_t^J}^{\frac{1}{\varphi-1}}} (\varphi - 1)$.

At the beginning of period t, total capital available is $\overline{K}_t^T$ and $\overline{K}_t^I$. New capital I_t^T and I_t^I is produced and sold in period t so that the aggregate dividends generated by the capital-producing sectors are

$$D_t^k = \pi_t^T + \pi_t^I. \tag{7}$$

During period t, tangible capital and intangible capital depreciate at the rates $0 \leq \delta < 1$.

And the law of motion of aggregate capital is

$$\overline{K}_{t+1}^{J} = I_t^J + (1-\delta)\overline{K}_t^J,$$

with $J \in \{T, I\}$.

4.2 Final Good Sector

There are two types of final-good-producing firms: high-productivity and low-productivity.

4.2.1 The High-Productivity Firms

There is a continuum of mass 1 of high-productivity firms.

Technology and financing opportunities

High-productivity firms produce a final good using a constant-returns-to-scale production function that is Cobb-Douglas in labor and capital. The firms use two different types of complementary capital, tangible and intangible. For simplicity, we assume that they are perfect complements. The production function takes the following form:

$$y_t^p = z_t(\mu)\, n_t^{(1-\alpha)} \left[\min\left(\frac{k_{T,t}}{1-\mu}, \frac{k_{I,t}}{\mu}\right)\right]^{\alpha}, \tag{8}$$

where $0 < \alpha \leq 1$ and $0 < \mu < 1$. The terms $k_{T,t}$ and $k_{I,t}$ represent tangible and intangible capital installed in period $t-1$ that produce output in period t, and n_t is labor. The Leontief production structure implies that, in equilibrium, intangible capital as a share of total capital in high-productivity firms is equal to μ. The productivity term $z_t(\mu)$ is increasing in the share of intangible capital and captures the higher productivity of more intangibles-intensive technologies. We drop from now on reference to the dependence of z_t on μ for ease of notation and defer discussion of their relationship to the calibration section.

The budget constraint for high-productivity firms is given by the following dividend equation:

$$d_t = y_t^p + (1+r_t)a_{f,t} - a_{f,t+1} - q_{T,t}(k_{T,t+1} - (1-\delta)k_{T,t}) - q_{I,t}(k_{I,t+1} - (1-\delta)k_{I,t}) - w_t n_t, \tag{9}$$

where r_t is the interest rate paid or received in date t; $q_{T,t}$, and $q_{I,t}$ are the prices of tangible and intangible capital, respectively; and w_t is the wage. The term $a_{f,t} > 0$ indicates that the firm is a net saver, and $a_{f,t} < 0$ indicates that the firm is a net borrower.

High-productivity firms are subject to frictions in their access to external finance. They are unable to issue equity, which means that dividends are subject to a non-negativity constraint:

$$d_t \geq 0. \tag{10}$$

They can issue one-period riskless debt, subject to the constraint that they can pledge, as collateral, the fractions θ^T and θ^I of tangible capital and intangible capital, respectively. This constraint translates into the following inequality:

$$a_{f,t+1} \geq -\frac{\theta^T q_{T,t+1} k_{T,t+1} + \theta^I q_{I,t+1} k_{I,t+1}}{1 + r_{t+1}}, \tag{11}$$

where $0 < \theta^T \leq 1$ and $0 < \theta^I < \theta^T$. In reality, firms finance part of their investment with equity issues, which could be captured in the model by assuming that dividends can be negative up to a fraction of the firm's value. However, rather than complicating the model further, in the calibration section we consider equity financing by assuming larger values of θ^T and θ^I than are normally assumed in the literature. This assumption is without loss of generality, because assuming instead negative dividends proportional to the firm's value and lower collateral values of capital would not change our qualitative and quantitative results.

From the Leontief structure of the production function, it follows that $k_{T,t} = \frac{1-\mu}{\mu} k_{I,t}$. Therefore, from now on, we use this result to express all equations as a function of intangible capital only. At the beginning of each period, both types of capital are predetermined and in their optimal ratio $k_{T,t} = \frac{1-\mu}{\mu} k_{I,t}$; therefore, the production function can be written as

$$y_t^p = z_t n_t^{(1-\alpha)} \left(\frac{k_{I,t}}{\mu} \right)^{\alpha}. \tag{12}$$

After producing, the firm's technology becomes obsolete with probability ψ. In this case, the firm liquidates all of its capital, pays out as dividends all of its savings, including the liquidation value of capital, and exits. We follow Kiyotaki and Moore (2012) and assume that high-productivity firms can only invest each period with probability η. This assumption, in addition to capturing the realistic feature that firms' investment is lumpy (Caballero (1999)), is meant to allow firms to have the opportunity to accumulate significant amounts of liquid savings, in line with the empirical evidence.[13]

Optimization

Firms choose their investment and savings in order to maximize the net present value of their dividends. Let λ_t and ϑ_t be the Lagrange multipliers of constraints (10) and (11), respec-

[13] In Section 6 we interpret ψ and η as shocks that generate creative destruction: η is the arrival probability of an investment opportunity to produce a new product, and ψ is the probability that the firm's technology becomes obsolete because a competing firm enters the market and produces an improved version of its product.

tively. We define the value function conditional on having an investment opportunity, denoted $V^+(k_{I,t}, a_{f,t})$, as follows:

$$V_t^+(k_{I,t}, a_{f,t}) \quad = \quad \max_{n_t, dt, a_{f,t+1}, k_{I,t+1}} (1+\lambda_t)d_t + \vartheta_t \left(a_{f,t+1} + \frac{\theta^T q_{T,t+1} k_{T,t+1} + \theta^I q_{I,t+1} k_{I,t+1}}{1+r_{t+1}} \right) + \frac{1}{1+r_{t+1}} \left[(1-\psi) V_{t+1}(k_{I,t+1}, a_{f,t+1}) + \psi d_{t+1}^{exit} \right], \tag{13}$$

where

$$d_t^{exit} = y_t^p + (1+r_t)a_{f,t} + (1-\delta) q_{T,t} \frac{1-\mu}{\mu} k_{I,t} + (1-\delta) q_{I,t} k_{I,t} - w_t, \tag{14}$$

and $V_{t+1}(k_{I,t+1}, a_{f,t+1})$ is the value function conditional on continuation but before the investment shock is realized:

$$V_{t+1}(k_{I,t+1}, a_{f,t+1}) = \eta V^+(k_{I,t+1}, a_{f,t+1}) + (1-\eta) V^-(k_{I,t+1}, a_{f,t+1}). \tag{15}$$

The value function of a non-investing firm, denoted $V^-(k_{I,t}, a_{f,t})$, is identical to $V^+(k_{I,t}, a_{f,t})$ but does not offer the opportunity to choose $k_{I,t+1}$.

The firm solves (13) (or its non-investing counterpart) subject to (9), (10), and (11). We next provide a characterization of high-productivity firms' optimal choice under the assumption that they are permanently financially constrained. We claim – and check later in our calibrated simulations – that, in equilibrium, the marginal return on capital for high-productivity firms is always higher than their user cost:

$$\frac{\partial y_{t+1}^p}{\partial k_{I,t+1}} = \frac{\alpha z_{t+1} n_{t+1}^{(1-\alpha)}}{\mu} \left(\frac{k_{I,t+1}}{\mu} \right)^{\alpha-1} > \left(q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} \right) - \frac{(1-\delta)\left(q_{T,t+1} \frac{1-\mu}{\mu} + q_{I,t+1} \right)}{1+r_{t+1}}. \tag{16}$$

The implication of assumption (16) for investing firms is that the borrowing constraint (11) is binding, and that firms choose not to pay dividends, so the equity constraint (10) is also binding. Making $d_t = 0$ in budget constraint (9), using (9) to substitute for $a_{f,t+1}$ in (11), assuming (11) is binding, and solving for $k_{I,t+1}$, we obtain their level of investment:

$$(k_{I,t+1} \mid \text{invest}) = \frac{y_t^p - w_t n_t + (1+r_t)a_{f,t} + (1-\delta)\left(q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} \right) k_{I,t}}{q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} - \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}} \right)}. \tag{17}$$

The right-hand side of equation (17) is the maximum feasible investment in intangible capital for a firm. The numerator is the total wealth available to invest. The denominator captures the downpayment necessary to purchase one unit of $k_{I,t+1}$ and $\frac{1-\mu}{\mu}$ units of $k_{T,t+1}$. The term

$q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}$ represents the total cost necessary to purchase these amounts of both types of capital, and the term $\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}$ is the amount that can be financed by borrowing.

Investing firms in equilibrium borrow as much as possible, and

$$(a_{f,t+1} \mid \text{invest}) = -\left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right) k_{I,t+1} < 0. \tag{18}$$

The implication of assumption (16) for non-investing firms is that they will not sell any of their capital, and, for these firms, the law of motion of capital is

$$(k_{I,t+1} \mid \text{not invest}) = (1-\delta)k_{I,t}. \tag{19}$$

Non-investing firms always retain all earnings and select $d_t = 0$ because they face a positive probability of being financially constrained in the future, and hence the value of cash inside the firm is always higher than its opportunity cost (see Appendix C for a formal proof). Substituting $d_t = 0$ and (19) in (9):

$$(a_{f,t+1} \mid \text{not invest}) = y_t^p + (1+r_t)a_{f,t} - w_t n_t. \tag{20}$$

Equations (18) and (20) determine the wealth dynamics of firms. A firm that invested in period $t-1$ but is not investing in period t has debt equal to $-a_{f,t} = \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right) k_{I,t+1}$. It uses current profits $y_t^p - w_t n_t$ to pay the interest rate on debt $-r_t a_{f,t}$ and to reduce the debt itself. As long as the firm is not investing, the debt $-a_{f,t}$ decreases until the firm becomes a net saver and has $a_{f,t} > 0$. At this point, wealth accumulation is driven both by profits $y_t^p - w_t n_t$ and by interest on savings $r_t a_{f,t}$, until the firm has an investment opportunity and its accumulated wealth $(1+r_t)a_{f,t}$ is used to purchase capital (see equation (17)). This discussion clarifies that a lower interest rate r_t helps the non-investing firm repay existing debt (the debt hangover channel), but it slows down the accumulation of savings after the firm has repaid the debt (the savings channel).

Finally, the first order condition for n_t, for both investing and non-investing firms, implies that given the wage w_t and its predetermined capital $k_{I,t}$, a firm will choose the profit-maximizing level of labor, which determines the optimal capital-labor ratio:

$$\frac{k_{I,t}}{n_t} = \mu \left[\frac{w_t}{(1-\alpha)\, z_t}\right]^{\frac{1}{\alpha}}. \tag{21}$$

4.2.2 The Low-Productivity Firms

There is a mass 1 of identical low-productivity firms that have access to two production functions. Each production function combines capital $k_{uJ,t}$ with specialized labor $n_{uJ,t}$ using a constant-returns-to-scale technology, where $J = \{I, T\}$ captures the tangibility of the capital used. The total amount $y_t^{\mathbf{u}}$ of the homogeneous final good produced is then

$$y_t^u = z_t^{u,I} n_{uI,t}^{1-\alpha} k_{uI,t}^{\alpha} + z_t^{u,T} n_{uT,t}^{1-\alpha} k_{uT,t}^{\alpha},$$

where α determine the capital share. We do not introduce the assumption of perfect complementarity between tangible and intangible capital (which we do introduce for the high-productivity firms) to gain tractability in the pricing of capital, as will become clear in the next section. This is without loss of generality.

This sector is assumed to be able to finance capital with equity from the household sector and to pay out all profits as dividends d_t^u to households every period:

$$d_t^u = y_t^u - w_t^{uI} n_{uI,t} - w_t^{uT} n_{uT,t} - q_{I,t}\left(k_{I,t+1}^u - (1-\delta)k_{I,t}^u\right) - q_{T,t}\left(k_{T,t+1}^u - (1-\delta)k_{T,t}^u\right). \quad (22)$$

In addition, the low-productivity firms sector is able to remunerate households for their labor services $(w_t^{uI} n_{uI,t} + w_t^{uT} n_{uT,t})$.

The first order conditions for the two types of labor imply that given wages w_t^{uI} and $w_t^{uT}$and a firm's predetermined capital stocks $k_{I,t}^u$ and $k_{T,t}^u$, a low-productivity firm will choose the profit-maximizing level of each type of labor, which determines the optimal capital-labor ratio:

$$\frac{k_{uJ,t}}{n_{uJ,t}} = \left[\frac{w_t^{uJ}}{(1-\alpha)\, z_t^{u,J}}\right]^{\frac{1}{\alpha}}. \quad (23)$$

Given that low-productivity firms are financially unconstrained, and provided that their marginal return on each of the two types of capital is lower than for high-productivity firms, low-productivity firms are willing to absorb all of the capital not demanded by high-productivity firms, at a price equal to their marginal return on capital.

4.2.3 Aggregation of the Firm Sector and Pricing of Assets

We assume (see Section 4.3) that the aggregate supply of all types of labor is normalized to $N = N_{uI} = N_{uT} = 1$. Since all high-productivity firms produce at the optimal capital-labor ratio determined by equation (21), and the production function is constant returns to scale , we can aggregate production across firms to obtain

$$Y_t^p = z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha}. \tag{24}$$

The wage is determined in competitive markets by the marginal return of labor:

$$w_t = (1-\alpha)\, z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha}. \tag{25}$$

And aggregate wealth W_t of the high-productivity firms at the beginning of period t is

$$W_t \equiv Y_t^p - w_t + (1+r_t)A_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right) K_{I,t}. \tag{26}$$

Aggregate capital is determined as follows. A fraction $(1-\psi)$ of high-productivity firms continue activity, and a fraction η of those have an investment opportunity. They have a fraction $(1-\psi)\,\eta$ of total wealth W_t, which they use to buy the amount of capital given by equation (17). A fraction ψ of high-productivity firms exit, and are replaced by an equal number of firms with an initial endowment of W_0 and no capital. A fraction η of new entrants invest. Therefore, we define total intangible capital in the hands of investing agents at the end of period t, expressed in aggregate terms, as $\eta K_{I,t+1}^{INV}$, where $K_{I,t+1}^{INV}$ is

$$K_{I,t+1}^{INV} = \frac{(1-\psi)W_t + \psi W_0}{\left(q_{T,t} - \theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\right)\frac{1-\mu}{\mu} + q_{I,t} - \theta^I \frac{q_{I,t+1}^I}{1+r_{t+1}}}. \tag{27}$$

The $(1-\eta)$ fraction of surviving firms that do not have an investment opportunity continue to hold their depreciated capital. Therefore, aggregate capital for the next period is equal to

$$K_{I,t+1} = \eta K_{I,t+1}^{INV} + (1-\delta)\,(1-\psi)\,(1-\eta)\, K_{I,t} \tag{28}$$

and

$$K_{T,t+1} = \frac{1-\mu}{\mu} K_{I,t+1}. \tag{29}$$

Furthermore, we can aggregate the output of low-productivity firms, substituting labor supply $N_{uI} = N_{uT} = 1$, and obtain

$$Y_t^u = z_t^{u,I}\left(\overline{K}^I - K_{I,t}\right)^{\alpha} + z_t^{u,T}\left(\overline{K}^T - K_{T,t}\right)^{\alpha}, \tag{30}$$

$$w_t^{uJ} = (1-\alpha)\, z_t^{u,J}\left(\overline{K}^J - K_{J,t}\right)^{\alpha}, \tag{31}$$

with $J = \{I, T\}$.

The marginal return of capital in the high-productivity firms is as follows. In order obtain a marginal increase $\frac{\partial Y_t^p}{\partial K_{I,t}} = \frac{\alpha}{\mu} z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha-1}$, these firms purchase one unit of intangible capital and $\frac{1-\mu}{\mu}$ units of tangible capital. The equilibrium described earlier requires that the high-productivity firms have the highest return on capital, or

$$\frac{\alpha}{\mu} z_t \left(\frac{K_{I,t+1}}{\mu} \right)^{\alpha-1} > z_t^{u,I} \alpha \left(\overline{K}^I - K_{I,t} \right)^{\alpha-1} + \frac{1-\mu}{\mu} z_t^{u,T} \alpha \left(\overline{K}^T - K_{T,t} \right)^{\alpha-1}, \tag{32}$$

where the right-hand side of this inequality captures the marginal return of one unit of tangible capital and $\frac{1-\mu}{\mu}$ units of intangible capital in the low-productivity firms.

If condition (32) is satisfied, then it follows immediately that the prices of capital are

$$q_{I,t} = z_I^{u,I} \alpha \left(\overline{K}^I - K_{I,t} \right)^{\alpha-\mathbf{1}} + \frac{1-\delta}{1+r_{t+1}} q_{I,t+1}, \tag{33}$$

and

$$q_{T,t} = z_t^{u,T} \alpha \left(\overline{K}^T - K_{T,t} \right)^{\alpha-\mathbf{1}} + \frac{1-\delta}{1+r_{t+1}} q_{T,t+1}. \tag{34}$$

If we substitute (33) and (34) into (32), it follows that

$$\frac{\alpha}{\mu} z_t \left(\frac{K_{I,t+1}}{\mu} \right)^{\alpha-1} > q_{I,t} - \frac{1-\delta}{1+r_{t+1}} q_{I,t+1} + \frac{1-\mu}{\mu} \left(q_{T,t} - \frac{1-\delta}{1+r_{t+1}} q_{T,t+1} \right), \tag{35}$$

which implies that the claim (16) is correct. Aggregate financial assets of the high-productivity firms ($A_{f,t+1}$) are equal to the assets saved from the previous period by continuing firms, $(1-\psi)\left((1+r_t)A_{f,t}\right)$, plus their current retained earnings, $(1-\psi)\left(Y_t^p - w_t\right)$, plus the endowments of new firms (ψW_0) minus total investment, $\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)\left(K_{I,t+1} - (1-\delta)(1-\psi)K_{I,t}\right)$:

$$\begin{aligned} A_{f,t+1} \quad = \quad & (1-\psi)\left(Y_t^p - w_t + (1+r_t)A_{f,t}\right) + \psi W_0 \\ & - \left(q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} \right) \left(K_{I,t+1} - (1-\delta)(1-\psi) K_{I,t} \right). \end{aligned} \tag{36}$$

Finally, total dividends paid out by exiting high-productivity firms to households are equal to

$$D_t^p = \psi \left(Y_t^p - w_t + (1+r_t)A_{f,t} + \left(q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} \right) K_{I,t} \right) - \psi W_0, \tag{37}$$

and the dividends paid by the low-productivity firms are:

$$D_t^u = Y_t^u - w_t^{uI} - w_t^{uT} - q_{I,t} \left[\left(\overline{K}^I - K_{I,t+1} \right) - \left(\overline{K}^I - K_{I,t} \right) \right] - q_{T,t} \left[\left(\overline{K}^T - K_{T,t+1} \right) - \left(\overline{K}^T - K_{T,t} \right) \right].$$

4.3 Households

We consider a life-cycle model with two types of households, young and old–with measures H^y and H^o, respectively–whose sum is normalized to 1. Young households supply three types of differentiated labor: high-productivity firm labor (in exchange for wage w_t), low-productivity intangible technology labor (in exchange for wage w_t^{uI}), and low-productivity tangible technology labor (in exchange for wage w_t^{uT}). There is an inelastic aggregate supply of one unit of each type of labor. Young households receive a fraction γ of the aggregate dividends. Households remain young for N periods and become old after $N+1$ periods, so that there is a constant fraction $\phi = \frac{1}{N}$ of young households for every age between 1 and N, and, every period, a measure ϕH^y of households becomes old. Old households cannot work, receive a fraction $(1-\gamma)$ of aggregate dividends, and die with probability ϱ. The measure of old households H^o is determined as follows:

$$H^o = (1-\varrho)H^o + \phi H^y. \tag{38}$$

At the same time, the measure of young households is

$$H^y = (1-\phi)H^y + N^y, \tag{39}$$

where N^y is the constant measure of newborn households. From the assumption that $H_t^o + H_t^y = 1$, it follows that $N^y = \frac{\phi\varrho}{\phi+\varrho}$, $H_t^o = \frac{\phi}{\phi+\varrho}$, and $H_t^y = \frac{\varrho}{\phi+\varrho}$.

We follow Blanchard (1985) and Yaari (1965) in assuming that households participate in a life insurance scheme when old. For the detailed solution of the households' maximization problem, see Appendix D.

5 Steady State

5.1 Equilibrium

We consider a steady state equilibrium and drop reference to the time subscript t. Total output of the high-productivity and low-productivity firms is, respectively,

$$Y^p = z\left(\frac{K_I}{\mu}\right)^{\alpha} \tag{40}$$

and

$$Y^u = z^{u,I}\left(\overline{K}^I - K_I\right)^{\alpha} + z^{u,T}\left(\overline{K}^T - K_T\right)^{\alpha}. \tag{41}$$

Dividends d are given by

$$d = D^p + D^u + D^k, \tag{42}$$

where

$$D^u = Y^u - w_t^{uI} - w^{uT} - q_I\delta\left(\overline{K}^I - K_I\right) - q_T\delta\left(\overline{K}^T - K_T\right),$$

$$D^p = \psi\left(\alpha z\left(\frac{K_I}{\mu}\right)^\alpha + (1+r)A_f + \left(q_T\frac{1-\mu}{\mu} + q_I\right)K_I\right) - \psi W_0,$$

and

$$D^k = \frac{q_{T,t}^{\frac{\varphi}{\varphi-1}}}{b^{T\frac{1}{\varphi-1}}}(\varphi-1) + \frac{q_{I,t}^{\frac{\varphi}{\varphi-1}}}{b^{I\frac{1}{\varphi-1}}}(\varphi-1).$$

Aggregate cash holdings of the high-productivity firms in the steady state can be obtained by combining (36), (24), and (25) to obtain

$$A_f = \frac{(1-\psi)\,\alpha z_t\left(\frac{K_I}{\mu}\right)^\alpha + \psi W_0 - \left(q_T\frac{1-\mu}{\mu} + q_I\right)[\psi + \boldsymbol{\delta}(1-\psi)]\,K_I}{[1-(1-\psi)(1+r)]}. \tag{43}$$

Aggregate borrowing is equal to aggregate savings, or

$$A_f = B, \tag{44}$$

where B is aggregate household borrowing, which we derive in detail in Appendix D. By Walras' Law, the aggregate resource constraint is satisfied. In order to determine the aggregate capital of the high-productivity firms, equation (28) in the steady state is equal to

$$K_I = \eta\frac{(1-\psi)W + \psi W_0}{\left[q_T\left(1-\frac{\theta^T}{1+r}\right)\frac{1-\mu}{\mu} + q_I\left(1-\frac{\theta^I}{1+r}\right)\right][1-(1-\boldsymbol{\delta})(1-\psi)(1-\eta)]}, \tag{45}$$

where W is defined using equation (26) in steady state:

$$W \equiv \alpha z_t\left(\frac{K_I}{\mu}\right)^\alpha + (1+r)A_f + (1-\boldsymbol{\delta})\left(q_T\frac{1-\mu}{\mu} + q_I\right)K_I. \tag{46}$$

We can also express (45) as

$$K_I = \frac{\eta(1-\psi)\left(\alpha z_t\left(\frac{K_I}{\mu}\right)^\alpha + (1+r)A_f\right) + \eta\psi W_0}{\left[q_T\left(1-\frac{\theta^T}{1+r}\right)\frac{1-\mu}{\mu} + q_I\left(1-\frac{\theta^I}{1+r}\right)\right][\delta + \psi(1-\delta)] - \left(q_T\frac{\theta^T}{1+r}\frac{1-\mu}{\mu} + q_I\frac{\theta^I}{1+r}\right)\eta(1-\delta)(1-\psi)}, \tag{47}$$

which has an intuitive explanation. The numerator is the aggregate amount of liquid resources of investing firms. The denominator is the downpayment necessary to support one unit of capital in the steady state. It requires the replacement of the depreciated capital and the lost capital of exiting firms (a fraction $\delta + \psi(1-\boldsymbol{\delta})$) and can benefit from using existing capital held by the investing firms as collateral (fraction $\eta(1-\delta)(1-\psi)$).

Finally, the prices of capital are determined by recursively iterating forward equations (33) and (34):

$$q_I = \frac{1}{r+\delta} z^{u,I} \alpha \left(\overline{K}^I - K_I \right)^{\alpha - \mathbf{1}} \tag{48}$$

and

$$q_T = \frac{1}{r+\delta} z^{u,T} \alpha \left(\overline{K}^T - K_T \right)^{\alpha - \mathbf{1}}, \tag{49}$$

where aggregate capital and investment are given by

$$\overline{K}^J = \frac{I^J}{\boldsymbol{\delta}} \tag{50}$$

and

$$I^J = \varphi \left(\frac{q_J}{b^J} \right)^{\frac{1}{\varphi - 1}}, \tag{51}$$

respectively, for $J \in \{I, T\}$.

The steady state values of W, A_f, B, K_I, q_I, q_T, and r are jointly determined by equations (43), (44), (45), (46), (48), (49), and (93).

5.2 Discussion

If we assume for simplicity that $q_T = q_I = q$, the collateral value of one unit of capital is $\frac{q}{1+r} \frac{1}{\mu} \left[(1-\mu) \theta^T + \mu \theta^I \right]$. Since $\theta^T > \theta^I$, a technology that relies more on tangible capital (lower μ) places a higher weight on the collateral value of tangible capital θ^T, thus increasing the overall collateral value of the firms' capital. Such an economy has a lower downpayment in the denominator of (47) and more capital K_I for a given total wealth in the numerator.

Equation (43) determines financial wealth A_f, which is equal to the net earnings of the high-productivity firms, in the numerator, multiplied by a multiplicative factor $\frac{1}{1-(1-\psi)(1+r)}$, which measures the future value of one unit of wealth saved today by these firms. The net earnings are the endowment of the new firms ψW_0 plus the net earnings of continuing firms. The term $(1-\psi) \alpha z_t \left(\frac{K_I}{\mu} \right)^{\alpha}$ is retained earnings, net of wage payments, and is concave in K_I. The term $\left(q_T \frac{1-\mu}{\mu} + q_I \right) [\psi + \boldsymbol{\delta}(1-\psi)] K_I$ is total expenditures to replace the depreciated capital of continuing firms $\delta(1-\psi) K_I$, and the capital liquidated by exiting firms ψK_I, and is linear in K_I. A high average collateral value of capital in a tangible economy increases K_I and makes it likely that the sum of the two last terms is negative, and since ψW_0 is very small, it also makes A_f negative: the high-productivity firms are, on aggregate, net borrowers. Conversely, in an intangible (high μ) economy, A_f is likely to be positive.

The previous discussion clarifies that the exogenous assumptions made in the simple model in Section 3 are endogenously derived in the full general equilibrium model. Moreover, even though a change in the interest rate affects aggregate capital K_I in (47) through the same four channels identified in the simple model in Section 3, it is important to emphasize that the endogeneity of financial assets amplifies the strength of the savings channel. When A_f is positive, a reduction in the interest rate reduces investment both through a reduction in the return on savings rA_f and through the multiplicative factor $\frac{1}{1-(1-\psi)(1+r)}$.

6 Calibration

[TABLE 1 ABOUT HERE]

For the purpose of evaluating the qualitative and quantitative importance of the channels explained earlier for the real economy, we calibrate the model to be broadly in line with recent U.S. data. We simulate the evolution of the economy from 1980 to the present as a sequence of steady states, and use this simulated time series to calculate the model-based moments. Our benchmark calibration is illustrated in Table 1. Our calibration strategy is twofold. We set most of our parameters to match key empirical moments of aggregate variables from 1980 to the present. A subset of parameters – those which are key to the mechanisms introduced in our model – are the basis of our comparative statics exercises and are set to change according to their observed variation or the observed variation of some direct moment they influence during the 1980-present period. In this latter group we include the share of intangibles (μ), the cost of producing capital (driven by parameters b^T and b^I), the rate of time preference of households (β), and the longevity of households (driven by ϱ).

We start discussing the calibration of parameters that remain constant across the different steady states. In the firm sector, the elasticity of output with respect to capital α is set equal to 0.4 for both types of firms, a common value used in most of the literature.[14] The measures of high-productivity and low-productivity firms are assumed to be equal. This assumed share of high-productivity firms, which are financially constrained in our model, matches the observed share of credit-constrained firms in the United States, estimated by Farre-Mensa and Ljungqvist (2014) to be roughly 50%.[15]

The pledgeability parameters of tangible capital θ^T and intangible capital θ^I are equal to 1.00 and 0.35, respectively. Thus, we assume tangible capital to be fully collateralizable, in line with Falato, Kadyrzhanova, and Sim (2014). Moreover, we calibrate θ^I to generate net leverage in the high-productivity firms on average equal to 6.4%, in line with the average net

[14] See King and Rebelo (1999) or Corrado, Hulten and Sichel (2009).

[15] They find that roughly three quarters of privately held firms are financially constrained. Within the sample of publicly listed firms, they report different estimates of the share of financially constrained firms that range between 10% and 45%. Given these estimates, we set the share of high productivity firms to be 50% in our simulations.

leverage ratio for Compustat publicly-listed firms.[16] We set θ^I relatively high compared with the literature to accommodate for the fact that we only allow firms to issue collateralized debt. As discussed in Section 2, in reality, firms finance their acquisitions in part with equity issues and other forms of external financing beyond collateralized debt.[17]

In order to calibrate the exit probability ψ and the investment probability η, we interpret them as shocks that generate creative destruction. Therefore, even though we do not model explicitly heterogeneous products, we interpret ψ as the probability that the firm's technology becomes obsolete because a competing firm enters the market and produces an improved version of its product. Moreover, we interpret η as the arrival probability of an investment opportunity to produce a new product. According to this interpretation, we set $\psi = \eta = 13\%$, which generates yearly capital reallocation of 6% of total capital (tangible plus intangible). This is consistent with David (2014), which measures reallocation of capital generated by mergers and acquisitions to be around 5% of total capital in the past few decades, and with the reallocation data from Eisfeldt and Rampini (2006).[18] The intuition is that when a firm's technology becomes obsolete, it sells its capital to the new and more productive firms.

The TFP of low-productivity firms, $z^{u,T}$ and $z^{u,I}$, is normalized to 10. The TFP of high-productivity firms z_t is modeled as:

$$z_t = [1 + (\mu - 0.2)\kappa]\, z, \tag{52}$$

so that for the early 1980s value of $\mu = 0.2$, $z_t = z$ for simplicity. We set $z = 25$ to match the average interquartile productivity differential of the firms, which in our simulations is 2.54 over the 1980-present period, a number consistent with the cross-sectional dispersion in productivity for U.S. firms identified in Syverson (2004) for a similar time period.[19] The parameter κ measures the increase in TFP associated with a stronger intensity of intangible capital in the production function. We choose $\kappa = 0.1$, which implies that an increase in μ is privately

[16] Bates et al. (2009) using data from 1980 to 2006, compute a value of 7.9%. They calculate net leverage as the ratio of total debt minus cash holdings to the book value of total assets, which maps in the model to $-A_f / (q_T K_T + q_I K_I)$.

[17] An alternative approach would have been to assume a value of θ_I much closer to zero, in line with Falato, Kadyrzhanova, and Sim (2014), and introduce equity issues by allowing dividends d_t to be negative, with an associated equity issuance cost proportional to the amount financed. This approach would have slightly complicated the model and yielded very similar quantitative results.

[18] Using capital reallocation data available at Andrea Eisfeldt's website (https://sites.google.com/site/andreaeisfeldt/reallocation_data_eisfeldt.xlsx), we compute an average capital reallocation of 5.8% of total capital over the 1980-2013 period.

[19] Syverson (2004) examines plant-level data from 1977 and finds an average interquartile difference in labor productivity around 2 for 4-digit U.S. manufacturing sectors. Since the dispersion of productivity is larger for less narrowly defined sectors, a value of 2.54 is probably a very conservative estimate of the dispersion of productivity across all firms.

optimal at the steady state equilibria obtained for most values of μ.[20] A positive value of κ is not necessary for our results. However, it is consistent with the notion of the rise of intangible capital as a privately optimal choice of firms, and allows us to be able to make conservative and robust statements about the potential for negative effects of the shift to intangibles.

The depreciation factor δ is set equal to 15%. This value is consistent with the depreciation rates used for the perpetual inventory method in Section 2.[21] The initial endowment of newborn firms W_0 is equal to 5, and is the only one not to be calibrated to match a specific moment due to a lack of a clear empirical counterpart. It corresponds to 2% of average firm annual output. Our results show very little sensitivity to variations in our choice of W_0 in the range 0.1%-20%.

The parameters associated to capital production are φ, b^T and b^I. The parameter φ determines the elasticity of the capital stock to the price of capital (see equation 51), and we calibrate it so that the elasticity of the stock of capital to the user cost of capital is in line with the empirical evidence. Caballero, Engel and Haltiwanger (1995) estimate the short run elasticity of the capital stock to the user cost of capital to be between 0 and -0.1, and the long run elasticity to be between -0.3 and -1 for most 2-digit sectors. Since we do not model taxes, and the price of the consumption good is normalized to 1, the user cost of tangible capital in our model is $(r + \delta) q^T$. We consider changes in the user cost of capital driven by exogenous changes in the interest rate. Our production sector implies that a decrease in r increases q^T(see equation (49)). However the user cost of capital falls in equilibrium, because the increase in q^T does not fully compensate the reduction in r. We choose a value of $\varphi = 9$, which generates an elasticity equal to -0.23. Given the value of φ, the initial values of b^T and b^I determine the aggregate supply of tangible and intangible capital and their equilibrium prices. We calibrate them so that the relative price of tangible to intangible capital is normalized to 1 in our early 1980s steady state simulation, and so that output of the high-productivity firms is roughly 50% of total output.

There are two household sector parameters that we keep constant across comparative statics. The share of dividends that are paid to the working-age population, γ, is set to 40% in order to target a real interest rate of $r = 6\%$ in our simulation of the early 1980s, consistent with the

[20] Our results are robust to setting κ high enough so that increases in μ are always privately optimal. Our benchmark calibration, however, reflects the possibility that some of the technological changes that have driven an increase in the intensity of intangible capital are not always endogenous firm choices but the consequence of structural economic changes, such as secular changes in the sectoral specialization of different countries.

[21] For tangible capital, this value is appropriate since we interpret it as a combination of more durable assets, such as equipment and structures, and less durable ones, such as inventories. For intangible capital, this value is consistent with existing literature regarding intangible and tangible capital, while possibly too low for other intangible assets such as computerised information and brand equity (Corrado, Hulten and Sichel 2006). Assuming higher depreciation rate for intangible capital does not significantly change the results presented in the following sections.

real rate in that period. We set the number of years households remain young to $N = 40$, which corresponds to a working-age period between the ages of 25 and 65 years.

Finally, we discuss the parameters that we vary in our comparative statics exercises. We follow Falato, Kadyrzhanova, and Sim (2014) in setting μ, the reliance on intangible capital of firms, at 0.2 in our exercise for the early 1980s, so that the share of intangible capital over total capital is 20%. We introduce a gradual linear shift in μ so that our simulation matches the observed intangible to total capital ratio of 60% ($\mu = 0.6$) in the 2010s (Corrado and Hulten (2010a), Falato, Kadyrzhanova, and Sim (2014), Döttling and Perotti (2015)).

We vary the parameters b^I and b^T in the capital production function (6) to capture the observed evolution of capital prices. This is important because capital prices matter for the mechanism we describe, and because it has been well documented that tangible capital has experienced a significant decrease in its relative price. Karabarbounis and Neiman (2014) estimate that the price of capital has fallen approximately by 30% between the late 1970s and the 2000s, and we match this trend by decreasing b^T accordingly. Reliable measures of the change in the relative price of intangible capital are not available, however.[22] Some authors have used instead the GDP deflator, which implies by construction no change in the relative price of intangible capital (Corrado, Hulten and Sichel (2009)). Other authors use an input cost approach. An important factor in the production of intangible capital is skilled labor (Dougherty, Inklaar, McGuckin, and van Ark (2007), Robbins, Belay, Donahoe, and Lee (2012)), which has experienced an important increase in its relative cost since the 1980s (Lemieux (2008)). An increase in input costs however might translate into lower intangible capital prices if the productivity of capital production increases substantially. This is the case for R&D, one of the types of intangible capital: Robbins, Belay, Donahoe, and Lee (2012) estimate an annual fall in the relative price of R&D of around 1.2% between 1998 and 2007 despite an increase in input costs. Computerised information, on the other hand, is estimated by Byrne and Corrado (2016) to have experienced an average annual real price change of -1% in the 1963-87 period, and of around -4% in the 1987-2015 period. Putting this evidence together, we change b^I over time so that the relative price of intangible capital remains roughly constant over time. It is important to note that what matters for our purpose is how much interest rates affect the path of relative prices of capital, and not what the precise level of capital prices would be absent the observed significant decrease in real rates.

The household sector parameters that we vary across our simulations are the discount factor β and the probability of death after the age of 65, ϱ. First, we vary ϱ so that we match changes

[22] See Corrado, Haskel, Iommi, and Jona Lasinio (2012) for a detailed description of the challenges in obtaining a general price deflator for intangible capital.

in the life expectancy in the U.S. between the 1980s and the present.[23] We vary the rate of time preference β to match the evolution of real interest rate from around $r = 6\%$ in the 1980s to around 0% in the present, and so that the value of β on average over our comparative statics exercises is in line with values used in the literature.[24]

7 Simulation Results

In this section, we introduce two comparative static exercises that capture how parallel developments in the household and the corporate sector have interacted to generate aggregate patterns consistent with the secular stagnation hypothesis. First, we explore how an expansion of households' savings affects economic outcomes in a tangibles-intensive economy compared to an intangibles-intensive one. Second, we introduce a simulation that replicates key trends in the United States between 1980 and 2015, a period characterized by an increase in households' incentive to save and a rise in the reliance on intangible capital.

7.1 The Effect of a Rise in Households' Propensity to Save

In order to clarify the different effects at play, we first conduct a counterfactual exercise in which households' propensity to save and life expectancy both gradually increase, reducing the equilibrium interest rate. We run two simulations: one in which the share of intangible capital is kept constant at $\mu = 0.05$ (a tangibles economy), and another in which it is kept constant at $\mu = 0.65$ (an intangibles economy). The expansion in household savings is achieved by decreasing the rate of household time preference (increasing β) and by increasing life expectancy (lowering ϱ) to generate a decline in the interest rate from 6% to around 1%.[25] The sequence of steady states associated to the set of different values of β, ϱ and μ is displayed in Figure 4.

[FIGURE 4 ABOUT HERE]

The left panel in the middle row of Figure 4 shows that the net leverage of high-productivity firms is positive in the tangibles economy and firms are on average net borrowers. Corporate net leverage is instead negative in the intangibles economy and firms are on average net savers.

[23] The Centers for Disease Control and Prevention (https://www.cdc.gov.htm) reports that life expentacy was around 70 years in 1970 and 78 years in 2016.

[24] Common values used in the literature range from 0.93 used in Jermann and Quadrini (2012) to 0.97 in Christiano, Eichembaum and Evans (2005). We set β to range from 0.9425 in the early 1980s to 0.9805 in recent years.

[25] All parameters are identical in the two cases except for the discount factor β, which is set so that in both cases the comparative statics exercise starts with a value of $r = 6\%$. Therefore, while in the tangibles economy β changes from 0.9425 to 0.9805, in the intangibles economy it changes from 0.9375 to 0.9755. To avoid confusion we do not report these different values of β on the x-axis.

Correspondingly, households are net savers (borrowers) in a tangibles (intangibles) economy, as shown in the top-right panel. Household sector developments encourage households to save more in a tangibles economy and borrow less in an intangibles economy, pushing down the interest rate in both cases. The drop in the interest rate increases the price of capital and encourages capital creation, so that aggregate tangible and intangible capital stocks increase.

The left and middle panels in the last row of Figure 4 analyze the changes in the allocation of capital and in efficiency. In the tangibles economy, capital allocation improves and there is an expansion of capital and output of high-productivity firms. High-productivity firms have a high leverage and the decline in the interest rate benefits them, both because it is easier to pay back debt (the debt hangover channel) and because they can borrow more when they invest (the collateral value channel).[26] These two channels prevail over the capital price channel, which operates in the opposite direction, and imply that the drop in r benefits high-productivity firms; they can absorb a higher share of existing capital, thus improving the allocation of resources.[27] Conversely, in the intangibles economy, firms are net savers. As explained in Section 3, in this case the decline in the interest rate hurts their accumulation of wealth (the savings channel), and the collateral value channel is very weak because firms' borrowing capacity is limited, so that a lower rate is strongly contractionary.

The last panel shows that, overall, output increases by around 1.5% in the tangibles economy, both because of the positive reallocation effect and because of the increase in the aggregate capital stock, while it declines by around 1.5% in the intangibles economy, because the contraction in the allocation of capital to the high productivity firms offsets the positive effect of the increase in aggregate capital.

[FIGURE 5 ABOUT HERE]

Figure 5 shows that the dispersion in the marginal productivity of capital increases with lower rates in the intangibles economy, while it falls moderately in the tangibles economy. The dispersion in TFP shows similar diverging trends as well. The values of μ chosen for the tangibles and intangibles economies correspond to the 5% and 95% percentiles, respectively, of the cross sectional distribution of the average share of intangible capital in 2-digit U.S. industrial sectors over the 1980-2015 period. Since interest rate movements are almost identical in both

[26] The drop in the interest rate increases corporate leverage in the tangibles economy via the collateral value channel and via a second, less intuitive, mechanism. As q_T and q_I go up and the productivity of capital in the high-productivity firms goes down because of the decreasing marginal product of capital and a fixed aggregate labor supply, the share of capital financed by debt as opposed to by the accumulation of past output goes up.

[27] Another indirect benefit of this positive reallocation is that since the low-productivity firms absorb less capital, their marginal return is higher, relative to the intangibles economy, driving up capital prices (middle row, middle graph), and stimulating capital production (middle row, left graph).

simulations, these can be interpreted as two sectors in an economy where capital and labor are sector specific. In this respect, the simulated trends shown in Figure 5 are fully consistent with the empirical trends shown in Figure 2.

7.2 The Simultaneous Rise in Households' Propensity to Save and in Intangible Capital (1980-2015)

In Section 7.1 we explored an expansion in household savings but kept the intensity of intangible capital constant. In this section, in contrast, we reproduce the simultaneous rise in the propensity of households to save and in the reliance on intangible capital observed during the period from 1980 to 2015. To increase our understanding of the interaction between both developments, we also describe a sequence of steady states in which we only increase the reliance on intangible capital. Our results are displayed in Figure 6. Since we abstract from long-run growth considerations, the graphs that show relative changes in total output should be interpreted as deviations from long-run trends.

We first focus on the exercise that explores the rise in intangibles in isolation. The gradual increase in μ pushes high-productivity firms to demand progressively more intangible capital and less tangible capital. Intangible capital attracts less external finance, which tightens firms' borrowing constraints significantly and decreases corporate leverage. High-productivity firms switch from being net borrowers to being net lenders, consistent with evidence in the United States for corporations (Armenter and Hnatkovska (2016), Quadrini (2016), Chen, Karabarbounis and Neiman (2016)). The increased reliance on a type of capital that firms cannot finance externally causes a contraction in the allocation of capital to the high productivity firms. Furthermore, the increase in net corporate savings reduces interest rates moderately, by about 1%, to ensure that households borrow more and absorb the excess savings. A lower interest rate also affects capital accumulation, but most of the misallocation is caused by high-productivity firms' lower ability to borrow. Aggregate output rises initially driven by higher productivity of intangible capital (see equation (52)), but eventually levels off and falls slightly as the negative effects of a decrease in corporate borrowing and lower interest rates dominate. Overall, a shift to intangibles is expansionary.

[FIGURE 6 ABOUT HERE]

When we consider corporate and household developments simultaneously, we observe instead a fall in aggregate output. The interest rate falls from 6% to around 0% and capital prices are generally higher than in the simulation of the rise in intangibles in isolation. Aggregate capital in the high-productivity firms (third row, middle panel) is roughly constant in the initial 1980-1990 period, while leverage is still positive and the increase in productivity driven by the rise of intangibles compensates the negative effects of the lower borrowing capacity. In this period, total output (bottom panel) expands by 2% until around the mid 1990s, thanks to the increase in productivity and aggregate capital.[28] During the 1990s and 2000s, however, capital and output of high-productivity firms both fall substantially because their borrowing capacity declines further and the economy becomes more similar to the intangibles economy described in Section 7.1, an economy in which a decline in the interest rate causes a large contraction of the output of high productivity firms. By 2015, their output has fallen by 18%, compared to a fall of 10% in the economy in which only the rise in intangibles occurs (third row, right panel). Lower rates damage the high productivity firms both because of the savings channel, which becomes stronger the larger are their net savings, and because low rates imply relatively higher capital prices, which hurt firms through the capital price channel. Thus, the reduction in interest rates, which is expansionary for highly leveraged high-productivity firms, hurts capital reallocation and growth once the economy relies more on intangible and less collateralizable capital.

It is important to note that while a decline in interest rates caused by household developments expands aggregate output in a tangibles economy (see Figure 4), and a shift to intangibles also expands it, the combination of the two developments is overall contractionary, with output in 2015 around 1% lower than in 1980 (bottom panel in Figure 6).

[FIGURE 7 ABOUT HERE]

The contraction in output happens despite our assumption that intangible capital is more productive, because of a strong misallocation effect in which too many resources are absorbed by low-productivity firms. Figure 7 quantifies the consequences of misallocation on productivity more precisely. It shows that, in a counterfactual scenario in which the allocation of resources does not worsen, the rise in intangibles increases aggregate TFP by almost 3%. The rise in intangibles in isolation, but allowing for an endogenous allocation of capital between high- and low-productivity firms, generates a TFP drop of -3.5%, which almost doubles to -6.5% when the rise of intangibles coincides with increased incentives to save in the household sector.

[FIGURE 8 ABOUT HERE]

[28] The values of b^T and b^I, which are calibrated to an empirically realistic evolution of capital prices in the simulation with both household and corporate developments, are also an important factor driving the increase in capital stock.

Finally, Figure 8 replicates the evolution of aggregate output in the benchmark case with both developments (bottom panel in Figure 6), and compares it to a counterfactual simulation in which the overhang/savings channel is eliminated. This counterfactual is constructed by assuming that interest rate changes can affect capital prices and the collateral constraint, but that firms' interest rate on debt or return on savings is kept constant at the initial value of 6%. In the simulated period in which firms are net borrowers (1980-1995), the debt overhang channel implies that lower rates benefit high productivity firms, and shutting it down (the dashed line in Figure 8) lowers output relative to the benchmark. However, once firms become net savers, the savings channel implies that lower rates worsen reallocation, and significantly contributes to the decline in aggregate output.

8 Conclusion

This paper highlights a novel misallocation effect of endogenously low interest rates that has potentially important policy implications. From a quantitative standpoint, our results are consistent with several developments that have taken place in the past 40 years: (i) net corporate savings increased as a fraction of GDP, (ii) household leverage increased as a fraction of GDP, (iii) the real interest rate fell, (iv) intra-industry dispersion in productivity has increased, and (v) output and productivity progressively declined relative to their previous trends. Interestingly, the model shows that even though the shift to intangible technologies was already taking place in the 1970s, its net negative effects on output growth only started to gather pace from the mid-1980s onward. This finding is consistent with studies that show a decline in dynamism of U.S. businesses starting in the mid 1980s and gathering speed especially from 2000 onward (Haltiwanger (2015)).

More broadly, our results suggest that the changes in firms' financing behavior brought about by technological evolution might help explain the subpar growth experienced in recent years, because they have occurred during a period of low interest rates. Our insights could be extended to develop interesting policy implications. On the one hand, the mechanisms described in this paper, operating mostly through the endogenous reaction of interest rates, suggest that the rise in intangibles might have important implications for monetary policy. On the other hand, the negative externality in households' and firms' excessive saving decisions might introduce a role for a fiscal policy that discourages such saving.

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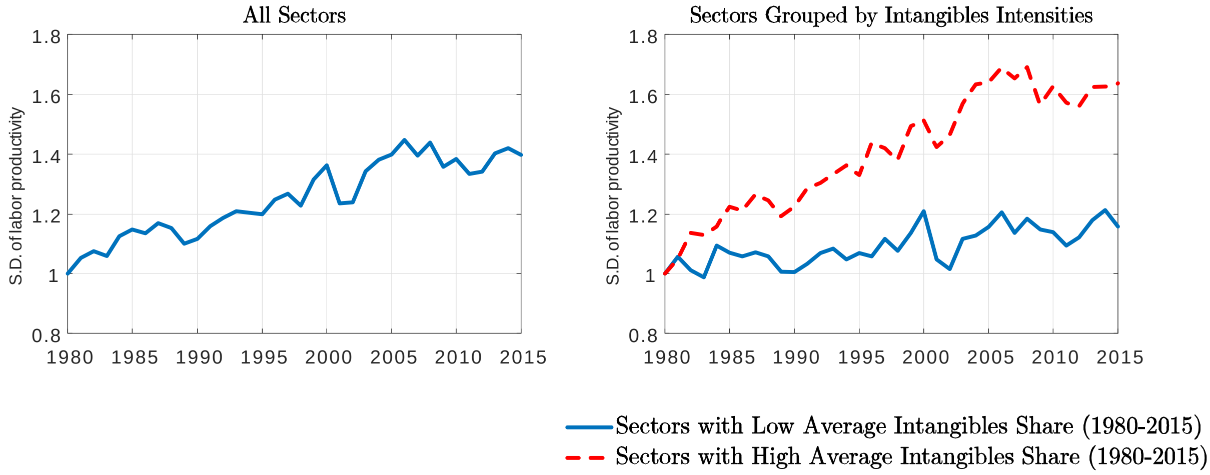

Figure 1: Within-Industry Dispersion in Firm-Level Labor Productivity (*Source: Compustat data, own calculations*)

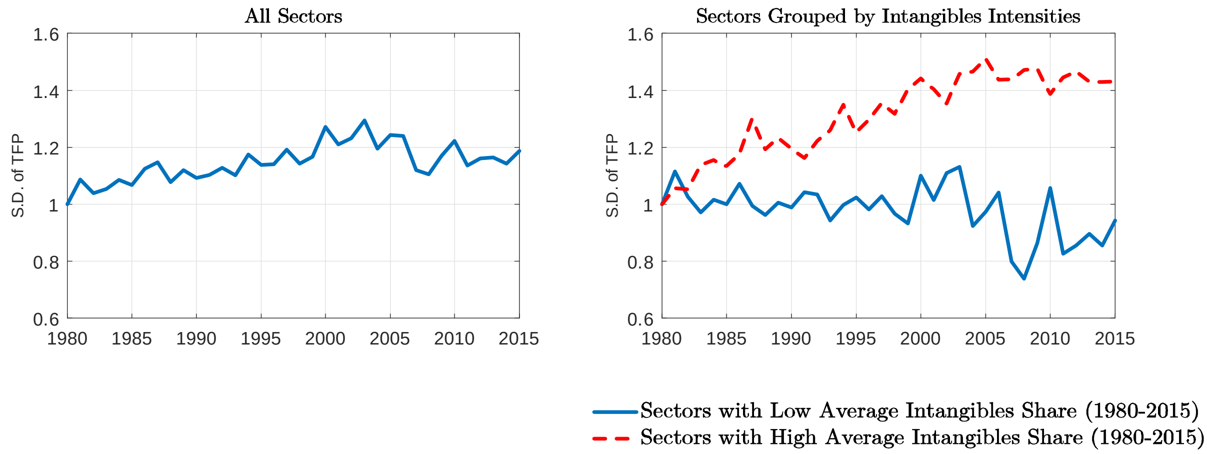

Figure 2: Within-Industry Dispersion in Firm-Level TFP (*Source: Compustat data, own calculations*)

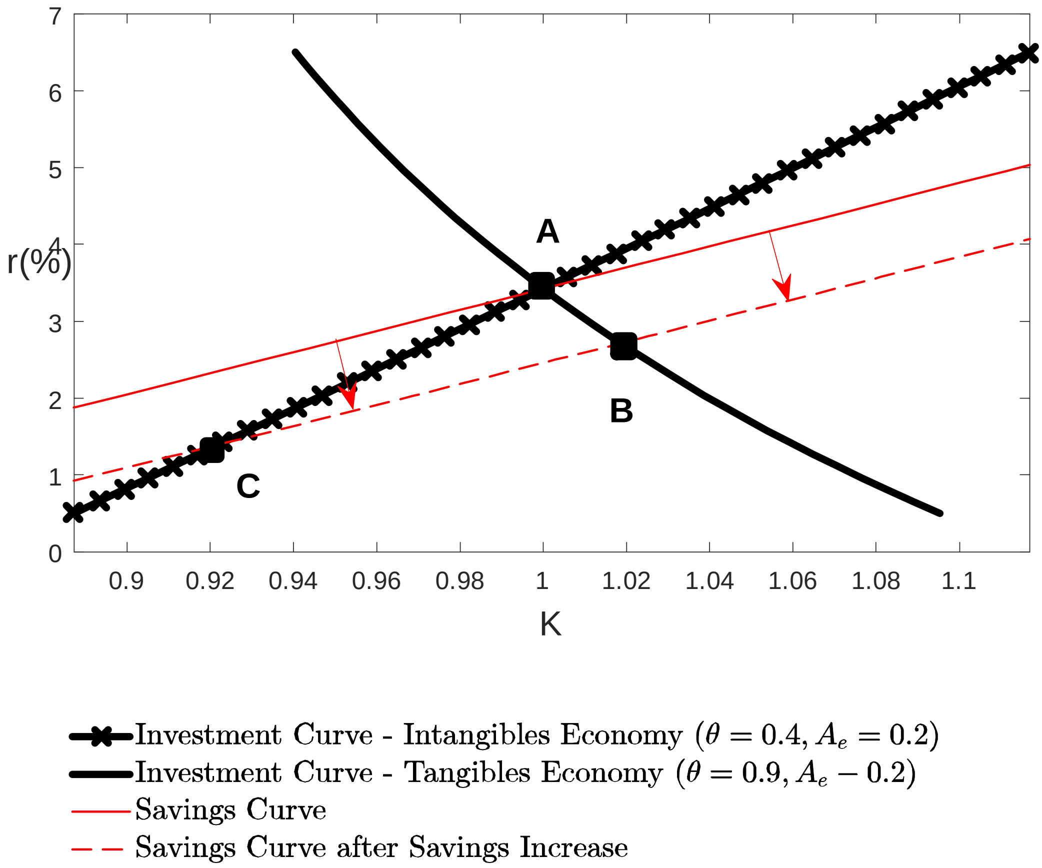

Figure 3: Credit Market Equilibrium in the Simple Model of Section 3

Parameter	Symbol	Value
Parameters that remain constant across comparative statics		
Parameter	**Symbol**	**Value**
Capital share, high-productivity firms	α	0.4
Capital share, low-productivity firms, tangible capital	χ_I	0.4
Capital share, low-productivity firms, intangible capital	χ_T	0.4
Low-productivity firms, TFP tangible technology	$z_t^{u,T}$	10
Low-productivity firms, TFP tangible technology	$z_t^{u,I}$	10
Years households remain young	N	40
High-productivity firms, TFP	z	25
Collateral value of tangible capital	θ^T	1
Collateral value of intangible capital	θ^I	0.35
Probability of an investment opportunity	η	0.13
Additional productivity of intangible capital	κ	0.1
Adjustment cost convexity	φ	9
Exit probability of high-productivity firms	ψ	0.13
Endowment of new firms	W_0	5
Depreciation of capital	δ	0.15
Share of dividends to young households	γ	40%
Parameters that change across comparative statics		
Parameter	**Symbol**	**Value in 1980 − in 2015**
Discount factor	β	$0.9425 - 0.9805$
Intangible share of total capital	μ	$0.2 - 0.6$
Probability of death of old households	ϱ	$0.170 - 0.075$
Adjustment cost parameter (intangible)	b_I	$3.2 * 10^{-6} - 15.3 * 10^{-6}$
Adjustment cost parameter (tangible)	b_T	$2.1 * 10^{-10} - 0.6 * 10^{-10}$

Table 1: Benchmark Calibration - Parameter Choices

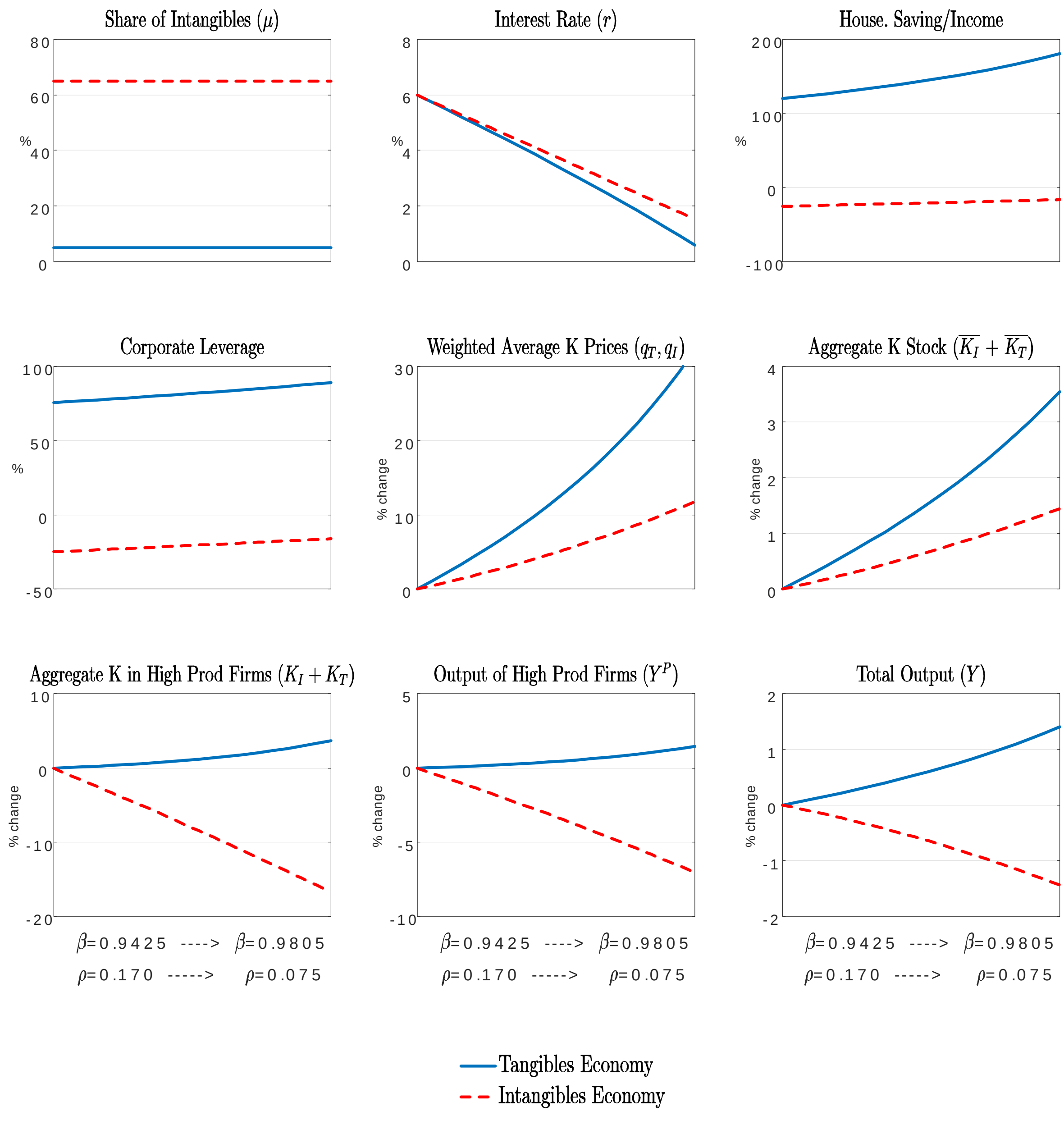

Figure 4: Simulation Exercise: households' propensity to save gradually increases because of (i) a decrease in the rate of time preference (β increases) and (ii) a decrease in the likelihood of death of old households (ϱ decreases) - comparison of the effects of the expansion in households savings in a tangibles economy ($\mu = 0.05$) and an intangibles economy ($\mu = 0.65$).

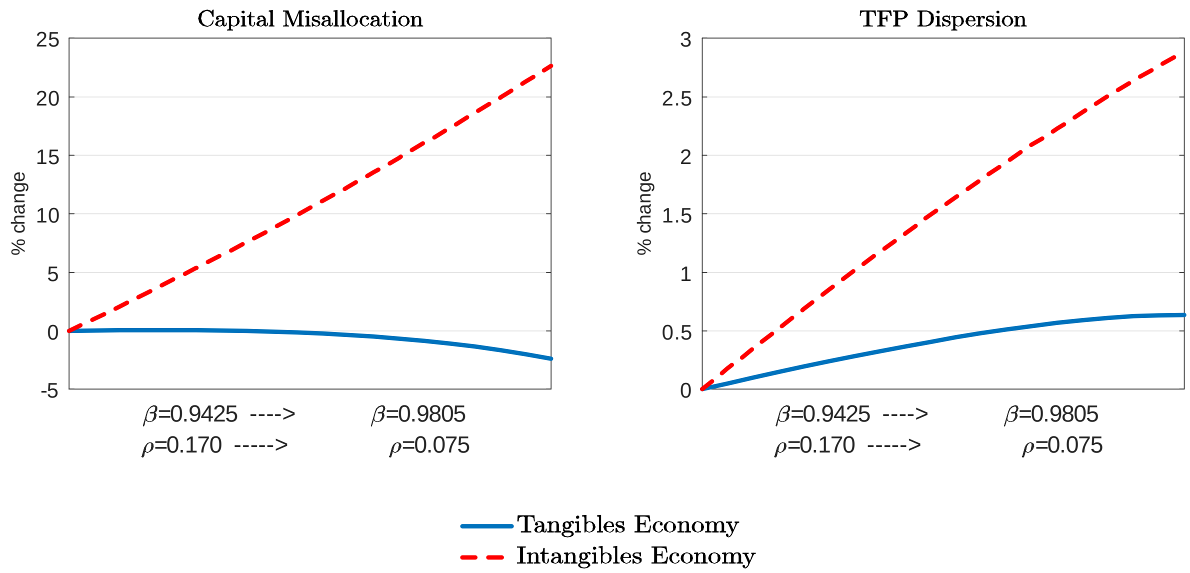

Figure 5: Simulation Exercise: households' propensity to save gradually increases because of (i) a decrease in the rate of time preference (β increases) and (ii) a decrease in the likelihood of death of old households (ϱ decreases) - comparison of capital misallocation and TFP dispersion in a tangibles economy ($\mu = 0.05$) and an intangibles economy ($\mu = 0.65$).

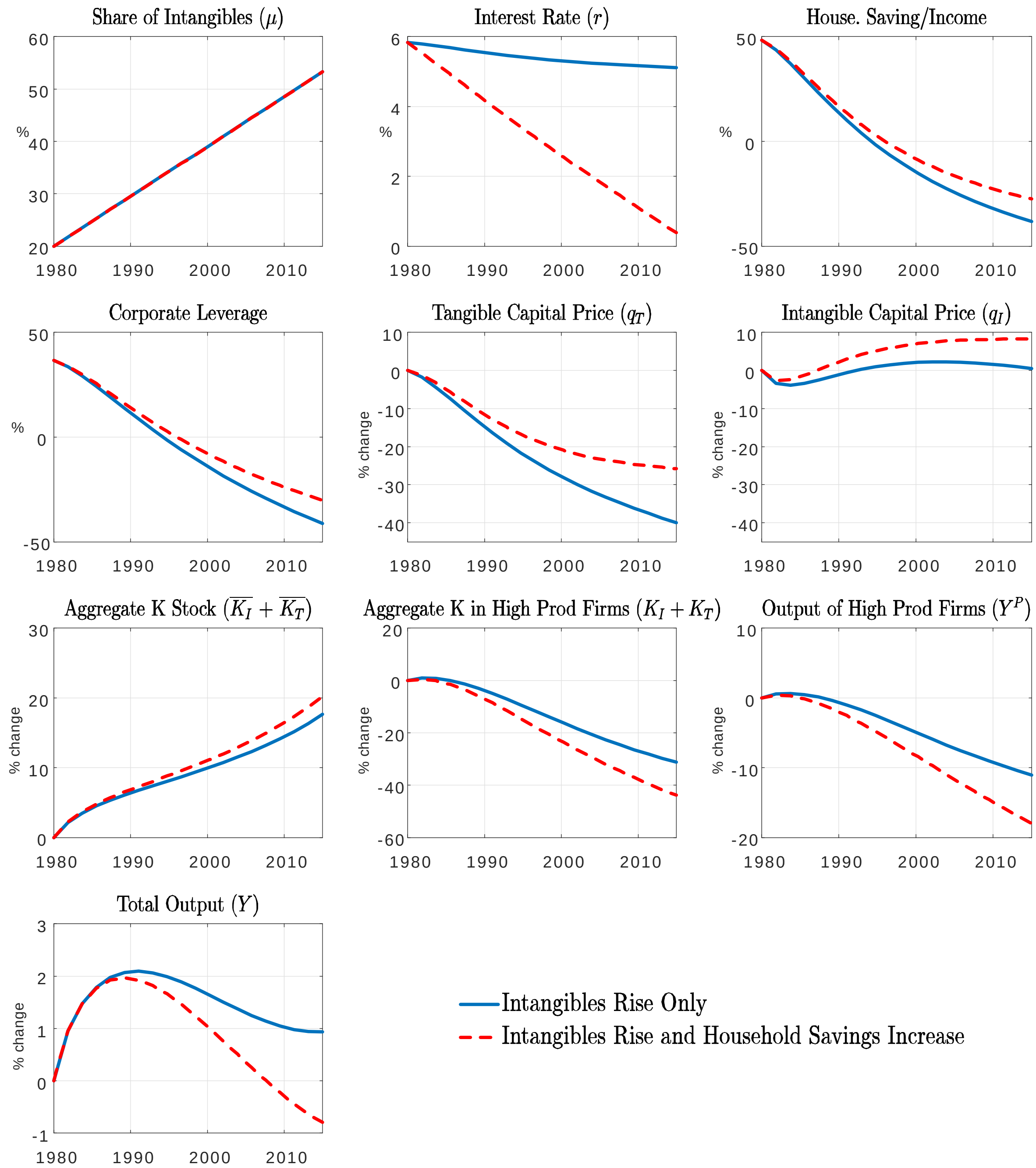

Figure 6: Simulation Exercise: households' propensity to save and the share of intangible capital both gradually increase - comparison of effects when both trends occur and when only the increase in the share of intangible capital occurs

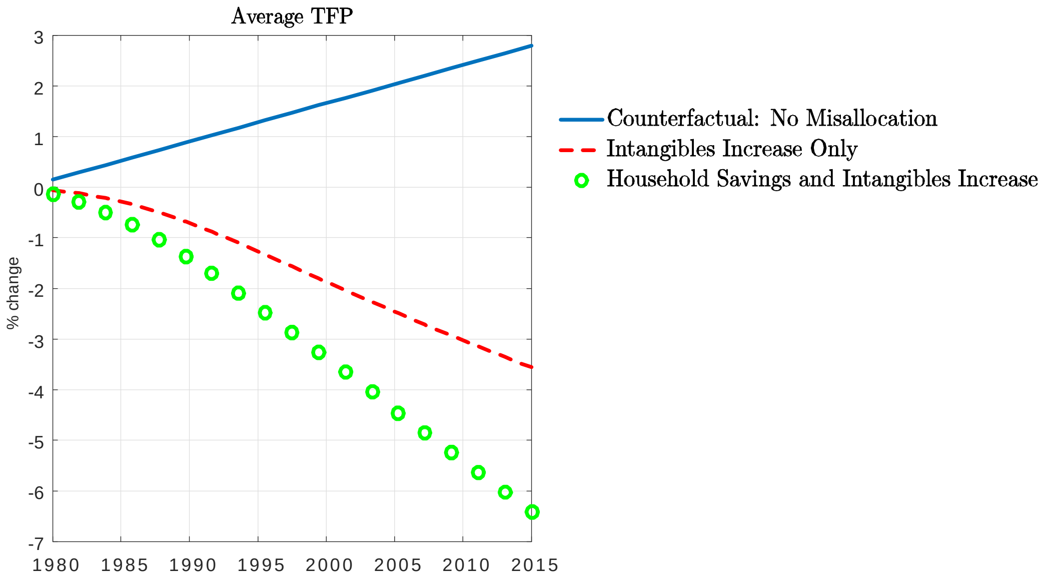

Figure 7: Simulation Exercise: households' propensity to save and the share of intangible capital both gradually increase - comparison of effects on TFP when both trends occur, when only the increase in the share of intangible capital occurs, and in a counterfactual partial equilibrium scenario

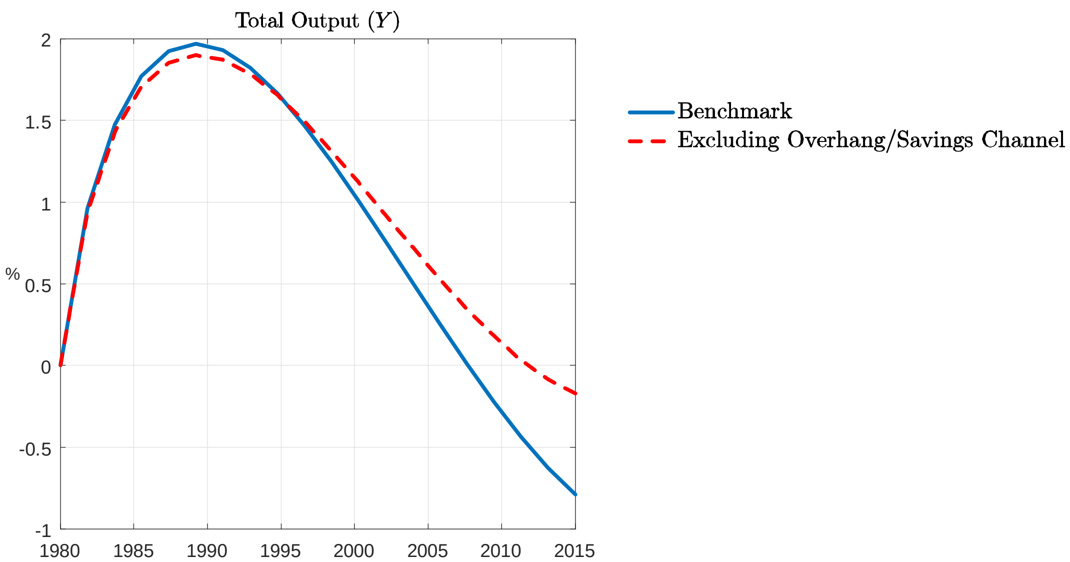

Figure 8: Simulation Exercise: households' propensity to save and the share of intangible capital both gradually increase - comparison of effects when we shut down the debt overhang/savings channel

APPENDICES
For Online Publication

A Robustness of Dispersion Evidence

A.1 Construction of the Intangible Capital Measure

We define intangible capital as the sum of knowledge capital and organizational capital.[29] Following Falato et al. (2014), we measure the former by capitalizing R&D expenses and the latter by capitalizing selling, general and administrative (SG&A) expenses weighted by 0.2.[30] The expenditures are capitalized by applying the perpetual inventory method with a depreciation rate of 15% for R&D and 20% for SG&A. In order to get a measure for tangible capital, we also use the perpetual inventory method to capitalize tangible capital expenses with a depreciation rate of 15%. We drop firms that are observed only once and firms that are not observed in a continuous time period, and we exclude regulated, financial, and public service firms. We consider sectors at the 2-digit Standard Industrial Classification (SIC) level and drop those with less than 500 firm-year observations over the sample period. We measure output by sales, labor input by the number of employees, and total capital by the sum of capitalized tangible and intangible capital.

TFP is defined as the residual of a Cobb-Douglas production function with a capital share of income equal to 0.40. To control for outliers, we drop firms in the 1st and 99th percentiles of the distribution of labor productivity.

A.2 Robustness Checks

One alternative explanation of the results shown in Figures 1 and 2 in Section 2 could be that the sectors with a high intangibles share do not have a worse allocation of resources, but rather are more dynamic and fast growing, and that the increase in dispersion of productivity reflects this higher dynamism. However, in Figure A, we show that sectors with high average sales growth have lower productivity dispersion in the whole sample period.

[FIGURE A ABOUT HERE]

[TABLE A ABOUT HERE]

Furthermore, Table A shows regression results where the dependent variable is a measure of productivity dispersion for each 2-digit sector-year observation. The regressors we consider are as follows: the dummy *High share*, which is equal to 1 if the sector belongs to the 50% 2-digit industries with the highest average intangible share and which is equal to 0 otherwise; a time trend; and year and sector fixed effects. In columns 1 and 2, the dependent variable is the dispersion in TFP. Column 1 includes year fixed effects and shows that the dispersion is significantly larger for sectors with higher intangible share. Column 2 includes a time trend, interacted with the *High share* variable, and both sector and time fixed effects. It shows that the

[29] Falato et al. (2014) also consider informational capital. However, they state that their results do not depend on its inclusion. As informational capital can be measured only at the industry level but not at the firm level using Compustat data, we choose not to include this type of capital.

[30] A portion of SG&A expenses captures expenditures that increase the value of intangible capital items such as brand names and knowledge capital. Part of SG&A expenditures, however, does not affect the value of intangible capital, so Falato et al. (2014) follow Corrado, Hulten, and Sichel (2009) and assume that the portion relevant to intangible capital is around 0.2.

trend in dispersion over time is significantly more positive in the 50% most intangible sectors than in the other sectors, confirming the significance of the result shown in Figures 1 and 2. Similar results, across the two groups of high and low intangibles sectors, are obtained using labor productivity, as shown in columns 3 and 4.

B Derivation of the Simple Model

This appendix solves a simple partial equilibrium model that delivers the equations introduced in Section 3. Consider an infinite-horizon, discrete-time model of the final good-producing sector of an economy. Firms use capital, which is in constant aggregate supply $\overline{K}$, to produce a homogeneous consumption good. There are two types of firms, *high-productivity* and *low-productivity*, each composed of a continuum of mass 1. High-productivity firms live for 2 periods, and there are overlapping generations of these types of firms. Efficiency in this economy is determined by the share of $\overline{K}$ allocated to high-productivity firms. Our main interest is studying how exogenous interest rate variations affect the allocation of capital and aggregate output depending on the degree of tangibility of capital. This framework is extended in Section 4 in a full-fledged general equilibrium setup that can be used for realistic quantitative analysis.

B.1 High-Productivity Firms

Technology and Financing

High-productivity firms live for two periods, which we denote with y (young) and o (old). An old firm that dies in period $t-1$ leaves a a financial endowment or liability a^e ($-\infty < a^e < \infty$), which translates into net worth $a^e(1+r_t)$ for the newborn young firm in period t. The young firm is able to produce y^e units of the final good in period t, and has access to a technology to produce the final good in period $t+1$ using the following linear production function:

$$y^p_{t+1} = z_{t+1}k_{t+1}, \tag{53}$$

where k_{t+1} represents capital purchased in t that produces output in $t+1$, and z_{t+1} is a productivity parameter. The firm can borrow b_{t+1} to purchase capital, subject to a constraint:

$$(1+r_{t+1})\, b_{t+1} \leq \theta q_{t+1}k_{t+1}, \tag{54}$$

where $0 < \theta \leq 1$. The collateral value of capital θ is the parameter in this stylized model that captures capital tangibility. A shift toward a stronger reliance on intangible capital will be captured as a decrease in θ. [31] Firms cannot issue equity.

The budget constraint for a high-productivity young firm is:

$$q_t k_{t+1} = a^e(1+r_t) + b_{t+1} + y^e. \tag{55}$$

[31] Note that a standard collateral constraint of the form

$$b_{t+1} \leq \frac{\theta q_{t+1}k_{t+1}}{1+r_{t+1}},$$

would not work when $W_0 < 0$. This is because in that case the firm would have to be assumed to borrow more than q_t per unit of capital:

$$q_t < \frac{\theta q_{t+1}}{1+r_{t+1}},$$

to be able to purchase capital and pay down the debt, and by making the collateral constraint a function of k_{t+1}, it would make the firm financially unconstrained, and the problem would not have a solution.

A mature firm realizes output, pays back any debts, sells its holdings of capital, and pays the residual, net of the endowment for the next generation a^e, as a dividend d_{t+1} to its shareholders:

$$d_{t+1} + a^e = y^p_{t+1} + q_{t+1}k_{t+1} - b_{t+1}\left(1 + r_{t+1}\right).$$

Optimal Solution

Productive young firms in $t = 0$ maximize the present value of the dividend d_{t+1}. We claim, and later verify, that their marginal product of capital is greater than its user cost:

$$z_t > q_t - \frac{q_{t+1}}{1 + r_{t+1}} \tag{56}$$

firms are credit constrained ((54) is binding), so that

$$k_{t+1} = \frac{a^e(1 + r_t) + y^e}{q_t - \theta\frac{q_{t+1}}{1+r_{t+1}}}. \tag{57}$$

in which investment is equal to the total wealth available to invest divided by the downpayment necessary to purchase one unit of capital.

B.2 Low-Productivity Firms

There is a mass 1 of identical firms in the unproductive sector that produce the same homogeneous final good as the high-productivity firms using a linear production function:

$$y^u_t = z^u_t k_{u,t}, \tag{58}$$

where $k_{u,t}$ represents capital installed in period $t-1$ that produces output in period t, and z^u_t is a productivity parameter. This sector is assumed to be financially unconstrained, and to pay out all profits as dividends:

$$d^u_t = y^u_t - q_t\left(k^u_{t+1} - k^u_t\right) \tag{59}$$

to their shareholders every period.

B.3 Aggregation

From (57) it follows that the aggregate stock of capital held by the high-productivity firms is

$$K_{t+1} = \frac{A^e(1 + r_t) + Y^e}{q_t - \theta\frac{q_{t+1}}{1+r_{t+1}}}.$$

and aggregate output is

$$Y_t = Y^p_t + Y^u_t + Y^e = z_t K_t + z^u_t\left(\overline{K} - K_t\right).$$

Under the assumption that the high-productivity firms have the highest return on capital ($z_t > z^u_t$), but their resources are insufficient to absorb all the capital, $K_{t+1} < \overline{K}$, it follows that the low-productivity firms are willing to absorb all the capital not demanded by the high-productivity firms at a price equal to their marginal return on capital. the price of capital is:

$$q_t = z^u + \frac{1}{1 + r_{t+1} + \xi}q_{t+1}, \tag{60}$$

which, together with the assumption that $z_t > z^u_t$, proves the claim (56). $\xi \geq 0$ is a wedge that reduces the sensitivity of the price of capital to the interest rate, and summarizes the effect of factors included in the full model developed later, such as decreasing marginal return to capital in the unproductive sector.

B.4 Steady State

We consider a steady state equilibrium and drop reference to the time subscript t. Total output is

$$Y = Y^p + Y^u + Y^e = zK + z^u \left(\overline{K} - K\right). \tag{61}$$

Aggregate capital holdings of the high-productivity firms in the steady state are:

$$K = \frac{A^e(1+r) + Y^e}{q\left(1 - \frac{\theta}{1+r}\right)}, \tag{62}$$

where

$$q = \frac{z^u}{r + \xi}. \tag{63}$$

Equations (1), (2), and (3) in the steady state equilibrium of Section 3 correspond to equations (61), (62) and (63) in this section.

C Optimal Dividend and Cash Accumulation Policy

Given equation (13), the first order condition for cash holdings $a_{f,t+1}$ for non investing firms is:

$$(1 + \lambda_t) = (1 - \psi)\left[\eta(1 + \lambda^+_{t+1} + \vartheta_t) + (1 - \eta)\left(1 + \lambda^-_{t+1} + \vartheta_t\right)\right] + \psi. \tag{64}$$

If we substitute (64) recursively forward, it is clear that if the firm expects ϑ_t to be positive now or in the future, then $\lambda_t > 0$, and a non-investing firm will always retain all earnings and $d_t = 0$. It is important to note that this is so because there is no cost of holding cash.

D Households

We derive below the solution of households' optimization problem under the steady state, so that wages, dividends and interest rate are constant. Households have log utility. A representative old household still living at time t maximizes the following objective function:

$$V^o_t\left(b^o_t\right) = \max_{c^o_t, b^o_{t+1}} \sum_{j=0}^{\infty} (1 - \varrho)^j \beta^j \log\left(c_{t+j}\right) \tag{65}$$

subject to

$$c^o_t = b^o_{t+1} + (1 - \gamma)d - \frac{(1 + r)}{(1 - \varrho)} b^o_t.$$

Working backward, we next consider the optimization problem of a young agent of age N in period t, who will become old in period $t + 1$[32]:

$$V^y_{t,N}\left(b^y_{t,N}\right) = \max_{c^y_{t,N}, b^o_{t+1}} u\left(c^y_{t,N}\right) + \beta(1 - \varrho)V^o_{t+1}\left(b^o_{t+1}\right) \tag{66}$$

[32] We assume that an agent can also die with probability ϱ in the transition between young and old.

subject to

$$c^y_{t,N} = \gamma d + w^{TOT} - (1+r)b^y_{t,N} + b^o_{t+1}. \tag{67}$$

where w^{TOT} is defined as:

$$w^{TOT} \equiv w + w^{uI} + w^{uT}$$

Then we consider the optimization problem for a young household of age $j < N$:

$$V^y_{t,j}\left(b^y_{t,j}\right) = \max_{c^y_{t,j}, b^y_{t+1,j+1}} u\left(c^y_{t,j}\right) + \beta V^y_{t+1,j+1}\left(b^y_{t+1,j+1}\right) \tag{68}$$

subject to

$$c^y_{t,j} = \gamma d + w^{TOT} - (1+r)b^y_{t,j} + b^y_{t+1,j+1} \tag{69}$$

D.1 Individual Problem of Old Households

We follow Blanchard (1985) and Yaari (1965) in assuming that households participate in a life insurance scheme when old. The insurance scheme works within a cohort so that the survivors within a cohort pay the debt of the dying (if they are in debt) or, alternatively, receive the savings of the dying. An old household begins a period with net debt $(1+r_t)b^o_t$. The insurance contract specifies that the ϱ fraction of old households that die transfer their assets (or debt) $(1+r_t)b^o_t$ to the life insurer. Among the fraction $(1-\varrho)$ of households that survive, if they are net savers ($b^o_t < 0$), then they receive a return $\frac{1}{1-\varrho}(1+r_t)b^o_t$ on their assets, while, if they are net debtors ($b^o_t > 0$), they make a payment of $\frac{1}{1-\varrho}(1+r_t)b^o_t$ to the life insurer.

The first order condition with respect to b^o_{t+1} is

$$c^o_{t+1} = \beta(1+r)c^o_t. \tag{70}$$

We guess a consumption policy rule:

$$c^o_t = \Delta d + \Theta b^o_t,$$

and plug it into the FOC

$$\Delta d + \Theta\left[c^o_t - (1-\gamma)\,d + \frac{(1+r)}{(1-\varrho)}b^o_t\right] = \beta(1+r)\left(\Delta d + \Theta b^o_t\right)$$
$$c^o_t = \left[\frac{\beta(1+r)\Delta}{\Theta} + (1-\gamma) - \frac{\Delta}{\Theta}\right]d + (1+r)\left[\beta - \frac{1}{(1-\varrho)}\right]b^o_t,$$

and then solve for the unknown coefficients

$$\Delta = \frac{\beta(1+r)\Delta}{\Theta} + (1-\gamma) - \frac{\Delta}{\Theta}$$
$$\Theta = (1+r)\left[\beta - \frac{1}{(1-\varrho)}\right]$$

$$\Delta = \frac{\beta\Delta}{\left[\beta - \frac{1}{(1-\varrho)}\right]} + (1-\gamma) - \frac{\Delta}{(1+r)\left[\beta - \frac{1}{(1-\varrho)}\right]}$$
$$\Delta = \frac{(1-\gamma)\,(1+r)\,[1-\beta(1-\varrho)]}{\varrho + r}$$

The policy rule is:

$$c_t^o = (1-(1-\varrho)\beta)\left[\frac{(1-\gamma)(1+r)}{(\varrho+r)}d - \frac{(1+r)}{(1-\varrho)}b_t^o\right]. \tag{71}$$

and the evolution of the wealth of old households is given by

$$b_{t+1}^o = \frac{(1-\varrho)\left[1-\beta(1+r)\right]}{(\varrho+r)}(1-\gamma)d + (1+r)\beta b_t^o, \tag{72}$$

which says that old households slowly consume their savings if $\beta(1+r) < 1$, and do so at a faster rate the higher the dividends. In our simulations typically $\beta(1+r) < 1$.[33]

D.2 Individual Problem of Young Households

We first consider the optimization problem of an agent of age N in period t, who will become old in period $t+1$:

$$V_{t,N}^y\left(b_{t,N}^y\right) = \max_{c_{t,N}^y, b_{t+1}^o} u\left(c_{t,N}^y\right) + \beta(1-\varrho)V_{t+1}^o\left(b_{t+1}^o\right) \tag{73}$$

such that:

$$c_{t,N}^y = \gamma d + w^{TOT} - (1+r)b_{t,N}^y + b_{t+1}^o. \tag{74}$$

The first order condition implies that

$$\frac{1}{c_{t,N}^y} + \beta\left(\frac{\partial V_{t+1}^o\left(b_{t+1}^o\right)}{\partial b_{t+1}^o} + \frac{\partial V_{t+1}^o\left(b_{t+1}^o\right)}{\partial b_{t+2}^o}\frac{\partial b_{t+2}^o}{\partial b_{t+1}^o}\right) = 0.$$

And applying the envelope theorem we obtain:

$$c_{t+1}^o = \beta\left(1+r\right)c_{t,N}^y.$$

We substitute c_{t+1}^o using (71) and we obtain:

$$c_{t,N}^y = \left(\frac{1}{\beta} - (1-\varrho)\right)\left(\frac{(1-\gamma)d}{(\varrho+r)} - \frac{b_{t+1}^o}{(1-\varrho)}\right). \tag{75}$$

Then we consider the optimization problem for a young household of age $j<N$:

$$V_{t,j}^y\left(b_{t,j}^y\right) = \max_{c_{t,j}^y, b_{t+1,j+1}^y} u\left(c_{t,j}^y\right) + \beta V_{t+1,j+1}^y\left(b_{t+1,j+1}^y\right), \tag{76}$$

such that:

$$c_{t,j}^y = \gamma d + w^{TOT} - (1+r)b_{t,j}^y + b_{t+1,j+1}^y, \tag{77}$$

which yields the standard Euler equation:

$$c_{t,j}^y = \left[\beta(1+r)\right]^{-(N-j)} c_{t+N-j,N}^y. \tag{78}$$

Equations (75) and (78) fully characterize the life-cycle path of consumption of a household

[33] To see this more clearly, denote $a = -b$ as savings, and write

$$a_{t+1}^o = (1+r)\beta a_t^o - \frac{(1-\varrho)\left[1-\beta(1+r)\right]}{(\varrho+r)}(1-\gamma)d.$$

as a function of its assets when entering old age in period $t+1$, b^o_{t+1}.

D.3 Value of Savings of Oldest Young: b^o_{t+1}

We use the above equations, the budget constraint (69), and the assumption that newborn households have no endowment ($b^y_{t,1} = 0$) to determine the value of savings for retirement $b^o_{t+1} \equiv b^y_{t+1,N+1}$.

We use the budget constraint for $j = 1$ (a young household of $age = 1$), in which the debt brought over, $b^y_{t,1}$, is zero:

$$b^y_{t,1} = \frac{\left(\gamma d + w^{TOT}\right) - c^y_{t,1} + b^y_{t+1,2}}{(1+r)} = 0,$$

and we solve forward:

$$\begin{aligned} b^y_{t,1} &= \frac{\left(\gamma d + w^{TOT}\right) - c^y_{t,1}}{(1+r)} + \frac{\left(\gamma d + w^{TOT}\right) - c^y_{t+1,2} + b^y_{t+2,3}}{(1+r)^2} \\ &= \left(\gamma d + w^{TOT}\right) \sum_{j=1}^{N} \frac{1}{(1+r)^j} - \sum_{j=1}^{N} \frac{c^y_{t+j-1,j}}{(1+r)^j} + \frac{b^y_{t+N,N+1}}{(1+r)^N} \end{aligned}$$

Making use of the FOC:

$$c^y_{t,j} = [\beta(1+r)]^{-(N-j)} c^y_{t+N-j,N} \tag{79}$$

we get

$$\begin{aligned} c^y_{t,j} &= [\beta(1+r)]^{-(N-j)} c^y_{t+N-j,N} \\ c^y_{t,1} &= [\beta(1+r)]^{-(N-1)} c^y_{t+N-1,N} \\ c^y_{t+1,2} &= [\beta(1+r)]^{-(N-2)} c^y_{t+1+N-2,N} = [\beta(1+r)]^{-(N-2)} c^y_{t+N-1,N} \\ c^y_{t+2,3} &= [\beta(1+r)]^{-(N-3)} c^y_{t+2+N-3,N} = [\beta(1+r)]^{-(N-3)} c^y_{t+N-1,N} \\ c^y_{t+N-1,N} &= [\beta(1+r)]^{-(N-N)} c^y_{t+N-1+N-N,N} = c^y_{t+N-1,N}, \end{aligned}$$

and plug in and simplify

$$\begin{aligned} \sum_{j=1}^{N} \frac{c^y_{t+j-1,j}}{(1+r)^j} &= \frac{c^y_{t,1}}{(1+r)} + \frac{c^y_{t+1,2}}{(1+r)^2} + \frac{c^y_{t+2,3}}{(1+r)^3} + ... + \frac{c^y_{t+N-1,N}}{(1+r)^N} \\ &= \frac{c^y_{t+N-1,N}}{(1+r)^N} \left[\sum_{j=0}^{N-1} \beta^{-(N-1-j)} \right] = -\frac{c^y_{t+N-1,N}}{(1+r)^N} \frac{\beta^N - 1}{\beta^{N-1}(\beta - 1)}. \end{aligned}$$

We substitute back in and keep simplifying

$$b^y_{t,1} = \left(\gamma d + w^{TOT}\right) \frac{1}{r} \left[1 - \frac{1}{(1+r)^N} \right] - \frac{c^y_{t+N-1,N}}{(1+r)^N} \frac{\beta^N - 1}{\beta^{N-1}(\beta - 1)} + \frac{b^y_{t+N,N+1}}{(1+r)^N}$$

$$0 = \left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[(1+r)^N - 1\right] - \left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)}\frac{(1-\gamma)d}{(\varrho+r)}$$
$$+\left[\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{1}{(1-\varrho)}\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)} + 1\right] b^y_{t+N,N+1}$$
$$\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)}\frac{(1-\gamma)d}{(\varrho+r)} - \left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[(1+r)^N - 1\right]$$
$$= \left[\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{1}{(1-\varrho)}\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)} + 1\right] b^y_{t+N,N+1}$$

Finally, we solve to get:

$$-b^y_{t+N,N+1} = \frac{\left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[(1+r)^N - 1\right] - \Psi\frac{(1-\gamma)d}{(\varrho+r)}}{\frac{\Psi}{1-\varrho} + 1} \tag{80}$$

$$\Psi \equiv \left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)}$$

Equation (80) is very intuitive. Savings for retirement $-b^o_{t+1}$ increase in the difference between income before and after retirement. Moreover, an increase in life expectancy (a drop in ϱ) reduces the value of the term Ψ and therefore increases $-b^o_{t+1}$.

D.4 Aggregate Savings of the Young

The previous section determines a sequence of optimal consumption at every age, $c^y_{t,1}, ..., c^y_{t,N}$, and applying the budget constraint (69) we can determine a sequence of assets for every age $b^y_{t,2}, ..., b^y_{t,N}$, which is constant for every period t. In equilibrium there is a measure 1 of households, a fraction $\frac{\phi}{\phi+\varrho}$ old, and a fraction $\frac{\varrho}{\phi+\varrho}$ young. Moreover there is a measure $\frac{\varrho}{\phi+\varrho}\frac{1}{N}$ of young households for each age. Therefore, after dropping the subscript t, we can define aggregate savings of the young households as:

$$B^y = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\sum_{j=1}^{N} b^y_{j+1}. \tag{81}$$

Savings of a young household are:

$$b^y_{j+1} = c^y_j - \gamma d - w^{TOT} + (1+r)b^y_j \tag{82}$$

We solve for b^y_N (from now on for simplicity omit the superscript y) :

$$b_N = \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}c_N + \frac{1}{1+r}b_{N+1}, \tag{83}$$

where both b_{N+1} and c_N are determined by (75) and (80) above. At age $N-1$ (we use $c_t = [\beta(1+r)]^{-1} c_{t+1}$):

$$b_{N-1} = \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}c_{N-1} + \frac{1}{1+r}\left[\frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}c_N + \frac{1}{1+r}b_{N+1}\right]$$

$$
\begin{aligned}
b_{N-1} &= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}\left[\beta(1+r)\right]^{-1} c_N \\
&\quad + \frac{1}{1+r}\left[\frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}c_N + \frac{1}{1+r}b_{N+1}\right] \\
&= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) + \frac{1}{(1+r)^2}\left(\gamma d + w^{TOT}\right) \\
&\quad - \frac{1}{1+r}\left[\beta(1+r)\right]^{-1} c_N - \frac{1}{(1+r)^2}c_N + \frac{1}{(1+r)^2}b_{N+1} \\
&= \left(\frac{1}{1+r} + \frac{1}{(1+r)^2}\right)\left(\gamma d + w^{TOT}\right) - \left(\frac{1}{\beta}+1\right)\frac{c_N}{(1+r)^2} + \frac{1}{(1+r)^2}b_{N+1},
\end{aligned}
$$

and therefore at age $N-2$:

$$
\begin{aligned}
b_{N-2} &= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}c_{N-2} + \frac{1}{1+r}b_{N-1} \\
&= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}\left[\beta(1+r)\right]^{-1}\left[\beta(1+r)\right]^{-1} c_N \\
&\quad + \frac{1}{1+r}\left(\left(\frac{1}{1+r} + \frac{1}{(1+r)^2}\right)\left(\gamma d + w^{TOT}\right) - \left(\frac{1}{\beta\,(1+r)^2} + \frac{1}{(1+r)^2}\right)c_N + \frac{1}{(1+r)^2}b_{N+1}\right) \\
&= \left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3}\right)\left(\gamma d + w^{TOT}\right) - \left(\frac{1}{\beta^2} + \frac{1}{\beta} + 1\right)\frac{c_N}{(1+r)^3} + \frac{1}{(1+r)^3}b_{N+1},
\end{aligned}
$$

and at a generic age $N-t$:

$$
b_{N-t} = \sum_{j=0}^{t}\frac{\gamma d + w^{TOT}}{(1+r)^{j+1}} - \frac{c_N}{(1+r)^{t+1}}\sum_{j=0}^{t}\frac{1}{\beta^j} + \frac{b_{N+1}}{(1+r)^{t+1}} \tag{84}
$$

We use general formulas: $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ and $\sum_{j=0}^{t} x^j = \left(1 - x^{t+1}\right)\frac{1}{1-x}$, or $\sum_{j=0}^{t}\frac{1}{(1+r)^j} = \left(1 - \frac{1}{(1+r)^{t+1}}\right)\frac{1+r}{r}$ and $\sum_{j=0}^{t}\frac{1}{(1+r)^{j+1}} = \left(1 - \frac{1}{(1+r)^{t+1}}\right)\frac{1}{r}$, so that

$$
\begin{aligned}
\sum_{j=0}^{t}\frac{\gamma d + w^{TOT}}{(1+r)^{j+1}} &= \left(1 - \frac{1}{(1+r)^{t+1}}\right)\frac{\gamma d + w^{TOT}}{r} \\
\frac{c_N}{(1+r)^{t+1}}\sum_{j=0}^{t}\frac{1}{\beta^j} &= \frac{c_N}{(1+r)^{t+1}}\left(1 - \frac{1}{\beta^{t+1}}\right)\frac{\beta}{\beta-1}
\end{aligned}
$$

hence:

$$
b_{N-t} = \left(1 - \frac{1}{(1+r)^{t+1}}\right)\frac{\gamma d + w^{TOT}}{r} - \frac{c_N}{(1+r)^{t+1}}\left(1 - \frac{1}{\beta^{t+1}}\right)\frac{\beta}{\beta-1} + \frac{b_{N+1}}{(1+r)^{t+1}} \tag{85}
$$

Now we add up the savings/borrowing over all ages from b_2 to b_N to get:

$$\begin{aligned}\sum_{t=0}^{N-2} b_{N-t} &= \frac{\gamma d + w^{TOT}}{r} \sum_{t=0}^{N-2} \left(1 - \frac{1}{(1+r)^{t+1}}\right) - \frac{\beta}{\beta-1} c_N \sum_{t=0}^{N-2} \left(\frac{1}{(1+r)^{t+1}} \left(1 - \frac{1}{\beta^{t+1}}\right)\right) \\ &+ b_{N+1} \sum_{t=0}^{N-2} \left(\frac{1}{(1+r)^{t+1}}\right)\end{aligned}$$

The value of each summation term is:

$$\begin{aligned}\sum_{t=0}^{N-2} \left(1 - \frac{1}{(1+r)^{t+1}}\right) &= -\frac{1}{r}\left(r - (r+1)^{-N+1} - Nr + 1\right) \\ &= -\frac{1}{r}\left(1 + r(1-N) - \frac{1}{(1+r)^{N-1}}\right) \\ &= \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}} + r(N-1) - 1\right)\end{aligned}$$

$$\begin{aligned}\sum_{t=0}^{N-2} \left(\frac{1}{(1+r)^{t+1}} \left(1 - \frac{1}{\beta^{t+1}}\right)\right) &= \sum_{t=0}^{N-2} \frac{1}{(1+r)^{t+1}} - \sum_{t=0}^{N-2} \left(\frac{1}{[(1+r)\,\beta]^{t+1}}\right) \\ &= \frac{1-(1+r)^{-N+1}}{(1+r)-1} - \frac{1-[(1+r)\,\beta]^{-N+1}}{[(1+r)\,\beta]-1}\end{aligned}$$

$$\sum_{t=0}^{N-2} \left(\frac{1}{(1+r)^{t+1}}\right) = \frac{1-(1+r)^{-N+1}}{(1+r)-1}$$

Substituting them back:

$$\sum_{t=0}^{N-2} b_{N-t} = \frac{\gamma d + w^{TOT}}{r} \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}} + r(N-1) - 1\right)$$
$$-\frac{\beta}{\beta-1} c_N \left[\frac{1-(1+r)^{-N+1}}{(1+r)-1} - \frac{1-[(1+r)\,\beta]^{-N+1}}{(1+r)\,\beta-1}\right] + b_{N+1} \frac{1-(1+r)^{-N+1}}{(1+r)-1}$$

We rename terms:

$$\sum_{t=0}^{N-2} b_{N-t} = A_1 \frac{\gamma d + w^{TOT}}{r} - A_2 c_N + A_3 b_{N+1}$$

$$\begin{aligned}A_1 &\equiv \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}} + r(N-1) - 1\right) \\ A_2 &\equiv \frac{\beta}{\beta-1}\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1} - \frac{1-[(1+r)\,\beta]^{-N+1}}{(1+r)\,\beta-1}\right] \\ A_3 &\equiv \frac{1-(1+r)^{-N+1}}{(1+r)-1}\end{aligned}$$

D.5 Aggregate Savings of the Old

In equilibrium there are $\frac{\varrho}{\phi+\varrho}\frac{1}{N}$ households that become old every period, and $\frac{\varrho}{\phi+\varrho}\frac{1}{N}(1-\varrho)^j$ households that survived for j periods . Therefore aggregate savings of the old households are:

$$B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\sum_{j=1}^{\infty}(1-\varrho)^j b_j^0 \tag{86a}$$

also note that b_1^0 is the initial savings from young age as defined in (80)

Recall that (from (72)):

$$b_{j+1}^o = A + \beta(1+r)b_j^o, \tag{87}$$

and

$$A \equiv \frac{(1-\varrho)\left[1-\beta(1+r)\right]}{(\varrho+r)}(1-\gamma)d,$$

so that:

$$b_2^o = A + \beta(1+r)b_1^o, \tag{88}$$

$$b_3^o = A + \beta(1+r)b_2^o = A + \beta(1+r)A + \beta^2(1+r)^2 b_1^o, \tag{89}$$

$$b_t^o = A + \beta(1+r)A + ... + \beta^{t-2}(1+r)^{t-2}A + \beta^{t-1}(1+r)^{t-1}b_1^o, \tag{90}$$

$$b_t^o = A\left[\sum_{j=0}^{t-2}\beta^j(1+r)^j\right] + \beta^{t-1}(1+r)^{t-1}b_1^o, \tag{91}$$

$$b_t^o = \frac{(\beta(1+r))^{t-1}-1}{\beta(1+r)-1}A + \left[\beta(1+r)\right]^{t-1}b_1^o, \tag{92}$$

$$\begin{aligned}
\sum_{t=1}^{\infty}(1-\varrho)^t b_t^0 &= \sum_{t=1}^{\infty}(1-\varrho)^t\left[\frac{(\beta(r+1))^{t-1}-1}{\beta(1+r)-1}A + \left[\beta(1+r)\right]^{t-1}b_1^o\right] \\
&= \frac{A}{\beta(1+r)-1}\sum_{t=1}^{\infty}(1-\varrho)^t\left[(\beta(r+1))^{t-1}-1\right] + b_1^o\sum_{t=1}^{\infty}(1-\varrho)^t\left[\beta(1+r)\right]^{t-1} \\
&= \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\sum_{t=1}^{\infty}(1-\varrho)^t(\beta(r+1))^{t-1} - \frac{A}{\beta+r\beta-1}\sum_{t=1}^{\infty}(1-\varrho)^t \\
&= \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\frac{1}{\beta(1+r)}\sum_{t=1}^{\infty}\left[(1-\varrho)\beta(1+r)\right]^t - \frac{A}{\beta+r\beta-1}\sum_{t=1}^{\infty}(1-\varrho)^t,
\end{aligned}$$

$$\sum_{t=1}^{\infty}\left[(1-\varrho)\beta(1+r)\right]^t = \frac{1}{1-(1-\varrho)\beta(1+r)} - 1 = \frac{(1-\varrho)\beta(1+r)}{1-(1-\varrho)\beta(1+r)},$$

and

$$\sum_{t=1}^{\infty}(1-\varrho)^t = \frac{1}{1-1+\varrho} - 1 = \frac{1-\varrho}{\varrho}.$$

Finally:

$$\sum_{t=1}^{\infty}(1-\varrho)^t b_t^0 = \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)} - \frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}$$

D.6 Summing Up Aggregate Household Borrowing

Aggregate household borrowing is:

$$B = B^o + B^y \tag{93}$$

where savings of the old is:

$$B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[\left(\frac{A}{\beta+r\beta-1}+b^{retirement}\right)\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)}-\frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}\right]$$

and

$$A \equiv \left(\frac{(1-\varrho)\left(1-(1+r)\beta\right)}{(\varrho+r)}\right)(1-\gamma)d,$$

$$b^{retirement} \equiv \frac{\Psi\frac{(1-\gamma)d}{(\varrho+r)}-\left(\gamma d+w^{TOT}\right)\frac{1}{r}\left[(1+r)^N-1\right]}{\frac{\Psi}{1-\varrho}+1},$$

and

$$\Psi \equiv \left(\frac{1}{\beta}-(1-\varrho)\right)\frac{\beta^N-1}{\beta^{N-1}(\beta-1)}.$$

And savings of the young is:

$$B^y = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\sum_{t=0}^{N-2}b_{N-t} = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[A_1\frac{\gamma d+w^{TOT}}{r}-A_2c_N+A_3b^{retirement}\right],$$

where

$$\begin{aligned}
A_1 &\equiv \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right),\\
A_2 &\equiv \frac{\beta}{\beta-1}\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-\left[(1+r)\beta\right]^{-N+1}}{(1+r)\beta-1}\right],\\
A_3 &\equiv \frac{1-(1+r)^{-N+1}}{(1+r)-1},
\end{aligned}$$

and

$$c_N = \left(\frac{1}{\beta}-(1-\varrho)\right)\left(\frac{(1-\gamma)d}{(\varrho+r)}-\frac{b^{retirement}}{(1-\varrho)}\right).$$

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(Appendix Tables and Figures)

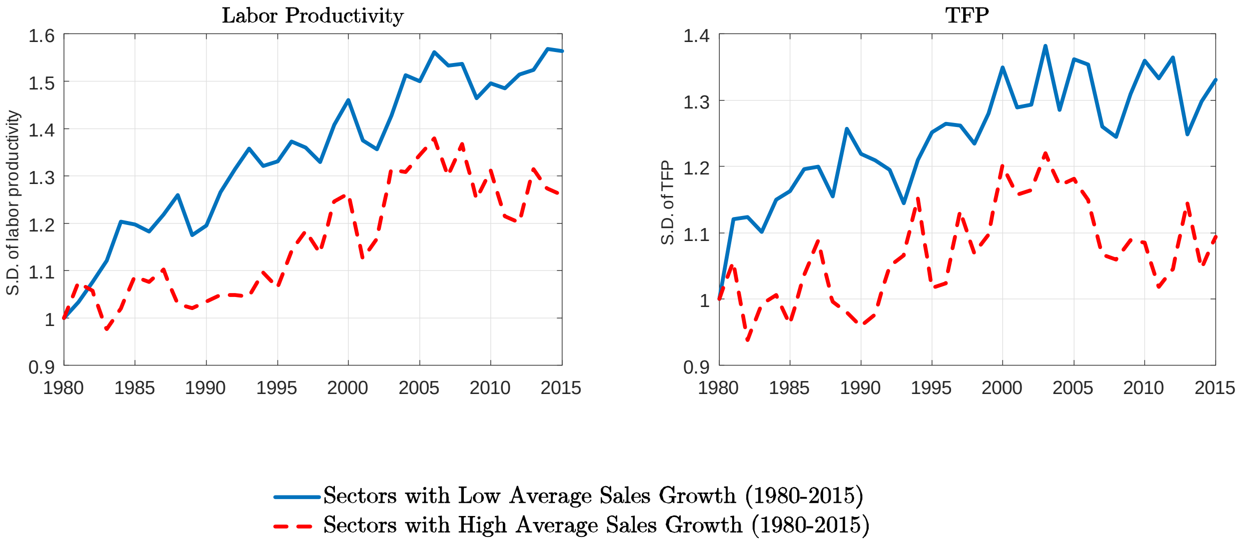

Figure A: Within-Industry Dispersion in Firm-Level Labor Productivity and TFP: Robustness Exercise that Groups Sectors According to Average Sales Growth Rates (*Source: Compustat data, own calculations*)

Variables	(1) *TFP*	(2) *TFP*	(3) y	(4) y
Time trend		−0.000892		0.00316 ∗ ∗∗
		(0.000837)		(0.000737)
Time trend*High share		0.00558 ∗ ∗∗		0.00370 ∗ ∗∗
		(0.000622)		(0.000548)
High share	0.0897 ∗ ∗∗		0.110 ∗ ∗∗	
	(0.00990)		(0.0144)	
Observations	828	828	828	828
R-squared	0.112	0.632	0.134	0.869
Industry FE	no	yes	no	yes
Year FE	yes	yes	yes	yes

Table A: Relationship Between the Intangible Share and the Dispersion in Productivity - Regression Analysis