Extracting Information from Trading Volume *

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Abstract

This paper shows how to extract information from equilibrium trading volume. The analysis is first carried out in a market-clearing framework with symmetrically (and later normally) distributed demands, and is then extended to include market-making models. The conclusions of this paper hence apply to the majority of the models developed in the noisy rational expectations literature. If a random variable is symmetrically distributed with the traders' demands around zero and the asset market clears, the volume-based conditional distribution of this variable is symmetric, and consequently its conditional expectation based on volume is zero. The random variable under consideration may be the true value of the asset, the price or, in a dynamic model, the price difference. The paper further proves that the covariance between the absolute value of any variable jointly normally distributed with the traders' demands and the equilibrium volume is positive, which agrees with the empirical evidence of a positive covariance between the absolute value of the price difference and volume. Furthermore, numerical examples indicate that, when the asset value is jointly normally distributed with the traders' demands, the probability of extreme realizations for the asset true value conditioned on volume is increasing in volume. The paper's proposition hold in market-making frameworks as long as the price, the asset value and the traders' demands are symmetrically (resp. jointly normally) distributed. Finally, the paper develops a simple static model where transaction costs can induce a positive covariance between price and volume.

The objective of this paper is to study how to extract information from equilibrium trading volume (defined here as the sum of buy orders) under very general assumptions, so that the propositions developed in the paper can apply to a broad class of models. The only assumptions are that the random variable about which one wants to extract information and the traders' demands for the asset be symmetrically distributed around zero (and later that this variable be jointly normally distributed with the traders' demands), and that the market clears. The results can be extended to most market-making environments, provided symmetry is preserved. The traders' utility functions, their motives to trade and their rationality can be left unexamined. The statistical properties described in the paper hold for any random variable symmetrically (and then jointly normally) distributed with the traders' demands. They hold in a static framework as well as in a dynamic one. The random variable in question can be the value of the asset, its price, or in a dynamic model, the price difference.

After a brief review of the recent empirical literature on the price-volume relation, [1] the major propositions are introduced. It is then shown that the results developed in market-clearing situations can be extended to most market-making frameworks, as long as the price, the asset value and the traders' demands are symmetrically (resp. jointly normally) distributed. Finally, a simple static model is introduced where transaction costs (taxes) can induce a positive covariance between price and volume. Proposition (1) shows that, if a mean-zero random variable is symmetrically distributed with the traders' demands around zero, and the asset market clears, the conditional distribution of this variable conditioned on volume is symmetric, and consequently its conditional expectation based on volume is zero. This result is quite intuitive. For example, as volume is the sum of the buy orders, a high volume indicates a large flow of *executed* buy orders. But as the market clears, this strong influx of executed buy orders corresponds necessarily to a equally strong flow of executed sell orders.

Proposition (4) shows that, if a mean-zero random variable is jointly normally distributed with

[1] For an extensive review of the literature before 1987, see Karpoff (1987).

the traders' demands and the asset market clears, the covariance between the absolute value of this variable and trading volume is positive, unless this random variable is independent of the traders' demands, in which case the covariance is zero. Taken together, propositions (1) and (4) show that a surge in volume could be associated with a large change in the asset value, without indicating the direction of the change. Proposition (4) is quite general; any dynamic model where the price change is jointly normally distributed with the traders' demands without being independent of them should yield a positive correlation between the absolute value of price change and trading volume. Models of trading volume in a market-clearing setup typically use normally distributed random variables. For those, propositions (1), and (4) apply, independently of the particular assumptions of each model. Furthermore, numerical examples based on normally distributed demands and asset values indicate that the volume-based conditional distribution of the true asset value has fatter tails than the unconditional distribution when volume is high, and thinner tails when volume is low, a finding that may help fine-tune risk management systems.

Most of the models in the noisy rational expectation literature, either in a market-clearing framework (as in Diamond and Verrechia (1981)), or in a market making (Kyle type frameworks), as in Wang (1992), (1994), Foster and Viswanathan (1994), He and Wang (1995), Michaely, Vila and Wang (1996), use normally distributed random variables. For them, propositions (1) and (4) in the paper hold. Alternatively, Blume et al (1994) construct a model where the precision of the signals observed by the traders are stochastic (hence the signals are not normally distributed). Simulations based on their model show a positive relation between volume and the absolute value of price change, and a symmetric relation between price change per se and volume (i.e., a large volume is associated with a negative or a positive movement in the asset price). The V-shaped form of the price-volume relation confirms the positive correlation between the magnitude of the price change and trading volume. This seems to indicate that properties similar to those shown in proposition (4) may still be valid for non-normal distributions. Furthermore, the symmetry argument behind proposition (1) also applies to some non-normal distribution, like the elliptically contoured distributions used by Foster and Viswanathan (1993).

1 Review of the recent empirical literature

In his extensive review of the literature, Karpoff (1987) points out that many empirical studies have found a positive correlation between the absolute value of price change and trading volume. Furthermore, Karpoff reports that price change *per se* and volume are found to be positively correlated in equity markets but not in futures markets, a pattern he attributes to the existence of short sale constraints in stock and bond markets. Recall that proposition (1), which implies a zero covariance between volume and price change per se, rests on the assumption that the traders' demands are symmetrically distributed. Short sale constraints would introduce asymmetries in the demands and could hence create a positive correlation between price change and volume, despite market clearing. Since 1987, many economists have studied the relation between trading volume and the magnitude of the price change, and between volume and price change per se, on stock data and futures data. These empirical studies tend to confirm Karpoff's conclusions. Let's review first the studies based on stock data, and turn later to studies based on futures.

Using hourly New York Stock Exchange data between 1979 and 1983, Jain and Joh (1988) find a significant positive relation between daily trading volume and the absolute value of the Standard & Poor 500 index returns. The relation between volume and returns is stronger for positive returns than for negative returns, and the difference between the two is statistically significant. Gallant et al (1992) use daily New York Stock Exchange data between 1928 and 1987 to compute an estimate of the joint density of current price change and volume conditional on past price changes and volume. They find that "the direction of the daily change in the stock market is unrelated to contemporaneous volume," (p. 223) and that price change are more volatile when volume is heavier. Before computing the conditional density, they plot price changes against adjusted volumes and point that unusually high volumes are associated with large price changes. Note that Gallant et al, using very long sample and sophisticated econometric methods found no relation between price change per se and trading volume, even though they study an equity market. Using daily Helsinki Stock Exchange data between 1977 and 1988, Martikainen et al (1994) find some positive and statistically significant cross-correlations between contemporaneous values of stock returns and volume, and between contemporaneous values of one and lagged values of the other. Assogbavi et al. (1995) study daily prices and trading volume for the Toronto Stock Exchange TOR35 composite index, the futures contracts on it, and 33 out of the 35 stocks making out the index, between May 1987 and November 1988 (with 1987 crash related observations deleted). A positive relation between price change per se and volume is found in 24 of the 33 stocks, but is statistically significant only for 10 stocks, while 20 stocks have a non-significant (positive or negative) relation. Their contribution is to show that "the greater the ratio of short positions to the number of shares outstanding, the greater the trading volume associated with price increases relative to that accompanying price decreases." Goodman (1996) uses a random sample of 50 stocks traded on the New York and the American stock exchanges between 1993 and 1994. His findings confirms that the absolute value of the price change is positively correlated with trading volume, and shows that strong volume is associated with extreme price movements, both positive and negative.

As for futures markets, Karpoff uses daily data on futures contracts for 9 commodities and 3 financial instruments (also called commodities thereafter) between January 1972 to December 1979. Out of the 442 individual contracts (contracts with different maturities are traded for each commodity), 353 (80 percent) display a positive relation between the absolute value of the price change and volume; this relation is statistically significant for 224 contracts (51 percent). About half of the contracts show a positive relation between price change per se and volume; this relation is statistically significant for only 25 contracts (6 percent). The analysis is repeated on 12 time series (one for each commodity) constructed from the futures contracts data. The relation between the magnitude of the price change and volume is positive for all the commodities, and statistically significant for 9 of them. None of the commodities shows a significant relation between price change per se and volume. Foster (1995) uses daily data on two oil futures contracts between January 1990 and June 1994, and one oil futures contract between January 1984 and June 1988. As in Gallant et al (1992), volume data are detrended and expressed in logarithms, and are first grouped in several classes per size. The relative price change is then plotted against the volume classes. The magnitude of the price change is an increasing function of volume, (except at the upper end of the volume spectrum for the WTI contract between 1984 and 1988) but the direction of the change is not related to trading volume. This conclusion still holds when actual volume data are used instead of volume classes.

2 Trading volume-based conditional distribution

Let y_i be trader's i demand for the asset $(i = 1, \ldots, n)$ and z be the aggregate trading volume, that is, the sum of the buy orders, $z = \Sigma_{i=1}^{n} y_i^+$, with $y_i^+ = y_i\, I[y_i > 0]$ where $I[]$ is indicator function. The asset market clears, hence $\Sigma_{i=1}^{n} y_i = 0$. Let $y = (y_i)_{i=1}^{n}$, y is assumed to have mean zero. The objective is to compute the conditional distribution and mean, conditioned on the trading volume, of random variables correlated with the traders' demands, such as the true value of the asset.

Proposition 1 *Let y be a n-dimensional vector and u a random variable so that (u, y) is symmetrically distributed around zero and $\Sigma_{i=1}^{n} y_i = 0$. Let $z = \Sigma_{i=1}^{n} y_i^+$, then the conditional distribution of u given z is symmetric around zero , i.e., $p(u < t||z) = p(-u < t||z)$, and $E[u||z] = 0$.*

The proof of proposition (1), done for three traders, is in the appendix . Although $E[u||z] = 0$ is a consequence of the symmetry of the conditional distribution, for expositional ease, we begin to deal with the conditional expectation. Proving the symmetry of the conditional distribution uses a lot of the same features with slightly heavier notations. The result in proposition (1) is quite intuitive. For example, as z is the sum of the buy orders, a high volume indicates a large flow of *executed* buy orders. But as the market clears, this strong influx of executed buy orders corresponds necessarily to a equally strong flow of executed sell orders. Because of the symmetry of the mean-zero vector constituted by u and the individual demands, one can not infer the value of u based on the aggregate volume, i.e., $E[u||z] = 0$. Proposition (1) implies that u and z are uncorrelated, for any random variable u symmetrically distributed with y. Propositions (2), (3), and lemmas (1) and (2) serve to introduce proposition (1). To see more easily the symmetric nature of aggregate volume in a market clearing model, consider the case of three traders. Let $z = y_1^+ + y_2^+ + y_3^+$ with $y_1 + y_2 + y_3 = 0$. Hence

$z = y_1^+ + y_2^+ - (y_1+y_2)^-$ and z depends on the signs of y_1, y_2 and y_1+y_2. Table (1) gives the definition of z in terms of y_1, y_2, and y_3 in the six different cases. $z = \Sigma_{i=1}^6 I[v_1^i > 0, v_2^i > 0](v_1^i + v_2^i)$, with $(v_1^1, v_2^1) = (y_1, y_2)$, $(v_1^2, v_2^2) = (-y_2, y_1 + y_2)$, $(v_1^3, v_2^3) = (y_1, -(y_1 + y_2))$, $(v_1^4, v_2^4) = (-y_1, y_1 + y_2)$, $(v_1^5, v_2^5) = (y_2, -(y_1 + y_2))$, $(v_1^6, v_2^6) = (-y_1, -y_2)$. From table (1), we get $(v_1^4, v_2^4) = -(v_1^3, v_2^3)$, $(v_1^5, v_2^5) = -(v_1^2, v_2^2)$, and $(v_1^6, v_2^6) = -(v_1^1, v_2^1)$. The symmetry between cases 1 and 6, 2 and 5, and 3 and 4 plays a key role in propositions (1) and (4). Let u be a zero mean random variable so that (u, y) is symmetrically distributed around zero. Before showing that $E[u||z] = 0$, we show that $cov(u, z) = 0$ because it is a much easier proof and uses the same basic market clearing-based intuition. Of course, $E[u||z] = 0$ implies $cov(u, z) = 0$. Table (1) shows that, for $i = 1, \ldots, 6$, $I[v_1^i > 0, v_2^i > 0]\, z] = I[v_1^i > 0, v_2^i > 0]\,(v_1^i + v_2^i)]$, and for $i = 1, \ldots, 3$,$(v_1^i, v_2^i) = -(v_1^{7-i}, v_2^{7-i})$. These identities and the symmetric distribution of the vector (u, y) imply that, for $i = 1, 2, 3$,

$$E[\,I[v_1^i > 0, v_2^i > 0]\,u\,z] = -\,E[\,I[v_1^{7-i} > 0, v_2^{7-i} > 0]\,u\,z] \tag{1}$$

We conclude that $E[u\,z] = 0$, which implies that $cov(u, z) = 0$ since $E[u] = 0$.

In the following, propositions (2) and (3) make it possible to compute the conditional expectation of a random variable conditioned on a sum of truncated random variable. The fact that $\Sigma_{i=1}^n y_i = 0$ is used in lemmas (1) and (3), and, together with propositions (2) and (3), will imply that $E[u||z] = 0$.

2.1 Conditioning on a sum of truncated random variables

Proposition 2 *Let (Ω, F, P) a probability space. Let x and z two random variables with z positive, y a k-dimensional random vector, and A a k-dimensional Borel set. Note $z_A = I[y \in A]\,z$, $z_{\bar{A}} = I[y \in \bar{A}]\,z$, and likewise for x. Let v be a random variable such that v coincides with z when $y \in A$, then*

$$E[x||z_A] = \begin{cases} \frac{E[x_A||v]}{p(y \in A||v)} & \text{if} \quad z_A > 0 \\ \frac{E[x_{\bar{A}}]}{p(y \in \bar{A})} & \text{if} \quad z_A = 0 \end{cases} \tag{2}$$

The proof is in the appendix.

Proposition 3 *Using the same notations as in proposition (2), we have;*

$$E[x||z] = p(y \in A||z)E[x|z_A, z_A > 0] + p(y \in \bar{A}||z)E[x|z_{\bar{A}}, z_{\bar{A}} > 0] \tag{3}$$

where $E[x|z_A, z_A > 0]$ is the restriction of $E[x||z_A]$ when $z_A > 0$, and likewise for $E[x|z_{\bar{A}}, z_{\bar{A}}]$ Furthermore, if A_i, $i = 1, \ldots, n$, form a partition of Ω, then;

$$E[x||z] = \Sigma_{i=1}^{n} p(y \in A_i||z)E[x|z_{A_i}, z_{A_i} > 0] \tag{4}$$

The proof is in the appendix. Proposition (2) can be used to compute the $E[x||z_{A_i}]$. Explicit formulas for the $p(y \in A_i||z)$ are not necessary to prove proposition (1). Later in the paper, studying the conditional distribution of a tail event conditioned on trading volume will call for explicit formulas for the $p(y \in A_i||z)$, which are given in lemma (4). We will make use of the following lemma

Lemma 1 *Let y be a n-dimensional random vector symmetrically distributed around zero, $\gamma \in R^n$, $v = \gamma' y$ and A a Borel set in R^n, then $P(y \in A||v)(t) = P(-y \in A||v)(-t)$ and $E[\,I[y \in A]\,y||v](t) = -E[\,I[-y \in A]\,y||v](-t)$, where t is a realization of the random variable v.*

Lemma (2) is taken from Billingsley's "Probability and measure", exercise 33.19.

Lemma 2 *Let $B(h, \varepsilon)$ be the open ball with center h and radius ε, then $p(y \in A||z) = lim_{\varepsilon \to 0} p(y \in A||z \in B(z(\omega), \varepsilon))$ In other words, $p(y \in A||z) = f(z(\omega))$, with $f(h) = lim_{\epsilon \to 0} \frac{p([y \in A] \cap [z \in (h-\epsilon, h+\epsilon)])}{p(z \in (h-\epsilon, h+\epsilon))}$.*

2.2 Using market clearing conditions

In table (1), call A_i the relevant set for $y = (y_1, y_2)$ in case i, for example $I[y \in A_1] = I[y_1 > 0, y_2 > 0]$, $I[y \in A_2] = I[y_1 > 0,\, y_2 < 0,\, y_1 + y_2 > 0]$, $\ldots$, $I[y \in A_6] = I[y_1 < 0, y_2 < 0]$, and let u be a random variable symmetrically distributed with y. Lemma (3) follows.

Lemma 3

$$
\begin{aligned}
p(y \in A_i || z) &= p(y \in A_{7-i} || z) \\
\frac{E[\,I[y \in A_i] u || v_1^i + v_2^i]}{P(y \in A_i || v_1^i + v_2^i)} &= -\frac{E[\,I[y \in A_{7-i}] u || v_1^{7-i} + v_2^{7-i}]}{P(y \in A_{7-i} || v_1^{7-i} + v_2^{7-i})}
\end{aligned} \tag{5}
$$

The proof is in the appendix. Lemma (3) together with propositions (2) and (3) proves that $E[u||z] = 0$, for any mean zero random variable u for which (u, y) is symmetrically distributed. To prove that the conditional distribution is symmetric, we note that $p(u < t||z) = E[\,I[u < t]\,||z]$ and proceed as above. If A_i, $i = 1, \ldots, n$, form a partition of Ω, then, with $z_{A_i} = I[\,y \in A_i]z$,

$$
E[\,I[u < t]\,||z] = \Sigma_{i=1}^n p(y \in A_i||z) E[\,I[u < t]\,|z_{A_i}, z_{A_i} > 0], \tag{6}
$$

with $E[\,I[u < t]\,|z_A, z_A > 0] = \frac{E[\,I[y \in A] I[u<t]\,||v]}{p(y \in A||v)}$, where the random variable v coincides with z_A when $y \in A$. Using the fact that $\frac{E[\,I[y \in A_i] I[u<t]\,||v]}{p(y \in A_i||v)} = \frac{E[\,I[y \in A_{7-i}] I[-u<t]\,||v]}{p(y \in A_{7-i}||v)}$, one concludes that $p(u < t||z) = p(-u < t||z)$.

3 Covariance between absolute value of a random variable and trading volume

In the preceding section, one needed only to assume that (u, y) was symmetrically distributed around a zero mean. With the added assumption that (u, y) is normally distributed, one obtains that the covariance between u and z is positive, except only when u and y are independent, in which case $cov(|u|, z) = 0$.

Proposition 4 *Assume y is a n-dimensional mean zero, normally distributed random vector, $\Sigma_{i=1}^n y_i = 0$, and u is a mean-zero random variable jointly normally distributed with y. Let $z = \Sigma_{i=1}^n y_i^+$, where $y_i^+ = I[y_i > 0]\,y_i$. Then, $cov(|u|, z) \geq 0$, and $cov(|u|, z) = 0$ if and only if u and y are independent.*

4 Volume-based conditional distribution for normal distributions

Assuming that (u, y) is normally distributed, we can derive formulas for the conditional distribution of u given z, and plot the conditional distribution as a function of the realized volume.

4.1 Computing the conditional distribution

Recall that, from equation (6) for any variable u, $p(u < t||z)$ is a weighted sum of $E[I[u < t]||z_{A_i}, z_{A_i} > 0]$ where the the A_i, $i = 1, \ldots, 6$, correspond to the cases presented in table (1). The weights are computed using lemma (4).

Lemma 4 *Let f^i be the density of the vector (v_1^i, v_2^i) defined in table (1); the conditional probability of y being in A_i knowing that $z = h$ is*

$$p(y \in A_i||z)(h) = \frac{\int_{t=0}^{h} f^i(t, h-t)dt}{\Sigma_{j=1}^{6} \int_{t=0}^{h} f^j(t, h-t)dt}$$

.

4.2 Qualitative characteristics of the conditional distribution

Using the formulas developed in the preceding section, we can plot the conditional distribution of the true asset value x given the volume z for various covariance structures. The upper panel of figure (1) displays the probability in the tail of the volume's distribution, $p(z > t)$, with t on the horizontal axis; the lower panel displays the volume-based conditional probability in the tails of the distribution $p(x > 1||z)$, with z on the horizontal axis, the dotted line in the lower panel corresponds to the unconditional probability $p(x > 1)$. As x is a standard normal, $p(x > 1)$ is the probability that the true asset value be one standard deviation higher than its mean. Figure (1) has been obtained using Dupont (1996) with two informed traders and one liquidity trader. The signals observed by the traders are independent and each has a correlation of .5 with the true value of the stock x. In this case, the mean volume is 0.678 and its standard deviation is 0.3717.

Different parameters values were tried without affecting the graph's overall form.

From figure (1), the volume-based conditional probability of 'large' realizations of the true asset value is increasing and convex in z. Figure (1) indicates that in sixty eight percent of the time, the conditional probability $p(x > 1||z)$ is inferior to the unconditional probability. [2] For the rare, high-volume events, the conditional probability surges rapidly above the unconditional probability. The volume mean exceeds its median, which reveals that the volume distribution is skewed by large volumes occurring with low probability. To sum, for low and "normal" levels of trading volume, the conditional probability of x given z is more concentrated around its mean than the unconditional one, whereas for uncommonly high levels of trading volume, the conditional distribution exhibits fat tails.

5 Market-making models

In the following section, propositions (1) and (4) are extended to market-clearing situations. Many financial economists use models where a market maker determines the price at which asset are exchanged. Trading volume is then defined as $V = \frac{1}{2}\Sigma_{i=1}^{n}|y_i| + \frac{1}{2}|\Sigma_{i=1}^{n} y_i|$, where y_i is the trader i's demand and n is the number of traders . The following shows that this is equivalent to the definition of volume used in this paper. By defining the market maker's demand as $y_{n+1} = -\Sigma_{i=1}^{n} y_i$, a market-making framework can always be recast into a market-clearing one. Using the definition of trading volume z introduced earlier in paper, $z = \Sigma_{i=1}^{n+1} y_i^+$, since $\Sigma_{i=1}^{n+1} y_i = 0$, $y_i = y_i^+ + y_i^-$ and $|y_i| = y_i^+ - y_i^-$, one gets $V = \frac{1}{2}\Sigma_{i=1}^{n+1}|y_i| = \Sigma_{i=1}^{n+1} y_i^+ = z$.

5.1 Glosten and Milgrom type models

In Glosten and Milgrom (1985), a competitive risk-neutral market maker facing better informed traders posts bid and ask prices. The equilibrium bid-ask spread reflects the informational asymmetry between the non-informed market maker and the traders. This structure was extended to a

[2] From the lower panel, for $z < .81$, $p(x > 1||z) < p(x > 1)$, from the upper panel $p(z > .81) = .32$.

monopolistic marker maker by Glosten (1989). In a comparable framework, Dupont (1996) introduces quantity limits, which are the maximum amounts the market maker is willing to exchange at the posted prices. In this model, the informed trader, who observes a signal correlated with the true value of the asset, buys (resp. sells) the asset if the quoted ask is below (resp. the bid is above) his valuation. The liquidity trader's demand is price-sensitive and stochastic. The informed trader is assumed to have a negative exponential utility function. Informed and liquidity traders' orders are lumped together and passed on to the specialist, who cannot distinguish informed from liquidity demand. Let a denote the ask price, z_a the ask quantity limit, b the bid price, z_b the bid quantity limit, x the true value of the traded asset, G the signal observed by the informed trader, $v = E[x|G]$ his valuation, $var(x|G)$ the conditional variance of x observing G, I the indicator function, $\tilde{d}(p)$ the liquidity trader's demand at price p, $\tilde{d}(p) = -p + \eta$, with η independent of x and G. The informed trader's demand $\tilde{q}(p)$ is $Min\{z_a, k(v-a)\}$ if $v > a$, $Max\{z_b, k(v-b)\}$ if $v < b$, and 0 if $b \leq v \leq a$, with $k = (\gamma\, var(x|G))^{-1}$. [3] To simplify notations, k is supposed to be equal to one. The vector (x, G, η) is normally distributed, its mean is zero.

Under these conditions, the optimal prices and quantity limits on the bid and ask sides are such that $b = -a$, $z_b = -z_a$. Define the transaction price when both bid and ask sides are hit by traders (see table (2)) as the average of the bid and ask prices, or a random draw between the two with probability 1/2. Define the market maker's demand as $\tilde{m} = -(\tilde{q} + \tilde{d})$. Let z be the trading volume. As $b = -a$ and v is symmetric around zero, looking at the values of $p(\tilde{q} < p)$ and $p(\tilde{q} > -p)$ shown for different p in table (3), and using the symmetry between bid and ask prices and quantity limits, one sees that $\tilde{q}$ is symmetrically distributed around zero. So is the liquidity demand $\tilde{d}$. The vector $(x, p, \tilde{q}, \tilde{d}, \tilde{m})$ is symmetrically distributed around zero. Hence, proposition

[3] Indeed, as shown below, the informed trader is always better off choosing $y > 0$ when $v > a$, $y < 0$ when $v < b$ and $y = 0$ when $b \leq v \leq a$. Let π the informed trader's profits. $\pi = I[y > 0]\, y\,(x-a) + I[y < 0]\, y\,(x-b) = \pi_1 + \pi_2$, with $\pi_1 = I[y > 0]\, I[v > a]\, y\,(x-a) + I[y < 0]\, I[v < b]\, y\,(x-b)$ and $\pi_2 = I[y > 0]\, I[v < a]\, y\,(x-a) + I[y < 0]\, I[v > b]\, y\,(x-b)$. $\pi_2 = \pi_{2,1} + \pi_{2,2}$, with $\pi_{2,1} = I[y > 0]\, I[v < a]\, y\,(v-a) + I[y < 0]\, I[v > b]\, y\,(v-b)$ and $\pi_{2,2} = I[y > 0]\, I[v < a]\, y\,(x-v) + I[y < 0]\, I[v > b]\, y\,(x-v)$. The informed trader chooses y to maximize $E[-\exp(-\gamma\,\pi)|G]$. Since $\pi_{2,1}$ is measurable with respect to G, $E[-\exp(-\gamma\,\pi)] = \exp(-\gamma\,\pi_{2,1})\, E[-\exp(-\gamma(\pi_1 + \pi_{2,2}))]$. As $\pi_{2,1} \leq 0$, $E[-\exp(-\gamma\,\pi)] \leq E[-\exp(-\gamma(\pi_1 + \pi_{2,2}))]$, and $E[-\exp(-\gamma\,\pi)] = E[-\exp(-\gamma(\pi_1 + \pi_{2,2}))]$ if $y \geq 0$ when $v > a$ and $y \leq 0$ when $v < b$. In that case $\pi_{2,2} = 0$, and π can be replaced by $I[v > a]\, y\,(x-a) + I[v < b]\, y\,(x-b)$ in the maximization problem.

(1) applies and $E[x|z] = cov(x, z) = cov(p, z) = 0$. Table (3) shows the values of $p(\tilde{q} < p)$ and $p(\tilde{q} > -p)$ for different p. Naturally, proposition (1) will apply to any market-making model which preserves the symmetry in the transaction price and the traders' demands.

The results of proposition (1), extended to market-making models above, seem to contrast with the models of Easley, Kiefer, and O'Hara (1996) and Easley, Kiefer, O'Hara and Paperman (1996), where the specialist uses the order flow to draw inference about the possibility of informed trading. In their model, the order flow is informative because the specialist takes into account the side on which he is hit. A large number of incoming orders on the ask side (resp. the bid side) indicates that the informed trader's valuation is superior to the ask (resp. inferior to the bid). But the volume per se is less informative since it could have been generated by informed traders' sales to the market maker or purchases from him. An outside observer can get more information in a Glosten-Milgrom market-making framework than in a pure market-clearing framework if he can separate sales to the specialist from purchases from him.

5.2 Kyle's type models

Kyle (1985) introduces a dynamic model of insider trading with a single risk-neutral informed trader, a noise trader and a competitive risk-neutral market maker. In contrast to the Glosten and Milgrom type models, there is no bid-ask spread and the price at which assets are exchanged is not posted but is determined after the traders submit their market orders. In equilibrium the market maker fixes the price to reflect the observed order flow, which aggregates the informed trader's and the liquidity demands (this could result from a unmodelled Bertrand game between multiple market makers). Writing $\tilde{v}$ the asset's liquidation value, observed only by the informed trader, $\tilde{u}$ the noise trading, $\tilde{x} = X(\tilde{v})$ the informed trader's demand, $\tilde{x} + \tilde{v}$ the order flow, the price $\tilde{p}$ is such that $\tilde{p} = P(\tilde{x} + \tilde{u}) = E[\tilde{v}|\tilde{x} + \tilde{u}]$. The vector $(\tilde{v}, \tilde{u})$ is normally distributed. Under these conditions, there is a unique linear Nash equilibrium where P and X are linear functions. The framework is applied to dynamic games with linear recursive equilibria, where the difference in the equilibrium price is a linear function of the difference in order flow. Kyle's framework, extended to multiple informed traders in Holden and Subrahmanyam (1992), has been adapted and used by many financial researchers. In these models, traders' demands, asset value and market prices are jointly normally distributed. The mean demands are zero, and the asset value mean (and hence the price mean) can be normalized to zero. Clearing the market by taking the market maker's net demand into account, propositions (1) and (4) apply. Consequently, with t representing the level or the first difference of the price or the value of the asset, $E[t|z] = cov(t, z) = 0$, $cov(|t|, z) > 0$ and $p(t > t|z)$ is increasing in z (the last result being based on numerical examples).

Foster and Viswanathan (1993) extend Kyle's framework by using distributions of the elliptically contoured class (ECC). Their framework includes multiple informed traders and some publicly available information. As they point out, ECC includes the normal distributions and many other interesting distributions, such as the multivariate t, the mixture of normals, and the multivariate double exponential. The linearity of decisions rules is preserved and the model can investigate the relations between price volatility and trading volume in a richer way. Demands, prices, and the asset value also follow a ECC joint distribution, and hence are symmetrically distributed. As a consequence, proposition (1) applies. Proposition (4) used the properties of the normal distribution and might not extent to all elliptically contoured distributions.

6 Model with asymmetric demands

The following section presents a simple, static model where the market clears and the covariance between price and volume can be positive. [4] Some models, like Eps (1975), rely on behavioral distinctions between different types of investors: optimistic traders (or "bulls") systematically ignore unfavorable information, pessimistic traders (or "bears") systematically ignore favorable information. These models deliver the positive volume-price change relation, but at the cost of imposing irrationality on the traders. Karpoff (1988) constructs a model where the demand and

[4] The relation between price change and volume cannot be investigated in this static model, which should be seen merely as a starting point.

supply curves are subject to parallel random shifts. The covariance between price change per se and trading volume has the same sign as the difference between the variance of the demand shift and the variance of the supply shift. Karpoff provides some justification for why costly short sales may result in the variance of the supply shift being smaller than the variance of the demand shift variance. However, one might prefer a more direct way of generating a positive volume-price change relation than a difference in the variance of the intercepts of the supply and demand curves.

The covariance between price and trading volume is studied in the model below, which features one noise trader, whose price-inelastic demand is ε, and an informed trader, who observes a signal G correlated with the liquidative value of the asset x. The vector (x, G, ε) is normally distributed, and $E[x] = \mu$, $E[G] = E[\varepsilon] = 0$. The informed trader must pay a proportional transaction tax whose rate may vary according to the direction of his trade and the sign of the price. [5] The equilibrium volume $|\varepsilon|$ is exogenous, and the equilibrium agents' demands ($+\varepsilon$ for the liquidity trader and $-\varepsilon$ for the informed trader) are symmetrically distributed around their zero means. However, the covariance between price and trading volume may be different from zero because of effect of taxes on the equilibrium price distribution.

The tax rate on purchases is τ_a when the price is positive and τ_d when it is negative; the tax rate on sales is τ_b when the price is positive and τ_c when it is negative. The tax rates τ_a, τ_b, τ_c, and τ_d are between zero and one. In each case, table (4) shows the after-tax prices the informed trader faces as a factor of the pre-tax prices. The informed trader has a negative exponential utility function with risk aversion coefficient γ, and submits a demand schedule as a function of the market clearing price p. Let $v = E[x|G]$ be the informed trader's valuation of the asset, $var(x|G) = 1 - \sigma^2$ the conditional variance of x given G, where $\sigma^2 = var(v)$, and write $k = \gamma\, var(x|G)$. To simplify computations, we assume that the informed trader considers the market price p as fixed although it is random. A rational agent should condition on p as well on G, but as the equilibrium price may not

[5] Given the distributional assumptions, the possibility of negative prices has to be considered. This is also the case in many microstructures models. Note that one would be willing to pay to sell the asset if one thinks that the asset price would slide further into the negative range.

be normally distributed, having the informed trader condition on it would make the optimization problem too difficult to solve. With this simplifying assumption, the informed trader's demand for the asset is

$$y = \begin{cases} k^{-1}\,(v-(1+\tau_a)\,p) & \text{if} \quad p>0 \quad \text{and} \quad v>(1+\tau_a)\,p \\ k^{-1}\,(v-(1-\tau_b)\,p) & \text{if} \quad p>0 \quad \text{and} \quad v<(1-\tau_b)\,p \\ 0 & \text{if} \quad p>0 \quad \text{and} \quad (1-\tau_b)\,p\le v\le(1+\tau_a)\,p \\ k^{-1}\,(v-(1-\tau_d)\,p) & \text{if} \quad p\le 0 \quad \text{and} \quad v>(1-\tau_d)\,p \\ k^{-1}\,(v-(1+\tau_c)\,p) & \text{if} \quad p\le 0 \quad \text{and} \quad v<(1+\tau_c)\,p \\ 0 & \text{if} \quad p\le 0 \quad \text{and} \quad (1+\tau_c)\,p\le v\le(1-\tau_d)\,p \end{cases} \tag{7}$$

The informed trader can buy (resp. sell) only if the liquidity demand ε is negative (resp. positive). Under these conditions, the market clearing price is $p = h(\varepsilon, v)[k\,\varepsilon\,+\,v]$, with

$$h(\varepsilon,v) = \begin{cases} (1+\tau_a)^{-1} & \text{if} \quad \varepsilon<0 \quad \text{and} \quad v>-k\,\varepsilon \\ (1-\tau_b)^{-1} & \text{if} \quad \varepsilon>0 \quad \text{and} \quad v>-k\,\varepsilon \\ (1-\tau_d)^{-1} & \text{if} \quad \varepsilon<0 \quad \text{and} \quad v\le-k\,\varepsilon \\ (1+\tau_c)^{-1} & \text{if} \quad \varepsilon>0 \quad \text{and} \quad v\le-k\,\varepsilon \\ 1 & \text{if} \quad \varepsilon=0 \end{cases} \tag{8}$$

In other words,

$$\begin{aligned} h(\varepsilon,v) \;&=\; I[\varepsilon<0]\,\{I[v+k\,\varepsilon>0](1+\tau_a)^{-1}+I[v+k\,\varepsilon\le 0](1-\tau_d)^{-1}\} \\ &+\; I[\varepsilon>0]\,\{I[v+k\,\varepsilon>0](1-\tau_b)^{-1}+I[v+k\,\varepsilon\le 0](1+\tau_c)^{-1}\}+I[\varepsilon=0] \end{aligned} \tag{9}$$

To compute $cov(p,z)$, recall that the trader's valuation v and the liquidity shock ε are normally and independently distributed, and $E[\varepsilon] = 0$, $E[v] = \mu$, $var(\varepsilon) = 1$, $var(v) = \sigma^2$. Let ϕ and Φ be the standard normal density and distribution functions. Let $m(\varepsilon, v) = ((1-\sigma^2)\,\varepsilon + v)\frac{1}{\sigma}\phi(\frac{v}{\sigma})$.

Then,

$$
\begin{aligned}
E[p\,\varepsilon^-] &= \int_{-\infty}^{0} \Big((1+\tau_a)^{-1}\,\varphi(\varepsilon) + (1-\tau_d)^{-1}\,\psi(\varepsilon)\Big)\,\varepsilon\,\phi(\varepsilon)\,d\varepsilon \\
E[p\,\varepsilon^+] &= \int_{0}^{+\infty} \Big((1-\tau_b)^{-1}\,\varphi(\varepsilon) + (1+\tau_c)^{-1}\,\psi(\varepsilon)\Big)\,\varepsilon\,\phi(\varepsilon)\,d\varepsilon
\end{aligned}
\tag{10}
$$

with

$$
\begin{aligned}
\varphi(\varepsilon) &= \int_{-(1-\sigma^2)\,\varepsilon}^{+\infty} m(\varepsilon,v)\,dv &&= ((1-\sigma^2)\,\varepsilon\,+\mu)\;\Phi(\tfrac{1-\sigma^2}{\sigma}\varepsilon+\tfrac{\mu}{\sigma}) + \sigma\,\phi(\tfrac{1-\sigma^2}{\sigma}\varepsilon+\tfrac{\mu}{\sigma}) \\
\psi(\varepsilon) &= \int_{-\infty}^{-(1-\sigma^2)\,\varepsilon} m(\varepsilon,v)\,dv &&= ((1-\sigma^2)\,\varepsilon\,+\mu)\;[1-\Phi(\tfrac{1-\sigma^2}{\sigma}\varepsilon+\tfrac{\mu}{\sigma})] - \sigma\,\phi(\tfrac{1-\sigma^2}{\sigma}\varepsilon+\tfrac{\mu}{\sigma})
\end{aligned}
\tag{11}
$$

Three cases are studied. In the first case, cash inflows are taxed at τ_a and cash outflows at τ_b. Hence, as selling at a negative price created an outflow, $\tau_c = \tau_a$; likewise, $\tau_d = \tau_b$ In the second case, no tax is levied when prices are negative: $\tau_c = \tau_d = 0$. In the third case, the informed trader's purchases are taxed at τ_a and his sales at τ_b whatever the sign of the price, i.e., $\tau_c = \tau_b$ and $\tau_d = \tau_a$. In each case, we study the covariance between price and trading volume when the mean asset value μ is zero and when it is positive. Except when indicated, the stated results are based on numerical examples and should be considered preliminary. When $\mu = 0$, the covariance between price and trading volume is zero in the first case [6], positive in the second, negative if $\tau_a > \tau_b$ and positive if $\tau_a < \tau_b$ in the third. When $\mu > 0$, the covariance is positive in the first two cases; in the third, it can be negative if $\tau_a > \tau_b$ and is positive if $\tau_a < \tau_b$. Note that in the second case, the covariance is always positive, whatever μ, τ_a and τ_b. In all cases, the covariance between price and trading volume is an increasing function of μ, so is the price.

When prices are positive, the intuition is immediate. When ε is positive, the equilibrium price must be such that the informed trader is willing to provide liquidity, but he sells at a discount, hence the price has to be even higher than it would have been in the absence of transaction tax. When ε is negative, the equilibrium price must be such that the informed trader is willing to absorb liquidity, but he buys at a premium, hence the price has to be even lower than it would have been in the absence of transaction tax. In this case, imposing a cost on purchases creates a

[6]This result is derived analytically.

positive covariance even without taxing the informed trader's sales. When prices are negative, the intuition is less obvious, especially as the results depend on whether the tax rate is based on the nature of the informed trader's transaction (sale or purchase) or the sign of the associated cash flow. Increasing the mean value of the asset lowers the probability of negative market prices and boosts the covariance between price and trading volume.

7 Conclusion and further research

The paper shows that the volume-based conditional expectation of any mean-zero random variable is zero provided that this random variable and the traders' demands are symmetrically distributed and the asset market clears. Furthermore, when the random variable and the traders' demands are jointly normally distributed, the covariance between the random variable's absolute value and volume is positive, and the tails of the volume-based conditional distribution seem to have fatter tails when volume is high. One of the paper's implications is that the covariance between the absolute value of price change and volume is positive (assuming normal distribution) whereas the covariance between price change per se and volume is zero. The first implication has verified in many empirical studies. As for the second, empirical studies have found a positive covariance between the price change per se and volume in equity markets but not in futures markets. Karpoff (1987) points out that this difference may be due to a key difference in the micro-structures of the two types of markets: short sale constraints. Different costs in taking long and short positions would introduce asymmetries in the demands, which could explain the positive covariance between price change and volume. A simple, static model was developed where the covariance between price and volume can be positive. Besides, the New York Stock Market (NYSE) is a complex mixed type where each market-maker (or specialist) acts alternatively as dealer and auctioneer. Moreover, specialists are required to maintain "a fair and orderly market in the securities assigned to them" [7], a responsibility that includes some form of market stabilization. A market where some

[7] NYSE Fact Book 1995, p. 5

agents are responsible for smoothing away short term imbalances may yield a non-zero covariance between volume and price change, even if those agents' trades represent only a small fraction of the activity. [8] Likewise, circuit breakers and other devices certainly affect the relation between price and volume. The results in this paper can be seen as a starting point from which further research taking into account the institutional features of each market can be developed.

[8] "The vast majority of NYSE volume is a result of public order meeting public orders" (NYSE Fact Book 1995).

Appendix

Before going in more details, it is useful to present the intuition on which the proves are built. Recall that for any given (x, y) random vector, $E[x] = p(y > 0)\, E[x|y > 0] + p(y \leq 0)\, E[x|y \leq 0]$, with $E[x|y > 0] = \frac{E[I[y>0]\,x]}{p(y>0)}$ and $E[x|y \leq 0] = \frac{E[I[y\leq 0]\,x]}{p(y\leq 0)}$, and

$$E[x|\, I[y > 0]\,] = \begin{cases} E[x|y > 0] & \text{if} \quad y > 0 \\ E[x|y \leq 0] & \text{if} \quad y \leq 0 \end{cases} \tag{12}$$

The paper basically extends this intuition to $E[x||z]$, where z is a positive random variable. In the remainder of the paper, call $(\Omega, \mathcal{F}, P)$ a probability space, and for any Borel set A and random variable y, write $[y \in A]$ the set $\{\omega \in \Omega, y(\omega) \in A\}$.

Proof of Proposition (2): To prove (2), we need only show that equations (13) and (14) hold.

$$E[x_A||z_{\bar{A}}] = \begin{cases} \frac{E[x_A]}{p(y\in A)} & \text{if} \quad z_{\bar{A}} = 0 \\ 0 & \text{if} \quad z_{\bar{A}} > 0 \end{cases} \tag{13}$$

$$E[x_A||z_A] = \begin{cases} \frac{E[x_A||v]}{p(y\in A||v)} & \text{if} \quad z_A > 0 \\ 0 & \text{if} \quad z_A = 0 \end{cases} \tag{14}$$

proof that equation (13) holds: Call $E^c[x_A||z_{\bar{A}}]$ the candidate conditional expectation defined by the right hand side of (13). Recall that, if G is a σ-field in F, $E[X||G]$ is a version of the conditional expectation of X given G if the properties (i) and (ii) are met.

(i) $E[X||G]$ is measurable with respect to G.

(ii) $\int_G E[X||G]\; dP = \int_G X\; dP.$

The candidate function is obviously measurable with respect to $\sigma(z_{\bar{A}})$. Note that this σ-algebra consists of the sets $[z_{\bar{A}} \in H]$, with H a Borel set. Let $G = [z_{\bar{A}} \in H]$ for a Borel set H. Recall that

$E^c[x_A||z_{\bar{A}}]$ is zero on $[y \in \bar{A}]$. Let's show that condition (ii) above is verified [9]. Two cases arise.

[1] $0 \in H$, then $[y \in A] \subset [z_{\bar{A}} \in H]$, hence $[z_{\bar{A}} \in H] \cap [y \in A] = [y \in A]$.

$$\int_G E^c[x_A||z_{\bar{A}}]\, dP = \int_{G\cap[y\in A]} E^c[x_A||z_{\bar{A}}]\, dP = \int_{[y\in A]} \frac{E[x_A]}{p(y \in A)}\, dP \tag{15}$$

The integrand in the right hand side is a constant, and $p(y \in A) = E[I(y \in A)]$, hence

$$\begin{aligned}
\int_G E^c[x_A||z_{\bar{A}}]\, dP &= E[x_A] \\
&= \int_{[y\in A]} x\, dP \\
&= \int_{[y\in A]\cap G} x\, dP \\
&= \int_G x\, I[y \in A]\, dP \\
&= \int_G x_A\, dP.
\end{aligned} \tag{16}$$

[2] H does not contain 0. For all the $\omega \in G$, $I[y \in \bar{A}] \neq 0$, i.e., $y(\omega) \in \bar{A}$. This means that $G \subset [y \in \bar{A}]$, hence $G \cap [y \in A]$ is empty.

$$\int_G E^c[x_A||z_{\bar{A}}]\, dP = 0. \tag{17}$$

On the other hand,

$$\int_G x_A\, dP = \int_{G\cap[y\in A]} x\, dP = 0 \tag{18}$$

Hence $\int_G E^c[x_A||z_{\bar{A}}]\, dP = \int_G x_A\, dP$.

proof that equation (14) holds: The candidate function is obviously measurable with respect to $\sigma(z_A)$. Recall that $z_A = I[y \in A]z$,and $z > 0$. Also, v and z coincide on $[y \in A]$. Note $G = [z_A \in H]$, for some Borelian H. Three cases arise.

[9] Note that $[z_{\bar{A}} = 0] = [z_A > 0] = [y \in A]$, and $[z_{\bar{A}} > 0] = [z_A = 0] = [y \in \bar{A}]$.

[1] H does not contain 0, then $[z_A \in H] = [v \in H] \cap [y \in A]$.

$$\begin{aligned}
\int_G E^c[x_A||z_A]\, dP &= \int_{[v\in H]\cap[y\in A]} \frac{1}{p(y \in A||v)} E[x_A||v]\, dP \\
&= E\{\ I[v \in H]\ I[y \in A]\ \tfrac{1}{p(y\in A||v)} E[x_A||v]\ \} \\
&= E\{E[\ I[v \in H]\ I[y \in A]\ \tfrac{1}{p(y\in A||v)} E[x_A||v]\ ||v]\ \} \\
&= E\{\ I[v \in H]\ E[\ I[y \in A]||v]\ \tfrac{1}{p(y\in A||v)} E[x_A||v]\ \} \\
&= E\{\ I[v \in H]\ E[x_A||v]\ \} \qquad (19) \\
&= E\{E[\ I[v \in H]\ x_A||v]\ \} \\
&= E\{\ I[v \in H]\ x_A\ \} \\
&= \int_{[v\in H]\cap[y\in A]} x_A\, dP \\
&= \int_G x_A\, dP.
\end{aligned}$$

[2] $H = 0$, then $[y \in \bar{A}] = [z_A \in H] = G$.

$$\int_G E^c[x_A||z_A]\, dP = \int_{[y\in\bar{A}]} E^c[x_A||z_A]\, dP = 0. \qquad (20)$$

On the other hand,

$$\int_G x_A\, dP = \int_{[y\in\bar{A}]} x_A\, dP = 0. \qquad (21)$$

Hence $\int_G E^c[x_A||z_A]\, dP = \int_G x_A\, dP$.

[3] H contains 0 but is not reduced to it. Let $H = H^+ \cup 0$, $G = z_A^{-1}(H) = z_A^{-1}(H^+) \cup z_A^{-1}(0) = [z_A \in H^+] \cup [y \in \bar{A}]$. Applying [1] and [2], we have $\int_G E^c[x_A||z_A]\, dP = \int_G x_A\, dP$.

Proof of Proposition (3): Let $I_A = I[y \in A]$, $\sigma(z, I_A)$ be the σ-field generated by the random vector (z, I_A), and $E[x||z, I_A]$ be the conditional expectation of x conditioned on $\sigma(z, I_A)$. The objective is to show that $E[x||z] = p(y \in A||z)\, E[x|z_A, z_A > 0] + p(y \in \bar{A}||z)\, E[x||z_{\bar{A}}, z_{\bar{A}} > 0]$, where $E[x|z_A, z_A > 0]$ is the restriction of $E[x||z_A]$ where $z_A > 0$, and similarly for $E[x||z_{\bar{A}}, z_{\bar{A}} > 0]$.

First, let's show that, for all $G \in \sigma(z, I_A)$, $G \cap [y \in A] \in \sigma(z_A)$. From theorem (20.1) in Billingsley (1986), $\sigma(z, I_A)$ consists exactly of the sets $[(z, I_A) \in H]$, with $H \in \mathcal{R}^2$, where $\mathcal{R}^2$ is the set of two-dimensional Borel sets. [10] Hence, for all $G \in \sigma(z, I_A)$, $G \cap [y \in A] = [z \in B] \cap [y \in A]$, where $B \in \mathcal{R}$. Let $B^+ = B \cap (0, +\infty)$, since $z > 0$, $[z \in B] \cap [y \in A] = [z \in B^+] \cap [y \in A]$. This set is empty or equal to $[z_A \in B^+] \cap [y \in A]$, as z coincides with z_A on $[y \in A]$. As $B^+ \subset (0, +\infty)$, $[z_A \in B^+] \subset [y \in A]$ and $[z_A \in B^+] \cap [y \in A] = [z_A \in B^+]$. In all cases $[z \in B] \cap [y \in A]$ is an element of $\sigma(z_A)$.

Now, let's call $E^c[x||z, I_A] = I_A\, E[x||z_A] + I_{\bar{A}}\, E[x||z_{\bar{A}}]$, and show that $E^c[x||z, I_A]$ is the conditional expectation of x conditioned on $\sigma(z, I_A)$. $E^c[x||z, I_A]$ is measurable with respect to $\sigma(z, I_A)$ since it is a function of z and I_A. Now, let $G \in \sigma(z, I_A)$.

$$\begin{aligned}
\int_G E^c[x||z, I_A]\, dP &= \int_{G \cap [y \in A]} E[x||z_A]\, dP + \int_{G \cap [y \in \bar{A}]} E[x||z_{\bar{A}}]\, dP \\
&= \int_{G \cap [y \in A]} x\, dP + \int_{G \cap [y \in \bar{A}]} x\, dP \\
&= \int_G x\, dP.
\end{aligned} \tag{22}$$

The second line of the equation comes from the fact that $G \cap [y \in A] \in \sigma(z_A)$ and that $E[x||z_A]$ is the conditional expectation of x on $\sigma(z_A)$, and similarly for $z_{\bar{A}}$. Hence $E[x||z, I_A] = I_A\, E[x||z_A] + I_{\bar{A}}\, E[x||z_{\bar{A}}]$. From proposition (2)

$$E[x||z_A] = \begin{cases} \frac{E[x_A||v]}{p(y \in A||v)} & \text{if} \quad z_A > 0 \\ \frac{E[x_{\bar{A}}]}{p(y \in \bar{A})} & \text{if} \quad z_A = 0 \end{cases} \tag{23}$$

where v coincides with z when $y \in A$. Let define $\psi_A = \frac{E[x_A||v]}{p(y \in A||v)}$, ψ_A is function of z and $E[x||z_A] =$

[10] Example (18.1) in Billingsley (1986) shows that $\mathcal{R}^2 = \mathcal{R} \times \mathcal{R}$, where $\mathcal{R}$ is the set of one-dimensional Borel sets.

ψ_A on $[y \in A]$. Hence, $E[x||z, I_A] = I_A\, \psi_A(z) + I_{\bar{A}}\, \psi_{\bar{A}}(z)$. Iterating expectations, we get

$$\begin{aligned} E[x||z] &= E[\, E[x||z, I_A] \,||\, z] \\ &= \psi_A(z)\, E[\, I[y \in A]||z] + \psi_{\bar{A}}(z)\, E[\, I[y \in \bar{A}]||z] \\ &= \psi_A(z)\, p(y \in A||z) + \psi_{\bar{A}}(z)\, p([y \in \bar{A}]||z) \end{aligned} \tag{24}$$

As ψ_A and $E[x||z_A]$ coincide on $[z_A > 0]$, writing $E[x|z_A, z_A > 0]$ the restriction of $E[x||z_A]$ to $[z_A > 0]$, we have $E[x||z, I_A] = p(y \in A||z)\, E[x||z_A, z_A > 0] + p([y \in \bar{A}]||z)\, E[x||z_{\bar{A}}, z_{\bar{A}} > 0]$.

Proof of lemma (1) : Let $P(y \in A||v)$ be the conditional probability of $y \in A$ conditioned on v. By definition, for all $G \in \sigma(v)$, $\int_G P(y \in A||v)\; dP = P([y \in A] \cap G)$ and $P(y \in A||v)$ is measurable with respect to $\sigma(v)$, i.e., there exists a (non-random) function φ_A so that for all $\omega \in \Omega$, $P(y \in A||v)(\omega) = \varphi_A[v(\omega)]$. Let A^* the Borel set so that $[y \in A^*] = [-y \in A]$. The objective is to show that $\varphi_A \circ v = \varphi_{A^*} \circ (-v)$. For example, we want to prove that if (y_1, y_2) is symmetrically distributed around zero, $p(y_1 < 0, y_2 < 0||y_1 + y_2)(-t) = p(y_1 > 0, y_2 > 0||y_1 + y_2)(t)$ where t is an observed value of $y_1 + y_2$.

Let $G \in \sigma(v)$, then $G = [v \in B]$, for a Borel set B. Now, as $v = \gamma' y$ and y are symmetrically distributed around zero, $P([y \in A] \cap [v \in B]) = P([-y \in A] \cap [-v \in B])$. Besides, $P(-y \in A|| - v) = P(-y \in A||v)$. Hence, $\int_{[v \in B]} P(y \in A||v)\, dP = \int_{[-v \in B]} P(-y \in A||v)\, dP$, that is $\int_{[v \in B]} (\varphi_A \circ v)\, dP = \int_{[-v \in B]} (\varphi_{A^*} \circ v)\, dP$. Call F the distribution function of v and assume that F is differentiable. Writing u the realization of the random variable v, we can write $\int_{[-v \in B]} (\varphi_{A^*} \circ v)\, dP = \int_{[-u \in B]} \varphi_{A^*}(u)\, dF(u)$. Applying the change of variable $t = -u$ and using the symmetry of v, we get $\int_{[-u \in B]} \varphi_{A^*}(u)\, dF(u) = \int_{[t \in B]} \varphi_{A^*}(-t)\, dF(t) = \int_{[v \in B]} (\varphi_{A^*}\, o\, (-v))\, dP$. Hence, for all $G \in \sigma(v)$, $\int_G (\varphi_A\, o\, v)\, dP = \int_G (\varphi_{A^*}\, o\, (-v))\, dP$; as $\varphi_A\, o\, v$ and $\varphi_{A^*}\, o\, (-v)$ are measurable with respect to $\sigma(v)$, we conclude that $\varphi_A\, o\, v = \varphi_{A^*}\, o\, (-v)$ a.e. . The proof for $E[\, I[y \in A]\, y||v]$ is similar.

Proof of lemma (3): The last line of equation (5) follows from lemma (1), since $v_j^{7-i} = -v_j^i$ and

$[y \in A_{7-i}] = [-y \in A_i]$, for $j = 1, 2$, and $i = 1, 2, 3$. Consider cases 1 and 6. From lemma (1), we know that $p(y_1 < 0, y_2 < 0||y_1 + y_2)(-t) = p(y_1 > 0, y_2 > 0||y_1 + y_2)(t)$, where t is a realization of $y_1 + y_2$. As $z = y_1^+ + y_2^+ + y_3^+$, the observed value for z is equal to t in case 1 and $-t$ in case 6. Similarly, $E[I[y_1 < 0, y_2 < 0]\, u||z, y_1 + y_2] = -E[I[y_1 > 0, y_2 > 0]\, u||z, y_1 + y_2]$. For the first line of equation (5), Lemma (2) implies that $p(y \in A_i|z \in B(h, \varepsilon)) = lim_{\varepsilon \to 0} \frac{p(y \in A_i, z \in B(h,\varepsilon))}{p(z \in B(h,\varepsilon))}$. Hence,

$$\begin{aligned} p(y \in A_1, z \in B(h,\varepsilon)) &= p(y_1 > 0, y_2 > 0, z \in B(h,\varepsilon)) \\ &= p(y_1 > 0, y_2 > 0, y_1 + y_2 \in B(h,\varepsilon)) \\ &= p(y_1 < 0, y_2 < 0, -(y_1 + y_2) \in B(h,\varepsilon)) \\ &= p(y_1 < 0, y_2 < 0, z \in B(h,\varepsilon)) \\ &= p(y \in A_6, z \in B(h,\varepsilon)) \end{aligned} \tag{25}$$

Proof of proposition (4): The number of traders, n, is assumed equal to 3. $cov(|u|, z) = 2\, cov(u^+, z)$, and $z = \Sigma_{i=1}^6 I[v_2^i > 0, v_3^i > 0](v_2^i + v_3^i)$, where v_2^i, v_3^i replace v_1^i and v_2^i defined in table (1). Noting $(\tilde{v}_2^i, \tilde{v}_3^i) = (v_2^{7-i}, v_3^{7-i}) = -(v_2^i, v_3^i)$, $i = 1, 2, 3$, one gets

$$z = \Sigma_{i=1}^3 \left(I[v_2^i > 0, v_3^i > 0](v_2^i + v_3^i) + I[\tilde{v}_2^i > 0, \tilde{v}_3^i > 0](\tilde{v}_2^i + \tilde{v}_3^i) \right). \tag{26}$$

Hence, with $v_1 = u$, one need only compute $cov\,(\, I[v_1 > 0]v_1\,,\, I[v_2 > 0, v_3 > 0](v_2 + v_3)\,) = E\,[I[v_1 > 0, v_2 > 0, v_3 > 0]v_1\; (v_2 + v_3)] - E\,[I[v_1 > 0]\, v_1]\;\; E\,[I[v_2 > 0, v_3 > 0]\,(v_2 + v_3)]$, where $v = (v_1, v_2, v_3)$ is normally distributed with mean zero. If (y_1, y_2) is a normally distributed vector with mean zero and variance S, noting $s_i = \sqrt{s_{ii}}$, $E[I[y_1 > 0]\, y_1] = \frac{s_1}{\sqrt{2\pi}}$ and $E[I[y_1 > 0, y_2 > 0]\, y_1] = \varphi(S)$, with $\varphi(S) = \frac{1}{2\sqrt{2\pi}}(s_1 + \frac{s_{12}}{s_2})$. Let $G(v_1, v_2, v_3) = v_1 f(v_1, v_2, v_3)$ where f the is density function of v, $G_1(v) = v_1 f_1(v) + f(v)$, then $G_2(v) = v_1 f_2$, $G_3 = v_1 f_3$. In the following, $\Sigma = (\sigma_{ij})_{i,j=1}^3$, $\Sigma^{-1} = (\sigma^{ij})_{i,j=1}^3$, $\sigma_i = \sqrt{\sigma_{ii}}$, $\sigma^i = \sqrt{\sigma^{ii}}$. Γ_i^* is the matrix obtained by deleting the i^{th} row and the i^{th} column of Σ^{-1}, $v = (v_1, v_2, v_3)$. The first derivative of f with respect to v is $-(\Sigma^{-1}.v)\, f(v)$, hence .

$$\begin{cases} (\sigma^{11}v_1^2+\sigma^{12}v_1v_2+\sigma^{13}v_1v_3)f(v) &= -G_1(v)+f(v) \\ (\sigma^{21}v_1^2+\sigma^{22}v_1v_2+\sigma^{23}v_1v_3)f(v) &= -G_2(v) \\ (\sigma^{31}v_1^2+\sigma^{32}v_1v_2+\sigma^{33}v_1v_3)f(v) &= -G_3(v) \end{cases} \tag{27}$$

Now, take expectation through the system (27), beginning with the first row.

$$\int_{t>0} G_1(t)dt = \int_{t_2>0,t_3>0} [G(t)]_{t_1=0}^{+\infty} dt_2 dt_3 = 0. \tag{28}$$

and

$$\int_{v>0} f(v)dv = p(v>0) = \frac{1}{8} + \frac{1}{4\pi}(arcsin(\rho_{12}) + arcsin(\rho_{13}) + arcsin(\rho_{23})) \tag{29}$$

Taking expectation through the second row, one gets equation (30).

$$\begin{aligned} \int_{t>0} -G_2 dt &= \int_{t_1>0}\int_{t_3>0} -t_1[f(t)]_{t_2=0}^{+\infty} dt_1 dt_3 \\ &= \int_{t_1>0}\int_{t_3>0} t_1 \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{|\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(\sigma^{11}t_1^2 + 2\sigma^{13}t_1t_3 + \sigma^{33}t_3^2)) dt_1 dt_3 \\ &= (2\pi|\Gamma_2^*||\Sigma|)^{-\frac{1}{2}} \int_{t_1>0}\int_{t_3>0} t_1 \frac{1}{2\pi}|\Gamma_2^*|^{\frac{1}{2}} exp(-\frac{1}{2}(t_1,t_3)\Gamma_2^* \begin{pmatrix} t_1 \\ t_3 \end{pmatrix}) dt_1 dt_3 \\ &= (2\pi|\Gamma_2^*||\Sigma|)^{-\frac{1}{2}} E[I[\eta_1>0,\eta_2>0]\eta_1] \\ &= \tfrac{1}{\sigma_2\sqrt{2\pi}} E[I[\eta_1>0,\eta_2>0]\eta_1] \\ &= \tfrac{1}{\sigma_2\sqrt{2\pi}} \varphi(\Gamma_2^{*-1}) \end{aligned} \tag{30}$$

where (η_1,η_2) is normally distributed with mean zero and variance Γ_2^{*-1}. The last but one line of (30) follows from the identity $\sigma_{22} = \frac{|\Gamma_2^*|}{|\Sigma^{-1}|}$, the last line from $E[I[\eta_1>0,\eta_2>0]\eta_1] = \varphi(\Gamma_2^{*-1})$. Likewise, $\int_{t>0} -G_3 dt = \frac{1}{\sigma_3\sqrt{2\pi}}\varphi(\Gamma_3^{*-1})$. Let $\rho_2^* = \frac{\sigma^{13}}{\sigma^1\sigma^3}$ and $\rho_3^* = \frac{\sigma^{12}}{\sigma^1\sigma^2}$. Using the definition of Γ_2^{*-1}

and noting that $\frac{\sigma^{33}}{|\Gamma_2^*|} = \frac{1}{\sigma^{11}(1-(\rho_2^*)^2)}$, one gets $\varphi(\Gamma_2^{*-1}) = \frac{1}{2\sigma^1\sqrt{2\pi}}\sqrt{\frac{1-\rho_2^*}{1+\rho_2^*}}$. Finally,

$$\begin{pmatrix} E[I[v>0]v_1^2] \\ E[I[v>0]v_1v_2] \\ E[I[v>0]v_1v_3] \end{pmatrix} = \Sigma \begin{pmatrix} \frac{1}{8}+\frac{1}{4\pi}(arcsin(\rho_{12})+arcsin(\rho_{13})+arcsin(\rho_{23})) \\ \frac{1}{4\pi}\frac{1}{\sigma_2}\frac{1}{\sigma^1}\sqrt{\frac{1-\rho_2^*}{1+\rho_2^*}} \\ \frac{1}{4\pi}\frac{1}{\sigma_3}\frac{1}{\sigma^1}\sqrt{\frac{1-\rho_3^*}{1+\rho_3^*}} \end{pmatrix}, \tag{31}$$

and

$$\begin{aligned} E[I[v>0]v_1(v_2+v_3)] &= (\sigma_{12}+\sigma_{13})(\tfrac{1}{8}+\tfrac{1}{4\pi}(arcsin(\rho_{12})+arcsin(\rho_{13})+arcsin(\rho_{23}))) \\ &+ \tfrac{1}{4\pi\sigma^1}\sqrt{\tfrac{1-\rho_2^*}{1+\rho_2^*}}(\sigma_2+\tfrac{\sigma_{23}}{\sigma_2})+\tfrac{1}{4\pi\sigma^1}\sqrt{\tfrac{1-\rho_3^*}{1+\rho_3^*}}(\sigma_3+\tfrac{\sigma_{23}}{\sigma_3}). \end{aligned} \tag{32}$$

Let $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) = (v_1, -v_2, -v_3)$ and $var(\tilde{v}) = \tilde{\Sigma} = (\tilde{\sigma}^{ij})_{i,j=1}^3$. Naturally, $\tilde{\rho}_{12} = -\rho_{12}$, $\tilde{\rho}_{13} = -\rho_{13}$, $\tilde{\rho}_{23} = \rho_{23}$. Writing $\tilde{\Sigma}^{-1} = (\tilde{\sigma}^{ij})_{i,j=1}^3$ and using the definition of the inverse, one also gets $\tilde{\sigma}^{ii} = \sigma^{ii}$, for $i = 1,2,3$, $\tilde{\sigma}^{12} = -\sigma^{12}$, $\tilde{\sigma}^{13} = -\sigma^{13}$, $\tilde{\sigma}^{23} = \sigma^{23}$, and consequently $\tilde{\rho}_2^* = -\rho_2^*$, $\tilde{\rho}_3^* = -\rho_3^*$. Hence,

$$\begin{aligned} E[I[\tilde{v}>0]\tilde{v}_1(\tilde{v}_2+\tilde{v}_3)] &= -(\sigma_{12}+\sigma_{13})(\tfrac{1}{8}+\tfrac{1}{4\pi}(-arcsin(\rho_{12})-arcsin(\rho_{13})+arcsin(\rho_{23}))) \\ &+ \tfrac{1}{4\pi\sigma^1}\sqrt{\tfrac{1+\rho_2^*}{1-\rho_2^*}}(\sigma_2+\tfrac{\sigma_{23}}{\sigma_2})+\tfrac{1}{4\pi\sigma^1}\sqrt{\tfrac{1+\rho_3^*}{1-\rho_3^*}}(\sigma_3+\tfrac{\sigma_{23}}{\sigma_3}), \end{aligned} \tag{33}$$

and

$$\begin{gathered} E[I[v>0]v_1(v_2+v_3)]+E[I[\tilde{v}>0]\tilde{v}_1(\tilde{v}_2+\tilde{v}_3)] = \\ (\sigma_{12}+\sigma_{13})(\tfrac{1}{2\pi}(arcsin(\rho_{12})+arcsin(\rho_{13}))+ \\ \tfrac{1}{4\pi\sigma^1}(\sigma_2+\tfrac{\sigma_{23}}{\sigma_2})[\sqrt{\tfrac{1-\rho_2^*}{1+\rho_2^*}}+\sqrt{\tfrac{1+\rho_2^*}{1-\rho_2^*}}]+\tfrac{1}{4\pi\sigma^1}(\sigma_3+\tfrac{\sigma_{23}}{\sigma_3})[\sqrt{\tfrac{1-\rho_3^*}{1+\rho_3^*}}+\sqrt{\tfrac{1+\rho_3^*}{1-\rho_3^*}}]. \end{gathered} \tag{34}$$

$$\begin{aligned} \frac{1}{2\sigma^1}\left(\sqrt{\tfrac{1-\rho_2^*}{1+\rho_2^*}}+\sqrt{\tfrac{1+\rho_2^*}{1-\rho_2^*}}\right) &= \frac{1}{\sigma^1\sqrt{1-(\rho_2^*)^2}} \\ &= \sqrt{\frac{\sigma^{33}}{|\Gamma_2^*|}} \\ &= \sqrt{\frac{\sigma^{33}|\Sigma|}{\sigma_{22}}} \\ &= \sqrt{\frac{\sigma_{11}\sigma_{22}-\sigma_{12}^2}{\sigma_{22}}} \\ &= \sigma_1\sqrt{1-\rho_{12}^2} \end{aligned} \tag{35}$$

$$\begin{aligned}
&E[I[v>0]v_1(v_2+v_3)] + E[I[\tilde{v}>0]\tilde{v}_1(\tilde{v}_2+\tilde{v}_3)] \ = \\
&\tfrac{1}{2\pi}(\sigma_{12}+\sigma_{13})(arcsin(\rho_{12})+arcsin(\rho_{13})) \ + \\
&\tfrac{1}{2\pi}\sigma_1(\sigma_2+\tfrac{\sigma_{23}}{\sigma_2})\sqrt{1-\rho_{12}^2} + \tfrac{1}{2\pi}\sigma_1(\sigma_3+\tfrac{\sigma_{23}}{\sigma_3})\sqrt{1-\rho_{13}^2}
\end{aligned} \tag{36}$$

Since

$$E[I[v_1>0]v_1] = \frac{1}{\sqrt{2\pi}}\sigma_1, \tag{37}$$

and

$$E[I[v_2>0,v_3>0](v_2+v_3)] = E[I[\tilde{v}_2>0,\tilde{v}_3>0](\tilde{v}_2+\tilde{v}_3)] = \frac{1}{2\sqrt{2\pi}}[\sigma_2+\frac{\sigma_{23}}{\sigma_3}+\sigma_3+\frac{\sigma_{23}}{\sigma_2}], \tag{38}$$

$$cov(I[v_1>0]v_1, I[v_2>0,v_3>0](v_2+v_3)) + cov(I[v_1>0]\tilde{v}_1, I[\tilde{v}_2>0,\tilde{v}_3>0](\tilde{v}_2+\tilde{v}_3)) = \frac{\sigma_1}{2\pi}(H(\Sigma)+G(\Sigma)), \tag{39}$$

with

$$H(\Sigma) = (\frac{\sigma_{12}}{\sigma_1}+\frac{\sigma_{13}}{\sigma_1})(arcsin(\rho_{12})+arcsin(\rho_{13})) \tag{40}$$

$$G(\Sigma) = (\sigma_2+\frac{\sigma_{23}}{\sigma_2})(\sqrt{1-\rho_{12}^2}-1) + (\sigma_3+\frac{\sigma_{23}}{\sigma_3})(\sqrt{1-\rho_{13}^2}-1) \tag{41}$$

Let $v_1 = u$, and (v_2, v_3) can take three values; in the first case $(v_2, v_3) = (y_1, y_2)$, in the second case, $(v_2, v_3) = (-y_2, y_1+y_2)$, in the third case $(v_2, v_3) = (y_1, -(y_1+y_2))$. Call Σ_i, the variance matrix of v in case i, $H = H(\Sigma_1)+H(\Sigma_2)+H(\Sigma_3)$, and $G = G(\Sigma_1)+G(\Sigma_2)+G(\Sigma_3)$. Then $cov(u^+, z) = \frac{\sigma_u}{2\pi}(H+G)$.

$$H = \sigma_u^{-1}\left(cov(u,y_1)arcsin(\rho_{u,y_1}) + cov(u,y_2)arcsin(\rho_{u,y_2}) + cov(u,y_1+y_2)arcsin(\rho_{u,(y_1+y_2)})\right) \tag{42}$$

This is because

$$\begin{aligned}
&\sigma_u\, H = cov(u,y_1+y_2)[arcsin(\rho_{u,y_1})+arcsin(\rho_{u,y_2})] \\
&+cov(u,y_1)[-arcsin(\rho_{u,y_2})+arcsin(\rho_{u,(y_1+y_2)})] \\
&-cov(u,y_2)[-arcsin(\rho_{u,(y_1+y_2)})+arcsin(\rho_{u,y_1})],
\end{aligned} \tag{43}$$

which rearranging terms, is equal to

$$cov(u,y_1)arcsin(\rho_{u,y_1}) + cov(u,y_2)arcsin(\rho_{u,y_2} + cov(u,y_1+y_2)arcsin(\rho_{u,y_1+y_2}) \tag{44}$$

Likewise,

$$G = \sigma_{y_1}(\sqrt{1-\rho^2_{u,y_1}} - 1) + \sigma_{y_2}(\sqrt{1-\rho^2_{u,y_2}} - 1) + \sigma_{y_1+y_2}(\sqrt{1-\rho^2_{u,y_1+y_2}} - 1) \tag{45}$$

Finally,

$$cov(|u|,z) = \frac{\sigma_u}{\pi}\left(\sigma_{y_1}h(\rho_{u,y_1}) + \sigma_{y_2}h(\rho_{u,y_2}) + \sigma_{y_1+y_2}h(\rho_{u,y_1+y_2})\right) \tag{46}$$

with $h(t) = t\ arcsin(t) + \sqrt{1-t^2} - 1$. For all $t \in [-1,1]$, $h(t) \geq 0$ and $h(t) = 0$ if and only if $t = 0$. Consequently, $cov(|u|,z)$ is non-negative and is zero only if u is independent of (y_1,y_2).

Proof of lemma (4) Take the case $i = 1$ and let $f^1 = f$. The other cases can be treated similarly. We know that, for a realization h of the random variable z, $p(y \in A_1||z) = lim_{\varepsilon\to 0}p(y \in A_1||z \in B(h,\varepsilon))$, with $B(h,\varepsilon)$ being the open ball with center $h > 0$ and of radius $\varepsilon > 0$, with $\varepsilon < h$, $p(y \in A_1||z \in B(h,\varepsilon)\,) = \frac{p(y\in A_1, z\in B(h,\varepsilon))}{p(z\in B(h,\varepsilon))}$, and $p(z \in B(h,\varepsilon)) = \Sigma_{i=1}^6 p(y \in A_i, z \in B(h,\varepsilon))$.

$$\begin{aligned} p(y \in A_1, z \in B(h,\varepsilon)) &= p(y_1 > 0, y_2 > 0, |y_1 + y_2 - h| < \varepsilon) \\ &= p(y_1 > 0, y_2 > 0, h - \varepsilon < y_1 + y_2 < h + \varepsilon) \\ &= p(0 < y_1 < h - \varepsilon, h - \varepsilon - y_1 < y_2 < h + \varepsilon - y_1) \\ &+ p(h - \varepsilon < y_1 < h + \varepsilon, 0 < y_2 < h + \varepsilon - y_1) \end{aligned} \tag{47}$$

Divide now the numerator and the denominator of $p(y \in A_1||z \in B(h,\varepsilon)\,)$ by 2ε, and take the limit when $\varepsilon \to 0$. With f being the density function of (y_1,y_2), we obtain:

$$\begin{aligned} \tfrac{1}{2\varepsilon}\,p(0 < y_1 < h - \varepsilon, h - \varepsilon - y_1 < y_2 < h + \varepsilon - y_1) &\to \int_{y_1=0}^{h} f(y_1, h - y_1)\,dy_1 \\ \tfrac{1}{2\varepsilon}\,p(h - \varepsilon < y_1 < h + \varepsilon, 0 < y_2 < h + \varepsilon - y_1) &\to 0 \end{aligned} \tag{48}$$

The first line of (48) comes from the fact that $p(0 < y_1 < h-\varepsilon, h-\varepsilon-y_1 < y_2 < h+\varepsilon-y_1) = \int_{y_1=0}^{h-\varepsilon}\int_{y_2=h-\varepsilon-y_1}^{h+\varepsilon-y_1} f(y_1,y_2)\,dy_1\,dy_2$. For the second line of (48), consider that

$$\begin{aligned} \tfrac{1}{2\varepsilon}p(h-\varepsilon < y_1 < h+\varepsilon, 0 < y_2 < h+\varepsilon-y_1) &= \int_{y_1=h-\varepsilon}^{h+\varepsilon}\frac{1}{2\varepsilon}\int_{y_2=0}^{h+\varepsilon-y_1} f(y_1,y_2)\,dy_2\,dy_1 \\ &\leq \int_{y_1=h-\varepsilon}^{h+\varepsilon}\frac{1}{2\varepsilon}\int_{y_2=0}^{2\varepsilon} f(y_1,y_2)\,dy_2\,dy_1 \end{aligned} \tag{49}$$

By the mean value theorem, there exists a $\theta \in [0, 2\varepsilon]$, so that $\frac{1}{2\varepsilon}\int_{y_2=0}^{2\varepsilon} f(y_1,y_2)\,dy_2 = f(y_1,\theta)$. Then, as $f(y_1,\theta)$ is bounded, when $\varepsilon \to 0$, $\int_{y_1=h-\varepsilon}^{h+\varepsilon} f(y_1,\theta(\varepsilon))\,dy_1 \to 0$. We conclude that $\frac{1}{2\varepsilon}p(y \in A_1, z \in B(h,\varepsilon)) \to \int_{y_1=0}^{h} f(y_1, h-y_1)\,dy_1$. Using table (1), the other cases are treated similarly.

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cases (i)	sign of y_1	sign of y_2	sign of y_1+y_2	z	v_1^i	v_2^i
1	+	+		y_1+y_2	y_1	y_2
2	+	-	+	y_1	$-y_2$	y_1+y_2
3	+	-	-	$-y_2$	y_1	$-(y_1+y_2)$
4	-	+	+	y_2	$-y_1$	y_1+y_2
5	-	+	-	$-y_1$	y_2	$-(y_1+y_2)$
6	-	-		$-(y_1+y_2)$	$-y_1$	$-y_2$

Table 1: Decomposition of trading volume

There are three traders; y_1 is the first trader's demand, y_2 is the second trader's demand, the last trader's demand is $y_3 = -(y_1+y_2)$. The trading volume is $z = y_1^+ + y_2^+ + y_3^+$. In each case i, $i = 1, \ldots, 6$, z can be written as $v_1^i + v_2^i$.

	$v < -a$	$-a \leq v \leq a$	$v > a$
$\eta < -a$	bid	bid	(bid, ask)
$-a \leq \eta \leq a$	bid	no trade	ask
$\eta > a$	(bid, ask)	ask	ask

Table 2: Sides where trading occurs

t	$t \leq -z$	$-z < t \leq 0$	$0 < t \leq z$	$t > z$
$p(\tilde{q} < t)$	0	$p(v \leq t)$	$p(v \leq b) + p(a \leq v \leq t + a)$	1
$p(\tilde{q} > -t)$	0	$p(v \geq -t)$	$p(v \geq a) + p(t + b \leq v \leq b)$	1

Table 3: Symmetry of informed trader's demand

v is the informed trader's valuation of the asset, η is the liquidity shock, the informed trader's demand is $v - p$, he liquidity trader's demand is $\eta - p$, a is the ask price, b is the bid price.

$\tilde{q}$ is the informed trader's demand, z is the maximum quantity the market maker is willing to sell at the ask price a or buy at the bid price b, v is the informed trader's valuation of the asset.

	$p > 0$	$p \leq 0$
purchase	$1 + \tau_a$	$1 - \tau_d$
sale	$1 - \tau_b$	$1 + \tau_c$

Table 4: Ratio of the after-tax price to the pre-tax price.

p is the market clearing price, the proportional tax is levied on the informed trader only. The tax rates τ_a, τ_b, τ_c, and τ_d are between zero and one.

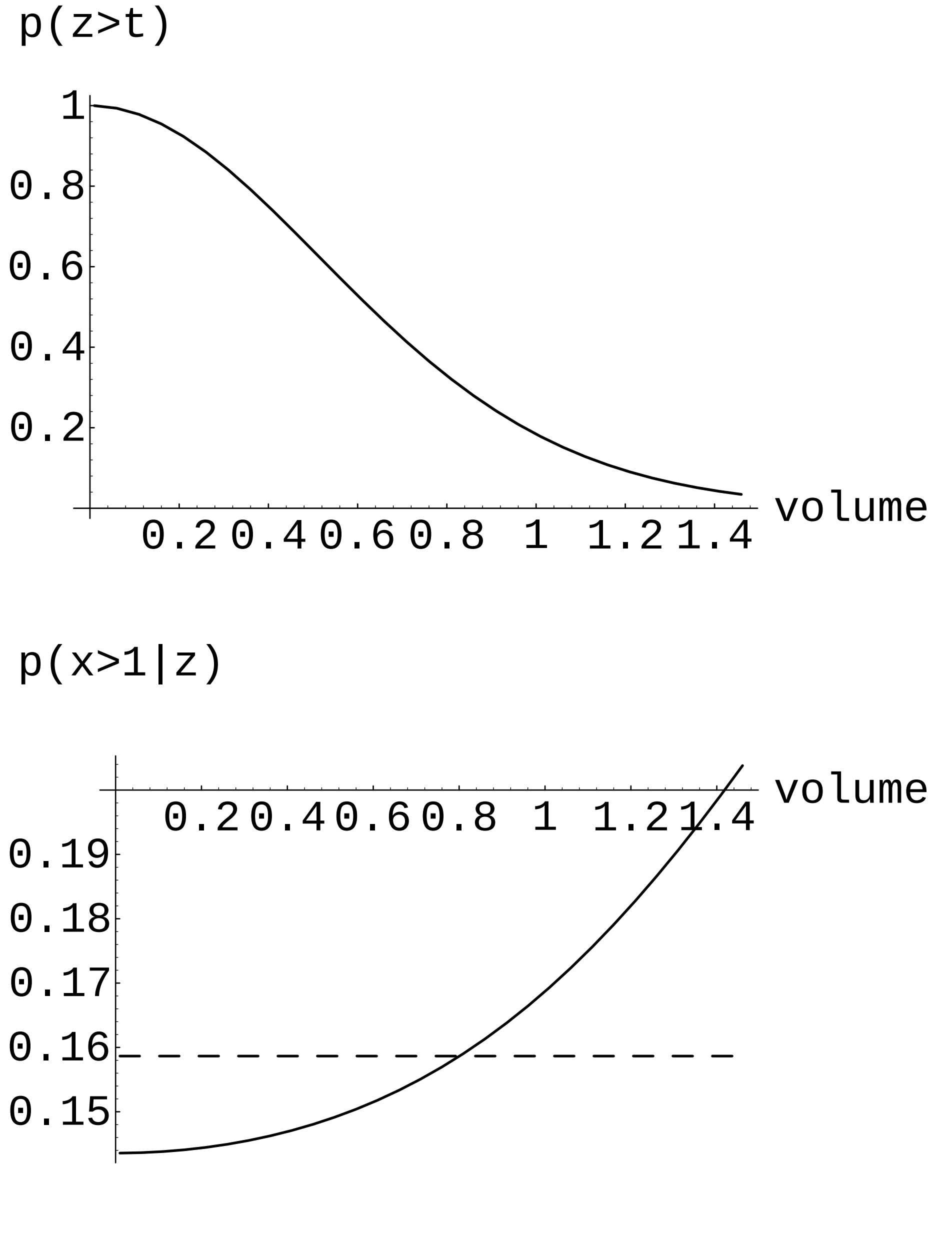

Figure 1: Volume-based conditional probability $P(x > 1|z)$.