Default Correlation: An Analytical Result

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Abstract

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Evaluating default correlations and the probabilities of multiple defaults is an important task in credit analysis and risk management, but it has never been an easy one because default correlations cannot be measured directly. This paper provides, for the first time, an analytical formula for calculating default correlations based on a first-passage-time model which can be easily implemented and conveniently used in a variety of financial applications. The result of this paper also provides a theoretical justification for many empirical results found in the literature and increases our understanding of the important features of default correlations.

Default Correlation and Risk Analysis: An Analytical Result

Evaluating correlations among firms' defaults is critical to the proper measurement of a wide variety of risks in financial markets. For example, the risks of bond portfolios, letters of credit, and credit default swaps are all functions of default correlations.

Currently, default correlations can be estimated in one of three ways. First, given two firms' asset values, their variance/covariance matrix, and their liability structures, there is an analytical solution under the assumption of Merton (1974) that default can only occur at a single point in time. This extremely restrictive assumption is relaxed in first-passage-time models of default risk, but to date, no analytical solution exists for default correlations in such models. Thus the second method is a Monte Carlo simulation of a specified model of default risk. Besides being extremely time consuming, this method provides only limited insight into the comparative statics of default correlations. The third method uses historical default data to estimate default correlations, an approach that cannot capture any firm-specific information. More importantly, because of the lack of reliable time series data, historical statistics are generally very inaccurate.

This paper provides an analytical solution to the default correlation based on a first-passage-time model. In comparison with other existing approaches, the solution is not only theoretically rigorous, but also practically implementable. Given the firms' asset values, their variance/covariance matrix, and the structure of firms' liabilities, the solution can be easily implemented in practice. These inputs can be estimated from firms' balance sheets and stock prices.

The rest of the paper is organized as follows: Section 1 defines the economy and presents a closed-form solution to the default correlation. Section 2 provides a couple of approaches to estimate the parameter values needed in the theoretical default correlation model so that the model can be readily used in practice. Section 3 uses the model to explain the observed empirical behavior of default correlations. Section 4 provides a brief comparison of our first-passage model with a Merton-style model. Section 5 concludes.

1 The Model

This section provides a basic theoretical framework to discuss default correlations. We consider the default correlation between two arbitrary firms, firm 1 and firm 2.

Assumption 1: Let V_1 and V_2 denote the total asset values of firm 1 and firm 2. The dynamics of V_1 and V_2 are given by the following vector stochastic process

$$\begin{bmatrix} d\ln(V_1) \\ d\ln(V_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \mathbf{\Omega} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{1}$$

where

μ_1 and μ_2 are constant drift terms,

z_1 and z_2 are two independent standard Brownian motions, and

$$\mathbf{\Omega} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

is a constant 2×2 matrix such that

$$\mathbf{\Omega} \cdot \mathbf{\Omega}' = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

The coefficient ρ reflects the correlation between the movements in the asset values of the two firms, which plays a critical role in determining the default correlation between the firms.

Assumption 2: The default of a firm is triggered by the decline in its value. For each firm i, there exist two positive constants K_i and λ_i such that the firm continues to operate and meets its contractual obligations as long as $V_i(t) > e^{\lambda_i t} K_i$. However, if its value $V_i(t)$ falls to the threshold level $e^{\lambda_i t} K_i$, it defaults on all of its obligations immediately and some form of corporate restructuring takes place.

This assumption follows Black and Cox (1976), Longstaff and Schwartz (1995), and Zhou (1996). By this assumption, determining the default event of a firm is equivalent to finding the first passage time of the firm's value to the trigger level. To simplify the mathematics, we assume that $\lambda_i = \mu_i$ in this paper.[1]

Denote $\tau_i := \min_{t \geq 0}\{t | e^{-\lambda_i t} V_{i,t} \leq K_i\}$ as the first time that firm i's value reaches its default threshold level. Then $D_i(t)$, the event that firm i defaults before some time $t > 0$, can be expressed as: $D_i(t) = \{\tau_i \leq t\}$. Using the result of Harrison (1990), we have

$$
\begin{aligned}
P(D_i(t)) &= P(\tau_i \leq t) \\
&= 2 \cdot N\left(-\frac{\ln(V_{i,0}/K_i)}{\sigma_i \sqrt{t}}\right).
\end{aligned} \tag{2}
$$

Define

$$
Z_i := \frac{\ln(V_{i,0}/K_i)}{\sigma_i}
$$

as the standardized distance of firm i to its default point. Eq. (2) then is simplified as:

$$
P(D_i(t)) = 2 \cdot N\left(-\frac{Z_i}{\sqrt{t}}\right). \tag{3}
$$

The default correlation between firm 1 and firm 2 over period $[0, t]$ is:

$$
\begin{aligned}
\rho_D(t) &= \text{Corr}[D_1(t), D_2(t)] \\
&= \frac{P[D_1(t) \cdot D_2(t)] - P[D_1(t)] \cdot P[D_2(t)]}{[P(D_1(t))(1 - P(D_1(t)))]^{1/2} \cdot [P(D_2(t))(1 - P(D_2(t)))]^{1/2}}.
\end{aligned} \tag{4}
$$

We know from basic probability theory that

$$
P(D_1 \cdot D_2) = P(D_1) + P(D_2) - P(D_1 + D_2). \tag{5}
$$

So given eq. (3), to determine the default correlation, the only remaining unknown we need to solve is the $P(D_1 + D_2)$, i.e., the probability that at least one default has occurred by

[1]This assumption makes it possible to remove the drift term from $\ln[e^{-\lambda_i t} V_i(t)]$.

time t. We will show that

$$P(D_1+D_2) = 1-\frac{2r_0}{\sqrt{2\pi t}}\cdot e^{-\frac{r_0^2}{4t}}\cdot\sum_{n=1,3,5,\cdots}\frac{1}{n}\cdot\sin\left(\frac{n\pi\theta_0}{\alpha}\right)\cdot \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right)+I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right)\right] \tag{6}$$

where $I_\nu(z)$ is the modified Bessel function I with order ν and $Z_i = b_i/\sigma_i$. As a matter of fact, one can easily verify that:

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{if } (.) > 0 \\ \pi+\tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{otherwise,} \end{cases} \tag{7}$$

$$r_0 = Z_2/\sin(\theta_0),$$

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi+\tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise.} \end{cases} \tag{8}$$

Mathematically, calculating the default correlation between the two firms now reduces to calculating the probability that at least one firm's value reaches the threshold level during time $[0, t]$, i.e.,

$$\begin{aligned} P(D_1+D_2) &= P(\tau_1 \le t \text{ or } \tau_2 \le t) \\ &= P(\tau \le t), \end{aligned} \tag{9}$$

where $\tau := \min(\tau_1, \tau_2)$.

Define

$$X_1(t) = -\ln[e^{-\lambda_i t}V_1(t)/V_1(0)],$$

$$X_2(t) = -\ln[e^{-\lambda_i t}V_2(t)/V_2(0)],$$

$$b_1 = -\ln[K_1/V_1(0)] = \ln[V_1(0)/K_1],$$

$$b_2 = -\ln[K_2/V_2(0)] = \ln[V_2(0)/K_2].$$

It is straightforward to verify that $[X_1(t), X_2(t)]$ follows a two dimensional Brownian motion:

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = -\mathbf{S} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}. \tag{10}$$

After this transformation, finding $P(\tau \leq t)$ is equivalent to finding the first passage time of the two dimensional Brownian motion $[X_1,\ X_2]'$ with initial condition $[X_1(0),\ X_2(0)]' = [0, 0]'$ to a boundary consisting of two intersecting lines $X_1 = b_1$ and $X_2 = b_2$. For notational convenience, this boundary will be denoted as $\partial(b_1, b_2)$ henceforth.

Suppose that the two dimensional Brownian motion process $[X_1(t), X_2(t)]'$ represents the position of a particle at time t and that $\partial(b_1, b_2)$ is a absorbing barrier. Let $f(x_1, x_2, t)$ be the transition probability density of the particle in the region $\{(x_1, x_2)|x_1 < b_1 \text{ and } x_2 < b_2\}$, i.e., the probability density that $[X_1(t), X_2(t)]' = [x_1, x_2]'$ and that the particle does not reach the barrier $\partial(b_1, b_2)$ in time interval $(0, t)$. We have

$$\begin{aligned} &P(X_1(s) < b_1 \text{ and } X_2(s) < b_2, \text{ for } \quad 0 < s < t,\ X_1(t) < y_1 \text{ and } X_2(t) < y_2) \\ &\qquad = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f(x_1, x_2, t)dx_1dx_2 := F(y_1, y_2, t). \end{aligned} \tag{11}$$

Thus $F(b_1, b_2, t)$ is the probability that absorption has not yet occurred by time t, i.e.,

$$F(b_1, b_2, t) = P(\tau > t) = 1 - P(\tau \leq t).$$

According to eq. (9), to estimate the joint probability of firms 1 and 2 as well as the default correlation between the two firms, we only need to calculate $F(b_1, b_2, t)$.

According to Cox and Miller (1965) and Karatzas and Shreve (1988), the transition probability density $f(x_1, x_2, t)$ satisfies the following Kolmogorov forward equation:

$$\begin{aligned} &\frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial f}{\partial t} \\ &\qquad (x_1 < b_1,\ x_2 < b_2), \end{aligned} \tag{12}$$

subject to certain boundary conditions.

Solving for the density function f from the above PDE and integrating, we obtain

Main Result 1 *The probability that no firm has defaulted by time t is given by*

$$\begin{aligned} F(b_1, b_2, t) &= 1 - P(D_1 + D_2) \\ &= \int_{-\infty}^{b_1}\int_{-\infty}^{b_2} f(x_1, x_2, t)dx_1 dx_2 \\ &= \frac{2r_0}{\sqrt{2\pi t}} \cdot e^{-\frac{r_0^2}{4t}} \cdot \sum_{n=1,3,5,\cdots} \frac{1}{n} \cdot \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \cdot \\ &\quad \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right)\right] \end{aligned} \tag{13}$$

where $I_\nu(z)$ is the modified Bessel function I with order ν and

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{b_1\sigma_2 - \rho b_2\sigma_1}\right) & \text{if } (.) > 0 \\ \pi + \tan^{-1}\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{b_1\sigma_2 - \rho b_2\sigma_1}\right) & \text{otherwise,} \end{cases} \tag{14}$$

$$r_0 = b_2/[\sigma_2 \sin(\theta_0)],$$

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise.} \end{cases} \tag{15}$$

All other notation is as previously defined.

It is obvious that the probability $F(b_1, b_2, t)$ at any given time horizon t is solely determined by standardized distances to default $Z_i = b_i/\sigma_i$. As a matter of fact, one can easily verify that:

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1 - \rho Z_2}\right) & \text{if } (.) > 0 \\ \pi + \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1 - \rho Z_2}\right) & \text{otherwise,} \end{cases} \tag{16}$$

$$r_0 = Z_2/\sin(\theta_0).$$

The above result offers a very tractable closed-form formula to calculate default correlations. Like the Black-Scholes model, this formula can be programmed into computers or calculators and be used to report results instantaneously.

2 How to Apply the Model in Practice

To apply the model in financial practice, we must estimate the following parameters: (V_i, σ_i, K_i, ρ) or (Z_i, ρ).

2.1 An Option Approach to Estimating Parameters

2.1.1 Estimating V_i, σ_i, and ρ

Typically, the total value of a firm's underlying assets is not observable because the market value of the firm's liabilities is not known. In practice, this problem can be circumvented by an option theoretic model of the firm, which treats the firm's equity as a call option on the firm's underlying assets. Denote S_i as the equity value of the firm, we have:

$$S_i = C(V_i, \sigma_i; \text{other values}), \tag{17}$$

where other values include the book value and the maturity of liabilities as well as the interest rate. These "other" values are generally observable.

Rewrite eq. (17) as

$$\ln(S_i) = G(\ln(V_i), \sigma_i; \text{other values}).$$

From Ito's lemma and eq. (1), we have

$$d\ln(S_i) = \eta_i(.;t)dt + \frac{\partial G}{\partial \ln(V_i)}[s_{i1},\ \ s_{i2}]\begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{18}$$

where $\eta_i(.;t)$ is a deterministic function of V_i, σ_i, etc. Eq. (18) implies that the volatility of log equity return $\sigma_{s,i} := \text{Var}_t[d\ln(S_i)/dt]$ satisfies

$$\begin{aligned} \sigma_{s,i} &= \frac{\partial G}{\partial \ln(V_i)} \cdot \sigma_i \\ &= H(V_i, \sigma_i; \text{other values}), \end{aligned} \tag{19}$$

Assume that we can observe stock price S_i and equity return volatility $\sigma_{s,i}$. Solving the joint equation system (17) and (19), we obtain V_i and σ_i. It follows from eq. (1) and eq.

(18) that

$$\rho = \mathrm{Corr}_t[d\ln(S_1),\ d\ln(S_2)].$$

Generally, firm value correlations estimated over finite horizons do not satisfy the above equation. However, the equation could be a good approximation for highly rated firms. Whether the equation is also a good approximation for highly leveraged (lowly rated) firms is still an unsolved practical issue.

2.1.2 Estimating K_i

The default point K_i is determined by the liability structure of the firm. It is the liabilities the firm must be able to meet by a chosen time horizon in order to stay in business. In practice, the relationship between K_i and the liability structure is often assessed by historical data. In the KMV Corporation's credit valuation model, the default point is defined as short-term liabilities plus a certain percentage (approximately 50%) of long term liabilities.

2.2 Statistical Approach to Estimating Parameters

One may also use a statistical approach to estimate Z. From equation (3), we know that one can easily calculate Z_i if one knows $P(D_i(t))$ for some t. Suppose that we want to estimate Z for the firm XYZ which has A-rated senior debt. From the historical data, we know the statistical cumulative default rates $\tilde{A}(t)$ for such firms at various investment horizons t. The parameter Z can then be chosen to fit the theoretical default probabilities $P(Z,t)$ to the historical default rates $\tilde{A}(t)$. One way to do this is with a least-squares approach:

$$Z = \arg\min_{Z} \sum_t \left(\frac{P(Z,t)}{t} - \frac{\tilde{A}(t)}{t} \right)^2.$$

In the above expression, cumulative default rates $P(Z,t)$ and $\tilde{A}(t)$ are divided by time horizon t so that they are transformed to average default rates per unit time.

Comparing with the option approach, the statistical approach is easier to use. However, because this approach is based solely on credit ratings, it does not effectively use all firm-specific information. In addition, since the default probability corresponding to a rating category is time-varying, historical default rates for firms in a given rating category may not reflect the true default probability of that rating category at any particular time.

3 Implications of the Model

In this section, we use numerical examples to investigate the implications of the model and to examine if the formula presented in the above section is consistent with the important empirical features of the historical default data.

Figure 1 plots the relationship between default correlation and investment horizon t as well as the underlying asset return correlation ρ. This plot offers the following results.

1) The default correlation and the underlying asset return correlation have the same sign. The higher is the underlying asset return correlation ρ, *ceteris paribus*, the higher is the default correlation. Generally, the default correlation is lower than the underlying asset return correlation.

This result is quite intuitive. For instance, if asset return correlation ρ is positive, when one firm defaults because of the drop in its value, it is likely that the value of the other firm has also declined and moved closer to its default boundary. The result may explain why firms in the same industry (region) often have higher default correlations than do the firms in different industries (regions).

2) Default correlations are generally very small over short investment horizons. They increase and then slowly decrease with time.

Over a short investment horizon, default correlations are low because quick defaults of firms are rare and are almost idiosyncratic. Default correlations eventually decrease with time because over a sufficiently long time horizon, the default of a firm is virtually inevitable and the non-default events become rare and idiosyncratic.[2] This result is consistent with an important phenomenon of historical default correlations reported in Lucas (1995). Lucas interprets this phenomenon as a result of business cycle fluctuations. Our result suggests that we do not need business cycle fluctuations to explain this phenomenon.

Figure 2 illustrates the relationship between default correlation and the credit quality of the firms, proxied by initial V/K. Like Figure 1, this plot also contains some interesting results.

1) The high credit quality of firms not only generates a low default probability of each firm, but also implies a low default correlation between firms for typical time horizons. The intuition behind this result is similar to that behind the low default correlation over short time horizon. For high credit quality firms, the conditional default probability $P(D_2|D_1)$ is small because even though the default of firm 1 signals that V_2 may have moved downward to the default boundary, it still has a long way to go to cause the default of firm 2. This result matches the well known empirical feature regarding the relationship between default correlation and credit ratings.

2) The time to reach the peak default correlation depends on the credit quality of the underlying firms. Generally, the high quality firms take a longer time to reach the peak. This result is consistent with the empirical finding of Lucas (1995) who thought that this finding was very puzzling. Now we know that this result is obtained because for high credit quality firms, it takes longer time for $P(D_1 \cdot D_2)$ to approach to $P(D_1) \cdot P(D_2)$.

3) Since the credit quality of firms is time-varying, the default correlation which depends on the credit quality is very dynamic.

The above results have many useful implications for credit analysis and risk management. Some examples are listed as follows.

1) The default correlation over a short horizon is often very small. As a result, portfolio

[2] The correlation between D_1 and D_2 is the same as the correlation between $1 - D_1$ and $1 - D_2$.

diversification can be very beneficial. However, diversifying the portfolio within an industry or diversifying across different industries has little impact on default risks.[3]

2) For long term investments (e.g., five to ten years), the default correlation can be quite a significant factor if the underlying firm values are highly correlated. In this case, concentration in one industry or one region could be very dangerous. Diversification across different regions or different industries is very desirable.

3) The dynamic nature of default correlations requires the active risk management of a portfolio. Consider a hypothetical loan portfolio which consists of two loans. Suppose that the annual default probability for each loan was 1% and the annual default correlation was 10% originally. Using the previous analysis, we know that the probability that both loans would default in a year was about 0.11%. Assume that the credit standings of the two loans have deteriorated recently so the annual default probability for each loan is now 2%. If the default correlation did not change, the probability that both loans default in a year would become 0.24%, about twice as large as the original probability. However, we know that declines in credit quality will also lead to increases in the default correlation. If the default correlation increases from 0.10 to 0.25, the joint probability of default will become 0.53%, five times as large as the original probability.

4) The dynamic nature of default correlations also provides some guidance for setting up capital requirements. Since the change in individual default risk may substantially affect the credit risk of a portfolio as shown above, the capital requirements must be adjusted accordingly.

We now use the historical data to test the theoretical model to see if the model can generate reasonable default correlations. The default data set here is obtained from Moody's default studies reported in Fons (1994). Using this data set, default correlations for various rating categories are estimated and are compared with the empirical results of Lucas (1995).

[3] Of course, different diversification strategies may still have important effects on the risk of price changes.

Table 1 reports cumulative default rates for 1 to 20 years for Moody's broad rating categories adopted from Fons (1994). These estimates are derived from Moody's default data covering the years 1970 through 1993.

Table 2 reports the standardized default distances Z derived from statistical default data using the approach described in Subsection 4.2. As expected, a high credit rating generally implies a high value of Z, or a long distance to default. The only exception is for Aaa and Aa rating categories. Table 2 shows that Z is 9.28 for Aaa rated firms and is 9.38 for Aa rated firms. This abnormal finding is mainly due to the statistical errors in default rates data. As we can see from Table 1, statistical default rates for Aaa rated firms are constantly higher than those for Aa rated firms after 15 years. Because of this anomaly, we combine the two rating categories in the following default correlation analysis. We use Aa to represent this combined rating category and use 9.30 as its Z-value.

The implied default correlations based on the Z-values in Table 2 are reported in Tables 3 through 7. These Tables show a similar pattern of default correlations to that reported in Lucas (1995) over short to middle investment horizons. Default correlations for highly rated firms are virtually zero at the short to middle investment horizons, but default correlations are pretty high for lowly rated firms even for short investment horizons. However, we do find some significant differences between the calibrated correlations reported here and those estimated default correlations in Fons (1994) for very long investment horizons, that is, the calibrated default correlations over long investment horizons (say 10 years) for highly rated firms are substantially higher than those estimated by Lucas. There are several potential explanations for these differences. One explanation is that the estimated long horizon default correlations in Fons (1994) contain large estimation errors. As noted by Fons himself, the fifteen overlapping time periods used in his study are possibly too short for the ten-year statistics, so the historical statistics describe only observed phenomena, not the true underlying correlation relationship. Another explanation is that Z and/or ρ used in calibrations are not their true values. The inappropriate choice of parameters may have a larger effect on calibrated correlations for some time horizons than on calibrated correlations for other horizons. Of course, it is also possible that the model itself is misspecified so that it cannot precisely estimate default correlations for certain kinds of firms over long horizons. But so far, we do not have evidence to prove that.

4 A Comparison with Merton-Style Model

Because of its simplicity, Merton's default model has been widely used by practitioners in credit risk analysis. Merton assumes that a firm has only one bond issue and can only default at the maturity of the bond. This assumption makes it hard to use the original Merton model to determine the default probability of a bond over any time horizon shorter than its remaining maturity. To overcome this limitation, practitioners simply assume that bond can only default at the end of any given time horizon. That is, to estimate the one-year default probability of a bond, they assume that bond can only default at the end of the year; to estimate the five-year default probability of a bond, they assume that default may only occur at the end of the fifth year. Can this approach yield good approximations for default correlations? We use some numerical simulations to answer this question.

Table 8 provides a comparison between default correlations implied by the Merton model and the first-passage model with given Z-scores. According to Table 2, $Z = 8$ roughly corresponds to A-rated firms and $Z = 3$ corresponds to certain low grade firms. Table 8 shows that the Merton approach generally underestimates default correlations. This is because for any given Z, the Merton approach always underestimates default probability because it only allows firm to default at a given point in time.

According to Table 8, with Z's being low or time horizons in consideration being relatively long, default correlations implied by the Merton approach are typically 20-30 percent lower than those implied by the first-passage model. For instance, with $Z = 3$ and a two-year investment horizon, the default correlation implied by the Merton approach is 9.6%, but the default correlation implied the first-passage approach is 12.2%. The two models generate similar default correlations for high grade firms over short time horizons. In this case, default correlations are virtually zero anyway.

The J.P. Morgan's CreditMetrics uses observed default rates for various credit ratings at a given time horizon (say one year) to back out Z-scores via the Merton model and then uses these obtained Z-scores to estimate default correlations between any two credit categories. This approach does not use firm specific information and relies solely on rough credit classification data. In principle, the same strategy can also be applied to the first-passage model. An interesting question is: How close are default correlations implied by the two different models? By answering this question, we may gain some insight about the performance of Merton's model to estimate default correlations.

It is straightforward to verify that the default correlation obtained by the above approach depends only on the default rate and the asset level correlation and not on the time horizon. More specifically, for any given ρ, the default correlation over a one year horizon with a 10% yearly default rate is just the same as the default correlation over a five year horizon with a 10% five-year cumulative default rate. For this reason, we just need to investigate default correlations obtained by the two models under different cumulative default rates.

Table 9 shows that for various default rates, the default correlations obtained by the Merton approach are only slightly higher than those obtained by the first-passage model. This is because on the one hand, the Merton model implies lower Z-scores and tends to overestimate default correlations; on the other hand, the model ignores possible early defaults and therefore tends to underestimate default correlations. The two effects offset each other and the net effect seems insignificant.

Interestingly, we find that the first-passage approach is even computationally more efficient than the Merton approach in estimating default correlations. That is, the analytical solution of the first-passage model is 8 to 10 times as efficient as the Merton approach for evaluating joint default probabilities and default correlations. This is mainly because the

Merton approach involves the numerical evaluation of multiple infinite integrals. For this reason, the first-passage model could be a better choice even if one just wants to estimate market-wide default correlations between various credit ratings.

5 Conclusion

This paper offers an analytical formula for calculating default correlations. Since the formula can be implemented very easily, it provides a convenient tool for credit analysis. The result of this paper also provides a theoretical justification for many empirical results found in the literature and increases our understanding about the important features of default correlations.

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Appendix

The appendix provides proof for the main technical result.

Lemma 1 *The density function $f(x_1, x_2, t)$ satisfies:*

$$f(b_1, x_2, t) = f(x_1, b_2, t) = 0.$$

Proof of Lemma 1: As an example, we just prove that $f(b_1, x_2, t) = 0$. The equality $f(x_1, b_2, t) = 0$ can be proved in the same way.

Suppose to the contrary that for $x_1 \in (b_1 - \Delta x_1, b_1)$ we have

$$f(x_1, x_2, t) > \eta > 0, \quad t_1 \leq t \leq t_2,\ x_{21} \leq x_2 \leq x_{22}.$$

Then for a small time interval $(t, t + \Delta t)$ contained in (t_1, t_2), the probability $p(t)\Delta t$ that the particle is absorbed is approximately the probability that the particle is near b_1 at time t and that the increment in x_1 carries the particle beyond b_1. Certainly we have

$$\begin{aligned} p(t)\Delta t \quad &\geq \quad Prob[x_1(t + \Delta t) - x_1(t) > \Delta x_1 | x_1(t) \in (b_1 - \Delta x_1, b_1), x_2(t) \in (x_{21}, x_{22})] \\ &\quad \cdot Prob[x_1(t) \in (b_1 - \Delta x_1, b_1), x_2(t) \in (x_{21}, x_{22})] \\ &> \quad \eta(x_{22} - x_{21})\Delta x_1 \cdot Prob(x_1(t + \Delta t) - x_1(t) > \Delta x_1). \end{aligned} \tag{20}$$

Denote $p_0 = Prob[x_1(t + \Delta t) - x_1(t) > \sigma_1 \sqrt{\Delta t}]$. Since $x_1(t + \Delta t) - x_1(t) \sim N(0, \sigma_1^2 t)$, we have $p_0 = 1 - N(1) > 0$. Hence if we take $\Delta x_1 = \sigma_1 \sqrt{\Delta t}$ in equation (20), we have

$$p(t)\Delta t > \eta(x_{22} - x_{21}) p_0 \sigma_1 \sqrt{\Delta t}.$$

This implies that $p(t)$ is infinite for $t \in (t_1, t_2)$. But $p(t)$ is in fact the probability density of the first passage time to certain barriers, so we have a contradiction. That is, $f(b_1, x_2, t)$ must be zero. □

As a result, the transition probability density $f(x_1, x_2, t)$ satisfies the PDE (eq. (12)):

$$\frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial f}{\partial t}$$
$$(x_1 < b_1,\ x_2 < b_2), \tag{21}$$

subject to the following boundary conditions:

$$\begin{aligned}
& f(-\infty, x_2, t) = f(x_1, -\infty, t) = 0, \\
& f(x_1, x_2, 0) = \delta(x_1)\delta(x_2), \\
& \int_{-\infty}^{b_1}\int_{-\infty}^{b_2} f(x_1, x_2, t)\,dx_1 dx_2 \le 1, \quad t > 0, \\
& f(b_1, x_2, t) = f(x_1, b_2, t) = 0.
\end{aligned} \tag{22}$$

Theorem 1 *The solution to P.D.E. (12) subject to conditions (22) is given by*

$$f(x_1, x_2, t) = \frac{2}{\sigma_1\sigma_2\sqrt{1-\rho^2}\alpha t} e^{-\frac{r^2+r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right)\sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right), \tag{23}$$

where

$$x_1 = b_1 - \sigma_1\left(\sqrt{1-\rho^2}r\cos(\theta) + \rho r \sin(\theta)\right),$$

$$x_2 = b_2 - \sigma_2 r \sin(\theta),$$

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{b_1\sigma_2 - \rho b_2\sigma_1}\right) & \textit{if } (.) > 0 \\ \pi + \tan^{-1}\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{b_1\sigma_2 - \rho b_2\sigma_1}\right) & \textit{otherwise,} \end{cases} \tag{24}$$

$$r_0 = b_2/[\sigma_2\sin(\theta_0)],$$

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \textit{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \textit{otherwise.} \end{cases} \tag{25}$$

Proof of Theorem 1: To solve PDE (12), we define

$$u_1 = \frac{x_1}{\sigma_1}$$

and

$$u_2 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_2}{\sigma_2} - \rho\frac{x_1}{\sigma_1}\right).$$

Accordingly, we have

$$\frac{1}{2}\frac{\partial^2 f}{\partial u_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial u_2^2} = \frac{\partial f}{\partial t}. \tag{26}$$

The absorbing barriers are now the lines

$$u_1 = \frac{b_1}{\sigma_1},\ u_2 = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{b_2}{\sigma_2} - \rho\frac{u_1}{\sigma_1}\right),$$

which are generally not at right angles. The following transformations will put the intersection point of the barriers to the origin

$$v_1 = u_1 - \frac{b_1}{\sigma_1}$$

$$v_2 = u_2 - \frac{1}{\sqrt{1-\rho^2}}\left(\frac{b_2}{\sigma_2} - \rho\frac{u_1}{\sigma_1}\right).$$

The rotation through angle

$$\beta = \pi + \tan^{-1}\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)$$

gives

$$w_1 = -\sqrt{1-\rho^2}v_1 + \rho v_2,$$

and

$$w_2 = -\rho v_1 - \sqrt{1-\rho^2}v_2.$$

Under these rigid transformations, P.D.E. (26) does not change form. The particle now starts from some point $[w_1(0), w_2(0)]'$ away from the origin and is absorbed at the boundaries

$$w_1 = 0,\ w_2 = -\frac{\rho}{\sqrt{1-\rho^2}}w_1.$$

Based on above transformations, we obtain:

$$\begin{aligned} x_1 &= b_1 - \sigma_1(\sqrt{1-\rho^2}w_1 + \rho w_2), \\ x_2 &= b_2 - \sigma_2 w_2. \end{aligned} \tag{27}$$

Letting

$$w_1 = r\cos(\theta)$$

$$w_2 = r\sin(\theta)$$

P.D.E. (26) becomes

$$\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = 2\frac{\partial f}{\partial t} \tag{28}$$

subject to boundary conditions:

$$\begin{aligned} & f(r,0,t) = f(r,\alpha,t) = f(\infty,\theta,t) = 0, \\ & f(r,\theta,0) = \delta(r - r_0)\delta(\theta - \theta_0), \\ & \int_{\theta=0}^{\alpha}\int_{r=0}^{\infty} J \cdot f(r,\theta,t)\,dr\,d\theta \le 1, \quad t > 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned} x_1(r,\theta) &= b_1 - \sigma_1\left(\sqrt{1-\rho^2}\,r\cos(\theta) + \rho r\sin(\theta)\right), \\ x_2(r,\theta) &= b_2 - \sigma_2 r\sin(\theta), \\ \alpha &= \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise,} \end{cases} \end{aligned} \tag{30}$$

and

$$J = r\sigma_1\sigma_2\sqrt{1-\rho^2}$$

is the Jacobian of the overall transformation from (x_1, x_2) to (r, θ).

Solving P.D.E. (28), we obtain[4]:

$$f = \frac{2r}{J \cdot \alpha \cdot t} e^{-\frac{r^2 + r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right). \tag{31}$$

[4]The process for solving P.D.E. (28) is very lengthy and complicated. We omit this process because the solution can be verified without resorting to the solving process. Those readers who are interested in knowing the process for solving this particular P.D.E. can contact us.

Theorem 1 then follows immediately. □

Proof of the Main Result: According to Theorem 1

$$
\begin{aligned}
F(b_1, b_2, t) &= \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} f(x_1, x_2, t) dx_1 dx_2 \\
&= \int_{\theta=0}^{\alpha} \int_{r=0}^{\infty} J \cdot f(r, \theta, t) d\theta dr \\
&= \frac{2}{\alpha \cdot t} e^{-\frac{r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \int_{\theta=0}^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta \int_{r=0}^{\infty} re^{-\frac{r^2}{2t}} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) dr,
\end{aligned}
$$

where $J = r\sigma_1\sigma_2\sqrt{1-\rho^2}$ is the Jacobian of the transformation as defined before.

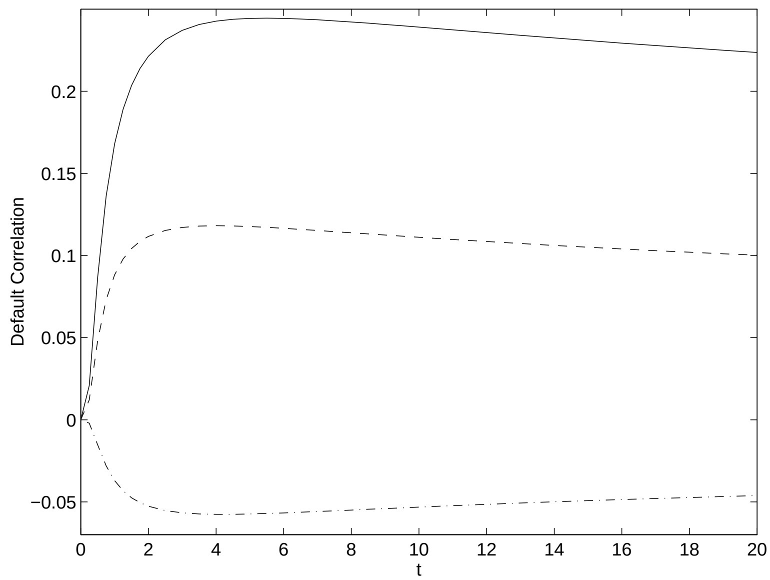
Using identities

$$
\int_{\theta=0}^{\alpha} \frac{n\pi}{\alpha} \sin(\frac{n\pi\theta}{\alpha}) d\theta = 1 - (-1)^n,
$$

and

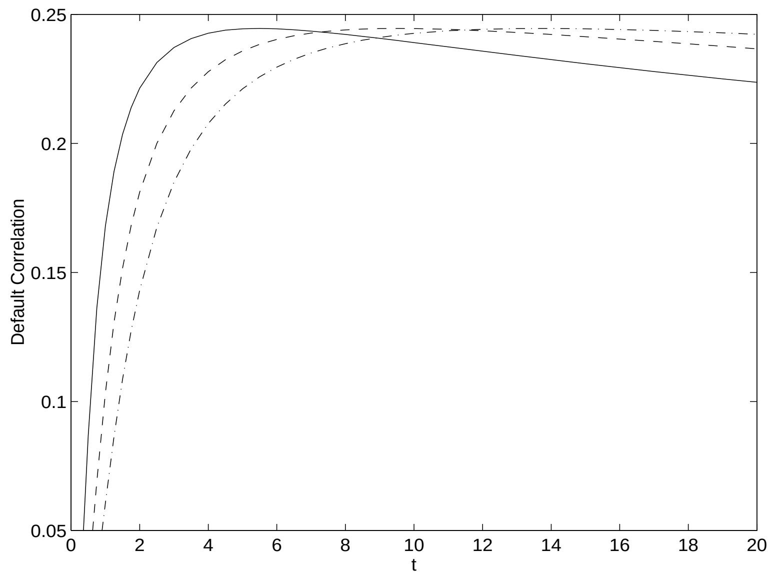
$$
\int_0^{\infty} re^{-c_1 r^2} I_v(c_2 r) dr = \frac{c_2}{8c_1} \sqrt{\frac{c_2^2}{8c_1}} e^{\frac{c_2^2}{8c_1}} \left[I_{\frac{1}{2}(v+1)}\left(\frac{c_2^2}{8c_1}\right) + I_{\frac{1}{2}(v-1)}\left(\frac{c_2^2}{8c_1}\right) \right],
$$

one obtains the main result immediately. □



—: $\rho = 0.4$ − −: $\rho = 0.2$ − ·: $\rho = -0.1$

Figure 1: **The Relationship between Default Correlation and the Correlation at the Asset level.** The parameter values used here are $V_1/K_1 = V_2/K_2 = 1.8$ and $\sigma_1 = \sigma_2 = 0.4$.



—: $\frac{V_1}{K_1} = \frac{V_2}{K_2} = 2.0$ – –: $\frac{V_1}{K_1} = \frac{V_2}{K_2} = 2.5$ – ·: $\frac{V_1}{K_1} = \frac{V_2}{K_2} = 3.0$

Figure 2: **The Relationship between Default Correlation and the Credit Quality.**

The parameter values used here are $\rho = 0.4$ and $\sigma_1 = \sigma_2 = 0.4$.

Table 1: **Historical Cumulative Default Rates (%), 1970-93.**

year	Aaa	Aa	A	Baa	Ba	B
1	0.00	0.02	0.01	0.16	1.79	8.31
2	0.00	0.04	0.09	0.51	4.38	14.85
3	0.00	0.08	0.28	0.91	6.92	20.38
4	0.04	0.20	0.46	1.46	9.41	24.78
5	0.12	0.32	0.62	1.97	11.85	28.38
6	0.22	0.43	0.83	2.46	13.78	31.88
7	0.33	0.52	1.06	3.09	15.33	34.32
8	0.45	0.64	1.31	3.75	16.75	36.71
9	0.58	0.76	1.61	4.39	18.14	38.38
10	0.73	0.91	1.96	4.96	19.48	39.96
11	0.90	1.09	2.30	5.56	20.84	41.08
12	1.09	1.29	2.65	6.19	22.22	41.74
13	1.30	1.51	2.99	6.77	23.54	42.45
14	1.55	1.76	3.29	7.44	24.52	43.04
15	1.84	1.76	3.62	8.16	25.46	43.70
16	2.18	1.76	3.95	8.91	26.43	44.43
17	2.38	1.89	4.26	9.69	27.29	45.27
18	2.63	2.05	4.58	10.45	28.06	45.58
19	2.63	2.24	4.96	11.07	28.88	45.58
20	2.63	2.48	5.23	11.70	29.76	45.58

Data Source: Fons (1994).

Table 2: **Z-values Implied by Historical Default Rates.**

year	Aaa	Aa	A	Baa	Ba	B
1	9.28	9.38	8.06	6.46	3.73	2.10

Table 3: **One Year Default Correlations (%).**

	Aa	A	Baa	Ba	B
Aa	0.00				
A	0.00	0.00			
Baa	0.00	0.00	0.00		
Ba	0.00	0.00	0.01	1.32	
B	0.00	0.00	0.00	2.47	12.46

Asset Level Correlation ρ=0.4.

Table 4: **Two Year Default Correlations (%).**

	Aa	A	Baa	Ba	B
Aa	0.00				
A	0.00	0.02			
Baa	0.01	0.05	0.25		
Ba	0.00	0.05	0.63	6.96	
B	0.00	0.02	0.41	9.24	19.61

Asset Level Correlation ρ=0.4.

Table 5: **Three Year Default Correlations (%).**

	Aa	A	Baa	Ba	B
Aa	0.04				
A	0.08	0.21			
Baa	0.13	0.44	1.32		
Ba	0.09	0.48	2.48	11.85	
B	0.05	0.28	1.81	13.82	22.25

Asset Level Correlation ρ=0.4.

Table 6: **Five Year Default Correlations (%).**

	Aa	A	Baa	Ba	B
Aa	0.59				
A	0.92	1.65			
Baa	1.24	2.60	5.01		
Ba	1.05	2.74	7.20	17.56	
B	0.65	1.88	5.67	18.43	24.01

Asset Level Correlation ρ=0.4.

Table 7: **Ten Year Default Correlations (%).**

	Aa	A	Baa	Ba	B
Aa	4.66				
A	5.84	7.75			
Baa	6.76	9.63	13.12		
Ba	5.97	9.48	14.98	22.51	
B	4.32	7.21	12.28	21.80	24.37

Asset Level Correlation ρ=0.4.

Table 8: **Default Correlations (%) Implied by Different Models**

Z-Scores Are Given

(Z_1, Z_2)	Model	Time Horizon (Years)					
		1	2	3	4	5	10
(8,8)	Merton	0.00	0.01	0.17	0.60	1.30	6.10
(8,8)	Current	0.00	0.02	0.23	0.80	1.72	7.93
(3,3)	Merton	3.25	9.61	13.6	16.2	17.9	21.7
(3,3)	Current	4.29	12.2	16.8	19.5	21.1	24.0

Asset Level Correlation ρ=0.4.

Table 9: **Default Correlations (%) Implied by Different Models**

Default Rates Are Given

Model	Default Rates (%)						
	0.1	0.5	1.0	5.0	10	20	40
Merton	2.85	5.77	7.74	14.58	18.50	22.63	25.86
Current	2.77	5.60	7.51	14.10	17.82	21.65	24.34

Asset Level Correlation ρ=0.4.