The Role of Diverse Beliefs in Asset Pricing and Equity Premia[1]

by Mordecai Kurz[2] and Maurizio Motolese[3]

A very preliminary draft: February 2, 2010

Summary:
Why do risk premia vary over time? We examine this problem theoretically and empirically by studying the effect of market belief on risk premia. Individual belief is taken as a fundamental, primitive, state variable. Market belief which is the distribution of individual beliefs is observable, it is central to the empirical evaluation and we show how to measure it. Our asset pricing model is familiar from the noisy REE literature but we adapt it to an economy with diverse beliefs. We derive equilibrium asset prices and implied risk premium. Our approach permits a closed form solution of prices hence we trace the exact effect of market belief on the time variability of asset prices and risk premia. We test empirically the theoretical conclusions.

We study premia on long positions of S&P500 for investment periods of 6 -12 months and find that, on average, a change in the mean market belief by 1 standard deviation changes equity premia by about 0.4 standard deviation of excess returns. We also find that the introduction of belief variables reduces the effect of the traditional Fama and French (1989) variables used to forecast excess returns. More specifically we find that given information about market belief the term premium, the default premium and Lattau and Ludvigson's (2001) CAY variables have no significant effect. As to the structure of the premium we show that when the market holds abnormally favorable belief about the future payoff on the S&P500 the market views the long position as less risky hence the equity risk premium declines. We also find that in a VAR model of asset and consumption growth, our market belief variables have a significant effect. We thus conclude that market belief has significant independent effect on economic fluctuations and risk premia.

JEL classification: C53, D8, D84, E27, E4, G12, G14.
Keywords: Risk premium; heterogenous beliefs; market state of belief; asset pricing; Bayesian rationality; Rational Beliefs.

0. Introduction

A large and growing literature in empirical finance has demonstrated that the predictability of asset returns is increased by using observed measures of market belief or sentiment. A sample of work using various measures includes Miller (1977), DeLong et al. (1990), Lee et al. (2002), Diether et al. (2002), Johnson (2004), Park (2005), Baker and Wurgler (2006), Fan (2006), Karakatsani and Salmon (2008), Campbell and Diebold (2009) and Kurz and Motolese (2011). Also, models with diverse beliefs have become the basis for recent work in Behavioral Finance. All this work is in contrast with the rational expectations view relating changes in risk premia *only* to changes in information about exogenous fundamentals which correctly forecast changes in risky events, the most

---

[1] We thank George Evans, Ken Judd, Volker Wieland and participants in the SITE summer 2009 meeting at Stanford University for helpful comments on an earlier draft. Howei Wu provided valuable technical assistance at various stage of the research and her joint work with Motolese is interrelated with the results reported in Section 4.

[2] Department of Economics, Serra Street at Galvez, Stanford University, Stanford, CA. 94305-6072, USA

[3] Istituto di Politica Economica, Università Cattolica di Milano, Via Necchi 5, 20123, Milano, Ital

Correspondence to: Mordecai Kurz, Email: mordecai@stanford.edu

important of which are business cycles. Volatility of asset prices is surely affected by real measures that forecast future business conditions, but the work cited holds that market volatility is driven also by how reality is perceived by investors. Leaving Fan (2006) and Kurz and Motolese (2011) aside for the moment, we stress that the finance literature cited has searched primarily for measures that can be used to forecast returns. From a conceptual perspective the literature takes the view that diverse beliefs reflect some irrational behavior which can be exploited by smart investors who use public information about irrational agents for better forecasting of future returns. The standard model inspiring this perspective has been the Noise Trading model of DeLong et al. (1990).

An alternative view holds that all agents are rational and equally smart but hold diverse beliefs due to their incomplete knowledge about the complex economy. This diversity is central since much of asset returns' volatility is caused by fluctuations in market belief which is an economy wide externality with a predictable component. The theoretical work on this approach developed over the last two decades. A sample of papers includes Harrison and Kreps (1978), Varian (1985), (1989), Harris and Raviv (1993), Detemple and Murthy (1994), Kandel and Pearson (1995), Kurz (1974), (1994), (1997), (2007), (2008) Kurz and Beltratti (1997), Kurz and Motolese (2001) (2011), Kurz and Schneider (1996), Kurz et al (2005a) (2005b), Kurz and Wu (1996), Motolese (2001), (2003), Fan (2006), Nakata (2007), Nielsen (1996), (2003) and Wu and Guo (2003), (2004). In particular, Kurz and Beltratti (1997), Kurz and Motolese (2001) and Kurz et al. (2005a) demonstrate that models with diverse beliefs can explain the equity premium. Most of these papers study market volatility via simulations. In contrast, Kurz and Motolese (2011) examine determinants of risk premia by studying analytical hypotheses which can then be tested directly with data on market belief.

The third line of research by Behavioral Finance, starts from the view that investors in financial markets are not fully rational. Although this is a very large literature, one can generalize by observing that the typical structure of a Behavioral Finance model assumes one population segment is fully knowledgeable and rational while a second segment exhibits some form of bounded rationality (e.g. Lee et al. (1991), Barberis et al. (1998), Daniel et al. (1998), Lee et al. (2002), Sheinkman and Xiong (2005)). But this is the structure of Noise Trading models. In such a world there are smart traders who know the truth and not-so-smart agents who trade aimlessly. Equilibrium reflects both the truth and the noise. Hence, a Behavioral Finance perspective is compatible with the empirical finance literature cited above. From a perspective of modeling asset pricing the Noise Trading

paradigm of DeLong et al. (1990) can then be taken to be the underlying formal structure of this literature.

One may observe that we are yet to find market data on noise trading or discover agents who provide data declaring themselves to be irrational noise traders. Hence, noise trading theory or irrationality in markets cannot be falsified with *direct* data measuring actions taken by individuals in the market who inform us about their irrationality. Standard empirical evidence such as orthogonality tests which are common in the Behavioral Finance literature reject rational expectations but do not prove irrational behavior. Rational Expectations is an extreme hypothesis which demands from agents to know what they cannot know. We thus reject both rational expectations and irrational noise trading as extreme hypotheses. We outline a middle ground theoretical framework for asset pricing under diverse beliefs which are rationalized and derive the asset pricing theory developed under these rationality restrictions. We then test the implications with data on market belief.

Briefly stated, our theoretical perspective proposes that all traders are smart but do not know the true stochastic dynamics of the economy. The true dynamics is non-stationary in the sense that it has a time varying structure reflected in changing products and technology, but the structure changes faster than can be learned with precision from data. Since investors know only long term empirical distributions deduced from data, equilibrium asset prices are determined only by what traders believe and by what they know. The *true* dynamics is irrelevant to asset pricing except for the way it reveals itself in the data agents know. In rejecting the false dichotomy of a noise trading paradigm between smart and not-so-smart traders, we observe that lack of full knowledge of the truth is the foundation of belief diversity but rational diversity is not noise; it has a structure and can be studied scientifically. The issue of rationality deserves a comment. It is obvious “rationality” of beliefs is different from rationality of actions. The Rational Expectations hypothesis was skillfully marketed by attaching the term “rational” to impossible requirements. This lead to a common view that one can either hold Rational Expectations or be irrational. Indeed, virtually all “evidence” for irrationality presumes only the false choice between these two norms. Rationalizing a belief is a complex problem and a theory which permits diverse but rationalizable beliefs cannot be based on too tight a rationality principle that would prevent diversity from arising in the first place. Belief diversity results from lack of some essential knowledge. Hence, to rationalize a belief one starts from that ignorance of the truth and ends by adopting principles of what is “reasonable” to believe rather than what is inevitable. In

short, with incomplete knowledge reasonable men can reach different conclusions using the same information!

An important dimension of the theory holds that agents' beliefs are primary causes of economic events and have their own independent dynamics. This view dictates some of the formal developments below. It does not negate the role of real fundamental factors in asset pricing but insists market belief is an independent force which affects market dynamics, economic performance and, ultimately, future economic reality. This idea, which is central to our work has gradually had an impact on the study of Economic History and Political Economy (e.g. see Mokyr (2010), Greif (2006) and Tabellini (2008)).

In this paper we study the role of market belief in asset prices and risk premia. Beliefs are diverse but rational in a sense to be discussed. An individual state of belief is an index which pins down an agent's conditional probability at a given date while a market state of belief is a *distribution* of individual beliefs. Although individual beliefs are not observed, we show how to extract market belief from forecast samples. In the analysis we focus on the first two moments of these distributions and establish the relation between risk premia and market belief. We derive analytical results which are then tested empirically with data on market belief.

Observations on market belief are extracted from three sources. First, semi-annual data of the Livingston Survey made available by the Philadelphia Federal Reserve. Second, monthly forecasts of future interest rates and macro economic variables compiled by the Blue Chip Financial Forecasts (BLUF) since 1983:1 and by the Blue Chip Economic Indicators (BLU) since 1980:1. Third, quarterly forecasts of macro economic variables by the Survey of Professional Forecasters (SPF) conducted since 1968:11 and made available by the Philadelphia Federal Reserve. Our analysis was conducted using either the Livingston files or the *combined* Blue Chip and SPF files as explained later.

The present paper is a companion to Kurz and Motolese (2011) where we focus on risk premia in debt markets: U.S. Treasuries and Federal Fund futures. This paper addresses the problem of the stock market hence our results are more directly comparable to the empirical finance literature cited earlier. Our results confirm that some traditional variables reported in Fama and French (1989) predict excess returns, particularly the dividend yield. However, we also find that for the investment periods of 6-12 studied here the Default Premium, the Term Spread and the variable CAY developed

by Lattau and Ludvigson (2001) have no effect once belief variables are introduced . We develop a measure of mean market belief and cross sectional standard deviation of beliefs and find that the mean market belief contributes significantly to explaining the equity risk premium. In a simple VAR model of wealth and consumption growth we also find that market belief has an independent effect on consumption growth. In contrast with our study of the bond market where we show the cross sectional standard deviation has a significant effect on risk premia, we cannot confirm this effect in the stock market.

Apart from showing market belief has an independent effect on fluctuations and risk premia we also show the effect on the risk premium takes a specific form which is predicted by the theory. We thus prove first a theorem which asserts that when the market holds abnormally favorable belief about future profits of an asset, the risk premium on long positions of that asset falls. More generally, market optimism about future economic conditions lowers the risk premium while pessimism about future economic conditions increases it. This is a strong result about market overshooting. We test this conclusion empirically and find the data are compatible with the theoretical findings.

1. Asset Pricing Under Heterogenous Beliefs

1.1 An Illustrative Decision Model

Consider an asset or a portfolio of assets whose market price is p_t, paying an exogenous risky sequence $\{D_t, t = 1, 2, ...\}$ under a true and unknown probability $\hat{\Pi}$ which is non-stationary due to structural changes over time. Let r_t be the riskless interest rate, $R_t = 1 + r_t$ and hence excess return over the riskless rate is $(1/p_t)(p_{t+1} + D_{t+1} - R_t p_t)$. The *risk premium* over the riskless rate is the conditional expectations of excess returns. Since it is a function of equilibrium prices, a risk premium - as a function of state variables - is best deduced from equilibrium prices. With this in mind, the model below is used to deduce a closed form solution of the asset price so as to enable a study of the factors determining the risk premium. To obtain closed form solutions we use a model which is very common in the literature on Noisy Rational Expectations Equilibrium (e.g. Brown and Jennings (1989), Grundy and McNichols (1989), Wang (1994), He and Wang (1995), Allen, Morris and Shin (2006) and others cited in Brunnermeier (2001)). Nevertheless, our key results are fully general and do not depend upon the model used. We now address two issues. Our agents do not know the true probability $\hat{\Pi}$ and this raises two questions that are at the basis of our later development. First, why

do agents not know $\hat{\Pi}$? Second, what is the common knowledge basis of all agents in an economy with diverse beliefs?

Starting with the second question, our answer is *past data on observables.* The economy has observable variables and D_t is one of them. Agents have a long history of the variables, allowing rich statistical analysis which leads all to compute the same relative frequencies of events or moments and finite dimensional distributions of observed variables. Using standard extension of measures methods they deduce from the data a unique empirical probability measure on infinite sequences denoted by $\hat{m}$. It can be shown that $\hat{m}$ is stationary (see Kurz (1994)) and we call it "the stationary measure." This is the *empirical knowledge shared by all agents*[4]. We assume in this paper the data reveals that under the measure $\hat{m}$, $\{D_t, t=1,2,...\}$ is a Markov process where D_{t+1} is distributed conditionally normal with means $\mu + \lambda_d(D_t - \mu)$ and variance σ_d^2. It follows that μ is the unconditional mean[5] of D_t. The unique probability $\hat{m}$ is then known to all. To simplify define $d_t = D_t - \mu$, hence the process $\{d_t, t=1,2,...\}$ is zero mean Markov process with unknown true probability Π and known stationary empirical probability m. We assume specifically that the dynamics of payoffs d_t *deduced from the empirical frequencies* is characterized by a first order Markov process with transition

(1) $$d_{t+1} = \lambda_d d_t + \rho^d_{t+1} \quad , \quad \rho^d_{t+1} \sim N(0, \sigma_d^2)$$

and since this characterizes the probability measure m, we write $E^m[d_{t+1}|d_t] = \lambda_d d_t$. We often use the term "empirical distribution" to refer to the probability m on infinite sequences deduced from data. *Why is m not equal to Π?* With this issue in mind we turn to the first question.

Our economy is undergoing changes in technology and social organization. These are rapid with major economic effects, making $\{d_t, t=1,2,...\}$ a non-stationary process. Although this means that the distributions of the d_t's are time dependent, it is more than viewing $\{d_t, t=1,2,...\}$ as a sequence of productivity "regimes." It also means that, although we measure the d_t in a single unit of account, over time the nature of assets and commodities change. Such variability makes it impossible

[4] We always have finite data and cannot estimate with certainty the measure $\hat{m}$ on sequences. However, if this measure has a simple representation such as a Markov transition function, then with adequate data it can be approximated so closely as to make the assumption in the text entirely reasonable. Estimation of $\hat{m}$ with an epsilon error only increases belief divergence as it reduces the scope of what is common knowledge and complicates the theory without adding much empirical substance.

[5] It would be more realistic to assume the values D_t grow and the growth rate of the values has a mean μ rather than the values themselves. This added realism is useful when we motivate the empirical model later but is not essential for the analytic development.

to learn the unknown Π. The probability m is then merely an average over an infinite number of regimes. Belief diversity starts from agents' disagreement over the meaning of public information. They believe Π is different from m and construct models to express the implications they see in the data. Being common knowledge, m is then a basic point of reference for any concept of rationality.

Turning now to our infinite horizon model, at date t agent i buys θ_t^i shares of stock and receives the payment $d_t + \mu$ for each unit of θ_{t-1}^i held. We assume the riskless rate is constant over time so there is a technology by which an agent can invest the amount B_t^i at date t and receive with certainty the amount $B_t^i R$ at date t+1. The definition of consumption is then standard

$$c_t^i = \theta_{t-1}^i [p_t + d_t + \mu] + B_{t-1}^i R - \theta_t^i p_t - B_t^i .$$

Equivalently, define wealth $W_t^i = c_t^i + \theta_t^i p_t + B_t^i$ and derive the familiar transition of wealth

(2a) $$W_{t+1}^i = (W_t^i - c_t^i)R + \theta_t^i Q_{t+1} \ , \quad Q_{t+1} = p_{t+1} + (d_{t+1} + \mu) - R p_t .$$

Q_t are excess returns. Given some initial values (θ_0^i, W_0^i) the agent maximizes the expected utility

(2b) $$U = \underset{(\theta^i, c^i)}{E_t^i} \left[\sum_{s=0}^{\infty} -\beta^{t+s-1} e^{-(\frac{1}{\tau} c_{t+s}^i)} \,|\, H_t\right]$$

subject to a vector of state variables ψ_t^i and their transitions, to be specified later. H_t consists of all past observable variables. We recognize the limitations of the exponential utility and use it as a vehicle to explain the main ideas, hence the term "illustrative" in this Section's title. After deducing the closed form solution of equilibrium risk premium we show how to generalize the key results.

We assumed (1) is the dynamics of d_t *deduced from the data* but explained that it may or may not be the true data generating process. We assume later that the true data generating process is

(3) $$d_{t+1} = \lambda_d d_t + b_t + \varepsilon_{t+1}^d \quad , \quad \varepsilon_{t+1}^d \sim N(0, \frac{1}{\beta})$$

where b_t is an infinite sequence of unobserved "regime" parameters. One may ask under what conditions can (1) be the empirical distribution of (3)? The answer is that for this computation one considers b_t to be a random variable and then require that the empirical distribution of $b_t + \varepsilon_{t+1}^d$ equals the distribution of $\rho_{t+1}^d \sim N(0, \sigma_d^2)$. Three examples will explain the required conditions.

<u>Example 1</u>: the sequence of b_t is a realization of i.i.d. draws from a random variable $Y \sim N(0, \sigma_y^2)$. In this case the required condition for (1) to be the empirical distribution of (3) is $\sigma_y^2 + \frac{1}{\beta} = \sigma_d^2$.

<u>Example 2:</u> b_t are realizations of a stochastic process $b_t = \lambda_b b_{t-1} + y_t$, $y_t \sim N(0, \sigma_y^2)$. The variance of b is $\sigma_y^2/(1-\lambda_b^2)$ hence the conditions are

$$\text{(i)} \quad \frac{\sigma_y^2}{1-\lambda_b^2} + \frac{1}{\beta} = \sigma_d^2 \quad , \quad \text{(ii)} \quad \mathrm{cov}(\varepsilon_t, \varepsilon_{t-1}) = -\frac{\lambda_b \sigma_y^2}{1-\lambda_b^2} .$$

That is, if the ρ_t^d sequence is i.i.d and the b_t exhibit serial correlation then for (1) to be the empirical distribution of (3) the ε_t sequence must exhibit serial correlation. Similarly, if the b_t exhibit serial correlation but the ε_t are i.i.d. then the empirical distribution will reveal serial correlation of the ρ_t^d.

<u>Example 3:</u> b_t are realizations of a stochastic process $b_t = \lambda_{b_t} b_{t-1} + y_t$, $y_t \sim N(0, \sigma_y^2)$ where λ_{b_t} are derived from a tent map on (-1, +1) hence are uniformly distributed in the interval. In this case the b_t are not serially correlated, we can show that $Var(b) = 3\sigma_y^2$ and the needed condition is $3\sigma_y^2 + \frac{1}{\beta} = \sigma_d^2$. One can modify the example and have the λ_{b_t} change only slowly at random dates.

The examples exhibit conditions on b_t which rationalize (3) with respect to (1). Indeed, suppose the truth was unknown and (3) was adopted as a belief by an agent. Then under the restrictions on b_t in each of the examples, (3) would have been a Rational Belief (see Kurz (1994)).

Since the truth is unknown what should agents do? Those who believe the economy is stationary accept (1) as the truth. Such belief should be rational since there is no empirical evidence against it. But the complexity of $\{ d_t, t=1,2,... \}$ lead most to hold the view that (1) is not adequate to forecast the future. All surveys of forecasters show that subjective judgment about the data contributes more than 50% to the final forecast (e.g. Batchelor and Dua (1991)). Hence, agents form their own beliefs about state variables. The structure of belief is then our next topic.

1.2 Modeling Diverse Beliefs I: Individual Belief as a State Variable and Rationality

As postulated earlier, we assume *forecast distributions are public observations over time*. This fact points to a difference between markets with and without private information. With asymmetric private information a market is secretive: agents do not reveal their forecasts since these provide real information and such revelation eliminates the small advantage they have. If an agent's forecast of state variables is revealed in our market - without private information - others do not consider it information. They view it as an expression of his opinion and do not update their own beliefs about state variables. On the other hand *the forecast of others is crucial information used to forecast endogenous variables*. This follows from the fact that future equilibrium prices depend upon future market belief hence to forecast future prices one must forecast future market belief (i.e. the belief of "others"). But then, how do we describe individual and market beliefs? Our response is to treat individual beliefs as primitive state variables. They are basic *cause* of market change like any

other state variable. Analytically we use the approach of Kurz, Jin and Motolese (2005a), (2005b) and Kurz and Motolese (2011) as adapted to the problem of this paper. This adaptation is outlined next.

In our Markov economy a probability measure on sequences of state variables is represented by a sequence of transition functions. Hence, an individual belief is described by a *personal* state of belief which uniquely pins down the agent's perceived transition functions of state variables. Hence, personal state variables and the economy-wide state variables are not the same. A personal state of belief may be thought of as analogous to an agent "type" at each date. However, at t he is not certain of his future belief types which are then determined by a transition of his own personal state of belief. The *distribution* of individual states of belief, which then defines "the market state of belief," is an economy-wide observable state variable. All moments of this distribution could matter in equilibrium, but due to the exponential utility we use, equilibrium endogenous variables depend only on the *mean market states of belief*. This is generalized in the empirical work reported later since we have already noted that market belief is observable. In equilibrium, endogenous variables (e.g. prices) are functions of the economy's state variables, including market belief. In a large economy an agent's *anonymity* implies that a personal belief state has a negligible effect on prices and past personal states are not observed. Finally, due to the effect of market belief on endogenous variables, to forecast future endogenous variables an agent must *forecast the beliefs of others*. We thus turn now to discuss the dynamics of individual beliefs and demonstrate that belief dynamics is dictated by rationality.

The theory of Rational Beliefs due to Kurz (1994), (1997) defines an agent to be *rational* if his model cannot be falsified by observed data and if simulated, it reproduces the stationary probability m deduced from past data. Here we do not use the detailed restrictions the theory imposes since we aim to test Theorem 3 below and this theorem is true under less restrictive conditions. We thus accept the idea that rationality should require a belief to be compatible with past data but translate it into three rationality principles used in the analysis to follow. We now explain them.

Rationality Principle 1: A belief cannot be a *constant transition* unless an agent believes the stationary transition (1) is the truth. This is so since if one holds a *constant* transition as his belief which is not (1) then the time average of his belief is different from (1). Since (1) is the time average in the data, it proves the agent is irrational. Simply stated, an agent who is always optimistic or always pessimistic relative to (1) is irrational since he holds beliefs

which are strongly rejected by the data. This leads to the observation *that rationality implies that beliefs must fluctuate* and thus must have inherent dynamics which contributes to market volatility.

Given the evidence for persistent *diverse* beliefs, one concludes that agents hold wrong beliefs (since there is only a single truth). But being rational and "wrong" are not in conflict. Rational agents hold wrong beliefs when there is no evidence against them. The term *"wrong"* is obviously used here relative to an unknowable standard. We then turn to a second rationality principle. To state it note that a belief index describes how an agent's perceived a transition *deviate*s from (1):

Rationality Principle 2: A belief does not deviate from (1) consistently and hence the belief index has an unconditional mean of zero.

Agent i's *state of belief is denoted by* g_t^i. It describes i's perception by pinning down his transition functions. Adding to "anonymity" we assume agent ℓ knows his own g_t^ℓ and the market *distribution* of g_t^i at t across i. In addition he observes past distributions of the g_τ^i for all $\tau < t$ hence he knows past values of all moments of the distributions of g_τ^i.

How is g_t^i used by the agents? If d_{t+1}^i is agent i's perception of t+1 payoff then g_t^i pins down $E_t^i d_{t+1}^i$ by specifying the *difference* between his date t forecast of all state variables and the forecasts under the empirical distribution m. Agent i's date t *perceived* distribution of d_{t+1} is then defined by

(4) $$d_{t+1}^i = \lambda_d d_t + \lambda_d^g g_t^i + \rho_{t+1}^{id} \quad , \quad \rho_{t+1}^{id} \sim N(0, \hat{\sigma}_d^2).$$

The assumption that $\hat{\sigma}_d^2$ is the same for all agents is made for simplicity. It follows that g_t^i measures

(5) $$E^i[d_{t+1}^i | H_t, g_t^i] - E^m[d_{t+1} | H_t] = \lambda_d^g g_t^i.$$

Rationality Principle 2 requires g_t^i to have a zero unconditional mean. But (5) also shows *how to measure* g_t^i *in practice*. For a state variable X_t, data on i's forecasts of X_{t+1} (in (4) it is d_{t+1}) are measured by $E^i[X_{t+1}^i | H_t, g_t^i]$. To compute (5) we use standard econometric techniques to construct $E^m[X_{t+1} | H_t]$. Such construction was used by Fan (2006), Kurz and Motolese (2010) and is later explained. We have $g_t^i = 0$ for an agent who believes m is the truth. Since a belief is about our changing society the g_t^i reflect belief about different economies. For example, in 1900 the g_t^i were related to electricity and combustion engines, while in 2000 they reflected beliefs about information technology. Success or failure of past g_τ^i tell you nothing about what present day g_t^i should be.

The two rationality principles we have adopted imply that in an economy with diverse beliefs an agent's belief (represented by his transition functions) must fluctuate and this means that the index

g_t^i must fluctuate over time. The third principle addresses the question of dynamics.

Rationality Principle 3: The transition functions of g_t^i are Markov, taking two possible forms which exhibit persistence

(6a) $$g_{t+1}^i = \lambda_Z g_t^i + \rho_{t+1}^{ig} \quad , \quad \rho_{t+1}^{ig} \sim N(0, \sigma_g^2)$$

(6b) $$g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^d d_t + \varrho_{t+1}^{ig} \quad , \quad \varrho_{t+1}^{ig} \sim N(0, \sigma_g^2)$$

where ρ_{t+1}^{ig} or ϱ_{t+1}^{ig} are correlated across i reflecting correlation of beliefs across individuals. We shall explain later why this correlation is a crucial component of the theory.

How can we justify (6a)-(6b) who play a central role in the model? Our first answer is that the data supports this specification and we demonstrate this fact below. Next, in Section 1.3 below we prove (6a)-(6b) as a result of Bayes rationality. Before doing so we comments on these conditions.

In a rapidly changing environment it is useful to describe belief diversity so that equilibrium analysis is *tractable*. As is shown below, the advantage of (6a)-(6b) is that they lead to a simple description of equilibrium pricing with diverse beliefs. It does not entail extraction of information from market prices, it needs each agent to have a distinct state space to describe his own uncertainty and requires an endogenous expansion of the economy-wide state space for equilibrium pricing.

(6a) is the dynamics of Example 2, Section 1.1 above and under the two conditions in that example it is a Rational Belief. We employed such dynamics in several papers (e.g. Kurz, Jin and Motolese (2005a), (2005b) and Kurz and Motolese (2009)). Here we develop the theory under both (6a) and (6b) and show that the main Theorem 3 below is true under either one. Persistence of beliefs is common to (6a) - (6b). To see the difference between them recall our assumption that agents have ample past data for learning (1). The question is how much they learn from *recent* data. Under (6a) individual models are based on other factors than recent quantitative data while under (6b) recent data contribute to their assessment. In the next section we deduce (6b) analytically from a Bayesian learning procedure. Since (6a) is a special case of (6b) Theorem 1 below provides conditions for (6a) or (6b) to be a rational dynamics of individual beliefs. We may note that for some readers the most compelling justification for the persistent dynamics of (6a) or (6b) is *the fact that it is supported by the data*. However, we now explore conditions for deducing (6a)-(6b) from Bayesian rationality.

1.3 Deducing (6b) from a Model of Bayesian Rationality

In a standard Bayesian model an agent faces data generated under a stationary structure but

with an unknown *fixed* parameter. The agent starts with a prior on the parameter and uses Bayesian inference for retrospective updating of his belief. The term "retrospective" stresses that inference is made *after* data is observed. In real time an agent uses the prior to forecast future variables while learning can only improve *future* forecasts of the variables. Under the true probability Π the value d_t has a sequence of transition functions of the form (4), i.e $d_{t+1} - \lambda_d d_t = b_t + \varepsilon^d_{t+1}$, $\varepsilon^d_{t+1} \sim N(0, \frac{1}{\beta})$. The fixed parameters are known as they are deduced from the empirical frequencies. We assume agents know λ_d and β but not the "regimes" b_t. The infinite number of time varying parameters b_t express the non stationarity of the economy. Changes reflect technologies and social organizations that define each era. In reality commodities change over time and b_t represent different objects hence a single commodity which is comparable over time is only a model simplification.

The structure of changing parameters requires us to supplement the standard Bayesian learning process. To explain why note that at t-1 an agent has a prior belief about b_{t-1} with which he forecasts d_t. After observing d_t he updates his prior into a sharper posterior estimate $E_t^i(b_{t-1}|d_t)$ of b_{t-1} which, as a random variable, we denote by $b_{t-1}(d_t)$. But at date t he needs to forecast d_{t+1}. For that he does not need a posterior estimate of b_{t-1} but rather, *a new prior on* b_t! Agents do not know if and when a parameter changes. If they knew b_t changes slowly or $b_t = b_{t-1}$ then an updated posterior of b_{t-1} is a good prior of b_t. Without such knowledge, they presume $b_{t-1} \neq b_t$ is possible and look for a new prior. They would seek additional information to arrive at an alternative subjective estimate of b_t. Public *qualitative* information is an important source which offers a route to such alternative estimate of b_t.

1.3.1 *Qualitative Information and Subjective Interpretation of Public Information*

Quantitative data like d_t arrive with *qualitative* information about unusual conditions under which the data was generated. For example, if d_t are profits of a firm then d_t is a number in a financial report which contains *qualitative* information about changing consumer fashion, new and competing products, technology, joint ventures, research & development etc. If d_t are profits of the S&P500 then *qualitative* information includes business conditions, productivity trends, monetary policy, taxes and other macroeconomic conditions. Qualitative information cannot, in general, be compared over time and does not constitute conventional "data." If a firm reports on research into something that did not exist before, no past data is available for comparison. When a new product

alters the nature of an industry, it is a unique event. Financial markets pay a great deal of attention to qualitative announcements which are often the focus of diverse opinions and investors' activity.

There is little modeling of qualitative information. Saari (2006) uses qualitative information in a competitive model of market shares. Toukan (2006) is a second example. Kandel and Pearson's (1995) model of diverse interpretation of public signals can be supported with an argument based on qualitative information. Kurz (2009) formulates a model where, in addition to d_t, date t qualitative information lead agents to form subjective probabilities about unobserved events which can impact d_{t+1} via b_t. These effects are open to diverse assessments: they can be positive or negative and their size and timing can be in doubt. A simple example will suffice. When a firm announced d_t it also announced that during the past year it completed a major project that includes an implementation of a long term research effort of major significance. The conclusions of the project are now being put into effect and these could have a big impact on next years profits. In this paper we avoid the problem of assessing qualitative information. Instead we focus only on the fact that it leads agents to formulate an alternate prior on b_t which, as a random variable, is denoted by B_t defined by $B_t^i \sim N(\Psi_t^i, \frac{1}{\gamma})$. One can say either that i "observes" Ψ_t^i and γ or that he assesses these values from qualitative public information. The main question is how to reconcile B_t^i with the posterior $b_{t-1}(d_t)$ formulated given the data d_t. To do that we need to specify the updating process.

1.3.2 *A Bayesian Inference: Beliefs are Markov State Variables with Transition (6b)*

Agents believe the true transition of profits is (4), that β is known but b_t are unknown. At date t-1 (say t -1 = -1) an agent needs to forecast d_t and uses for that a prior belief about b_{t-1} described by $b_{t-1}^i \sim N(b, \frac{1}{\alpha})$. Now we move to date t (here t = 0) and after observing the data d_t (recall $\rho_t^d \sim N(0, \frac{1}{\beta})$) the posterior on b_{t-1} is updated to be

(7) $$E_t^i(b_{t-1}|d_t) = \frac{\alpha b + \beta[d_t - \lambda_d d_{t-1}]}{\alpha + \beta} \quad , \quad b_{t-1}(d_t) \sim N[E_t^i(b_{t-1}|d_t), \frac{1}{\alpha + \beta}]$$

Now, using qualitative data, agent i makes the assessment $B_t^i \sim N(\Psi_t^i, \frac{1}{\gamma})$ independently of the random variable $b_{t-1}(d_t)$ and we have two alternative priors. Our key assumption is:

<u>Assumption (A)</u>: Agent i uses a subjective probability μ to form date t prior belief about b_t as a random variable defined by

(8a) $$b_t(d_t, \Psi_t^i) = \mu\, b_{t-1}(d_t) + (1-\mu) B_t^i \qquad 0 < \mu < 1$$

Generally, if $b_t(d_{t+1}, \Psi_t^i)$ is a posterior given data d_{t+1} only, a revised prior given B_{t+1}^i is given by

(8b) $$b_{t+1}(d_{t+1}, \Psi_{t+1}^i) = \mu b_t(d_{t+1}, \Psi_t^i) + (1-\mu) B_{t+1}^i \quad ,$$

(8c) $$E_{t+1}^i(b_{t+1} | d_{t+1}, \Psi_{t+1}^i) = \mu E_{t+1}^i(b_t | d_{t+1}, \Psi_t^i) + (1-\mu)\Psi_{t+1}^i \qquad 0 < \mu < 1.$$ [6]

with the notation $\Gamma(b_t) = \text{Precision}(b_t | d_t, \Psi_t^i)$.

<u>Theorem 1</u> : If Assumption (A) holds then $\Gamma(b_t)$ converges for large t to a constant Γ^* but the Bayes estimate $E_t^i(b_t | d_t, \Psi_t^i)$ fluctuates indefinitely. Let $g_t^i = E_t^i(b_t | d_t, \Psi_t^i) - [\mu\beta/(\Gamma^* + \beta)] d_t$ then this index is a Markov state variable and (6b) holds with $\varrho_{t+1}^{ig} = (1-\mu)\Psi_{t+1}^i$: Assumption (A) implies (6b).

<u>Proof</u> : See Appendix B.

The random component explaining both diversity and dynamics is $\varrho_{t+1}^{ig} = (1-\mu)\Psi_{t+1}^i$. It arises from random arrival of qualitative information which is interpreted differently by different agent.

There are two ways to justify (6a). From Theorem 1 it follows that (6a) is a special case of (6b) when $\lambda_Z^d = 0$. Condition (B.5) in Appendix B shows that $\lambda_Z^d = 0$ if $\mu = \lambda_d(1 + (\beta/\Gamma^*))$. Second, an alternate Bayesian learning can be developed in which persistence in the arrival of qualitative information implies the Markov property in (6a).

1.4 Modeling Diverse Beliefs II: Market Belief and the Central Role of Correlation

The analysis below and Appendix A are developed for the (6a) dynamics but can be easily adapted to (6b). Hence we deduce the equilibrium and state Theorem 3 for both (6a) and (6b). Averaging (6a) we denote by Z_t the mean of the cross sectional distribution of g_t^i and refer to it as " average market belief." It is observable. Due to correlation across agents' ρ_t^{ig}, the law of large numbers does not apply and the average of ρ_t^{ig} over i does not vanish. We write it in the form

(9) $$Z_{t+1} = \lambda_Z Z_t + \hat{\rho}_{t+1}^Z .$$

The true distribution of $\hat{\rho}_{t+1}^Z$ is unknown. The random term in (9) is import and shows the dynamics of Z_t depends upon the *correlation* across agents' beliefs rather than on their actual beliefs. If the

[6] Note: we use a notation of $E_t^i(b_t | d_t, \Psi_t^i)$ for date t *prior* belief about the parameter b_t used to forecast d_{t+1}. We then use the notation $E_{t+1}^i(b_t | d_{t+1}, \Psi_t^i)$ for the posterior belief *about the same* b_t given the observation of d_{t+1} but without changing the estimate of Ψ_t^i. Assumption (A) uses this posterior belief as a building block in revising the prior $E_{t+1}^i(b_{t+1} | d_{t+1}, \Psi_{t+1}^i)$ about the new parameter b_{t+1}.

random terms ρ_t^{ig} or ϱ_t^{ig} in (6a) or (6b) were independent, then by the law of large numbers we would have $\hat{\rho}_t^Z = 0$. Correlation then ensures market belief does not degenerate into $Z_t = 0$ in case of (6a) or into a deterministic relation $Z_{t+1} = \lambda_Z Z_t + \lambda_Z^d d_t$ for (6b). In either case correlation is why market belief does not possess deterministic dynamics and becomes irrelevant to asset pricing. Since correlation is not determined by individual rationality it becomes *an important belief externality*.

Correlation may exhibit non stationarity inherited by the Z_t. Since they are observable, market participants have data on $\{(d_t, Z_t), t = 1, 2, \ldots\}$ hence they know the *joint empirical distribution* of these variables. For simplicity we assume this distribution is described by the system of equations

(10a) $$d_{t+1} = \lambda_d d_t + \rho_{t+1}^d$$

(10b) $$Z_{t+1} = \lambda_Z Z_t + \rho_{t+1}^Z$$

$$\begin{pmatrix} \rho_{t+1}^d \\ \rho_{t+1}^Z \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_d^2, & 0, \\ 0, & \sigma_Z^2 \end{bmatrix} = \Sigma \right), \quad \text{i.i.d.}$$

Now, an agent who does not believe that (10a)-(10b) is the truth, formulates his own model\belief. We have seen in (4) how agent i's belief state g_t^i pins down his forecast of d_{t+1}^i. We now broaden this idea to an agent's perception model of the two state variables (d_{t+1}^i, Z_{t+1}^i). Keeping in mind that before observing (d_{t+1}, Z_{t+1}) agent i knows d_t and Z_t, his belief takes the general form

(11a) $$d_{t+1}^i = \lambda_d d_t + \lambda_d^g g_t^i + \rho_{t+1}^{id}$$

(11b) $$Z_{t+1}^i = \lambda_Z Z_t + \lambda_Z^g g_t^i + \rho_{t+1}^{iZ}$$

(11c) $$g_{t+1}^i = \lambda_Z g_t^i + \rho_{t+1}^{ig}$$

$$\begin{pmatrix} \rho_{t+1}^{id} \\ \rho_{t+1}^{iZ} \\ \rho_{t+1}^{ig} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}, \begin{bmatrix} \hat{\sigma}_d^2, & \hat{\sigma}_{Zd}, & \hat{\sigma}_{gd} \\ \hat{\sigma}_{Zd}, & \hat{\sigma}_Z^2, & \hat{\sigma}_{Zg}, \\ \hat{\sigma}_{gd}, & \hat{\sigma}_{Zg}, & \sigma_g^2 \end{bmatrix} = \Sigma^i \right)$$

(11a)-(11b) show that, as required, g_t^i pins down the transition of both state variables (d_{t+1}^i, Z_{t+1}^i). This simplicity ensures that one state variable pins down agent i's subjective belief of how conditions at date t are different from normal as reflected by the empirical distribution:

(12) $$E_t^i \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} - E_t^m \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_d^g g_t^i \\ \lambda_Z^g g_t^i \end{pmatrix}.$$

Note: in (11a)-(11c) random terms are not required to be i.i.d. Also, adapting (10b), (11b)-(11c) to the dynamics (6b) is simple. Instead of (10b) we write $Z_{t+1} = \lambda_Z Z_t + \lambda_Z^d d_t + \rho_{t+1}^Z$ and instead of (11b)-(11c) we have $Z_{t+1}^i = \lambda_Z Z_t + \lambda_Z^d d_t + \lambda_Z^g g_t^i + \rho_{t+1}^{iZ}$ and $g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^d d_t + \rho_{t+1}^{ig}$.

1.4.1 No Representative Agent: The Centrality of Belief Heterogeneity

Is belief heterogeneity central to (11a)-(11c)? For exponential utility only average market

belief matters so why could the model not be reduced to a representative agent? For *actually* heterogenous economy the model cannot be reduced to a representative agent for several reasons.

(i) The first answer is in (11b). One element of the theory requires an agent to forecast Z_{t+1} which is the belief of "others" and for that he must perceive the market as different from himself. Without this reality a representative agent identifies himself with the average market and there is nothing to forecast: <u>*equation (11b) disappears*</u>. Formally it leads the average market belief Z_t to disappear as a state variable from the optimization of the agent. This fact is also seen when we define average market expectation operator to be $\bar{E}_t(\bullet) = \frac{1}{N}\sum_i^N E_t^i(\bullet)$. From (11c) it is

$$\bar{E}_t\begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} - E_t^m\begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_d^g Z_t \\ \lambda_Z^g Z_t \end{pmatrix}. \tag{13}$$

In heterogenous market with a representative agent, the agent forecasts with $\bar{E}_t(\bullet)$. Such forecasts would hold for the dividend process but not for Z_{t+1}. One could formally replace (11b) with the mean of (11c) and put a 0 in the second component of (13) but this is rejected by reason 4 below.

(ii) Heterogeneity plays a key role in the dynamics of Z_t via ρ_t^d which depends upon the *correlation* across beliefs. Since the correlation is not determined by considerations of individual rationality, aggregate dynamics is a consequence of the heterogeneity externality. Changes in heterogeneity without changes in the mean matter but are missed by a representative agent.

(iii) When utility is not exponential then the entire distribution matters. Many papers cited earlier (e.g. Lee et al. (2002), Diether et al. (2002), Johnson (2004), Park (2005), Baker and Wurgler (2006), Karakatsani and Salmon (2008), Kurz and Motolese (2011) find that fluctuations in the cross-sectional variance of their measures of belief contributes to the predictability of returns. Hence, the effect of cross sectional variance is a null hypothesis we also test below.

(iv) If the economy is heterogenous but there is a representative agent the belief of that agent would be some average of the beliefs of individuals. But the average belief cannot even be a probability. Indeed, the perception models (11a)-(11c) show that properties of conditional probabilities do not apply to the market belief operator $\bar{E}_t(\bullet)$ since it is *not a proper conditional expectation*. To see why let $X=D\times Z$ be a space where (d_t, Z_t) take values and G^i be the space of g_t^i. Since i conditions on g_t^i, his unconditional probability is a measure on the space $((D\times Z\times G^i)^\infty, \mathscr{F}^i)$ where $\mathscr{F}^i$ is a sigma field. The market conditional belief operator is an average over conditional probabilities, each conditioned on a *different* state variable. Hence, this averaging does not even permit one to write a

probability space for the market belief. The market belief is neither a probability nor rational and the following result summarizes this fact:

<u>Theorem 2</u>: The market belief operator violates iterated expectations: $\bar{E}_t(d_{t+2}) \neq \bar{E}_t\bar{E}_{t+1}(d_{t+2})$.

<u>Proof</u>: Since $E_t^i(d_{t+2}) = \lambda_d E_t^i(d_{t+1}) + \lambda_d^g E_t^i(g_{t+1}^i) = \lambda_d[\lambda_d d_t + \lambda_d^g g_t^i] + \lambda_d^g \lambda_Z g_t^i$ it follows that

(14) $$\bar{E}_t(d_{t+2}) = \lambda_d^2 d_t + \lambda_d^g(\lambda_d + \lambda_Z)Z_t.$$

But we also have from (11a) that $\bar{E}_{t+1}(d_{t+2}) = \lambda_d d_{t+1} + \lambda_d^g Z_{t+1}$ and using (11b) we can compute

$$E_t^i\bar{E}_{t+1}(d_{t+2}) = \lambda_d[\lambda_d d_t + \lambda_d^g g_t^i] + \lambda_d^g[\lambda_Z Z_t + \lambda_Z^g g_t^i].$$

Aggregating we now conclude that

(15) $$\bar{E}_t\bar{E}_{t+1}(d_{t+2}) = \lambda_d^2 d_t + \lambda_d^g(\lambda_d + \lambda_Z + \lambda_Z^g)Z_t.$$

Comparison of (14) and (15) shows that $\bar{E}_t(d_{t+2}) \neq \bar{E}_t\bar{E}_{t+1}(d_{t+2})$. ■

1.4.2 Market Belief, Information and Human Exuberance

We address two issues. First, the question of *belief and Information***.** For an agent, Z_t is a state variable like others. News about Z_t are used to forecast prices in the same way data such as a leading indicator is used to assess the risk of a recession. How do agents update beliefs when they observe the mean belief Z_t of others? In contrast with private information models, agents do not revise their beliefs about the state variable d_{t+1}: (11a) *does not* depend upon Z_t. They do not view Z_t as information about d_{t+1} since it is not a "signal" about unobserved private information they do not have. However, Z_t is crucial "news" about *what the market thinks* about d_{t+1}! Hence, the importance of Z_t is it's great value for forecasting future *endogenous* variables. Date t endogenous variables depend upon Z_t and future endogenous variables depend upon future Z's. Since market belief exhibits *persistence*, agents know that today's market belief is useful for forecasting future prices.

Finally, in considering economies populated by fully rational agents who hold diverse beliefs, is there a role for human "exuberance?" The answer is yes! If all agents are individually rational we still have the mechanism of *correlation* which is not subject to any principle of individual rationality. We have seen that with diverse beliefs such correlation plays a central role in market dynamics hence such correlation is actually a belief externality, taken by all as given. In the model above there would be no aggregate dynamics in (9) without correlation and such correlation can entail market herding, bubbles and other forms of excess volatility which, in our opinion, are often offered incorrectly as

evidence for market irrationality.

1.4.3 The Empirical Evidence Regarding Belief Dynamics (6a)-(6b)

We later use two data sources to extract time series on Z_t for six month horizon from GDP forecast data. We briefly examine now the question of which of the two models (6a) or (6b) fit the data better. We test the following regression model for two data sources and different periods

(16) $$Z_{t+1} = a_0 + a_1 Z_t + a_2 y_t + \epsilon^Z_{t+1}$$

where y_t is the annualized six month growth rate of GDP (final release). The results are reported in Table 1. It is clear the persistence parameter a_1 is always significant and takes values around 0.60, a result which is virtually the same as the result in Kurz and Motolese (2011) where beliefs are derived from of interest rate forecasts. As to a_2 the results show it is small and statistically significant for some periods. Hence, the results are inconclusive but favor the dynamic model of (6a).

Table 1: Estimated Coefficients of Market Belief Dynamics

Data Source and Period		Coefficient	Std. Error	p-value	R^2
Livingston – 1968:S2-2007:S2	a_0 a_1 a_2	-0.275 0.842 0.048	0.260 0.052 0.060	0.293 0.000 0.426	0.732
Livingston – 1988:S1-2007:S2	a_0 a_1 a_2	-1.326 0.515 0.128	0.248 0.097 0.055	0.000 0.000 0.026	0.385
SPF – 1968:S2-2007:S2	a_0 a_1 a_2	-0.173 0.811 0.040	0.283 0.054 0.074	0.543 0.000 0.586	0.684
SPF – 1988:S1-2007:S2	a_0 a_1 a_2	-1.131 0.587 0.176	0.205 0.051 0.051	0.000 0.000 0.002	0.628
SPF/BLU – 1968:S2-2007:S2	a_0 a_1 a_2	-0.005 0.838 -0.008	0.321 0.051 0.084	0.988 0.000 0.929	0.697
SPF/BLU – 1988:S1-2007:S2	a_0 a_1 a_2	-1.032 0.633 0.171	0.172 0.054 0.036	0.000 0.000 0.000	0.719

1.5 Combining the Elements: the Implied Asset Pricing Under Diverse Beliefs

We now derive equilibrium prices and the risk premium. We return to (2a)-(2b) where the state variables are defined $\psi^i_t = (1, d_t, Z_t, g^i_t)$. We maximize (2b) subject to the budget constraint (2a) and transitions (11a)-(11c). A full analysis of the problem is found in Appendix A. A simple way to approach the problem is to specify an assumption and conjecture. First, assume the agent believes

$\{d_t, t = 1, 2, ...\}$ are conditionally normal. Second, conjecture that equilibrium price p_t is also conditionally normal. Theorem 3 will then confirm the conjectures. In Appendix A we show that for an optimum of (2a)-(2b), there is a constant vector u so the stock demand function is

(17) $$\theta_t^i(p_t) = \frac{R\tau}{r\hat{\sigma}_Q^2}[E_t^i(Q_{t+1}) + u\psi_t^i] \ , \ u = (u_0, u_1, u_2, u_3) \ , \ \psi_t^i = (1, d_t, Z_t, g_t^i).$$

$\hat{\sigma}_Q^2$ is an *adjusted* conditional variance (see Appendix A for details) of excess stock returns, assumed constant and the same for all agents. The term $u\psi_t^i$ is the intertemporal hedging demand which is linear in agent i's state variables.

For an equilibrium to exist we need some stability conditions. First we require the interest rate r to be positive, R = 1 + r > 1 so that $0 < \frac{1}{R} < 1$. Now we add:

(18) **<u>Stability Conditions</u>**: We require that $0 < \lambda_d < 1 \ , \ \lambda_Z < 1 \ , \ 0 < \lambda_Z + \lambda_Z^g < 1$.

The first requires $\{d_t, t = 1, 2, ...\}$ to be stable and have an empirical distribution. The second is *a* stability of *belief* condition. It requires i to believe (d_t, Z_t) is stable. To see why, take expectations of (11b), average over the population and recall that Z_t are market averages of the g_t^i. This implies that $\bar{E}_t[Z_{t+1}] = (\lambda_Z + \lambda_Z^g)Z_t$.

<u>Theorem 3:</u> Consider the model with heterogenous beliefs under the stability conditions specified with supply of shares which equals N. Then either under (6a) or (6b) there is a unique equilibrium price function which takes the form $p_t = a_d d_t + a_Z Z_t + P_0$.

<u>Proof:</u> Average (17), use the fact that the aggregate stock supply is N and rearrange to have

(19) $$\frac{r\hat{\sigma}_Q^2}{R\tau} = [\bar{E}_t(p_{t+1} + d_{t+1} + \mu) - Rp_t + (u_0 + u_1 d_t + (u_2 + u_3)Z_t)].$$

Now use the perception models (11a)-(11b) about the state variables, average them over the population and use the definition of Z_t to deduce the following relationships under (6a)

(20) $$\bar{E}_t[d_{t+1} + \mu] = \lambda_d d_t + \mu + \lambda_d^g Z_t$$

(21) $$\bar{E}_t[Z_{t+1}] = (\lambda_Z + \lambda_Z^g)Z_t.$$

Using these to solve for date t price we deduce

(22) $$p_t = \frac{1}{R}[\bar{E}_t(p_{t+1})] + \frac{1}{R}[(\lambda_d + u_1)d_t + (\lambda_d^g + u_2 + u_3)Z_t] + \frac{1}{R}[\mu + u_0] - \frac{r\hat{\sigma}_Q^2}{R^2\tau}$$

(22) shows that equilibrium price is the solution of a linear difference equation in the two state variables (d_t, Z_t). Hence, a standard argument (see Blanchard and Kahn(1980), *Proposition 1*, page 1308) shows that the solution is

(23a) $$p_t = a_d d_t + a_Z Z_t + P_0$$

To match coefficients use (11a)-(11b) to insert (20) - (21) into (22) and conclude that

(23b) $$a_d = \frac{\lambda_d + u_1}{R - \lambda_d}, \quad a_Z = \frac{(a_d + 1)\lambda_d^g + (u_2 + u_3)}{R - (\lambda_Z + \lambda_Z^g)}, \quad P_0 = \frac{(\mu + u_0)}{r} - \frac{\hat{\sigma}_Q^2}{R\tau}.$$

The stability conditions ensure that (23a) - (23b) is the unique solution as asserted.

Under the (6b) dynamics the conditions (20)-(21) are written as

(20') $$\bar{E}_t[d_{t+1} + \mu] = \lambda_d d_t + \mu + \lambda_d^g Z_t$$

(21') $$\bar{E}_t[Z_{t+1}] = (\lambda_Z + \lambda_Z^g)Z_t + \lambda_Z^d d_t.$$

Computing the corresponding stochastic difference equation we find the solution is the same as (23a) with coefficients which are now defined by

(23b') $$a_d = \frac{a_Z\lambda_Z^d + (\lambda_d + u_1)}{R - \lambda_d}, \quad a_Z = \frac{(a_d + 1)\lambda_d^g + (u_2 + u_3)}{R - (\lambda_Z + \lambda_Z^g)}, \quad P_0 = \frac{(\mu + u_0)}{r} - \frac{\hat{\sigma}_V^2}{R\tau}.$$ ■

We do not have closed form solutions for hedging demand parameters $u = (u_0, u_1, u_2, u_3)$ and computed numerical Monte Carlo solutions instead. For all model parameters values given the relevant case $(\lambda_d^g > 0, \lambda_Z^g > 0)$ we find $a_d > 0$ and $a_Z > 0$. These are reasonable: p_t *increases with higher* d_t *and with higher* Z_t*- today's market belief in unusually higher future dividends*.

1.6 Equilibrium Risk Premium Under Heterogenous Beliefs

1.6.1 The Main Equilibrium Results

Under heterogenous beliefs we have many risk premia and one chooses a concept which is appropriate for an application. As a random variable the risk premium on a long position is

(24) $$\pi_{t+1} = \frac{p_{t+1} + d_{t+1} + \mu - Rp_t}{p_t}.$$

(24) is the actual excess returns of stocks over the riskless bond. The need is to measure the premium as *a known expected quantity*, recognized by all. There are many such measures. One is the subjective

premium of agent i. We computed it by using the equilibrium map (23a) and perception model (11a) - (11c) to have

(25) $$\frac{1}{p_t}E_t^{\,i}(p_{t+1}+d_{t+1}+\mu-Rp_t) = \frac{1}{p_t}[(a_d+1)(\lambda_d d_t+\lambda_d^g g_t^{\,i})+a_Z(\lambda_Z Z_t+\lambda_Z^g g_t^{\,i})+\mu+P_0-Rp_t]$$

By taking the mean of (25) we can also compute the average market risk premium. We seek an objective measure, computed by all agents who study the long term premium using to the empirical measure m. By (23a) and the stationary transition (10a)-(10b) we can deduce that it is

(26) $$E_t^{\,m}[\pi_{t+1}] = \frac{1}{p_t}E_t^{\,m}[p_{t+1}+d_{t+1}+\mu-Rp_t] = \frac{1}{p_t}[(a_d+1)(\lambda_d d_t)+a_Z(\lambda_Z)Z_t+\mu+P_0-Rp_t].$$

The concept in (26) is the way Econometricians and all researchers cited above have been measuring the risk premium. For this reason we refer to it as "the" risk premium.

We then arrive at two conclusions. First, the risk premium is different from the market perceived premium when $Z \neq 0$ since

(27) $$\frac{1}{p_t}E_t^{\,m}(p_{t+1}+d_{t+1}+\mu-Rp_t) - \frac{1}{p_t}\bar{E}_t(p_{t+1}+d_{t+1}+\mu-Rp_t) = -\frac{1}{p_t}[(a_d+1)\lambda_d^g+a_Z\lambda_Z^g]Z_t.$$

A second conclusion is derived by studying (26). From (23c) we can compute the condition $-(u_2+u_3) = -a_Z(R-\lambda_Z)+[(a_d+1)\lambda_d^g+a_Z\lambda_Z^g]$ and hence we can deduce the main result:

<u>Theorem 3:</u> The equilibrium risk premium has the following analytical expression

(28a) $$\frac{1}{p_t}E_t^{\,m}(p_{t+1}+d_{t+1}+\mu-Rp_t) = \frac{1}{p_t}[(\frac{r\hat{\sigma}_Q^2}{R\tau} - u_0 - u_1 d_t) - a_Z(R-\lambda_Z)Z_t]$$

Since $a_z > 0$, $R > 1$ and $\lambda_Z < 1$ it follows that

(28b) the Risk Premium $E_t^{\,m}[\pi_{t+1}]$ is decreasing in the mean market belief Z_t.

Conclusions (28a) -(28b) are central to this paper. (28a) exhibits the endogenous component of the risk premium which we call "The Market Belief Risk Premium." It shows that market belief has a complex effect on market risk premia. It consists of two parts

(I) The first is a direct effect on the permanent mean premium $\frac{r\hat{\sigma}_Q^2}{R\tau}$. It is shown in Appendix A that there exist weights $(\omega_1,\omega_{12},\omega_2)$ such that

$$\hat{\sigma}_Q^2 = Var_t^{\,i}((\omega_1(\lambda_d d_t + \lambda_d^g g_t^{\,i} + \omega_{12}\rho_{t+1}^{id}) + \omega_2(\lambda_Z Z_t + \lambda_Z^g g_t^{\,i} + \omega_{12}\rho_{t+1}^{iZ})).$$

Volatility of belief contributes directly to the volatility of excess returns and increases permanently the risk premium.

(II) The second is the effect of market belief on the *time variability* of the risk premium,

reflected in $-a_Z(R-\lambda_Z)Z_t$ with a negative sign when $Z_t > 0$.

To explain this second result we note that it says that if one runs a regression of excess returns on the observable variables, the effect of market belief on long term excess return *is negative*. This sign is surprising since when $Z_t > 0$ the market expects *above normal* future dividends but in that case the risk premium on the stock *is lower*. When $Z_t < 0$ the market holds bearish belief about future dividend but the risk premium *is higher*. Since we have data on Z_t and on the distribution of belief the result will be empirically tested. Before proceeding to the empirical test we discuss some ramifications of this result.

1.6.2 The Market Belief Risk Premium is General

The main result (28b) was derived for the exponential utility function. We argue that this result is more general and depends only on the positive coefficient a_z of Z_t in the price map. To show this, assume any additive utility function over consumption and a risky asset which pays a "dividend" or any other random payoff d_t. Denote the price map by $p_t = \Phi(d_t, Z_t)$. We are interested in the slope of $E_t^m[\pi_{t+1}]$ with respect to Z_t. Focusing only on the numerator in (26), linearize the price around 0 and write $p_t = \Phi_d d_t + \Phi_Z Z_t + \Phi_0$. The desired result depends only upon the condition that $\Phi_Z > 0$. It is reasonable as it requires current price to increases if the market is more optimistic about the asset's future payoffs. To prove the point note that

$$\begin{aligned} E_t^m[p_{t+1} + (d_{t+1} + \mu) - Rp_t] &\approx E_t^m[\Phi_d d_{t+1} + \Phi_Z Z_{t+1} + \Phi_0 + (d_{t+1} + \mu) - R(\Phi_d d_t + \Phi_Z Z_t + \Phi_0)] \\ &= [(\Phi_d + 1)\lambda_d - R\Phi_d]d_t - \Phi_Z(R - \lambda_Z)Z_t + [\mu + \Phi_0(1 - R)]. \end{aligned}$$

The desired result follows from $\Phi_Z > 0$, $R > 1$ and $\lambda_Z < 1$. This result may be altered if the interest rate is not fixed and an endogenously determined R_t also changes with Z_t in (28a).

1.6.3 Interpretation of the Market Belief Risk Premium

Why is the effect of Z_t on the risk premium *negative*? Since this result applies to any asset with risky payoffs, we offer a general interpretation. Our result shows that when the market holds abnormally favorable belief about future asset payoffs the market views the long position as less risky and consequently the risk premium on a long position falls. Hence, in the long run fluctuations in risk premia are *inversely* related to degree of market optimism about future prospects of an asset's payoff.

To explore the result, it is important to explain *what it does not say*. One could interpret it to

confirm a common claim that to maximize excess returns it is optimal to be a "contrarian" to the market consensus. To understand why *this is a false interpretation* note that when an agent holds a belief about future dividends, the market belief Z_t does not offer him new information for altering his belief about dividends. If the agent believes future dividends will be abnormally high but $Z_t < 0$, the agent *does not change his forecast* of d_{t+1}. Z_t is an important input to forecasting returns since it is used to forecast future prices. Keep in mind that given available information and a probability belief, denoted say by Γ^i, an optimizing agent is already on his demand function. In response to a changed Z_t he does not just abandon his demand by replacing Γ^i with the empirical measure m. This argument is analogous to the one showing why it is not optimal to adopt log utility as your utility even though it maximizes the growth rate of your wealth. Yes, it does that, but you dislike the sharp declines which you expect to occur in the value of your assets if you follow the strategy called for by the log utility. By analogy, following a "contrarian" policy implies a high long run average return in accord with m since this is what (28a) says. But if your subjective model disagrees with the probability m you will dislike being short when your optimal position should be long. This argument explains why most people do not systematically bet against the market, as a "contrarian" strategy (28a) would dictate.

Taking a positive view, our results show fluctuations in market belief are important for the time variability of risk premia and the market pricing of risk. Market optimism in bull markets or pessimism in bear markets have added effects on market risk perception above and beyond any real information about the economy. (28a) shows that in the long run market belief has an inverse effect on market risk premia. To see how individually perceived premia are affected by market belief use (25) and (26) to find $E_t^i(\pi_{t+1}) = E_t^m(\pi_{t+1}) + \frac{1}{p_t}[(a_d+1)\lambda_d^g + \lambda_Z^g]g_t^i$. Hence, optimizing agents use Z_t in calculating their subjective premia in the same way they use any state variable which describes the stat of the economy. We turn now to an empirical test of our theory.

2. Testing the Time Variability of Stock Market Risk Premia: The Data

2.1 The Forecast Data

We use data on the distribution of commercial forecasts and take them as proxies for the forecasts of future business conditions made by the general public. Several different sources of data are employed: the Livingston survey (LIV), the Survey of Professional Forecasters (SPF), the Blue

Chip Financial Forecasts (BLUF) and the Blue Chip Economic Indicators (BLUE). In all cases forecast data of future real gross domestic product (GDP) *growth rates* are taken as a proxy for expected business conditions in the future.

Livingston GDP Growth Rate Forecasts. The Livingston survey is conducted biannually in June and December since June 1946. Our sample begins in December 1968 and continues until December 2007. LIV provides forecast data on the level of nominal GDP in the following June and December rather than forecast data of real GDP growth. Hence, we had to construct growth rate data from individual expectated nominal GDP and consumer price index (CPI) levels. Also, LIV does not ask participants to provide expected current levels for nominal GDP and CPI and this deprives us of a common baseline data. Hence, like others who use LIV, we cannot compute individual 6-month-ahead forecasts for real GDP growth rates. Instead, for all individuals and survey dates we compute 12-month-ahead annualized real GDP growth rate forecasts based on the implied expected growth rate between 6 months ahead and 12 months ahead. We then take the average forecast, obtaining a series of 2 semester ahead forecasts spanning 1968:S2-2007:S2.

Survey of Professional Forecasters GDP Growth Rate Forecasts. Among other data, SPF provides forecasts of quarterly real GDP levels from which growth rate forecasts are derived. The survey is currently conducted by the Federal Reserve Bank of Philadelphia, continuing the American Statistical Association\NBER survey started in 1968. The survey is conducted quarterly and released in the middle month of each quarter. The sample available extends from 1968:Q4 to 2007:Q4. Semi-annual forecasts of real GDP growth rates which are comparable to those deduced from LIV are computed by taking geometric averages of forecasts from SPF for two consecutive quarters.

Blue Chip Financial Forecasts. BLUF reports quarterly forecasts of U.S. interest rates and quarterly forecasts of real GDP growth rates and inflation. Forecasts reported in BLUF are collected on the 24th and 25th of each month and released to subscribers on the first day of the following month. Since January 1983 BLUF has published individual and mean forecasts for each variable. The mean is taken over forecasters participating in that month. The sample available extends from 1984:M7 to 2007:M12. As is the case for SPF data, to obtain semi-annual forecasts comparable to those from LIV we compute geometric averages of forecasts of two consecutive quarters.

Blue Chip Economic Indicators. BLUE reports quarterly forecasts of U.S. economic growth, inflation, interest rates, and other macroeconomic indicators of future business activity. Forecasts

reported in BLUE are collected on the 2nd and 3rd day of each month and released to subscribers on the 10th of the month. Since 1976 BLUE publishes, for each variable, individual and mean forecasts. The mean is taken over all forecasters participating in that month. The sample available to us extends from 1980:M1 to 2007:M12. As is the case for SPF and BLUF, to obtain semi-annual forecasts comparable with those from LIV we compute geometric averages of forecasts of two consecutive quarters.

Forecasts reported in the four surveys above are labeled by their release date. We assume these forecasts are conditional on information available at the moment the forecasts were collected. In the case of LIV forecasts are conditional on information available at the end of May and November respectively for 1st and 2nd semester. For SPF, which does not follow a strict pattern of dates for collection and release of forecasts, forecast are conditional on information available at the end of the middle month of each quarter (May and November respectively for 1st and 2nd semester). Finally, for BLUE and BLUF, forecasts are conditional on information available at the end of the month prior to the month of release (June and December respectively for 1st and 2nd semester). In what follows we take into account the above calendar dates so that the analysis is based on forecasts which are made conditional on the same set of information. However, the reader must keep in mind that the underlying data has a rather complex structure. The Livingston forecasts are for *semester* data based on information provided one month prior to release; forecast data of SPF, BLUE and BLUF are given by *quarters* into the future although much of it is released *monthly* based on information available at the end of the month prior to release. Estimation of excess returns will later be done on semester basis but some background procedures, such as the construction of belief data, employ models based on monthly or quarterly data. All this points out to the reader that the time unit in our empirical investigation is rather complex and a perfect notation could be very cumbersome. We have thus selected our notation with care but regret in advance if we repeat too often the explanation of the symbols employed. The basic principle we adopted is that *all data is monthly based but models may be estimated using semester or quarterly dates*. Hence the index t refers to months.

We study forecasts of GDP growth rates, mainly two semesters ahead of date t, from date t +6 to date t+12. Hence, we adopt a specialized notation for this specific variable. We denote by $y^{t,2}$ the annualized growth rate of GDP over an interval of six months, 2 semesters ahead between the end of t+6 and t +12. Similarly, $y^{t,h}$ denotes the same variable h semesters into the future. In Table 2 we

list our notation, time spans and frequencies of forecast data and market states of belief about future GDP growth which are extracted from the survey data outlined above. We use the notation $\bar{E}_t^{J}\{y^{t,2}\}$ and Z_t^{J} to indicate, respectively, the J source average expected six-month real GDP growth rate 2 semesters ahead, which is from date t+6 to date t +12 and the corresponding extracted market state of belief. Note that we omit from Z_t^{J} the symbols y and 2 since these are fixed throughout the paper. All forecasts and extracted market states of beliefs are *annualized*. A description of the procedure for extracting data on beliefs follows below.

Table 2: Notation for Forecasts and Extracted Market States of Belief

Notation		Time Span	Frequency
Average Forecast	Market State of Belief		
$\bar{E}_t^{LIV}\{y^{t,2}\}$	Z_t^{LIV}	1968:S2-2007:S2	Semi-annual
$\bar{E}_t^{SPF}\{y^{t,2}\}$	Z_t^{SPF}	1968:S2-2007:S2	Semi-annual
$\bar{E}_t^{BLU}\{y^{t,2}\}$	Z_t^{BLU}	1980:S1-2007:S2	Semi-annual
$\bar{E}_t^{SPF/BLU}\{y^{t,2}\}$	$Z_t^{SPF/BLU}$	1968:S2-2007:S2	Semi-annual

In Table 2 the superscript BLU indicates data obtained by combining the BLUE and BLUF files. The resulting BLU data consist at each date of the weighted average of the average forecasts from BLUE and BLUF where the weights are given by the number of individuals participating in each survey. The superscript SPF/BLU refers to a combined BLU and SPF forecasts where SPF data is added before the beginning date of the BLU data in order to extend the time span to 1968:S2-2007:S2.

Place Figure 1 Here

Figure 1 traces graphs of the average forecasts listed in Table 2. The Figure reveals a high correlation among the three sources. The one between $\bar{E}_t^{LIV}\{y^{t,2}\}$ and $\bar{E}_t^{SPF}\{y^{t,2}\}$ for the entire time span of 1968:S2-2007:S2 is 0.75 while the correlations among the three sources for the common period of time 1980:S1-2007:S2 is as reported in Table 3.

Table 3: Correlation among Market Forecasts

Note: the table reports correlation for the common sample 1979:S2-2007:S2, while the full sample correlation between LIV and SPF is reported in the text.

	$\bar{E}_t^{LIV}\{y^{t,2}\}$	$\bar{E}_t^{SPF}\{y^{t,2}\}$	$\bar{E}_t^{BLU}\{y^{t,2}\}$
$\bar{E}_t^{LIV}\{y^{t,2}\}$	1.00		
$\bar{E}_t^{SPF}\{y^{t,2}\}$	0.64	1.00	
$\bar{E}_t^{BLU}\{y^{t,2}\}$	0.93	0.76	1.00

Strong similarities among market forecasts from LIV, SPF and BLU can also be seen from Table 4 where we report some of their descriptive statistics.

Table 4: Descriptive Statistics of Market Forecasts

Note: the first three columns in the table report statistics of annualized growth rates for 1968:S2-2007:S2, while the last one covers the sample 1979:S2-2007:S2.

	$\bar{E}_t^{LIV}\{y^{t,2}\}$	$\bar{E}_t^{SPF}\{y^{t,2}\}$	$\bar{E}_t^{SPF/BLU}\{y^{t,2}\}$	$\bar{E}_t^{BLU}\{y^{t,2}\}$
Mean	3.024	2.977	3.098	2.938
Median	2.897	3.000	3.083	2.912
Max	5.912	5.560	5.560	5.300
Min	-0.015	-0.889	0.440	1.295
Std. Dev.	1.133	1.155	1.113	0.804
Skewness	0.426	-0.262	0.264	0.538
Kurtosis	3.580	4.186	2.855	3.372
Autocorrelation	0.631	0.693	0.673	0.665
N. Obs.	79	79	79	57

2.2 Extracting Mean Market States of Belief

We now explain how mean market belief is extracted from forecast data. Here we ignore the identification of data sources. Let $y^{t,h}$ be the annualized growth rate of real GDP known h semesters ahead of date t, by $E_t^i\{y^{t,h}\}$ agent i's conditional forecast of $y^{t,h}$, and by $E_t^m\{y^{t,h}\}$ the forecast under the stationary probability m. Agent i's state of belief at date t about $y^{t,h}$ is then defined by

(29) $$Z_t^{y,h,i} = E_t^i\{y^{t,h}\} - E_t^m\{y^{t,h}\}.$$

This does not render $Z_t^{y,h,i}$ m-orthogonal to date t information but it removes from $E_t^i\{y^{t,h}\}$ the effect of all state variables, as measured by $E_t^m\{y^{t,h}\}$. Note that $Z_t^{y,h,i}$ is a *deviation* from the stationary forecast: when $Z_t^{y,h,i} > 0$ agent i has an unusually favorable view of future GDP growth but this does not mean he believes output will go up. He believes output will grow *faster than usual* where "usual" is defined by growth rate expected under m. The market state of belief is defined by

(30a) $$Z_t^{y,h} = \frac{1}{N}\sum_{i=1}^{N}[E_t^i\{y^{t,h}\} - E_t^m\{y^{t,h}\}] = \bar{E}_t\{y^{t,h}\} - E_t^m\{y^{t,h}\}$$ [7]

and the cross sectional variance of beliefs is

(30b) $$(\sigma_t^{(y,h)})^2 = \frac{1}{N}\sum_{i=1}^{N}\left([E_t^i\{y^{t,h}\} - E_t^m\{y^{t,h}\}] - [\bar{E}_t\{y^{t,h}\} - E_t^m\{y^{t,h}\}]\right)^2 = \frac{1}{N}\sum_{i=1}^{N}\left(E_t^i\{y^{t,h}\} - \bar{E}_t\{y^{t,h}\}\right)^2.$$

Since $\bar{E}_t\{y^{t,h}\}$ is the mean forecast it reflects the market's view on abnormal conditions h semesters into the future. These conditions are the reason why the market forecasts $\bar{E}_t\{y^{t,h}\}$ and not $E_t^m\{y^{t,h}\}$: if $Z_t^{y,h} > 0$ it is "optimistic" about unusually high GDP growth in t+h.

[7] In this general section, the mean belief is denoted by $Z_t^{y,h}$ whereas in the specific empirical applications below the variable y and the forecast horizon h are fixed. Hence, we ignore these and define Z_t^J with respect to the data source J only.

To measure $Z_t^{y,h}$ we need data on the two components which define it. The LIV, SPF, BLUF and BLUE files provide direct data on $E_t^i\{y^{t,h}\}$ and $\bar{E}_t\{y^{t,h}\}$ as discussed. The problem is then the construction of the forecasts $E_t^m\{y^{t,h}\}$. In Kurz and Motolese (2011) we employed the Stock and Watson's (2002) method of diffusion indices to compute stationary forecasts of interest rates. However, when forecasting future growth rate of GDP such a method leads to highly volatile stationary forecasts. It is well known that real GDP growth is difficult to forecast (see, for example, Romer and Romer (2000)). Kurz (2005) estimates correlation between the Fed's Green book forecasts 1965-1995 and the realized growth rates of GDP. The table below shows that forecast accuracy falls sharply after a horizon of six months:

Accuracy of Fed's GDP Growth Forecasts

Horizon (Quarters)	R^2
0	0.53
1	0.25
2	0.18
3	0.08
4	0.16

Models with many independent variables tend to generate unstable out of sample forecasts (see Stock and Watson (2002)) hence the practice is to forecast GDP growth with models employing a small number of independent variables. To compute the stationary forecasts $E_t^m\{y^{t,h}\}$ we thus opt for a simple model *which is estimated with monthly data* with the following four categories of variables: Leading indicator expressed by the non-farm-payroll; state of monetary policy expressed by the Federal Funds rate; inflation expressed by the consumer price index and general persistence of growth expressed by the moving average of past GDP growth rates.

The information used to compute $E_t^m\{y^{t,h}\}$ should consist of variables that *correspond to those which are available in real time to forecasters at date t, when forecasts are made*. This requires us to check on the dates for data release. The Bureau of Economic Analysis produces GDP quarterly and releases the data towards the end of the last month of the following quarter. For example when agents forecast future growth at the end of June they know the GDP of the first quarter of the year, hence they can compute all six-month growth rates and moving averages up to that quarter. As for the NFP and CPI data, they are compiled monthly by the Bureau of Labor Statistics and made available during the following month. Thus, at the time the forecasts are made agents know NFP and CPI data lagged by one month. Furthermore, NFP data are revised twice after the initial release and undergo an

annual benchmark revision every June. We therefore compiled vintage real-time NFP data available at the Federal Reserve Bank of Philadelphia and used the first release of NFP for month t-1 and the revised value of NFP for month t-13 to compute the year over year growth rate. To sum up, the stationary model for computing $E_t^m\{y^{t,h}\}$ is estimated with the following variables:

$\bar{y}_t$ a 4 quarters moving average of GDP growth rates known at date t, as explained above;

NFP_{t-1} the monthly year over year growth rate of Non-Farm-Payroll known at the forecast month t which is the one dated one month prior to t;

CPI_{t-1} the monthly year over year rise of Consumer Price Index in the month before the one in which the forecast is made;

F_t the average Federal Funds rate during the month at the end of which the forecast is made.

Real Time vs. A Single Estimate. Had our data set been long, the stationary forecast $E_t^m\{y^{t,2}\}$ could have been constructed from any single, fixed long time interval and it would be time invariant. Since our data set is short and we study excess returns, we decided not to use the parameters of a *single* model estimated for the entire period 1948:M2 to 2007:M12[8]. Instead, all estimates of $E_t^m\{y^{t,2}\}$ and $Z_t^{y,2}$ are made <u>*by using real time forecasts*</u>. For each semester date t we use monthly data from 1948:M2 up to t in order to re-estimate parameters $(\hat{\alpha}^t, \hat{\beta}^t, \hat{\delta}^t, \hat{\gamma}^t, \hat{\eta}^t)$ of a stationary model with which we compute $E_t^m\{y^{t,2}\}$ and deduce the value $Z_t^{y,2}$. That is, at each semester date t we compute the stationary forecast for 2 semesters in the future with a model of the form

$$E_t^m\{y^{t,2}\} = \hat{\alpha}^t + \hat{\beta}^t\bar{y}_{t-1} + \hat{\delta}^t NFP_{t-1} + \hat{\gamma}^t F_t + \hat{\eta}^t CPI_{t-1}$$

and to estimate the equation we use only data for {t, t-1, t-2, ..., 1}. We lose 2 semester observations (or 12 months) since at t = 1 we use data for t = 1 on the right and for t = 13 on the left. Parameters $\hat{\alpha}^t, \hat{\beta}^t, \hat{\delta}^t, \hat{\gamma}^t, \hat{\eta}^t$ are estimated at date t with an OLS regression in which the date τ runs from 1 to t:

(31) $$y^{\tau,2} = \alpha^t + \beta^t\bar{y}_{\tau-1} + \delta^t NFP_{\tau-1} + \gamma^t F_\tau + \eta^t CPI_{\tau-1} + \varepsilon^t_{\tau+12}.$$

In (31) the parameters vary with t but as t rises they stabilize. It is clear our estimates for the early part of the sample have an added variance which arises from the time variability of the estimates.

Figure 2 Place Here

Figure 2 traces graphs of mean market beliefs listed in Table 2. The most important fact to

[8] Though we forecast semi-annual data at the end of the 1st and 2nd semester, in order to benefit from the information flow in between we estimate the parameters in (31) on a monthly basis and then pick the forecasts of interest.

stress is that the time series in Figure 1 and those in Figure 2 are *different*: mean forecast data and mean market belief data are different time series. This is an important since one contribution of this paper is the case made by the theory we developed earlier which stresses that the primitive variables to be used in the analysis are the time series in Figure 2, not those in Figure 1. Here we show they are different. High correlation among the different measures is also seen here. Correlation between Z_t^{LIV} and Z_t^{SPF} for the entire span 1968:S2-2007:S2 is 0.94 while the correlations among the three market states of belief for the common span of 1980:S1-2007:S2 is reported in Table 5.

Table 5: Correlation among Mean Market Belief

Note: the table reports correlation for the common sample 1979:S2-2007:S2, while the full sample correlation between LIV and SPF is reported in the text.

	Z_t^{LIV}	Z_t^{SPF}	Z_t^{BLU}
Z_t^{LIV}	1.00		
Z_t^{SPF}	0.92	1.00	
Z_t^{BLU}	0.96	0.95	1.00

In Table 6 we report some descriptive statistics of the data series of extracted market states of belief from different sources which also show significant similarities.

Table 6: Descriptive Statistics of Mean Market Belief

Note: the first three columns in the table report statistics for the period 1968:S2-2007:S2, while the last one covers the sample 1979:S2-2007:S2.

	Z_t^{LIV}	Z_t^{SPF}	$Z_t^{SPF/BLU}$	Z_t^{BLU}
Mean	-0.631	-0.121	0.010	-0.488
Median	-1.209	-0.576	-0.549	-1.043
Max	4.615	4.438	5.837	5.837
Min	-3.748	-2.924	-2.714	-2.714
Std. Dev.	1.759	1.751	1.867	1.883
Skewness	0.635	0.561	0.919	1.721
Kurtosis	2.753	2.431	3.524	5.734
Autocorrelation	0.839	0.814	0.826	0.836
N. Obs.	79	79	79	57

As we noted, Z_t^J are not necessarily orthogonal to date t macroeconomic variables. To understand their behavior we report in Table 7 their correlation with the semiannual growth rate of GDP known at date t, the Federal Fund Rate at date t and the year over year growth rate of Non Farm Payroll at date t-1. It is noteworthy to observe that Z_t^J are correlated with the Federal Funds rate. The same observation is made in Kurz and Motolese (2011) although Z_t^J is extracted there from interest rate forecasts. The correlation of Z_t^J with F_t suggests that one cause of belief diversity is associated with

monetary policy and disagreement among market participants about the effect and future course of monetary policy. This is actually not surprising and we return to this issue later.

Table 7: Correlation of Mean Market Belief with Macro Variables

Note: the first three columns in the table report correlation for the period 1968:S2-2007:S2, while the last one covers the sample 1979:S2-2007:S2.

	Z_t^{LIV}	Z_t^{SPF}	$Z_t^{SPF/BLU}$	Z_t^{BLU}
y_{t-1}	0.146	0.244	0.133	0.040
F_t	0.699	0.627	0.719	0.880
NFP_{t-1}	0.284	0.352	0.253	0.156
DP_t	0.667	0.591	0.640	0.679
DEF_t	0.422	0.345	0.452	0.642
$TERM_t$	-0.447	-0.411	-0.440	-0.466
CAY_t	-0.334	-0.330	-0.330	-0.223

2.3 The Excess Returns Data

Stock Market Excess Returns are computed employing the Center for Research and Security Prices (CRSP) data. In particular we compute $R_{t,t+6}$, the annualized 1-semester (6-month) return on the CRSP value weighted index net of the return on a 90 day Treasury Bill between date t and t+6, and $R_{t,t+12}$, the annualized 2-semester (12-month) return on the CRSP value weighted index net of the return on a 90 day Treasury Bill between date t and t+12. The Excess Returns are of semiannual frequency with starting dates on January and July 1st covering the sample period of 1968:S2-2007:S2. The graphs of excess returns are reported in Figure 3.

Figure 3 Place Here

2.4 Baseline Model's Financial and Macro Predictors

Predictability of excess returns was studied by Fama and French (1989) from which a large literature emanated and hence our reference model is set to their specification. The key variables Fama and French (1989) found useful for forecasting excess returns are (i) the dividend yield, (ii) the default premium in bond pricing and (iii) the term premium. They explained that these variables predict excess returns because they are leading indicators for recessions and future business conditions. We employ the specification of Campbell and Diebold (2009) who argued that instead of variables which forecast future business conditions we should use survey forecast data. Indeed, they show that by using the mean forecasts of the Livingston survey they are able to offer better forecast functions of

excess returns and these render some of the Fama and French (1989) variables insignificant. Following Campbell and Diebold (2009) we use financial and macro variables in our baseline model relative to which we test the effect of market belief. The financial predictors known at semester date t which we use are the annualized percentages of the following:

- DP_t – the dividend yield, calculated for the CRSP value-weighted portfolio;
- DEF_t – the default premium, calculated as the yield difference between a broad corporate bond portfolio and the AAA yield;
- $TERM_t$ – the term premium, calculated as the yield difference between a 10-year Treasury bond and a one-month Treasury bill.

In our baseline model we also us a single macro predictor:

- CAY_t the Lettau and Ludvigson's (2001) consumption wealth ratio.

All the variables above are of semiannual frequency and cover the sample period 1968:S2-2007:S2.

3. Testing the Time Variability of Stock Risk Premia: The Estimated Functions

For both the 1-semester and the 2-semester stock excess returns we estimate linear functions of the following general form

(32) $$R_{t,t+h} = b^h M_t + c^h Z_t + d^h \sigma_t^Z + \varepsilon_{t+h}$$

where M_t is a vector of financial and macro predictors DP_t, DEF_t, $TERM_t$, CAY_t. Z_t is one of the extracted market state of belief indexes discussed in Section 2.2 and σ_t^Z is the corresponding cross section standard deviation of individual belief states at date t. We estimate (32) for the three different sources of market belief data LIV, SPF and SPF/BLU. Our testing procedure will take several steps.

3.1 Testing the Effect of $(Z_t^J, \sigma_t^{Z,J})$ vs. mean forecast $\bar{E}_t^J\{y^{t,2}\}$ and CAY_t

Campbell and Diebold (2009) showed $\bar{E}_t^{LIV}\{y^{t,2}\}$ has predictive power in explaining stock market excess returns. Here we show that, as put forward in the theoretical part of our paper, the primitive variable to consider is Z_t^J, not $\bar{E}_t^J$. We employ the procedure of estimating equation (32) and testing the two alternative specifications Z_t^J and $\bar{E}_t^J\{y^{t,2}\}$. In addition, as shown by Lettau and Ludvigson (2001) the variable CAY is also constructed to be a proxy for market belief and hence we need to examine it in relation to our measures (Z_t^J, σ_t^Z). To facilitate comparison of coefficient

magnitudes we standardize both returns as well as predictors to have zero mean and unit variance in all regression estimates of (32). Furthermore, we compare the results with the basic estimates in which no belief variable is included. We report parameter estimates of (32) in Table 8A-8B.

Table 8A: $R_{t,t+6}$ - Excess Returns: Mean Forecast vs. Mean Belief

Note: regressions (32) to explain 1-semester ahead Stock Market Excess Returns including the cross-sectional standard deviation of the market state of belief. * and † denote significance respectively at the 10% and 5% level. All $\mathbf{R^2}$ are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	(1)	LIV		SPF		SPF/BLU	
$\bar{E}_t^J\{y^{t,2}\}$	---	-0.24 (0.13)*	---	-0.17 (0.10)*	---	-0.29 (0.15)*	---
Z_t^J	---	---	-0.59 (0.17)†	---	-0.38 (0.13)†	---	-0.52 (0.15)†
$\sigma_t^{Z,J}$	---	---	-0.04 (0.14)	---	-0.13 (0.15)	---	-0.12 (0.14)
DP_t	0.10 (0.14)	0.16 (0.17)	0.52 (0.19)†	0.17 (0.17)	0.43 (0.20)†	0.14 (0.16)	0.48 (0.19)†
DEF_t	0.13 (0.12)	0.22 (0.11)†	0.11 (0.09)	0.15 (0.11)	0.09 (0.10)	0.24 (0.12)†	0.16 (0.08)*
$TERM_t$	0.11 (0.12)	0.19 (0.13)	-0.02 (0.10)	0.18 (0.13)	0.03 (0.11)	0.21 (0.12)*	-0.03 (0.11)
CAY_t	0.24 (0.12)†	0.16 (0.14)	0.01 (0.13)	0.15 (0.12)	0.07 (0.12)	0.11 (0.12)	0.03 (0.12)
$\bar{R}^2$(%)	**7.94**	**10.50**	**18.27**	**8.85**	**13.32**	**11.57**	**17.15**

Table 8B: $R_{t,t+12}$ - Excess Returns: Mean Forecast vs. Mean Belief

Note: regressions (32) to explain 2-semester ahead Stock Market Excess Returns including the cross-sectional standard deviation of the market state of belief. * and † denote significance respectively at the 10% and 5% level. All $\mathbf{R^2}$ are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	(1)	LIV		SPF		SPF/BLU	
$\bar{E}_t^J\{y^{t,2}\}$	---	-0.14 (0.15)	---	-0.14 (0.12)	---	-0.27 (0.16)*	---
Z_t^J	---	---	-0.40 (0.17)†	---	-0.28 (0.14)†	---	-0.46 (0.15)†
$\sigma_t^{Z,J}$	---	---	-0.09 (0.15)	---	-0.12 (0.14)	---	-0.11 (0.12)
DP_t	0.20 (0.16)	0.28 (0.18)	0.52 (0.19)†	0.26 (0.17)	0.46 (0.17)†	0.24 (0.16)	0.53 (0.18)†
DEF_t	0.01 (0.14)	0.07 (0.14)	0.01 (0.12)	0.03 (0.13)	-0.01 (0.12)	0.12 (0.13)	0.04 (0.11)
$TERM_t$	0.20 (0.14)	0.22 (0.15)	0.09 (0.12)	0.24 (0.15)*	0.13 (0.13)	0.29 (0.14)†	0.07 (0.12)
CAY_t	0.26 (0.14)*	0.22 (0.17)	0.11 (0.16)	0.19 (0.16)	0.13 (0.16)	0.15 (0.17)	0.08 (0.16)
$\bar{R}^2$(%)	**12.69**	**13.22**	**17.28**	**12.69**	**15.09**	**15.48**	**19.58**

It is clear from Tables 8A-8B that, as reported by Campbell and Diebold (2009), the mean forecast $\bar{E}_t^J\{y^{t,2}\}$ does improve the predictability of returns but our belief variables (Z_t^J, σ_t^Z) are much superior both in terms of statistical significance and in terms of adjusted $\mathbf{R^2}$. This result is true for all data sources and for sub-samples not reported here. Tables 8A-8B also show that when we use our measures $(Z_t^J, \sigma_t^{Z,J})$ the approximate CAY variable is never significant. This remains true for all sub-samples that we examined. Our conclusions are compatible with Motolese and Wu's (2009) results which show that for all sub-samples after 1980 the CAY variable fails to predict changes of wealth.

Table 9A: $R_{t,t+6}$ - Excess Returns: Mean Forecast vs. Mean Belief (excluding CAY)

Note: regressions (32) without variable CAY to explain 1-semester ahead Stock Market Excess Returns and including the cross-sectional standard deviation of the market state of belief. * and † denote significance respectively at the 10% and 5% level. All $\mathbf{R^2}$ are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	(1)	LIV		SPF		SPF/BLU	
$\bar{E}_t\{y^{t,2}\}$	---	-0.30 (0.11)†	---	-0.23 (0.09)†	---	-0.33 (0.12)†	---
Z_t^J	---	---	-0.60 (0.14)†	---	-0.41 (0.12)†	---	-0.54 (0.14)†
$\sigma_t^{Z,J}$	---	-0.08 (0.15)	-0.03 (0.14)	-0.16 (0.18)	-0.14 (0.15)	-0.13 (0.17)	-0.13 (0.14)
DP_t	0.21 (0.14)	0.22 (0.18)	0.53 (0.19)†	0.24 (0.17)	0.48 (0.18)†	0.20 (0.15)	0.51 (0.18)†
DEF_t	0.02 (0.13)	0.18 (0.11)†	0.11 (0.09)	0.11 (0.12)	0.07 (0.10)	0.22 (0.12)*	0.15 (0.09)*
$TERM_t$	0.19 (0.11)*	0.26 (0.11)	-0.02 (0.10)	0.24 (0.11)	0.04 (0.11)	0.26 (0.10)†	-0.03 (0.11)
$\bar{R}^2$(%)	**4.47**	**9.78**	**19.37**	**8.54**	**14.22**	**11.87**	**18.20**

Table 9B: $R_{t,t+12}$ - Excess Returns: Mean Forecast vs. Mean Belief (excluding CAY)

Note: regressions (32) without variable CAY to explain 2-semester ahead Stock Market Excess Returns and including the cross-sectional standard deviation of the market state of belief. * and † denote significance respectively at the 10% and 5% level. All $\mathbf{R^2}$ are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	(1)	LIV		SPF		SPF/BLU	
$\bar{E}_t\{y^{t,2}\}$	---	-0.22 (0.12)*	---	-0.21 (0.10)†	---	-0.32 (0.12)†	---
Z_t^J	---	---	-0.48 (0.15)†	---	-0.35 (0.12)†	---	-0.50 (0.13)†
$\sigma_t^{Z,J}$	---	-0.12 (0.17)	-0.08 (0.15)	-0.15 (0.17)	-0.12 (0.15)	-0.12 (0.16)	-0.12 (0.13)
DP_t	0.32 (0.15)	0.37 (0.18)†	0.60 (0.16)†	0.35 (0.17)†	0.56 (0.14)†	0.31 (0.16)*	0.60 (0.14)†
DEF_t	-0.11 (0.14)	0.01 (0.15)	-0.03 (0.12)	-0.03 (0.14)	-0.07 (0.12)	0.08 (0.15)	0.02 (0.12)
$TERM_t$	0.28 (0.12)†	0.32 (0.11)†	0.10 (0.12)	0.32 (0.12)†	0.15 (0.12)	0.34 (0.11)†	0.08 (0.12)
$\bar{R}^2$(%)	**8.20**	**10.88**	**17.67**	**11.43**	**15.21**	**15.12**	**20.25**

Sharpening the comparison between the effect of mean forecast and mean market belief Z_t^J we exclude in Tables 9A-9B the CAY variable and while controlling for all financial variables we see that the effect of mean forecast is small compared to the contribution of Z_t^J. We shall thus focus in the rest of this paper on the effect of $(Z_t^J, \sigma_t^{Z,J})$.

Turning to the effect of the cross sectional standard deviation $\sigma_t^{Z,J}$ we note first that only in the case of Z_t^{LIV} we have a genuine measure of its correspondent $\sigma_t^{Z,J}$. In the case of Z_t^{SPF} we use the cross-sectional standard deviation of the second quarter among the two consecutive quarterly forecasts used to compute semi-annual data. Furthermore, we use $\sigma_t^{Z,SPF}$ as proxy for $\sigma_t^{Z,SPF/BLU}$ where the computation of its cross-sectional standard deviation is quite cumbersome due to the fact that BLU data is the result of the weighted average of the average forecasts from BLUE and BLUF. Regardless of these fine issues Tables 8A-8B and 9A-9B also show that the cross-sectional standard deviation of market belief Z_t^J does not add anything to the estimates of (32). The parameter estimates of $\sigma_t^{Z,J}$ are consistently negative, which is the sign we find in the bond market model of Kurz and Motolese (2009). However, contrary to the results for the bond market, these parameters are never significant.

Our best explanation is that the effect of $\sigma_t^{Z,J}$ is complex. We know from the bond market model that for short investment horizons the sign of the $\sigma_t^{Z,J}$ parameter *is either insignificant or is positive*. It becomes negative and very significant for investment horizons longer than 6-9 months. Hence it is not entirely surprising that in the stock market model under consideration here these parameters are not statistically significant for investment horizons of 6-12 months.

3.2 Testing the Effect of other variable vs. $(Z_t^J, \sigma_t^{Z,J})$

We now turn to the correlation between the belief variables $(Z_t^J, \sigma_t^{Z,J})$ and other variables as reported in Table 7 and test how robust the simple model (32) is. Our question is simple: is the effect of $(Z_t^J, \sigma_t^{Z,J})$ a reflection of other variables correlated with $(Z_t^J, \sigma_t^{Z,J})$ or does it represent an independent effect of belief? We study the variables DP_t, DEF_t, $TERM_t$, NFP_{t-1}, and F_t by using a baseline model and then adding each of these variables one at a time. We also introduce them in pairs. The test is conducted over the three data sources we have and for the two investment horizons of 6 and 12 months. The results are reported in the sequence of Tables 10A-10F. We report first the results for the Livingston survey then for SPF and finally for the combination SPF\BLU.

To sum up our key result we find that although the parameters of $(Z_t^J, \sigma_t^{Z,J})$ change due to correlation, the conclusion remains the same and points to an independent effect of belief, represented by $(Z_t^J, \sigma_t^{Z,J})$, which is not explained by other variables. Moreover, the order of magnitude of the effect of $(Z_t^J, \sigma_t^{Z,J})$ remains essentially the same regardless of data source or investment period.

Table 10A: $R_{t,t+6}$ - Stock Excess Returns (Livingston)

Note: testing the contribution of Z_t^J in regressions (33) to explain 1-semester ahead Stock Market Excess Returns by adding correlated macro variables. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	Livingston Z_t								
Z_t	-0.56 (0.15)†	-0.57 (0.13)†	-0.59 (0.14)†	-0.58 (0.14)†	-0.54 (0.14)†	-0.60 (0.13)†	-0.57 (0.14)†	-0.61 (0.13)†	-0.59 (0.13)†
σ_t	-0.03 (0.13)	-0.03 (0.13)	-0.03 (0.13)	-0.03 (0.14)	-0.03 (0.13)	-0.03 (0.13)	-0.03 (0.13)	-0.03 (0.14)	-0.03 (0.14)
DP_t	0.62 (0.19)†	0.58 (0.18)†	0.53 (0.19)†	0.59 (0.18)†	0.60 (0.18)†	0.52 (0.21)†	0.55 (0.21)†	0.52 (0.21)†	0.59 (0.22)†
F_t	-0.07 (0.13)	---	---	---	-0.06 (0.12)	---	-0.08 (0.11)	---	-0.21 (0.16)
NFP_{t-1}	---	-0.05 (0.09)	---	---	-0.04 (0.09)	0.02 (0.15)	0.04 (0.14)	0.02 (0.15)	0.05 (0.14)
DEF_t	---	---	0.10 (0.10)	---	---	0.12 (0.17)	0.14 (0.17)	0.13 (0.16)	0.20 (0.16)
$TERM_t$	---	---	---	0.01 (0.10)	---	---	---	-0.02 (0.10)	-0.13 (0.15)
$\bar{R}^2$(%)	**19.90**	**19.91**	**20.41**	**19.71**	**19.00**	**19.36**	**18.54**	**18.29**	**18.05**

Table 10B: $R_{t,t+12}$ - Stock Excess Returns (Livingston)

Note: testing the contribution of Z_t^J in regressions (33) to explain 2-semester ahead Stock Market Excess Returns by adding correlated macro variables. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	Livingston Z_t								
Z_t	-0.50 (0.16)†	-0.52 (0.14)†	-0.53 (0.15)†	-0.48 (0.15)†	-0.49 (0.15)†	-0.52 (0.14)†	-0.49 (0.15)†	-0.47 (0.14)†	-0.47 (0.14)†
σ_t	-0.08 (0.15)	-0.08 (0.15)	-0.08 (0.15)	-0.08 (0.15)	-0.08 (0.15)	-0.08 (0.15)	-0.08 (0.15)	-0.08 (0.15)	-0.08 (0.15)
DP_t	0.63 (0.17)†	0.59 (0.15)†	0.59 (0.16)†	0.58 (0.15)†	0.63 (0.17)†	0.60 (0.18)†	0.63 (0.18)†	0.61 (0.18)†	0.59 (0.19)†
F_t	-0.08 (0.14)	---	---	---	-0.08 (0.14)	---	-0.08 (0.15)	---	0.04 (0.23)
NFP_{t-1}	---	-0.02 (0.12)	---	---	-0.02 (0.12)	-0.04 (0.20)	-0.02 (0.20)	-0.02 (0.19)	-0.03 (0.19)
DEF_t	---	---	0.01 (0.12)	---	---	-0.02 (0.20)	-0.01 (0.21)	-0.05 (0.19)	-0.06 (0.21)
$TERM_t$	---	---	---	0.09 (0.11)	---	---	---	0.10 (0.12)	0.12 (0.19)
$\bar{R}^2$(%)	**18.28**	**18.03**	**17.99**	**18.71**	**17.20**	**16.94**	**16.07**	**16.57**	**15.44**

Table 10C: $R_{t,t+6}$ - Stock Excess Returns (SPF)

Note: testing the contribution of Z_t^J in regressions (33) to explain 1-semester ahead Stock Market Excess Returns by adding correlated macro variables. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	SPF Z_t								
Z_t	-0.37 (0.11)†	-0.40 (0.13)†	-0.43 (0.12)†	-0.40 (0.12)†	-0.35 (0.11)†	-0.42 (0.13)†	-0.37 (0.11)†	-0.41 (0.13)†	-0.38 (0.11)†
σ_t	-0.15 (0.15)	-0.15 (0.14)	-0.15 (0.15)	-0.13 (0.15)	-0.16 (0.14)	-0.15 (0.15)	-0.16 (0.15)	-0.14 (0.15)	-0.18 (0.17)
DP_t	0.61 (0.19)†	0.52 (0.17)†	0.48 (0.18)†	0.52 (0.17)†	0.60 (0.19)†	0.48 (0.20)†	0.56 (0.21)†	0.48 (0.20)†	0.61 (0.24)†
F_t	-0.17 (0.12)	---	---	---	-0.16 (0.12)	---	-0.18 (0.11)	---	-0.32 (0.19)
NFP_{t-1}	---	-0.06 (0.10)	---	---	-0.04 (0.10)	-0.02 (0.17)	0.02 (0.16)	-0.01 (0.17)	0.02 (0.17)
DEF_t	---	---	0.08 (0.10)	---	---	0.06 (0.17)	0.10 (0.18)	0.06 (0.18)	0.16 (0.19)
$TERM_t$	---	---	---	0.06 (0.11)	---	---	---	0.04 (0.11)	-0.14 (0.18)
$\bar{R}^2$(%)	**16.07**	**15.09**	**15.23**	**15.11**	**15.12**	**14.10**	**14.31**	**13.05**	**13.80**

Table 10D: $R_{t,t+12}$ - Stock Excess Returns (SPF)

Note: testing the contribution of Z_t^J in regressions (33) to explain 2-semester ahead Stock Market Excess Returns by adding correlated macro variables. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	SPF Z_t								
Z_t	-0.35 (0.13)†	-0.40 (0.13)†	-0.41 (0.13)†	-0.37 (0.12)†	-0.34 (0.12)†	-0.38 (0.12)†	-0.34 (0.12)†	-0.33 (0.11)†	-0.33 (0.12)†
σ_t	-0.17 (0.14)	-0.17 (0.15)	-0.17 (0.15)	-0.15 (0.14)	-0.18 (0.15)	-0.17 (0.15)	-0.18 (0.15)	-0.15 (0.15)	-0.15 (0.15)
DP_t	0.63 (0.17)†	0.53 (0.13)†	0.54 (0.14)†	0.52 (0.13)†	0.62 (0.16)†	0.58 (0.18)†	0.64 (0.19)†	0.58 (0.18)†	0.60 (0.21)†
F_t	-0.17 (0.14)	---	---	---	-0.17 (0.15)	---	-0.16 (0.15)	---	-0.05 (0.25)
NFP_{t-1}	---	-0.03 (0.13)	---	---	-0.02 (0.13)	-0.07 (0.21)	-0.04 (0.21)	-0.05 (0.21)	-0.04 (0.21)
DEF_t	---	---	0.01 (0.12)	---	---	-0.07 (0.20)	-0.04 (0.21)	-0.10 (0.20)	-0.09 (0.22)
$TERM_t$	---	---	---	0.13 (0.11)	---	---	---	0.14 (0.13)	0.11 (0.21)
$\bar{R}^2$(%)	**15.97**	**14.72**	**14.66**	**16.08**	**14.86**	**13.73**	**13.75**	**14.15**	**13.00**

As we have seen before, the accuracy of SPF data seems to be lower than that of Livingston or BLU and we associate it with the fact that SPF forecasting dates have been less precise as to timing and this introduces an error for which we cannot control.

Table 10E: $R_{t,t+6}$ - Stock Excess Returns (SPF/BLU)

Note: testing the contribution of Z_t^J in regressions (33) to explain 1-semester ahead Stock Market Excess Returns by adding correlated macro variables. * and † denote

significance respectively at the 10% and 5% level. All $\mathbf{R^2}$ are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	SPF/BLU Z_t								
Z_t	-0.47 (0.14)†	-0.48 (0.12)†	-0.52 (0.12)†	-0.50 (0.13)†	-0.45 (0.14)†	-0.52 (0.12)†	-0.49 (0.14)†	-0.53 (0.14)†	-0.49 (0.13)†
σ_t	-0.13 (0.13)	-0.13 (0.13)	-0.13 (0.13)	-0.12 (0.14)	-0.14 (0.13)	-0.13 (0.13)	-0.13 (0.13)	-0.13 (0.14)	-0.16 (0.16)
DP_t	0.61 (0.19)†	0.58 (0.17)†	0.51 (0.18)†	0.58 (0.18)†	0.60 (0.18)†	0.51 (0.20)†	0.53 (0.20)†	0.51 (0.20)†	0.59 (0.22)†
F_t	-0.07 (0.13)	---	---	---	-0.06 (0.12)	---	-0.07 (0.11)	---	-0.22 (0.18)
NFP_{t-1}	---	-0.08 (0.09)	---	---	-0.08 (0.09)	-0.00 (0.16)	0.01 (0.15)	-0.00 (0.16)	0.01 (0.16)
DEF_t	---	---	0.14 (0.09)	---	---	0.14 (0.17)	0.15 (0.17)	0.15 (0.17)	0.21 (0.18)
$TERM_t$	---	---	---	0.02 (0.11)	---	---	---	-0.03 (0.11)	-0.15 (0.17)
$\bar{R}^2$(%)	**18.14**	**18.60**	**19.24**	**17.98**	**17.66**	**18.15**	**17.24**	**17.07**	**16.84**

Table 10F: $R_{t,t+12}$ - Stock Excess Returns (SPF/BLU)

Note: testing the contribution of Z_t^J in regressions (33) to explain 2-semester ahead Stock Market Excess Returns by adding correlated macro variables. * and † denote significance respectively at the 10% and 5% level. All $\mathbf{R^2}$ are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	SPF/BLU Z_t								
Z_t	-0.50 (0.13)†	-0.52 (0.11)†	-0.54 (0.12)†	-0.50 (0.12)†	-0.49 (0.13)†	-0.53 (0.12)†	-0.50 (0.13)†	-0.50 (0.12)†	-0.49 (0.13)†
σ_t	-0.14 (0.13)	-0.14 (0.12)	-0.13 (0.13)	-0.12 (0.13)	-0.14 (0.13)	-0.14 (0.13)	-0.14 (0.13)	-0.13 (0.13)	-0.13 (0.14)
DP_t	0.65 (0.16)†	0.62 (0.14)†	0.59 (0.14)†	0.60 (0.14)†	0.64 (0.15)†	0.60 (0.18)†	0.63 (0.18)†	0.61 (0.18)†	0.61 (0.20)†
F_t	-0.08 (0.14)	---	---	---	-0.07 (0.14)	---	-0.08 (0.14)	---	-0.00 (0.25)
NFP_{t-1}	---	-0.04 (0.12)	---	---	-0.04 (0.12)	-0.03 (0.20)	-0.02 (0.20)	-0.02 (0.20)	-0.02 (0.20)
DEF_t	---	---	0.05 (0.11)	---	---	0.02 (0.20)	0.03 (0.21)	-0.00 (0.20)	0.00 (0.21)
$TERM_t$	---	---	---	0.08 (0.11)	---	---	---	0.07 (0.13)	0.07 (0.22)
$\bar{R}^2$(%)	**20.99**	**20.93**	**20.90**	**21.29**	**20.08**	**19.88**	**19.01**	**19.18**	**18.05**

We now address the question of effect of the two Fama-French financial variables DEF_t, $TERM_t$ which do not seem to exhibit statistical significance. A careful examination of Fama and French (1989) reveals that the effects of the variables DEF_t and $TERM_t$ are significant only for longer investment period of over a year, and mostly up to four years. Hence, we do not find it surprising that in the shorter investment period of 6-12 months the standard errors of the estimates are too high to judge their actual contribution to explaining risk premia. We thus do not draw any conclusions about the efficacy of these variables in models with longer investment periods.

3.3 Interaction of Z with DP

Table 7 reveals a correlation between Z_t^J and DP_t which we examine now. The hypothesis is that the effect of belief is not well approximated by a liner term since it may depend on Z_t^J or DP_t. To that end we first study the following model with the product variables $Z_t^J \times DP_t$

(33) $$R_{t,t+h} = b^h Z_t^J + c^h \sigma_t + d^h DP_t + e^h (Z_t^J \times DP_t) + \varepsilon_{t+h}.$$

The results are reported in Tables 11A-11B and show a drastic increase in the standard errors of the

estimates due to the correlation between Z_t^J and $Z_t^J \times DP_t$.

Table 11A: $R_{t,t+6}$ - Stock Market Excess Returns -- Interactions

Note: regressions (33) to explain 1-semester ahead Stock Market Excess Returns. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	LIV	SPF	SPF/BLU
Z_t^J	-0.75 (0.48)	-0.30 (0.38)	-0.48 (0.39)
σ_t^J	-0.03 (0.13)	-0.15 (0.15)	-0.12 (0.13)
DP_t	0.62 (0.21)†	0.52 (0.18)†	0.58 (0.18)†
$(Z_t^J)^2 \times DP_t$	0.15 (0.40)	-0.13 (0.35)	-0.02 (0.35)
$\bar{R}^2(\%)$	**19.89**	**14.95**	**17.93**

Table 11B: $R_{t,t+12}$ - Stock Market Excess Returns -- Interactions

Note: regressions (34) to explain 2-semester ahead Stock Market Excess Returns. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	LIV	SPF	SPF/BLU
Z_t^J	-1.18 (0.43)†	-0.77 (0.47)*	-1.00 (0.43)†
σ_t^J	-0.08 (0.16)	-0.15 (0.15)	-0.12 (0.12)
DP_t	0.71 (0.16)†	0.56 (0.14)†	0.63 (0.14)†
$(Z_t^J)^2 \times DP_t$	0.61 (0.38)	0.36 (0.47)	0.45 (0.42)
$\bar{R}^2(\%)$	**21.02**	**15.64**	**22.07**

An alternative specification of the effect of Z_t^J is expressed in the model

(34) $$R_{t,t+h} = b^h Z_t^J + c^h \sigma_t + d^h DP_t + e^h((Z_t^J)^2 \times DP_t) + \varepsilon_{t+h}$$

and the results are reported in Tables 12A-12B. In these tables the standard errors remain small and the non linear effect is significant.

Table 12A: $R_{t,t+6}$ - Stock Market Excess Returns (equation (34a))

Note: regressions (34) to explain 1-semester ahead Stock Market Excess Returns. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	LIV	SPF	SPF/BLU
Z_t^J	-0.54 (0.13)†	-0.29 (0.10)†	-0.37 (0.14)†
σ_t^J	-0.03 (0.13)	-0.18 (0.16)	-0.16 (0.14)
DP_t	0.59 (0.17)†	0.54 (0.17)†	0.57 (0.17)†
$(Z_t^J)^2 \times DP_t$	-0.13 (0.07)*	-0.22 (0.09)†	-0.16 (0.10)*
$\bar{R}^2(\%)$	**21.29**	**18.16**	**19.36**

Table 12B: $R_{t,t+12}$ - Stock Market Excess Returns (equation (34a))

Note: regressions (34) to explain 2-semester ahead Stock Market Excess Returns. * and † denote significance respectively at the 10% and 5% level. All R^2 are adjusted for degrees of freedom and Newey-West robust standard errors are reported in parenthesis. The sample is over 1968:S2-2007:S2.

	LIV	SPF	SPF/BLU
Z_t^J	-0.52 (0.16)†	-0.33 (0.14)†	-0.46 (0.16)†
σ_t^J	-0.08 (0.15)	-0.18 (0.15)	-0.15 (0.13)
DP_t	0.60 (0.15)†	0.55 (0.14)†	0.61 (0.14)†
$(Z_t^J)^2 \times DP_t$	-0.04 (0.10)	-0.13 (0.13)	-0.09 (0.11)
$\bar{R}^2(\%)$	**18.12**	**15.71**	**21.20**

4. Effect of Mean Market Belief on Aggregate Consumption

In a paper which used the same belief data we used here, Motolese and Wu (2009) re-examined the effect of the CAY variable (consumption\wealth ratio) proposed by Lattau and Ludvigson (2001) as a proxy for market belief in order to explain the equity risk premium. They found that given the belief variables, CAY is insignificant. This is not surprising in light of the results we presented above. Motolese and Wu (2009) also estimated the cointegrated VAR model of Lattau and Ludvigson (2001) in which the growth rate of consumption and wealth are deduced from log linearization of the optimizing agents budget constraints. A simple two variable version of their model is as follows

(35a) $$\triangle a_t = \alpha_0 + \alpha_1 \triangle a_{t-1} + \alpha_2 Z_t + \alpha_3 \sigma_t^Z + \alpha_4 F_t + \epsilon_t$$

(35b) $$\triangle c_t = \beta_0 + \beta_1 \triangle c_{t-1} + \beta_2 \triangle a_{t-1} + \beta_3 Z_t + \beta_4 \sigma_t^Z + \beta_5 F_t + \epsilon_t$$

where

$\triangle a_t$ – annual growth rate of assets in quarter preceding semester date t

$\triangle c_t$ – annual growth rate of consumption in quarter preceding semester date t

F_t – Fed funds rate at semester date t.

(Z_t, σ_t^Z) – measures of market belief.

In estimating (35a)-(35b) we used the data provided by Lattau and Ludvigson (2001) but the definition of wealth we use here does not incorporate their estimated values of human capital. In estimating these equations we have not standardized the variables and this facilitates the interpretation of the parameters. Table 8 reports the estimated coefficients of (35a)-(35b) using the earlier semester data 1968:S2-2007S2.

Table 8: VAR Model for Asset and Consumption Growth

Note: Estimated parameters of (35a) -(35b) to explain asset and consumption growth rates. * and † denote significance respectively at 10% and 5% levels. All $\mathbf{R}^2$ are adjusted for degrees of freedom and Newey-West robust standard errors are in parenthesis. The sample is over 1968:S2-2007:S2.

	LIV		SPF/BLU	
	Δa_t	Δc_t	Δa_t	Δc_t
Constant	0.040 (0.012)†	0.008 (0.002)†	0.028 (0.006)†	0.007 (0.002)†
Δa_{t-1}	-0.094 (0.151)	0.313 (0.119)†	-0.095 (0.138)	0.228 (0.123)*
Δc_{t-1}	----	0.051 (0.022)†	----	0.056 (0.020)†
Z_t	**0.366 (0.172)**†	**0.103 (0.043)**†	**0.238 (0.133)***	**0.118 (0.035)**†
$\sigma_{Z,J}$	**- 0.622 (0.396)**	**-0.149 (0.071)**†	**- 0.074 (0.151)**	**-0.060 (0.036)***
F_t	- 0.211 (0.095)†	-0.011 (0.021)	- 0.257 (0.091)†	-0.035 (0.018)*
$\bar{R}^2$(%)	9.16	26.17	4.47	25.42

It is seen the fit of the consumption equation is much better than of wealth, a well known result. We also find that both mean market belief as well as the cross sectional standard deviation contribute significantly to the explanation of consumption growth. Indeed, since all variables are in terms of annualized growth rates, 1 percentage change in Z causes a 0.1 percentage change in the growth rate of consumption. As seen in Table 6, a change of *two standard deviations* in Z would then cause a 0.35 percentage point change in the growth rate of consumption . The direction of change is also as one should expect from the theory: consumer expectations of abnormally high growth rate of income in the future would lead to an increased consumption. On the other hand an increase in dispersion of belief causes a decrease in consumption. Finally, the mean market belief have a significant effect on the growth rate of wealth and the effect is also in the direction predicted by the theory: a rise in the mean market belief about better future business conditions increases the present prices of assets and the growth rate of wealth.

References

Aiolfi, M., Favero, C.A. (2005): "Model Uncertainty, Thick Modeling and the Predictability of Stock Returns." *Journal of Forecasting*, **24**, 233 - 254.

Allen, F., Morris, S., Shin, H.S. (2006) : "Beauty Contests and Iterated Expectations in Asset Markets." *Review of Financial Studies*, **19**, 719-752.

Baker, M., Wurgler, J. (2006): "Investor Sentiment and Cross Section of Stock Returns." *Journal of Finance*, **61**, 1645 - 1680.

Barberis, N., Schleifer, A., Vishnay, R. (1998): "A Model of Investor Sentiment." *Journal of Financial Economics*, **49**, 307 - 343.

Batchelor, R., Dua, P. (1991): "Blue Chip Rationality Tests." *Journal of Money, Credit and Banking*, **23**, 692 - 705.

Blanchard, O.J., Kahn, C.M. (1980): "The Solution of Linear Difference Models Under Rational Expectations." *Econometrica*, **48**, 1305 - 1311.

Brown, D., Jennings, R. (1989): "On Technical Analysis." *Review of Financial Studies*, **2**, 527-551.
Brunnermeier, M.K. (2001): Asset Pricing Under Asymmetric Information. Oxford: Oxford University Press.
Campbell, S.D., Diebold, F.X. (2009): "Stock Returns and Expected Business Conditions: Half a Century of Direct Evidence." *Journal of Business and Economic Statistics*, **27** , 266-278.
Daniel, K.D., Hirshleifer, D., Subrahmanyam, A. (1998): "A Theory of Overconfidence, Self-Attribution, and Security Market Under- and Over- Reactions." *Journal of Finance*, **53**, 1839 - 1885.
DeLong, B., Shleifer, A., Summers, L., Waldman, R.J., (1990): "Noise Trader Risk in Financial Markets." *The Journal of Political Economy*, **98**, 703-738.
Detemple, J., Murthy S.(1994): "Intertemporal Asset Pricing with Heterogeneous Beliefs." *Journal of Economic Theory* **62**, 294-320.
Diether, K.B., Malloy, C.J., Scherbina, A. (2002): "Differences of Opinion and the Cross Section of Stock Returns." *Journal of Finance*, **57**, 2113 - 2141.
Fan, M., (2006): "Heterogeneous Beliefs, the Term Structure and Time-Varying Risk Premia." *Annals of Finance*, **2**, 259-285.
Fama, E.F., French, K.R. (1989): "Business Conditions and Expected Returns on Stocks and Bonds." *Journal of Financial Economics*, **25**, 23 - 49.
Greif, A. (2006): *Institutions and the Path to Economic Modernity: Lessons from Medieval Trade.* Cambridge: Cambridge University Press.
Grundy, B., McNichols, M. (1989): "Trade and Revelation of Information Through Prices and Direct Disclosure." *Review of Financial Studies*, **2**, 495-526.
Hansen, L.P., Hodrick, R.J. (1980): "Forward Exchange Rates as Optimal Predictors of Future Spot Rates: An Econometric Analysis." *Journal of Political Economy*, **88**, 829-853.
Harris, M., Raviv, A. (1993) : "Differences of Opinion Make a Horse Race." *Review of Financial Studies* **6**, 473-506
Harrison, M., Kreps, D.(1978): "Speculative Investor Behavior in a Stock Market with Heterogenous Expectations." *Quarterly Journal of Economics* **92**, 323-336.
He, H., Wang, J. (1995): " Differential Information and Dynamic Behavior of Stock Trading Volume." *Review of Financial Studies*, **8**, 914 - 972.
Hodrick, R.J., (1992): "Dividend Yields and Expected Stock Returns: Alternative Procedures for Influence and Measurement." *Review of Financial Studies*, **5**, 357-386.
Johnson, T. C. (2004): " Forecast Dispersion and the Cross Section of Expected Returns." *The Journal of Finance*, **59**, 1957 - 1978.
Kandal, E., Pearson, N. (1995): "Differential Interpretation of Public Signals and Trade in Speculative Markets." *Journal of Political Economy*, **103**, 831-872.
Karakatsani, N. V., Salmon, M. (2008): " Sentiment and Price Formation: Interaction and Regime Shifts." Working Paper, Warwick Business School, University of Warwick, March.
Kurz, M.(1974): "The Kesten-Stigum Model and the Treatment of Uncertainty in Equilibrium Theory." In Balch, M.S., McFadden, D.L., Wu, S.Y., (ed.) *Essays on Economic Behavior Under Uncertainty*. Amsterdam: North-Holland, 389-399.
Kurz, M. (1994): "On the Structure and Diversity of Rational Beliefs." *Economic Theory* **4**, 877 - 900 . (An edited version appears as Chapter **2** of Kurz, M. (ed.) (1997)).
Kurz, M. (ed) (1997): *Endogenous Economic Fluctuations: Studies in the Theory of Rational Belief.* Studies in Economic Theory, No. **6**, Berlin and New York: Springer-Verlag.

Kurz, M. (2005): “Diverse Beliefs, Forecast Errors and Central Bank Policy.” Working Paper, Stanford University, July.

Kurz, M. (2008): “Beauty Contests Under Private Information and Diverse Beliefs: How different?” *Journal of Mathematical Economics*, **44,** *762-784.*

Kurz, M. (2009): “Rational Diverse Beliefs and Economic Volatility.” Chapter 8 in Hens, T. and Schenk-Hoppé, K.R. (ed.) *Handbook Of Financial Markets: Dynamics and Evolution*.” Amsterdam: North-Holland, 440- 506.

Kurz, M., Beltratti, A. (1997): “The Equity Premium is No Puzzle.” Chapter **11** in Kurz, M. (ed.) (1997) *Endogenous Economic Fluctuations: Studies in the Theory of Rational Belief.* Studies in Economic Theory, No. **6**, Berlin and New York: Springer-Verlag, 283 -316.

Kurz, M., Jin, H., Motolese, M. (2005a): “Determinants of Stock Market Volatility and Risk Premia.” *Annals of Finance*, **1**, 109-147.

Kurz, M., Jin, H., Motolese, M. (2005b): “The Role of Expectations in Economic Fluctuations and the Efficacy of Monetary Policy.” *Journal of Economic Dynamics and Control*, **29**, 2017 - 2065.

Kurz, M., Motolese, M. (2001): “Endogenous Uncertainty and Market Volatility.” *Economic Theory*, **17**, 497 - 544.

Kurz, M., Motolese, M. (2011): “Diverse Beliefs and Time Variability or Risk Premia.” *Economic Theory*, **(forthcoming)**

Kurz, M., Schneider, M.(1996): Coordination and Correlation in Markov Rational Belief Equilibria. *Economic Theory* **8**, 489 - 520. (Appears as Chapter **10** of Kurz, M. (ed.) (1997a))

Kurz, M., Wu, H.M. (1996): “Endogenous Uncertainty in a General Equilibrium Model with Price Contingent Contracts.” *Economic Theory*, **8**, 461 -488. (Appears as Chapter **2** of Kurz, M. (ed.) (1997a)).

Lee, W.Y., Jiang, C.X., Indro, D.C. (2002): “Stock Market Volatility, Excess Returns, and the role of Investor Sentiment.” *Journal of Banking & Finance*, **26**, 2277 - 2299.

Lee, C.A., Shleifer, A., Thaler, R. (1991): “Investor Sentiment and the Closed-End Fund Puzzle.” **Journal of Finance**, **46**, 75 - 109.

Miller, E.M (1977): “Risk, Uncertainty and Divergence of Opinion.” *Journal of Finance*, **32**, 1151-1168.

Mokyr, J. (2010): *The Enlightened Economy: An Economic History of Britain 1700-1850.* London: New Haven: Yale University Press.

Motolese, M. (2001): “Money Non-Neutrality in a Rational Belief Equilibrium with Financial Assets.” *Economic Theory*, **18**, 97 - 16.

Motolese, M. (2003): “Endogenous Uncertainty and the Non-Neutrality of Money.” *Economic Theory*, **21**, 317 - 345.

Nakata, H. (2007): “A Model of Financial Markets with Endogenously Correlated Rational Beliefs.” *Economic Theory*, **30**, 431 - 452.

Nielsen, C.K. (1996): “Rational Belief Structures and Rational Belief Equilibria.” *Economic Theory*, **8**, 339 - 422 .

Nielsen, C.K. (2003): “Floating Exchange Rates vs. A Monetary Union Under Rational Beliefs: The Role of Endogenous Uncertainty.” *Economic Theory*, **21**, 347 - 398.

Park, C. (2005): “Stock Return Predictability and the Dispersion of Earning Forecasts.” *Journal of Business*, **78**, 2351 - 2375.

Romer, C. D., Romer, D. H. (2000):“Federal Reserve Information and the Behavior of Interest

Rates." *American Economic Review*, June, pp. 429-457.
Scheinkman, J.A., Xiong, W. (2003): "Overconfidence and Speculative Bubbles." *Journal of Political Economy*, **111**, 1183-1219
Stock, H.J., Watson, W.M. (1999): "Forecasting Inflation." Department of Economics, Princeton University.
Stock, H.J., Watson, W.M. (2001): "Forecasting Output and Inflation: The Role of Asset Prices." Department of Economics, Princeton University, February.
Stock, H.J., Watson, W.M. (2002): "Macroeconomic Forecasting Using Diffusion Indexes." *Journal of Business and Economic Statistics*, The American Statistical Association, **20**, 147-162.
Stock, H.J., Watson, W.M. (2005): "An Empirical Comparison of Methods for Forecasting Using Many Predictors." Department of Economics, Harvard University, January.
Tabellini, G. (2008): "Institutions and Culture." *Journal of the European Economic Association Papers and Proceedings*, **6**, 255 - 294.
Varian, H.R. (1985): "Divergence of Opinion in Complete Markets: A Note." *Journal of Finance* **40**, 309-317
Varian, H.R. (1989) : "Differences of Opinion in Financial Markets." In *Financial Risk: Theory, Evidence and Implications*, Proceeding of the 11th Annual Economic Policy Conference of the Federal Reserve Bank of St. Louis, Stone, C.C. ed. Boston: Kluwer Academic Publishers.
Wang, J. (1994): "A Model of Competitive Stock Trading Volume." *Journal of Political Economy*, 102, 127 - 168.
Wu, H.M., Guo, W.C. (2003): "Speculative Trading with Rational Beliefs and Endogenous Uncertainty." *Economic Theory*, **21** , 263 292.
Wu, H.M., Guo, W.C. (2004): "Asset Price Volatility and Trading Volume with Rational Beliefs." *Economic Theory*, **23** , 461- 488.

APPENDIX A: Derivation of the Value Function

For simplicity we ignore in this Appendix the index i identifying the agent who carries out the optimization. Hence, the dynamic programming problem is as follows. Given initial values (θ_0, W_0), maximize

$$U_t = \underset{(\theta, C)}{E_t} \left[\sum_{s=0}^{\infty} -\beta^{t+s-1} e^{-(\frac{1}{\tau}C_{t+s})} \,|\, H_t\right]$$

subject to the following definitions

$$W_{t+1} = (W_t - C_t)R + \theta_t Q_{t+1}$$
$$Q_{t+1} = p_{t+1} + (d_{t+1} + \mu) - p_t R$$
$$\psi_t = (1, d_t, z_t, g_t)$$

and stochastic transition functions

$$\begin{aligned} d_{t+1} &= \lambda_d d_t + \lambda_g^d g_t + \varepsilon_{t+1}^d \\ Z_{t+1} &= \lambda_z Z_t + \lambda_g^z g_t + \varepsilon_{t+1}^z, \\ g_{t+1} &= \lambda_z g_t + \varepsilon_{t+1}^g \end{aligned} \quad \Lambda_\psi = \begin{pmatrix} 1, 0, 0, 0 \\ 0, \lambda_d, 0, \lambda_g^d \\ 0, 0, \lambda_z, \lambda_g^z \\ 0, 0, 0, \lambda_z \end{pmatrix}, \quad \hat{\varepsilon}_t = (1, \varepsilon_t^d, \varepsilon_t^z, \varepsilon_t^g), \ (\varepsilon_t^d, \varepsilon_t^z, \varepsilon_t^g) \sim N(0, \Sigma).$$

Step 1: simplification. We thus define, for the unknown matrix V

$$\Lambda = \begin{pmatrix} \lambda_d\,,\,0\,,\,\lambda_g^d \\ 0\,,\,\lambda_z\,,\,\lambda_g^z \\ 0\,,\,0\,,\,\lambda_z \end{pmatrix} \quad , \quad V = \begin{pmatrix} v_{00}\,,\,v_{01}\,,\,v_{02}\,,\,v_{03} \\ v_{01} \\ v_{02} \qquad V_{11} \\ v_{03} \end{pmatrix} = \begin{pmatrix} v_{00}\,, & \hat{v}_0^{\,T} \\ \hat{v}_0 & ,V_{11} \end{pmatrix}$$

We now have $\psi_{t+1} = \Lambda_\psi \psi_t + \Lambda_\varepsilon \hat{\varepsilon}_{t+1}$, where $\Lambda_\varepsilon = \begin{pmatrix} 0\,,\,0 \\ 0\,,\,I_{(3\times3)} \end{pmatrix}$ is a 4×4 matrix

We assume that $p_t = a_d d_t + a_z z_t + P_0$ and verify it later when we solve for equilibrium. Using this price map we can compute excess return in terms of the state variables we have that

$$Q_{t+1} = (a_d + 1)[\lambda_d d_t + \lambda_g^d g_t + \varepsilon_{t+1}^d] + a_z[\lambda_z Z_t + \lambda_g^z g_t + \varepsilon_{t+1}^z] + P_0 - [a_d d_t + a_z Z_t + P_0]R + \mu$$

Hence

$$Q_{t+1} = [(a_d + 1)\lambda_d - R a_d] d_t + [a_z \lambda_z - R a_z] Z_t + [(a_d + 1)\lambda_g^d + a_z \lambda_g^z] g_t + [P_0(1 - R) + \mu] + [(a_d + 1)\varepsilon_{t+1}^d + a_z \varepsilon_{t+1}^d]$$

Or,

$$Q_{t+1} = a^T \psi_t + \hat{b}^T \varepsilon_{t+1} \text{ , hence } E_t[Q_{t+1}] = a^T \psi_t$$

where

$$a^T = ([P_0(1 - R) + \mu]\,,\,[(a_d + 1)\lambda_d - R a_d]\,,\,[a_z \lambda_z - R a_z]\,,\,[(a_d + 1)\lambda_g^d + a_z \lambda_g^z]) \quad , \quad \hat{b}^T = (0, (a_d + 1), a_z, 0).$$

Also, we shall use the notation $b^T = ((a_d + 1), a_z, 0)$. Now compute the expression

$$-\alpha W_{t+1} - \frac{1}{2}\psi_{t+1}^T V \psi_{t+1} = -\alpha(W_t - C_t)R - \alpha\theta_t[a^T \psi_t + \hat{b}^T \hat{\varepsilon}_{t+1}] - \frac{1}{2}\psi_t^T \Lambda_\psi^T V \Lambda_\psi \psi_t - \psi_t^T \Lambda_\psi^T V \Lambda_\varepsilon \hat{\varepsilon}_{t+1} - \frac{1}{2}\hat{\varepsilon}_{t+1}^T \Lambda_\varepsilon^T V \Lambda_\varepsilon \hat{\varepsilon}_{t+1}$$

Algebra and simplification leads to the conclusion that we have

$$-\alpha W_{t+1} - \frac{1}{2}\psi_{t+1}^T V \psi_{t+1} = -A_t - e_t^T \varepsilon_{t+1} - \frac{1}{2}\varepsilon_{t+1}^T V_{11} \varepsilon_{t+1}$$

where

$$A_t = \alpha(W_t - C_t)R + \alpha\theta_t a^T \psi_t + \frac{1}{2}\psi_t^T \Lambda_\psi^T V \Lambda_\psi \psi_t$$

$e_t^T = [\alpha\theta_t b^T + \psi_t^T \Lambda_0^T]$ (this is a 3 vector) where $\Lambda_0^T = \begin{pmatrix} \hat{v}_0^{\,T} \\ \Lambda^T V_{11} \end{pmatrix}$ (3x4) matrix, $\Lambda_0 = (v_0, V_{11}\Lambda)$

Step 2: The Bellman Equation. It is well known (see, for example, the Appendix of Wang (1994)) that the Bellman Equation for this problem with $\gamma = \frac{1}{\tau}$ is written in the form

$$J_t = \underset{(\theta_t, C_t)}{\text{Max}}\,[-\beta^{t-1}\exp\{-\gamma C_t\} - \beta^t E_t \exp\left\{-A_t - e_t^T \varepsilon_{t+1} - \frac{1}{2}\varepsilon_{t+1}^T V_{11} \varepsilon_{t+1}\right\} \quad \text{for some parameter matrix V}$$

But we know that

$$E_t \exp\left\{-A_t - e_t^T \varepsilon_{t+1} - \frac{1}{2}\varepsilon_{t+1}^T V_{11} \varepsilon_{t+1}\right\} = |1 + \Sigma V_{11}|^{-\frac{1}{2}} \exp[\frac{1}{2} e_t^T (I + \Sigma V_{11})^{-1} \Sigma e_t - A_t].$$

Also

$$\frac{1}{2} e_t^T (1 + \Sigma V)^{-1} \Sigma e_t = \frac{1}{2}[\alpha\theta_t b^T + \psi_t^T \Lambda_0^T]^T (I + \Sigma V_{11})^{-1} \Sigma [\alpha\theta_t b + \Lambda_0 \psi_t]$$

$$= \frac{1}{2}\alpha^2 \theta_t^2 b^T \Omega b + \alpha\theta_t b^T \Omega \Lambda_0 \psi_t + \frac{1}{2}\psi_t^T \Lambda_0^T \Omega \Lambda_0 \psi_t \quad \text{, where } \Omega = (I + \Sigma V_{11})^{-1}\Sigma.$$

Hence, we have an expression for the expectations

$$\frac{1}{2} e_t^T (1 + \Sigma V_{11})^{-1} \Sigma e_t - A_t = -\alpha(W_t - C_t)R - \alpha\theta_t[a^T - b^T \Omega \Lambda_0]\psi_t + \frac{1}{2}\alpha^2 \theta_t^2 b^T \Omega b - \frac{1}{2}\psi_t^T [\Lambda_\psi^T V \Lambda_\psi - \Lambda_0^T \Omega \Lambda_0]\psi_t.$$

The first order conditions are then stated as follows. Equating the derivative with respect to θ to zero leads to

$$-\alpha[a^T - b^T\Omega\Lambda_0]\psi_t + \alpha^2\theta_t b^T\Omega b = 0$$

And this proves equation (11) in the text which we can write in the more explicit form (since $E_t[Q_{t+1}] = a^T\psi_t$)

$$\theta_t = \frac{1}{\alpha b^T\Omega b}\left\{[a^T - b^T\Omega\Lambda_0]\psi_t\right\} \equiv \frac{1}{\alpha b^T\Omega b}\left\{[E_t(Q_{t+1}) + u^T\psi_t\right\} \quad , \quad u^T = -b^T\Omega\Lambda_0 .$$

This last equation determines the parameter vector u. It also shows that this vector is the same for all agents since the assumption made in the text is that all agents are identically the same except for their belief states g_t. The last equation shows that the vector u depends only upon parameters of the stochastic structure.

Step 3: The Adjusted Variance and Constants. We can also explain the "adjustment" to the variance in (11) since

$$\hat{\sigma}_Q^2 = b^T\Omega b$$

which is the variance of the excess return function where the covariance matrix used is not Σ but rather Ω.
We now have

$$\alpha^2\theta_t^2 b^T\Omega b = \frac{1}{b^T\Omega b}\left\{\psi_t^T[a^T - b^T\Omega\Lambda_0]^T[a^T - b^T\Omega\Lambda_0]\psi_t\right\}.$$

Hence the optimized value of the exponent is simply

$$\frac{1}{2}e_t^T(1+\Sigma V_{11})^{-1}\Sigma e_t - A_t = -\alpha(W_t - C_t)R - \frac{1}{2}\psi_t^T M\psi_t$$

Where

$$M = \frac{1}{b^T\Omega b}[a^T - b^T\Omega\Lambda_0]^T[a^T - b^T\Omega\Lambda_0] + [\Lambda_\psi^T V\Lambda_\psi - \Lambda_0^T\Omega\Lambda_0].$$

Now take the derivative with respect to C and equate to zero to obtain

$$\gamma\exp\{-\gamma C_t\} = \alpha R\beta|1+\Sigma V_{11}|^{-\frac{1}{2}}\exp\left\{-\alpha(W_t - C_t)R - \frac{1}{2}\psi_t^T M\psi_t\right\} \quad , \quad \text{let} \quad G = |1+\Sigma V_{11}|^{-\frac{1}{2}}.$$

Hence the solution for C must satisfy (with *log* being the logarithm to base e)

$$\gamma C_t = -\log[\frac{\beta\alpha RG}{\gamma}] + \alpha(W_t - C_t)R + \frac{1}{2}\psi_t^T M\psi_t$$

hence we finally have

$$C_t = -\frac{1}{\gamma + \alpha R}\log[\frac{\beta\alpha RG}{\gamma}] + \frac{\alpha R}{\gamma + \alpha R}W_t + \frac{1}{2(\gamma + \alpha R)}\psi_t^T M\psi_t.$$

The final details of showing that the value function is indeed the solution of the Bellman Equation requires the insertion of the optimal solutions into the Bellman Equation and deducing the unknown parameters. Doing this leads to the following conclusion. First define the term $\hat{G} = \frac{1}{\gamma R}\log(r\beta G)$. Then it is demonstration that the unknown parameter α and matrix V are determined by the conditions

(i) $\alpha = \frac{r\gamma}{R}$.

(ii) $\frac{M}{R} - V + 2[\gamma\hat{G} + \log(\frac{r}{R})]i_{11} = 0$

where i_{11} is a 4×4 matrix with the (1,1) element being 1 and all others being 0.

APPENDIX B: Proof of Theorem 1

Proof : Using Assumption A we combine the two sources to have that

$$b_t(d_t, \Psi_t^i) = \mu b_{t-1}(d_t) + (1-\mu)B_t^i$$

with a mean of $E_t^i(b_t|d_t,\Psi_t^i) = \mu E_t^i(b_{t-1}|d_t) + (1-\mu)\Psi_t^i \qquad 0<\mu<1$

and conditional variance $Var(b_t|d_t,\Psi_t^i) = \frac{\mu^2}{(\alpha+\beta)} + \frac{(1-\mu)^2}{\gamma}$.

Let $\zeta = \frac{1}{\mu^2}$ and $\xi = \frac{1}{(1-\mu)^2}$ and we write the precision of the distribution of this new posterior as

$$\text{Precision}(b_t|d_t,\Psi_t^i) = \Gamma(b_t) = 1/[\frac{1}{\zeta(\alpha+\beta)} + \frac{1}{\xi\gamma}] = \frac{\zeta(\alpha+\beta)\xi\gamma}{\zeta(\alpha+\beta)+\xi\gamma}$$

At date t+1 the agent observes d_{t+1}. By (7) in the text it follows that updating $E_t^i(b_t|d_t,\Psi_t^i)$ the agent has

$$E_{t+1}^i(b_t|d_{t+1},\Psi_t^i) = \frac{\Gamma(b_t)E_t^i(b_t|d_t,\Psi_t^i) + \beta[d_{t+1} - \lambda_d d_t]}{\Gamma(b_t)+\beta} \quad , \quad b_t(d_{t+1},\Psi_t^i) \sim N[E_t^i(b_t|d_{t+1},\Psi_t^i), \frac{1}{\Gamma(b_t)+\beta}].$$

After assessing the mean Ψ_{t+1}^i he formulates the new posterior which is

$$b_{t+1}(d_{t+1},\Psi_{t+1}^i) = \mu b_t(d_{t+1},\Psi_t^i) + (1-\mu)B_{t+1}^i$$

with mean

(B.1) $$E_{t+1}^i(b_{t+1}|d_{t+1},\Psi_{t+1}^i) = \mu E_{t+1}^i(b_t|d_{t+1},\Psi_t^i) + (1-\mu)\Psi_{t+1}^i \qquad 0<\mu<1.$$

conditional variance

(B.2) $$Var(b_{t+1}|d_{t+1},\Psi_{t+1}^i) = \frac{1}{\zeta(\Gamma(b_t)+\beta)} + \frac{1}{\xi\gamma}.$$

and precision

(B.3) $$\Gamma(b_{t+1}) = 1/[\frac{1}{\zeta(\Gamma(b_t)+\beta)} + \frac{1}{\xi\gamma}] = \frac{\zeta(\Gamma(b_t)+\beta)\xi\gamma}{\zeta(\Gamma(b_t)+\beta)+\xi\gamma}.$$

We can now deduce the full symmetry of the process. For large t we then have

$$b_t(d_{t+1},\Psi_t^i) \sim N[E_t^i(b_t|d_{t+1},\Psi_t^i), \frac{1}{\Gamma(b_t)+\beta}]$$

$$E_t^i(b_t|d_{t+1},\Psi_t^i) = \frac{\Gamma(b_t)E_t^i(b_t|d_t,\Psi_t^i) + \beta[d_{t+1} - \lambda_d d_t]}{\Gamma(b_t)+\beta}.$$

After observing Ψ_{t+1}^i the new posterior is

$$b_{t+1}(d_{t+1},\Psi_{t+1}^i) = \mu b_t(d_{t+1},\Psi_t^i) + (1-\mu)B_{t+1}^i.$$

The mean, conditional variance and precision are then as in (B1), (B.2) and (B.3) and hence we have an equation for the precision

$$\Gamma_{t+1} = \frac{\zeta(\Gamma_t+\beta)\xi\gamma}{\zeta(\Gamma_t+\beta)+\xi\gamma} .$$

It is well defined for $1<\zeta<\infty$ (i.e. $0<\mu<1$) and in that case it has the unique positive solution

$$\zeta = \frac{1}{\mu^2} \text{ and } \xi = \frac{1}{(1-\mu)^2} \quad , \quad \Gamma^* = \frac{(\beta+\xi\gamma(1-\frac{1}{\zeta})) + \sqrt{(\beta+\xi\gamma(1-\frac{1}{\zeta}))^2 + 4\beta}}{2}.$$

The negative root has no economic meaning. When $\zeta = 1$, $\xi = \infty$ there is no solution, and Γ_t diverges for large t, which is the classical case. With $\Gamma = \Gamma^*$ we rewrite the equations above as

$$E_{t+1}^i(b_{t+1}|d_{t+1},\Psi_{t+1}^i) = \mu E_{t+1}^i(b_t|d_{t+1},\Psi_t) + (1-\mu)\Psi_{t+1}^i$$

$$= \mu \frac{\Gamma^* E_t^i(b_t | d_t, \Psi_t^i) + \beta[d_{t+1} - \lambda_d d_t]}{\Gamma^* + \beta} + (1-\mu)\Psi_{t+1}^i$$

Hence,

(B.4) $$E_{t+1}^i(b_{t+1} | d_{t+1}, \Psi_{t+1}^i) - \frac{\mu\beta}{\Gamma^*+\beta} d_{t+1} = \frac{\mu\Gamma^*}{\Gamma^*+\beta}[E_t^i(b_t | d_t, \Psi_t^i) - \frac{\mu\beta}{\Gamma^*+\beta} d_t] + \frac{\mu\beta}{\Gamma^*+\beta}[\frac{\mu\Gamma^*}{\Gamma^*+\beta} - \lambda_d] d_t + (1-\mu)\Psi_{t+1}^i.$$

If we now define

(B.5) $$g_{t+1}^i = E_{t+1}^i(b_{t+1} | d_{t+1}, \Psi_{t+1}^i) - \frac{\mu\beta}{\Gamma^*+\beta} d_{t+1} \;,\; \varrho_{t+1}^{ig} = (1-\mu)\Psi_{t+1}^i \;,\; \lambda_Z = \frac{\mu\Gamma^*}{\Gamma^*+\beta} \;,\; \lambda_Z^d = \frac{\mu\beta}{\Gamma^*+\beta}[\frac{\mu\Gamma^*}{\Gamma^*+\beta} - \lambda_d],$$

then (B.4) is exactly (6b): $$g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^d d_t + \varrho_{t+1}^{ig}$$ ■

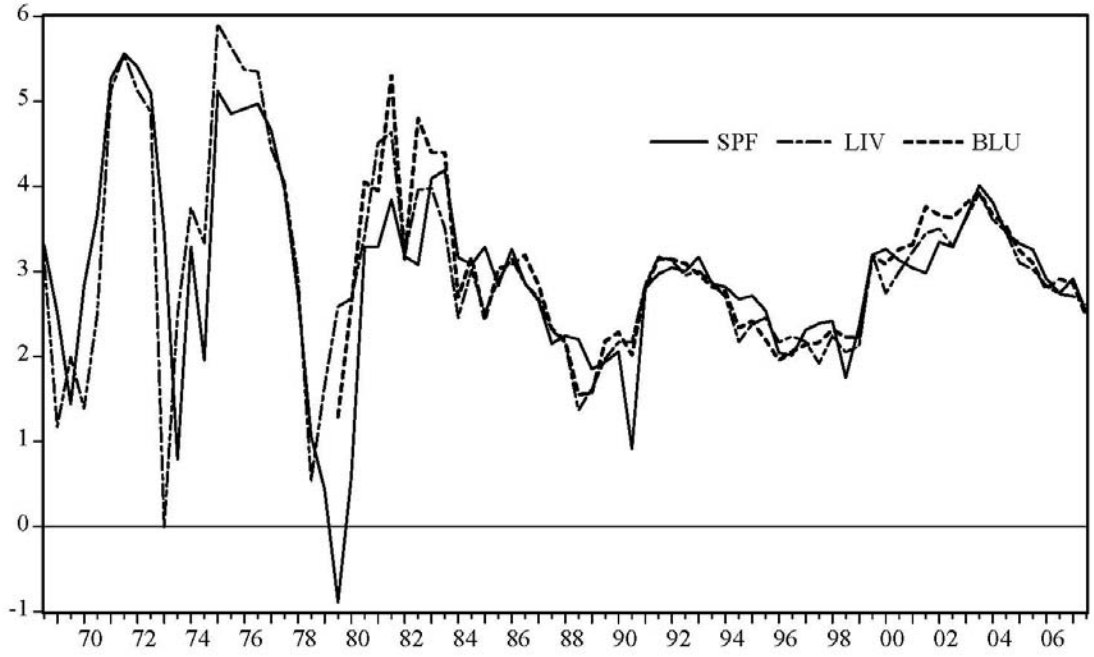

Figure 1: the 12-month-ahead forecasts from LIV, SPF and BLU

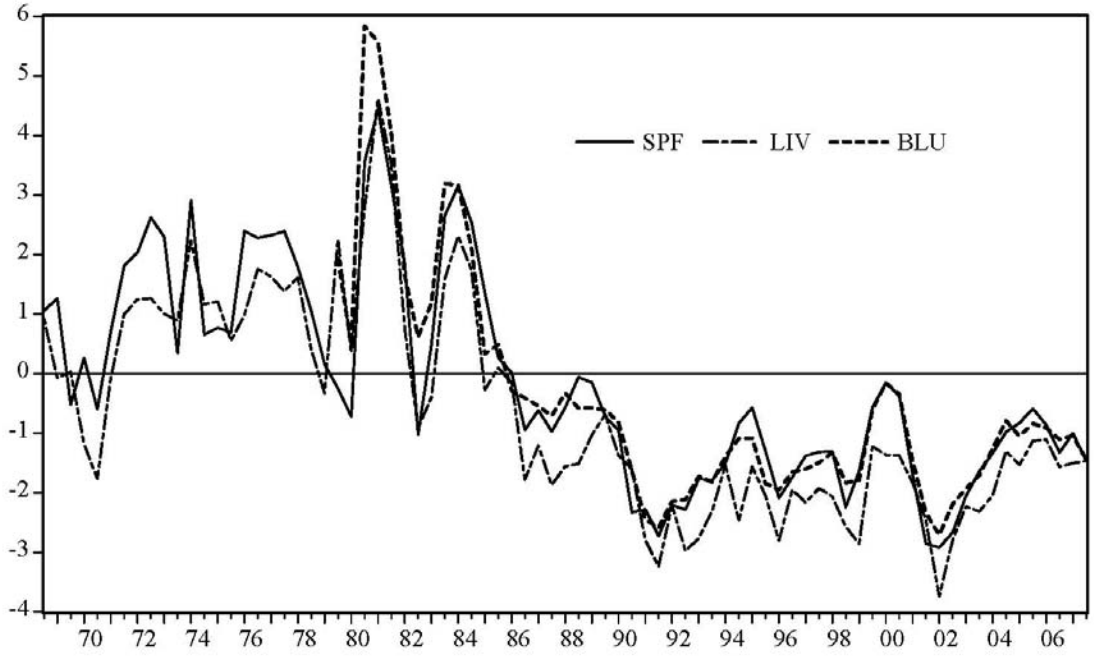

Figure 2: the 12-month-ahead Market States of Belief from LIV, SPF and BLU

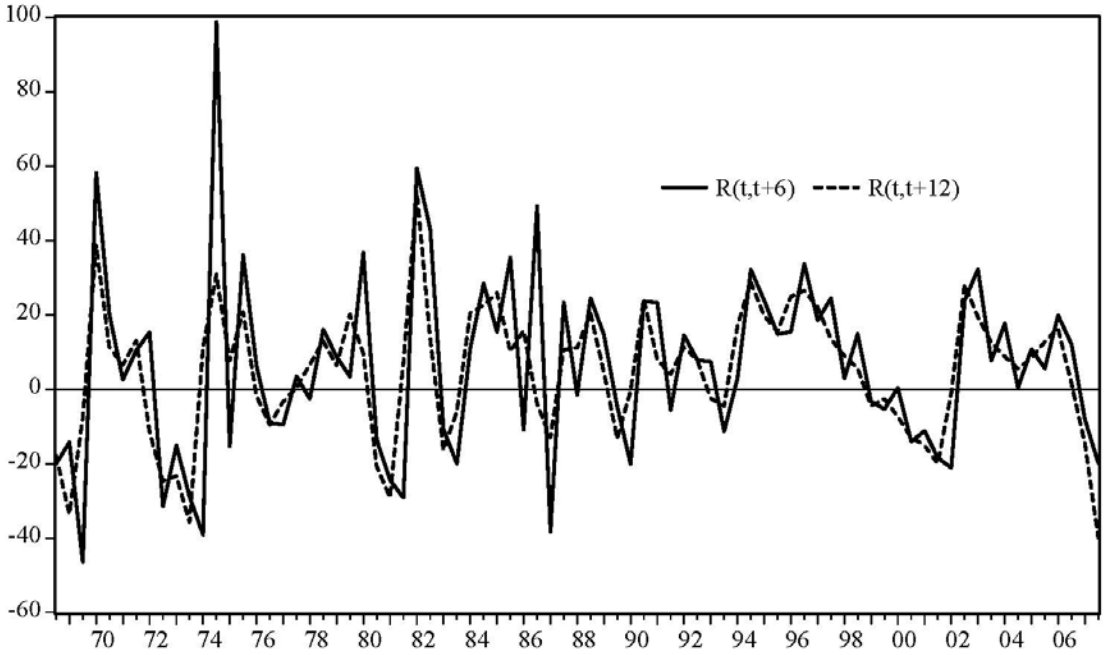

Figure 3: The 6-month and 12-month Stock Market Excess Returns