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# Uncertainty at the Zero Lower Bound

**Taisuke Nakata**

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Taisuke Nakata†
Federal Reserve Board

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**Abstract**

This paper examines how the presence of uncertainty alters allocations and prices when the nominal interest rate is constrained by the zero lower bound. I conduct the analysis using a standard New Keynesian model in which the nominal interest rate is determined according to a truncated Taylor rule. I find that an increase in the variance of shocks to the discount factor process reduces consumption, inflation, and output by a substantially larger amount when the zero lower bound is binding than when it is not. Due to the zero lower bound constraint, policy functions for the real interest rates and the marginal costs of production are highly convex and concave, respectively. As a result, a mean-preserving spread in the shock distribution increases the expectation of future real interest rates and decreases the expectation of future real marginal costs, which lead forward-looking households and firms to reduce consumption and set lower prices today. The more flexible prices are, the larger the effects of uncertainty are at the zero lower bound.

JEL: E32, E52, E61, E62, E63

Keywords: Occasionally Binding Constraints, Liquidity Trap, Zero Lower Bound, Uncertainty.

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†Division of Research and Statistics, Federal Reserve Board, 20th Street and Constitution Avenue N.W. Washington, D.C. 20551; Email: taisuke.nakata@frb.gov.

# 1 Introduction

The recovery from the Great Recession has been sluggish. In the United States, five years after the beginning of the recession, output continues to be substantially below its trend level and unemployment rate remains elevated. Many expect the process of the recovery to remain slow over the coming years.

While a myriad of factors have likely contributed to this slow recovery, two factors have attracted the attention of economists and policymakers. The first is the zero lower bound constraint on the short-term nominal interest rate, which is widely seen as the standard policy tool for macroeconomic stabilization. In late 2008, the Federal Reserve effectively lowered the federal funds rate to zero. In a typical recession, the Fed could further reduce the nominal interest rate to stimulate the economy in response to deteriorating economic circumstances. However, having already lowered the policy rate to zero, it cannot reduce the rate further. The policy rate has been at zero since then, and the term structure of interest rates suggests it is expected to remain so for a few more years.

Another factor likely to have contributed to the sluggish recovery is uncertainty. Recessions are typically associated with an elevated level of uncertainty. In any recession, households, firms, and policymakers are uncertain as to its severity and duration. Since the Great Recession was more severe than any other post-WWII recession, uncertainty about the future course of the economy might have been particularly high. There is ample anecdotal evidence suggesting that uncertainty has slowed investment and hiring by firms, and many believe that it has played a key role in the slow recovery.[1]

Many authors have recently examined the implications of these two factors for the aggregate economy using macroeconomic models. On the one hand, there is now a large literature documenting how the effects of various exogenous shocks are amplified when the economy is constrained at the zero lower bound.[2] On the other hand, there is an emerging literature examining the effects of an increase in uncertainty on the aggregate economy using macroeconomic models.[3] However, these two factors—the zero lower bound and uncertainty—have been examined largely in isolation thus far.[4]

Accordingly, this paper examines how the presence of uncertainty alters allocations and prices when the nominal interest rate is constrained at the zero lower bound. It does so in the context of a standard New Keynesian model in which the central bank sets the nominal interest rate according to a truncated Taylor rule. Following the literature, I use the exogenous variation in the household's discount rate as the device that pushes the economy into the zero lower bound region. However, unlike most of the previous literature studying the dynamics of the economy

[1] See Kocherlakota (2010) for an example.

[2] See Christiano, Eichenbaum, and Rebelo (2011), Erceg and Linde (2010), Eggertsson (2010), Woodford (2011).

[3] See Bloom (2009), Bloom, Floetotto, and Jaimovich (2010), and Fernandez-Villaverde, Guerron-Quintana, Kuester, and Rubio-Ramirez (2012).

[4] Since the first draft of this paper, several new papers came out that analyze the dynamics of stochastic New Keynesian models at the zero lower bound with truncated Taylor rules. See Basu and Bundick (2012) and Gordon, Fernndez-Villaverde, Guerrn-Quintana, and Rubio-Ramirez (2012). However, none of the papers compares the decision rules in the stochastic economy with those in the deterministic economy.

along a specific deterministic path of the household's discount rate, the discount rate in this paper is stochastic. I then compare allocations and prices in the deterministic economy with those in the stochastic economy in order to understand the effect of uncertainty.

The main finding of the paper is that the presence of uncertainty reduces consumption, inflation, and output by a substantially larger amount when the nominal interest rate is constrained at the zero lower bound than when it is not. The zero lower bound constraint creates kinks in the policy functions for real interest rates and real marginal costs of production, making them highly convex and concave, respectively. As a result, when the nominal interest rate is zero, a mean-preserving spread in the shock distribution increases the conditional expectation of future real interest rates and decreases the conditional expectation of future marginal costs. Since households and firms are forward-looking, such changes in expectations lead them to reduce consumption and set lower prices today. While there are other factors in the model that make policy functions nonlinear (the curvature in the household's utility function and the quadratic price adjustment cost), their effects are quantitatively much less important than those arising from the zero lower bound.

Associated with nonlinearity in policy functions is a key feature of the model—that the agent's forecasts are asymmetric when the nominal interest rate is near or at the zero lower bound. A realization of a shock that would lead to an increase in consumption/output and reduction in deflation will be partially offset by an increase in the nominal interest rate if the shock is sufficiently large. However, a realization of a shock in the opposite direction cannot be offset by a further decline in the nominal interest rate if the rate is already zero. Thus, there are discrepancies between median forecasts and average forecasts. This is a key feature of the stochastic economy with the zero lower bound as well as the reason why the presence of uncertainty has non-neutral effects on the decisions of households and firms.

In addition to describing *how* and *why* uncertainty matters at the zero lower bound, I also contribute to the literature by studying *when* it matters by conducting an extensive sensitivity analysis. I show that the additional declines in consumption, output, and inflation due to uncertainty are larger (i) when prices are more flexible, and (ii) when shocks are more persistent. In the appendix, I also demonstrate that the effects of uncertainty are larger in the model with inflation inertia and consumption habits, and verify that the result is not specific to having the discount rate process as the driver to push the nominal interest rate to zero.

The observation that the presence of uncertainty alters allocations and prices at the zero bound is not new. Adam and Billi (2007) and Nakov (2008) first made this observation in the context of optimal discretionary nominal interest rate policy. However, as their focus was on describing optimal nominal interest rate policy near the zero bound, they did not elaborate on how and why the presence of uncertainty affects allocations and prices. The major contribution of this paper is to present a detailed and transparent account of how and why uncertainty alters allocations and prices in the presence of the zero lower bound. For that purpose, I work in a simpler environment where the nominal interest rate is determined according to a truncated Taylor rule. A simpler environment allows us to understand the mechanism in a more transparent way by abstracting from the optimal response of the central bank's nominal interest rate policy to an increase in uncertainty. Also, because many researchers have studied the dynamics of New Keynesian economy at the zero lower bound in which central bank sets the nominal interest rates by a truncated Taylor rule, instead of optimally, it is useful to know the effect of uncertainty in this context.

The rest of the paper is organized as follows. The next section describes the model and defines the equilibrium. Section 3 discusses parametrization and the solution method. Section 4 discusses the main result and key features of the model, while section 5 explains the mechanism behind the main result. Section 6 conducts sensitivity analysis with respect to the model's parameters, and Section 7 discusses the effect of having an additional source of uncertainty. A final section concludes. In the Appendix, I consider the robustness of the result in various models featuring inflation inertia, consumption habits, inertia in the nominal interest rate policy, and alternative shocks.

# 2 Model

This section describes the model and defines the equilibrium. The model is formulated in discrete time with an infinite horizon. There are four main actors; a representative household, a final good producer, a continuum of intermediate good producers with unit measure, and the government.

## 2.1 Household

The representative household chooses consumption, labor supply, and bond holdings to maximize the expected discounted sum of the future period utilities. The household likes consumption and dislikes labor. The period utility is assumed to be separable. The household problem is given by

$$\max_{C,N,B} E_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[\prod_{s=0}^{t-1} \delta_s\Big] \Big[\frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n}\Big]$$

subject to

$$P_t C_t + R_t^{-1} B_t \leq W_t N_t + B_{t-1} + P_t \Phi_t$$

$C_t$ is consumption and $N_t$ is labor supply. $P_t$ is the price of consumption good, $W_t$ is the nominal wage, and $\Phi_t$ is the profit from the intermediate goods producers. $B_t$ is a one-period risk free bond that pays one unit of money at t+1, and $R_t^{-1}$ is the price of the bond.

The discount rate at time t is given by $\beta\delta_t$ where $\delta_t$ is the discount factor shock that alters the weight of the future utility at time t+1 relative to the period utility at time t. $\delta_t$ follows an AR(1) process.

$$(\delta_t - 1) = \rho(\delta_{t-1} - 1) + \epsilon_t \ \forall \ t \geq 2$$

and $\delta_1$ is given. $\epsilon_t$ is an innovation to the discount factor shock and is distributed as normal with mean 0 and standard deviation $\sigma_\epsilon$.[5] An increase in $\delta_t$ means that the household increases the relative valuation of the future utility flows. In the absence of any changes in the nominal interest rate, the household accordingly decreases the consumption today.

## 2.2 Producers

There is a representative final good producer and a continuum of intermediate goods producers indexed by $i \in [0, 1]$. The representative final good producer purchases the intermediate goods, combines them into the final good using CES technology, and sells it to the household and the government.

$$\max_{Y_{i,t}, i \in [0,1]} \quad P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di$$

subject to the CES production function, $Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}}$.

Intermediate-good producers use labor to produce imperfectly substitutable intermediate goods according to a linear production function. Each firm sets the price of its own good in order to maximize the expected discounted sum of future profits. Price changes are subject to quadratic adjustment costs.

$$\max_{P_{i,t}} E_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[\prod_{s=0}^{t-1} \delta_s\right] \lambda_t \left[P_{i,t} Y_{i,t} - W_t N_{i,t} - P_t \frac{\varphi}{2} \left[\frac{P_{i,t}}{P_{i,t-1}} - 1\right]^2 Y_t\right]$$

subject to $Y_{i,t} = \left[\frac{P_{i,t}}{P_t}\right]^{-\theta} Y_t$, and $Y_{i,t} = N_{i,t}$. $\lambda_t$ is the Lagrange multiplier on the household's budget constraint at time t, and $\beta^{t-1}\left[\prod_{s=0}^{t-1} \delta_s\right]\lambda_t$ measures the marginal value of an additional profit to the household. There is no heterogeneity in the time zero prices across firms. That is, $P_{i,0} = P_0$ for some given constant $P_0 > 0$.

## 2.3 Government's Policy

Throughout the paper, I assume that the nominal interest rate is determined according to a truncated Taylor rule.

$$R_t = \max[1, \frac{1}{\beta} \Pi_t^{\phi}]$$

[5]It is standard to model $\log(\delta_t)$ as AR(1) process to make sure that $\delta_t$ is positive for all t. However, in such a setting, an increase in $\sigma_\epsilon$ increases not only the variance of $\delta$, but also the expected value of $\delta$. Since the goal of my analysis is to study the effect of mean-preserving spread in $\delta_t$, I specify $\delta_t$ as AR(1). As the variance of shocks we consider is small, the probability of $\delta_t$ becoming negative is very small. I also conducted the analysis using the standard specification, and policy functions turn out to be very close to those reported in this paper.

where $\Pi_t \equiv \frac{P_t}{P_{t-1}}$. In the appendix, I will consider the consequence of introducing inertia in the policy rule.

### 2.4 Market Clearing Conditions

Market clearing conditions for the final good, labor, and the government bond are given by:

$$\begin{aligned} Y_t &= C_t + \int \frac{\varphi}{2}\Big[\frac{P_{i,t}}{P_{i,t-1}} - 1\Big]^2 Y_t di \\ N_t &= \int_0^1 N_{i,t} di \\ B_t &= 0 \end{aligned}$$

### 2.5 An Equilibrium

Given $P_0$ and a stochastic process for $\delta_t$, an equilibrium consists of allocations $(\{C_t, N_t, N_{i,t}, Y_t, Y_{i,t}\}_{t=1}^{\infty})$, prices $(\{W_t, P_t, P_{i,t}\}_{t=1}^{\infty})$, and a policy instrument $(\{R_t\}_{t=1}^{\infty})$ such that (i) allocations solve the problem of the household given prices and policies, (ii) $P_{i,t}$ solves the problem of firm i, (iii) $P_{i,t} = P_{j,t}$ for all $i \neq j$, (iv) $R_t$ follows a truncated Taylor rule, and (v) all markets clear.

It is straightforward to show that a symmetric equilibrium can be characterized recursively by $\{C_t, N_t, Y_t, w_t, \Pi_t, R_t\}_{t=1}^{\infty}$ satisfying the following set of equilibrium conditions.

$$C_t^{-\chi_c} = \beta\delta_t R_t E_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1} \tag{1}$$

$$w_t = N_t^{\chi_n} C_t^{\chi_c} \tag{2}$$

$$\frac{N_t}{C_t^{\chi_c}}\big[\varphi(\Pi_t - 1)\Pi_t - (1-\theta) - \theta w_t\big] = \beta\delta_t E_t \frac{N_{t+1}}{C_{t+1}^{\chi_c}} \varphi(\Pi_{t+1} - 1)\Pi_{t+1} \tag{3}$$

$$Y_t = C_t + \frac{\varphi}{2}\big[\Pi_t - 1\big]^2 Y_t \tag{4}$$

$$Y_t = N_t \tag{5}$$

$$R_t = \max[1, \frac{1}{\beta}\Pi_t^{\phi}] \tag{6}$$

Equation (1) is the consumption Euler Equation and equation (2) is the intratemporal optimality condition of the household. Equation (3) is the optimality condition of the intermediate good producing firms, often referred to as the forward-looking Phillips Curve. It relates today's inflation to real marginal cost today and expected inflation tomorrow. Equation (4) is the aggregate resource constraint. The last term of equation (4) captures the resource cost of price adjustment. Equation (5) is the aggregate production function.

# 3 Parametrization and Solution Method

## 3.1 Parametrization

**Table** 1 lists the baseline parameter values selected. The discount rate $\beta$ is set to $\frac{1}{1+0.0075}$, which implies an annualized steady-state real interest rate of 3 percent. The values chosen for the household's preference parameters, the elasticity of substitution among intermediate goods, and the price adjustment cost are within the range considered in the literature. The baseline value of the price adjustment cost parameter, $\varphi = 175$, implies the the slope of the log-linearized Phillips curve that is approximately the same to the one in the Calvo model with 80 percent probability of price non-adjustment. Two parameters describing the evolution of time-preference ($\rho$ and $\sigma_\epsilon$) have important influences on the equilibrium. For the persistence parameter, $\rho$, I choose 0.8 as a benchmark, which is the value considered in Adam and Billi (2006), Adam and Billi (2007), and Nakov (2008). For the variance of shock, $\sigma_\epsilon$, I choose a value of $\frac{0.17}{100}$.[6] This value makes the frequency of hitting the zero lower bound around 4 percent. The implied unconditional standard deviation of $\delta_t$, which will be denoted by $\sigma_\delta$, is 0.0028. In Section 6, I will consider the robustness of the results to alternative parameter values.

## 3.2 Solution Method

The model is solved globally by a time-iteration method of Coleman (1991). The time-iteration method starts from a guess of policy functions. Assuming that the guessed policy functions are in use for the next period, the first order necessary conditions of the government problem are solved to find the policy functions in the current period in a finite number of grids. This process is repeated until the policy function today becomes arbitrarily close to the policy function tomorrow. For all specifications of the model, I use 5001 grids on $[1-5\sigma_\delta, 1+5\sigma_\delta]$ for $\delta_t$ where $\sigma_\delta$ is the unconditional standard deviation of $\delta_t$.

# 4 Results

## 4.1 The effect of uncertainty at vs. away from the zero lower bound

**Figure** 1 shows the policy functions in the stochastic economy along with those in the deterministic economy. Dashed black lines are for the deterministic model, and solid black lines are for the stochastic model.

An increase in the discount factor shock, $\delta$, means that the household becomes more patient. With more patience, the household wants to save more for tomorrow and spend less today. This decline in the demand for consumption good leads to lower output and inflation. The truncated Taylor rule dictates that a reduction in inflation be accompanied by a reduction in the nominal

[6] The larger the $\sigma_\epsilon$ is, the more frequently the discount rate $\beta\delta_t$ exceeds one. An equilibrium does not exist if the discount rate exceeds one sufficiently frequently. Mendes (2011) provides a proof that no equilibrium exists when the variance of ths discount rate shock exceeds a certain threshold value.

interest rate, which partially offsets the household's desire to save. In equilibrium, the nominal interest rate, inflation, consumption, and output all decline as $\delta$ becomes larger. For a sufficiently large increase in the discount factor shock, the nominal interest rate cannot decline further due to the zero lower bound. As a result, in *both* deterministic and stochastic economies, the declines in consumption, output, and inflation in response to an increase in the discount factor shock are larger when the nominal interest rate is constrained at the zero lower bound than otherwise.

Notice that, while declines in consumption, output, inflation, and the real wage are approximately the same in deterministic and stochastic economies when the nominal interest rate is above zero, they are substantially different at the zero lower bound. In particular, consumption, output, inflation, and the real wage decline by a larger amount in the stochastic economy than in the deterministic economy when the zero bound constraint is binding. This can be seen in Figure 1 where solid black lines lie below dashed black lines when the nominal interest rate is zero. On the other hand, the differences between two lines are negligible when the nominal interest rate is above zero, showing that the presence of uncertainty leaves allocations and prices essentially unchanged when the nominal interest rate is away from the zero lower bound. This is the main result of the paper—the presence of uncertainty alters allocations and prices to a much greater extent when the zero lower bound is binding than when it is not.

**Figure 2** gives an alternative look at this result through impulse response functions of the model's variables in response to a large initial increase in the discount factor shock ($\delta_1 = 1 + 3\sigma_\delta$) so that the nominal interest rate is initially zero. In the deterministic economy, this increase should be interpreted as unexpected. The dashed black lines depict the impulse response functions for the deterministic model. For any variable X, solid black line shows the average response of X in the stochastic economy ($E_1[X_t|\delta_1 = 1 + 3\sigma_\delta]$) for all $t \geq 1$, expressed as a percentage deviation from its deterministic steady-state value. Shaded gray areas are used to show X's probability distribution at time t. This figure shows that the declines in consumption, output, real wage, and inflation are larger in the stochastic economy than in the deterministic economy. The differences are quantitatively important. Consumption initially declines by 3 percent in the stochastic economy while it declines by 2 percent in the deterministic economy. Initial deflation is 5 percent in the stochastic economy versus 3 percent in the deterministic economy. The differences in the expected paths are large in the first several periods in which the nominal interest rate is expected to stay at zero, while they become smaller as the forecast horizon increases and the nominal interest rate is expected to be above zero. That is, the presence of uncertainty depresses consumption, output, and inflation at the zero lower bound to a much greater degree when the zero lower bound is binding than when it is not.

## 4.2 Nonlinearity of policy functions

As already mentioned, in *both* the deterministic and stochastic economies, an additional increase in the discount rate reduces allocations and prices by more when the nominal interest rate is zero than when it is not, making policy functions nonlinear. However, the kink is not the only nonlinearity in the policy functions. While policy functions are almost linear in the region where the zero lower bound does not bind (see the part of policy functions to the left-hand side of solid or dashed blue vertical lines in Figure 1), they exhibit nonlinearity in the region where the zero lower bound binds (see the part of policy functions to the right-hand side of solid or dashed blue vertical lines in Figure 1). In particular, when the zero lower bound is binding, the additional declines in consumption, inflation, and real wage in response to an increase in the discount rate are larger when the discount rate is larger.

This result arises because the expected duration of the zero nominal interest rate in the near future affects the household's consumption decision. Even if the nominal interest rate is zero today, when the discount rate is small (i.e. when $\delta$ is close to the solid or dashed blue vertical line), the household expects the nominal interest rate to be above zero tomorrow where the policy function for inflation is almost linear. Thus, even though today's inflation declines steeply in response to a marginal increase in the discount rate, the additional decline in the expected inflation tomorrow is the same as when the zero lower bound is not binding. Consumption declines by more than it would in absence of the zero lower bound constraint, but since the additional decline in the expected inflation tomorrow is not affected, the decline in consumption is limited. On the other hand, if the discount rate is very large and the nominal interest rate is expected to be zero tomorrow, the additional decline in the expected inflation tomorrow in response to a marginal increase in the discount rate is larger. Thus, the additional decline in the real interest rate is also larger, and the household reduces consumption by more. As a result, in the region where the zero lower bound is binding (i.e. in the region left to the solid or dashed blue vertical line), the larger the discount rate is, the larger the additional decline in consumption is in response to a marginal increase in the discount rate. On the other hand, in the region where the zero lower bound is not binding (i.e. in the region right to the solid or dashed blue vertical line), the additional decline in consumption in response to a marginal increase in the discount rate does not depend on the level of the discount rate.

This feature of the policy function for consumption is transmitted into the policy functions for real wage and inflation through the household's intratemporal optimality condition and the Phillips curve, creating curvatures in these policy functions in the region of the state space where the nominal interest rate is zero. These curvatures, in addition to the kinks described earlier, make policy functions nonlinear.

In the stochastic economy, nonlinearity of policy functions manifests itself in the asymmetry in agent's forecasts when the nominal interest rate is at or near zero, which can be seen by comparing the darkest part of the fan chart (the median or modal forecasts) and the solid black lines (the average forecasts) in the first several quarters in Figure 2. In the density forecasts of consumption, inflation, real wage, and output, mean forecasts lie below medium forecasts. In the density forecasts of the nominal interest rate and real interest rates, mean forecasts lie above medium forecasts. This feature will be an important factor in understanding why the presence of uncertainty reduces allocations and prices at the zero lower bound, the task we will turn to in Section 5.

### 4.3 Relative (un)importance of other nonlinearities

In addition to the lower bound constraint on the nominal interest rate, there are other features in the model that make policy functions nonlinear, such as the curvature in the household's utility function and the quadratic nature of price adjustment costs. To isolate the role of other nonlinearities in generating the results above, this subsection repeats the same exercise of comparing deterministic and stochastic economies, but using a system of log-linearized equilibrium conditions—log-linearized except for the max operator that truncates the nominal interest rate.[7] Equilibrium conditions of this partially log-linearized economy are given by

$$\hat{C}_t = E_t\hat{C}_{t+1} - \frac{1}{\chi_c}(\hat{R}_t - E_t\hat{\Pi}_{t+1}) - \frac{1}{\chi_c}(\delta_t - 1) \tag{7}$$

$$\hat{w}_t = \chi_n\hat{N}_t + \chi_c\hat{C}_t \tag{8}$$

$$\hat{\Pi}_t = \kappa\hat{w}_t + \beta E_t\hat{\Pi}_{t+1} \tag{9}$$

$$\hat{Y}_t = \hat{C}_t \tag{10}$$

$$\hat{Y}_t = \hat{N}_t \tag{11}$$

$$\hat{R}_t = \max[1 - \frac{1}{\beta}, \phi\hat{\Pi}_t] \tag{12}$$

where $\kappa := \frac{\theta-1}{\varphi}$ and $\hat{X}_t := log(X_t/X_{ss})$ for any variable X with $X_{ss}$ being the deterministic steady state value of X.[8] Using the time-iteration method, I solve for a set of policy functions $\{\hat{C}_t, \hat{Y}_t, \hat{N}_t, \hat{w}_t, \hat{\Pi}_t, \hat{R}_t\}$ that satisfies these equilibrium conditions.

Figure 3 presents policy functions from this partially log-linearized economy. The dashed black lines are the policy functions in the deterministic version of the economy, and the solid black lines are those in the stochastic version of the economy. The figure shows that the effects of uncertainty in the partially log-linearized economy are qualitatively similar to those in the fully nonlinear economy: While the presence of uncertainty has very small effects on allocations and prices away from the zero lower bound, it leads to large declines in allocations and prices at the zero lower bound. By construction, if not for the lower bound constraint on the nominal interest rate, the presence of uncertainty would not change allocations and prices at all. Thus, these figures show that a key nonlinearity that makes uncertainty have an important effect at the zero lower bound is the one coming from the zero lower bound constraint, not the curvature in the utility function nor the quadratic nature of price adjustment costs.

Notice that, while the policy functions from the partially log-linearized economy are qualitatively similar to those from the fully nonlinear economy, the differences between them are quantitatively large. In particular, in both deterministic and stochastic environments, the declines in consump-

[7] The vast majority of the literature uses log-linearized equilibrium conditions with truncated Taylor rules. Exceptions include Braun and Waki (2010), Gordon, Fernndez-Villaverde, Guerrn-Quintana, and Rubio-Ramirez (2012), and Gust, Lopez-Salido, and Smith (2012).

[8] Notice that I linearize—instead of log-lineaze—the first equation with respect to $\delta_t$ in order to facilitate the comparison with the policy functions from the fully nonlinear economy.

tion, output, inflation, and the real wage at the zero lower bound are substantially larger in the partially log-linearized economy than in the fully nonlinear economy. For example, the declines in consumption, inflation, output, and the real wage are 6, 8, 6, and 12 percent at $\delta = 1 + 4\sigma_\delta$ in the partially log-linearized stochastic economy (see the solid black lines in Figure 3), while they are 5, 6, 3, and 7.5 percent in the fully nonlinear stochastic economy (see the solid black lines in Figure 1). Large approximation errors generated by partially log-linearizing the model should caution us against the use of partially log-linearized economy for quantitative analyses of the model at the zero lower bound.

# 5 Mechanism

This section describes the mechanism generating the main result discussed in the previous section—namely, that the presence of uncertainty reduces allocations and prices by a substantially larger amount when the zero lower bound is binding than when it is not.

I start by considering the effect of uncertainty on the decisions of the household and firms in a partial equilibrium environment. In this exercise, the household takes as given the policy functions for the nominal interest rate, real wage, and inflation from the deterministic economy described above. Firms similarly take as given the policy function for the real wage from the deterministic economy. In the first subsection, I will show that, in this partial equilibrium setup, an increase in variance of shocks to the discount factor process leads the household to consume less. In the second subsection, I turn to the firms' problem, and show that, in this partial equilibrium setup, an increase in the variance leads firms to set lower prices. In general equilibrium, changes in the decisions of the household and firms caused by uncertainty lead to changes in the policy functions for the nominal interest rate, inflation, and real marginal cost, which in turn affect the agents' decisions. I describe these general equilibrium effects in the third subsection.

Throughout this section, I illustrate the mechanism using the consumption Euler Equation and forward-looking Phillips curve from the partially log-linearized economy. The partially log-linearized economy can serve as a tractable environment to explain the impact of uncertainty because, as discussed in Section 4.3, other nonlinearities in the model do not play an important role in generating the main result.

## 5.1 Household's Problem in Partial Equilibrium

In order to understand the effect of uncertainty on the household's consumption decision, notice first that what determines the household's consumption today is the expected real interest rates in the future. This can be seen by iterating the consumption Euler Equation forward to obtain

$$\hat{C}(\delta_t) = \lim_{s\to\infty} E_t\hat{C}(\delta_{t+s}) - \frac{1}{\chi_c}E_t\sum_{s=0}^{\infty}(\hat{R}(\delta_{t+s}) - \hat{\Pi}(\delta_{t+s+1})) - \frac{1}{\chi_c}E_t\sum_{s=0}^{\infty}(\delta_{t+s} - 1) \tag{13}$$

This expression says that consumption at time t is a function of the expected sum of future real interest rates as well as the expected sum of future discount factor shocks.

### 5.1.1 The effect of uncertainty at the zero lower bound

Suppose that there is an unexpected large increase in the discount factor shock ($\delta_t \gg 1$) so that the nominal interest rate is zero at time t. Consider an increase in the standard deviation of the discount factor shock. Assume that the increase is small enough that the first term in the equation (13) is unchanged. How does this increase in uncertainty affect the last two terms—the expected sum of future real interest rates and the expected sum of future discount factor shocks?

An increase in uncertainty does not change the expected future sum of the discount factor shocks because the sum depends on $\delta_{t+s}$ linearly. However, the increase in uncertainty affects the second term of equation (13), the expected future sum of real interest rates. The expected future sum of real interest rates can be decomposed into two parts: the expected future sum of nominal interest rates and the expected future sum of inflation. Let us examine the effect of uncertainty on each of the two parts.

Consider first the effect of uncertainty on the expected future sum of nominal interest rates. Since the policy function for the nominal interest rate is convex due to the kink generated by the zero lower bound constraint, by Jensen's inequality, we have

$$\hat{R}(E_t[\delta_{t+s}]) \leq E_t\hat{R}(\delta_{t+s}) \tag{14}$$

for any $s \geq 1$. Since $E_t[\delta_{t+s}]$ does not change with the variance of shocks, the left-hand side of this inequality is equal to the (expected) nominal interest rate at $t+s$ in the deterministic economy. Thus, this inequality means that the expected future nominal interest rate is higher when $\sigma_\epsilon > 0$ than when $\sigma_\epsilon = 0$. Since this inequality holds for all $s \geq 1$, an increase in uncertainty increases the expected future *sum* of the nominal interest rate as well.

Top-left panel in **Figure** 4 shows how a mean-preserving spread in the shock distribution affects expected future nominal interest rates when the initial discount factor shock is large and the nominal interest rate is zero at time one. The dashed black lines are the (expected) nominal interest rates over the first ten quarters when $\sigma_\epsilon = 0$. The solid black lines are the expected nominal interest rates when $\sigma_\epsilon > 0$. Fan chart shows the probability distributions of future nominal interest rates when $\sigma_\epsilon > 0$. Solid black lines are indeed above dashed black lines, confirming the argument based on Jensen's inequality that an increase in uncertainty increases the expected nominal interest rate path.

In the case of the nominal interest rate, it is relatively intuitive to understand this effect because the policy function for the nominal interest rate is convex in a particular way: it is *truncated from below*. When the variance of shocks is zero, the (expected) nominal interest rates in the first few quarters are zero (see the dashed line). With positive variance, the most likely nominal interest rate remains zero in the first few quarters (see the darkest part of the fan chart), but the expected nominal interest rates becomes positive (see the solid line). While negative realizations of future shocks will lead the nominal interest rate to rise, positive realizations will not be met by a decrease in the nominal interest rate due to the zero lower bound constraint. Thus, an increase in the variance raises the expected nominal interest in the near future from zero to some positive value.

We can similarly understand the partial equilibrium effect of an increase in the shock variance on the expected future sum of inflation at the zero lower bound, the second ingredient of the expected real interest rate. Since the policy function for inflation is a concave function of the discount factor shock (see Figure 1 or 3), Jensen's inequality implies that a mean-preserving spread in the shock distribution leads to a reduction in the expected inflation. A combination of the increase in the expected future sum of nominal interest rates and the reduction in the expected future sum of inflation means an increase in the expected future sum of real interest rates. The middle-left panel of Figure 4 indeed shows that solid black line lies above dashed black line, confirming that an increase in uncertainty pushes up the expected path of real interest rates when the nominal interest is at the zero lower bound.

As we can see in equation (13), consumption today is a decreasing function of the expected sum of future real interest rates. When the economy is at the zero lower bound, an increase in the variance of shocks leads to an increase in the expected real interest rates in the future. Faced with higher expected real interest rates in the future, the household reduces consumption today.

#### 5.1.2 The effect of uncertainty away from the zero lower bound

As discussed earlier, policy functions are almost linear when the nominal interest rate is above zero. Thus, these effects of uncertainty become smaller when the nominal interest rate is away from the zero lower bound. Top-right and middle-right panels of **Figure** 4 plot the forecasts of nominal and real interest rates with zero and positive variances when the initial discount factor shock is one standard deviation away from the steady state and the nominal interest rate is above zero at time one. The conditional expected paths of future nominal and real interest rates are not significantly affected by an increase in the shock variance, which can be seen by the negligible difference between solid black and dashed black lines in the figure. This is true because the economy is on average reverting to the steady state level, and only very large shocks in the future can push the nominal interest rates to zero where the policy functions for relevant prices are nonlinear. With the expected real interest rates essentially unchanged, the increase in the variance of shocks does not alter the household's consumption decision.

### 5.2 Firms' Problem in Partial Equilibrium

The presence of uncertainty not only influences the consumption decision of households, but also affects the price setting decision of firms. To understand why, notice first that what matters for firms in setting their prices today is the expected future real wage. This can be seen by iterating forward the Phillips Curve to obtain

$$\hat{\Pi}(\delta_t) = \lim_{s\to\infty} E_t\hat{\Pi}(\delta_{t+s}) + \kappa E_t \sum_{s=0}^{\infty} \beta^s \hat{w}(\delta_{t+s}) \tag{15}$$

This expression says that the firms' price setting decision today is a function of the expected discounted sum of future real wages.

#### 5.2.1 The effect of uncertainty at the zero lower bound

Suppose again that there is an unexpected large increase in the discount factor shock, $\delta$, so that the nominal interest rate is zero at time one. Consider the effect of an increase in the variance of the discount rate shock. As the dashed black line in the bottom-right panel of Figure 1 or 3 shows, the policy function for the real wage is concave due to the zero lower bound constraint. Therefore, by Jensen's inequality, an increase in uncertainty leads the expected discounted sum of future real wages to decline. The bottom-left panel of **Figure** 4 shows how a mean-preserving spread in the shock distribution affects the expected future real wage in the partial equilibrium environment. Consistent with the prediction of Jensen's inequality, the solid black line lies below the dashed black line, meaning that the presence of uncertainty reduces the expected future real wages. Faced with lower expected discounted sum of future real marginal costs, forward-looking firms set lower prices today.

#### 5.2.2 The effect of uncertainty away from the zero lower bound

As before, this effect of uncertainty on the expected real wage is weaker when the initial increase in the discount rate is small and the nominal interest rate is above zero. The bottom-right panel of Figure 4 shows that an increase in uncertainty has negligible effects on the expected path of future real wages when the nominal interest rate is above zero, as reflected in the negligible difference between the solid and dashed black lines. As the policy function for real wage is almost linear when the nominal interest rate is not at the zero lower bound, a change in uncertainty does not alter the expected future real wages. Only when the economy is at the zero lower bound, does an increase in uncertainty have quantitatively important effects on the expected real wage, and thus on today's inflation.

### 5.3 General Equilibrium Effects

The previous two subsections have shown that, when at the zero lower bound, the household reduces consumption and firms set lower prices today in response to an increase in uncertainty, *taking policy functions for the nominal interest rate, inflation, and real wage as fixed.* In general equilibrium, such changes in the behaviors of the household and firms induce changes in the policy functions for prices, and those changes in turn influence the behavior of the household and firms.

**Figure** 5 shows the partial equilibrium effect of an increase in uncertainty on the demand for the government bond by the household. An increase in uncertainty leads the household to reduce consumption today. In the partial equilibrium environment in which the policy function for real wage is given, a reduction in consumption comes with an increase in labor supply (i.e. reduction in leisure) and an increase in the demand for bond, which is represented by the shift in the demand curve. As discussed earlier, this effect is larger when the economy is at the zero lower bound. Thus, the figure shows that the shift in the demand curve is small when the economy is away from the zero lower bound while it is large when the economy is at the zero lower bound.

In this model, the government's supply of the bond is fixed at zero. If the economy is away from the zero lower bound, an equilibrium can be attained by a lower nominal interest rate. However, when the economy is at the zero lower bound, a further decline in the nominal interest rate is not feasible. Other forces have to do the work of reducing the demand for the government bond. There are two forces in the economy that can make the household save less. One force is an increase in the expected inflation, which lowers the expected real interest rate. The other is a decline in today's income relative to the future income, which makes the household today want to borrow more from the future (i.e., save less today).

As discussed in the previous subsection, firms respond to an increase in uncertainty by setting lower prices today in the partial equilibrium environment. This reduction in today's price increases the relative price tomorrow and serves as the first force that induces the household to save less. The second force—the decline in the relative income today—comes from a reduction in output. In general equilibrium, the household's real income today is given by the output minus the resource costs of price adjustment cost in the economy. The wage does not matter for the household's income because a higher wage means a higher labor income for the household *and* lower profits they receive from the intermediate good producers, which cancel out in equilibrium. The consequence of the decline in output, the second force to induce the household save less, is the reduced demand for labor, which leads to a decline in the real wage. Thus, in general equilibrium, declines in consumption and inflation come together with declines in output and real wage. Figures 1 and 2 indeed show that consumption, output, inflation, and real wage all decline in the presence of uncertainty when the economy is at the zero lower bound.

Since a decline in inflation pushes up the real interest rate, it gives a further incentive for the household to spend less. Similarly, a decline in the real wage gives a further incentive for firms to lower prices. Thus, the declines in consumption and inflation described earlier in partial equilibrium settings will be amplified in general equilibrium, leading to large differences in allocations and prices between the deterministic and stochastic economies.

# 6 Sensitivity Analysis

Now that we have understood why the presence of uncertainty has adverse effects on the economy at the zero lower bound, we will now analyze when such effects of uncertainty are large.

### 6.1 Sensitivity to alternative structural parameter values

In order to understand how structural parameters affect the magnitude of additional declines due to uncertainty, recall the analysis in Section 5 that shows that the presence of uncertainty reduces consumption through its effect on expected future real interest rates, and that the presence of uncertainty leads to a decline in inflation through its effect on expected future real marginal costs. According to the consumption Euler Equation iterated forward (eqn. 13), the same decline in the expected sum of future real interest rates leads to a larger decline in consumption today when $\chi_C$ is smaller, i.e. when the intertemporal elasticity of substitution (IES) is large. Similarly, according to the Phillips curve iterated forward (eqn. 15), the same decline in the expected discounted sum of future real wages leads to a larger decline in inflation today when the slope of the Phillips curve ($\kappa \equiv \frac{\theta-1}{\varphi}$) is larger. The slope of the Phillips curve is larger when prices are more flexible (i.e., smaller $\varphi$) or when intermediate goods are more substitutable (i.e., larger $\theta$).

**Figure 6** confirms these predictions. The left panel in the first row of Figure 6 shows the declines in consumption when $\delta_t = 1 + 3\sigma_\delta$, which is three standard deviations away from the deterministic steady-state level in the stochastic economy, for various values of price adjustment cost parameter ($\varphi$). A larger $\varphi$ means that prices are more sticky. The solid black line corresponds to the stochastic economy, and the dashed black line to the deterministic economy. The right panel in the first row of Figure 6 shows the decline in inflation when $\delta_t = 1 + 3\sigma_\delta$ in a similar manner.

In the deterministic economy, the declines in consumption and inflation at $\delta_t = 1 + 3\sigma_\delta$ from the steady-state level do not vary much for the range of price flexibility shown. However, in the stochastic economy, the magnitude of the declines depends importantly on price flexibility. Consistent with the aforementioned prediction, the more flexible the price is (i.e. the smaller the price adjustment cost is), the larger the additional declines in consumption and inflation due to uncertainty. While the additional declines in consumption and inflation due to uncertainty are both about 0.5 percent at $\varphi = 200$, they are about 1.5 percent and 2 percent at $\varphi = 160$. The second and third rows of Figure 6 show how the substitutability of intermediate goods and the household's risk aversion, respectively, affect the quantitative importance of uncertainty on consumption and inflation at the zero lower bound. Again, consistent with the observations made above, the more substitutable intermediate goods are, or the larger the IES is, the larger the additional declines in consumption and output due to uncertainty.

### 6.2 Sensitivity to alternative degrees of shock persistence

How do the parameters of the discount factor shock process, $\sigma_\epsilon$ and $\rho$, affect the quantitative effects of uncertainty? It is perhaps obvious that the smaller $\sigma_\epsilon$ is, the smaller the additional reductions in consumption, output, and inflation are. What is less obvious is how alternative degrees of persistence affect the quantitative significance of uncertainty.

**Figure 7** shows policy functions from three economies with alternative degrees of persistence. The second column shows policy functions from the baseline model with $\rho = 0.8$, and the first and third columns show the policy function from the model with low ($\rho = 0.6$) and high persistence

($\rho = 0.825$), respectively. In each figure, solid black and dashed black lines are respectively policy functions for the stochastic and deterministic economies. In each model, the standard deviation of the shock is chosen so that the frequency of being at the zero lower bound is 4 percent.[9]

These figures show that the more persistent the process is, the more adverse the effects of uncertainty are at the zero lower bound. At four standard deviations away from the steady-state, the additional declines in consumption due to uncertainty are 0.5 percent, 1 percent, and 2 percent when persistence is low, medium, and high, respectively. The additional declines in inflation due to uncertainty are 0.5 percent, 1.5 percent, and 4 percent when persistence is low, medium, and high, respectively.

Why are the effects of uncertainty larger when shocks are more persistent? Recall that the key factor that generates large adverse effects of uncertainty at the zero lower bound is nonlinearity in the policy functions for real interest rates and real marginal costs. If the discount rate shock is not persistent, even if the nominal interest rate is zero today, the economy to be away from the zero lower bound region in the near future where policy functions for relevant prices are almost linear. On the other hand, if the shock is persistent, the household and firms expect the nominal interest rate to be at the zero lower bound for a long period where policy functions exhibit nonlinearity. Thus, the adverse effects of uncertainty is larger when the process driving the economy into the zero lower bound is more persistent.

# 7 Additional Source of Uncertainty

In this section, I introduce another exogenous variable—government spending—to the model in order to study the effect of having an additional source of uncertainty on allocations and prices. Throughout this section, government spending is assumed to follow the following AR(1) process.[10]

$$G_t = G_{ss} + \rho_g(G_{t-1} - G_{ss}) + \sigma_g \epsilon_{g,t}$$

I consider three economies. In the first economy, both discount factor shock and government spending processes are deterministic ($\sigma_\delta = \sigma_g = 0$). In the second economy, the discount factor shock process is stochastic, but the government spending process is deterministic ($\sigma_\delta > 0$ and $\sigma_g = 0$). In the third economy, both processes are stochastic ($\sigma_\delta > 0$ and $\sigma_g > 0$). I set $\rho_g = 0.8$ and $\sigma_g = \frac{0.3}{100}$.[11]

**Figure** 8 shows policy functions for the model's variables in the three economies with different degrees of uncertainty. As discussed earlier, an increase in the variance of discount factor shocks reduces allocations and prices at the zero lower bound. This is confirmed by comparing the

[9] I find that the larger the persistence is, the smaller the maximum frequency of being at the zero lower bound consistent with the existence of an equilibrium. When persistence is larger than 0.825, no equilibrium exists in which the probability of being at the zero lower bound is more than 4 percent.

[10] See footnote 6 in Section 2 for a discussion of why the level specification, instead of log specification is used.

[11] These parameter values imply much less persistent and less volatile government spending process than in the U.S. data. The point is to show that introducing another source of uncertainty can affect allocations and prices at the zero lower bound, even when the variance of a newly introduced variable is quite small.

dashed black and red lines. Comparing the dashed and solid black lines shows that introducing an additional layer of uncertainty—uncertainty on future government spending—further reduces allocations and prices when the economy is at the zero lower bound. While allocations and prices are not so different between these two economies when they are away from the zero lower bound, declines in consumption, inflation, and the real wage at the zero lower bound are larger with uncertainty about government spending than without it.[12] The same mechanism generating the main result explains this phenomenon. Policy functions for allocations and prices in this model with exogenously varying government spending are functions of two state variables: the discount factor shock and government spending. The zero lower bound constraint not only creates kinks in these policy functions in the dimension of discount factor shock, but also does so in the dimensions of government spending shock. Thus, at the zero lower bound, an increase in uncertainty about government spending also has non-neutral effects on the expected future real interest rates and real marginal costs.

While this section focused on the additional source of uncertainty coming from the government spending process, in unreported exercises, I also considered two other sources of additional uncertainty—variations in markup and total factor productivity—and confirmed that adding these alternative exogenous processes similarly reduces allocations and prices at the zero lower bound. Also, to check whether the result is dependent on having the discount factor shock process in the model, I considered models without the discount factor shock, but with either one of government spending, markup, and total factor productivity shocks (see the Appendix D). I confirmed that the presence of uncertainty also decreases allocations and prices at the zero lower bound in these alternative models. Given the generality of the mechanism, it is likely that other sources of uncertainty would similarly reduce allocations and prices at the zero lower bound.

# 8 Conclusion

This paper investigated how the presence of uncertainty alters allocations and prices when the nominal interest rate is constrained by the zero lower bound in a New Keynesian economy. I find that uncertainty reduces consumption, inflation, and output by a larger amount when the zero lower bound is binding than when it is not. This result arises because policy functions for the real interest rate and the real wage are highly convex and concave due to the zero lower bound. Such nonlinear policy functions imply that a mean-preserving spread in the shock distribution increases the expected real interest rates and reduces expected real wages in the future. These changes in expectations then lead the forward-looking household and firms to reduce consumption and set lower prices today.

[12] The decline in output is little affected. This is because, in the model with additional uncertainty, the decline in inflation is larger, and more output is needed to cover the increased resource cost of price adjustment.

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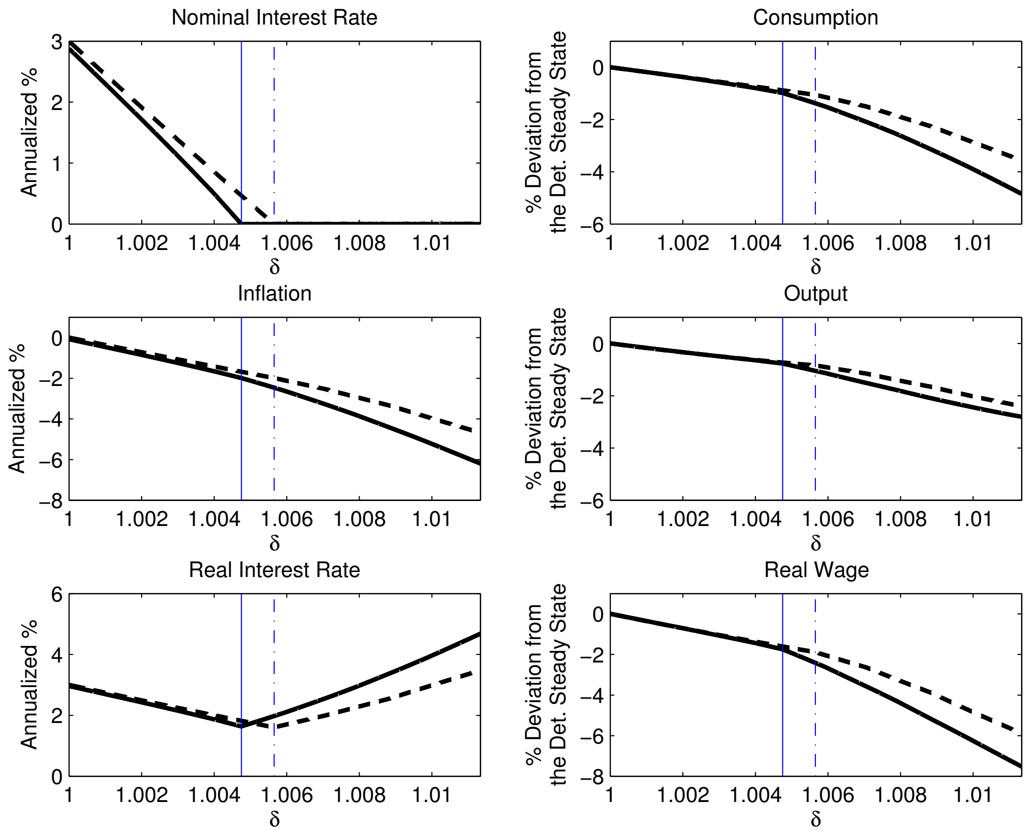
Table 1: Parameter Values

| Parameter | Description | Parameter Value |
|---|---|---|
| $\beta$ | Discount rate | $\frac{1}{1+0.0075} \approx 0.9925$ |
| $\chi_c$ | Inverse intertemporal elasticity of substitution for $C_t$ | 1.0 |
| $\chi_n$ | Inverse labor supply elasticity | 1.0 |
| $\theta$ | Elasticity of substitution among intermediate goods | 10 |
| $\varphi$ | Price adjustment cost | 175 |
| $\phi$ | Coefficient for inflation in the Taylor rule | 1.5 |
| $\rho$ | AR(1) coefficient for the discount factor | 0.8 |
| $\sigma_\epsilon$ | The standard deviation of shocks to the discount factor | $[0, \frac{0.17}{100}]$ |
| $\sigma_\delta$ | The implied unconditional standard deviation of $\delta$ | 0.0028 |

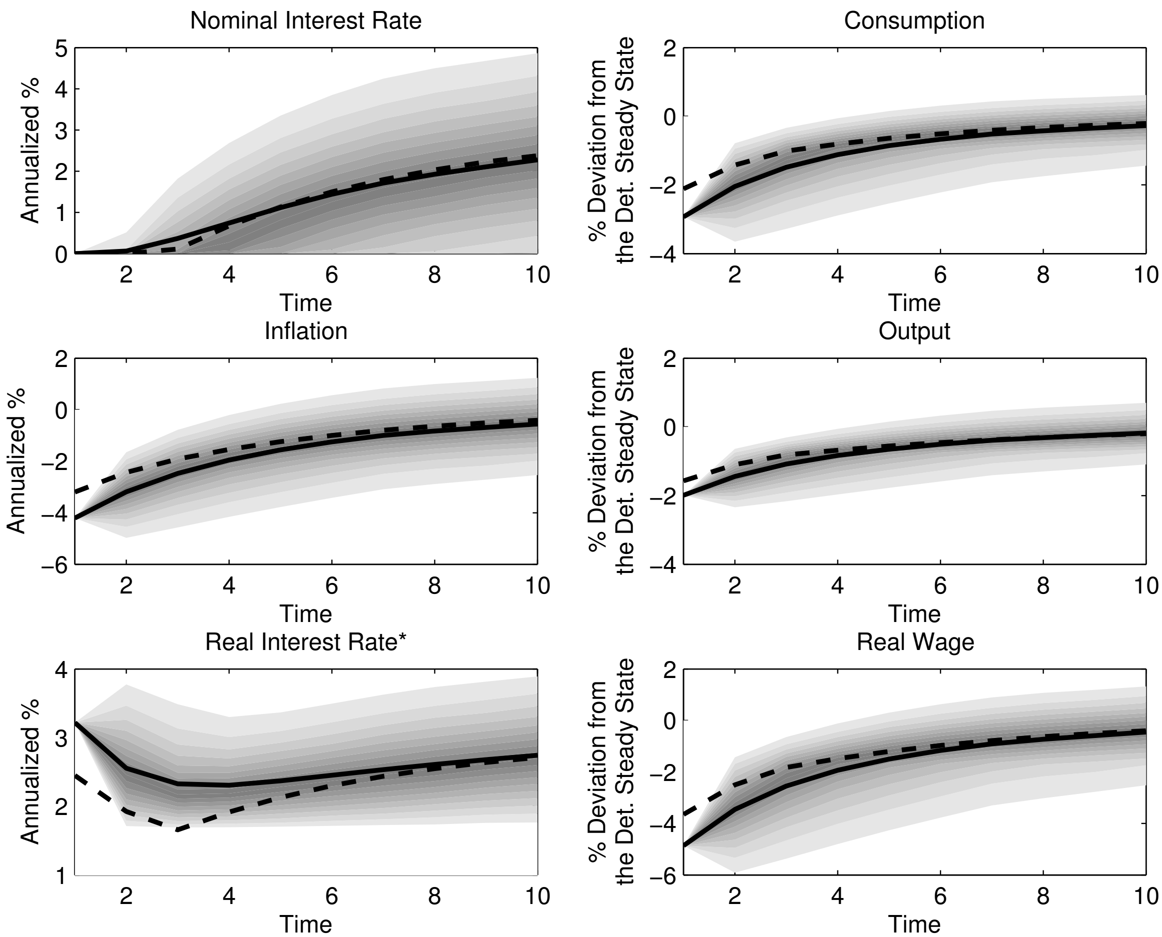
Figure 1: Policy Functions For Allocations and Prices: Deterministic vs. Stochastic Economies




Dashed black lines: Deterministic Model ($\sigma_\epsilon = 0$)
Solid black lines: Stochastic Model ($\sigma_\epsilon = \frac{0.17}{100}$)

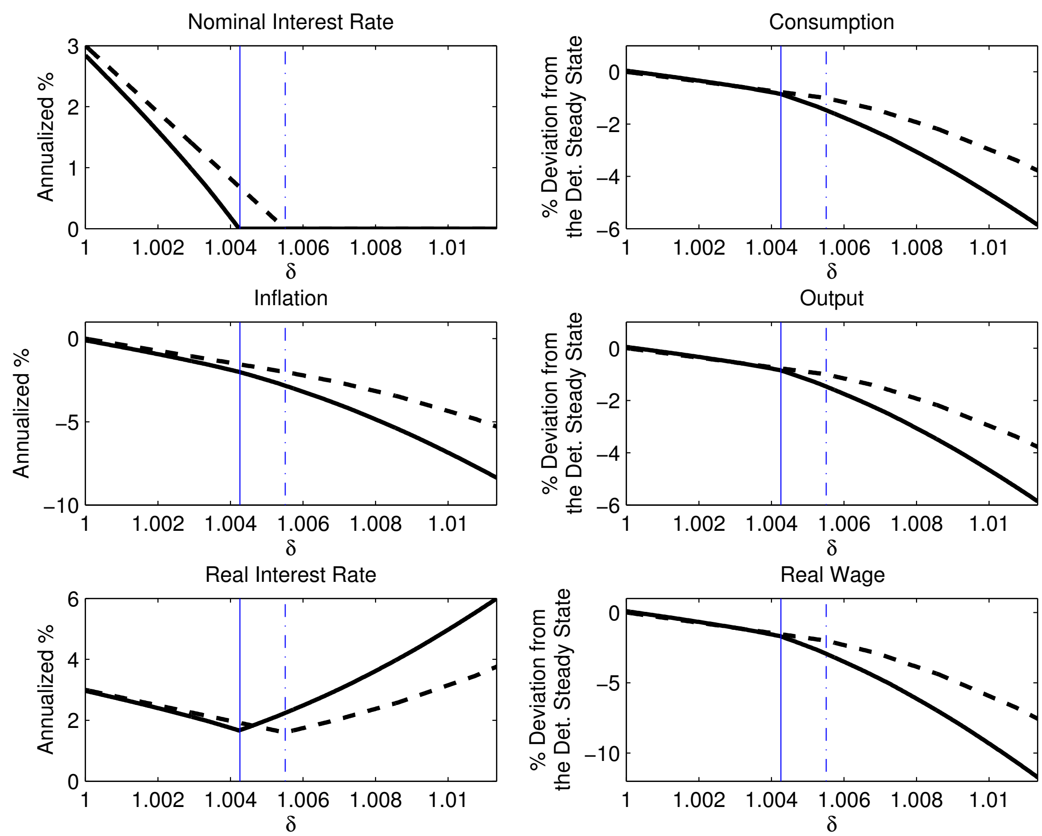
*Policy functions are shown for the range of $\delta$ that covers its steady-state level ($\delta = 1$) to the level that is 4 standard deviations away from the steady-state ($\delta = 1 + 4\sigma_\delta = 1.0113$).

Figure 2: Impulse Response Functions: Deterministic vs. Stochastic Economies




Dashed black lines: Impulse response functions in the deterministic model ($\sigma_\epsilon = 0$).
Solid black lines: Nonlinear impulse response functions (i.e., $E_1[X_t|\delta_1 = 1 + 3\sigma_\delta]$ for a variable X) in the stochastic model ($\sigma_\epsilon = \frac{0.17}{100}$).
Shaded Grey Areas: Fan charts for the stochastic model ($\sigma_\epsilon = \frac{0.17}{100}$).

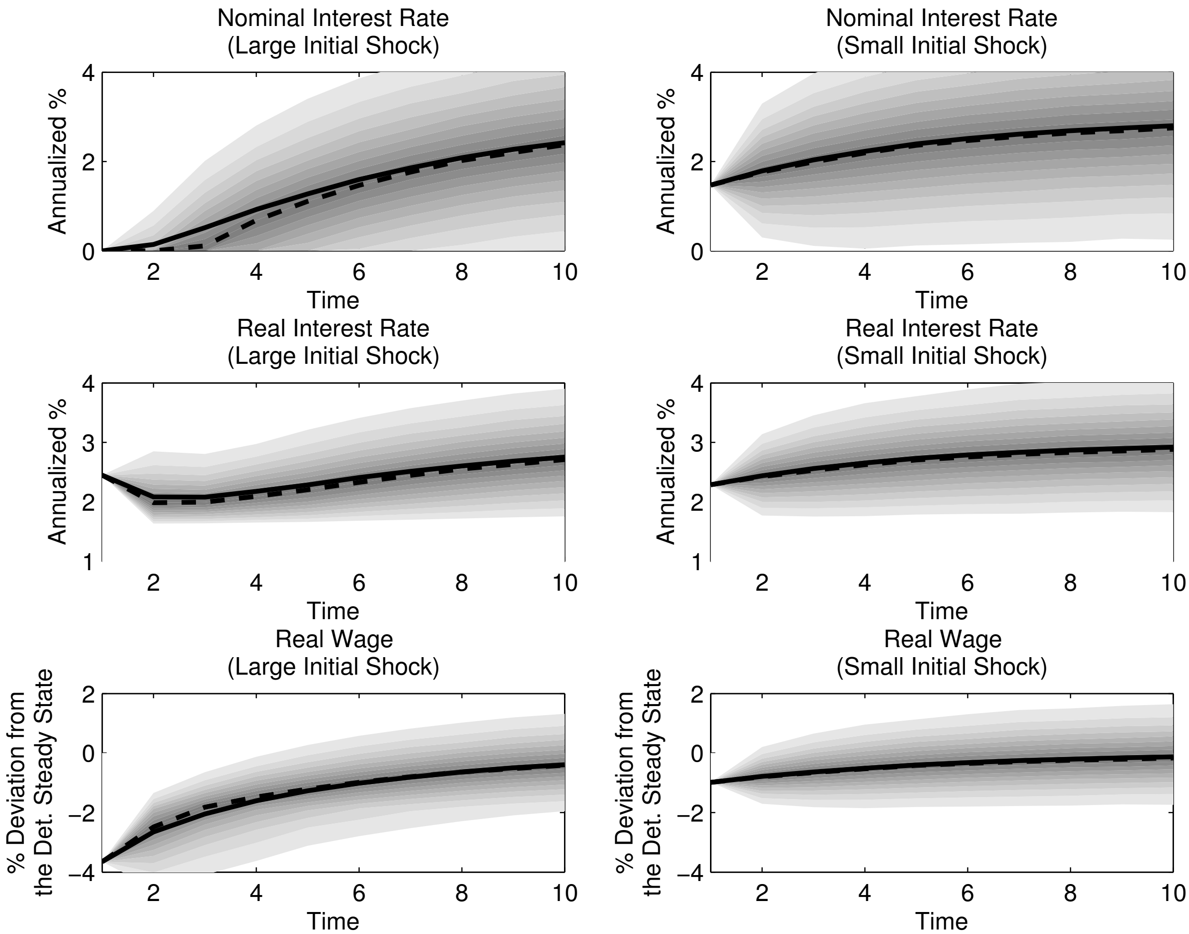
Figure 3: Policy Functions From A Partially Loglinearized Model




Dashed black lines: Deterministic model ($\sigma_\epsilon = 0$)
Solid black lines: Stochastic model ($\sigma_\epsilon = \frac{0.17}{100}$)

*Policy functions are shown for the range of $\delta$ that covers its steady-state level ($\delta = 1$) to the level that is 4 standard deviations away from the steady-state ($\delta = 1 + 4\sigma_\delta = 1.0113$).

Figure 4: Forecasts of Prices in Partial Equilibrium:
At vs. Away From the Zero Lower Bound




Dashed black lines: Forecasts in the partial equilibrium model with $\sigma_\epsilon = 0$.
Shaded Grey Areas: Density forecasts in the partial equilibrium model with $\sigma_\epsilon = \frac{0.17}{100}$.
Solid black lines: Point forecasts (i.e., $E_1[X_t|\delta_1 = 1 + 3\sigma_\delta]$ for any variable X)
in the partial equilibrium model with $\sigma_\epsilon = \frac{0.17}{100}$.

Figure 5: The Effect of an Increase in Uncertainty

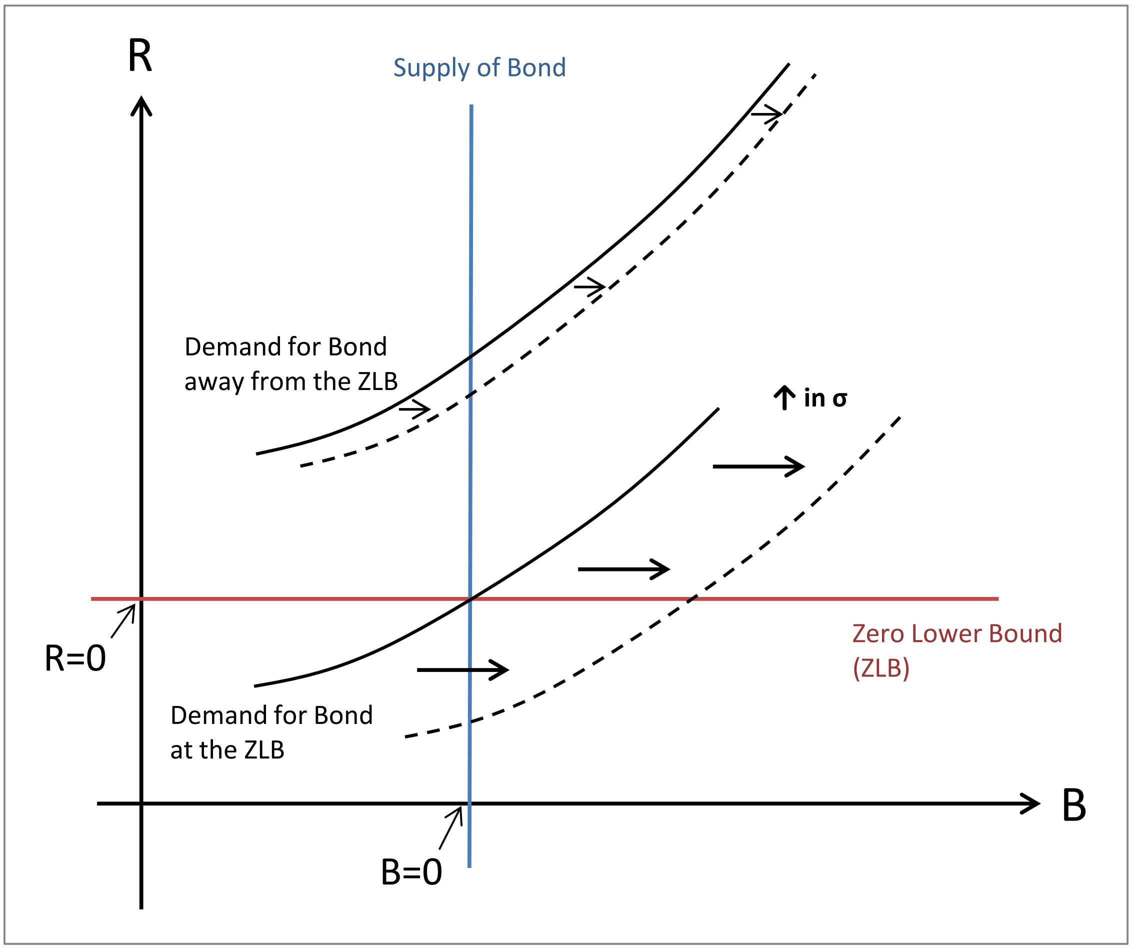
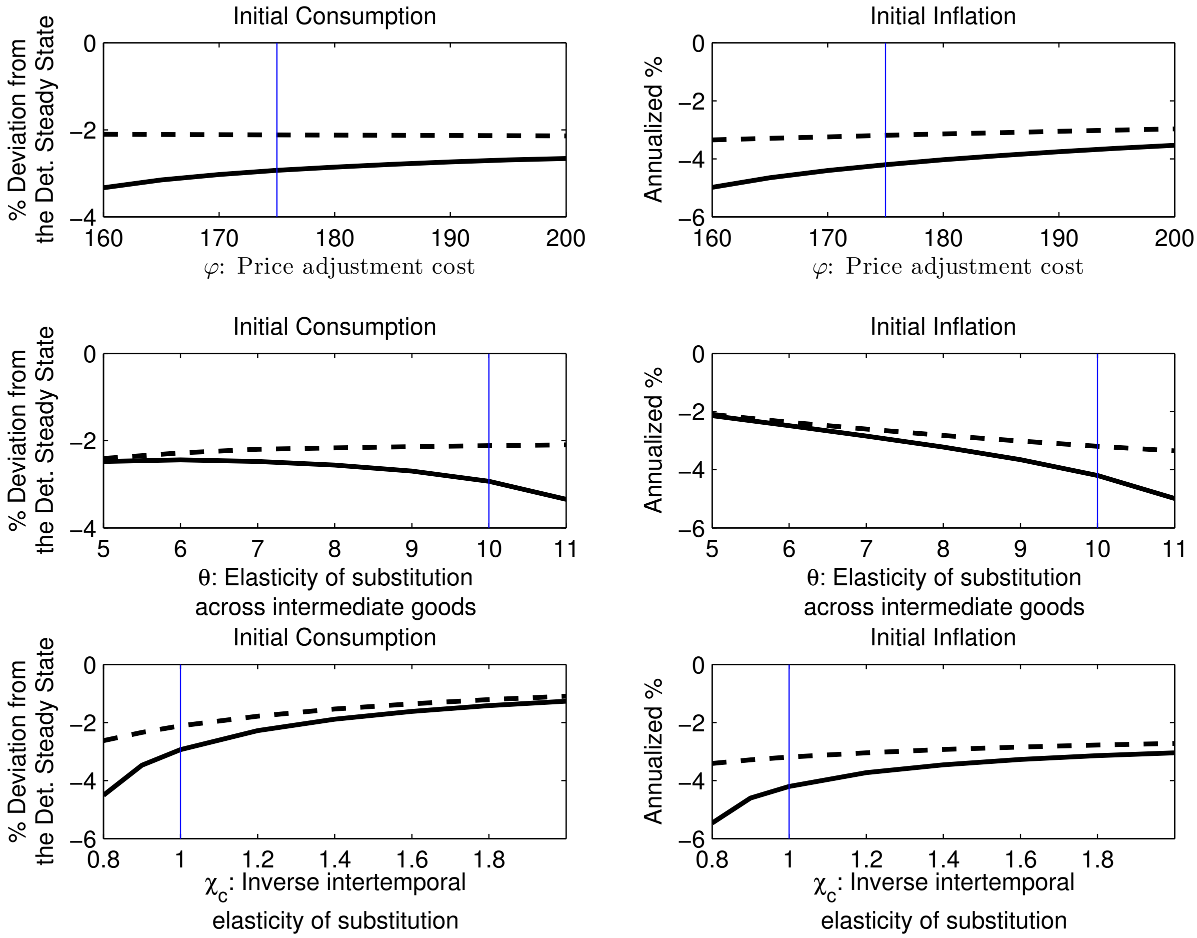


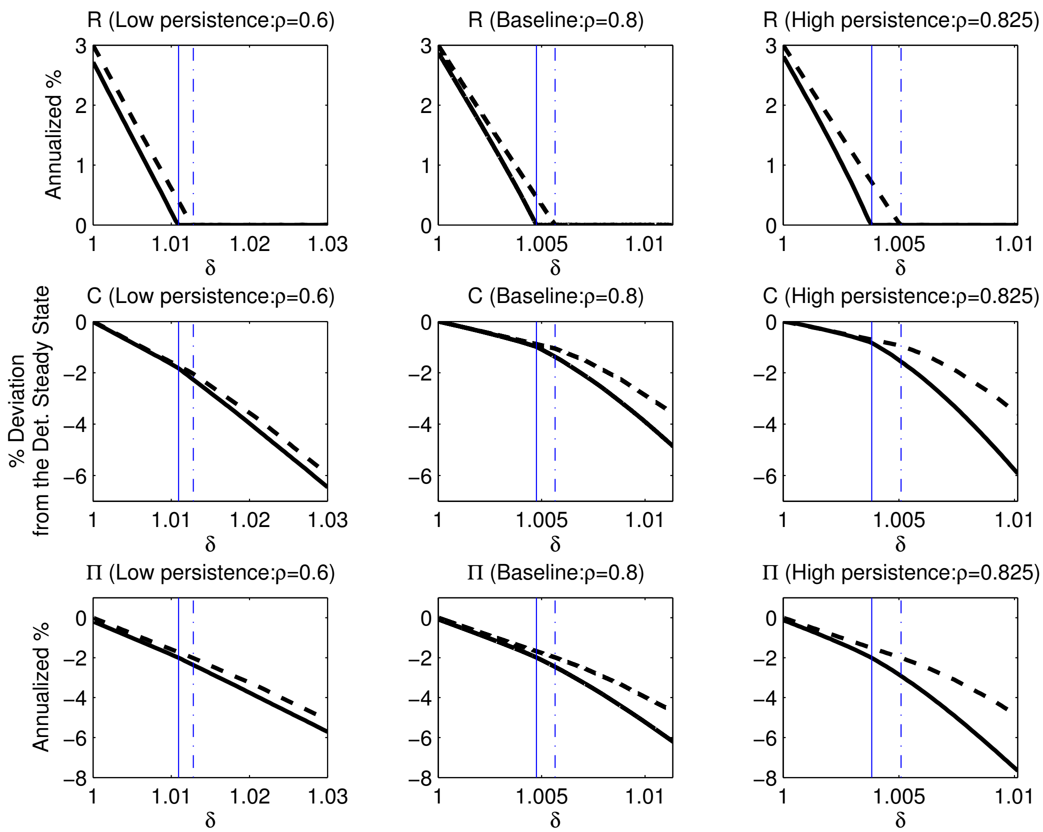
Figure 6: Sensitivity Analysis (I):
Declines in Consumption and Inflation at $\delta_1 = 1 + 3\sigma_\delta$




Dashed black line: Deterministic economy ($\sigma_\epsilon = 0$).
Solid black line: Stochastic economy ($\sigma_\epsilon = \frac{0.17}{100}$).

*For the price adjustment cost, $\varphi = 160$ implies the slope of the log-linearized Phillips curve that is equivalent to the one in the Calvo model with 78 percent chance of no price adjustment. $\varphi = 200$ corresponds to the Calvo model with 82 percent chance of no price adjustment.

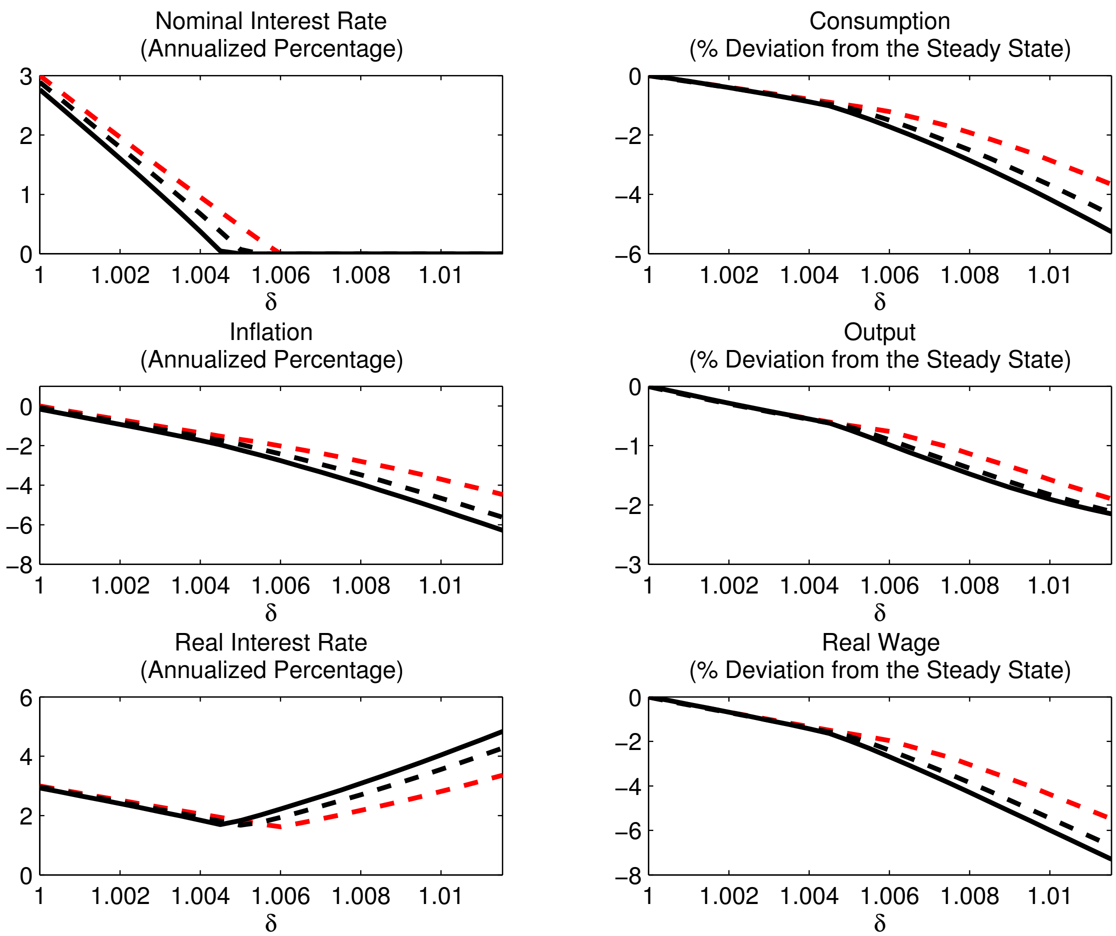
Figure 7: Sensitivity Analysis (II):
Policy Functions With Alternative Persistence of Shocks




Dashed black lines: Deterministic model
Solid black lines: Stochastic model

*For each model, policy functions are shown for the range of $\delta$ that covers its steady-state level to the level that is 4 standard deviations away from the steady-state. In each model, the standard deviation of the shock is chosen so that the frequency of being at the zero lower bound is 4 percent.

Figure 8: Policy Functions For Allocations and Prices: Additional Source of Uncertainty




Dashed red lines: $\delta$ and G are both deterministic ($\sigma_\delta = \sigma_g = 0$).
Dashed black lines: $\delta$ is stochastic and G is deterministic ($\sigma_\delta = \frac{0.17}{100}$ and $\sigma_g = 0$).
Solid black lines: $\delta$ and G are both stochastic ($\sigma_\delta = \frac{0.17}{100}$ and $\sigma_g = \frac{0.3}{100}$).

*Policy functions are shown for the range of $\delta$ that covers its steady-state level ($\delta = 1$) to the level that is 4 standard deviations away from the steady-state ($\delta = \delta + 4\sigma_\delta = 1.0113$). The second argument of the policy function is fixed at $G_t = G_{ss}$.

# Appendix

Appendices A through D present various results documenting the sensitivity of the paper's main finding—an increase in uncertainty reduces consumption, output, and inflation at the zero lower bound in a quantitatively important way—to alternative specifications of the model.

## A A model with price indexation

This section studies the effect of uncertainty at the zero lower bound when there is a backward-looking element in the firms' price setting decision. Following Ireland (2007), I modify the price adjustment cost function to penalize firms for deviating from the lagged aggregate inflation as follows.

$$P_t \frac{\varphi}{2}\Big[\frac{P_{i,t}}{\Pi_{t-1}^{\alpha} P_{i,t-1}} - 1\Big]^2 Y_t$$

where $\alpha$ measures the degree of price indexation. $\alpha = 0$ corresponds to the benchmark case without any indexation considered in the main text. This modification leads to a so-called hybrid Phillips curve in which today's inflation is a function of both expected inflation tomorrow and realized inflation yesterday.

The first and second columns in Figure 9 show the impulse response functions from the baseline economy without indexation and an economy with $\alpha = 0.5$. In both economies, $\sigma_\epsilon$ is set to $\frac{0.12}{100}$ as the maximum $\sigma_\epsilon$ consistent with the existence of equilibrium is lower than the original value of $\frac{0.17}{100}$ when the degree of price indexation is large. The initial $\delta$ is set to four standard deviations away from the steady state.

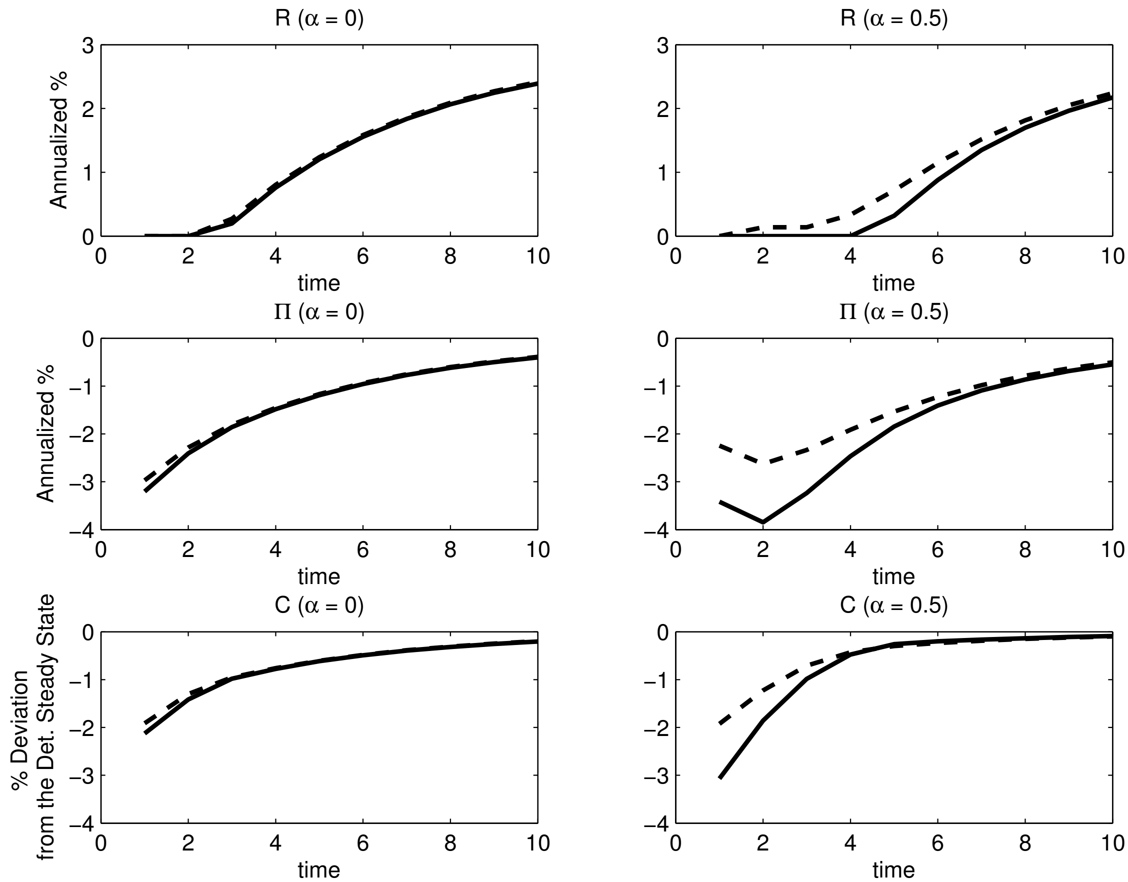
These figures show that the additional declines in consumption and inflation due to uncertainty are larger in an economy with price indexation. In the model without price indexation, the presence of uncertainty reduces consumption and inflation by about 0.2 percent at time one. In the model with price indexation, the presence of uncertainty reduces consumption and inflation by about one percent at time one. Also, in the model with price indexation, the additional decline in inflation due to uncertainty remains large for a longer period.

To understand why inertia in the price setting behavior magnifies the impact of uncertainty, let us examine the following log-linearized optimality condition of the firm.

$$\hat{\Pi}(\delta_t) - \alpha\hat{\Pi}(\delta_{t-1}) = \kappa\hat{w}(\delta_t) + \beta E_t\big[\hat{\Pi}(\delta_{t+1}) - \alpha\hat{\Pi}(\delta_t)\big]$$

To the extent that the discount rate is persistent, we can approximate this equilibrium condition as follows.

Figure 9: Impulse Response Functions with Alternative Degrees of Price Indexation




Dashed black lines: Deterministic model ($\sigma_\epsilon = 0$)
Solid black lines: Stochastic model ($\sigma_\epsilon = \frac{0.12}{100}$)

*Solid black lines show the evolution of endogenous variables when the realizations of $\epsilon_t$ is zero for all $t > 1$ in the stochastic economy (i.e. median responses).

$$(1-\alpha)\hat{\Pi}(\delta_t) \cong \kappa\hat{w}(\delta_t) + \beta E_t\big[(1-\alpha)\hat{\Pi}(\delta_{t+1})\big]$$

Iterating forward and dividing both sides by $1-\alpha$, we obtain

$$\hat{\Pi}(\delta_t) \cong \lim_{s\to\infty}\frac{E_t\hat{\Pi}(\delta_{t+s})}{1-\alpha} + \frac{\kappa}{1-\alpha}E_t\sum_{s=0}^{\infty}\beta^s\hat{w}(\delta_{t+s})$$

As described in the main text, an increase in uncertainty reduces the expected discounted sum of future real marginal costs, $E_t\sum_{s=0}^{\infty}\beta^s\hat{w}(\delta_{t+s})$, due to the concavity of the policy function for the real wage. The coefficient $\frac{\kappa}{1-\alpha}$ determines how sensitive today's inflation is to changes in the expected discounted sum of future real marginal costs. The larger $\alpha$ is, the more sensitive today's inflation is to the change in the expected future real marginal costs. Thus, an increase in uncertainty reduces today's inflation by a larger amount when the degree of indexation, $\alpha$, is higher.

# B A model with consumption habits

In this section, I will consider the effect of introducing consumption habits in the household's preference. The period utility function of the household is given by

$$\frac{(C_t-\gamma C_{t-1})^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n}$$

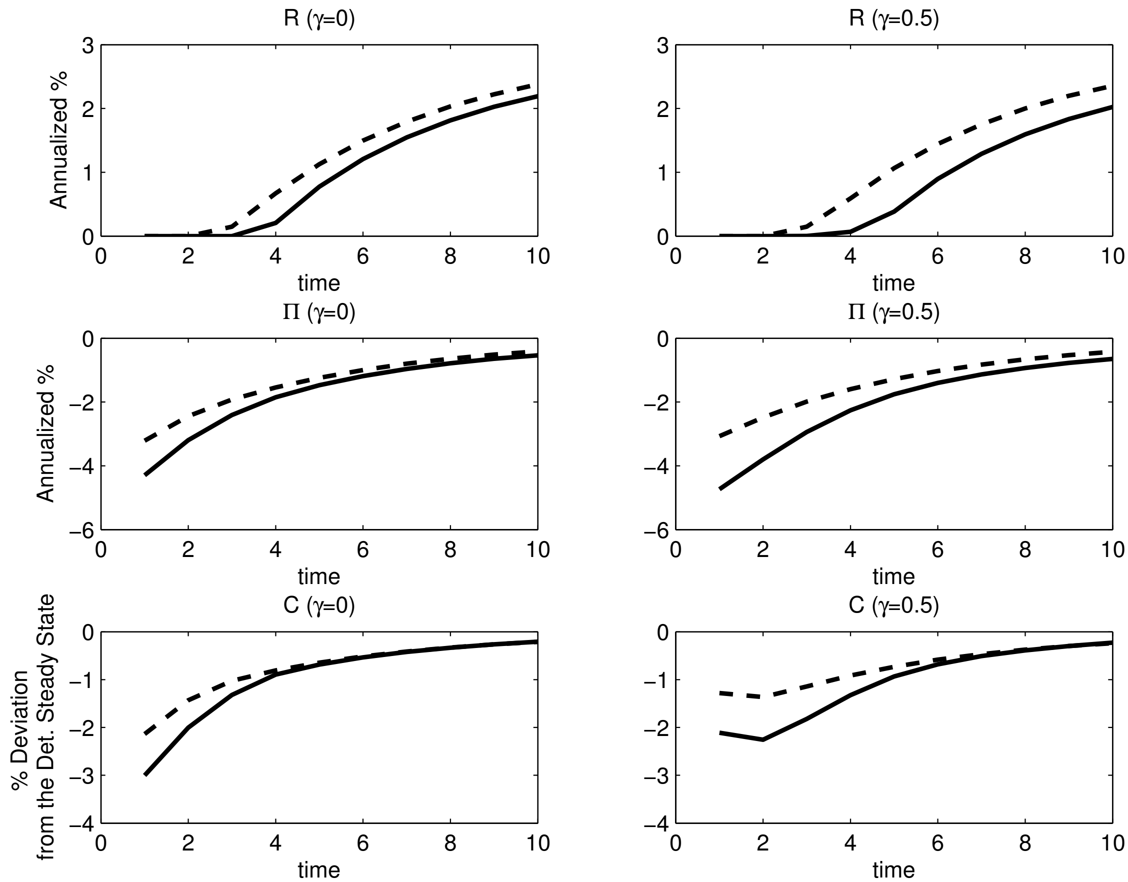
The first and second columns in Figure 10 show the impulse response functions for nominal interest rate, inflation, and consumption in the baseline economy without consumption habits and an economy with $\gamma = 0.5$. The figure shows that the effect of uncertainty is larger in the model with consumption habit. In particular, the additional decline in inflation due to uncertainty is about 1 percent in the baseline economy while it is about 2 percent in the economy with consumption habit. For consumption, even though the additional decline due to uncertainty is not so different at time one across two economies as measured by *percentage* deviation from the steady-state, the absolute decline is actually larger in the model with consumption habits because the steady-state consumption level is larger. Also, the effect of uncertainty persists longer in the economy with consumption habits.

We can again use the equilibrium condition from the partially log-linearized economy to understand why the effects of uncertainty are larger with consumption habits. The log-linearized consumption Euler equation in the model with consumption habit is given by

$$\hat{C}(\delta_t) - \gamma\hat{C}(\delta_{t-1}) = E_t\big[\hat{C}(\delta_{t+1}) - \gamma\hat{C}(\delta_t)\big] - \frac{1}{\chi_c}(\hat{R}_t - E_t\hat{\Pi}_{t+1}) - \frac{1}{\chi_c}(\delta_t - 1)$$

To the extent that the discount rate is persistent, we can approximate this equilibrium condition as follows.

Figure 10: Impulse Response Functions with Alternative Degrees of Consumption Habit




Dashed black lines: Deterministic model ($\sigma_\epsilon = 0$)
Solid black lines: Stochastic model ($\sigma_\epsilon = \frac{0.17}{100}$)

*Solid black lines show the evolution of endogenous variables when the realizations of $\epsilon_t$ is zero for all $t > 1$ in the stochastic economy (i.e. median responses).

$$(1-\gamma)\hat{C}(\delta_t) \cong E_t\big[(1-\gamma)\hat{C}(\delta_{t+1})\big] - \frac{1}{\chi_c}(\hat{R}_t - E_t\hat{\Pi}_{t+1}) - \frac{1}{\chi_c}(\delta_t - 1)$$

Iterating forward and dividing both sides by $1-\gamma$, we obtain

$$\hat{C}(\delta_t) = \lim_{s\to\infty}\frac{E_t\hat{C}(\delta_{t+s})}{1-\gamma} - \frac{1}{\chi_c(1-\gamma)}E_t\sum_{s=0}^{\infty}(\hat{R}(\delta_{t+s}) - \hat{\Pi}(\delta_{t+s+1})) - \frac{1}{\chi_c(1-\gamma)}E_t\sum_{s=0}^{\infty}(\delta_{t+s} - 1)$$

As described in the main text, an increase in uncertainty reduces the expected sum of future real interest rates, $E_t\sum_{s=0}^{\infty}(\hat{R}(\delta_{t+s}) - \hat{\Pi}(\delta_{t+s+1}))$. The coefficient $\frac{1}{\chi_c(1-\gamma)}$ determines how sensitive today's consumption is to changes in the expected sum of future real interest rates. The larger $\gamma$ is, the more sensitive today's consumption is to the change in the expected future real interest rates. Thus, an increase in uncertainty reduces today's consumption by a larger amount when the degree of consumption habit, $\gamma$, is higher. A larger decline in consumption leads to larger declines in the real wage and inflation through the intratemporal optimality condition of the household and the Phillips curve.

# C A model with inertia in the Taylor rule

In this section, I document the effect of having inertia in the truncated Taylor rule. I consider two versions of the inertial Taylor rule, both of which appeared in the literature. The first version is given by

$$\begin{aligned} R_t^* &= \frac{1}{\beta}\Big[\frac{R_{t-1}}{R_{ss}}\Big]^{\rho_R}\big[\Pi_t\big]^{\phi(1-\rho_R)}\Big] \\ R_t &= \max[1, R_t^*] \end{aligned}$$

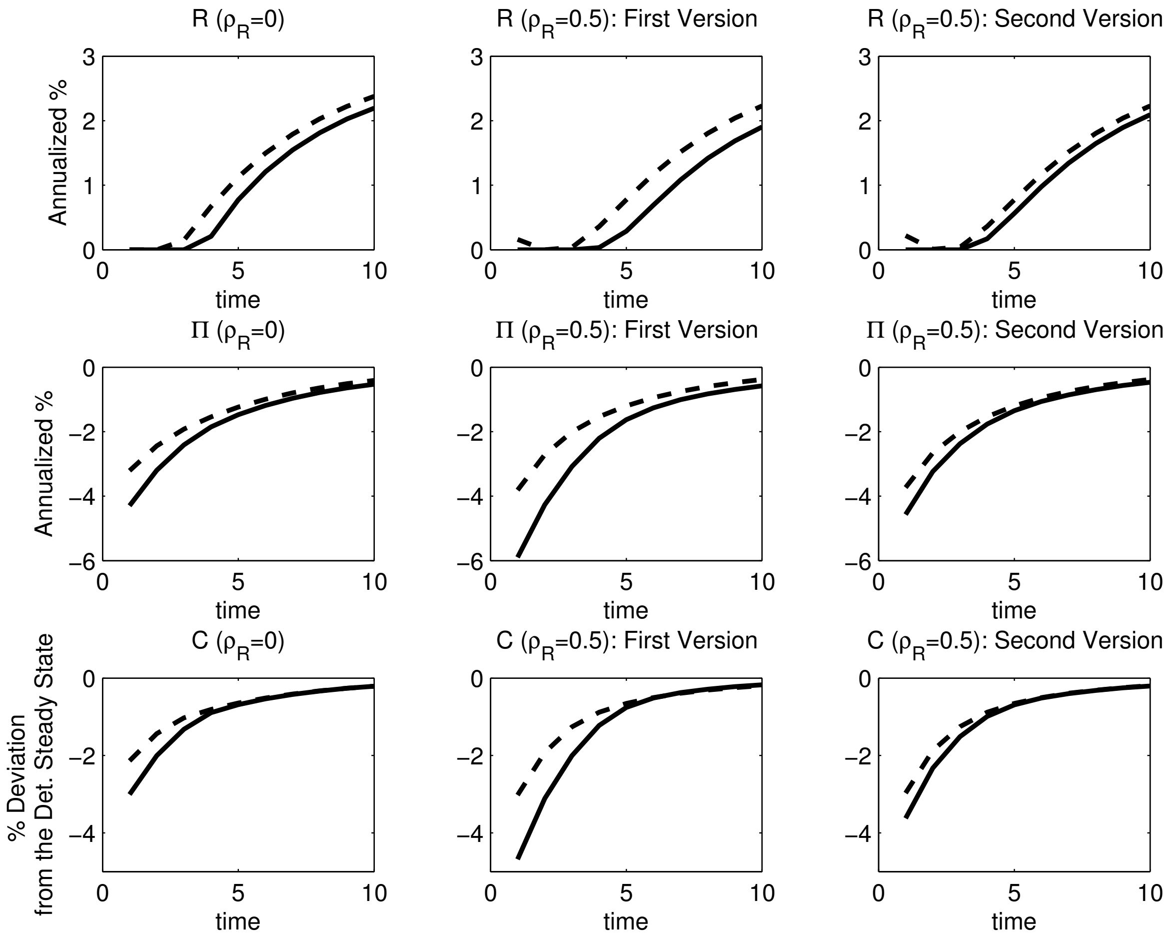
In this version, $R_t^*$, the notional nominal interest rate, is a function of the lagged actual nominal interest rate. This version of the inertial Taylor rule has been considered by Nakov (2008) and Billi (2011). The second version is given by

$$\begin{aligned} R_t^* &= \frac{1}{\beta}\Big[\frac{R_{t-1}^*}{R_{ss}}\Big]^{\rho_R}\big[\Pi_t\big]^{\phi(1-\rho_R)}\Big] \\ R_t &= \max[1, R_t^*] \end{aligned}$$

In this version, $R_t^*$ is a function of the lagged notional nominal interest rate. This version has been considered by Basu and Bundick (2012) and Gust, Lopez-Salido, and Smith (2012). These two versions would be identical in the absence of the zero lower bound, but lead to different dynamics in the presence of the zero lower bound.

Figure 11 shows the impulse response functions of nominal interest rate, consumption, and

Figure 11: Impulse Response Functions with Alternative Degrees of Policy Inertia




Dashed black lines: Deterministic model ($\sigma_\epsilon = 0$)
Solid black lines: Stochastic model ($\sigma_\epsilon = \frac{0.17}{100}$ in the first column and $\sigma_\epsilon = \frac{0.22}{100}$ in the second and third columns)

*Solid black lines show the evolution of endogenous variables when the realizations of $\epsilon_t$ is zero for all $t > 1$ in the stochastic economy (i.e. median responses).

inflation for the two versions of the inertial Taylor rule with $\rho_R = 0.5$. The second and third columns are respectively for the first and second versions, while the first column shows the baseline model with no inertia for a comparison purpose. Inertia in the truncated Taylor rule prevents the nominal interest rate from immediately falling in response to an increase in the discount factor shock, and the nominal interest rate falls to zero very rarely with the baseline value of the shock variance. To see the effect of uncertainty in the model with inertial policy rules in an environment in which there is a reasonable probability of hitting the zero bound, the variance of shocks is increased to $\sigma_\epsilon = \frac{0.22}{100}$ so that the frequency of hitting the zero bound is 4 percent in the second and third columns.

The figure shows that the presence of uncertainty reduces consumption and inflation at the zero lower bound in the model with inertial Taylor rules. In both versions, solid black lines lie below dashed black lines at the zero lower bound in a quantitatively important way, even though the differences are much smaller when the nominal interest rate is above zero. However, the effects of uncertainty are quantitatively different between two alternative specifications. In particular, the additional decline due to uncertainty is smaller in the second version than in the first version. The additional declines in consumption and inflation at time one are about 1 percent in the first version while they are about 0.5 percent in the second version. As such, this exercise illustrates the importance of taking uncertainty into account when one evaluates the performance of alternative policy rules.

# D A model with alternative shocks

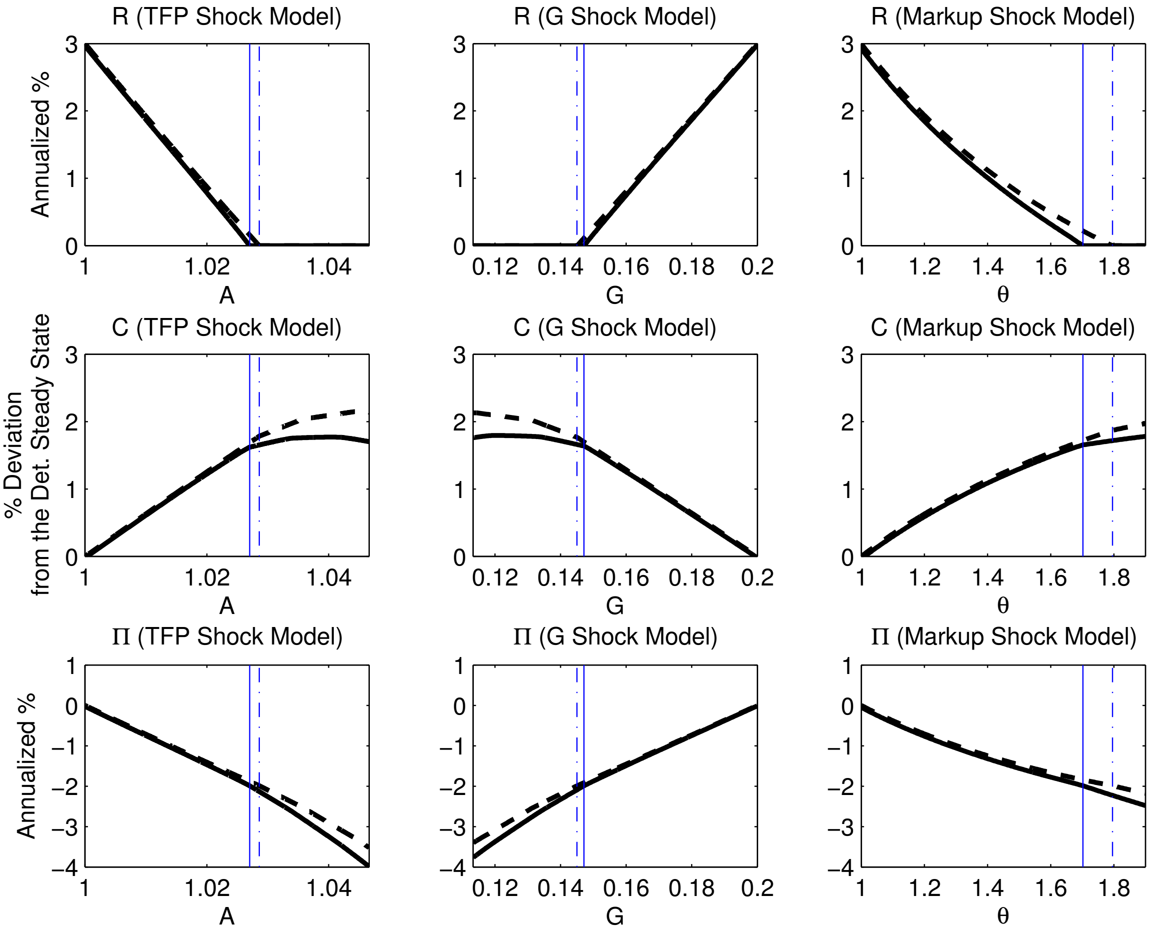
Throughout the paper, I used an exogenous variation in the household's discount rate as the device that pushes the nominal interest rate to zero. One may wonder whether or not the importance of uncertainty arises only in the presence of this specific exogenous force. In this section, I will consider models without the discount rate shock, but with either one of the following three shocks—government spending, mark-up, and technology shocks—, in order to show that an increase in the variance of *any* shocks will reduce allocations and price at the zero lower bound.

In the model with TFP shock, the production function of intermediate goods producers is given by $Y_{i,t} = A_t N_{i,t}$ where $A_t$ follows an AR(1) process. In the model with markup shock, the parameter governing the degree of imperfect competition, $\theta$, is replaced by $\theta_t \bar{\theta}$ where $\log(\theta_t)$ follows an AR(1) process.[13] In the model with government spending shock, the government spending is assumed to follow an AR(1) process, as in Section 7. In all three models, persistence of the exogenous variables is set to 0.8 and the standard deviation of innovations to the processes is chosen so that the probability of being at the zero lower bound is 1 percent.[14]

[13] It turns out that a large variance is needed to make the probability of being at the zero lower bound 1 percent in the model with markup shock, and I needed to use log specification in order to ensure that $\theta_t$ stays positive.

[14] The implied unconditional variance of these shocks are unrealistically large. In fact, several papers have suggested that these three shocks are unlikely to send the nominal interest rates to zero. The point of this exercise is to simply show the main finding of the paper is robust even if shocks other than the discount rate ever pushes the nominal interest rate to zero.

Figure 12: Policy Functions With Alternative Shocks




Dashed black lines: Deterministic model
Solid black lines: Stochastic model

*For each model, policy functions are shown for the range of $\delta$ that covers its steady-state level to the level that is 4 standard deviations away from the steady-state. In each model, the standard deviation of the shock is chosen so that the frequency of being at the zero lower bound is 1 percent.

Figure 12 shows the policy functions for nominal interest rate, consumption, and inflation from the three models. The first, second, and third columns are for the economy with TFP shock, the economy with government spending shock, and the economy with mark-up shock, respectively.

In response to an increase in total factor productivity, consumption rises, and inflation and nominal interest rate fall. For a sufficiently large increase in total factor productivity, the nominal interest rate cannot fall further due to the zero bound constraint, and this slows down an increase in consumption and accerelates decline in inflation. In the presence of uncertainty, the slow down in consumption and decline in inflation are magnified.

In the model with government spending shock, a reduction in government spending causes consumption to rise, and inflation and nominal itneerst rate to fall. For a sufficiently large decline in government spending, the zero lower bound constraint prevents the nominal interest rate from falling further, which slows down the rise in consumption and accelerates the decline in inflation. The presence of uncertainty further depresses consumption and inflation when the nominal interest rate is zero.

Finally, in the model with mark-up shock, an increase in the substitutability of intermediate goods (i.e., a reduction in markup) leads inflation and the nominal interest rate to fall and consumption to rise. For a sufficiently large shock, the nominal interest rate hits the zero lower bound, which leads inflation to decline more rapidly and slows down the increase in consumption. In the presence of uncertainty, these adverse effects of the zero lower bound constraints on consumption and inflation are amplified. A rise in consumption slows down by more and inflation declines by more.

Thus, the mechanism described in the main text is not a specific feature of the model with discount factor shocks. An increase in the variance of *any* shocks reduces allocations and prices when the nominal interest is constrained by the zero lower bound.