"Interest Rates as Options:" Assessing the markets' view of the liquidity trap

Antulio N. Bomfim*
Federal Reserve Board

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Abstract

Nominal short term interest rates have been low in the United States, so low that some have wondered whether the federal funds rate is likely to hit its lower bound at 0 percent. Such a scenario, which some economists have called the liquidity trap, would imply that the Federal Reserve could no longer lower short-term interest rates to counter any deflationary tendencies in the economy. In this paper, I use an affine term structure model to infer what interest rates tell us about the probability, as assessed by financial market participants, of such an event taking place. I also examine whether U.S. short-term rates have been low enough to distort the shape of the yield curve.

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*Board of Governors of the Federal Reserve System, Washington, DC 20551; E-mail: abomfim@frb.gov; Fax: (202) 452-2301, Tel.: (202) 736-5619. I thank Geert Bekaert, Michael Gibson, and Athanasios Orphanides for helpful suggestions. I have also benefited from comments provided by seminar participants at the Federal Reserve Board. The opinions expressed in this paper are not necessarily shared by the Board of Governors of the Federal Reserve System or any other members of its staff.

1 Introduction

Nominal short-term interest rates have been low in the United States, and, with inflation also running at very low levels, many readings on real short-term rates have actually drifted below zero. Nominal rates have been so low that some market observers have wondered about the possibility that the federal funds rate—the key interest rate controlled by the Federal Reserve—may reach its lower bound at 0 percent.[1] Such a scenario would be reminiscent of what economists as far back as Keynes (1936) have called a liquidity trap, a situation where the Federal Reserve would no longer be able to lower short-term interest rates to react to an eventual pick-up in deflationary forces.

In this paper I use an affine two-factor model of the yield curve to examine the term structure of swap rates and infer what observed swap rates have to say about the likelihood, as perceived by investors, that the lower bound on the federal funds rate will become a binding constraint for U.S. monetary policymakers.[2] I also use the model to assess whether short-term rates have been close enough to zero to distort the usual role of the yield curve as an indicator of market participants' expectations of where the short rate—and potentially the economy—is headed.

The analytical framework follows the seminal work of Black (1995), from

[1]One might argue that the lower bound for the overnight federal funds rate should be positive given that the associated loans are not entirely free of risk. The methodology and main conclusions discussed in this paper would be little changed if I had imposed a slightly positive lower bound, as would be suggested by the average spread of the overnight federal funds rate over the Treasury general collateral repo rate—a proxy for the riskless short rate. (That spread has averaged close to 3 basis points over the past couple of years.)

[2]Focusing on the interest rate swap market has the advantage that swap rates essentially embed expectations of future short-term LIBOR, and the credit quality of participants in the LIBOR and fed funds markets is essentially the same.

which I also borrowed part of the title of this paper. Black proposed an interpretation of a nominal short-term interest rate as a call option on the "equilibrium" or "shadow" interest rate, where the option is struck at zero percent. He noted that when short rates are close to such a "strike rate" usual term-structure relationships can be significantly affected by the value of the options embedded in current and expected short rates. In this paper, I build on the Black framework along three dimensions. First, I examine the indicator properties of the yield curve when the short rate is close to zero and argue that, under such circumstances, the typical relationship between the slope of the yield curve and future economic activity can be substantially distorted. Second, I extend the Black approach and show how to extract, from observed yields, market-implied probabilities that the economy will slip into the liquidity trap over given time horizons. Lastly, while Black carried out his analysis in the context of a highly stylized theoretical model that was arbitrarily parameterized, the main focus of this paper is to propose and use an estimated term structure model to assess the quantitative importance of Black's original theoretical results and of the extensions developed in this paper.

The empirical model suggests that the recent configuration of interest rates in the United States does imply some probability that the zero bound on nominal short rates will be binding in the near term, but such a probability appears to be very small. Indeed, market expectations that the recent lows reached by short-term rates are only a temporary phenomenon seem sufficiently prevalent in the U.S. that the yield curve is likely not being distorted by the zero-bound constraint. I assess the sensitivity of these results to alternative parameterizations of the model and find that my findings are quite robust to reasonable variation in the parameters. On the whole, the results suggest that the option-like feature of short-term nominal rates uncovered by Black (1995) did not play a quantitatively important role in shaping the U.S yield curve during the interest rate lows of late 2002 and early 2003.

The paper is organized as follows. In Section 2, I provide a brief review of Black's (1995) main points, discuss the informational content of the yield curve when short rates are near zero, and show how to extend the Black analysis to compute market-implied probabilities that a given economy will slip into a liquidity-trap situation. I discuss the empirical model and estimation approach in Section 3 and report the main results of the paper in Section 4. I run a battery of sensitivity analysis exercises in Section 5. Concluding remarks are presented in Section 6.

2 Interest Rates as Options

The analysis in this paper is motivated by Black's (1995) insight that the fact that nominal interest rates are bounded at zero gives them an option-like feature. In particular, one can write the observed nominal short rate, $r(t)$, as follows

$$r(t) = \max[0, \rho(t)] = \rho(t) + \max[0, -\rho(t)] \tag{1}$$

where $\rho(t)$ is the equilibrium value of the short rate, defined as the one where the market for loanable funds clears. Alternatively, in the monetary economics literature, one can think of $\rho(t)$ as the value of the nominal short-term interest rate that is consistent with the prescription of the policy rule followed by the central bank, or, in the traditional Keynesian economics literature, as that value of the short rate that corresponds to the point of intersection between the IS and LM curves.

The first equality in (1) is the essence of Black's analysis. It tells us that one can think of the observed short rate as a call option on $\rho(t)$, struck at zero percent. The second equality tells us that the short rate observed in the marketplace can also be expressed as the sum of two components, the equilibrium rate $\rho(t)$ and an option-like term that provides a lower bound for the market rate at zero when $\rho(t)$ is negative. This last term corresponds to the payoff of a floor written on $\rho(t)$ with a strike rate of 0 percent, and thus one can think of the holder of a money market instrument as being long this floor and a bank deposit that pays $\rho(t)$. Intuitively, the floor is akin to an option to switch one's money funds holdings into currency if $\rho(t)$ falls below zero. When the market value of the floor is positive, the observed rate $r(t)$ is too high relative that rate that would clear the market for loanable funds and the resulting excess supply of loanable funds means that consumers and firms are borrowing less than the socially optimal amount. The end-result is that the economy may find itself "trapped" in a low equilibrium, where productive activity persistently falls short of the economy's potential, a situation that is commonly dubbed "the liquidity trap."

During most times, the equilibrium nominal rate $\rho(t)$ is sufficiently above zero that the value of the embedded floor shown in the second equality in (1) can be safely ignored. Indeed, most term structure models disregard the distinction between market and equilibrium interest rates. Under certain circumstances, however, the equilibrium nominal rate can become negative—for instance, either the short-run equilibrium real rate or expected inflation can fall below zero. Examples of such episodes include the United States during the great depression (Bernanke, 2002) or Japan in the 1990s (Krugman, 1998). In such cases, Black (1995) noted, usual approaches of modeling the yield curve have to be modified to take into account the binding zero constraint on nominal rates.

Black's approach to modeling the yield curve when short rates are close to zero is intuitively appealing. Instead of modeling the market rate directly, as is common in the term structure literature, he proposes that one should model the equilibrium short rate first and then use (1) to model the evolution of the market rate. Black illustrated his main points in the context of a risk-neutral random walk model for the equilibrium rate,

$$d\rho(t) = b dW(t) \tag{2}$$

where b is the volatility of the equilibrium short rate, and $dW(t)$ is an infinitesimal increment to a standard Brownian motion.

Using standard valuation techniques—see, e.g., Bjork (1998)—it can be shown that the time-t arbitrage-free price, $P(t,T)$, of a zero-coupon bond that matures at time T satisfies the following partial differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial \rho^2} b^2 - rP = 0 \tag{3}$$

provided shocks to $\rho(t)$ are the only source of uncertainty about bond prices, and assuming that $r(t)$ evolves according to (1) and (2). Let the face value of the bond be \$1 and impose the final condition

$$P(T, T) = 1 \tag{4}$$

and one can solve this pricing problem numerically using, for instance, either finite differences or Monte Carlo methods.

Black (1995) used the above setup to show that when observed short rates are at or near zero, the option-like feature of the nominal short rate can have potentially large effects both on the yield curve and on the term structure of yield volatilities. To see this, note that the same logic that led Black to write the spot short-term interest rate as the sum of an equilibrium rate and the payoff of a floor, equation (1), also applies to forward rates. In particular, one can write the instantaneous time-t forward rate that corresponds to the future period s, $f(t, s)$, as

$$f(t, s) = E_t[r(s)] + \text{forward premium} + \text{floor value} \tag{5}$$

where $E_t[r(s)]$ is the expected value of $r(s)$, based on information as of time t and computed under the objective probability measure. Thus, given that bond yields can be written as the mean value of all instantaneous forward rates over the life of the bond, the option-like feature of the nominal short rate can affect the entire yield curve. Black illustrated this point in the context of an artificial economy where the short rate evolves according to (1)–(4) and where $r(t)$ and b are 1 percent and 100 basis points, respectively. Figure 1 shows two zero-coupon curves derived from the solution to (3) and (4), one where the zero-bound constraint is explicitly incorporated into the pricing problem—the solid line—and one where it is not—the dashed line. The latter can be thought of as the term structure of equilibrium zero-coupon yields, or the zero-coupon yields that would clear the money and capital markets and prevent the model economy from falling into the liquidity trap. The former corresponds to the observed yield curve. As shown in the figure, the two curves differ substantially with the gap between them corresponding to the value of the option-like feature of future short rates.

2.1 The informational content of the yield curve

One point not explicitly made by Black (1995), but that bears being emphasized here is that, when short-term rates are sufficiently close to zero, investors and economic policymakers would be ill-advised if they were to follow a conventional rule of thumb regarding the (risk-adjusted) slope of the yield curve and future economic activity. In particular, it can be shown that the zero-coupon yields that would prevail in the absence of the zero bound on the short rate—the dashed line in Figure 1—can be written as

$$R(t,T)^* = \frac{E_t\left[\int_t^T \rho(s)ds\right]}{T-t} + \Phi(t,T) \tag{6}$$

where $E_t[.]$ denotes an expectation computed under the objective probability measure and $\Phi(t,T)$ is the associated term premium, which can be derived from the solution to (3) and (4). Thus, controlling for the term premium and ignoring the distinction between ρ and r, one can interpret $R(t,T)^*$ as an indicator of where future values of the equilibrium short rate will be, on average, over the life of the bond. Indeed, a vast literature has examined the properties of longer-term interest rates, especially of their spreads over short-term rates, as indicators of future movements in short rates and, in many economic models, of future levels of economic activity—see, e.g., Bernanke and Blinder (1992).

When the short rate is at or near zero, however, long rates embed more than just the usual term premium and expectations regarding future movements in short rates. When the zero bound on the short rate is explicitly taken into account, one can look back at equation (5) and think of a longer-dated zero-coupon yield as incorporating three components,

$$R(t,T) = \frac{E_t\left[\int_t^T \rho(s)ds\right]}{T-t} + \Phi(t,T) + \Psi(t,T) \tag{7}$$

where the last term captures the option value associated with the zero bound. This term is potentially negligible when current and expected values of ρ are sufficiently above zero, but it could be significantly positive otherwise. Indeed, when $\Psi(t,T)$ is large, $R(t,T)$ will be higher than what would be suggested by expected values of ρ and the term premium alone, and, thus, as we saw in Figure 1, the observed yield curve will be steeper than what would otherwise be the case. Under such circumstances, investors and monetary policymakers alike would be considerably misled if they were to take the positive slope of the observed (solid) yield curve in Figure 1 as an indication that market participants expect that economic activity is likely to increase in the future. In reality, the negative slope of the equilibrium (dashed) yield curve in Figure 1 suggests that the economy is expected to remain trapped in its low-activity equilibrium well into the future.[3] Thus, we have a situation where an upward sloping (observed) yield curve is signaling expectations of a prolonged slump in the economy. This is exactly the opposite of the usual indicator property attributed to the yield curve!

2.2 Computing liquidity-trap probabilities

As discussed above, by liquidity trap I mean a situation where the equilibrium nominal short rate $\rho(t)$ becomes nonpositive, leaving the observed rate at its lower bound at zero and potentially substantially reducing the effectiveness of conventional monetary policy actions. One may therefore be interested in assessing, at any point in time, the likelihood that such a scenario will emerge. Relying on well-known tools from statistics and stochastic calculus, I propose here a novel approach to using the yield curve to answer this question. I illustrate the approach by applying it to the simple model used thus far, but I shall use it later with the richer estimated model described in the next section.

Let $C(\rho(t), t; t')$, be the time-t probability that ρ will not hit zero before some specified time $t' > t$. A priori, one does not know the functional form of $C(\rho(t), t; t')$, but this unknown function can be shown to satisfy the following partial differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial \rho^2} b^2 = 0 \tag{8}$$

[3] Baz, Prieul, and Toscani (1998) suggest that this phenomenon may have been at play in Japan in the mid-1990s.

which is akin to the backward Kolmogorov equation that relates to (2). Given the definition of $C(.)$, the boundary and final conditions associated with (8) are given by:

$$C(0, t, t') = 0 \tag{9}$$

$$C(\rho(t'), t', t') = 1 \quad \text{if} \quad \rho(t') > 0 \tag{10}$$

Equation (9) simply says that the time-t probability that ρ will not hit zero before time t' is zero if ρ is already on the boundary at time t. Equation (10) states that if ρ is still in positive territory at t', then it has no time to become nonpositive and thus $C(\rho(t'), t; t') = 1$.

It can be shown that the solution to the above problem is given by

$$C(\rho(t), t, t') = N\left(\frac{\rho(t)}{b\sqrt{t'-t}}\right) - N\left(\frac{-\rho(t)}{b\sqrt{t'-t}}\right) \tag{11}$$

where $N(.)$ is the standard normal CDF—see, e.g., Cox and Miller (1965).

Equation (11) is a formula for the time-t probability that ρ will not hit zero before time t' when ρ is assumed to evolve according to (2). Table 1 shows risk-neutral liquidity-trap probabilities, $1 - C(\rho(t), t; t')$, computed for various time horizons based on the same parameterization used in Figure 1. The table indicates that the substantial differences between the yield curves shown in Figure 1 are associated with very high probabilities that the model economy will slip into the liquidity trap in the not too distant future.

3 A Model for Empirical Analysis

The simple model depicted in the previous section was an abstraction used solely for illustrative purposes. That model did not capture several important stylized facts regarding the yield curve, such as the observation that the short rate tends to display some mean reversion. In addition, the model allowed for only one factor, the short rate, to affect the entire term structure of interest rates, but empirical evidence suggests that at least one more common factor may lie behind movements in the yield curve. I should also point out that the short rate process in the simple model was specified only in terms of the risk-neutral probability measure, and thus the liquidity-trap probabilities that we computed do not necessarily correspond to the physical (objective) probabilities of ρ hitting zero percent. Lastly the model was arbitrarily parameterized, rather than either calibrated or estimated based on observable data. Together, these limitations of the simple model make it ill-suited to assess the quantitative importance of the theoretical results obtained thus far. To address this topic, I now turn my attention to a two-factor affine model that was estimated from the term structure of U.S. swap rates.

Consider the following model for the evolution of the equilibrium short rate, $\rho(t)$, under the objective probability measure,

$$d\rho(t) = k[\theta(t) - \rho(t)]dt + v dW_\rho(t) \quad (12)$$

$$d\theta(t) = \alpha[\beta - \theta(t)]dt + \eta dW_\theta(t) \quad (13)$$

$\rho(t)$ is assumed to error-correct toward its time-varying central tendency

$\theta(t)$ with a mean reversion coefficient k. $\theta(t)$ is also assumed to follow a mean-reverting process, with α denoting its speed of mean reversion and β its long-run value. $dW_\rho(t)$ and $dW_\theta(t)$ are uncorrelated stochastic shocks (infinitesimal increments to standard Brownian motions) hitting the short rate and its central tendency, respectively, and v and η are their volatilities.

The above model is a two-factor extension of the one-factor model originally proposed by Vasicek (1977) and a variant of the double-mean reverting framework discussed by Balduzzi, Das, and Foresi (1998), Babbs and Nowman (1999) and Bomfim (2003). In terms of the economics of the model, $\theta(t)$ can be thought of as where market participants think the equilibrium short rate will be in the future, and β is the steady-state value of the equilibrium short rate. As for the actual short rate, $r(t)$, I continue to assume that it evolves according to (1).

Equations (12) and (13) comprise a Gaussian model of the equilibrium short rate, in that it can be shown that unexpected movements in $\rho(t)$ and $\theta(t)$ are normally distributed. A common criticism levied at Gaussian models is that they have the counterfactual implication that negative short rates are possible. This limitation of this class of models, however, is actually a desirable feature in the context of this paper, given that I am modeling the equilibrium short rate, which could well be negative, in conjunction with the actual short rate, which is nonnegative as prescribed by equation (1).

To derive the yield curve relationship implied by (12) and (13), assuming that stochastic movements in $\rho(t)$ and $\theta(t)$ are the only sources of uncertainty regarding movements in the yield curve, we can again rely on standard valuation techniques to write down the partial differential equation that the price, $P(t,T)$, of a zero-coupon bond must satisfy in the absence of arbitrage opportunities:

$$\frac{\partial P}{\partial t}+\frac{\partial P}{\partial \rho}[k(\theta-\rho)-\tilde{\lambda}_\rho v]+\frac{\partial P}{\partial \theta}[\alpha(\beta-\theta)-\tilde{\lambda}_\theta \eta]+\frac{1}{2}\left[\frac{\partial^2 P}{\partial \rho^2}v^2+\frac{\partial^2 P}{\partial \theta^2}\eta^2\right]-rP=0 \tag{14}$$

where $\tilde{\lambda}_\rho$ and $\tilde{\lambda}_\theta$ are the market prices of risk associated with the element of uncertainty in $\rho(t)$ and $\theta(t)$, respectively. I assume that the market prices of risk are proportional to the volatilities of the underlying factors:

$$\tilde{\lambda}_\rho = \lambda_\rho v \tag{15}$$

$$\tilde{\lambda}_\theta = \lambda_\theta \eta \tag{16}$$

The particular functional form of $P(t,T)$ that satisfies (14) subject to (1) and (4) corresponds to the bond pricing formula of the two-factor model.

3.1 Model estimation

Because of the additional complications associated with the zero-bound constraint, I will generally solve the PDE problem described in the previous subsection numerically, using both Monte Carlo and finite differences methods. For the purpose of estimating the parameters of the model, however, I shall rely on the analytical solution to a version of the model that ignores the zero-bound constraint on the short rate. In particular, I estimate the model using weekly data from 1989 to 2001, a period when nominal short-term interest rates were safely above zero, and thus I assume that $r(t)=\rho(t)$

over the entire estimation period.[4] This allows me not only to write down analytical expressions relating the prices of zero-coupon bonds to the factors $\rho(t)$ and $\theta(t)$, but also to rely on well-known likelihood estimation procedures based on the Kalman filter—see, e.g., Harvey (1990). Assuming that the factors $\rho(t)$ and $\theta(t)$ are not directly observable, the transition equation for the Kalman filter is simply the solution to the system of stochastic differential equations given by (12) and (13):

$$X(T) = e^{-K(T-t)}X(t) + [I - e^{-K(T-t)}]\Theta + \int_t^T e^{-K(u-t)}\sigma dW(u) \qquad (17)$$

where $X(t) \equiv [\rho(t), x(t)]'$, $x(t) \equiv \theta(t) - \beta$, $K \equiv \begin{bmatrix} k & -k \\ 0 & \alpha \end{bmatrix}$, $\Theta \equiv [\beta, 0]'$, $\sigma \equiv \begin{bmatrix} v & 0 \\ 0 & \eta \end{bmatrix}$, I is a 2x2 identity matrix, and $dW(u)$, defined as $[dW_\rho(u), dW_\theta(u)]'$, is a normally distributed vector with zero mean and variance-covariance matrix $V \equiv \int_t^T e^{-K(T-u)}\sigma\sigma' e^{-K'(T-u)}du$.

To arrive at the measurement equations for the Kalman filter, I use the solution to (14) and (4), again assuming that $\rho(t) = r(t)$:

$$P(t,T) = e^{-A(t,T)-B(t,T)r(t)-C(t,T)x(t)} \qquad (18)$$

where, in addition to being maturity-dependent, $A(t,T)$, $B(t,T)$, and $C(t,T)$ are functions of the parameters of the short-rate model and of the market prices of risk.

[4] As reported later in this paper, I checked to see whether the values of the options embedded in the yield curve were nonzero over this period. They were not.

Using the fact that $P(t,T) = e^{-R(t,T)(T-t)}$, where $R(t,T)$ is the time-t yield on a zero-coupon bond that matures at T, I can write $R(t,T)$ as an affine function of the factors

$$R(t,T) = \frac{1}{T-t}[A(t,T) + B(t,T)r(t) + C(t,T)x(t)] \tag{19}$$

and the vector of measurement equations is simply (19) expressed in matrix form for various maturities and augmented with maturity specific measurement errors

$$y(t) = D_0(\Omega, \Lambda) + D_X(\Omega, \Lambda)X(t) + \epsilon(t) \tag{20}$$

where $y(t)$ is an n-by-1 vector of zero-coupon bond yields implied by the LIBOR and swap curves, and $\epsilon(t)$ is the corresponding vector of measurement errors—$\epsilon(t)$ is assumed to have zero mean and a diagonal variance-covariance matrix H. (Ω is the vector of parameters of the short rate model, and Λ is the vector of market prices of risk.)

3.2 Model estimation results

I used zero-coupon yields implied by swap rates and short-term LIBOR to estimate the model. These yields are shown in Figure 2. They are based on six- and twelve-month LIBOR and on swap rates corresponding to two-, three-, four-, five-, seven-, and ten-year maturities.[5] Although the data extend from January 1989 through January 2003, the sample used in the

[5]To obtain the implied zero-coupon yields, I first computed zero-coupon bond prices from the LIBOR/swap curve using a standard bootstrapping procedure and then generated a zero-coupon curve using the cubic hermite interpolation method. (I also experimented with a cubic spline, and the results were largely unaffected.)

estimation contains only the 667 weekly observations ranging from January 1989 through December 2001. As noted, I did not use data from 2002 and 2003 in the estimation of the model, as short-term nominal rates dipped very close to zero during that period. Using those data could potentially have made problematic my decision to disregard the zero bound constraint when estimating the model.

Estimation results are summarized in Figures 3 through 6. Figure 3 shows the model's ability to fit the average shape of the yield curve during each of the past twelve years. Figure 4 shows actual and model-implied yield curves for the last week in the entire sample. Both figures suggest that the two-factor model generally does a good job accounting for both the average level and shape of the curve during the past several years. Figure 5, which shows estimated model residuals, provides some perspective on how the model captures the time-series variation in the zero-coupon yields. The model residuals are very small, indicating that the Kalman filter was able to ascribe most of the movements in zero-coupon yields to changes in the two estimated factors and suggesting that the assumption that the zero-bound constraint was not binding during the estimation period was a reasonable one. The validity of this assumption is also suggested by Figure 6, which shows the estimated factors, both of which remained in positive territory during the estimation sample and into 2002 and 2003. In addition, I also computed the value of option-like feature of nominal rates—$\Psi(t, T)$ in equation (7)—and found that it was essentially zero throughout the estimation period across all maturities.

Table 2 summarizes the parameter estimates underlying the model-implied yields shown in Figures 3 through 6. According to the model, both the short rate and its tendency have a statistically significant degree of mean reversion, the k and α parameters, respectively. The estimated value of k, 0.42, implies a half life of about 1.6 years, suggesting that $\theta(t)$ can be thought of as where the short rate will be, in the absence of any unanticipated shocks, over the next few years. The estimated value of α—about 0.05—implies a much slower degree of mean reversion of $\theta(t)$ towards its long-term value, β, but β itself was not precisely estimated by the model and its point estimate seems high. The model did allow me to estimate more precisely the standard errors of the shocks to the short rate and central tendency equations—v and η—which, expressed on an annual basis, are estimated to be 84 and 110 basis points, respectively.

Similar to the long-run value of the short rate, β, the market price of risk parameters, λ_r and λ_θ, were imprecisely estimated in the context of the model. In Section 5, I examine the sensitivity of my main results to alternative parameterizations regarding β, λ_r, and λ_θ.

4 Results from the Estimated Model

I extended the analysis carried out with the stylized model used in Section 2 to the estimated two-factor model proposed and estimated in Section 3. As noted, the two-factor model fits the data well and thus constitutes a potentially useful framework for quantifying any effects that the zero bound constraint may have on the yield curve, as well as for estimating probabilities that real-world economies will face liquidity-trap situations.

4.1 Market-based liquidity-trap probabilities

The analysis here mirrors that of Section 2.2. In particular, the time-t market-implied probability $C(\rho(t), t; t')$ that the nominal short rate will not hit zero before time t' can be shown to satisfy a backward partial differential equation that is similar to (8) and subject to the same boundary and final conditions (9) and (10). Instead of solving that problem to derive an analytical solution for $C(.)$, however, I relied on numerical solutions based on finite difference methods, which are quite accurate and relatively straightforward to implement.[6]

Thus far the empirical work has focused on the pre-2002 period, which allowed me to estimate the two-factor model while implicitly assuming that the zero-bound constraint on nominal rates could be ignored. I shall focus now on the behavior of the yield curve in the U.S. during 2002 and early 2003, a period when short-term nominal rates became very close to zero. How does the estimated model parse out the level and shape of the U.S. yield curve in 2002 into liquidity-trap probabilities? To start answering this question, for each data point in 2002 and early 2003, I computed the model-implied probability that the nominal short rate will become nonpositive sometime during the ensuring two-year period. The resulting liquidity-trap probabilities, as seen from the perspective of the yield curve, are shown as the solid line in Figure 7, which also depicts the evolution of swap-implied zero-coupon yields corresponding to the two-, five-, and ten-year maturities. Unlike the

[6] I assessed the accuracy of the numerical algorithm using the stylized model and found that it delivered solutions that were very close to the analytical solution discussed in Section 2.

risk-neutral probabilities derived in Section 2, these are “real-world,” objective probabilities implied by the yield curve and the estimated parameters of the model.

As can be seen in Figure 7, the model would interpret the low levels of market interest rates over the second half of 2002 and in early 2003 as implying some odds that the U.S. economy might slow enough to fall into a liquidity-trap type scenario over the next two years, but these market-implied odds are very small. For instance, as of early 2003, the model would place the probability of short rate hitting its lower bound at zero within the next two years at close to 4 percent. It is noteworthy, however, that, while small, this probability was higher then than at any period included in the sample.

Table 3 shows liquidity-trap probabilities implied by the model over various horizons, inferred from the observed swap curve early in 2003. These market-implied probabilities increase with the time horizon, but remain relatively low even for the five-year horizon. One might contrast the shape of term structure of liquidity-trap probabilities associated with the two-factor model with that of the artificial economy, which has a much more pronounced upward slope. The fact that the two-factor model detects a significant degree of mean reversion in the short rate process, while the stylized model assumed no mean reversion in the artificial economy, accounts for the flatter pattern of liquidity-trap probabilities in the estimated model.

Table 3 also contrasts objective and risk-neutral probabilities implied by the estimated model. Because the risk-neutral process for the short rate has a higher drift than does the objective process, the risk-neutral probabilities generally tend to underestimate the likelihood that the economy will fall into a liquidity-trap scenario.

4.2 Any distortions to the U.S. yield curve?

Black (1995) highlighted the potential for the shape of the yield curve to be affected when short rates are close enough to zero percent, and this paper has taken that insight further by cautioning policymakers and other market participants about changes in the informational content of the yield curve during such times. The question then arises: Are rates close enough to zero that their option-like feature is distorting the yield curve?

I shall continue to focus on 2002 and early 2003. Similar to the approach illustrated for the stylized model of Section 2, I derive the bond pricing equation for the two-factor model under two assumptions—with and without explicitly imposing the zero bound constraint on the short rate $r(t)$—in order to estimate the size of the option value associated with the floor. Analytical solutions to the first case, no zero bound constraint, are relatively straightforward and, indeed, I outlined the derivation of those in Section 3 in order to obtain the measurement equations for the Kalman filter—see equation (18). While this solution method can be defended for the pre-2002 period, however, it may not be appropriate for 2002 and 2003 and hence my interest in solving equation (14) while explicitly incorporating the zero bound constraint.[7]

To answer the question of whether the proximity of short rates to zero has affected the shape of the U.S. yield curve, I compare, in Figure 8,

[7]I solved the version of the model that incorporates the zero bound constraint using the same numerical methods outlined in Sections 2 and 3.

two curves: a model-generated yield curve that does not impose the zero bound constraint—the dashed line—and a model-generated curve that explicitly takes into account the non-negativity condition on the short rate—the dashed-dotted line. As shown in the figure, one can hardly tell the difference between these two model-generated curves, suggesting that market participants see the odds of the U.S. economy slipping into the liquidity trap as low enough that one can still ignore the zero bound constraint when pricing bonds or when interpreting movements in the yield curve. The curves shown in Figure 8 are for the week of January 17, 2003. Analyses of other weeks in 2002 would lead to conclusions similar to the ones just reported.

5 Sensitivity Analysis

The results reported in Section 4 are, of course, functions of the estimated parameters of the two-factor model. While the model does a very good job capturing the behavior of the yield curve over the past decade, I was unable to estimate some of the underlying parameters with the desired precision, particularly, as reported in Section 3, the long-term value of the short rate, β, and the market price of risk parameters, λ_ρ and λ_θ. Thus, I shall start this section by assessing the robustness of my results to variations in these parameters.

Long-run value of the nominal short rate. At 8.3 percent, the point estimate of β is likely high. For instance, it is well above the sample average of the six-month zero-coupon yield, which is about 5-1/4 percent. To assess the sensitivity of my results to a lower value of β, I reran the analysis discussed in Section 4 with β set at this lower value. The results were little changed from the benchmark case. While one might argue that the lower value of β would help keep short-term rates lower, and thus potentially closer to the zero bound, I would make two points. First, although lower than the estimated value, 5-1/4 percent is still sufficiently above zero that it may help anchor the short rate in positive territory. Second, the degree of mean reversion in the medium-term trend of the short rate, $\theta(t)$, is very low, though statistically significant, and that should help mitigate the effect of changes in β on the computed liquidity-trap probabilities.

Market prices of risk. As long as my main interest lies on objective probabilities, the estimated values of the market price of risk parameters have no direct effect on computed liquidity-trap probabilities. Where these parameters do matter explicitly is on the computation of risk-neutral probabilities as they enter into the adjustment to the drift of the short rate process when one goes from the objective to the risk-neutral measure. I examined the effects of doubling the market price of risk parameters, which had the expected result of driving a wider wedge between objective and risk neutral probabilities. As a result, the risk-neutral probability went from about 1.8 percent in the benchmark case to close to 1 percent. Reducing the market price of risk parameters would have the unsurprising effect of bringing risk-neutral and objective probabilities closer.

Short rate volatilities. Unlike the parameters examined thus far, the volatility of the short rate and its central tendency, v and η, respectively, were quite precisely estimated in the context of the two-factor model. Nonetheless, despite the model's simplifying assumption that v and η are constant, volatilities in the real world do display some time variation.

As a first attempt to gauge the sensitivity of my results to changes in volatility, I computed liquidity-trap probabilities under two alternative scenarios. In the high-volatility scenario, I increased v and η by 3 times the standard errors of estimate associated with these parameters. In the low-volatility scenario, I reduced v and η by the same amount. The main results were relatively robust to these variations in volatility. The probability that ρ will hit the zero bound during the ensuing two-year period rose to about 6 percent in the high-volatility specification and fell to close to 3 percent in the low-volatility scenario.

I also experimented with an additional high-volatility case, one that I dubbed the "extreme-volatility scenario." In this scenario, I increased v and η 50 percent, about 20 times the standard errors of estimate associated with these parameters. Not surprisingly, such a large increase in volatility had a noticeable effect on the model-implied liquidity-trap probability, which rose to about 22 percent. In the face of such an outsized rise in volatility, however, one might find some comfort in the fact that the model would suggest more than 3 in 4 odds that the economy will not fall into the liquidity trap over the next two-year period. Moreover, Bomfim (2002) used swaption prices to generate time-varying estimates of short rate volatility over the 1994-2001 period, and the resulting series never reached a level as high as in the extreme-volatility scenario.

The extreme-volatility scenario can be examined further to assess whether the corresponding liquidity-trap probability would be elevated enough to distort the shape of the yield curve. Figure 9 shows two hypothetical yield curves, one where the zero-bound constraint is ignored—the dashed line—and another where it is explicitly taken into account—the solid line. The starting values for $r(t)$ and $\theta(t)$ are the ones suggested by the model at the beginning of 2003, and all parameter values are as in Table 2, except for v and η, which are set as in the extreme-volatility scenario. Figure 9 shows that even with a liquidity-trap probability as high as 22 percent, the shape yield curve is mostly unaffected by the zero bound constraint. It is only for maturities beyond 15 years that one starts seeing the option value associated with the zero bound having some effect on the slope of the curve, and even then the effect is relatively small. Figure 9 reinforces the finding that the qualitative effects uncovered by Black (1995) regarding the effect of the zero bound on the shape of the yield curve were likely not of great quantitative importance in the United States in 2002 and early 2003.

6 Concluding Remarks

This paper had two main motivations. On the theoretical front, the work of Fisher Black (1995) provided a useful and intuitive framework for analyzing the yield curve when short-term nominal rates are close to zero. On the practical front, nominal rates were indeed close to zero in the United States in 2002 and early 2003. My goal was to use the Black framework as a starting point for the analysis and then to extend it to be able to make empirical statements about observed yield curves.

The paper has three main intended contributions to the term structure literature. First, it makes explicit that Black's (1995) insight that nominal interest rates have an option-like feature that can affect the shape of the yield curve should be taken as a cautionary note for financial market participants. For instance, those who might feel tempted, say, to use term structure spreads to assess market sentiment regarding future states of the economy might want to take notice of the fact that, when rates are sufficiently close to zero, a positive slope of the yield curve could have as much, if not more, to do with expectations of a prolonged liquidity-trap situation than with forecasts of improved economic conditions.

The second main intended contribution of the paper was to show how to use the observed yield curve to assess the likelihood, as assessed by market participants, that the economy will slip into the liquidity trap. The approach outlined in this paper has the advantage of requiring fewer structural assumptions than one that would be based, for instance, on a macroeconometric model. In addition, we were able to obtain not just the risk-neutral liquidity-trap probabilities that one could compute, say, from options prices, but also the objective probabilities that presumably underlie the evolution of the short rate in the real world.

Third, I wanted to apply my extensions of the Black framework to an estimated term-structure model that would allow me to assess the quantitative significance of the theoretical results. I did so in the context of a two-factor affine model of the U.S. swap curve, which I estimated using conventional maximum-likelihood methods. When applied to the estimated model, the analytical framework described in this paper suggested that the configuration of market interest rates in the United States during 2002 and early 2003 did imply some odds that the U.S. economy could slip into a liquidity-trap scenario in the not too distant future. But these market-implied odds were very small; indeed, small enough that the results suggested that the shape of the yield curve during that period was likely not significantly affected by the proximity of short-term nominal rates to their lower bound at zero. The model would suggest that financial market participants saw the low levels of short-term rates observed in 2002 and in the beginning of 2003 as a temporary phenomenon, believing that rates will mean revert to more normal levels in due course.

I should close while pointing out that the methods described in this paper have applications well beyond the interest rate environment prevailing in the United States in 2002 and early 2003. They are, for instance, potentially equally applicable to assessing market participants' expectations of future economic conditions in other countries or in the United States during other times. In addition, the extended interest-rate-as-options framework introduced in this paper has implications both for market practitioners pricing fixed-income products and for researchers interested in the determinants of interest rate dynamics.

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Table 1
Risk-Neutral Liquidity-trap Probabilities for the Artificial Economy[a]

Horizon	Prob[$\rho \leq 0$] within horizon (percent)
one month	0.1
three months	4.5
six months	15.7
one year	31.7
two years	47.9
five years	65.5

[a]Table entries show the probability that ρ will become nonpositive in the artificial economy during the horizons indicated. ρ is assumed to evolve according to (2), with $b = 100$ basis points and with the initial value of ρ set at 1 percent.

Table 2 —Estimated Parameters for the Two-factor Model[a]

$$d\rho(t) = k[\theta(t) - \rho(t)]dt + v dW_\rho(t)$$
$$d\theta(t) = \alpha[\beta - \theta(t)]dt + \eta dW_\theta(t)$$

Parameter	Estimated value	Standard error
k	0.4186	0.0039
α	0.0458	0.0015
β	0.0838	0.0588
η	0.0110	0.0003
v	0.0084	0.0002
λ_ρ	40.9367	38.951
λ_θ	0.1273	23.389
$h_{1/2}$	0.1310	0.0086
h_1	0.0001	0.0003
h_2	0.1254	0.0123
h_3	0.1122	0.0116
h_5	0.0608	0.0029
h_7	0.0001	0.0067
h_{10}	0.0671	0.0021

[a]Estimation period: 1989 to 2001 (667 weekly observations). The model was estimated using zero-coupon yields derived from LIBOR and swap rates for the following maturities: six-months, one, two, three, five, seven, and ten years. h_i is the standard error of the residuals of the measurement equation associated with a zero-coupon bond with i years to maturity.

Table 3
Estimated Liquidity-trap Probabilities for the U.S. Economy[a]
(as of January 17, 2003)

Horizon	Prob[$\rho \leq 0$] within horizon	
	objective	risk-neutral
six months	2.0	1.2
one year	3.0	1.6
two years	3.6	1.8
five years	4.8	2.4

[a]Table entries show the probability that the equilibrium short rate, ρ, will become nonpositive in the U.S. economy during the horizons indicated. ρ is assumed to evolve according to the estimated two-factor model. Model parameter values are shown in Table 2.

Figure 1
Observed and Actual Yield Curves in the Random−Walk Model

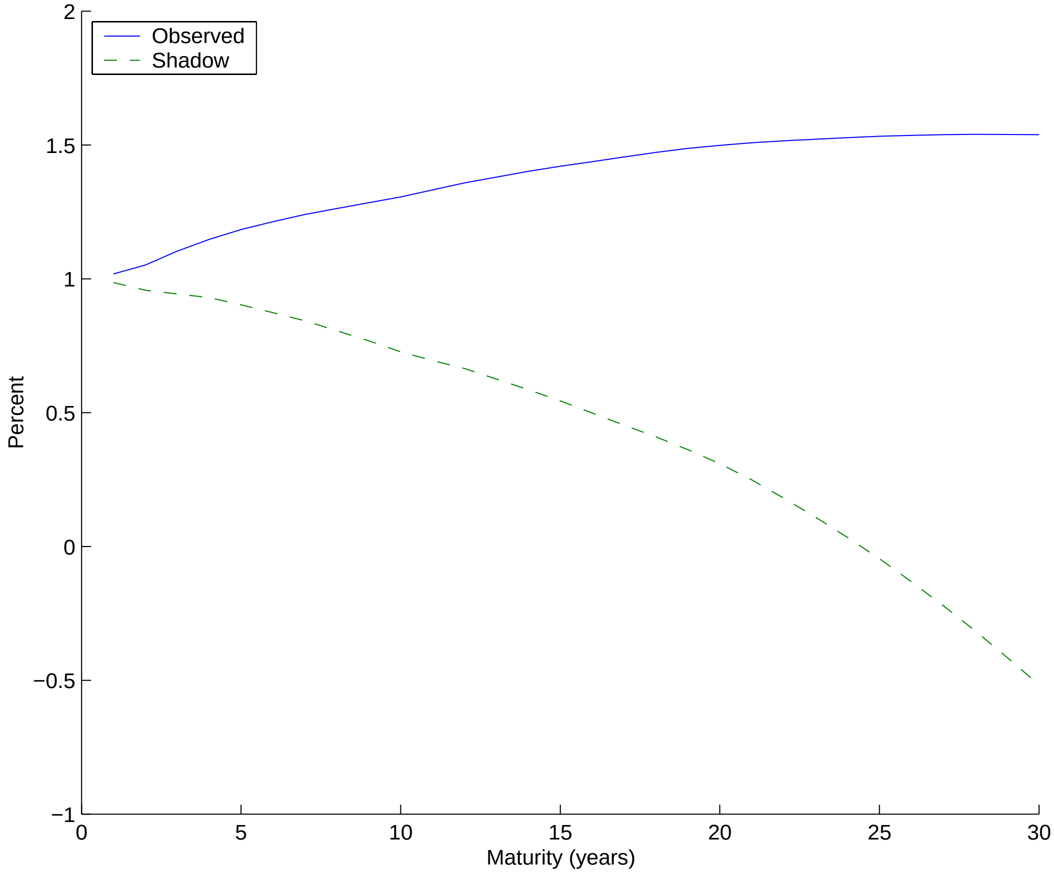


Figure 2
Swap−implied zero−coupon yields

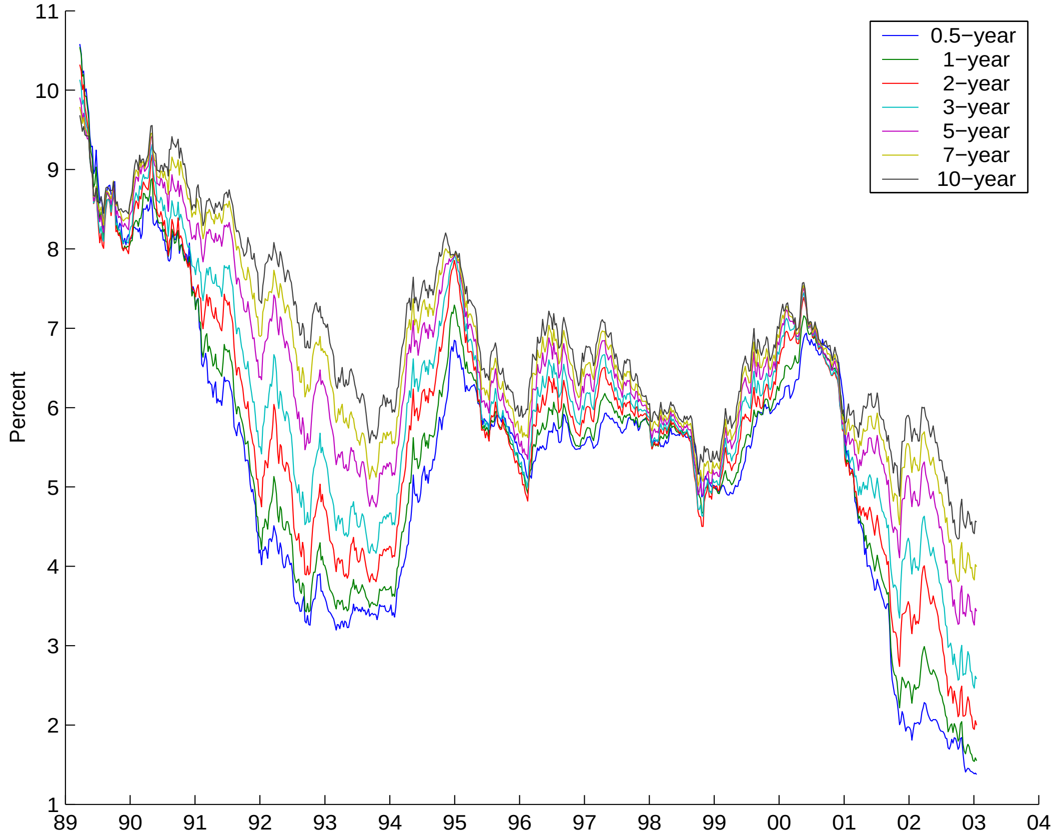


Figure 3
Actual (solid) & Model−implied (dotted) Yield Curves
Annual Averages

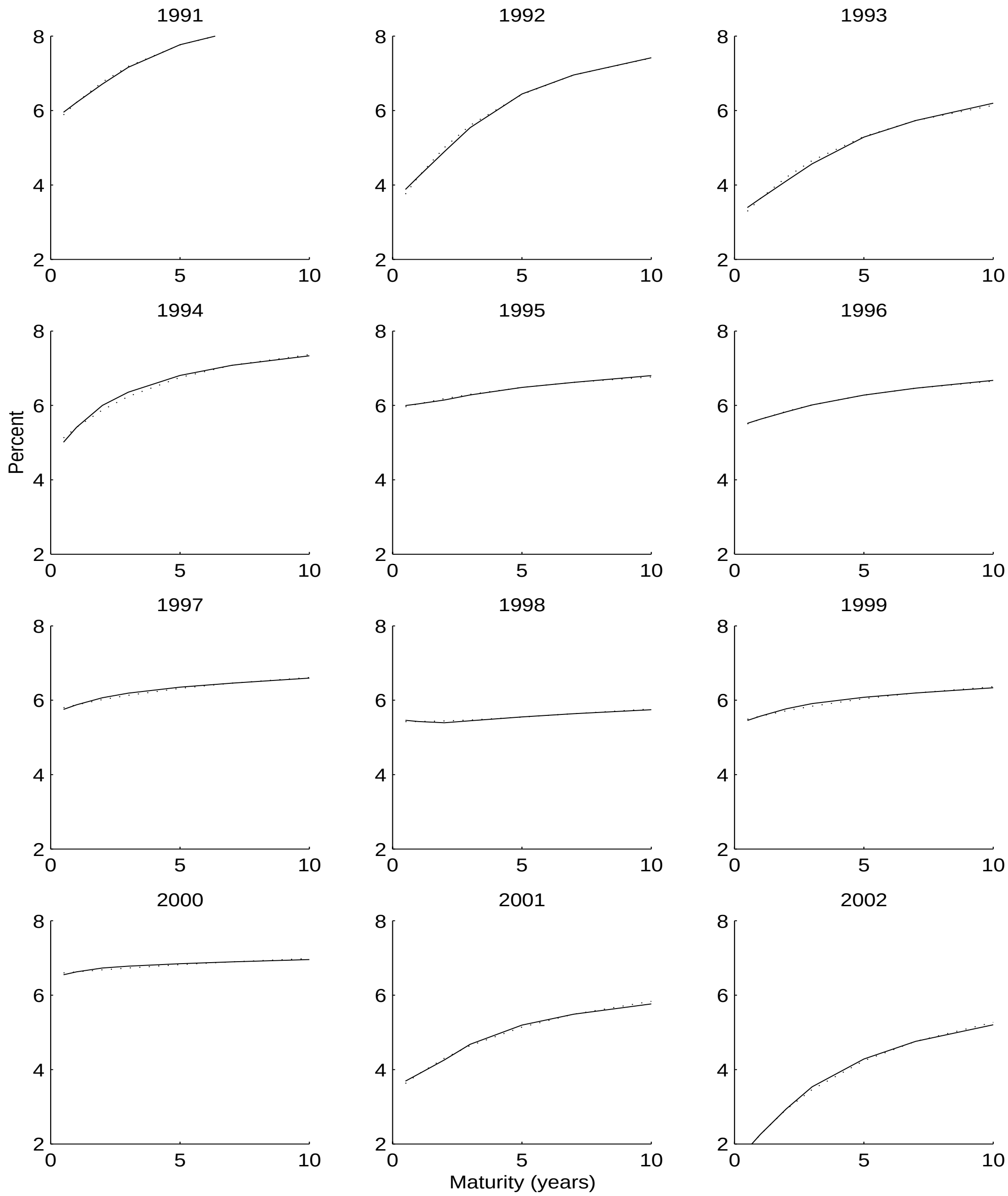


Figure 4
Swap−implied zero yield curve, week ending January 17, 2003

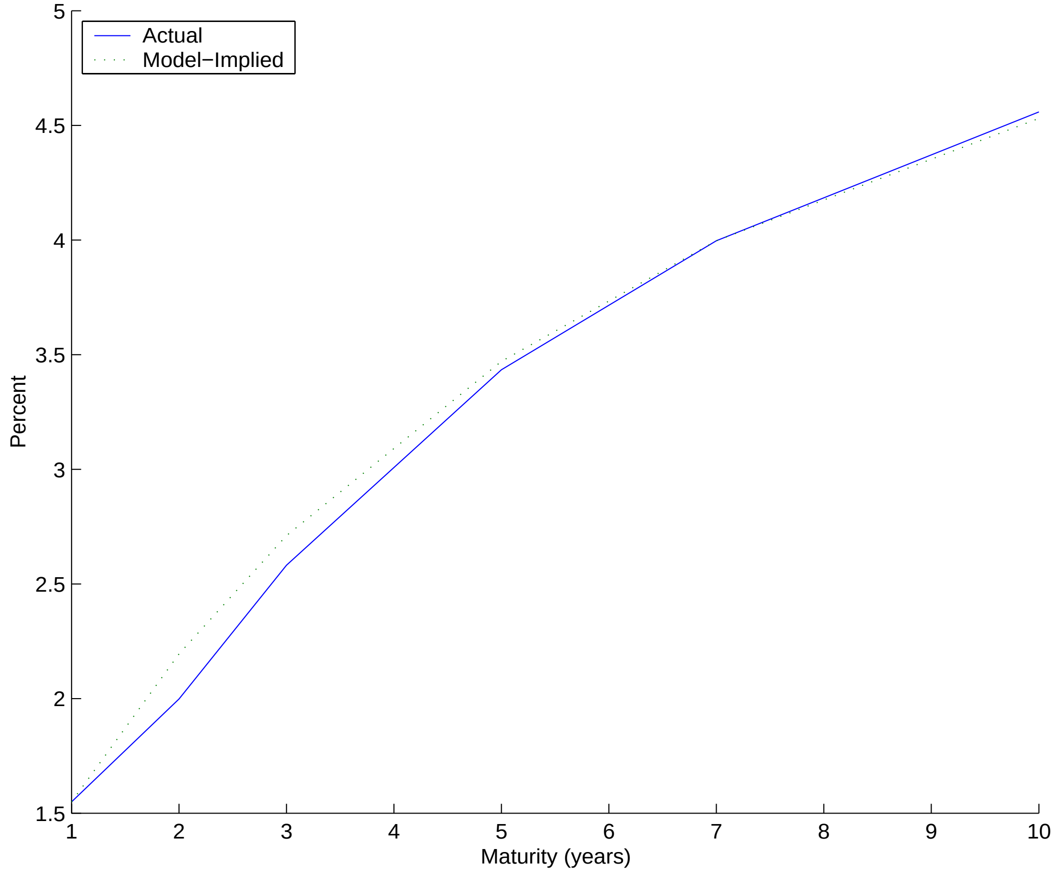


Figure 5
Measurement Equation Residuals

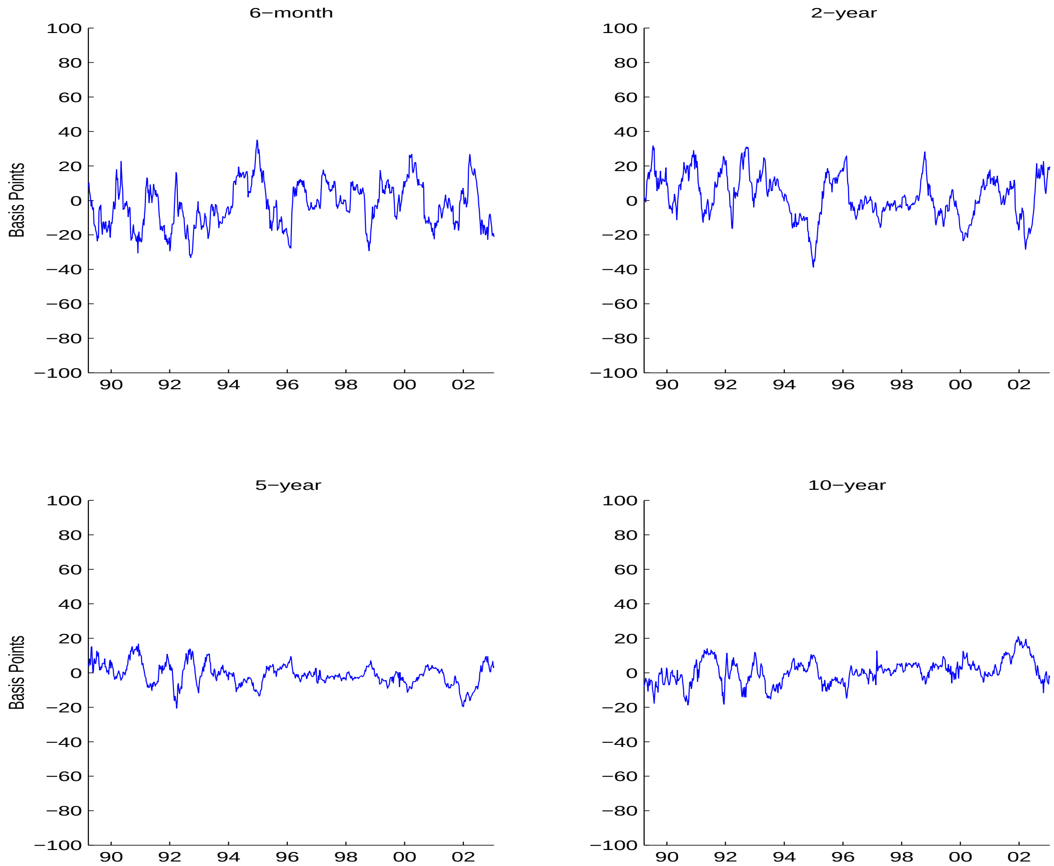


Figure 6
Estimated Factors

Figure 7
Probability of Zero Short Rate within Ensuing Two−year Period

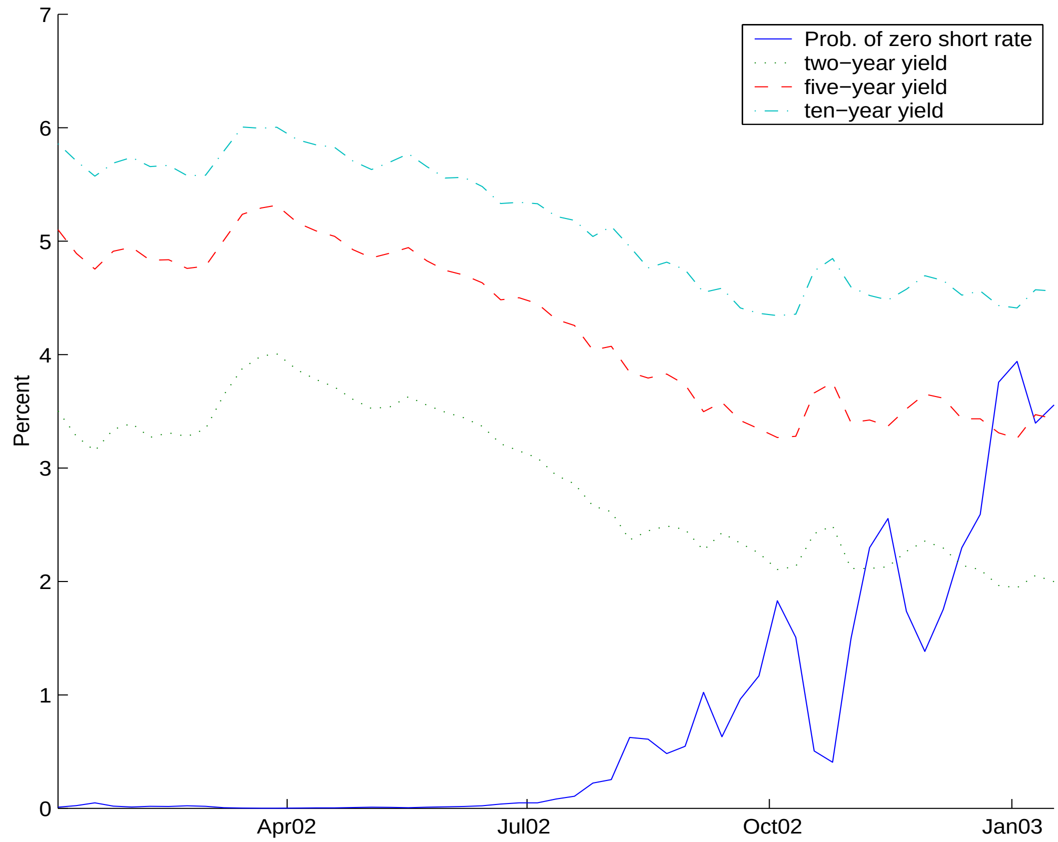


Figure 8

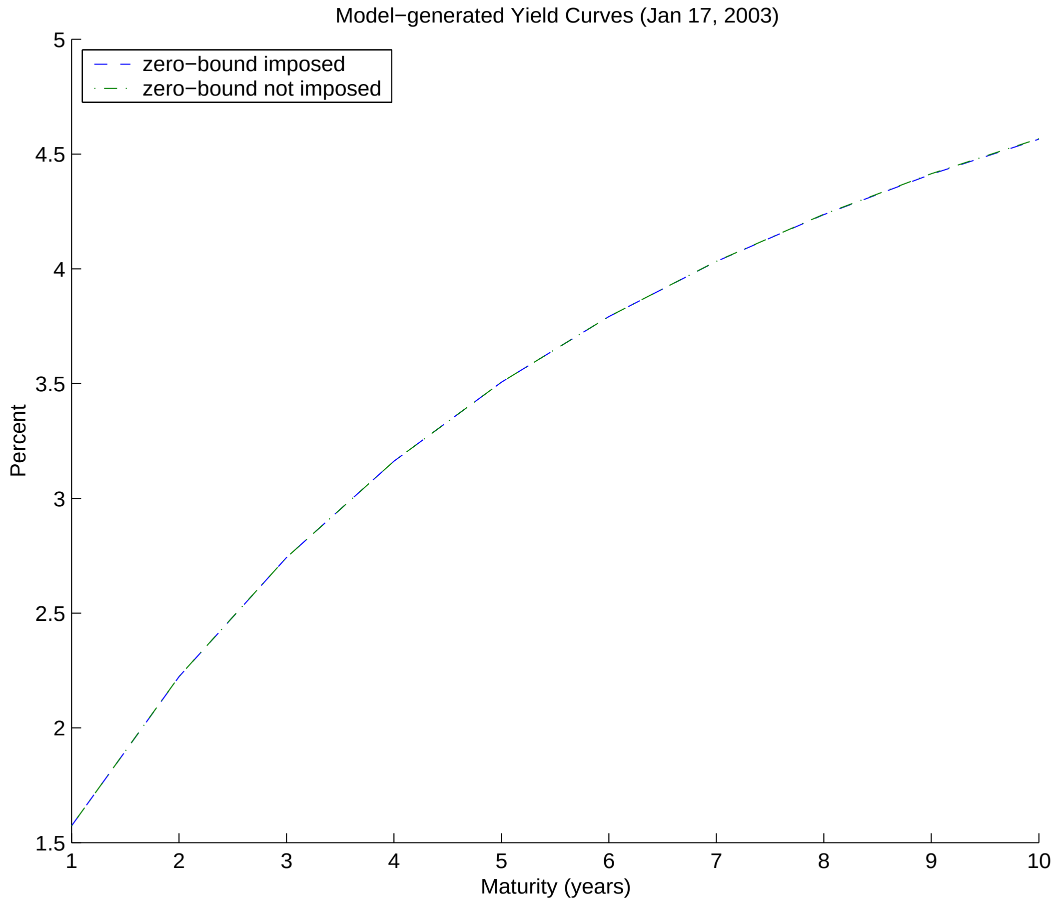
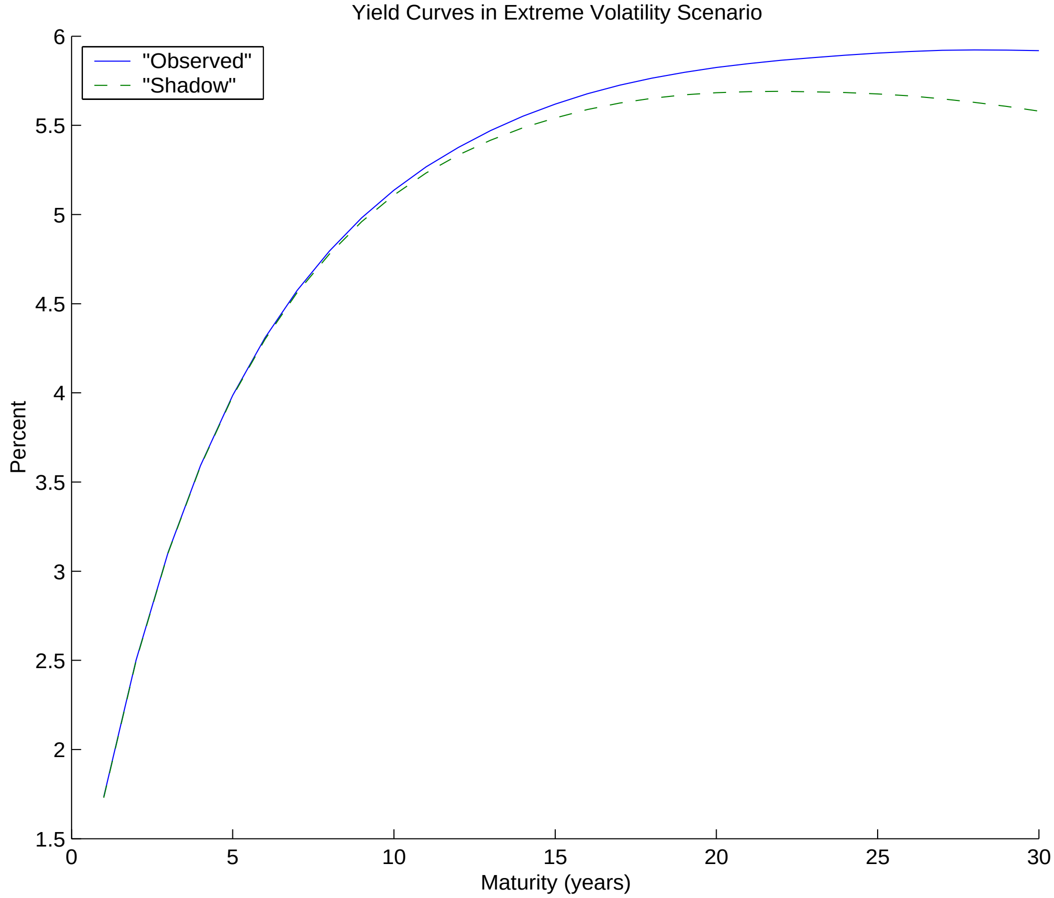

Model−generated Yield Curves (Jan 17, 2003)
zero−bound imposed
zero−bound not imposed
Percent
Maturity (years)
5
4.5
4
3.5
3
2.5
2
1.5
1
2
3
4
5
6
7
8
9
10

Figure 9


Yield Curves in Extreme Volatility Scenario
"Observed"
"Shadow"
Percent
Maturity (years)
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
0
5
10
15
20
25
30