Finance and Economics Discussion Series
Divisions of Research & Statistics and Monetary Affairs
Federal Reserve Board, Washington, D.C.

Using a Projection Method to Analyze Inflation Bias in a Micro-Founded Model

Gary S. Anderson, Jinill Kim, and Tack Yun

2010-18

Using a Projection Method to Analyze Inflation Bias in a Micro-Founded Model

Gary S. Anderson, Jinill Kim, and Tack Yun

Federal Reserve Board

Abstract

Since Kydland and Prescott (1977) and Barro and Gordon (1983), most studies of the problem of the inflation bias associated with discretionary monetary policy have assumed a quadratic loss function. We depart from the conventional linear-quadratic approach to the problem in favor of a projection method approach. We investigate the size of the inflation bias that arises in a microfounded nonlinear environment with Calvo price setting. The inflation bias is found to lie between 1% and 6% for a reasonable range of parameter values, when the bias is defined as the steady-state deviation of the discretionary inflation rate from the optimal inflation rate under commitment.

Keywords: Inflation Bias, Discretionary Monetary Policy, Projection Methods,
JEL: E31, E52, C61, C63

1. Introduction

Since Kydland and Prescott (1977) initiated the literature of rules versus discretion, improvement upon discretionary equilibria by reducing inflation bias has long been a research theme in policy circles as well as academia, including Barro and Gordon (1983), Clarida, Gali and Gertler (1999), M. King (1997) and Woodford (2003). In most of the existing papers on the inflation bias, the one-period loss function assigned to the central bank is quadratic in inflation and the level of output relative to its target. It is well known that Rotemberg and Woodford (1997) and Benigno and Woodford (2006) have provided a microfoundation for the use of such a loss function by showing that this simple quadratic function can be derived as the second-order approximation to the non-linear social welfare function in a Calvo model.

However, as discussed in Woodford (2003), such a derivation does not hold under discretion unless the steady-state level of output under flexible prices is sufficiently close to its efficient level; these papers approximate the model around the deterministic steady

[1]The authors appreciate comments by Andrew Levin, Edward Nelson, David Lopez-Salido, Victor Ríos-Rull, Alex Wolman, and participants at 2008 SCE Conference, as well as encouragements from Mike Woodford. The views in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or any other person associated with the Federal Reserve System.

state with zero inflation, but the optimal allocation under discretion leads to an unknown positive inflation under monopolistic distortion. In light of this observation, this paper does not follow the conventional linear-quadratic approach to studying the inflation bias induced by discretion. Instead, we use a projection method to analyze the inflation bias in a microfounded non-linear model with a Calvo price-setting environment. In our model, since the optimal inflation rate under commitment is zero, the inflation bias is defined as the (optimal) discretionary inflation rate. To do so, we characterize a set of conditions for the optimal allocation under discretion without any approximations. We then use Chebyshev polynomials to approximate policy functions that link inflation and output to a set of state variables, thereby converting optimization conditions into a set of non-linear equations for the coefficients of Chebyshev polynomials. The results on inflation bias based on the global projection method are compared with those based on the linear-quadratic approximation method.

We would like to note that perturbation methods can be modified and used to analyze this problem. For example, Dotsey and Hornstein (2003) and Klein, Krusell and Ríos-Rull (2008) have employed a perturbation method, with an iterative procedure to compute numerical solutions: Dotsey and Hornstein (2003) solve an optimal discretion problem with an iteration of the linear-quadratic approximation, while Klein, Krusell and Ríos-Rull (2008) apply a perturbation procedure to a nonlinear Generalized Euler Equation. These methods can be used to compute the optimal inflation rate at a deterministic steady state. But we have chosen to use the projection method since this method can conveniently be extended to a stochastic setting with technology shocks.

Our paper is not the only one to analyze the discretionary equilibrium in a nonlinear Calvo model. Wolman and Van Zandweghe (2008) use a fixed-point algorithm to solve for the optimal policy instrument and investigate whether multiple Markov Perfect Equilibria can arise in the Calvo model—as compared to the results of King and Wolman (2004) for the Taylor pricing contract. In addition, Adam and Billi (2007) work on optimal discretion in a model that is linear in every aspect except for the zero lower bound for the nominal interest rate.

The rest of this paper is organized as follows. In section 2, we describe a discretionary equilibrium in the Calvo (1983) pricing model where the planner is not allowed to make any commitment about his or her future behavior. Section 3 contains numerical results based on the projection method. In section 4, we conclude.

2. Economic Structure and Discretionary Equilibrium

This section describes the economic structure in our model and the discretionary equilibrium of the planner's problem.

2.1. Economic Structure

The economy is populated by households and firms.

2.1.1. Households

At period 0, the preference ordering of the representative household is summarized by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\upsilon H_t^{1+\chi}}{1+\chi} \right], \tag{1}$$

where C_t denotes consumption, H_t denotes hours worked. The parameter β denotes the time-discount factor, σ measures the degree of relative risk aversion, χ controls the labor supply elasticity, and υ plays the role of fixing the steady-state level for labor. Households purchase differentiated goods in retail markets and combine them into a single composite good using the Dixit-Stiglitz aggregator, and utils of households depend upon the amount of the composite good. The demand curve for each good z can be derived from the following cost-minimization problem:

$$\min \int_0^1 P_t(z) C_t(z)\, dz \quad s.t. \quad C_t = \left(\int_0^1 C_t(z)^{\frac{\epsilon-1}{\epsilon}} dz \right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1, \tag{2}$$

where $P_t(z)$ represents the nominal price of good z and $C_t(z)$ is its demand. The first-order condition for this cost-minimization problem yields the demand curve for firm z:

$$C_t(z) = \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} C_t. \tag{3}$$

The parameter ϵ represents the elasticity of demand, and the aggregate price level P_t is

$$P_t = \left(\int_0^1 P_t^{1-\epsilon}(z)\, dz \right)^{\frac{1}{1-\epsilon}}. \tag{4}$$

The household's dynamic budget constraint at period t is given by

$$C_t + E_t \left[Q_{t,t+1} \frac{B_{t+1}}{P_{t+1}} \right] = \frac{B_t}{P_t} + (1-\tau_W) \frac{W_t}{P_t} H_t + \Xi_t - T_t, \tag{5}$$

where B_{t+1} is the nominal payoff at period $t+1$ of the bond-portfolio held at period t, W_t is nominal wage, and Ξ_t is the real dividend income, T_t is the real lump-sum tax, and τ_W denotes a constant employment tax rate (or subsidy when negative) that is applied to labor income. In addition, $Q_{t,t+1}$ is the stochastic discount factor used for computing the real value at period t of one unit of the consumption good at period $t+1$. Hence, if R_t represents the risk-free nominal (gross) rate of interest at period t, the absence of arbitrage in equilibrium leads to

$$\frac{1}{R_t P_t} = E_t \left[\frac{Q_{t,t+1}}{P_{t+1}} \right]. \tag{6}$$

The representative household maximizes (1) subject to the flow budget constraints (5) in each period. The first-order conditions are given by

$$\upsilon C_t^{\sigma} H_t^{\chi} = (1-\tau_W) \frac{W_t}{P_t}, \tag{7}$$

$$Q_{t,t+1} = \beta \left(\frac{C_t}{C_{t+1}} \right)^{\sigma}, \tag{8}$$

and substitution of (8) into (6) yields the consumption Euler equation:

$$\beta R_t E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{P_t}{P_{t+1}} \right] = 1. \tag{9}$$

2.1.2. Firms

Each firm produces a differentiated good z using a constant returns to scale production function:

$$Y_t(z) = A_t H_t(z), \tag{10}$$

where $Y_t(z)$ is the output of firm z, and $H_t(z)$ denotes the hours hired by the firm and A_t is an exogenous aggregate productivity shock at period t. Firms set prices as in the sticky price model of Calvo (1983). Specifically, each period a fraction of firms $(1-\alpha)$ are allowed to change prices, whereas the other fraction, α, keeps prices the same. Let P_t^* be the new price charged by a firm resetting its price. Then, resetting firms choose a new optimal price in order to maximize the following expected discounted sum of profits:

$$\sum_{k=0}^{\infty} \alpha^k E_t \left[Q_{t,t+k} \left((1-\tau_P) \frac{P_t^*}{P_{t+k}} - \frac{W_{t+k}}{A_{t+k} P_{t+k}} \right) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \right], \tag{11}$$

where τ_P denotes the amount of proportional revenue tax (or subsidy when negative). Differentiating this expression with respect to P_t^* gives rise to the first-order condition:

$$\sum_{k=0}^{\infty} \alpha^k E_t \left[Q_{t,t+k} \left((1-\tau_P) \frac{P_t^*}{P_{t+k}} - \frac{\epsilon}{\epsilon-1} \frac{W_{t+k}}{A_{t+k} P_{t+k}} \right) P_{t+k}^{\epsilon} Y_{t+k} \right] = 0. \tag{12}$$

Furthermore, the Calvo type staggering transforms equation (4) into

$$P_t = \left[(1-\alpha) \left(P_t^* \right)^{1-\epsilon} + \alpha P_{t-1}^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}. \tag{13}$$

Next, we will show that the profit maximization condition (12) can be rewritten in a recursive way. In order to see this, note that substituting (8) into (12) and then rearranging, we have

$$\begin{aligned} & (1-\tau_P) \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{Y_{t+k}}{C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon-1} \right] \frac{P_t^*}{P_t} \\ = & \frac{\epsilon}{\epsilon-1} \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{W_{t+k} Y_{t+k}}{A_{t+k} P_{t+k} C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon} \right]. \end{aligned} \tag{14}$$

It is now useful to define two variables, F_t and S_t, as follows.

$$\begin{aligned} F_t &= (1-\tau_P) \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{Y_{t+k}}{C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon-1} \right], \\ S_t &= \frac{\epsilon}{\epsilon-1} \sum_{k=0}^{\infty} (\alpha\beta)^k E_t \left[\left(\frac{W_{t+k} Y_{t+k}}{A_{t+k} P_{t+k} C_{t+k}^{\sigma}} \right) \left(\frac{P_{t+k}}{P_t} \right)^{\epsilon} \right]. \end{aligned} \tag{15}$$

We then have the following recursive representations of the two variables F_t and S_t:

$$F_t = (1-\tau_P) \frac{Y_t}{C_t^{\sigma}} + \alpha\beta E_t \left[\Pi_{t+1}^{\epsilon-1} F_{t+1} \right], \tag{16}$$

$$S_t = \frac{\epsilon}{\epsilon - 1}\left(\frac{W_t}{P_t A_t}\right)\left(\frac{Y_t}{C_t^{\sigma}}\right) + \alpha\beta E_t\left[\Pi_{t+1}^{\epsilon} S_{t+1}\right], \tag{17}$$

with two terminal conditions,

$$\lim_{T\to\infty}(\alpha\beta)^T E_t\left[\left(\prod_{k=1}^{T}\Pi_{t+k}^{\epsilon-1}\right)F_{t+T}\right] = 0, \quad \lim_{T\to\infty}(\alpha\beta)^T E_t\left[\left(\prod_{k=1}^{T}\Pi_{t+k}^{\epsilon}\right)S_{t+T}\right] = 0,$$

where $\Pi_t = P_t/P_{t-1}$. We now substitute the definitions of F_t and S_t specified above in (15) into the profit maximization condition (14), to yield

$$\frac{P_t^*}{P_t} = \frac{S_t}{F_t}. \tag{18}$$

In addition, substituting equation (18) into (13) leads to

$$1 = (1-\alpha)\left(\frac{S_t}{F_t}\right)^{1-\epsilon} + \alpha\Pi_t^{\epsilon-1}. \tag{19}$$

We have thus expressed the profit maximization condition (14) and the price level definition (13) in terms of F_t and S_t with their intertemporal evolution equations (16) and (17).

2.1.3. Social Resource Constraint

In any model with staggered price setting, relative prices can differ across firms. Furthermore, if firms have different relative prices, there are distortions that create a wedge between the aggregate output measured in terms of production factor inputs and the aggregate demand measured in terms of the composite goods. In order to see the relative price distortions, let us aggregate individual outputs:

$$A_t H_t = Y_t \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} dz,$$

where $H_t = \int_0^1 H_t(z)dz$. By defining a measure of relative price distortion as

$$\Delta_t = \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} dz, \tag{20}$$

the aggregate production function can be written as follows:

$$Y_t = \frac{A_t}{\Delta_t} H_t. \tag{21}$$

In order to obtain a law of motion for the measure of relative price distortion described above, note that the Calvo-type staggering allows one to rewrite the measure of relative price distortions specified in equation (20) as

$$\Delta_t = (1-\alpha)\left(\frac{P_t^*}{P_t}\right)^{-\epsilon} + \alpha\Pi_t^{\epsilon}\Delta_{t-1}. \tag{22}$$

Then, substituting (13) into (22), one can derive an expression for how the measure of relative price distortions evolves over time:

$$\Delta_t = (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} + \alpha\Pi_t^{\epsilon}\Delta_{t-1}. \tag{23}$$

Finally, the aggregate market clearing condition is given by

$$Y_t = C_t, \tag{24}$$

so the social resource constraint in period t is therefore given by

$$\frac{A_t}{\Delta_t}H_t = C_t. \tag{25}$$

2.2. The Planner's Problem under Discretion

In this section, following Woodford (2003), we interpret a planner's problem without commitment as an optimal planning problem. In his book (p. 465), the optimal allocation under discretion is defined as "a procedure under which at each time that an action is to be taken, the central bank evaluates the economy's current state and hence its possible future paths from now on, and chooses the optimal current actions in the light of this analysis, with no advance commitment about future actions, except that they will similarly be the ones that seem best in whatever state may be reached in the future."

Before proceeding, it is worth discussing implementability constraints, which restrict the feasible allocations of the social planner. First, the household budget constraint is not included as a constraint for the optimal allocation problem because of the lump-sum tax. Second, the size of the employment subsidy rate determines whether the profit maximization condition is binding or not as an implementability constraint in the optimal allocation problem.

In order to gain some insights about the role of the employment subsidy, we describe the equilibrium conditions for the flexible price model and then compare them with those for the first-best equilibrium. Since $\alpha = 0$ corresponds to the flexible-price model, it follows from (12) that the profit maximization condition for the flexible-price model turns out to be

$$\frac{W_{f,t}}{P_{f,t}} = (1-\tau_P)\left(1-\epsilon^{-1}\right)A_t. \tag{26}$$

where $W_{f,t}$ and $P_{f,t}$ are the nominal wage rate and the price level in the flexible price model. Combining (7) with (26), we can see that the relationship between MRS and MPL in the flexible price model is given by

$$\upsilon C_{f,t}^{\sigma}H_{f,t}^{\chi} = (1-\Phi)A_t, \tag{27}$$

where $C_{f,t}$ and $H_{f,t}$ denote consumption and labor input in the flexible-price model and Φ measures the overall distortion in the steady-state output level as a result of taxes/subsidies and market power:

$$\Phi = 1-(1-\tau_P)(1-\tau_W)\left(1-\epsilon^{-1}\right).$$

We can see from (27) that when we set $\Phi = 0$, the flexible price model can achieve the efficient level of output—which would be attained at the perfectly competitive equilibrium.

We now characterize the planner's problem under discretion, which is similar to the setup of Adam and Billi (2007) except for their imposition of the zero lower bound and our more disaggregated nonlinear constraints. The government at period 0 chooses a set of decision rules for $\{ C_t, H_t, F_t, S_t, \Pi_t, \Delta_t \}_{t=0}^{\infty}$ in order to maximize

$$V_t(\Delta_{t-1}, A_t) = \max \left\{ \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\upsilon H_t^{1+\chi}}{1+\chi} + \beta E_t \left[V_{t+1}(\Delta_t, A_{t+1}) \right] \right\}, \quad (28)$$

subject to the following equilibrium conditions in each period $t = 0, \cdots, \infty$:

$$C_t = \frac{A_t}{\Delta_t} H_t, \quad (29)$$

$$F_t = (1 - \tau_P) \frac{A_t H_t}{\Delta_t C_t^{\sigma}} + \alpha\beta E_t \left[\Pi_{t+1}(\Delta_t, A_{t+1})^{\epsilon-1} F_{t+1}(\Delta_t, A_{t+1}) \right], \quad (30)$$

$$S_t = \frac{\upsilon H_t^{1+\chi}}{(1 - \tau_W)(1 - \epsilon^{-1}) \Delta_t} + \alpha\beta E_t \left[\Pi_{t+1}(\Delta_t, A_{t+1})^{\epsilon} S_{t+1}(\Delta_t, A_{t+1}) \right], \quad (31)$$

$$\Delta_t = (1 - \alpha) \left(\frac{1 - \alpha \Pi_t^{\epsilon-1}}{1-\alpha} \right)^{\frac{\epsilon}{\epsilon-1}} + \alpha \Pi_t^{\epsilon} \Delta_{t-1}, \quad (32)$$

$$S_t = F_t \left(\frac{1 - \alpha \Pi_t^{\epsilon-1}}{1-\alpha} \right)^{\frac{1}{1-\epsilon}}. \quad (33)$$

Here, the absence of commitment leads us to express the values of period $t + 1$ of the planner's choice variables in terms of the values at period $t + 1$ of state variables such as $F_{t+1}(\Delta_{t+1}, A_{t+1})$, $S_{t+1}(\Delta_{t+1}, A_{t+1})$, and $\Pi_{t+1}(\Delta_{t+1}, A_{t+1})$; that is, future variables are taken as given by the planner, instead of chosen optimally as under commitment. In addition, we allow for the possibility that the value function on the right-hand side differs from that on the left-hand side while the system is away from a stationary equilibrium.[2] The same principle is applied to the notation of functions $F_{t+1}(\Delta_{t+1}, A_{t+1})$, $S_{t+1}(\Delta_{t+1}, A_{t+1})$, $\Pi_{t+1}(\Delta_{t+1}, A_{t+1})$ so that we do not record $F(\Delta_{t+1}, A_{t+1})$, $S(\Delta_{t+1}, A_{t+1})$, and $\Pi(\Delta_{t+1}, A_{t+1})$. In our numerical implementation of the projection method, however, we employ the assumption that functional forms of these functions are invariant over time.

3. Projection Methods and Numerical Results

This section starts with a description of a projection method to obtain numerical solutions for the discretionary equilibrium. We will also present our numerical results regarding the size of optimal inflation under discretion that are compared with those from a linear-quadratic approximation analysis (e.g. Woodford, 2003).

[2] The characterization of optimal policy conditions and our application of the projection method are described in the appendix.

3.1. Projection Method with Homotopy Procedure: A Nontechnical Guide

We employ a projection method to compute numerical solutions that approximate the nonlinear dynamic system of implementablity conditions of the planner's problem and its first-order conditions. In order to deal with a feature of the generalized Euler equation that future variables should be expressed as functions of current variables, we approximate policy functions by a set of Chebyshev polynomials because functional forms of derivatives of Chebyshev polynomials are analytically known. We also adopt a homotopy procedure to improve on our initial guesses for the nonlinear solution. Our use of a homotopy procedure is motivated by our finding that in the course of obtaining valid solutions over the relevant range of state variables, it was important to have flexibility in setting and readjusting the range of these variables.[3]

In our computation, we begin by characterizing the full set of dynamic equilibrium conditions in a non-linear state-space representation. In order to do this, we define a new function Γ in order to collect the policy functions of endogenous variables as follows:

$$\Gamma(s_t) = \Gamma(\Delta(s_t), \Pi(s_t), C(s_t), F(s_t), H(s_t), S(s_t), \phi_1(s_t), \phi_2(s_t), \phi_3(s_t), \phi_4(s_t), \phi_5(s_t))$$

where the realized values at period t of these functions are determined by the following state at period t: s_t=$[\Delta_{t-1} A_t]'$. Here, functions ($\phi_1(s_t)$, $\phi_2(s_t)$, $\phi_3(s_t)$, $\phi_4(s_t)$ and $\phi_5(s_t)$) represent Lagrange multipliers of 5 constraints of the planner's problem (29)–(33). Given the specification of the function Γ, the equilibrium conditions lead to a system of equations satisfying $\mathcal{N}(\Gamma()) = 0$ where $\mathcal{N}(\Gamma()) = 0$ represents our system of 11 equations: equations (29)–(33) and (36)–(41) for endogenous variables. We also assume that the logarithm of the aggregate productivity disturbance follows an AR(1) process:

$$a_t = \rho a_{t-1} + \theta_t,$$

where $a_t = \log A_t$ and the mean-zero Gaussian white noise, θ_t, is identically and independently distributed over time.

Turning to the solution method, we adopt a projection method to approximate the functions. Furthermore, since we allow for random technology shocks, we express each of the functions, Γ_i, as a linear combination of an outer product of orthogonal polynomials in Δ_{t-1} and A_t:

$$\hat{\Gamma}_i(\Delta_{t-1}, A_t) = \sum_{j_1=0}^{k_1} \sum_{j_2}^{k_2} \mu_{j_1 j_2} \varphi_{j_1 j_2}(\Delta_{t-1}, A_t)$$

[3] Judd (1998) provides an exposition about homotopy continuation methods as a part of his discussion on numerical solutions to nonlinear equations. The idea behind continuation methods is to examine and solve a series of problems, beginning with a problem for which we know the solution and ending with the problem of interest. The reason why a continuation method is of our interest is that we can obtain an analytic solution for the discretionary equilibrium in the Calvo pricing model when fiscal policy eliminates the distortion associated with monopolistic competition in retail goods markets. We then take this case as a problem whose solution is known. With this known solution as a starting point, we can solve a series of problems until we reach solution under our target parameter values. In addition, our procedure is comparable to a linear homotopy among his examples. A description of our use of the homotopy method can be found in the appendix.

where $\varphi_{j_1 j_2}(\Delta_{t-1}, A_t)$ express the value at period t of the product of j_1th order Chebyshev polynomial for Δ_t and j_2th order Chebyshev polynomial for A_t.[4] We then use a collocation method to determine the coefficient $\mu_{j_1 j_2}$. In particular, we employ Newton's method to find $\mu_{j_1 j_2}$ such that $\mathcal{N}(\hat{\Gamma}(\Delta_{t-1}, A_t)_l) = 0$ at each point $(\Delta_{t-1},\, A_t)_l$ of Chebyshev nodes $\{(\Delta_{t-1},\, A_t)_l\}_{l=1}^{M}$ with the use of Gauss-Hermite integration for computing the expected values of future variables.

3.2. Some Implementation Issues

Initially, we investigated adapting existing code for solving the problem. We have located freely available FORTRAN code from Judd (1992) and MATLAB code from Gapen and Cosimano (2005). We found that the code was very useful for benchmarking and validation but difficult to modify to solve our particular problem.

For the problem at hand, we have found that—to obtain convergence for a given degree of approximation—it is important to start with a narrow range of values of Δ_{t-1} in the definition of the Chebyshev polynomials, and then gradually extend the range. Thus, our code systematically adjusts the range of Chebyshev polynomials from narrow to wide, for a given set of parameters and a given degree of approximation.

We have implemented the projection method software in Java. The object oriented nature of Java and the availability of the open-source Eclipse IDE (Geer, 2005) for Java greatly facilitated developing the software.[5] Furthermore, because of the notoriously slow "for" loops in MATLAB, the Java code runs much faster than it would have if we had used generic MATLAB routines.[6] The Java code can run on both Linux and Windows machines. We currently use Mathematica as a user-interface to the Java Code. Both JBENDGE and JMulTi are based on the JStatCom which provides a standardized application interface which we hope to adopt in the future.[7]

There are a number of improvements in the code that could be addressed in the future. We envision developing a generic open source tool, but currently the code depends on Mathematica; we would like to develop a Dynare interface. The program uses a simple operator overloading while it would be preferable to use more efficient automatic differentiation techniques.

3.3. Numerical Results

To determine the optimal inflation rate under discretion, we must assign numerical values to the parameters. Although we experimented with many different values, the

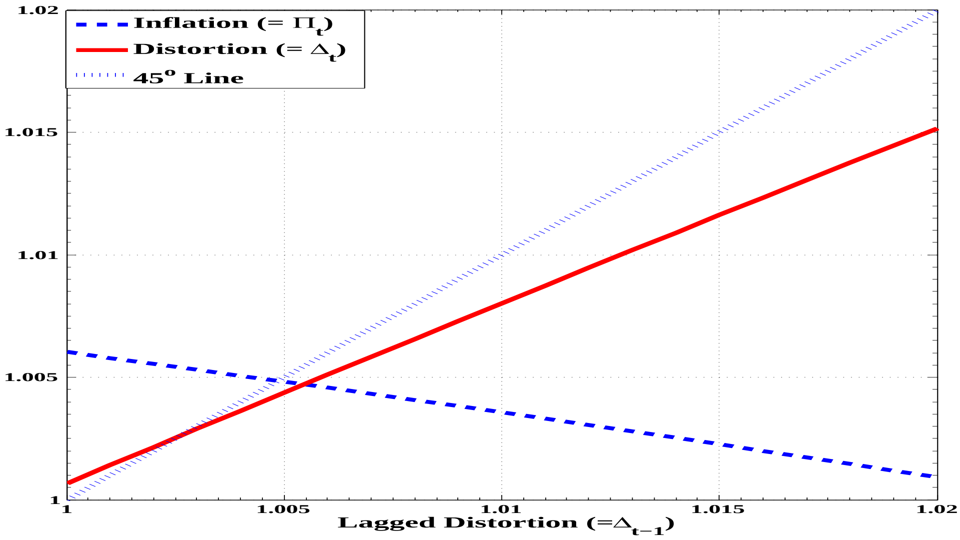
[4]Chebyshev polynomials are a sequence of orthogonal polynomials important in approximation theory. The Chebyshev nodes are the roots of the Chebyshev polynomials. Chebyshev polynomial approximations that interpolate at the Chebyshev nodes provide an approximation that is close to the polynomial of the best approximation to a continuous function under the maximum norm.

[5]Unlike traditional approaches to developing software, an Integrated Development Environment (IDE) brings all of the programmers tools into one convenient place. In the past, programmers had to edit files, save the files out, run the compiler, then the linker, build the application then run it through a debugger. Today's IDEs bring editor, compiler, linker and debugger into one place to increase programmer productivity. MATLAB also provides an object oriented capability, but object creation and method invocation are much slower for object oriented MATLAB than for Java.

[6]MATLAB code consisting of vector-based operations can be faster than Java code, but it would be difficult to construct a set of programs implementing a projection method that relies solely on these types of operations.

[7]For more information on JBENDGE and JMulTi, see Winschel (2008) and Lütkepohl and Krätzig (2004), respectively.

Figure 1: Phase diagram for relative price distortion



Note: This figure expresses the current level of relative price distortion as a function of its lagged level in order to demonstrate how the measure of the relative price distortion distortion converges to its steady-state level.

benchmark parameter values are taken from Yun (2005). For example, we assumed that utility is logarithmic in consumption ($\sigma = 1$) and quadratic in labor ($\chi = 1$). We also set $\epsilon = 11$, $\alpha = 0.75$, and $\beta = 0.99$. We depart from Yun (2005) along one dimension: there is no subsidy nullifying the monopolistic distortion so the degree of monopolistic distortion is kept at $\Phi = \epsilon^{-1}$ in the benchmark. Table 1 summarizes our benchmark parameter values.

Using this benchmark specification, we solve the model via a projection method contemplating values for the relative price distortion in the range of $(1, 1.2)$. Figure 1 illustrates the solution of this discretionary equilibrium. The solid line represents the values of Δ_t as a function of Δ_{t-1}, shown for a narrow range of $(1, 1.02)$ to focus on the area around the steady state. This line crosses the 45-degree line (dotted) at around 1.0026, which is the steady-state value for the dispersion measure. At this steady state, the value of $\bar{\Pi}$ is about 1.0054 (dashed line). In terms of the annualized rate for net inflation, this steady-state inflation rate corresponds to 2.2%.[8]

Since the results in Figure 1 are based on a global solution method, it would be instructive to provide some measure of error in the approximation. As a heuristic measure,

[8]We define the annualized inflation rate in percent as $100 \times \left(\bar{\Pi}^4 - 1\right)$.

Table 1: Calibration of Parameters

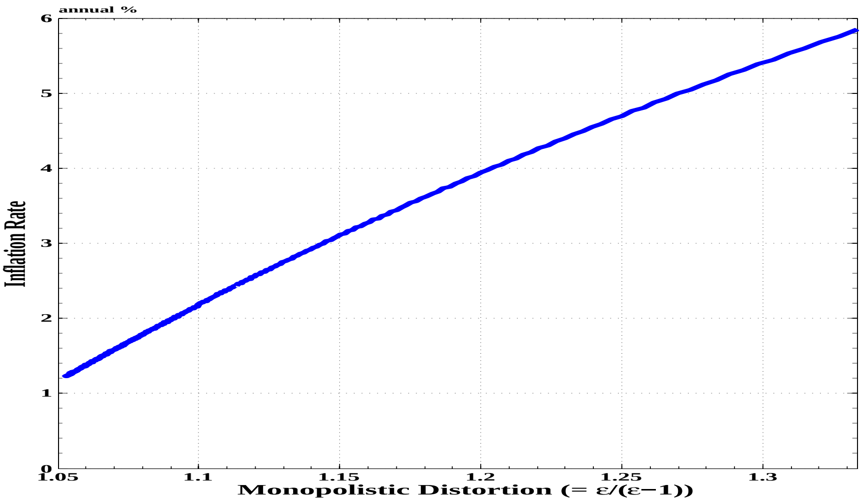
Parameter	Value	Definition
σ	1	Relative risk-aversion coefficient
χ	1	Inverse of labor supply elasticity
β	0.99	Time-discount factor
ϵ	11	Demand elasticity
α	0.75	Probability of fixing prices in each period
Φ	0.091	Degree of distortion
ρ	0.95	AR(1) coefficient of the logarithm of labor productivity
σ_θ	0.01	Standard deviation of technology shock

we compare two ways of computing the relative price distortion. One is the approximate solution for the distortion as reported in Figure 1, and the other is the right-hand side of (32) with inflation set to the values reported in this figure. We then compute the relative difference between these two ways of computing the size of relative price distortion. Over the full collocation range of distortion, $(1, 1.2)$, the maximum percentage difference is on the order of 10^{-8}.

It is widely known that the size of the inflation bias depends on the degree of monopolistic competition, since imperfect competition makes equilibrium flexible-price output lower than the socially efficient output. In our benchmark specification used in Figure 1, the elasticity of substitution (ϵ) determines how monopolistically competitive the economy is, and the size of distortion (Φ) is equal to its reciprocal. Figure 2 shows the size of inflation bias changes as we change the markup by varying ϵ. Our benchmark of $\epsilon = 11$ corresponds to the markup of 1.1. As we decrease ϵ, the markup and inflation bias increase. When we choose $\epsilon = 4$, the markup is 1.33, and the inflation bias is about 6% annually.

To bring out how other parameters affect the size of the inflation bias, we do some comparative statics with the model as shown in Table 2. First, according to the results based on the projection method using our benchmark parameter values, increasing α increases the inflation bias for values of α below 0.85. Increasing α decreases steady

Figure 2: Impact of monopolistic distortion on the inflation bias



Note: The monopolistic distortion means the markup of firms at the deterministic steady-state with zero inflation rate. This figure depicts how changes in the size of the monopolistic distortion in retail goods market affects on the inflation bias.

state inflation for values of α above 0.85.[9] Second, the smaller the curvature parameters (σ and χ), the bigger the inflation bias. When the utility function moves closer to being linear in consumption and labor, the size of the inflation bias increases significantly.

3.4. Comparison with the Linear-Quadratic Approximations

We now compare our results with those from the conventional linear-quadratic approach (e.g. Woodford, 2003). The inflation bias expression that emerges from this approach is

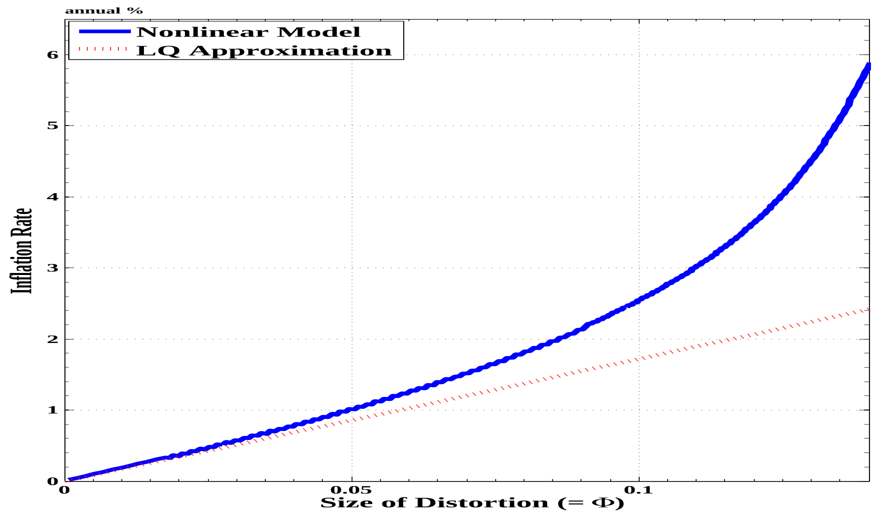
$$\bar{\pi} = \left(\frac{\kappa\lambda}{(1-\beta)\,\lambda + \kappa^2} \right) \frac{\Phi}{\sigma + \chi}, \tag{34}$$

where $\kappa = (1-\alpha)\,(1-\alpha\beta)\,(\sigma+\chi)\,/\alpha$ and $\lambda = \kappa/\epsilon$. This formula produces an inflation bias of 1.6% per annum, under our benchmark calibration including the size of monopolistic competition.[10] Compared to the size of the inflation bias based on the projection

[9] We thank Alex Wolman for pointing out this non-monotonicity.

[10] Monopolistic competition is an indispensable feature in our steady-state analysis of the inflation bias. Even without this deterministic inflation bias, there is a difference between discretion and commitment in a stochastic environment that can arise in the absence of monopolistic competition. This stabilization bias has been studied in a recent paper by Söderström et al. (2005). In such a model, the deterministic steady state is close to the efficient equilibrium, so linear-quadratic approximations would be valid.

Figure 3: Impact of total distortion on the inflation bias



Note: The "total distortion" refers to the wedge between the marginal rate of substitution between consumption and leisure and the marginal product of labor at the steady-state with zero inflation rate. This figure depicts the dependence of the inflation bias on the size of the total steady-state distortion, $\Phi = 0$ corresponding to the absence of distortion.

method (2.2%), the linear-quadratic approach underestimates the inflation bias by about a third.

Figure 3 depicts how much the linear-approximation underestimates the inflation bias as we alter the value of Φ by varying the value of τ_W or τ_P. Note that the range of Φ in this figure covers our benchmark parametrization of $\Phi = 0.091$. The solid line represents the size of inflation bias under our projection method. The magnitude of the inflation bias increases faster than linearly with respect to the amount of monopolistic distortion. The dashed line represents the level of inflation bias under the linear-quadratic approximation and becomes tangent to the solid line as Φ gets close to 0. The difference between the two lines increases as monopolistic distortion moves the economy farther away from the efficient outcome.

Finally, we compare the results of the sensitivity analysis that are obtained from the two approaches. In particular, equation (34) implies that since the discount factor is close to unity, we can approximate the inflation bias under the linear-quadratic approach by using $\bar{\pi} \approx \Phi / [\epsilon (\sigma + \chi)]$. According to this formula, the inflation bias is inversely related to σ, χ, and ϵ (given the size of the total distortion), while the bias is approximately proportional to the size of monopolistic distortion (given the size of the markup). These predictions of the linear-quadratic approach on the sensitivity analysis are confirmed by numerical results in the final column of Table 2. Furthermore, as noted earlier, the nonlinear projection method indicates that the smaller the curvature parameters (σ and χ), the bigger the inflation bias. The linear-quadratic and nonlinear projection approaches therefore appear to produce a similar relationship between parameter values and the size of the inflation bias. Based on these numerical results, one might argue that the simplicity and transparency of the linear-quadratic approach would be very helpful in relating the size of the inflation bias to the values for the parameters, though the linear-quadratic approach underestimates the size of inflation bias. But it should be noted that such an interpretation could potentially be misleading. For example, when the discount factor is less than unity, the expression for $\bar{\pi}$ indicates that an increase in α would imply a decrease in $\bar{\pi}$. However, according to the results based on the projection method using our benchmark parameter values, the size of inflation bias is not a monotone function of α.

4. Conclusion

We have demonstrated how a projection method can be used to compute the inflation bias in a full nonlinear version of the Calvo model. The annual inflation bias is between 1% and 6% under plausible parameter values.

In a recent paper, Schmitt-Grohe and Uribe (2009) report that the optimal inflation rate under commitment predicted by leading theories of monetary nonneutrality ranges from minus the real rate of interest to numbers insignificantly above zero. They also argue that the zero bound on nominal interest rates does not represent an impediment to setting inflation targets near or below zero. Meanwhile, our results indicate that the optimal inflation rate turns out to be substantially higher than zero in the absence of commitment.

In particular, we expect that the larger the "degree" of commitment, the smaller the size of the inflation bias. It would thus be interesting to see how the change in the "degree" of discretion affects the size of the inflation bias. In this vein, the format of Debortoli and Nunes (2007) provides an interesting starting point because they have modelled an imperfect commitment setting in which there is a continuum of loose commitment possibilities ranging from full commitment to full discretion.[11] In addition, we note that it would be possible to use the same projection method to analyze the effects of loose commitment on the inflation bias.

[11] Schaumburg and Tambalotti (2007) also discussed intermediate cases between discretion and commitment using a linear-quadratic model.

Table 2: Sensitivity Analysis

Parameter	Values				Numerical Results		
	α	σ	χ	ϵ	Price Distortion	Nonlinear Solution	*LQ* Solution
	0.5	1	1	11	1.001	1.004	1.004
α	0.75	1	1	11	1.003	1.005	1.004
	0.95	1	1	11	1.093	1.003	1.003
	0.75	0.16	1	11	1.048	1.016	1.007
σ	0.75	1	1	11	1.003	1.005	1.004
	0.75	5	1	11	1.001	1.003	1.001
	0.75	1	0.25	11	1.027	1.013	1.008
χ	0.75	1	1	11	1.003	1.005	1.004
	0.75	1	4.75	11	1.001	1.002	1.001
ϵ	0.75	1	1	11	1.003	1.005	1.004
	0.75	1	1	21	1.001	1.001	1.001

Note: The last two columns represent quarterly (gross) inflation for each set of parameter values. Specifically, the nonlinear solution corresponds to $\bar{\Pi}$ and the linear-quadratic solution represents $(1+\bar{\pi})$. In addition, the price distortion measures $\bar{\Delta}$, where $\bar{\Delta}$ denotes the steady-state level of the relative price distortion. The sensitivity analysis for ϵ is carried out by setting the value of Φ at its benchmark.

References

Adam, K. and R. Billi (2007) “Discretionary Monetary Policy and the Zero Lower Bound on Nominal Interest Rates,” *Journal of Monetary Economics*, 54, 728-752.

Barro, R. and D. Gordon (1983) “A Positive Theory of Monetary Policy in a Natural Rate Model,” *Journal of Political Economy*, 91, 589–610.

Benigno, P. and M. Woodford. (2006) “Optimal Taxation in an RBC Model: A Linear-Quadratic Approach,” *Journal of Economic Dynamics and Control*, 30, 1445-1489.

Calvo, G. (1983) “Staggered Prices in a Utility Maximizing Framework” *Journal of Monetary Economics*, 12, 383-398.

Clarida, R., J. Gali, and M. Gertler (1999) “The Science of Monetary Policy: A New Keynesian Perspective,” *Journal of Economic Literature*, 37, 1661-1707.

Debortoli, D. and R. Nunes. (2007) “Loose Commitment,” Federal Reserve Board, International Finance Discussion Paper 916.

Dotsey M. and A. Hornstein. (2003) “Should a Monetary Policy Maker Look at Money?” *Journal of Monetary Economics*, 50, 547-579.

Gapen, M. and T. Cosimano. (2005) “Solving Ramsey Problems with Nonlinear Projection Methods,” *Studies in Nonlinear Dynamics & Econometrics* 9.

Geer, D. (2005) “Eclipse Becomes the Dominant Java IDE.” *Computer*, 38, 16-18.

Judd, K. (1992) “Projection Methods for Solving Aggregate Growth Models,” *Journal of Economic Theory*, 58, 410-452

Judd, K. (1998) Numerical Methods in Economics, MIT Press, Cambridge, MA.

Klein, P., P. Krushell, and J. V. Ríos-Rull. (2008) “Time-Consistent Public Policy,” *Review of Economic Studies*, 75(3), 789-808.

King, M. (1997) “Changes in UK Monetary Policy: Rules and Discretion in Practice,” *Journal of Monetary Economics*, 39, 81-97.

King, R. and A. Wolman (2004) “Monetary Discretion, Pricing Complementarity, and Dynamic Multiple Equilibria,” *Quarterly Journal of Economics*, 119, 1513-1553.

Kydland, F. and E. Prescott (1977) “Rules rather than Discretion: The Inconsistency of Optimal Plans,” *Journal of Political Economy*, 85, 473-491.

Schaumburg, E. and A. Tambalotti (2007) “An investigation of the gains from commitment in monetary policy,” *Journal of Monetary Economics*, 54, 302-324.

Schmitt-Grohe, S. and M. Uribe (2009) “The Optimal Rate of Inflation,” *Forthcoming in Handbook of Monetary Economics.*

Wolman, A. and W. Van Zandweghe (2008) “Discretionary Monetary Policy in the Calvo Model,” Unpublished Manuscript.

Woodford, M. (2003) Interest and Prices, Princeton University Press, Princeton, NJ.

Yun, T. (2005) “Optimal Monetary Policy with Relative Price Distortions,” *American Economic Review*, 95, 89-108.

Appendices

This appendix provides additional detail about the model specification and our solution technique. Section Appendix A presents a full description of the Lagrangian of the government's planning problem when the planner cannot make commitment about his or her future behavior. Section Appendix B provides additional detail about our implementation of the residual function for the projection method. Section Appendix C describes how we use homotopy methods to obtain solutions.

Appendix A. Lagrangian

In the presence of technology shocks, the Lagrangian of this problem can be written as

$$\begin{aligned}
\pounds \quad = \quad & \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{\upsilon H_t^{1+\chi}}{1+\chi} + \beta E_t\left[V\left(\Delta_t, A_{t+1}\right)\right] \\
& +\phi_{1t}\left[\frac{A_t H_t}{\Delta_t} - C_t\right] \\
& -\phi_{2t}\left[\left(1-\tau_P\right)\frac{A_t H_t}{\Delta_t C_t^{\sigma}} + \alpha\beta E_t\left[L\left(\Delta_t, A_{t+1}\right)\right] - F_t\right] \\
& -\phi_{3t}\left[\frac{\upsilon\left(1-\tau_P\right)H_t^{1+\chi}}{\left(1-\Phi\right)\Delta_t} + \alpha\beta E_t\left[M\left(\Delta_t, A_{t+1}\right)\right] - S_t\right] \\
& +\phi_{4t}\left[\left(1-\alpha\right)\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} + \alpha\Pi_t^{\epsilon}\Delta_{t-1} - \Delta_t\right] \\
& -\phi_{5t}\left[F_t\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{1-\epsilon}} - S_t\right].
\end{aligned}$$

where auxiliary functions $L\left(\Delta_t, A_{t+1}\right)$ and $M\left(\Delta_t, A_{t+1}\right)$ are defined as

$$L\left(\Delta_t, A_{t+1}\right) = \Pi_{t+1}^{\epsilon-1}F_{t+1}, \tag{A.1}$$

$$M\left(\Delta_t, A_{t+1}\right) = \Pi_{t+1}^{\epsilon}S_{t+1}. \tag{A.2}$$

An noted earlier, the absence of commitment leads us to express the values of period $t+1$ of the planner's choice variables in terms of the values at period $t+1$ of state variables such as $F_{t+1}(\Delta_{t+1}, A_{t+1})$, $S_{t+1}(\Delta_{t+1}, A_{t+1})$, and $\Pi_{t+1}(\Delta_{t+1}, A_{t+1})$. In order to simplify the characterization of the first-order conditions of the planner's problem, we introduce two new functions $L\left(\Delta_t, A_{t+1}\right)$ and $M\left(\Delta_t, A_{t+1}\right)$ as composite functions of $F_{t+1}(\Delta_{t+1}, A_{t+1})$, $S_{t+1}(\Delta_{t+1}, A_{t+1})$, and $\Pi_{t+1}(\Delta_{t+1}, A_{t+1})$ respectively.

Having described the optimal policy problem under discretion, the first-order conditions can be summarized as follows:

$$1 + \sigma\frac{A_t H_t}{\Delta_t C_t}\phi_{2t} = \phi_{1t}C_t^{\sigma}, \tag{A.3}$$

$$\upsilon\Delta_t C_t^{\sigma}H_t^{\chi} + A_t\phi_{2t} + \frac{\upsilon\left(1+\chi\right)}{\left(1-\Phi\right)}\phi_{3t}C_t^{\sigma}H_t^{\chi} = \phi_{1t}A_t C_t^{\sigma}, \tag{A.4}$$

$$\phi_{2t} = \phi_{5t}\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{1-\epsilon}}, \tag{A.5}$$

$$\phi_{3t} = -\phi_{5t}, \tag{A.6}$$

$$\epsilon\left(\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{\epsilon-1}} - \Pi_t\Delta_{t-1}\right)\phi_{4t} = \frac{-1}{1-\alpha}\left(\frac{1-\alpha\Pi_t^{\epsilon-1}}{1-\alpha}\right)^{\frac{-\epsilon}{\epsilon-1}} F_t\phi_{5t}, \tag{A.7}$$

$$\frac{A_tH_t}{\Delta_t^2C_t^\sigma}\phi_{2t} + \phi_{3t}\frac{\upsilon H_t^{1+\chi}}{(1-\Phi)\Delta_t^2} - \phi_{4t} + \alpha\beta E_t\left[\Pi_{t+1}(\Delta_t, A_{t+1})^\epsilon\phi_{4t+1}\right] = \phi_{1t}\frac{A_tH_t}{\Delta_t^2} + \alpha\beta E_t\left[\phi_{2t}L_1(\Delta_t, A_{t+1}) + \phi_{3t}M_1(\Delta_t, A_{t+1})\right], \tag{A.8}$$

where ϕ_{1t}, ϕ_{2t}, ϕ_{3t}, ϕ_{4t}, and ϕ_{5t} are Lagrange multipliers for (29), (30), (31), (32), and (33) respectively, and τ_P is assumed to be zero for simplicity.

Appendix B. Projection Method with Collocation

We will approximate 11 policy functions by using Chebyshev polynomials as follows:

$$\Gamma_i(\Delta_{t-1}, a_t) = \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\omega_{j_1j_2}\varphi_{j_1j_2}(\Delta_{t-1}, a_t), \quad i = C, H, \Delta, \Pi, S, F, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5$$

where the function φ is defined as

$$\varphi_{j_1j_2}(\Delta_{t-1}, a_t) = T_{j_1-1}\left(\frac{2(\Delta-\Delta_{\min})}{\Delta_{\max}-\Delta_{\min}} - 1\right)T_{j_2-1}\left(\frac{2(a-a_{\min})}{a_{\max}-a_{\min}} - 1\right)$$

Here $T_j(x)$ denotes jth order Chebyshev polynomials. In this appendix, we use the logarithm of labor productivity as an argument of policy functions.

Having determined functional forms of approximate policy functions, we will determine a nonlinear system of equations for weights of 11 approximate policy functions. Specifically we use 11 equilibrium conditions to define 11 residual functions as follows. Each equilibrium condition generates a residual function as can be seen below:

$$R_1 = \frac{\exp(a_t)\Gamma_H(s_t)}{\Gamma_\Delta(s_t)} - \Gamma_C(s_t)$$

$$R_2 = \Gamma_C(s_t)^{1-\sigma} + \alpha\beta E_t[L(s_{t+1})] - \Gamma_F(s_t)$$

$$R_3 = \frac{\nu\Gamma_H(s_t)^{1+\chi}\Gamma_\Delta(s_t)}{1-\Phi} + \alpha\beta E_t[M(s_{t+1})] - \Gamma_S(s_t)$$

$$R_4 = \Gamma_\Delta(s_t) - (1-\alpha)\left(\frac{1-\alpha\Gamma_\Pi(s_t)^{\epsilon-1}}{1-\alpha}\right)^{\frac{\epsilon}{\epsilon-1}} - \alpha\Gamma_\Pi(s_t)^\epsilon\Delta_{t-1}$$

$$R_5 = \Gamma_F(s_t)\left(\frac{1-\alpha\Gamma_\Pi(s_t)^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{1-\epsilon}} - \Gamma_S(s_t)$$

$$R_6 = \sigma\Gamma_{\phi_2}(s_t) - \Gamma_C(s_t)^{\sigma}\Gamma_{\phi_1}(s_t) + 1$$

$$R_7 = \Gamma_{\phi_2}(s_t)\exp(a_t) + \nu\Gamma_C(s_t)^{\sigma}\Gamma_H(s_t)^{\chi}(\Gamma_\Delta(s_t) + \zeta\Gamma_{\phi_3}(s_t)) - \Gamma_C(s_t)^{\sigma}\Gamma_{\phi_1}(s_t)\exp(a_t)$$

$$R_8 = \Gamma_{\phi_2}(s_t) - \left(\frac{1-\alpha\Gamma_\Pi(s_t)^{\epsilon-1}}{1-\alpha}\right)^{\frac{1}{1-\epsilon}}\Gamma_{\phi_5}(s_t)$$

$$R_9 = \Gamma_{\phi_3}(s_t) + \Gamma_{\phi_5}(s_t)$$

$$\begin{aligned} R_{10} = \quad & \epsilon\left(\left(\tfrac{1-\alpha\Gamma_\Pi(s_t)^{1-\epsilon}}{1-\alpha}\right)^{\frac{1}{\epsilon-1}} - \Gamma_\Pi(s_t)\Gamma_\Delta(s_t)\right)\Gamma_{\phi_4}(s_t) \\ & + \tfrac{1}{1-\alpha}\left(\tfrac{1-\alpha\Gamma_\Pi(s_t)^{\epsilon-1}}{1-\alpha}\right)^{\frac{-\epsilon}{\epsilon-1}}\Gamma_F(s_t)\Gamma_{\phi_5}(s_t) \end{aligned}$$

$$\begin{aligned} R_{11} = \quad & \tfrac{\Gamma_C(s_t)^{1-\sigma}}{\Gamma_\Delta(s_t)}\Gamma_{\phi_2}(s_t) + \Gamma_{\phi_3}(s_t)\tfrac{\nu(1+\chi)\Gamma_H(s_t)^{1+\chi}}{(1-\Phi)\Gamma_\Delta(s_t)} - \Gamma_{\phi_4}(s_t) \\ & + \alpha\beta E_t\left[\Gamma_\Pi(s_{t+1})^{\epsilon}\Gamma_{\phi_4}(s_{t+1})\right] - \Gamma_{\phi_1}(s_t)\Gamma_C(s_t) - \Gamma_\Delta(s_t) \\ & - \alpha\beta E_t\left[\Gamma_{\phi_2}(s_t)L_1(s_{t+1}) + \Gamma_{\phi_3}(s_t)M_1(s_{t+1})\right] \end{aligned}$$

where $\zeta = \frac{1+\chi}{1-\Phi}$, functions $L(s_t)$ and $M(s_t)$ are defined as $L(s) = \Gamma_\Pi(s_t)^{\epsilon}\Gamma_F(s_t)$ and $M(s) = \Gamma_\Pi(s_t)^{\epsilon}\Gamma_S(s_t)$. Hence, the partial derivatives of these two functions with respect to Δ_t can be written as

$$L_1(s_t) = (\epsilon-1)\Gamma_\Pi(s_t)^{\epsilon-2}\Gamma_F(s_t)\partial\Gamma_\Pi(s_t) + \Gamma_\pi(s_t)^{\epsilon-1}\partial\Gamma_F(s_t)$$

$$M_1(s_t) = \epsilon\Gamma_\Pi(s_t)^{\epsilon-1}\Gamma_S(s_t)\partial\Gamma_\Pi(s_t) + \Gamma_\Pi(s_t)^{\epsilon}\partial\Gamma_S(s_t)$$

The derivatives of policy functions $\partial\Gamma_\Pi(s_t)$, $\partial\Gamma_F(s_t)$ and $\partial\Gamma_S(s_t)$ can be derived as follows:

$$\partial\Gamma_i(s_t) = \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\frac{2\omega_{j_1j_2}}{\Delta_{\max}-\Delta_{\min}}T'_{j_1-1}(\frac{2(\Delta-\Delta_{\min})}{\Delta_{\max}-\Delta_{\min}}-1)T_{j_2-1}(\frac{2(a-a_{\min})}{a_{\max}-a_{\min}}-1)$$

for Π, F, and S and where $T'_{j_1-1}(x)$ denotes the derivative of the jth order Chebyshev polynomials. The derivatives of the Chebyshev polynomials are easy to compute using the following relation:

$$T'_j(x) = j\ U_{j-1}(x)$$

for $j = 1, \cdots, \infty$ and sequences of two polynomials $\{T_j(x)\}_{j=0}^{\infty}$ and $\{U_j(x)\}_{j=0}^{\infty}$ are recursively defined as

$$\begin{array}{lll} U_{j+1}(x) = 2xU_j(x) - U_{j-1}(x) & U_1(x) = 2x & U_0(x) = 1 \\ T_{j+1}(x) = xT_j(x) - T_{j-1}(x) & T_1(x) = x & T_0(x) = 1. \end{array}$$

We now move onto the characterization of the integrals appearing in the residual functions. Fortunately, the expectation operator only involves three terms.

$$\alpha\beta E_t[L(s_{t+1})]$$
$$\alpha\beta E_t[M(s_{t+1})]$$
$$\alpha\beta E_t\left[\Gamma_{\phi_2}(s_t)L_1(s_{t+1}) + \Gamma_{\phi_3}(s_t)M_1(s_{t+1})\right]$$

It is assumed in the paper that the technology shock follows a normal distribution with mean zero and standard deviation σ_θ. Hence, we standardize the shock by using $\theta_t = \sigma_\theta z_t$ where $z_t \sim N(0,1)$. The integral of an expression involving our approximated policy functions $I(\Delta, a, \mathcal{W}, z)$ is then approximated using the following finite sum

$$\int_{-\infty}^{\infty} I(\Delta, a, \mathcal{W}, z)\frac{\exp(-\frac{z^2}{2})}{\sqrt{2}}dz = \sum_{l=1}^{k_z} I(\Delta, a, \mathcal{W}, \sqrt{2}z_l)m_l$$

where m_l and z_l are Gauss-Hermite quadrature weights and points and $\mathcal{W}$ is the set that includes all weights of 11 approximate policy functions. Given this approximation of the integrals, conditional expectations of functions $L(s_{t+1})$ and $M(s_{t+1})$ can be written as

$$E_t[M(s_{t+1})] = \sum_{l=1}^{k_z} \hat{M}(\Delta, a, \mathcal{W}, \sqrt{2}z_l)m_l; E_t[L(s_{t+1})] = \sum_{l=1}^{k_z} \hat{L}(\Delta, a, \mathcal{W}, \sqrt{2}z_l)m_l$$

where functions $\hat{M}(s, \omega, z)$ and $\hat{L}(s, \omega, z)$ are defined as

$$\begin{aligned} \hat{M}(s, \omega, z) &= \Gamma_\Pi(\Gamma_\Delta(\Delta, a), \rho_a a + z)\Gamma_F(\Gamma_\Delta(\Delta, a), a\rho_a + z) \\ \hat{L}(s, \omega, z) &= \Gamma_\Pi(\Gamma_\Delta(\Delta, a), a\rho_a + z)\Gamma_S(\Gamma_\Delta(\Delta, a), a\rho_a + z) \end{aligned}$$

We will consider collocation. Under orthogonal collocation, we choose $k_{j_1} \times k_{j_2}$ zeros of $\varphi_{j_1 j_2}(\Delta, a)$ and then substitute them into residual functions. Since all of 11 residual functions should become zero for each point of (Δ_{j_1}, a_{j_2}), it means that $R(\Delta_{j_1}, a_{j_2}) = 0_{11\times 1}$ holds where (Δ_{j_1}, a_{j_2}) represents a collocation point among $k_{j_1} \times k_{j_2}$ zeros of $\varphi_{j_1 j_2}(\Delta, a)$ and $R(\Delta_{j_1}, a_{j_2})$ represents a vector function that contain residual functions:

$$R(\Delta_{j_1}, a_{j_2}) = [R_1(\Delta_{j_1}, a_{j_2}), \cdots R_{11}(\Delta_{j_1}, a_{j_2})]'.$$

In addition, the set of zeros of Chebyshev polynomials can be written as follows.

$$\Delta_{j_1} = (z_{j_1} + 1)\frac{\Delta_{\max} - \Delta_{\min}}{2} + \Delta_{\min};\ a_{j_2} = (z_{j_2} + 1)\frac{a_{\max} - a_{\min}}{2} + a_{\min}$$

where z_{j_1} and z_{j_2} are defined as $z_{j_1} = \cos(\frac{(2j_1-1)\pi}{2k_1})$ and $z_{j_2} = \cos(\frac{(2j_2-1)\pi}{2k_2})$. As a result, we have a nonlinear system of equations for weights of approximate policy functions. We then use Newton's method in order to find a numerical solution to this nonlinear system of equation.

Finally we discuss how we choose ranges of the aggregate productivity and the relative price distortion. Following Judd (1992), the maximum of log productivity is set equal to the long-run value of a that would occur if $\theta = 2\sigma_\theta$ for all t: $a_{\max} = 2\sigma_\theta/(1-\rho)$.[12] The minimum of log productivity is the negative of maximum of log productivity. The minimum value of the relative price distortion is 1. However, it is hard to make an appropriate choice of the maximum of the relative price distortion. In particular, this issue is closely related to our application of homotopy method that will be explained in the next section.

[12] Judd actually used 3σ, but, for our benchmark parameter settings, the use of a smaller range had no impact on the results.

Appendix C. Projection Method with Homotopy Procedure

In the course of developing our model, we created routines for several heuristics for solving the nonlinear-equation system determining the collocation-polynomial weights. This section characterizes these heuristics using the homotopy formalism described in Judd (1998).

To summarize the basic idea, we begin by solving the model for a set of parameters that makes the model easy to solve. We use this solution to facilitate the solution of a "nearby" model that has parameters set closer to the parameter settings that we are really interested in. We repeat this process, solving a sequence of similar models en route to solving the model with our benchmark parametrization.

It might prove useful, in general, to use information provided by perturbation solutions as a basis for initial weights for the collocating polynomials. However, in our model we found the following heuristics easily implementable and capable of reliably producing accurate approximations with the appropriate dynamic properties. Although we did not do so, in the general case it seems likely to be worthwhile to investigate how to reliably exploit high order perturbation solutions to provide initial weights for projection calculations.

There are two distinct phases in this solution process. In the first phase, we solve the model using $0th$ order polynomials, i.e. constant functions varying the parameters from the easy values to the benchmark values.[13] This phase employs the parameter homotopy described in Algorithm 1 below.[14]

In the second phase, we increase the order of the Chebyshev polynomials. Prior to this phase, we are collocating with constant functions, and the range of the Chebyshev polynomials plays no role. In the second phase, the specific range of the Chebyshev polynomials can have a dramatic effect on the solvability of the system. Fortunately, experience with our model supports the following conjecture:

Conjecture 1 *Let $\mathcal{M}_0$ correspond to a basis consisting entirely of 0th order Chebyshev polynomials. Suppose we can solve the 0th order problem so that there exists $\mathcal{W}_0^*$ such that*

$$\mathcal{W}_0^* = \mathcal{N}(\hat{\Lambda}(s_t, \Upsilon, \mathcal{I}, \mathcal{M}_0), \mathcal{W}_0^*)$$

then

$$\exists\, \gamma > 0 \ni \mathcal{W}^* = \mathcal{N}(\hat{\Lambda}(s_t, \Upsilon, \mathcal{I}(\gamma), \mathcal{M}), \mathcal{W}_0^*) \neq \emptyset$$

■

Thus, when the Chebyshev polynomial domain is small enough, the nonlinear system we must solve is similar to the system for $\mathcal{W}_0$. As a result, Newton's method will also converge for the problem with higher order polynomials when the domain of the

[13] In our model, since setting $\tau_P = 0, \tau_W = \frac{1}{\epsilon - 1}$ leads to a non-distorted steady state with $\bar{\Delta} = \bar{\Pi} = 1$ this parametrization is easy to solve.

[14] We also use parameter homotopy to generate graphs of the steady state values of variable vis a vis parameters.

Chebyshev polynomials is small enough. Consequently, when necessary, we can employ a homotopy on the range to extend the range from a small range to the full range. Although, in practice, the algorithms rarely solve the problem on such a small domain, the existence of such a domain guarantees that the algorithms will terminate. We have automated this heuristic using the algorithms described in Algorithm 2 below.

We express a projection method using the function $\hat{\Lambda}(s_t, \Upsilon, I, M)$ where s_t is the state at period t, Υ is the set of parameters, I is the set of ranges for the Chebyshev polynomials, and M is the set of orders for the Chebyshev polynomials. We use Newton's method to update the Chebyshev polynomials' weights in the following way:

$$\mathcal{N}(\hat{\Lambda}(s_t, \Upsilon, I, M), \mathcal{W}_i) = \begin{cases} W_{i+1}, & \text{if Newton method converges;} \\ \emptyset, & \text{if Newton method fails} \end{cases}$$

where $\mathcal{N}$ represents an application of Newton's method and $\mathcal{W}_i$ is a set of weights. We now describe the two types of homotopy. In the case of the range homotopy, we predetermine a set of ranges of the Chebyshev polynomials. For example, we use a tensor product of finite number of bounded and closed intervals: $I = [l_1, u_1] \otimes \cdots \otimes [l_m, u_m]$ where m represents the number of state variables and each interval specifies the minimum and maximum of a state variable. In order to implement the range homotopy, we define a nested range for each range $[l_k,\, u_k]$ by using a parameter γ:

$$\mu(\gamma, [l_k, u_k]) = [l_k + (1-\gamma)\frac{l_k + u_k}{2}, u_k - (1-\gamma)\frac{l_k + u_k}{2}] \quad \text{for} \qquad k = 1, \cdots, m$$

where $\mu(\gamma, [l_k, u_k])$ denotes the nested range for each interval $[l_k, u_k]$. Hence, we have a new set of ranges of the Chebyshev polynomials: $I(\gamma) = \mu([l_1, u_1]) \otimes \cdots \otimes \mu([l_m, u_m])$. As a result, a particular Newton method step can be described as follows:

$$\mathcal{N}(\hat{\Lambda}(s_t, \Upsilon, I(\gamma), M), \mathcal{W}_i, \gamma) = \begin{cases} W_{i+1}, & \text{if Newton method converges;} \\ \emptyset, & \text{if Newton method fails.} \end{cases}$$

In applying a homotopy, we seek a $\mathcal{W}_0$ and a sequence $\{\gamma_n\}_{n=0}^N$ such that there is no failure of Newton's method for each value of $\{\gamma_n\}_{n=0}^N$. Thus, any algorithm applying a homotopy method must implement strategies for adjusting the value of γ when Newton's method fails. In our code, we choose a recursive updating rule for the value of γ. For example, suppose that we failed the Newton's method at $\gamma = \gamma_n$. In this case, we shrink the value of γ_n by setting a new value of γ_n as follows: $\gamma_n^1 = \nu\ \gamma_n^0$, where γ_n^1 denotes the new trial value of γ, γ_n^0 is the old trial value of γ at the last round of Newton's method and ν is a shrink factor that is a positive constant between 0 and 1. If the Newton's method does not fail, we use the current value of γ_n as a new value of γ_n. The following pseudo-code characterizes algorithms that work for our model.

Algorithm 1 Parameter Homotopy Procedures

procedure MoveLowerEndToUpper($\mathcal{N}(\hat{\Lambda}(s_t, \cdot, \mathcal{I}, \mathcal{M}), \mathcal{W}), \Upsilon^*, \{\Upsilon_0, \mathcal{W}\}$)
    $Q :=$ **true**
    $\Upsilon := \Upsilon_0$
    $\mathcal{W} = \mathcal{W}_0$
    while Q **do**
        $\{\Upsilon, \mathcal{W}\} := \mathit{FindBetterParams}(\mathcal{N}(\hat{\Lambda}(s_t, \cdot, \mathcal{I}, \mathcal{M}), \mathcal{W}), \Upsilon^*, \{\Upsilon, \mathcal{W}\})$
        if $\Upsilon = \Upsilon^*$ **then**
            $Q :=$ **false**
        end if
    end while
end procedure

procedure FindBetterParams($\mathcal{N}(\hat{\Lambda}(s_t, \cdot, \mathcal{I}, \mathcal{M}), \cdot), \Upsilon^*, \{\Upsilon_0, , \mathcal{W}_0\}$)
    $\nu \in (0, 1)$
    $\gamma := 1$
    $Q :=$ **true**
    while Q **do**
        $\mathcal{W}^* := \mathcal{N}(\hat{\Lambda}(s_t, \eta(\gamma, \Upsilon_0, \Upsilon_*), \mathcal{I}, \mathcal{M}), \mathcal{W})$
        if $\mathcal{W}^* = \emptyset$ **then**
            $\gamma := \nu \times \gamma;$
        else
            $Q :=$ **false**
        end if
    end while
    return($\{\eta(\gamma, \Upsilon_0, \Upsilon_*), \mathcal{W}^*\}$)
end procedure

Algorithm 2 Range Homotopy Procedures

```
procedure WidenRangeToFull(𝒩(Λ̂(s_t, ϒ, ℐ(·), ℳ), 𝒲_0))
    Q := true
    𝒲 = 𝒲_0
    while Q do
        {γ, 𝒲} := FindWiderRange(𝒩(Λ̂(s_t, ·, ℐ, ℳ), 𝒲), α*, 𝒲)
        if γ := 1 then
            Q := false
        end if
    end while
end procedure
```

```
procedure FindWiderRange(𝒩(Λ̂(s_t, ϒ, ℐ(·), ℳ), 𝒲_0))
    ν ∈ (0, 1)
    γ := 1
    Q := true
    while Q do
        𝒲* := 𝒩(Λ̂(s_t, ϒ, ℐ(γ), ℳ), 𝒲_0))
        if 𝒲* = ∅ then
            γ := ν × γ
        else
            Q := false
        end if
    end while
    return(γ, 𝒲*}))
end procedure
```