A generalization of generalized beta distributions

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Abstract

This paper introduces the "compound confluent hypergeometric" (CCH) distribution. The CCH unifies and generalizes three recently introduced generalizations of the beta distribution: the Gauss hypergeometric (GH) distribution of Armero and Bayarri (1994), the generalized beta (GB) distribution of McDonald and Xu (1995), and the confluent hypergeometric (CH) distribution of Gordy (forthcoming). In addition to greater flexibility in fitting data, the CCH offers two useful properties. Unlike the beta, GB and GH, the CCH allows for conditioning on explanatory variables in a natural and convenient way. The CCH family is conjugate for gamma distributed signals, and so may also prove useful in Bayesian analysis. Application of the CCH is demonstrated with two measures of household liquid assets. In each case, the CCH yields a statistically significant improvement in fit over the more restrictive alternatives.

JEL Codes: C10, D91, G11

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The beta distribution is widely used in statistical modeling of bounded random variables. It is easily calculated, can take on a variety of shapes, and, perhaps as importantly, none of the other commonly used distribution functions have compact support.[1] However, its application is limited in important ways. First, as a two parameter distribution, it can provide only limited precision in fitting data. It is desirable to have more parametrically flexible versions of the beta to allow a richer empirical description of data while still offering more structure than a nonparametric estimator. Second, the beta does not offer a natural and convenient means of introducing explanatory variables. In a Beta(p,q) distribution, the parameters (p,q) jointly determine both the shape and moments of the distribution. There is no satisfactory way of conditioning the mean by specifying p and q as functions of explanatory variables and regression coefficients. Third, the beta is inconvenient for use in Bayesian analysis. It is conjugate for binomial signals, but not for signals of any continuous distribution.

Recent research has contributed three generalizations of the beta which address one or more of these limitations. Armero and Bayarri (1994) define the Gauss hypergeometric (GH) distribution by the density function

$$GH(x;p,q,r,\lambda) = \frac{x^{p-1}(1-x)^{q-1}(1+\lambda x)^{-r}}{\mathrm{B}(p,q)\,{}_2F_1(r,p;p+q;-\lambda)} \quad \text{for } 0 < x < 1 \tag{1}$$

with $p > 0$, $q > 0$ and where ${}_2F_1$ denotes the Gauss hypergeometric function. The GH collapses to the ordinary beta if either $r = 0$ or $\lambda = 0$, and to the beta-prime if $q = \lambda = 1$. Armero and Bayarri apply the GH to a Bayesian queuing theory problem.

A related distribution is introduced by McDonald and Xu (1995) as the "generalized beta"

[1] Except, of course, the uniform distribution, which is a special case of the beta.

(GB) distribution. The GB is defined by the pdf

$$GB(x;a,b,c,p,q) = \frac{|a|x^{ap-1}(1-(1-c)(x/b)^a)^{q-1}}{b^{ap}\mathrm{B}(p,q)(1+c(x/b)^a)^{p+q}} \quad \text{for } 0 < x^a < b^a/(1-c) \tag{2}$$

and zero otherwise with $0 \leq c \leq 1$ and b, p and q positive. As in the ordinary beta distribution, the parameters p and q control shape and skewness. Parameters a and b control "peakedness" and scale, respectively. Given $a = b = 1$, the parameter c shifts the GB from the ordinary beta distribution ($c = 0$) to the beta-prime distribution ($c = 1$).

Gordy (forthcoming) generalizes the beta in an unrelated direction. The "confluent hypergeometric" distribution $\mathrm{CH}(p, q, s)$ is defined by the pdf

$$CH(x;p,q,s) = \frac{x^{p-1}(1-x)^{q-1}\exp(-sx)}{\mathrm{B}(p,q)\ {}_1F_1(p,p+q,-s)} \quad \text{for } 0 < x < 1. \tag{3}$$

where ${}_1F_1$ is the confluent hypergeometric function defined in Abramowitz and Stegun, eds (1968, 13.1.2) (hereafter cited as "AS").[2] Gordy (forthcoming) shows that a beta prior and gamma signal gives rise to a CH posterior and applies this property to auction theory.

In this paper, I unify and further generalize these three distributions. The constant of proportionality in the new pdf is the product of a beta function and a compound confluent hypergeometric function. Therefore, I denote this distribution the "compound confluent hypergeometric" (CCH). The CCH is defined and described in Section 1. Special cases are discussed in Section 2. In particular, I show that the beta, GB, GH, CH and gamma distributions are all special cases of the CCH. Empirical applications to measures of household liquid assets are provided in Section 3.

[2] This function is denoted there as M and referred to as the "degenerate" hypergeometric function Φ in Gradshteyn and Ryzhik (1965, 9.210.1). In *Mathematica*, it is the `Hypergeometric1F1`.

1 Definition of the CCH

I define the CCH by the density function

$$CCH(x; p, q, r, s, \nu, \theta) = \frac{x^{p-1}(1-\nu x)^{q-1}(\theta + (1-\theta)\nu x)^{-r}\exp(-sx)}{\mathrm{B}(p,q)H(p,q,r,s,\nu,\theta)} \quad \text{for } 0 < x < 1/\nu \tag{4}$$

for $0 < p$, $0 < q$, $r \in \Re$, $s \in \Re$, $0 \leq \nu \leq 1$, and $0 < \theta$.[3] The function H is given by

$$H(p, q, r, s, \nu, \theta) = \nu^{-p}\exp(-s/\nu)\Phi_1(q, r, p+q, s/\nu, 1-\theta) \tag{5}$$

where Φ_1 is the confluent hypergeometric function of two variables defined in Gradshteyn and Ryzhik (1965, 9.261.1) by

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_n}{(\gamma)_{m+n}m!n!}x^m y^n \tag{6}$$

and where $(a)_k$ is Pochhammer's notation, i.e., $(a)_0 = 1$, $(a)_1 = a$, $(a)_k = (a)_{k-1}(a+k-1)$. For convenience in exposition, I refer to H as a "compound confluent hypergeometric" function rather than as a "confluent hypergeometric function of two variables."

In Appendix A, I show that equation (4) integrates to one. To guarantee that the CCH distribution is well-defined everywhere on the parameter space, I prove in Appendix C the theorem

Theorem 1

For all $(p, q, r, s, \nu, \theta)$ such that $p > 0$, $q > 0$, $r \in \Re$, $s \in \Re$, $0 < \nu \leq 1$ and $\theta > 0$, $H(p, q, r, s, \nu, \theta)$ is a finite positive real number.

It is straightforward to check that the moment generating function for the CCH is given by

$$M(t) = \frac{H(p, q, r, s-t, \nu, \theta)}{H(p, q, r, s, \nu, \theta)}$$

[3] $\nu = 0$ is handled as a special case. See the UH distribution in Section 2.

and the kth order moments are given by

$$E(X^k) = \frac{(p)_k}{(p+q)_k} \frac{H(p+k,q,r,s,\nu,\theta)}{H(p,q,r,s,\nu,\theta)}. \tag{7}$$

Theorem 1 is sufficient to guarantee that all moments of X exist.

Given the restrictions on the parameters, the Φ_1 function in equation (5) can always be expressed as an infinite series in which all terms are non-negative (see Appendix B). Therefore, the H function can be calculated without numerical round-off problems. Despite its apparent complexity, it is quickly calculated over most of the relevant parameter space.[4] Computation time decreases with p, ν and θ and increases with q, $|r|$ and $|s|$. It appears that ν and θ jointly have the largest effect. To take an easy example, computation of $H(p=20, q=2, r=5, s=0, \nu=1, \theta=1)$ to 14 places accuracy takes 0.0002 seconds on a SparcStation10. To take a more difficult example, $H(2, 20, 15, 10, 0.01, 0.01)$ requires 0.6 seconds. To the extent that the CCH is employed to generalize the beta distribution, rather than the beta-prime distribution, computationally easy cases will predominate in empirical applications.

Figures 1a, 1b and 1c plot the CCH pdf for a variety of parameter values. The figures show that the role of parameters p and q in the CCH is much the same as in the ordinary beta distribution. Parameter ν rescales the distribution for longer or shorter support. The remaining parameters r, s and θ "squeeze" the density function to the left or right. While the shapes portrayed in these figures are qualitatively familiar from the beta distribution, the CCH can also take on a wide range of multi-modal or long-tailed shapes which the beta cannot. Examples are presented in Figure 2.

The parameter s allows for a convenient method of conditioning the CCH distribution on exogenous variables. The bottom panel of Figure 1b shows that increasing (decreasing) s squeezes the distribution to the left (right). In Appendix D it is proved that the mean changes monotonically

[4] Software in `C` and `MATLAB` is available upon request.

Figure 1a: Compound Confluent Hypergeometric PDFs

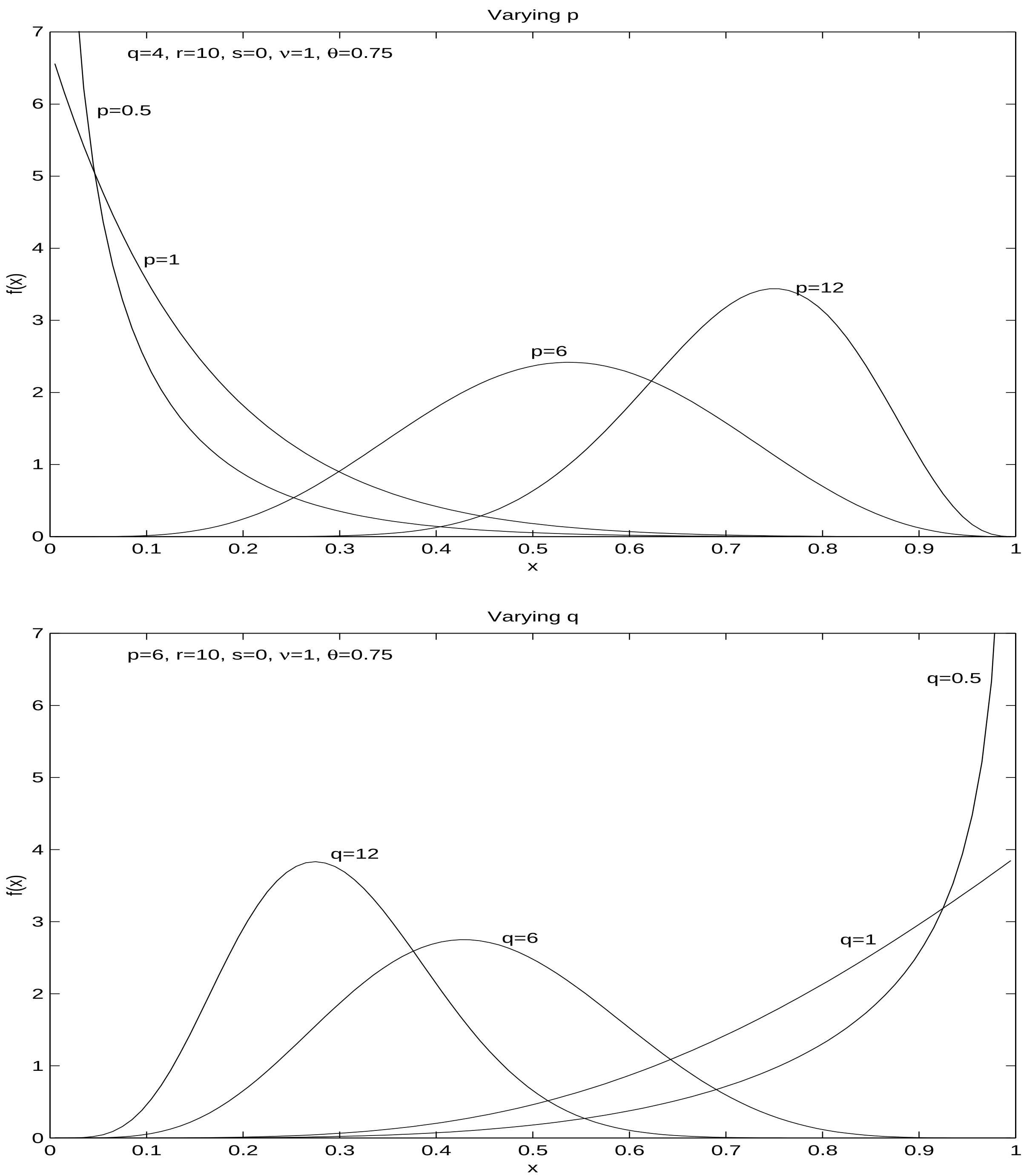
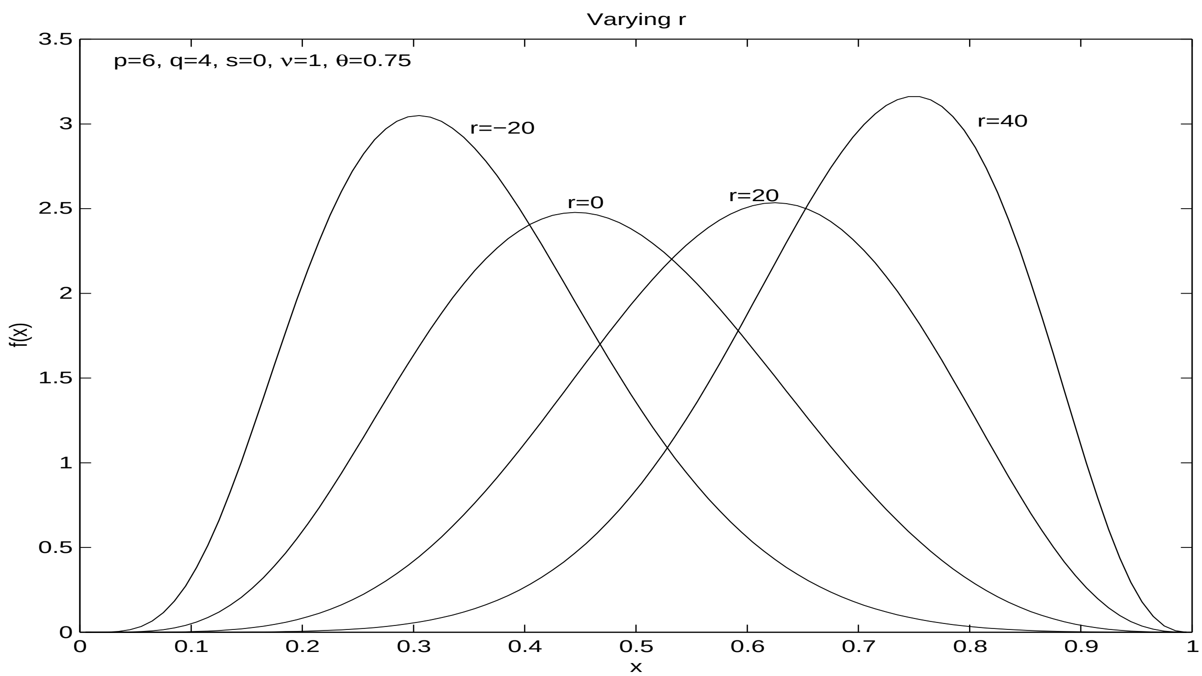


Figure 1b: Compound Confluent Hypergeometric PDFs (continued)



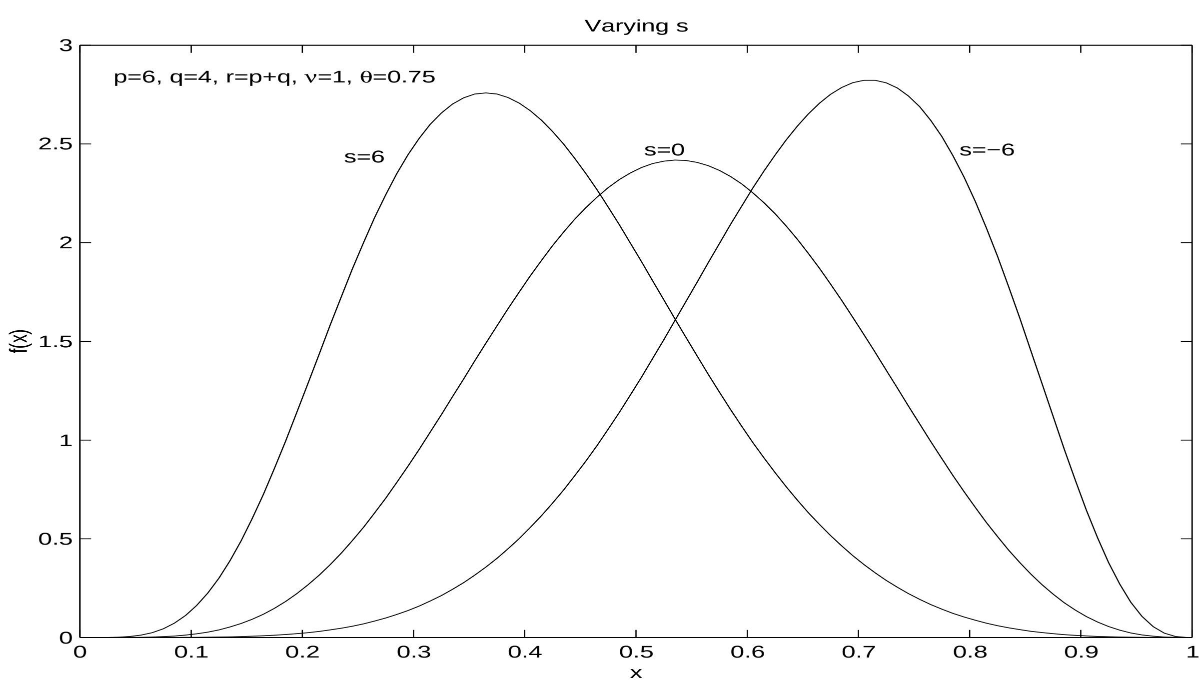


Figure 1c: Compound Confluent Hypergeometric PDFs (continued)

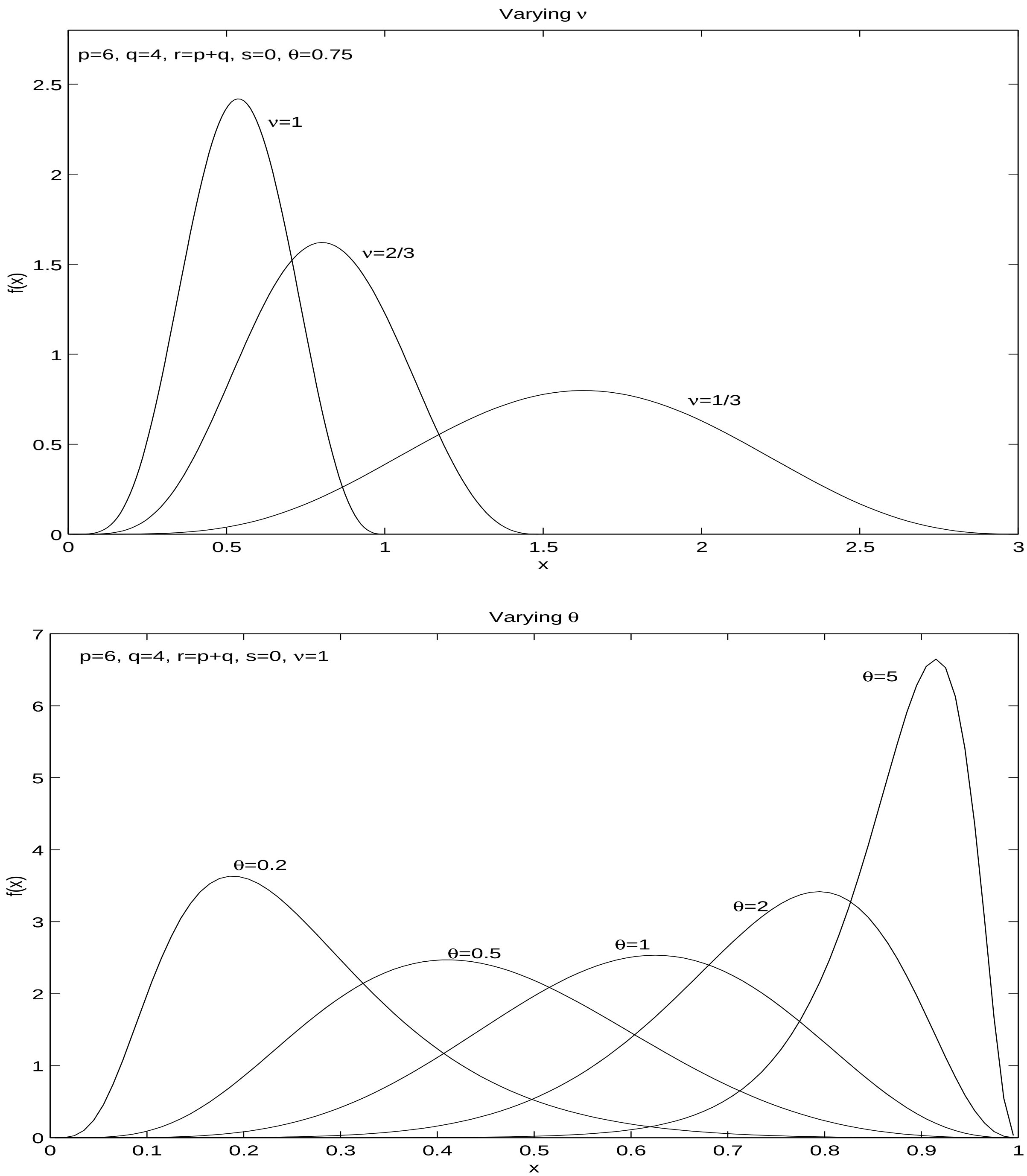
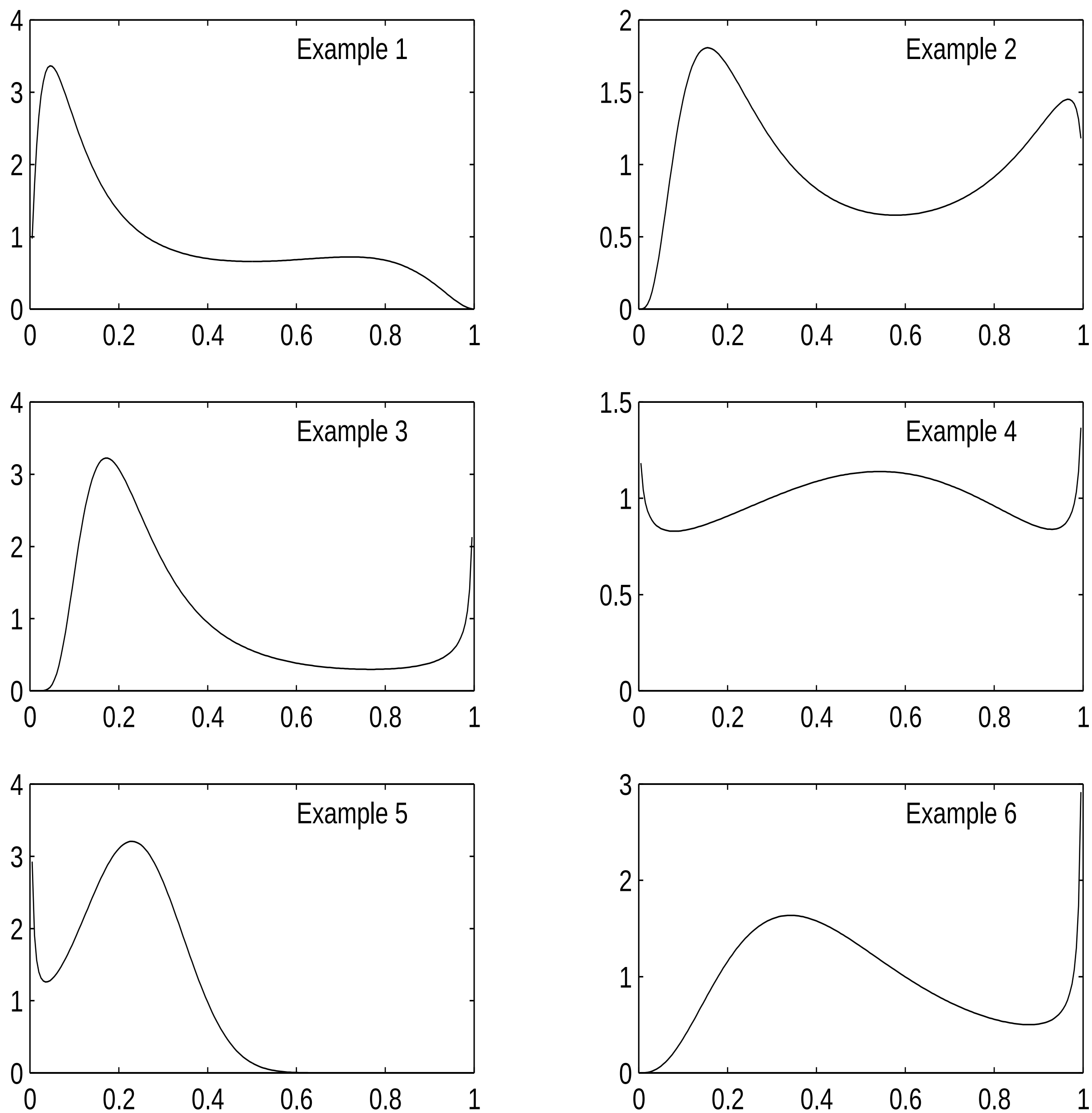


Figure 2: Multi-modal and Long-Tailed CCH PDFs



Parameter values:
Example 1: $p = 2$, $q = 3$, $r = 15$, $s = -20$, $\nu = 1$, $\theta = 0.25$.
Example 2: $p = 5$, $q = 1.2$, $r = 37$, $s = -26$, $\nu = 1$, $\theta = 0.36$.
Example 3: $p = 12$, $q = 0.4$, $r = 25$, $s = -10$, $\nu = 1$, $\theta = 0.14$.
Example 4: $p = 0.8$, $q = 0.7$, $r = -15$, $s = -10$, $\nu = 1$, $\theta = 2$.
Example 5: $p = 0.2$, $q = 25$, $r = -40$, $s = 10$, $\nu = 1$, $\theta = 0.4$.
Example 6: $p = 4$, $q = 0.2$, $r = 0$, $s = 10$, $\nu = 1$, $\theta = 2$.

with s, i.e.,

Proposition 2 $\frac{dE(X)}{ds} = -\text{Var}(X) < 0$.

Therefore, if z_t is a vector of exogenous variables, it might be assumed that $X_t \sim \text{CCH}(p, q, r, s_t, \nu, \theta)$ for $s_t = z_t\delta$. The regression coefficients δ are easily estimated along with the fixed parameters (p, q, r, ν, θ) by maximum likelihood. The derivative of the function H with respect to s has a simple analytic expression, so the computational burden of adding variables to z_t is not especially large.

Lemma 3 $\frac{d}{ds}H(p, q, r, s, \nu, \theta) = -\frac{p}{p+q}H(p+1, q, r, s, \nu, \theta)$.

The proof is in Appendix D.

Like the beta, the CCH displays a simple reflection property. If $X \sim \text{CCH}(p, q, r, s, \nu, \theta)$ and if $Y = 1/\nu - X$, then $Y \sim \text{CCH}(q, p, r, -s, \nu, 1/\theta)$. Like the CH, the CCH may be useful in Bayesian decision problems, because it is conjugate for gamma distributed signals. If the prior for an unknown random variable X is the $\text{CCH}(p, q, r, s, \nu, \theta)$ and signal $W|(X{=}x) \sim \text{Ga}(\tau, x)$, then the posterior distribution of $X|(W{=}w)$ is $\text{CCH}(p+\tau, q, r, s+w, \nu, \theta)$.

2 Special cases

Generalized Beta: When $s = 0$, $r = p + q$, $\nu = (1-c)/b$, and $\theta = 1 - c$, the CCH simplifies to the GB with $a = 1$. The CCH distribution can easily be extended to accommodate a peakedness parameter a as well, which is ignored here to simplify exposition. The restriction is relatively minor, because if $X \sim GB(a, b, c, p, q)$ then X can be rescaled as $\tilde{X} \equiv X^a \sim CCH(p, q, r{=}p+q, s{=}0, \nu{=}(1-c)/b, \theta{=}1-c)$.

Gauss Hypergeometric: When $s = 0$ and $\nu = 1$, the $\mathrm{CCH}(p, q, r, s{=}0, \nu{=}1, \theta)$ simplifies to the $\mathrm{GH}(p, q, r, \lambda)$ for $\lambda = (1 - \theta)/\theta$. The function H simplifies to

$$H(p, q, r, s{=}0, \nu{=}1, \theta) = \theta^r {}_2F_1(r, q; p + q; 1 - \theta) = {}_2F_1(r, p, p + q; -(1 - \theta)/\theta) \tag{8}$$

where the last transformation follows from AS15.3.4.

Confluent Hypergeometric: When $\nu = 1$ and $\theta = 1$, $\mathrm{CCH}(p, q, r, s, \nu{=}1, \theta{=}1)$ simplifies to the confluent hypergeometric distribution $\mathrm{CH}(p, q, s)$. It is clear from transformation (T4) in Appendix B that $H(p, q, r, s, \nu{=}1, \theta{=}1) = {}_1F_1(p, p + q, -s)$.

Gamma: When $\nu = 0$ and $s > 0$, $\mathrm{CCH}(p, q, r, s, \nu{=}0, \theta)$ simplifies to the gamma distribution $\mathrm{Ga}(p, s)$. Whereas the GB distribution includes the gamma distribution as a limiting case, the CCH includes the gamma as a special case.

Confluent HypergeometricU: An interesting limiting case of the CCH arises as $\nu \to 0$ and $\theta \to 0$ such that $\nu(1 - \theta)/\theta \to \lambda$ for a constant value λ such that $s/\lambda \geq 0$. The limiting distribution is based on the "U" confluent hypergeometric function defined in AS13.1.3.[5] Let the "UH" distribution be defined by the pdf

$$UH(x; p, r, s, \lambda) = \frac{x^{p-1}(1 + \lambda x)^{-r} \exp(-sx)}{\Gamma(p) U(p, p + 1 - r, s/\lambda)} \quad \text{for } 0 < x < \infty. \tag{9}$$

In Appendix A it is shown that this density integrates to one. Moments are given by

$$E(X^k) = (p)_k \frac{U(k + p, k + p + 1 - r, s/\lambda))}{U(p, p + 1 - r, s/\lambda))}.$$

[5] This function is denoted as the "degenerate" hypergeometric function Ψ in Gradshteyn and Ryzhik (1965, 9.210.2). In *Mathematica*, it is the `HypergeometricU`.

When $s = 0$, $\lambda = 1$ and $r = p + q$, the UH($p, r = p + q, s = 0, \lambda = 1$) simplifies to the beta-prime. When $\lambda = 0$ or $r = 0$, the UH(p, r, s, λ) for $s > 0$ simplifies to the gamma distribution Ga(p, s).

3 Empirical applications

To demonstrate the use of the CCH distribution, I model two measures of household liquid assets: the ratio of household liquid assets to total household financial assets, and the log of household liquid assets. The distributions of these variables across households are important in understanding motives for saving and the determinants of porfolio allocation. For example, the buffer-stock model of saving (Carroll 1992) and the life-cycle model (see, e.g., Poterba and Samwick 1997) differ in implications for the cross-sectional distribution of liquid assets. Cross-sectional patterns in portfolio share assigned to liquid, low risk assets may permit tests or calibrations of models of portfolio choice, such as Laibson (1997) and Bodie, Merton and Samuelson (1992). More directly, the distributions of these variables may shed light on the effects of income distribution and demographic transitions on savings and on market risk-premia, and on the importance of liquidity constraints in aggregate consumption.

All data come from the 1995 Survey of Consumer Finances. The SCF provides data on the assets and liabilities of U.S. households (see Kennickell, Starr-McCluer and Sundén (1997) for a general description). In order to provide adequate representation of the highly skewed distribution of assets, the SCF oversamples wealthy households, so in all calculations below I apply SCF sample weights to obtain unbiased estimates of the population distribution. Weights are normalized to mean one to avoid biasing the size of test statistics.[6]

I define liquid assets, LIQ, as the sum of checking deposits, savings deposits, money market and

[6]Parameters estimated on the unweighted samples do differ from the results reported below, but are qualitatively similar. In particular, LR test statistics yield the same conclusions.

money market mutual fund deposits, and call accounts at brokerages. Total financial assets, FIN, includes LIQ, stocks, bonds, trusts, cash value of whole life insurance, and other financial assets.[7] Weighted median and mean sample values are $1,600 and $13,261 for LIQ and $10,000 and $88,629 for FIN. Roughly 9% of households have no financial assets and roughly 13% have no liquid assets. Roughly 20% have positive liquid assets but no other financial assets. To bound the liquidity ratio in $(0, 1)$, I assume household i has an additional c_i dollars in "pocket cash," and another f_i dollars in small loans to friends, where c_i and f_i are iid draws from a Uniform$(0, 150)$ distribution. I then define $LIQRAT_i \equiv (LIQ_i + c_i)/(FIN + c_i + f_i)$.[8] The top panel of Figure 3 shows a U-shaped histogram for LIQRAT. Roughly half of U.S. households maintain either a very low or very high percentage of household financial assets in liquid deposits.

Table 1 presents the results of maximum likelihood estimation of distribution parameters for LIQRAT using the beta, confluent hypergeometric, Gauss hypergeometric and CCH distributions.[9] Because 0 and 1 are natural bounds for LIQRAT, ν is fixed to 1.[10] The estimated distributions are plotted in the bottom panel of Figure 3. All of the estimated distributions are U-shaped, but they do differ visibly. Furthermore, LR tests strongly reject (at well beyond 1% levels) the equivalence of the restricted distributions to the CCH. The CCH thus offers a significantly better fit to the data than beta, CH or GH.

It might be expected that LIQRAT will depend on household characteristics. Higher income households may have less need to keep savings in a highly liquid form. Higher levels of education

[7]These are standard definitions used in analysis of the SCF data. See the URL <http://www.bog.frb.fed.us/pubs/oss/oss2/95/codebk95pt5.html> at the Federal Reserve Board's web site.

[8]This procedure causes estimated parameters to vary with each set of random draws. I have run the program many times, however, and find that results appear to be stable.

[9]Standard errors are obtained using the method of Berndt, Hall, Hall and Hausman (1974). I abstract from variation associated with the inclusion of imputed values in the SCF by using only the first set of implicates. Rubin (1987) discusses correction to the standard errors for multiply imputed data. In this application, the correction should be quite small and is ignored.

[10]The GB is not included in this comparison because it is equivalent to the beta when restrictions $a = 1$ and $c/b = 1 - \nu = 0$ are imposed.

Figure 3: Distribution of Liquid Asset Ratio

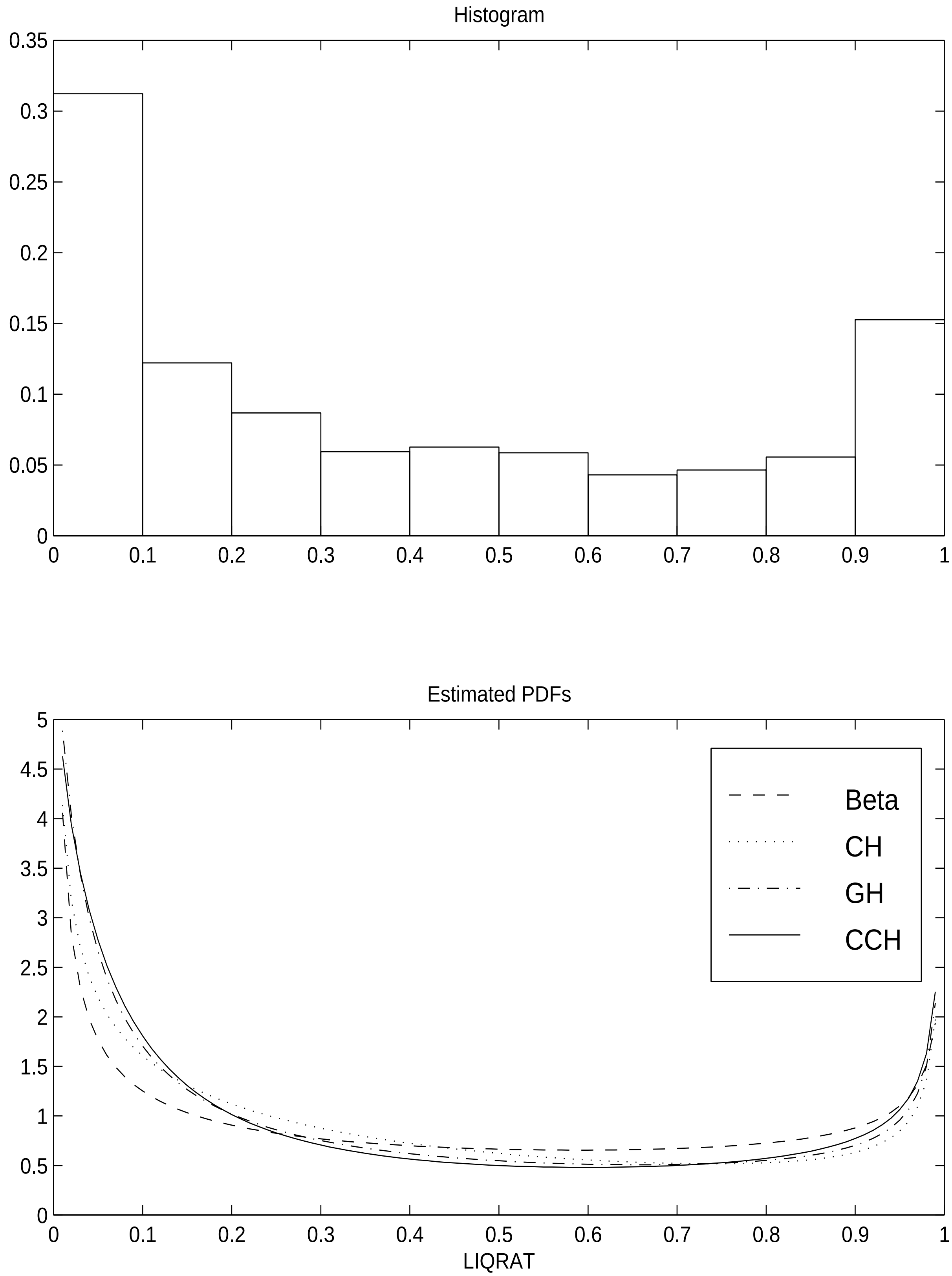


Table 1: Estimated Distributional Parameters for Liquidity Ratio[†]

	Beta	CH	GH[‡]	CCH
p	0.501	0.646	1.062	0.939
	(0.011)	(0.017)	(0.051)	(0.034)
q	0.687	0.477	0.526	0.649
	(0.011)	(0.018)	(0.015)	(0.026)
r			1.052	6.296
			(0.050)	(2.819)
s		1.585		−5.732
		(0.109)		(1.867)
θ			0.032	0.285
			(0.007)	(0.086)
LogLik:	838.08	939.87	1022.1	1042.5

†: Standard errors in parentheses. 4299 observations in sample. Sample weights are normalized to have mean one. ν is fixed at 1.

‡: The parameter λ in the standard form for the GH is given by $\lambda = (1-\theta)/\theta$.

may be associated with greater sophistication in financial matters, and so a greater propensity to hold assets other than liquid deposits. Compared to households in the 35-65 age group, older and younger households are likely to have higher LIQRAT. Older households may be more risk averse, and so prefer safer asset allocations. Younger households may be more likely to face income uncertainty and to have near-term goals for saving (e.g., downpayment for home purchase), and thus to have greater demand for both safety and liquidity. In Table 2, I present results from the estimation of the conditional distribution $LIQRAT_i \sim \text{CCH}(p, q, r, s_i, 1, \theta)$. I assume $s_i = z_i\delta$, where the z_i include household income, and dummy variables for the age and level of education of the head of household, and δ is a vector of coefficients.

Results are presented in Table 2. Estimates for p, q, r and θ are close to those for the unconditional CCH (last column of Table 1). The δ coefficients are individually and jointly different from zero at the 1% confidence level. As predicted, higher income and higher level of education shift the distribution to the left. The young (AGE$<$ 35) and the old (AGE$>$ 65) maintain higher liquidity ratios than those in between.

Table 2: Estimated Conditional Distribution for Liquidity Ratio†

	Coeff	S.E.
p	0.9450	0.0347
q	0.6497	0.0262
r	4.8953	2.0656
θ	0.2485	0.0796
CONSTANT	−7.6773	1.5430
LOG(1+INCOME)	0.3779	0.0243
AGE< 35	−0.5675	0.0973
AGE> 65	−0.2441	0.0920
EDU1 (NO HS DIPLOMA)‡	−1.1686	0.1146
EDU2 (HS DIPLOMA)‡	−0.6659	0.0987
EDU3 (SOME COLLEGE)‡	−0.4702	0.1131
LogLik:	1194.75	

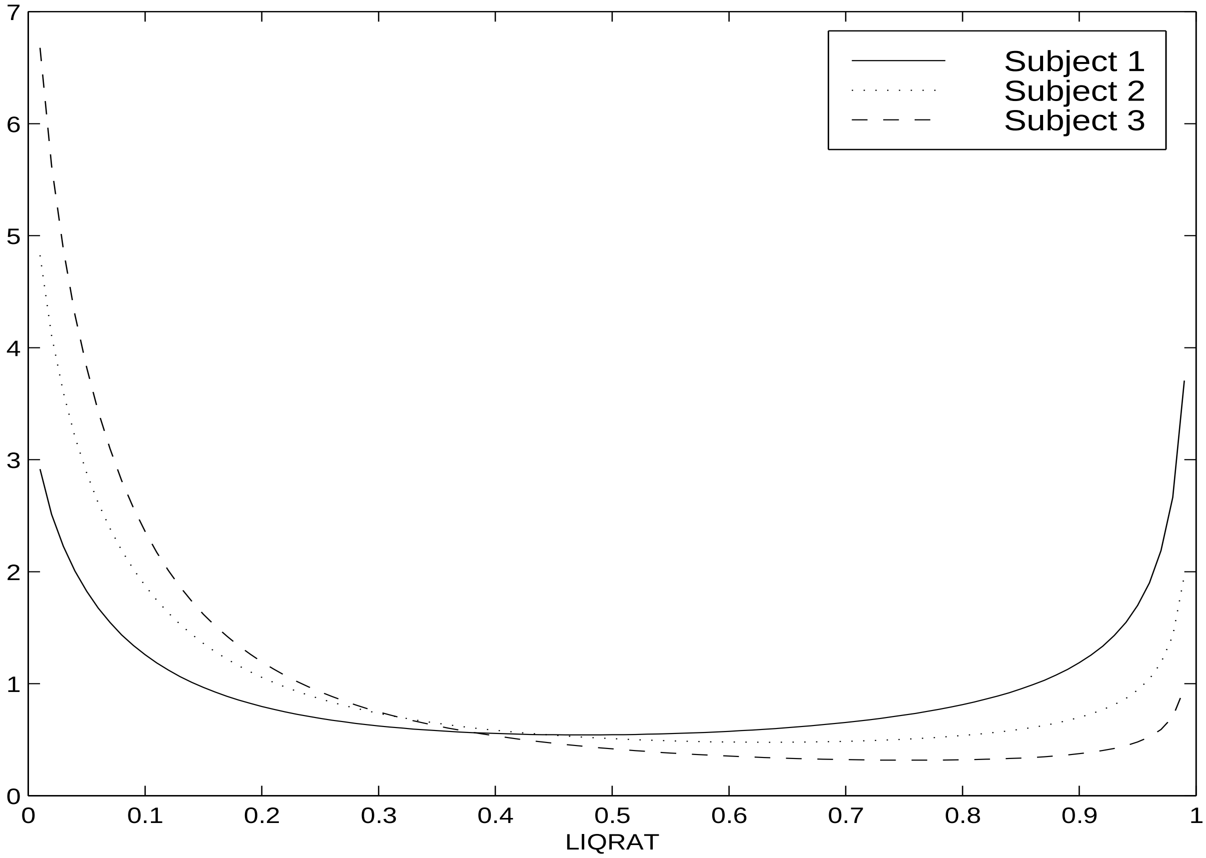
†: 4299 observations in sample. Sample weights are normalized to be mean one. ν is fixed at 1.
‡: COLLEGE DIPLOMA is the reference category for the education dummies.

Figure 4 demonstrates the effect of conditioning through s on the CCH distribution. Subject 1 is a 25 year old high school drop-out earning \$20,000 per year; subject 2 is 70 years old, had some college, and earns \$40,000 per year; and subject 3 is a 45 year old college graduate earning \$100,000 per year. Parameters are taken from Table 2. All three distributions are U-shaped, but that of Subject 1 is most tilted towards high LIQRAT and that of Subject 3 is most tilted towards low LIQRAT. The means of these three distributions are 0.31, 0.22, and 0.16, respectively.

The second measure of household liquidity is the log of total household liquid assets (including the randomly drawn amount of pocket cash). This is defined as $LOGLIQ_i \equiv \log(1 + LIQ_i + c_i)$. The top panel of Figure 5 shows a single-peaked histogram for LOGLIQ.

Table 3 presents the results of maximum likelihood estimation of distribution parameters for LOGLIQ using the gamma, generalized beta and CCH distributions. In principle, LOGLIQ is unbounded (so $\nu = 0$), but for the purposes of this example I allow ν to be estimated freely. I also

Figure 4: Conditional CCH PDFs for Liquid Asset Ratio



include in Table 3 estimates for two special cases of the CCH. The first, labelled GB(s), is the GB extended to include an $\exp(-sx)$ term. The second, labelled CH(ν), is the CH with the $(1-x)^{q-1}$ term generalized to $(1-\nu x)^{q-1}$.

The estimated distributions are plotted in the bottom panel of Figure 5. All are single-peaked, but differences are clearly visible. Once again, LR tests strongly reject (at well beyond 1% levels) the equivalence of the restricted distributions to the CCH. The CH(ν) is closest to the CCH, both in likelihood and to the eye. The GB and GB(s) appear to underperform even the gamma distribution.

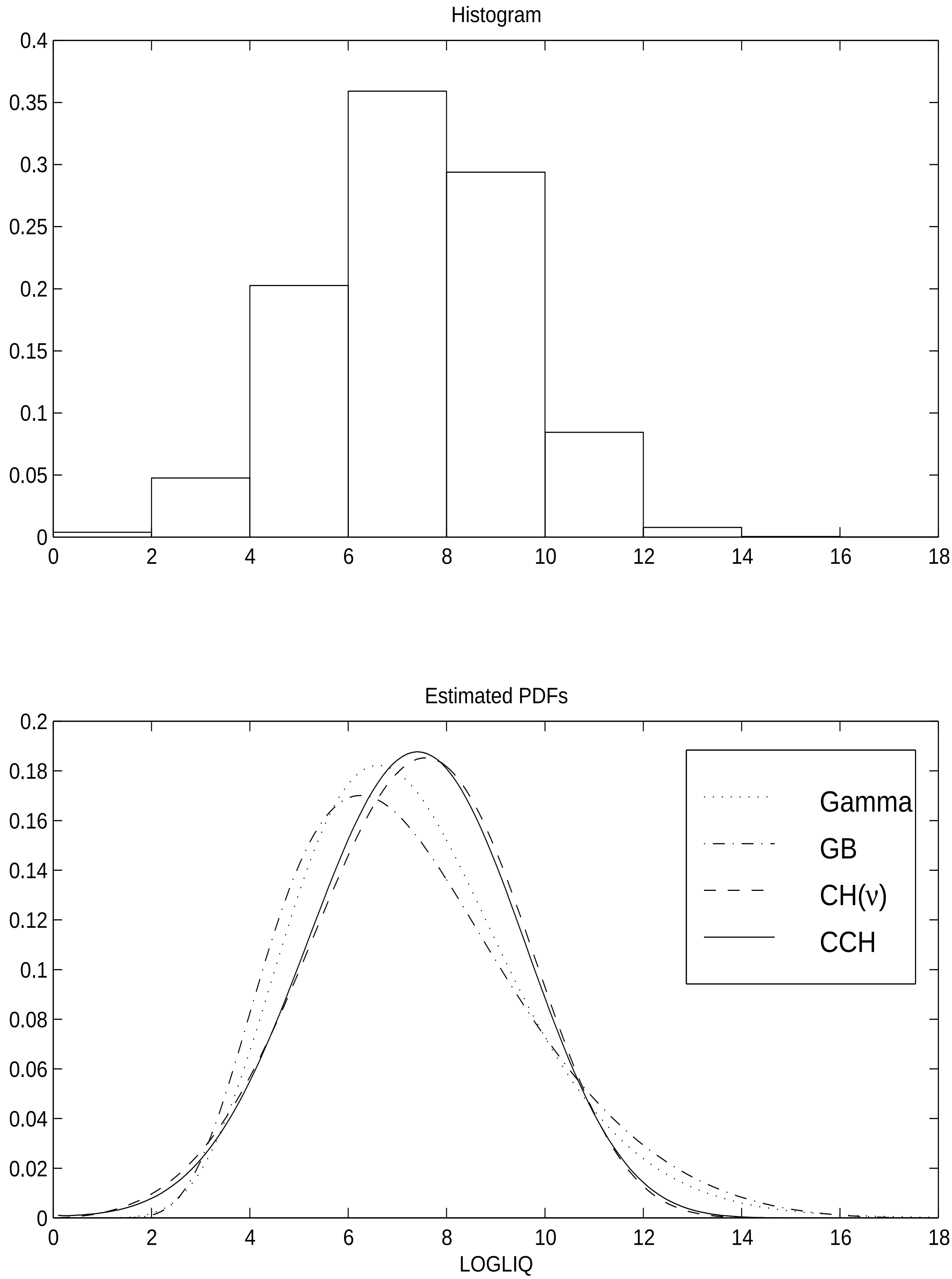
In the estimation of the full CCH, r has an especially large standard deviation. This can be seen in the estimation of the CCH distribution for LIQRAT in Table 1 as well, but is more pronounced here. One reason is simply that fairly large changes in r are needed to produce moderate changes in the distribution (see Figure 1b). However, it appears that joint identification of r, ν and s may sometimes be weak.

Table 3: Estimated Distributional Parameters for Log(Liquidity)†

	Gamma	GB‡	GB(s)‡	CH(ν)‡	CCH
p	10.284	37.562	49.417	2.480	0.446
	(0.038)	(0.309)	(0.169)	(0.150)	(0.387)
q		3.589	17.836	20.741	14.326
		(0.168)	(0.750)	(0.791)	(2.848)
r			$[p+q]$	[1]	−15.797
					(15.476)
s	1.405		−2.954	−1.686	−1.079
	(0.012)		(0.083)	(0.086)	(1.064)
ν		0.055	0.055	0.055	0.055
		(0.002)	(0.001)	(0.001)	(0.001)
θ			$[\nu]$	[1]	0.214
					(0.161)
LogLik:	−9506.4	−9779.0	−9674.7	−9260.5	−9247.9

†: Standard errors in parentheses. 4299 observations in sample. Sample weights are normalized to have mean one.
‡: The GB is estimated with restriction $a=1$; the parameter ν is equivalent to c/b in the standard form for the GB. The GB(s) distribution is the GB extended to include an $\exp(-sx)$ term. The CH(ν) distribution is the CH with the $(1-x)^{q-1}$ term generalized to $(1-\nu x)^{q-1}$. Imposed parameter restrictions are shown in hard brackets.

Figure 5: Distribution of Log Liquid Assets



Conclusion

Recent work on extending the beta distribution has given rise to three non-nested generalizations: the Gauss hypergeometric distribution of Armero and Bayarri (1994), the generalized beta distribution of McDonald and Xu (1995), and the confluent hypergeometric distribution of Gordy (forthcoming). In this paper, these divergent strands are unified into a new six parameter distribution, which I denote the "compound confluent hypergeometric" distribution. Plots of the CCH for a variety of parameter values show that the CCH density can take not only the familiar variety of U-shaped and single-peaked forms of the beta distribution, but also a variety of multi-modal and long-tailed forms.

Use of the CCH is demonstrated with two measures of household liquid assets. In each example, estimated CCH distributions are qualitatively similar to estimates using more restrictive distributions, but the additional precision in fit is highly significant statistically and clearly visible in plots of the fitted densities.

A Integrating the CCH density

Let A be the integral

$$A \equiv \int_0^{1/\nu} x^{p-1}(1-\nu x)^{q-1}(\theta+(1-\theta)\nu x)^{-r}\exp(-sx)dx \tag{10}$$

where $0<p$, $0<q$, $r\in\Re$, $s\in\Re$, $0<\nu\le 1$, and $0<\theta$. Make the change of variable $u=1-\nu x$ to get

$$A=\nu^{-p}\exp(-s/\nu)\int_0^1 (1-u)^{p-1}u^{q-1}\left(1-(1-\theta)u\right)^{-r}\exp((s/\nu)u)du. \tag{11}$$

Take the series expansion of the exponential function to get

$$\begin{aligned}
A &= \nu^{-p}\exp(-s/\nu)\sum_{m=0}^{\infty}\frac{(s/\nu)^m}{m!}\int_0^1 u^{q+m-1}(1-u)^{p-1}\left(1-(1-\theta)u\right)^{-r}du \\
&= \nu^{-p}\exp(-s/\nu)\sum_{m=0}^{\infty}\frac{(s/\nu)^m}{m!}\frac{\Gamma(q+m)\Gamma(p)}{\Gamma(p+q+m)}{}_2F_1(r,q+m;p+q+m;1-\theta) \\
&= \mathrm{B}(p,q)\nu^{-p}\exp(-s/\nu)\sum_{m=0}^{\infty}\frac{(q)_m}{(p+q)_m\,m!}\left(\frac{s}{\nu}\right)^m\sum_{n=0}^{\infty}\frac{(r)_n(q+m)_n}{(p+q+m)_n\,n!}(1-\theta)^n \\
&= \mathrm{B}(p,q)\nu^{-p}\exp(-s/\nu)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q)_{m+n}\,(r)_n}{(p+q)_{m+n}\,m!n!}\left(\frac{s}{\nu}\right)^m(1-\theta)^n
\end{aligned}$$

where $(a)_i$ is Pochhammer's notation, i.e., $(a)_0=1$, $(a)_1=a$, $(a)_i=(a)_{i-1}(a+i-1)$. The second line follows from the first by AS15.3.1, and the third line from the second by AS15.1.1. By the definition of $H(p,q,r,s,\nu,\theta)$ in equation (5), the last line gives $A=\mathrm{B}(p,q)H(p,q,r,s,\nu,\theta)$, which guarantees that equation (4) integrates to unity.

For the UH limiting case, solve

$$A=\int_0^{\infty} x^{p-1}(1+\lambda x)^{-r}\exp(-sx)dx$$

by the change of variable $u = \lambda x$. The solution is

$$\begin{aligned} A &= \lambda^{-r} \int_0^\infty u^{p-1}(1+u)^{-r} \exp(-(s/\lambda)u)du \\ &= \lambda^{-r}\Gamma(p)U(p, p+1-r, s/\lambda) \qquad (12) \end{aligned}$$

where U is the "U" confluent hypergeometric function (see AS13.1.3 and AS13.2.5).

B Transformation rules for the Φ_1 function

In the remaining appendixes, it will be convenient to make use of alternative expressions for the nested infinite series in the definition equation (6) for Φ_1. For $0 < \alpha < \gamma$ and $0 \le y < 1$,

$$\begin{aligned} \Phi_1(\alpha, \beta, \gamma, x, y) &= \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m} \frac{x^m}{m!} {}_2F_1(\beta, \alpha+m; \gamma+m; y) \qquad \text{(T1)} \\ &= \exp(x) \sum_{m=0}^{\infty} \frac{(\gamma-\alpha)_m}{(\gamma)_m} \frac{(-x)^m}{m!} {}_2F_1(\beta, \alpha; \gamma+m; y) \qquad \text{(T2)} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{y^n}{n!} {}_1F_1(\alpha+n, \gamma+n; x) \qquad \text{(T3)} \\ &= \exp(x) \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{y^n}{n!} {}_1F_1(\gamma-\alpha, \gamma+n; -x). \qquad \text{(T4)} \end{aligned}$$

These transformations are derived using straightforward manipulations of the nested infinite series. All that is needed is the rule $(a)_{m+n} = (a)_m(a+m)_n$ and the Kummer transformation ${}_1F_1(a, b, x) = \exp(x){}_1F_1(b-a, b, -x)$ (see AS13.1.27).

When $y < 0$, apply the transformation rule (AS15.3.4)

$${}_2F_1(a, b; c; y) = (1-y)^{-a}{}_2F_1(a, c-b; c; y/(y-1))$$

to the form (T2) to get

$$\begin{aligned}
\Phi_1(\alpha,\beta,\gamma,x,y) &= \exp(x)\sum_{m=0}^{\infty}\frac{(\gamma-\alpha)_m}{(\gamma)_m}\frac{(-x)^m}{m!}(1-y)^{-\beta}{}_2F_1(\beta,\gamma-\alpha+m;\gamma+m;y/(y-1)) \\
&= \exp(x)(1-y)^{-\beta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_m}{(\gamma)_m\, m!}\frac{(\beta)_n(\gamma-\alpha+m)_n}{(\gamma+m)_n\, n!}(-x)^m\left(\frac{y}{y-1}\right)^n \\
&= \exp(x)(1-y)^{-\beta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma-\alpha)_{m+n}(\beta)_n}{(\gamma)_{m+n}\, m!n!}(-x)^m\left(\frac{y}{y-1}\right)^n \\
&= \exp(x)(1-y)^{-\beta}\Phi_1(\gamma-\alpha,\beta,\gamma,-x,y/(y-1)). \qquad \text{(T5)}
\end{aligned}$$

If $y < 0$, then $0 < y/(y-1) < 1$, so forms (T1)–(T4) can be applied to the Φ_1 term in the last line.

C Bounding the function H

In this appendix, I prove Theorem 1. From the definition of H in equation (5), it is clear that H is finite and positive everywhere on the parameter space if and only if the nested infinite series given by $\Phi_1(q, r, p+q, s/\nu, 1-\theta)$ converges to a positive real number for all $p > 0$, $q > 0$, $r \in \Re$, $s \in \Re$, $0 < \nu \leq 1$ and $0 < \theta$. Therefore, it is sufficient to show that $\Phi_1(\alpha,\beta,\gamma,x,y)$ converges to a positive real number for all $0 < \alpha < \gamma$, $\beta \in \Re$, $x \in \Re$ and either $0 \leq y < 1$ (i.e., for $\theta \leq 1$) or $y < 0$ (i.e., for $\theta > 1$).

The techniques needed to bound Φ_1 depend on the signs of β, x and y. Assume first that $\beta > 0$, $0 \leq y < 1$, and $x \geq 0$. For this case, we need the lemma

Lemma 4 *For all $x \geq 0$ and $0 < \alpha < \gamma$, $1 \leq {}_1F_1(\alpha,\gamma,x) \leq \exp(x)$.*

Proof: Expand the ${}_1F_1$ series as

$${}_1F_1(\alpha,\gamma,x) = \sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\gamma)_m}\frac{x^m}{m!}.$$

The first term ($m = 0$) equals 1 and remaining terms are non-negative, so the summation must be greater than or equal to 1. Since $\alpha < \gamma$, each term must be less than $x^m/m!$, so

$$ {}_1F_1(\alpha, \gamma, x) < \sum_{m=0}^{\infty} \frac{x^m}{m!} = \exp(x). \quad \blacksquare $$

Consider the form for Φ_1 given by (T3). Given $0 \le y < 1$, each term in the expansion is non-negative, so

$$ \begin{aligned} \Phi_1(\alpha, \beta, \gamma, x, y) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{y^n}{n!} {}_1F_1(\alpha + n, \gamma + n; x) \\ &< \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{y^n}{n!} \exp(x) = \exp(x)\, {}_2F_1(\alpha, \beta; \gamma; y) \end{aligned} $$

where the inequality follows from Lemma 4. The ${}_2F_1$ series converges for all y inside the unit circle (AS15.1.1), so this expression is finite. Taking the lower bound,

$$ \begin{aligned} \Phi_1(\alpha, \beta, \gamma, x, y) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{y^n}{n!} {}_1F_1(\alpha + n, \gamma + n; x) \\ &\ge \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{y^n}{n!} = {}_2F_1(\alpha, \beta; \gamma; y) \ge 1 \end{aligned} $$

where the first inequality follows from the lower bound in Lemma 4. The first term ($n = 0$) in the ${}_2F_1$ series expansion equals 1 and remaining terms are non-negative, so the summation must be greater than or equal to 1.

Next, assume $x < 0$. We need the lemma:

Lemma 5 *For $x < 0$, $0 < \alpha < \gamma$ and $n \ge 0$,*

$$ 1 \le {}_1F_1(\gamma - \alpha, \gamma + n, -x) \le {}_1F_1(\gamma - \alpha, \gamma, -x) < \exp(-x). $$

The proof is similar to that of Lemma 4.

Consider the form for Φ_1 given by (T4). Given $0 \le y < 1$, each term in the expansion is non-negative, so

$$\begin{aligned}\Phi_1(\alpha,\beta,\gamma,x,y) &= \exp(x)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{y^n}{n!}\,{}_1F_1(\gamma-\alpha,\gamma+n;-x)\\ &\le \exp(x)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{y^n}{n!}\,{}_1F_1(\gamma-\alpha,\gamma;-x)\\ &= \exp(x)\,{}_1F_1(\gamma-\alpha,\gamma;-x)\,{}_2F_1(\alpha,\beta;\gamma;y) < {}_2F_1(\alpha,\beta;\gamma;y)\end{aligned}$$

where both inequalities follow from Lemma 5. To get a lower bound,

$$\begin{aligned}\Phi_1(\alpha,\beta,\gamma,x,y) &= \exp(x)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{y^n}{n!}\,{}_1F_1(\gamma-\alpha,\gamma+n;-x)\\ &\ge \exp(x)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{y^n}{n!} = \exp(x)\,{}_2F_1(\alpha,\beta;\gamma;y).\end{aligned}$$

Thus, for $\beta > 0$ and $0 \le y < 1$, Φ_1 converges to a positive finite number for all $x \in \Re$.

When $\beta \le 0$, use forms (T1) and (T2) for Φ_1 and the transformation for the ${}_2F_1$ given in AS15.3.3:

$${}_2F_1(\beta,\alpha,\gamma,y) = (1-y)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\beta,\gamma-\alpha,\gamma,y). \tag{13}$$

The following lemma bounds the values of the ${}_2F_1$ terms in Φ_1.

Lemma 6 *For* $0 \le y < 1$, $0 < \alpha < \gamma$, *and* $\beta \le 0$,

$$0 < {}_2F_1(\beta,\alpha,\gamma,y) \le 1.$$

Proof: Apply equation (13). The first term in the expansion of ${}_2F_1(\gamma-\beta,\gamma-\alpha,\gamma,y)$ is 1 and the rest are non-negative, so ${}_2F_1(\gamma-\beta,\gamma-\alpha,\gamma,y) \ge 1$. Since $\gamma-\alpha-\beta > 0$ and $0 < (1-y) \le 1$, the right hand side of equation (13) must be strictly positive, so $0 < {}_2F_1(\beta,\alpha,\gamma,y)$. To show the upper bound, I show first that the derivative of ${}_2F_1(\beta, \alpha, \gamma, y)$ with respect to y is non-positive for $\beta \leq 0$. AS15.2.1 gives:

$$\frac{d}{dy}{}_2F_1(\beta, \alpha, \gamma, y) = \frac{\beta\alpha}{\gamma}{}_2F_1(\beta + 1, \alpha + 1, \gamma + 1, y).$$

If $\beta+1 > 0$, then the first term in the expansion of ${}_2F_1(\beta+1, \alpha+1, \gamma+1, y)$ is 1 and the rest are non-negative. If $\beta + 1 \leq 0$, then the argument just used to show $0 < {}_2F_1(\beta, \alpha, \gamma, y)$ applies here as well. In either case, ${}_2F_1(\beta+1, \alpha+1, \gamma+1, y) > 0$. Since $\beta\alpha/\gamma \leq 0$, the derivative $\frac{d}{dy}{}_2F_1(\beta, \alpha, \gamma, y) \leq 0$ for all $0 \leq y < 1$. Therefore, ${}_2F_1(\beta, \alpha, \gamma, 0) = 1$ is the upper bound on ${}_2F_1(\beta, \alpha, \gamma, y)$ for $0 \leq y < 1$.■

The remaining arguments parallel those used earlier. For $x \geq 0$, use (T1) and Lemma 6 to show

$$0 < \Phi_1(\alpha, \beta, \gamma, x, y) \leq {}_1F_1(\alpha, \gamma, x).$$

For $x < 0$, use (T2), Lemma 6 and the Kummer transformation to demonstrate the same bounds.

Thus far, it has been assumed that $0 \leq y < 1$. For the case $y < 0$, rule (T5) gives

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \exp(x)(1 - y)^{-\beta}\Phi_1(\tilde{\alpha}, \beta, \gamma, -x, \tilde{y})$$

where $\tilde{\alpha} \equiv \gamma - \alpha > 0$, $\tilde{\alpha} < \gamma$ and $0 \leq \tilde{y} \equiv y/(y - 1) < 1$. It has already been established that the right hand side Φ_1 converges to a finite positive number, so $\Phi_1(\alpha, \beta, \gamma, x, y)$ must as well.■

D Proposition 2 and Lemma 3

To show Lemma 3, substitute the expression for Φ_1 in rule (T2) into equation (5) to get

$$H(p, q, r, s, \nu, \theta) = \nu^{-p} \sum_{i=0}^{\infty} \frac{(p)_i}{(p + q)_i \, i!} \left(\frac{-s}{\nu}\right)^i {}_2F_1(r, q; p + q + i; 1 - \theta).$$

Take the derivative with respect to s:

$$\begin{aligned}
\frac{d}{ds}H(p,q,r,s,\nu,\theta) &= \nu^{-p}\sum_{i=1}^{\infty}\frac{(p)_i}{(p+q)_i\, i!}\left(\frac{-i}{\nu}\right)\left(\frac{-s}{\nu}\right)^{i-1}{}_2F_1(r,q;p+q+i;1-\theta) \\
&= -\nu^{-(p+1)}\sum_{i=0}^{\infty}\frac{(p)_{i+1}}{(p+q)_{i+1}\, i!}\left(\frac{-s}{\nu}\right)^{i}{}_2F_1(r,q;p+q+i+1;1-\theta) \\
&= -\nu^{-(p+1)}\sum_{i=0}^{\infty}\frac{p(p+1)_i}{(p+q)(p+q+1)_i\, i!}\left(\frac{-s}{\nu}\right)^{i}{}_2F_1(r,q;p+q+i+1;1-\theta) \\
&= -\frac{p}{p+q}H(p+1,q,r,s,\nu,\theta)
\end{aligned}$$

where the second line follows from a shift of index in the first line.■

To show Proposition 2, take the derivative of equation (7) with respect to s and use Lemma 3 to substitute for derivatives of H. This gives

$$\begin{aligned}
\frac{dE(X^k)}{ds} &= \frac{(p)_k}{(p+q)_k}\left(\frac{p}{p+q}\frac{H(p+k,q,r,s,\nu,\theta)}{H(p,q,r,s,\nu,\theta)}\frac{H(p+1,q,r,s,\nu,\theta)}{H(p,q,r,s,\nu,\theta)} - \frac{p+k}{p+q+k}\frac{H(p+k+1,q,r,s,\nu,\theta)}{H(p,q,r,s,\nu,\theta)}\right) \\
&= E(X^k)E(X) - E(X^{k+1}).
\end{aligned}$$

Proposition 2 is a special case of this result.■

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