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# Expectations of functions of stochastic time with application to credit risk modeling

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# Expectations of functions of stochastic time with application to credit risk modeling[*]

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**Abstract**

We develop two novel approaches to solving for the Laplace transform of a time-changed stochastic process. We discard the standard assumption that the background process $(X_t)$ is Lévy. Maintaining the assumption that the business clock $(T_t)$ and the background process are independent, we develop two different series solutions for the Laplace transform of the time-changed process $\tilde{X}_t = X(T_t)$. In fact, our methods apply not only to Laplace transforms, but more generically to expectations of smooth functions of random time. We apply the methods to introduce stochastic time change to the standard class of default intensity models of credit risk, and show that stochastic time-change has a very large effect on the pricing of deep out-of-the-money options on credit default swaps.

*JEL* Codes: G12, G13
*Keywords*: time change, default intensity, credit risk, CDS options

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# 1 Introduction

Stochastic time-change offers a parsimonious and economically well-grounded device for introducing stochastic volatility to simpler constant volatility models. The constant volatility model is assumed to apply in a latent "business time." The speed of business time with respect to calendar time is stochastic, and reflects the varying rate of arrival of news to the markets. Most applications of stochastic time-change in the finance literature have focused on the pricing of stock options. Log stock prices are naturally modeled as Lévy processes, and it is well known that any Lévy process subordinated by a Lévy time-change is also a Lévy process. The variance gamma (Madan and Seneta, 1990; Madan et al., 1998) and normal inverse Gaussian (Barndorff-Nielsen, 1998) models are well-known early examples. To allow for volatility clustering, Carr, Geman, Madan, and Yor (2003) introduce a class of models in which the background Lévy process is subordinated by the time-integral of a mean-reverting CIR activity-rate process, and solve for the Laplace transform of the time-changed process. Carr and Wu (2004) extend this framework to accommodate dependence of a general form between the activity rate and background processes, as well as a wider class of activity rate processes.

In this paper, we generalize the basic model in complementary directions. We discard the assumption that the background process is Lévy, and assume instead that the background process $(X_t)$ has a known Laplace transform, $S(u;t) = \mathrm{E}\left[\exp(-uX(t))\right]$. Maintaining the requirement that the business clock $(T_t)$ and the background process are independent, we develop two different series solutions for the Laplace transform of the time-changed process $\tilde{X}_t = X(T_t)$ given by $\tilde{S}(u;t) = \mathrm{E}\left[\exp(-uX(T_t))\right] = \mathrm{E}\left[S(u;T_t)\right]$. In fact, our methods apply generically to a very wide class of smooth functions of time, and in no way require $S$ to be the Laplace transform of a stochastic process. Henceforth, for notational parsimony, we drop the auxilliary parameter $u$ from $S(t)$.

Our two series solution are complementary to one another in the sense that the restrictions imposed by the two methods on $S(t)$ and on $T_t$ differ substantively. The first method requires that $T_t$ be a Lévy process, but imposes fairly mild restrictions on $S(t)$. The second method imposes fairly stringent restrictions on $S(t)$, but very weak restrictions on $T_t$. In particular, the second method allows for volatility clustering through serial dependence in the activity rate. Thus, the two methods may be useful in different sorts of applications.

Our application is to modeling credit risk. Despite the extensive literature on stochastic volatility in stock returns, the theoretical and empirical literature on stochastic volatility in credit risk models is sparse. Empirical evidence of stochastic volatility in models of corporate bond and credit default swap spreads is provided by Jacobs and Li (2008), Alexander and Kaeck (2008), Zhang et al. (2009) and Gordy and Willemann (2012). To introduce stochastic volatility to the class of default intensity models pioneered by Jarrow and Turnbull (1995) and Duffie and Singleton (1999), Jacobs and Li (2008) replace the widely-used single-factor CIR specification for the intensity with a two-factor specification in which a second CIR process controls the volatility of the intensity process. The model is formally equivalent to the Fong and Vasicek (1991) model of stochastic volatility in interest rates. An important limitation of this two-factor model is that there is no region of the parameter space for which the default intensity is bounded nonnegative (unless the volatility of volatility is zero).[1]

In this paper, we introduce stochastic volatility to the default intensity framework by time-

[1]The structural models of Merton (1974) and Black and Cox (1976) have also been extended to allow for stochastic volatility. See Fouque et al. (2006), Hurd (2009), and Gouriéroux and Sufana (2010).

changing the firm's default time. Let $\tilde{\tau}$ denote the calendar default time, and let $\tau = T_{\tilde{\tau}}$ be the corresponding time under the business clock. Define the background process $X_t$ as the time-integral (or "compensator") of the default intensity and $S(t)$ as the business-time survival probability function $S(t) = \mathrm{E}[\exp(-X_t)]$. If we impose independence between $X_t$ and $T_t$, as we do throughout this paper, then time-changing the default time is equivalent to time-changing $X_t$, and the calendar-time survival probability function is

$$\tilde{S}(t) = \Pr(\tilde{\tau} > t) = \Pr(\tau > T_t) = \mathrm{E}[\exp(-X(T_t))] = \mathrm{E}[\mathrm{E}[\exp(-X(T_t))|T_t]] = \mathrm{E}[S(T_t)].$$

The time-changed model inherits important properties of the business time model. In particular, when the default intensity is bounded nonnegative in business time, the calendar-time default intensity is also bounded nonnegative. However, analytical tractability in the business time model is not, in general, inherited. If we allow for serial dependence in the default intensity, the compensator $X_t$ cannot be a Lévy process, so the method of Carr and Wu (2004) cannot be applied.[2] We show that both of our series methods are applicable and, indeed, both can be implemented efficiently.

The idea of time-changing default times appears to have first been used by Joshi and Stacey (2006). Their model is intended for pricing collateralized debt obligations, so makes the simplifying assumption that firm default intensities are deterministic.[3] Mendoza-Arriaga et al. (2010) apply time-change to a credit-equity hybrid model. If we strip out the equity component of their model, the credit component is essentially a time-changed default intensity model. Unlike our model, however, their model does not nest the CIR specification of the default intensity, which is by far the most widely used specification in the literature and in practice. Most closely related to our paper is the time-changed intensity model of Mendoza-Arriaga and Linetsky (2012).[4] They obtain a spectral decomposition of the subordinate semigroups, and from this obtain a series solution to the survival probability function. As in our paper, the primary application in their paper is to the evolution of survival probabilities in a model with a CIR intensity in business time and a tempered stable subordinator. When that CIR process is stationary, their solution coincides with that of our second solution method. However, our method can be applied in the non-stationary case as well and generalizes easily when the CIR process is replaced by a basic affine process. Empirically, the default intensity process is indeed non-stationary under the risk-neutral measure for the typical firm (Duffee, 1999; Jacobs and Li, 2008).

Our two expansion methods are developed for a general function $S(t)$ and wide classes of time-change processes in Sections 2 and 3. An application to credit risk modeling is presented in Section 4. The properties of the resulting model are explored with numerical examples in Section 5. In Section 6, we show that stochastic time-change has a very large effect on the pricing of deep out-of-the-money options on credit default swaps. In Section 7, we demonstrate that our expansion methods can be extended to a much wider class of multi-factor affine jump-diffusion business time models.

[2]As we will discuss in Section 4, the compensator $X_t$ can be expressed as a time-changed Lévy process, but not in a way that allows the Laplace transform of $\tilde{X}_t$ to be obtained as in Carr and Wu (2004).

[3]Ding et al. (2009) solve a more sophisticated model in which the default intensity is self-exciting, but is constant in between default arrival times.

[4]We became aware of this paper only upon release of the working paper on SSRN in September 2012. Our research was conducted independently and contemporaneously.

## 2 Expansion in derivatives

The method of this section imposes weak regularity conditions on $S(t)$, but places somewhat strong restrictions on $T_t$. Throughout this section, we assume

**Assumption 1.** *(i) $T_t$ is a subordinator. (ii) The Laplace exponent $\Psi(u)$ of $T_t$ exists for all $u < u_0$ for a threshold $u_0 > 0$ and is real analytic about the origin.*

A subordinator is an almost surely increasing Lévy process (see Proposition 3.10 in Cont and Tankov, 2004, for a formal definition). The Laplace exponent solves $\mathrm{E}[\exp(uT_t)] = \exp(t\Psi(u))$. Since $t\Psi(u)$ is the cumulant generating function of $T_t$, part (ii) of the assumption guarantees that all cumulants (and moments) of $T_t$ are finite, and that we can expand $\Psi(u)$ as

$$\Psi(u) = \psi_1 u + \frac{1}{2}\psi_2 u^2 + \frac{1}{3!}\psi_3 u^3 \dots \tag{2.1}$$

The $n^{\text{th}}$ cumulant of $T_t$ is $t\psi_n$. Carr and Wu (2004) normalize $\psi_1 = 1$ so that the business clock is an unbiased distortion of the calendar, i.e., $\mathrm{E}[T_t] = t$. We assume $\psi_1 > 0$ but otherwise leave it unconstrained. The moments of $T_t$ can be obtained from the cumulants:

$$\mathrm{E}[T_t^n] = \sum_{m=0}^{n} Y_{n,n-m}(\psi_1, \dots, \psi_{m+1}) t^{n-m} \tag{2.2}$$

where $Y_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ is the incomplete Bell polynomial. For notational compactness, we may write $Y_{n,n-m}(\psi)$ to mean $Y_{n,n-m}(\psi_1, \dots, \psi_{m+1})$. In the analysis below, we will manipulate Bell polynomials in various ways. Unless otherwise noted, the transformations can easily be verified using the identities collected in Appendix A.

We assume that $S(t)$ is a smooth function of time. Imposing Assumption 1, we expand $S(t)$ as a formal series and integrate:

$$\begin{aligned}
\tilde{S}(t) = \mathrm{E}[S(T_t)] = E\left[\sum_{n=0}^{\infty} \frac{\beta_n}{n!} T_t^n\right] &= \sum_{n=0}^{\infty} \frac{\beta_n}{n!} \mathrm{E}[T_t^n] \\
&= \sum_{n=0}^{\infty} \frac{\beta_n}{n!} \sum_{m=0}^{n} Y_{n,n-m}(\psi) t^{n-m} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{\beta_n}{n!} Y_{n,n-m}(\psi) t^{n-m} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta_{n+m}}{(n+m)!} Y_{n+m,n}(\psi) t^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta_{n+m}}{n!} \left(\frac{n!}{(n+m)!} Y_{n+m,n}(\psi)\right) t^n
\end{aligned} \tag{2.3}$$

From equation (A.1) and the recurrence rule (A.3), it follows immediately that

**Lemma 1.** *Under Assumption 1,*

$$\frac{n!}{(n+m)!} Y_{n+m,n}(\psi) = \frac{\psi_1^{n+m}}{m!} \sum_{j=0}^{m} (n)_j Y_{m,j}\left(\frac{\psi_2}{2\psi_1^2}, \frac{\psi_3}{3\psi_1^3}, \frac{\psi_4}{4\psi_1^4}, \dots\right)$$

where $(z)_j$ denotes the falling factorial $(z)_j = z \cdot (z-1) \cdots (z-j+1)$. To handle the special case of $m = 0$, we have $Y_{n,n}(\psi) = \psi_1^n$ for $n \geq 0$.

Defining the constants

$$\gamma_{m,j} = \frac{1}{m!} Y_{m,j}\left(\frac{\psi_2}{2\psi_1^2}, \frac{\psi_3}{3\psi_1^3}, \frac{\psi_4}{4\psi_1^4}, \ldots\right)$$

for $m \geq j \geq 0$, we can write

$$\frac{n!}{(n+m)!} Y_{n+m,n}(\psi) = \psi_1^{n+m} \sum_{j=0}^{m} \gamma_{m,j} (n)_j.$$

Observe that $\gamma_{m,j}$ depends on $m$, $j$, and $\psi_1, \ldots, \psi_{m+1}$, but not on $n$. We substitute into equation (2.3) to get

$$\tilde{S}(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta_{n+m}}{n!} \psi_1^{n+m} t^n \sum_{j=0}^{m} \gamma_{m,j} (n)_j = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \gamma_{m,j} \sum_{n=0}^{\infty} \frac{\beta_{n+m}}{n!} (n)_j \psi_1^{n+m} t^n \tag{2.4}$$

Observing that

$$\frac{(n)_j}{n!} = \begin{cases} 1/(n-j)! & \text{if } j \leq n, \\ 0 & \text{if } j > n, \end{cases}$$

we have

$$\sum_{n=0}^{\infty} \frac{\beta_{n+m}}{n!} (n)_j \psi_1^{n+m} t^n = \sum_{n=j}^{\infty} \frac{\beta_{n+m}}{(n-j)!} \psi_1^{n+m} t^n = \sum_{n=0}^{\infty} \frac{\beta_{n+m+j}}{n!} \psi_1^{n+m+j} t^{n+j} = t^j D_t^{m+j} S(\psi_1 t)$$

where $D_t$ is the differential operator $\frac{d}{dt}$. Substituting into equation (2.4) delivers

$$\tilde{S}(t) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \gamma_{m,j} t^j D_t^{m+j} S(\psi_1 t). \tag{2.5}$$

To obtain a generating function for the constants $\gamma_{m,j}$, we substitute $\exp(ut/\psi_1)$ for $S(t)$ and then divide each side by $\exp(ut/\psi_1)$.

$$\exp(t\Psi(u/\psi_1) - tu) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \gamma_{m,j} t^j u^{m+j}. \tag{2.6}$$

By Assumption 1(ii), $\Psi(u)$ is analytic in the neighborhood of the origin, and $t\Psi(u/\psi_1) - tu$ is linear in $t$. Therefore, $t\Psi(u/\psi_1) - tu$ is analytic in $t$ and locally analytic in $u$. The exponential of a convergent series gives rise to a convergent series, so the series in (2.6) is convergent for any $t \geq 0$ and for $u$ near zero.[5] This will be helpful in the analysis below. We also note that the constants $\gamma_{m,j}$ can easily be computed via the recurrence rule (A.4).

To guarantee that the series expansion in equation (2.5) is convergent, we would require rather strong conditions. The function $S(t)$ must be entire, the coefficients $\beta_n$ in equation (2.3) must decay faster than geometrically, and the coefficients $\gamma_{m,j}$ must vanish at a geometric rate in $m, j$. In application, it may be that none of these assumptions hold. If $S(t)$ is analytic but non-entire, then $D_t^{m+j} S(\psi_1 t) = O((m+j)!)$, so geometric behavior in the $\gamma_{m,j}$ would not be sufficient for

[5] By Hartog's theorem, a function which is analytic in a number of variables separately, for each of them in some disk, is jointly analytic in the product of the disks (Narasimhan, 1971).

convergence. Furthermore, we will provide a practically relevant specification below in which the $\gamma_{m,j}$ are *increasing* in $m$ for fixed $j$. Even if the series expansion in equation (2.5) is, in general, divergent, we will see that it may nonetheless be computationally effective.

We now provide an alternative justification for equation (2.5) to clarify the convergence behavior of our expansion. We introduce a regularity condition on $S(t)$:

**Assumption 2.** *There exists a finite signed measure $\mu$ on $[0, \infty)$ such that*

$$S(t) = \int_0^\infty e^{-ut} d\mu(u)$$

This regularity condition is roughly equivalent to imposing analyticity and restrictions on tail behavior in the complex plane. It is an assumption that is often made in asymptotics and often satisfied. The condition could be relaxed at the expense of making the analysis more cumbersome.[6]

Assumption 2 implies

$$S(\psi_1 t) = \int_0^\infty \exp(-\psi_1 ut) d\mu(u) \tag{2.7}$$

so

$$D_t^{m+j} S(\psi_1 t) = \psi_1^{m+j} \int_0^\infty (-u)^{m+j} \exp(-\psi_1 ut) d\mu(u)$$

Assumption 2 guarantees that this integral is convergent for all $m + j \geq 0$, which implies that $S$ is smooth. Thus we have for all $M \geq 0$

$$\sum_{m=0}^{M} \sum_{j=0}^{m} \gamma_{m,j} t^j D_t^{m+j} S(\psi_1 t) = \sum_{m=0}^{M} \sum_{j=0}^{m} \gamma_{m,j} \psi_1^{m+j} t^j \int_0^\infty (-u)^{m+j} \exp(-\psi_1 ut) d\mu(u) \tag{2.8}$$

Let $R_M$ be the remainder function from the generating equation (2.6), that is,

$$R_M(t, u) = \exp(t\Psi(u/\psi_1)) - e^{tu} \sum_{m=0}^{M} \sum_{j=0}^{m} \gamma_{m,j} t^j u^{m+j} \tag{2.9}$$

and let $\tilde{S}_M(t)$ be the approximation to $\tilde{S}(t)$ up to term $M$ in the expansion (2.5), i.e.,

$$\tilde{S}_M(t) = \sum_{m=0}^{M} \sum_{j=0}^{m} \gamma_{m,j} t^j D_t^{m+j} S(\psi_1 t). \tag{2.10}$$

The following proposition formalizes our approximation:

**Proposition 1.** *Under Assumptions 1 and 2,*

$$\tilde{S}(t) = \tilde{S}_M(t) + \int_0^\infty R_M(t, -\psi_1 u) d\mu(u)$$

[6] Our analysis indicates that Assumption 2 could be replaced altogether with the much weaker condition that $S(t)$ is analyzable, i.e., that the function admits a Borel summable transseries at infinity (see Écalle, 1993).

*Proof.* By (2.7) we have

$$\tilde{S}(t) = \mathrm{E}\left[\int_0^\infty e^{-uT_t} d\mu(u)\right] = \int_0^\infty \mathrm{E}\left[e^{-uT_t} d\mu(u)\right] = \int_0^\infty e^{t\Psi(-u)} d\mu(u) \tag{2.11}$$

where Assumption 2 guarantees the change of the order of integration and the last equality follows from the fact that $t\Psi(u)$ is the cumulant generating function of $T_t$. We obtain from (2.8), (2.9) and (2.11) that

$$\tilde{S}(t) = \int_0^\infty \left( R_M(t, -\psi_1 u) + e^{-t\psi_1 u} \sum_{m=0}^{M} \sum_{j=0}^{m} \gamma_{m,j} \psi_1^{m+j} t^j (-u)^{m+j} \right) d\mu(u)$$

$$= \sum_{m=0}^{M} \sum_{j=0}^{m} \gamma_{m,j} t^j D_t^{m+j} S(\psi_1 t) + \int_0^\infty R_M(t, -\psi_1 u) d\mu(u)$$

which implies the conclusion. □

Since $M$ can be arbitrarily large, Proposition 1 provides a rigorous meaning for equation (2.5). However, it does not by itself explain why we should expect $\tilde{S}_M(t)$ to provide a good approximation to $\tilde{S}(t)$. Equation (2.8) shows that the divergent sum (2.5) comes from the Laplace transform of the locally convergent sum in (2.6) (with $u$ replaced by $-\psi_1 u$). It has been known for a long time that a divergent power series obtained by Laplace transforming a locally convergent sum is computationally very effective when truncated close to the numerically least term. In recent years, this classical method of "summation to the least term" has been justified rigorously in quite some generality for various classes of problems.[7] The analysis of Costin and Kruskal (1999) is in the setting of differential equations, but their method of proof extends to much more general problems. Although the series in our analysis is not a usual power series, the procedure is conceptually similar and therefore expected to yield comparably good results.

For an interesting class of processes for $T_t$, the sequence $\psi_1, \psi_2, \ldots$ takes a convenient form. Let $\xi = \psi_1$ be the *scaling parameter* of the process, and let $\alpha = \psi_1^2/\psi_2$ be the *precision parameter*. We introduce the assumption

**Assumption 3.** *$\psi_n = a_{n-1}\xi^n/\alpha^{n-1}$ where $a_0 = a_1 = 1$ and $a_2, a_3, \ldots$ do not depend on $(\alpha, \xi)$.*

Assumption 3 implies $\psi_n/\psi_1^n = a_{n-1}/\alpha^{n-1}$, so we use transformation (A.1) to get

$$Y_{m,j}\left(\frac{\psi_2}{2\psi_1^2}, \frac{\psi_3}{3\psi_1^3}, \frac{\psi_4}{4\psi_1^4}, \ldots\right) = \alpha^{-m} Y_{m,j}\left(\frac{1}{2}, \frac{a_2}{3}, \frac{a_3}{4}, \ldots\right)$$

Thus, under Assumptions 1 and 3, Lemma 1 implies

$$\frac{n!}{(n+m)!} Y_{n+m,n}(\psi) = \frac{\xi^{n+m}}{\alpha^m} \frac{1}{m!} \sum_{j=0}^{m} (n)_j Y_{m,j}\left(\frac{1}{2}, \frac{a_2}{3}, \frac{a_3}{4}, \ldots\right) \tag{2.12}$$

[7] The earliest use of optimal truncation of divergent series was a proof by Cauchy (1843) that the least term truncation of the Gamma function series is optimal, giving rise to errors of the same order of magnitude as the least term. Stokes (1864) took the method further, less rigorously, but applied it to many problems and used it to discover what we now call the Stokes phenomenon in asymptotics.

Define a new set of constants

$$c_{m,j} = \alpha^m \gamma_{m,j} = \frac{1}{m!} Y_{m,j}\left(\frac{1}{2}, \frac{a_2}{3}, \frac{a_3}{4}, \ldots\right)$$

so that

$$\frac{n!}{(n+m)!} Y_{n+m,n}(\psi) = \frac{\xi^{n+m}}{\alpha^m} \sum_{j=0}^{m} c_{m,j} (n)_j.$$

Under Assumptions 1, 2 and 3, Proposition 1 holds with

$$\tilde{S}_M(t) = \sum_{m=0}^{M} \alpha^{-m} \sum_{j=0}^{m} c_{m,j} t^j D_t^{m+j} S(\xi t) \tag{2.13}$$

$$R_M(t,u) = \exp(t\Psi(u/\xi)) - e^{tu} \sum_{m=0}^{M} \alpha^{-m} \sum_{j=0}^{m} c_{m,j} t^j u^{m+j} \tag{2.14}$$

This solution is especially convenient for two reasons. First, when the precision parameter $\alpha$ is large, the expansion will yield accurate results in few terms. The variance $\mathrm{V}[T_t]$ is inversely proportional to $\alpha$, so $T_t$ converges in probability to $\xi t$ as $\alpha \to \infty$. This implies that $\tilde{S}(t) \approx S(\xi t)$ for large $\alpha$. Since the expansion constructs $\tilde{S}(t)$ as $S(\xi t)$ plus successive correction terms, it is well-structured for the case in which $T_t$ is not too volatile. The same remark applies to the more general case of Proposition 1, but the logic is more transparent when a single parameter controls the scaled higher cumulants. Second, in the special case of Assumption 3, the coefficients $c_{m,j}$ depend only on the chosen family of processes for $T_t$ and not on its parameters $(\alpha, \xi)$. In econometric applications, there can be millions of calls to the function $\tilde{S}(t)$, so the ability to pre-calculate the $c_{m,j}$ can result in significant efficiencies.

The three-parameter tempered stable subordinator is a flexible and widely-used family of subordinators. We can reparameterize the standard form of the Laplace exponent given by Cont and Tankov (2004, §4.2.2) in terms of our precision and scale parameters ($\alpha$ and $\xi$) and a stability parameter $\omega$ with $0 \leq \omega < 1$. We obtain

$$\Psi(u) = \begin{cases} \alpha \frac{1-\omega}{\omega} \left\{1 - (1 - \xi u/(\alpha(1-\omega)))^{\omega}\right\} & \text{if } \omega \in (0,1) \\ -\alpha \log(1 - \xi u/\alpha) & \text{if } \omega = 0 \end{cases} \tag{2.15}$$

It can easily be verified that

**Proposition 2.** *If $T_t$ is a tempered stable subordinator, then Assumption 1 is satisfied with $u_0 = (1-\omega)\alpha/\xi$ and Assumption 3 is satisfied with*

$$a_n = \frac{(1-\omega)^{(n)}}{(1-\omega)^n}$$

We denote by $(z)^{(n)}$ the rising factorial $(z)^{(n)} = z \cdot (z+1) \cdots (z+n-1)$.

Two well-known examples of the tempered stable subordinator are the gamma subordinator ($\omega = 0$) and the inverse Gaussian subordinator ($\omega = 1/2$). For the gamma subordinator, the constants $a_n$ simplify to $a_n = n!$, so

$$c_{m,j} = \frac{1}{m!} Y_{m,j}\left(\frac{1!}{2}, \frac{2!}{3}, \frac{3!}{4}, \ldots\right)$$

We can calculate the $c_{m,j}$ efficiently via recurrence. Rządkowski (2012) shows that

$$c_{m,j} = \frac{1}{m+j}\left(c_{m-1,j-1} + (m+j-1)c_{m-1,j}\right) \tag{2.16}$$

for $m \geq j \geq 1$. The recurrence bottoms out at $c_{0,0} = 1$ and $c_{m,0} = 0$ for $m > 0$.

In the inverse Gaussian case ($\omega = 1/2$), the $a_n$ parameters are

$$a_n = \frac{1}{(1/2)^n}\frac{1}{2}\frac{3}{2}\cdots\frac{2n-1}{2} = \prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{2^n n!}$$

so

$$\frac{a_n}{n+1} = \frac{1}{2^n}\frac{(2n)!}{(n+1)!} = \frac{1}{2^n}(2n)_{n-1}.$$

Thus, for $1 \leq j \leq m$, we have

$$\begin{aligned} c_{m,j} = \frac{1}{m!}Y_{m,j}\left((2)_0/2^1, (4)_1/2^2, (6)_2/2^3, (8)_3/2^4, \ldots\right) &= \frac{1}{2^m m!}Y_{m,j}\left((2)_0, (4)_1, (6)_2, (8)_3, \ldots\right) \\ &= \frac{1}{2^m m!}\binom{m-1}{j-1}(2m)_{m-j} \qquad (2.17) \end{aligned}$$

where the last equality follows from identity (A.2).

# 3 Expansion in exponential functions

The method of this section relaxes the assumption that $T_t$ is a Lévy process, but is more restrictive on $S(t)$. In the simplest case, we require that

**Assumption 4.** *$S(t)$ has a series expansion of the form*

$$S(t) = \exp(at)\sum_{n=0}^{\infty}\beta_n \exp(-n\gamma t)$$

*for constants $a \leq 0$ and $\gamma \geq 0$. The series $\sum_{n=0}^{\infty}|\beta_n|$ is convergent.*

The convergence of $\sum|\beta_n|$ implies uniform convergence for the expansion of $S(t)$. Note here that we are redeploying symbols $a$, $\gamma$ and $\beta$, which were defined differently in Section 2. When this assumption is satisfied, Assumption 2 is satisfied with

$$\mu(u) = \sum_{n=0}^{\infty}\beta_n\mu_{n\gamma-a}(u) \tag{3.1}$$

where $\mu_x$ is the point measure of mass one at $u = x$. Since $\sum_{n=0}^{\infty}|\beta_n|$ is convergent, $\mu$ is a finite measure.

Let $M_t(u)$ denote the moment generating function for $T_t$. We assume

**Assumption 5.** *$M_t(u)$ exists for $u \leq a$.*

Many time-change processes of empirical interest have known moment generating functions that satisfy Assumption 5. When $T_t$ is a Lévy process satisfying Assumption 1, Assumption 5 is immediately satisfied.

Assumption 5 can accommodate non-Lévy specifications as well. Since volatility spikes are often clustered in time, it may be desirable to allow for serial dependence in the rate of time change. Following Carr and Wu (2004) and Mendoza-Arriaga et al. (2010), we let a positive process $\nu(t)$ be the instantaneous activity rate of business time, so that

$$T_t = \int_0^t \nu(s_-)\, ds.$$

If we specify the activity rate as an affine process, the moment generating function for $T_t$ will have the tractable form $M_t(u) = \exp(A^\nu_t(u) + B^\nu_t(u)\nu_0)$ for known functions $A^\nu_t(u)$ and $B^\nu_t(u)$. A widely-used special case is the *basic affine process*, which has stochastic differential equation

$$d\nu_t = \kappa_\nu(\theta_\nu - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}dW^\nu_t + dJ^\nu_t \tag{3.2}$$

where $J^\nu$ is a compound Poisson process, independent of the diffusion $W^\nu_t$. The arrival intensity of jumps is $\zeta_\nu$ and jump sizes are exponential with mean $\eta_\nu$. In Appendix B, we review the solution of functions $A^\nu_t(u)$ and $B^\nu_t(u)$ under this specification. Carr and Wu (2004, Table 2) list alternative specifications of the activity rate with known $M_t(u)$.

Under Assumptions 4 and 5, we have

$$\tilde{S}(t) = \mathrm{E}\left[S(T_t)\right] = \sum_{n=0}^{\infty} \beta_n \mathrm{E}\left[\exp((a-n\gamma)T_t)\right] = \sum_{n=0}^{\infty} \beta_n M_t(a-n\gamma)$$

which leads to the following proposition:

**Proposition 3.** *Under Assumptions 4 and 5,*

$$\tilde{S}(t) = \sum_{n=0}^{\infty} \beta_n M_t(a-n\gamma)$$

*converges uniformly in* $t$.

*Proof.* Since $a - n\gamma \leq 0$ for all $n$, we have $|M_t(a-n\gamma)| \leq 1$ for all $n$ and $t$. Thus, we have

$$\left|\tilde{S}(t) - \sum_{n=0}^{n_1} \beta_n M_t(a-n\gamma)\right| = \left|\sum_{n=n_1+1}^{\infty} \beta_n M_t(a-n\gamma)\right| \leq \sum_{n=n_1+1}^{\infty} |\beta_n| \to 0$$

as $n_1$ goes to $\infty$. □

When $S(t)$ is a Laplace transform of the time-integral of a nonnegative diffusion and when $T_t$ is a Lévy subordinator, Proposition 3 is equivalent to the eigenfunction expansion of Mendoza-Arriaga and Linetsky (2012).[8] However, because our approach is agnostic with respect to the interpretation of $S(t)$, it can be applied in situations when spectral decomposition is unavailable, e.g., when the

[8] The tractability of time-changing an expansion in exponential functions of time was earlier exploited by Mendoza-Arriaga et al. (2010, Theorem 8.3) for the special case of the JDCEV model.

background process is a time-integral of a process containing jumps. All that is needed is that $S(t)$ has a convergent Taylor series expansion as specified in Assumption 4. Moreover, our approach makes clear that $T_t$ need not be a Lévy subordinator.

As we will see in the next section, there are situations in which Assumption 4 does not hold, so neither Proposition 3 nor the corresponding eigenfunction expansion of Mendoza-Arriaga and Linetsky (2012) pertains. However, our method can be adapted so long as $S(t)$ has a suitable expansion in powers of an affine function of $\exp(-\gamma t)$. We will make use of this alternative in particular:

**Assumption 4'.** *$S(t)$ has a series expansion of the form*

$$S(t) = \exp(at) \sum_{n=0}^{\infty} \beta_n (1 - 2\exp(-\gamma t))^n$$

*for constants $a \leq 0$ and $\gamma \geq 0$. The series $\sum_{n=0}^{\infty} |\beta_n|$ is convergent.*

Under Assumptions 4' and 5, we have

$$\begin{aligned} \tilde{S}(t) &= \mathrm{E}\left[S(T_t)\right] = \sum_{n=0}^{\infty} \beta_n \mathrm{E}\left[\exp(aT_t)(1 - 2\exp(-\gamma T_t))^n\right] \\ &= \sum_{n=0}^{\infty} \beta_n \sum_{m=0}^{n} \binom{n}{m} (-2)^m \mathrm{E}\left[\exp((a - m\gamma)T_t)\right] = \sum_{n=0}^{\infty} \beta_n \sum_{m=0}^{n} \binom{n}{m} (-2)^m M_t(a - m\gamma) \end{aligned} \tag{3.3}$$

This leads to

**Proposition 3'.** *Under Assumptions 4' and 5,*

$$\tilde{S}(t) = \sum_{n=0}^{\infty} \beta_n \sum_{m=0}^{n} \binom{n}{m} (-2)^m M_t(a - m\gamma)$$

*converges uniformly in $t$.*

*Proof.* The proof is similar to that of Proposition 3. Observe that

$$\left| \sum_{m=0}^{n} \binom{n}{m} (-2)^m M_t(a - m\gamma) \right| = \left|\mathrm{E}\left[\exp(aT_t)(1 - 2\exp(-\gamma T_t))^n\right]\right| \leq \left|\mathrm{E}\left[\exp(aT_t)\right]\right| \leq 1$$

Thus,

$$\left| \tilde{S}(t) - \sum_{n=1}^{n_1} \beta_n \sum_{m=0}^{n} \binom{n}{m} (-2)^m M_t(a - m\gamma) \right| = \left| \sum_{n=n_1+1}^{\infty} \beta_n \sum_{m=0}^{n} \binom{n}{m} (-2)^m M_t(a - m\gamma) \right| \leq \sum_{n=n_1+1}^{\infty} |\beta_n| \to 0$$

as $n_1$ goes to $\infty$. $\square$

Although Assumption 4' is not a sufficient condition for Assumption 2, it is sufficient for purposes of approximating $\tilde{S}(t)$ by the expansion in derivatives of Section 2. Let $S_n(t)$ denote the approximation to $S(t)$ given by the finite expansion

$$S_n(t) = \exp(at) \sum_{m=0}^{n} \beta_m (1 - 2\exp(-\gamma t))^m$$

for $n \geq 0$. This function by construction satisfies Assumption 4', and furthermore satisfies Assumption 2 with

$$\mu(u) = \sum_{m=0}^{n} \beta_m \sum_{j=0}^{m} \binom{m}{j} (-2)^j \mu_{j\gamma - a}(u)$$

where $\mu_x$ is the point measure of mass one at $u = x$. Therefore, expansion in derivatives can be applied to $S_n(t)$. Let $\tilde{S}_{n,M}(t)$ be the approximation to $\tilde{S}_n(t)$ up to term $M$ in the expansion in derivatives, and let

$$\delta_{n,M}(t) = |\tilde{S}_n(t) - \tilde{S}_{n,M}(t)| = \int_0^\infty R_{n,M}(t, -\psi_1 u) d\mu(u)$$

be the corresponding remainder term in Proposition 1. By Proposition 3', for any $\epsilon > 0$ there exists $n'$ such that for all $n > n'$, $|\tilde{S}(t) - \tilde{S}_n(t)| < \epsilon$. Thus, we can bound the residual in expansion in derivatives by

$$|\tilde{S}(t) - \tilde{S}_{n,M}(t)| < \delta_{n,M}(t) + \epsilon$$

for $n > n'$.

# 4 Application to credit risk modeling

We now apply the two expansion methods to the widely-used default intensity class of models for pricing credit-sensitive corporate bonds and credit default swaps. In these models, a firm's default occurs at the first event of a non-explosive counting process. Under the business-time clock, the intensity of the counting process is $\lambda_t$. The intuition driving the model is that $\lambda_t dt$ is the probability of default before business time $t + dt$, conditional on survival to business time $t$. We define $X_t$ as the time-integral of $\lambda_t$, which is also known as the compensator.

Let $\tilde{\tau}$ denote the calendar default time, and let $\tau = T_{\tilde{\tau}}$ be the corresponding time under the business clock. Current time is normalized to zero under both clocks. The probability of survival to future business time $t$ is

$$S(t; \ell) = \Pr(\tau > t | \lambda_0 = \ell) = \mathrm{E}\left[\exp(-X_t) | \lambda_0 = \ell\right] = \mathrm{E}\left[\exp\left(-\int_0^t \lambda_s ds\right) \middle| \lambda_0 = \ell\right]$$

Maintaining our assumption that $X_t$ and $T_t$ are independent, the calendar-time survival probability function is

$$\tilde{S}(t; \ell) \equiv \Pr(\tilde{\tau} > t | \lambda_0 = \ell) = \Pr(\tau > T_t | \lambda_0 = \ell) = \mathrm{E}\left[\Pr(\tau > T_t | T_t, \lambda_0 = \ell) | \lambda_0 = \ell\right] = \mathrm{E}\left[S(T_t; \ell)\right] \tag{4.1}$$

It is easily seen that time-changing the default time is equivalent to time-changing the compensator, i.e., that the survival probability in calendar time is equal to $\mathrm{E}\left[\exp(-\tilde{X}_t) | \lambda_0 = \ell\right]$, where $\tilde{X}(t) =$

$X(T_t)$. In application, we are often interested in the calendar time default intensity. When $T_t$ is the time-integral of an absolutely continuous activity rate process $\nu(t)$, we can apply a change of variable as in Mendoza-Arriaga et al. (2010, §4.2):

$$\tilde{X}_t = \int_0^{T(t)} \lambda(s)\, ds = \int_0^t \lambda(T_s)\nu(s)\, ds$$

from which it is clear that the default intensity in calendar time is simply $\tilde{\lambda}(t) = \nu(t)\lambda(T_t)$. Observe that if $\nu_t$ and $\lambda_t$ are both bounded nonnegative, then $\tilde{\lambda}(t)$ is bounded nonnegative as well.

When $T_t$ is a Lévy subordinator, the $T_t$ process is not differentiable, so the change of variable cannot be applied. We have so far fixed the current time to zero to minimize notation. To accommodate analysis of time-series behavior, let us define

$$\tilde{S}_t(s;\ell) = \Pr(\tilde{\tau} > t+s | T_t, \lambda(T_t) = \ell, \tau > T_t).$$

The default intensity under calendar time can then be obtained as $\tilde{\lambda}(t) = -\tilde{S}'_t(0;\lambda(T_t))$.[9] Assume that $\lambda_t$ is bounded nonnegative. Since $T_t$ is nondecreasing,

$$\tilde{X}(t+\delta) - \tilde{X}(t) = \int_{T(t)}^{T(t+\delta)} \lambda_s ds$$

must be nonnegative for any $\delta \geq 0$, so

$$\begin{aligned}\tilde{S}_t(\delta;\lambda(T_t)) &= \Pr(\tilde{\tau} > t+\delta | T_t, \lambda(T_t), \tau > T_t) \\ &= \mathrm{E}\left[\exp\left(-(\tilde{X}(t+\delta) - \tilde{X}(t))\right) | T_t, \lambda(T_t)\right] \leq 1 = \tilde{S}_t(0;\lambda(T_t)).\end{aligned}$$

Since this holds for any nonnegative $\delta$, we must have $\tilde{S}'_t(0;\lambda(T_t)) \leq 0$ which implies $\tilde{\lambda}(t) \geq 0$. Thus, the bound on $\lambda_t$ is preserved under time-change.

We acknowledge that the assumption of independence between $X_t$ and $T_t$ may be strong. In the empirical literature on stochastic volatility in stock returns, there is strong evidence for dependence between the volatility factor and stock returns (e.g., Andersen et al., 2002; Jones, 2003; Jacquier et al., 2004). In the credit risk literature, however, the evidence is less compelling. Across the firms in their sample, Jacobs and Li (2008) find a median correlation of around 1% between the default intensity diffusion and the volatility factor. Nonetheless, for a significant share of the firms, the correlation appears to be material. We hope to relax the independence assumption in future work.

We re-introduce the *basic affine process*, which we earlier defined in Section 3. Under the business clock, $\lambda_t$ follows the stochastic differential equation

$$d\lambda_t = \kappa_x(\theta_x - \lambda_t)dt + \sigma_x\sqrt{\lambda_t}dW_t^x + dJ_t^x \tag{4.2}$$

where $J^x$ is a compound Poisson process, independent of the diffusion $W_t^x$. The arrival intensity of jumps is $\zeta_x$ and jump sizes are exponential with mean $\eta_x$. We assume $\kappa_x\theta_x > 0$ to ensure that the default intensity is nonnegative. The generalized Laplace transform for the basic affine process is

$$\mathrm{E}\left[\exp\left(wX_t + u\lambda_t\right)\right] = \exp\left(\mathfrak{A}_t^x(u,w) + \mathfrak{B}_t^x(u,w)\lambda_0\right) \tag{4.3}$$

[9] The negative derivative $-S'_t(s;\lambda_t)$ is the density of the distribution of business-clock default time $\tau$ at business time $t+s$ given information at $t$, so $-S'_t(0;\lambda_t)$ is the instantaneous hazard rate in business time, i.e., $\lambda_t = -S'_t(0;\lambda_t)$. By the same logic, the instantaneous hazard rate at calendar time $t$ is $-\tilde{S}'_t(0;\lambda(T_t))$.

for functions $\{\mathfrak{A}^x_t(u,w), \mathfrak{B}^x_t(u,w)\}$ with explicit solution given in Appendix B. Defining the functions $A^x(t) \equiv \mathfrak{A}^x_t(0,-1)$ and $B^x(t) \equiv \mathfrak{B}^x_t(0,-1)$, we arrive at the survival probability function

$$S(t;\lambda_0) = \exp(A^x(t) + B^x(t)\lambda_0).$$

We digress briefly to consider whether the method of Carr and Wu (2004) can be applied in this setting. The compensator $X(t)$ is not Lévy, but can be expressed as a time-changed time-integral of a constant intensity, where the time change in this case is the time-integral of the basic affine process in (4.2). Thus, we can write $\tilde{X}(t)$ as $X^*(T^*(t))$ where $X^*(t) = t$ is trivially a Lévy process and $T^*(t)$ is a *compound* time change. However, this approach leads nowhere, because $T^*(t)$ is equivalent to $\tilde{X}(t)$. Put another way, we are still left with the problem of solving the Laplace transform for $\tilde{X}(t)$.

To apply our expansion in exponential functions, we show that Assumption 4 is satisfied when $\kappa_x > 0$ and Assumption 4' is always satisfied. In Appendix C, we prove

**Proposition 5.** *Assume $\lambda_t$ follows a basic affine process. Then $S(t)$ has the series expansion*

$$S(t;\ell) = \exp(at)\sum_{n=0}^{\infty}\beta_n(\ell)\left(1 - 2\exp(-\gamma t)\right)^n \tag{i}$$

*For the case $\kappa_x > 0$, $S(t)$ has the series expansion*

$$S(t;\ell) = \exp(at)\sum_{n=0}^{\infty}\beta_n(\ell)\,\exp(-n\gamma t) \tag{ii}$$

*In each case, $a < 0$ and $\gamma > 0$, and the series $\sum_{n=0}^{\infty}|\beta_n|$ is convergent.*

The appendix provides closed-form solutions for $a$, $\gamma$, and the sequence $\beta_n$.

When $\kappa_x > 0$ and in the absence of jumps ($\zeta_x = 0$), the expansion in (ii) is equivalent to the eigenfunction expansion in Davydov and Linetsky (2003, §4.3).[10] Therefore, the associated solution to $\tilde{S}(t)$ under a Lévy subordinator is identical to the solution in Mendoza-Arriaga and Linetsky (2012). Our result is more general in that it permits non-stationarity (i.e., $\kappa_x \leq 0$) in expansion (i) and accommodates the presence of jumps in the intensity process in expansions (i) and (ii). Furthermore, it is clear in our analysis that our expansions can be applied to non-Lévy specifications of time-change as well, such as the mean-reverting activity rate model in (3.2).

Subject to the technical caveat at the end of Section 3, Assumptions 1 and 4' are together sufficient for application of expansion in derivatives without restrictions on $\kappa_x$. To implement, we need an efficient algorithm to obtain derivatives of $S(t)$. Let $\Omega_n(t)$ be the family of functions defined by

$$\Omega_n(t) = \mathrm{E}\left[\exp(-X_t)\lambda_t^n\right] = \frac{\partial^n}{\partial u^n}\exp\left(\mathfrak{A}^x_t(u,-1) + \mathfrak{B}^x_t(u,-1)\lambda_0\right)\Big|_{u=0}$$

These functions have closed-form expressions, which we provide in Appendix D. Using Itô's Lemma, we prove in Appendix E:

[10]Specifically, our $\beta_n$ is equal to $c_n\varphi_n(\lambda_0)$ in the notation of Davydov and Linetsky (2003, Proposition 9).

**Proposition 6.** *For all $n \geq 0$,*

$$\Omega_n'(t) = \left(n\kappa_x\theta_x + \frac{1}{2}n(n-1)\sigma_x^2\right)\Omega_{n-1}(t) - n\kappa_x\Omega_n(t) - \Omega_{n+1}(t) + \zeta_x\Xi_n(t)$$

*where* $\Omega_{-1}(t) \equiv 0$, $\Xi_0(t) = 0$ *and*

$$\Xi_{n+1}(t) = (n+1)\eta(\Xi_n(t) + \Omega_n(t)).$$

Proposition 6 points to a general strategy for iterative computation of the derivatives of $S(t)$. We began with $S(t) = \Omega_0(t)$. We then apply Proposition 6 to obtain

$$S'(t) = \Omega_0'(t) = -\Omega_1(t).$$

We differentiate again to get

$$S''(t) = DS'(t) = -(\kappa_x\theta_x\Omega_0(t) - \kappa_x\Omega_1(t) - \Omega_2(t) + \zeta_x\Xi_1(t)) = \Omega_2(t) + \kappa_x\Omega_1(t) - (\zeta_x\eta_x + \kappa_x\theta_x)\Omega_0(t)$$

and so on. In general, $D^n S(t)$ can be expressed as a weighted sum of $\Omega_0(t), \Omega_1(t), \ldots, \Omega_n(t)$. While the higher derivatives of $S(t)$ would be tedious to write out, the recurrence algorithm is easily implemented. The incremental cost of computing $D^n S(t)$ is dominated by the cost of computing $\Omega_n(t)$, assuming that the lower order $\Omega_j(t)$ have been retained from computation of lower order derivatives of $S(t)$.

# 5 Numerical examples

We explore the effect of time-change on the behavior of the model, as well as the efficacy of our two series solutions. To fix a benchmark model, we assume that $\lambda_t$ follows a CIR process in business time with parameters $\kappa_x = 0.2$, $\theta_x = 0.02$ and $\sigma_x = 0.1$. This calibration is consistent in a stylized fashion with median parameter values under the physical measure as reported by Duffee (1999). Our benchmark specification adopts inverse Gaussian time change. In all the examples discussed below, the behavior under gamma time-change is quite similar.

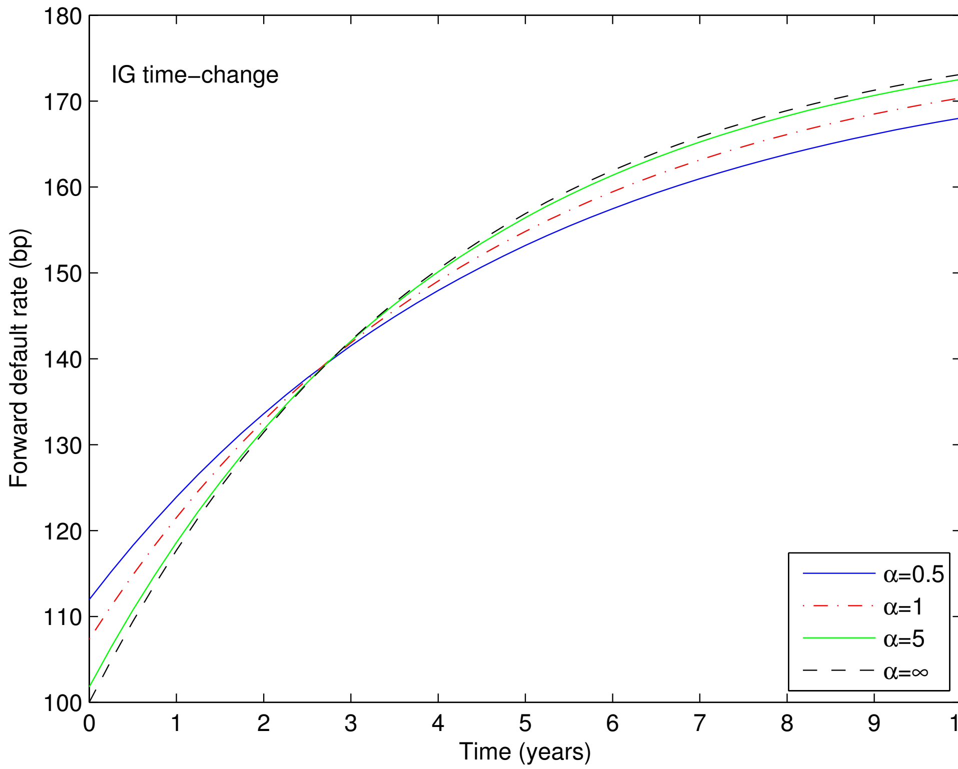
The survival probability function is falling monotonically and almost linearly, so is not scaled well for our exercises. Instead, following the presentation in Duffie and Singleton (2003, §3), we work with the forward default rate, $\tilde{h}(t) \equiv -\tilde{S}'(t)/\tilde{S}(t)$.[11] In our benchmark calibration, we set starting condition $\lambda_0 = 0.01$ well below its long-run mean $\theta_x$ in order to give reasonable variation across the term structure in the forward default rate. Both $X(t)$ and $T(t)$ are scale-invariant processes, so we fix the scale parameter $\xi = 1$ with no loss of generality.

Figure 1 shows how the term structure of the forward default rate changes with the precision parameter $\alpha$. We see that lower values of $\alpha$ flatten the term-structure, which accords with the intuition that the time-changed term-structure is a mixture across time of the business-time term-structure. Above $\alpha = 5$, it becomes difficult to distinguish $\tilde{h}(t)$ from the term structure $h(t)$ for the CIR model without time-change.

Finding that time-change has negligible effect on the term structure $\tilde{h}(t)$ for moderate values of $\alpha$ does *not* imply that time-change has a small effect on the time-series behavior of the default intensity. For a given time-increment $\delta$, we obtain by simulation the kurtosis of calendar time

[11] In a deterministic intensity model, the forward default rate would equal the intensity.

Figure 1: Effect of time-change on forward default rate

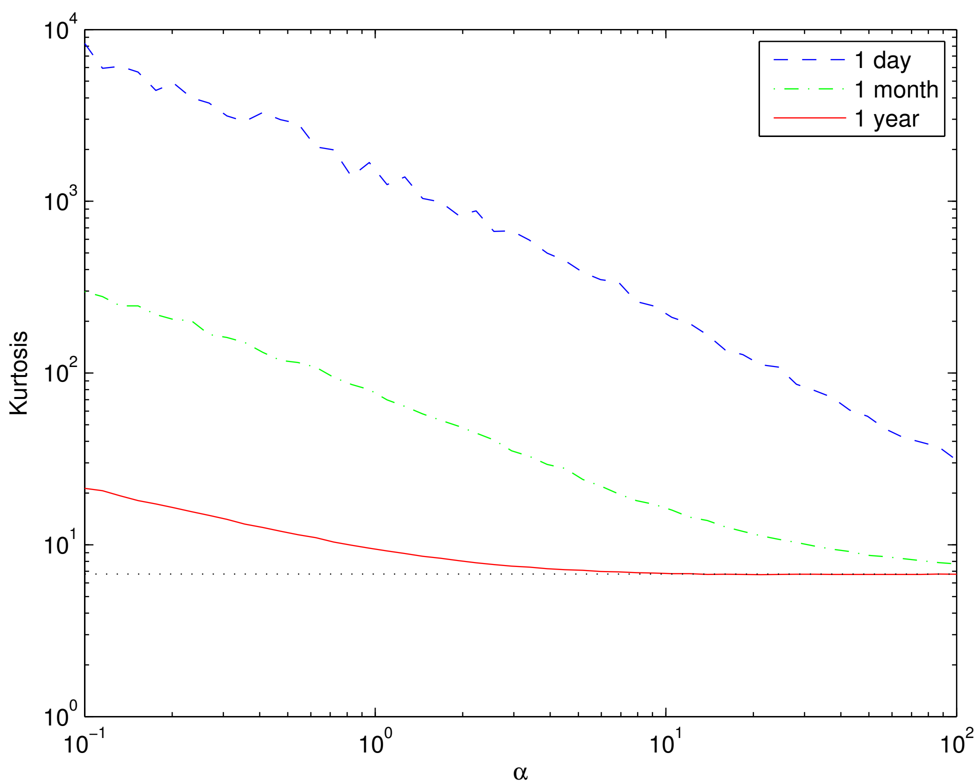



CIR model under business time with parameters $\kappa_x = .2, \theta_x = .02, \sigma_x = .1$, starting condition $\lambda_0 = \theta_x/2 = .01$, and inverse Gaussian time-change with $\xi = 1$. When $\alpha = \infty$, the model is equivalent to the CIR model without time-change. Term-structures calculated with the series expansion in exponential functions of Proposition 3 with 12 terms.

increments of the default intensity (that is, $\tilde{\lambda}(t+\delta) - \tilde{\lambda}(t)$) under the stationary law. For the limiting CIR model without time-change, moments for the increments $\lambda(t+\delta) - \lambda(t)$ have simple closed-form solutions provided by Gordy (2012). The kurtosis is equal to $3(1 + \sigma_x^2/(2\kappa_x\theta_x))$, which is invariant to the time-increment $\delta$.

In Figure 2, we plot kurtosis as a function of $\alpha$ on a log-log scale. Using the same baseline model specification as before, we plot separate curves for a one day horizon ($\delta = 1/250$, assuming 250 trading days per year), a one month horizon ($\delta = 1/12$), and an annual horizon ($\delta = 1$). As we expect, kurtosis at all horizons tends to its asymptotic CIR limit (dotted line) as $\alpha \to \infty$. For fixed $\alpha$, kurtosis also tends to its CIR limit as $\delta \to \infty$. This is because an unbiased trend stationary time-change has no effect on the distribution of a stationary process far into the future. For intermediate values of $\alpha$ (say, between 1 and 10), we see that time-change has a modest impact on kurtosis beyond one year, but a material impact at a one month horizon, and a very large impact at a daily horizon.

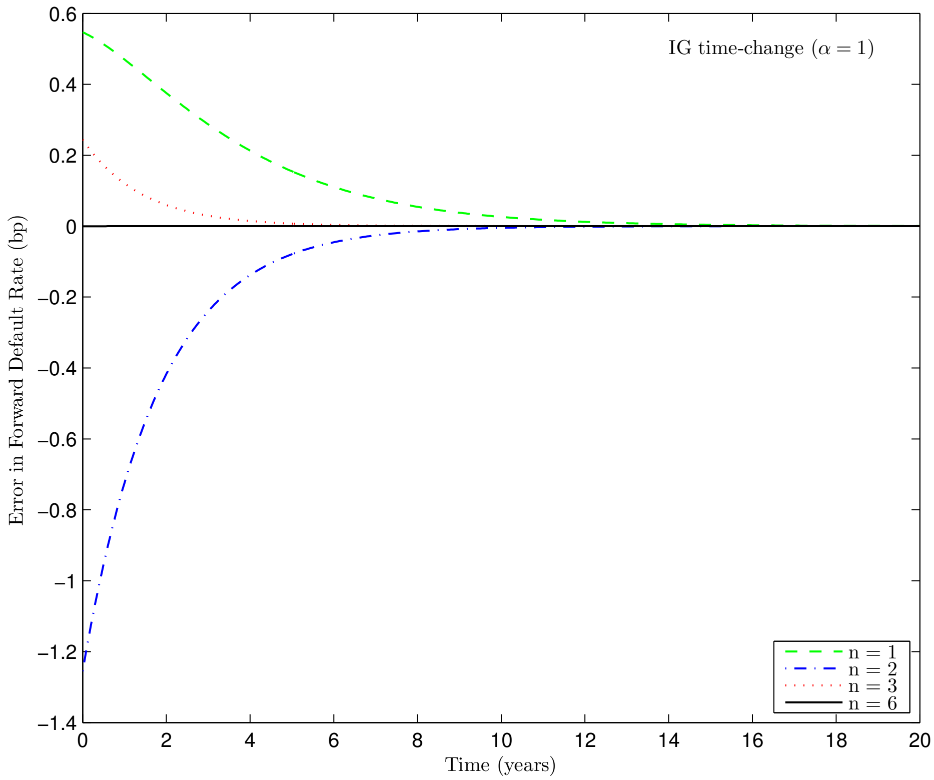
Figure 2: Kurtosis of increments under IG time-change




Stationary CIR model under business time with parameters $\kappa_x = .2, \theta_x = .02, \sigma_x = .1$, and inverse Gaussian time-change with $\xi = 1$. Dotted line plots the limiting CIR kurtosis. Both axes on log-scale. Moments of the calendar time increments are obtained by simulation with 5 million trials.

Next, we explore the convergence of the series expansion in exponentials. Let $\tilde{h}_n(t)$ denote the estimated forward default rate using the first $n$ terms of the series for $\tilde{S}(t)$ and the corresponding expansion for $\tilde{S}'(t)$. Figure 3 shows that the convergence of $\tilde{h}_n(t)$ to $\tilde{h}(t)$ is quite rapid. We proxy the series solution with $n = 12$ terms as the true forward default rate, and plot the error $\tilde{h}_n(t) - \tilde{h}(t)$ in basis points (bp). The error is decreasing in $t$, as the series in Proposition 5 is an asymptotic expansion. With only $n = 3$ terms, the error is 0.25bp at $t = 0$, which corresponds to a relative error under 0.25%. With $n = 6$ terms, relative error is negligible (under 0.0005%) at $t = 0$.

Figure 3: Convergence of expansion in exponentials

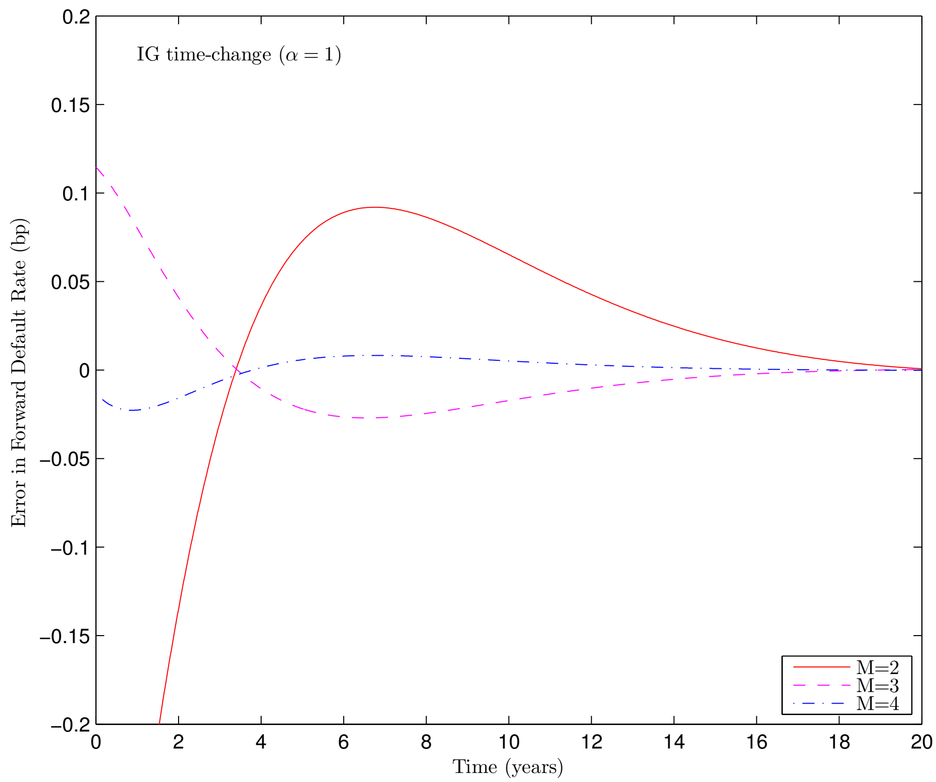



CIR model under business time with parameters $\kappa_x = .2, \theta_x = .02, \sigma_x = .1$, starting condition $\lambda_0 = \theta_x/2 = .01$, and inverse Gaussian time-change with $\alpha = 1$ and $\xi = 1$.

We turn now to the convergence of the expansion in derivatives. In Figure 4, we plot the error against the benchmark for $M = 2, 3, 4$ terms in expansion (2.10). The benchmark curve is calculated, as before, using the series expansion in exponential functions with 12 terms. The magnitude of the relative error is generally largest at small values of $t$. For $M = 2$, the forward default rate is off by nearly 0.5bp at $t = 0$. Observed bid-ask spreads in the credit default swap market are an order of magnitude larger, so this degree of accuracy is already likely to be sufficient for empirical application. For $M = 4$, the gap is never over 0.025bp at any $t$.

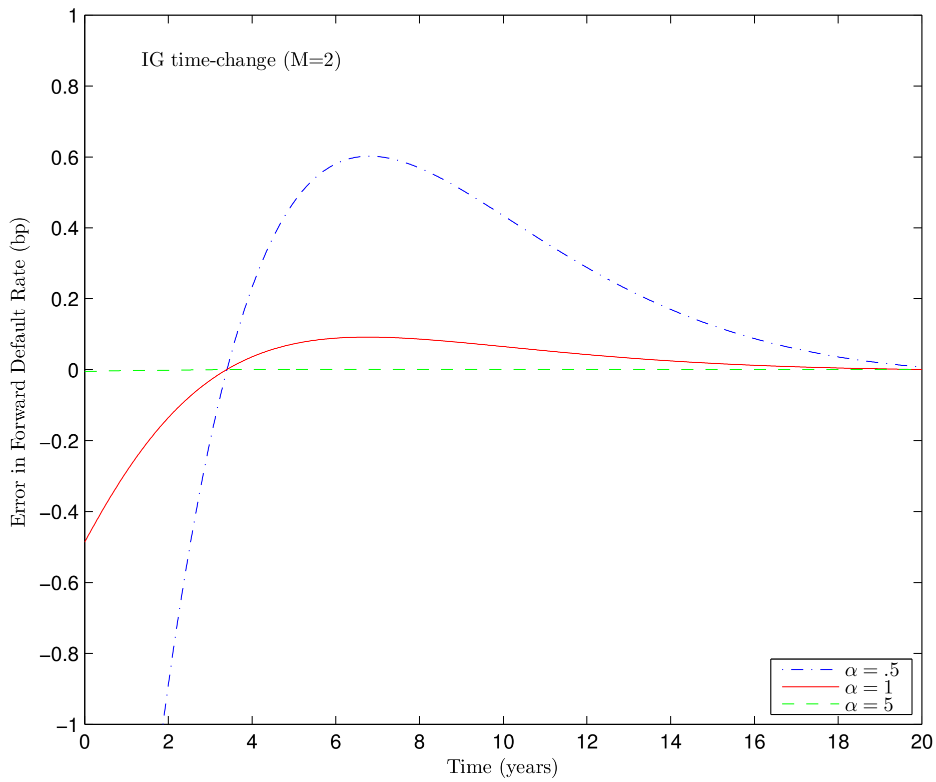
In Figure 5, we hold fixed $M = 2$ and explore how error varies with $\alpha$. As the expansion is in powers of $1/\alpha$, it is not surprising that error vanishes as $\alpha$ grows, and is negligible (under 0.005bp in absolute magnitude) at $\alpha = 5$. Experiments with other model parameters suggest that absolute relative error increases with $\sigma_x$ and $\theta_x$ and decreases with $\kappa_x$.

Figure 4: Expansion in derivatives: Varying $M$




CIR model under business time with parameters $\kappa_x = .2, \theta_x = .02, \sigma_x = .1$, starting condition $\lambda_0 = \theta_x/2 = .01$, and inverse Gaussian time-change with $\alpha = 1, \xi = 1$.

Figure 5: Expansion in derivatives: Convergence in $1/\alpha$




CIR model under business time with parameters $\kappa_x = .2, \theta_x = .02, \sigma_x = .1$, starting condition $\lambda_0 = \theta_x/2 = .01$, and inverse Gaussian time-change with $\xi = 1$. Number of terms in expansion is fixed to $M = 2$.

# 6 Options on credit default swaps

In the previous section, we observe that stochastic time-change has a negligible effect on the term structure of default probability for moderate values of $\alpha$, which implies that introducing time-change should have little impact on the term-structure of credit spreads on corporate bonds and credit default swaps (CDS). Nonetheless, time-change has a large effect on the forecast density of the default intensity at short horizon. Consequently, introducing time-change should have material impact on the pricing of short-dated options on credit instruments.[12] In this section, we develop a pricing methodology for European options on single-name CDS in the time-changed CIR model.

At present, the CDS option market is dominated by index options. The market for single-name CDS options is less liquid, but trades do occur. A *payer option* gives the right to buy protection of maturity $y$ at a fixed spread $K$ (the "strike" or "pay premium") at a fixed expiry date $\delta$. A payer option is in-the-money if the par CDS spread at date $\delta$ is greater than $K$. A *receiver option* gives the right to sell protection. We focus here on the pricing of payer options, but all results extend in an obvious fashion to the pricing of receiver options. An important difference between the index and single-name option markets is that single-name options are sold with knock-out, i.e., the option expires worthless if the reference entity defaults before $\delta$. As we will see, this complicates the analysis. Willemann and Bicer (2010) provide an overview of CDS option trading and its conventions.

To simplify the analysis and to keep the focus on default risk, we assume a constant risk free interest rate $r$ and a constant recovery rate $R$. In the next section, we generalize our methods to accommodate a multi-factor model governing both the short rate and default intensity. The assumption of constant recovery can be relaxed by adopting the stochastic recovery model of Chen and Joslin (2012) in business time. We assume that $\lambda_t$ follows a mean-reverting ($\kappa_x > 0$) CIR process in business time, and that the clock $T_t$ is a Lévy process satisfying Assumption 1 with Laplace exponent $\Psi(u)$. All probabilities and expectations are under the risk-neutral measure.

In the event of default at date $\tilde{\tau}$, the receiver of CDS protection receives a single payment of $(1-R)$ at $\tilde{\tau}$. Therefore, the value at date $s$ of the protection leg of a CDS of maturity $y$ is

$$(1-R)\int_0^y e^{-rt} q(t;\lambda_{T(s)})dt$$

where $\tilde{q}(t;\ell) = -\tilde{S}'(t;\ell)$ is the density of the remaining time to default (relative to date $s$) conditional on $\lambda_{T(s)} = \ell$. From Proposition 3, we have

$$\tilde{S}(t;\ell) = \sum_{n=0}^{\infty} \beta_n(\ell)\exp(t\Psi(a-n\gamma))$$

for $a < 0$ and $\gamma > 0$. We differentiate, insert into the expression for the protection leg, apply Fubini's theorem, and integrate term-by-term to get

$$(1-R)\sum_{n=0}^{\infty} \beta_n(\lambda_{T(s)})\frac{\Psi(a-n\gamma)}{\Psi(a-n\gamma)-r}(1-\exp((\Psi(a-n\gamma)-r)y)) \tag{6.1}$$

[12] We abstract here from the distinction between the risk-neutral measure that governs pricing and the physical measure that governs the empirical time-series of returns. Our statement continues to hold if one adopts the change of measure for the CIR process that is most commonly seen in the literature (called "drift change in the intensity" by Jarrow et al., 2005).

To price the premium leg, we make the simplifying assumption that the spread $\varsigma$ is paid continuously until default or maturity. The value at date $s$ of the premium leg of a CDS of maturity $y$ is then

$$\varsigma \int_0^y e^{-rt} \tilde{S}(t; \lambda_{T(s)}) dt$$

We again substitute the expansion for $\tilde{S}(t)$ and integrate to get

$$\varsigma \sum_{n=0}^{\infty} \beta_n(\lambda_{T(s)}) \frac{1}{r - \Psi(a - n\gamma)} (1 - \exp((\Psi(a - n\gamma) - r)y)) \tag{6.2}$$

The par spread $\varsigma^{par}$ equates the protection leg value (6.1) to the premium leg value (6.2).

To simplify exposition, we assume that CDS are traded on a running spread basis.[13] Let $p(\ell, \varsigma)$ be the net value of the CDS for the buyer of protection at time $s$ given $\lambda_{T(s)} = \ell$ and the spread $\varsigma$, i.e., $p$ is the difference in value between the protection leg and the premium leg. Note that $p$ does not depend directly on time $s$. This simplifies to

$$p(\ell, \varsigma) = \sum_{n=0}^{\infty} \beta_n(\ell) \frac{(1 - R)\Psi(a - n\gamma) + \varsigma}{\Psi(a - n\gamma) - r} (1 - \exp((\Psi(a - n\gamma) - r)y))$$

Given the strike spread $K$ and default intensity $\lambda_{T(\delta)}$, the payoff to the payer option at expiry is

$$1_{\{\tilde{\tau} > \delta\}} \max\{0, p(\lambda_{T(\delta)}, K)\} \tag{6.3}$$

The value at time 0 of the payer option is the expectation of expression (6.3) over the joint distribution of $(\lambda_{T(\delta)}, 1_{\{\tilde{\tau} > \delta\}})$ under the risk-neutral measure:

$$\begin{aligned} G(K, \delta; \lambda_0) &= \mathrm{E}\left[1_{\{\tilde{\tau} > \delta\}} \max\{0, p(\lambda_{T(\delta)}, K)\} | \lambda_0\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[1_{\{\tau > T(\delta)\}} \max\{0, p(\lambda_{T(\delta)}, K)\} | T(\delta), \lambda_0\right] | \lambda_0\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\mathrm{E}\left[1_{\{\tau > T(\delta)\}} | T(\delta), \lambda_{T(\delta)}, \lambda_0\right] \cdot \max\{0, p(\lambda_{T(\delta)}, K)\} | T(\delta), \lambda_0\right] | \lambda_0\right] \end{aligned}$$

Let $\mathcal{L}_t(u; \lambda_0, \lambda_t)$ be the Laplace transform of $X_t$ conditional on $(\lambda_0, \lambda_t)$. This is given by Broadie and Kaya (2006, Eq. 40) for the CIR process in business time. Since

$$\mathrm{E}\left[1_{\{\tau > t\}} | \lambda_0, \lambda_t\right] = \mathcal{L}_t(1; \lambda_0, \lambda_t),$$

we have

$$G(K, \delta; \lambda_0) = \mathrm{E}\left[\mathrm{E}\left[\mathcal{L}_{T(\delta)}(1; \lambda_0, \lambda_{T(\delta)}) \cdot \max\{0, p(\lambda_{T(\delta)}, K)\} | T(\delta), \lambda_0\right] | \lambda_0\right] \tag{6.4}$$

This expectation is most easily obtained by Monte Carlo simulation. In each trial $i = 1, \ldots, I$, we draw a single value of the business clock expiry date $\Delta_i$ from the distribution of $T(\delta)$. Next, we draw $\Lambda_i$ from the noncentral chi-squared transition distribution for $\lambda_{\Delta(i)}$ given $\Delta_i$ and $\lambda_0$. The transition law for the CIR process is given by Broadie and Kaya (2006, Eq. 8). The option value is estimated by

$$\hat{G}(K, \delta; \lambda_0) = \frac{1}{I} \sum_{i=1}^{I} \mathcal{L}_{\Delta(i)}(1; \lambda_0, \Lambda_i) \cdot \max\{0, p(\Lambda_i, K)\} \tag{6.5}$$

[13] That is, the quoted spread specifies the coupon paid by buyer of protection to seller. Since the so-called Big Bang of April 2009, CDS have been traded at standard coupons of 100 and 500 basis points with compensating upfront payments between buyer and seller. See Leeming et al. (2010) on the recent evolution of the CDS market.

Observe that we can efficiently calculate option values across a range of strike spreads with the same sample of $\{\Delta_i, \Lambda_i\}$.

Figure 6 depicts the effect of $\alpha$ on the value of a one month payer option on a five year CDS. Model parameters are taken from the baseline values of Section 5. The riskfree rate is fixed at 3% and the recovery rate at 40%. Depending on the choice of $\alpha$, the par spread is in the range of 115–120bp (marked with circles). For deep out-of-the-money options, i.e., for $K \gg \varsigma^{par}$, we see that option value is decreasing in $\alpha$. Stochastic time-change opens the possibility that the short horizon to option expiry will be greatly expanded in business time, and so increases the likelihood of extreme changes in the intensity. The effect is important even at large values of $\alpha$ for which the term-structure of forward default rate would be visually indistinguishable from the CIR case in Figure 1. For example, at a strike spread of 200bp, the value of the option is nearly 700 times greater for the time-changed model with $\alpha = 10$ than for the CIR model without time-change.

Perhaps counterintuitively, the value of the option is increasing in $\alpha$ for near-the-money options. Because the transition variance of $\lambda_t$ is concave in $t$, introducing stochastic time-change actually reduces the variance of the default intensity at option expiry even as it increases the higher moments. Relative to out-of-the-money options, near-the-money options are more sensitive to the variance and less sensitive to higher moments.

The effect of time to expiry on option value is depicted in Figure 7. The solid lines are for the model with stochastic time-change ($\alpha = 1$), and the dashed lines are for the CIR model without time-change. Relative to the case of the short-dated (one month) option, stochastic time-change has a small effect on the value of the long-dated (one year) option. This is consistent with our observation in Figure 2 that the kurtosis of $\tilde{\lambda}(t+\delta) - \tilde{\lambda}(t)$ converges to that of $\lambda(t+\delta) - \lambda(t)$ as $\delta$ grows large. Because the Lévy subordinator lacks persistence, stochastic time-change simply washes out at long horizon.

## 7 Multi-factor affine models

We have so far taken the business-time default intensity to be a single-factor basic affine process. In this section, we show that our methods of Sections 2 and 3 can be applied to a much wider class of multi-factor affine jump-diffusion models. For the sake of brevity, we limit our analysis here to stationary models.

Let $\mathbf{Z}_t$ be a $d$-dimensional affine jump-diffusion, and let the default intensity at business time $t$ be given by an affine function $\lambda(\mathbf{Z}_t)$. We now obtain a convergent series expansion of

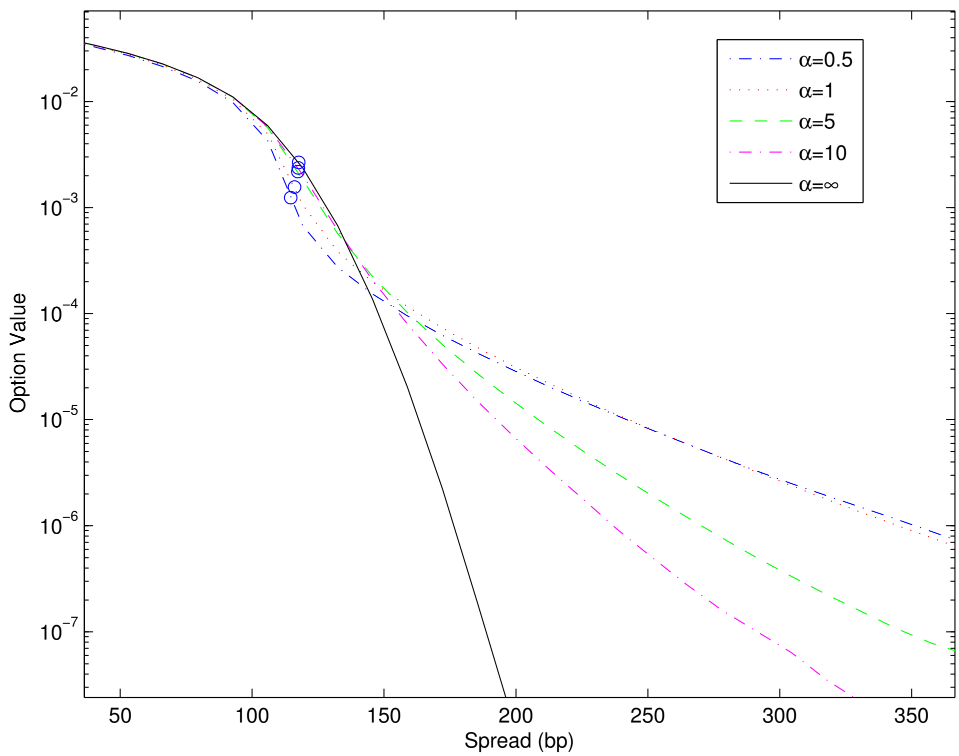
$$S(t; \mathbf{z}) = \mathrm{E}\left[\exp\left(-\int_0^t \lambda(\mathbf{Z}_s) ds\right) \middle| \mathbf{Z}_0 = \mathbf{z}\right]$$

As in Duffie et al. (2000, §2), we assume that the jump component of $\mathbf{Z}_t$ is a Poisson process with time-varying intensity $\zeta(\mathbf{Z}_t)$ that is affine in $\mathbf{Z}_t$, and that jump sizes are independent of $\mathbf{Z}_t$.

By Proposition 1 of Duffie et al. (2000, §2), $S(t; \mathbf{z})$ has exponential-affine solution $S(t; \mathbf{z}) = \exp(A(t) + \mathbf{B}(t) \cdot \mathbf{z})$, where $\cdot$ denotes the inner product. Functions $A(t)$ and $\mathbf{B}(t)$ satisfy complex-valued ODEs which we represent simply as

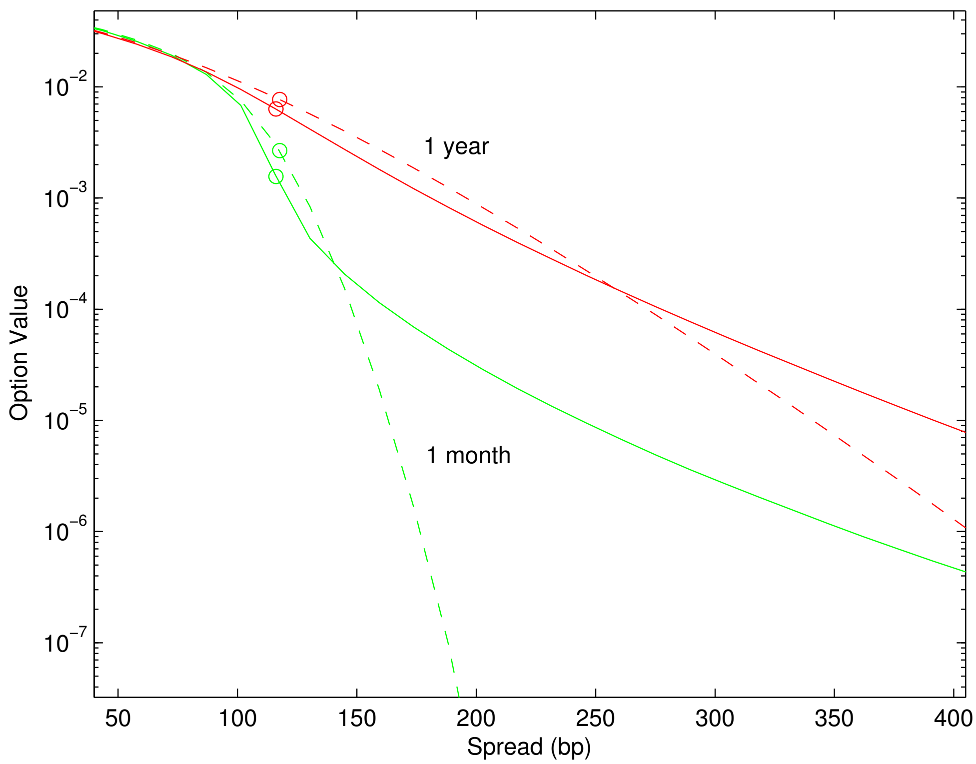
$$\begin{cases} \dot{\mathbf{B}}(t) = \mathbf{G}_1(\mathbf{B}(t)) \\ \dot{A}(t) = G_0(\mathbf{B}(t)); \quad A(0) = 0 \end{cases} \tag{7.1}$$

Figure 6: Effect of $\alpha$ on option value




Value of one-month ($\delta = 1/12$) payer option on a 5 year CDS as function of strike spread. CIR model under business time with parameters $\kappa_x = .2, \theta_x = .02, \sigma_x = .1$, starting condition $\lambda_0 = \theta_x/2 = .01$, and inverse Gaussian time-change with $\xi = 1$. Riskfree rate $r = 0.03$. Recovery $R = 0.4$. Circles mark par spread at date 0. When $\alpha = \infty$, the model is equivalent to the CIR model without time-change.

Figure 7: Effect of time to expiry on option value




Value of CDS payer option. CIR model under business time with parameters $\kappa_x = .2, \theta_x = .02, \sigma_x = .1$, starting condition $\lambda_0 = \theta_x/2 = .01$. Solid lines for model with inverse Gaussian time-change with $\alpha = 1$ and $\xi = 1$, and dashed line for the CIR model without time-change. Riskfree rate $r = 0.03$. Recovery $R = 0.4$. Circles mark par spread at date 0.

where $t \geq 0$; $A(t) \in \mathbb{C}, \mathbf{B}(t) \in \mathbb{C}^d$. In typical application, $\mathbf{B}(t)$ tends to zero as $t \to \infty$, which indicates the existence of an attracting critical point. Less restrictively, we assume there is an attracting critical point $\mathbf{B}_0 \in \mathbb{C}^d$ such that $\mathbf{G}_1(\mathbf{B}_0) = 0$, and analyze the system in a neighborhood of such a point. The functions $G_0 : \mathbb{C}^d \to \mathbb{C}$ and $\mathbf{G}_1 : \mathbb{C}^d \to \mathbb{C}^d$ are assumed to be analytic in a neighborhood of $\mathbf{B}_0$, which is a mild restriction of the setting in Duffie et al. (2000).[14] With the changes of variables

$$\mathbf{B}(t) = \mathbf{B}_0 + \mathbf{y}(t);\ \mathbf{G}_1(\mathbf{B}_0 + \mathbf{y}) = \Xi\mathbf{y} + \mathbf{F}(\mathbf{y})$$

where $\Xi$ is the Jacobian of $\mathbf{G}_1$ at $\mathbf{B}_0$, the first equation in (7.1) becomes

$$\dot{\mathbf{y}} = \Xi\mathbf{y} + \mathbf{F}(\mathbf{y});\ t \geq 0;\ \ \mathbf{y} \in \mathbb{C}^d;\ \mathbf{F}(\mathbf{y}) = o(\mathbf{y}) \text{ as } \mathbf{y} \to 0 \tag{7.2}$$

which we study under assumptions guaranteeing stability of the equilibrium.

**Assumption 7.**

*(i)* $\mathbf{F}$ *is analytic in a polydisk centered at zero.*

*(ii)* $\Xi$ *is a diagonalizable matrix of constants. Its eigenvalues* $\xi_1, ..., \xi_d$ *are nonresonant, i.e., for* $k_1, ..., k_d$ *nonnegative integers with* $|\mathbf{k}| := k_1 + k_2 + ... + k_d \geq 2$, *we have* $\xi_j - \mathbf{k} \cdot \boldsymbol{\xi} \neq 0$ *for* $j = 1, ..., d$.

*(iii)* $\xi_1, ..., \xi_d$ *are in the left half plane,* $\Re(\xi_i) < 0, i = 1, 2, ..., d$.

Part (i) is a fairly weak restriction on the Laplace transform of the jump size distribution. Part (ii) is quite weak, as it holds everywhere on the parameter space except on a set of measure zero. Part (iii) is a stationarity condition. Under this assumption, the eigenvalues of $\Xi$ are in the *Poincaré domain*, i.e., the domain in $\mathbb{C}^d$ in which zero is not contained in the closed convex hull of $\xi_1, ..., \xi_d$. It ensures that the solutions of the linearized part, $\dot{\mathbf{y}} = \Xi\mathbf{y}$, decay as $t \to \infty$.

The following is a classical theorem due to Poincaré (see e.g., Ilyashenko and Yakovenko, 2008).

**Theorem 1** (Poincaré)**.** *Under Assumption 7, there is a positive tuple* $\delta_i > 0$ *s.t. in the polydisk* $\mathbb{D}_\delta = \{\mathbf{y} : |y_i| < \delta_i, i = 1, ..., d\}$ (7.2) *is analytically equivalent to*

$$\dot{\mathbf{w}} = \Xi\mathbf{w} \tag{7.3}$$

*with a conjugation map tangent to the identity.*

Analytic equivalence means that there exists a function $\mathbf{h}$ analytic in $\mathbb{D}_\delta$ with $\mathbf{h} = O(\mathbf{w}^2)$ such that $\mathbf{y}$ satisfies (7.2) if and only if $\mathbf{w}$ defined by

$$\mathbf{y} = \mathbf{w} + \mathbf{h}(\mathbf{w}) \tag{7.4}$$

satisfies (7.3). Tangent to the identity simply means the fact that the linear part of the conjugation map is the identity, as seen in (7.4).

Let $\mathbb{D}^*$ denote the common polydisk of analyticity of $\mathbf{h}$ and $\mathbf{F}$.

[14] Comparing our system to the corresponding ODE system (2.5) and (2.6) in Duffie et al. (2000), our restriction merely imposes that the "jump transform" $\theta(\mathbf{u})$ is analytic in a neighborhood of $\mathbf{B}_0$. Note here that we have reversed the direction of time. Duffie et al. (2000) fix the horizon $T$ and vary current time $t$, whereas we fix current time at 0 and vary the horizon $t$.

**Proposition 7.** *The general solution of* (7.2) *with initial condition*

$$\mathbf{y}(0) = \mathbf{c} \tag{7.5}$$

*where* $\mathbf{c} = (c_1, ..., c_d) \in \mathbb{D}^*$, *is*

$$\mathbf{y}(t) = \sum_{|\mathbf{k}|>0} \boldsymbol{\Upsilon}_{\mathbf{k}} \mathbf{c}^{\mathbf{k}} e^{(\mathbf{k}\cdot\boldsymbol{\xi})t} \tag{7.6}$$

*where* $\mathbf{c}^{\mathbf{k}} = c_1^{k_1} c_2^{k_2} \cdots c_d^{k_d}$ *and* $\boldsymbol{\Upsilon}_{\mathbf{k}} \in \mathbb{C}^d$ *are the Taylor coefficients of* $\mathbf{w} + \mathbf{h}(\mathbf{w})$.

*Proof.* The general solution of (7.3) is

$$\mathbf{w} = c_1 e^{\xi_1 t}\mathbf{e}_1 + c_2 e^{\xi_2 t}\mathbf{e}_2 + ... + c_d e^{\xi_d t}\mathbf{e}_d \tag{7.7}$$

where $c_i \in \mathbb{C}$ are arbitrary. The rest follows from Theorem 1, (7.4), the fact that $\mathbf{c} \in \mathbb{D}^*$, and the analyticity of $\mathbf{h}$ which implies that its Taylor series at zero converges. □

Let $\hat{\mathbb{B}}$ be a ball in which $\mathbf{F}$ is analytic and

$$\|\mathbf{F}(\mathbf{y})\| < -\|\mathbf{y}\| \max_i \Re(\xi_i) \tag{7.8}$$

The existence of $\hat{\mathbb{B}}$ is guaranteed by Assumption 7(i) and by the property $\mathbf{F}(\mathbf{y}) = o(\mathbf{y})$ as $\mathbf{y} \to 0$ in (7.2). Condition (7.8) implies that $\hat{\mathbb{B}}$ is an invariant domain under the flow. This is an immediate consequence of the much stronger Proposition 8 below, but it has an elementary proof: By Cauchy-Schwartz and the assumption on $\mathbf{F}$,

$$\Re\langle \mathbf{y}, \mathbf{F}(\mathbf{y})\rangle \le \|\mathbf{y}\| \cdot \|\mathbf{F}(\mathbf{y})\| < -\|\mathbf{y}\|^2 \max_i \Re(\xi_i).$$

Since

$$\Re\langle \mathbf{y}, \Xi\mathbf{y}\rangle = \Re\left(\sum_{i=1}^d \xi_i |y_i|^2\right) \le \|\mathbf{y}\|^2 \max_i \Re(\xi_i),$$

there exists some $\epsilon > 0$ for which

$$\langle \mathbf{y}, \dot{\mathbf{y}}\rangle = \Re\left(\langle \mathbf{y}, \Xi\mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{F}(\mathbf{y})\rangle\right) < -\epsilon\|\mathbf{y}\|^2$$

We can also write the inner product as

$$\langle \mathbf{y}, \dot{\mathbf{y}}\rangle = \frac{1}{2}\frac{d}{dt}\|\mathbf{y}\|^2 = \frac{1}{2}\Re\left(\frac{d}{dt}|\mathbf{y}|^2\right)$$

which implies $\|\mathbf{y}(t)\|^2 \le \|\mathbf{y}(0)\|^2 e^{-2\epsilon t}$. Thus, if the initial condition $\mathbf{c}$ is in $\hat{\mathbb{B}}$, then the solution $\mathbf{y}(t)$ remains in $\hat{\mathbb{B}}$ for all $t \ge 0$.

**Proposition 8.** *Under Assumption 7, the domain of analyticity of* $\mathbf{h}$ *includes* $\hat{\mathbb{B}}$ *and the exponential expansion* (7.6) *converges absolutely and uniformly for all* $t \ge 0$ *and* $\mathbf{c} \in \hat{\mathbb{B}}$.

*Proof.* Condition (7.8) ensures that the Arnold (1969, §5) transversality condition and Theorem 2.1 of Carletti et al. (2005) apply, which guarantees the convergence of the Taylor series of $\mathbf{h}$ in $\hat{\mathbb{B}}$. The result follows in the same way as Proposition 7. □

We turn now to the second equation in the ODE system (7.1). Assume, without loss of generality, that $g_0(\mathbf{y}) = G_0(\mathbf{B}_0 + \mathbf{y})$ is analytic in the same polydisk as $\mathbf{G}_1$, i.e., in $\mathbb{D}^*$, which implies that the expansion converges uniformly and absolutely inside $\hat{\mathbb{B}}$. Imposing Assumption 7, we substitute (7.6) to obtain a uniformly and absolutely convergent expansion in $t$, and integrate term-by-term to get

$$A(t) = G_0(\mathbf{B}_0)t + \sum_{|\mathbf{k}|>0} \boldsymbol{\upsilon}_{\mathbf{k}}(\mathbf{k}\cdot\boldsymbol{\xi})^{-1}\mathbf{c}^{\mathbf{k}}e^{(\mathbf{k}\cdot\boldsymbol{\xi})t} \tag{7.9}$$

where $\boldsymbol{\upsilon}_{\mathbf{k}}$ are the Taylor coefficients of $g_0$.

The absolute and uniform convergence of the expansions of $A(t)$ and $B(t)$ extends to the expansion of $\exp(A(t) + \mathbf{B}(t)\cdot\mathbf{z})$, which is the multi-factor extension of Proposition 5(ii). Thus, subject to Assumption 7 and $\mathbf{c} \in \hat{\mathbb{B}}$, expansion in exponentials can be applied to the multi-factor model. Furthermore, the construction in (3.1) of a finite signed measure satisfying Assumption 2 extends naturally, so expansion in derivatives also applies.

The multi-factor extension can be applied to a joint affine model of the riskfree rate and default intensity. Let $r_t$ be the short rate in business time and let $R_t$ be the recovery rate as a fraction of market value at $\tau_-$. We assume that $Y_t \equiv r_t + (1 - R_t)\lambda_t$ is an affine function of the affine jump-diffusion $\mathbf{Z}_t$, and solve for the business-time default-adjusted discount function $\mathrm{E}\left[\exp\left(-\int_0^t Y(\mathbf{Z}_s)ds\right)\middle|\mathbf{Z}_0 = \mathbf{z}\right]$. Subject to the regularity conditions in Assumption 7, we can thereby introduce stochastic time-change to the class of models studied by Duffie and Singleton (1999) and estimated by Duffee (1999). The possibility of handling stochastic interest rates in our framework is also recognized by Mendoza-Arriaga and Linetsky (2012, Remark 4.1).

# Conclusion

We have derived and demonstrated two new methods for obtaining the Laplace transform of a stochastic process subjected to a stochastic time change. Each method provides a simple way to extend a wide variety of constant volatility models to allow for stochastic volatility. More generally, we can abstract from the background process, and view our methods simply as ways to calculate the expectation of a function of stochastic time. The two methods are complements in their domains of application. Expansion in derivatives imposes strictly weaker conditions on the function, whereas expansion in exponentials imposes strictly weaker conditions on the stochastic clock. We have found both methods to be straightforward to implement and computationally efficient.

Relative to the earlier literature, the primary advantage of our approach is that the background process need not be Lévy or even Markov. Thus, our methods are especially well-suited to application to default intensity models of credit risk. Both of our methods apply to the survival probability function under the ubiquitous basic affine specification of the default intensity. The forward default rate is easily calculated as well. Therefore, we can easily price both corporate bonds and credit default swaps in the time-changed model. In a separate paper, a time-changed default intensity model is estimated on panels of CDS spreads (across maturity and observation time) using Bayesian MCMC methods.

In contrast to the direct approach of modeling time-varying volatility as a second factor, stochastic time-change naturally preserves important properties of the background model. In particular, so long as the default intensity is bounded nonnegative in the background model, it will be bounded nonnegative in the time-changed model. In numerical examples in which the business-time default intensity is a CIR process, we find that introducing a moderate volatility in the stochastic clock has hardly any impact on the term-structure of credit spreads, yet a very large impact on the intertemporal variation of spreads. Consequently, the model preserves the cross-sectional behavior of the standard CIR model in pricing bonds and CDS at a fixed point in time, but allows for much greater flexibility in capturing kurtosis in the distribution of changes in spreads across time. The model also has a first-order effect on the pricing of deep out-of-the-money CDS options.

# A Identities for the Bell polynomials

Bell polynomial identities arise frequently in our analysis, so we gather the important results together here for reference. In this appendix, $a$ and $b$ are scalar constants, and $x$ and $y$ are infinite sequences $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$. Unless otherwise noted, results are drawn from Comtet (1974, §3.3), in some cases with slight rearrangement.

We begin with the incomplete Bell polynomials, $Y_{n,k}(x)$. The homogeneity rule is

$$Y_{n,k}(abx_1, ab^2x_2, ab^3x_3, \ldots) = a^k b^n Y_{n,k}(x_1, x_2, x_3, \ldots). \tag{A.1}$$

From Mihoubi (2008, Example 2), we can obtain the identity

$$Y_{n,k}((z)_0, (2z)_1, (3z)_2, (4z)_3, \ldots) = \binom{n-1}{k-1}(zn)_{n-k} \tag{A.2}$$

for any $z \in \mathbb{R}^+$. Recall here that $(z)_j$ denotes the falling factorial $(z)_j = z \cdot (z-1) \cdots (z-j+1)$. We also make use of the recurrence rules

$$\frac{n!}{(n+m)!} Y_{n+m,n}(x_1, x_2, \ldots) = \frac{1}{m!} \sum_{j=1}^{m} (n)_j x_1^{n-j} Y_{m,j}\left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right) \tag{A.3}$$

$$k\, Y_{n,k}(x_1, x_2, \ldots) = \sum_{j=1}^{n-k+1} \binom{n}{j} x_j Y_{n-j,k-1}(x_1, x_2, \ldots) \tag{A.4}$$

For $n \geq 0$, the complete Bell polynomials, $Y_n(x)$, are obtained from the incomplete Bell polynomials as

$$Y_n(x) = \sum_{k=0}^{n} Y_{n,k}(x). \tag{A.5}$$

Note that $Y_0(x) = Y_{0,0}(x) = 1$ and that $Y_{n,0}(x) = 0$ for $n \geq 1$. Riordan (1968, §5.2) provides the recurrence rule

$$Y_{n+1}(x_1, x_2, \ldots) = \sum_{k=0}^{n} \binom{n}{k} x_{k+1} Y_{n-k}(x_1, x_2, \ldots) \tag{A.6}$$

# B Generalized transform for the basic affine process

In this appendix, we set forth the closed-form solution to functions $\mathfrak{A}_t(u, w), \mathfrak{B}_t(u, w)$ in the generalized transform

$$\mathrm{E}\left[\exp\left(w \int_0^t \lambda_s ds + u\lambda_t\right)\right] = \exp\left(\mathfrak{A}_t(u, w) + \mathfrak{B}_t(u, w)\lambda_0\right) \tag{B.1}$$

The process $\lambda_t$ is assumed to follow a basic affine process as in equation (4.2). If we replace $\lambda_t$ by $\nu_t$, then the results here apply to equation (3.2) as well with $A_t^\nu(u) = \mathfrak{A}_t^\nu(0,u)$ and $B_t^\nu(u) = \mathfrak{B}_t^\nu(0,u)$. In this appendix we drop the subscripts in the BAP parameters $(\kappa, \theta, \sigma, \zeta, \eta)$.

We follow the presentation in Duffie (2005, Appendix D.4), but with slightly modified notation. All functions and parameters associated with the generalized transform are written in Fraktur script. Let

$$
\begin{aligned}
\breve{\mathfrak{c}}_1 &= \frac{1}{2w}(\kappa + \sqrt{\kappa^2 - 2w\sigma^2}) \\
\breve{\mathfrak{f}}_1 &= \sigma^2 u^2 - 2\kappa u + 2w \\
\breve{\mathfrak{d}}_1 &= (1 - \breve{\mathfrak{c}}_1 u)\frac{\sigma^2 u - \kappa + \sqrt{(\sigma^2 u - \kappa)^2 - \sigma^2 \breve{\mathfrak{f}}_1}}{\breve{\mathfrak{f}}_1} \\
\breve{\mathfrak{a}}_1 &= (\breve{\mathfrak{d}}_1 + \breve{\mathfrak{c}}_1)u - 1 \\
\breve{\mathfrak{b}}_1 &= \frac{\breve{\mathfrak{d}}_1(-\kappa + 2w\breve{\mathfrak{c}}_1) + \breve{\mathfrak{a}}_1(\sigma^2 - \kappa\breve{\mathfrak{c}}_1)}{\breve{\mathfrak{a}}_1\breve{\mathfrak{c}}_1 - \breve{\mathfrak{d}}_1} \\
\breve{\mathfrak{a}}_2 &= \frac{\breve{\mathfrak{d}}_1}{\breve{\mathfrak{c}}_1} \\
\breve{\mathfrak{b}}_2 &= \breve{\mathfrak{b}}_1 \\
\breve{\mathfrak{c}}_2 &= 1 - \frac{\eta}{\breve{\mathfrak{c}}_1} \\
\breve{\mathfrak{d}}_2 &= \frac{\breve{\mathfrak{d}}_1 - \eta\breve{\mathfrak{a}}_1}{\breve{\mathfrak{c}}_1}
\end{aligned}
$$

We next define for $i = \{1, 2\}$

$$\breve{\mathfrak{h}}_i = \frac{\breve{\mathfrak{a}}_i\breve{\mathfrak{c}}_i - \breve{\mathfrak{d}}_i}{\breve{\mathfrak{b}}_i\breve{\mathfrak{c}}_i\breve{\mathfrak{d}}_i} \tag{B.2}$$

and the functions

$$\breve{\mathfrak{g}}_i(t) = \frac{\breve{\mathfrak{c}}_i + \breve{\mathfrak{d}}_i \exp(\breve{\mathfrak{b}}_i t)}{\breve{\mathfrak{c}}_i + \breve{\mathfrak{d}}_i}. \tag{B.3}$$

Then the functions $\breve{\mathfrak{A}}$ and $\breve{\mathfrak{B}}$ are

$$\breve{\mathfrak{A}}_t(u,w) = \kappa\theta\breve{\mathfrak{h}}_1 \log(\breve{\mathfrak{g}}_1(t)) + \frac{\kappa\theta}{\breve{\mathfrak{c}}_1}t + \zeta\breve{\mathfrak{h}}_2 \log(\breve{\mathfrak{g}}_2(t)) + \zeta\frac{1 - \breve{\mathfrak{c}}_2}{\breve{\mathfrak{c}}_2}t \tag{B.4a}$$

$$\breve{\mathfrak{B}}_t(u,w) = \frac{1 + \breve{\mathfrak{a}}_1 \exp(\breve{\mathfrak{b}}_1 t)}{\breve{\mathfrak{c}}_1 + \breve{\mathfrak{d}}_1 \exp(\breve{\mathfrak{b}}_1 t)}, \tag{B.4b}$$

The discount function is $S(t) = \exp(A(t) + B(t)\lambda_0)$ where $A(t) = \breve{\mathfrak{A}}_t(0,-1)$ and $B(t) = \breve{\mathfrak{B}}_t(0,-1)$. To obtain these, let $\mathfrak{a}_i$ be the value of $\breve{\mathfrak{a}}_i$ when $u = 0$ and $w = -1$ for $i = \{1, 2\}$, and similarly define $\mathfrak{b}_i$, $\mathfrak{c}_i$, etc. These simplify to

$$\begin{aligned}
\mathfrak{a}_1 &= -1 \\
\mathfrak{b}_1 &= \mathfrak{b}_2 = -\sqrt{\kappa^2 + 2\sigma^2} \\
\mathfrak{c}_1 &= \frac{1}{2}(\mathfrak{b}_1 - \kappa) \\
\mathfrak{d}_1 &= \frac{1}{2}(\mathfrak{b}_1 + \kappa) \\
\mathfrak{a}_2 &= \mathfrak{d}_1/\mathfrak{c}_1 \\
\mathfrak{c}_2 &= 1 - \frac{\eta}{\mathfrak{c}_1} \\
\mathfrak{d}_2 &= \frac{\mathfrak{d}_1 + \eta}{\mathfrak{c}_1}
\end{aligned}$$

For the special case of $u = 0$ and $w = -1$, the $\breve{\mathfrak{h}}_i$ simplify to

$$\mathfrak{h}_1 = -2/\sigma^2 \tag{B.5a}$$

$$\mathfrak{h}_2 = -2\eta/(\sigma^2 - 2\eta(\kappa + \eta)) \tag{B.5b}$$

The $\mathfrak{g}_i(t)$ do not simplify dramatically. We obtain

$$A(t) = \kappa\theta\mathfrak{h}_1 \log(\mathfrak{g}_1(t)) + \frac{\kappa\theta}{\mathfrak{c}_1} t + \zeta\mathfrak{h}_2 \log(\mathfrak{g}_2(t)) + \zeta\frac{1 - \mathfrak{c}_2}{\mathfrak{c}_2} t \tag{B.6a}$$

$$B(t) = \frac{1 - \exp(\mathfrak{b}_1 t)}{\mathfrak{c}_1 + \mathfrak{d}_1 \exp(\mathfrak{b}_1 t)}. \tag{B.6b}$$

# C Expansion of the BAP survival probability function

We draw on the notation and results of Appendix B, and begin with the expansion in (i) of Proposition 5. Let $\gamma = -\mathfrak{b}_1 = -\mathfrak{b}_2$, and introduce the change of variable $y = 1 - 2\exp(-\gamma t)$. Then for $i = 1, 2$, we can expand

$$\begin{aligned}
\log(\mathfrak{g}_i(t)) = \log\left(\frac{\mathfrak{c}_i + \mathfrak{d}_i \exp(-\gamma t)}{\mathfrak{c}_i + \mathfrak{d}_i}\right) &= \log\left(\frac{\mathfrak{c}_i + \mathfrak{d}_i(1 - y)/2}{\mathfrak{c}_i + \mathfrak{d}_i}\right) \\
&= \log\left(\frac{\mathfrak{c}_i + \mathfrak{d}_i/2 - y\mathfrak{d}_i/2}{\mathfrak{c}_i + \mathfrak{d}_i/2}\right) + \log\left(\frac{\mathfrak{c}_i + \mathfrak{d}_i/2}{\mathfrak{c}_i + \mathfrak{d}_i}\right) \\
&= \log(1 - \varphi_i y) + \log\left(1 - \frac{\mathfrak{d}_i/2}{\mathfrak{c}_i + \mathfrak{d}_i}\right)
\end{aligned} \tag{C.1}$$

where

$$\varphi_i = \frac{\mathfrak{d}_i}{2\mathfrak{c}_i + \mathfrak{d}_i}$$

For $i = 1$, we find $\varphi_1 = (\gamma - \kappa)/(3\gamma + \kappa)$. Since $\sigma^2 > 0$, we have $\gamma > |\kappa| \geq 0$, which implies $0 < \varphi_1 < 1$. For $i = 2$, we find

$$\varphi_2 = \frac{\mathfrak{d}_1 + \eta}{2\mathfrak{c}_1 + \mathfrak{d}_1 - \eta}$$

Since $\eta \geq 0$, we have $-1 < \varphi_2 \leq \varphi_1$. Since $|y| \leq 1$, and since $\log(1+x)$ is analytic for $|x| < 1$, the expansion in (C.1) is absolutely convergent. Finally, since $\mathfrak{c}_1 < 0$ and $\mathfrak{d}_1 < 0$ for all $\kappa$ and $\eta \geq 0$, we have

$$1 - \frac{\mathfrak{d}_2/2}{\mathfrak{c}_2 + \mathfrak{d}_2} = 1 - \frac{1}{2}\frac{\mathfrak{d}_1 + \eta}{\mathfrak{c}_1 + \mathfrak{d}_1} \geq 1 - \frac{1}{2}\frac{\mathfrak{d}_1}{\mathfrak{c}_1 + \mathfrak{d}_1} > 0$$

so the logged constant in (C.1) is real-valued.

Using the same change of variable, the function $B(t)$ has expansion

$$\begin{aligned} B(t) = \frac{1+y}{2\mathfrak{c}_1 + \mathfrak{d}_1 - \mathfrak{d}_1 y} &= \frac{1}{2\mathfrak{c}_1 + \mathfrak{d}_1}\left(\frac{1+y}{1-\varphi_1 y}\right) \\ &= \frac{\varphi_1}{\mathfrak{d}_1}(1+y)\sum_{n=0}^{\infty} \varphi_1^n y^n = \frac{\varphi_1}{\mathfrak{d}_1} + \frac{1}{\mathfrak{d}_1}(1+\varphi_1)\sum_{n=1}^{\infty}\varphi_1^n y^n \end{aligned} \tag{C.2}$$

Again, since $1/(1-x)$ is analytic for $|x| < 1$, this expansion is absolutely convergent.

We combine these results to obtain

$$A(t)+B(t)\lambda_0 = at - \kappa\theta\mathfrak{h}_1 \log\left(1 - \frac{\mathfrak{d}_1/2}{\mathfrak{c}_1 + \mathfrak{d}_1}\right) - \zeta\mathfrak{h}_2 \log\left(1 - \frac{\mathfrak{d}_2/2}{\mathfrak{c}_2 + \mathfrak{d}_2}\right) + \frac{\varphi_1}{\mathfrak{d}_1}\lambda_0 + \sum_{n=1}^{\infty} q_n(1-2\exp(-\gamma t))^n \tag{C.3}$$

where

$$a = \frac{\kappa\theta}{\mathfrak{c}_1} + \zeta\frac{1-\mathfrak{c}_2}{\mathfrak{c}_2} < 0$$

and

$$q_n = \left[\frac{1+\varphi_1}{\mathfrak{d}_1}\lambda_0 - \frac{\kappa\theta\mathfrak{h}_1}{n}\right]\varphi_1^n - \frac{\zeta\mathfrak{h}_2}{n}\varphi_2^n$$

The expansion in (C.3) is absolutely convergent for $t \geq 0$.

Since the composition of two analytic functions is analytic, a series expansion of $S(t)$ in powers of $y$ is absolutely convergent for $|y| \leq 1$ (equivalently, $t \geq 0$). Thus, Proposition 5 holds with

$$\beta_n = \left(1 - \frac{\mathfrak{d}_1/2}{\mathfrak{c}_1 + \mathfrak{d}_1}\right)^{-\kappa\theta\mathfrak{h}_1}\left(1 - \frac{\mathfrak{d}_2/2}{\mathfrak{c}_2 + \mathfrak{d}_2}\right)^{-\zeta\mathfrak{h}_2} \exp\left(\frac{\varphi_1}{\mathfrak{d}_1}\lambda_0\right)\frac{1}{n!}Y_n(q_1 1!, q_2 2!, \ldots, q_n n!) \tag{C.4}$$

These coefficients are most conveniently calculated via a recurrence rule easily derived from (A.6):

$$\beta_n = \sum_{k=1}^{n} \frac{k}{n} q_k \beta_{n-k} \tag{C.5}$$

We now assume $\kappa > 0$ and derive the expansion in (ii) of Proposition 5. Here we introduce the change of variable $z = \exp(-\gamma t)$. Following the same steps as above, we find that $\log(\mathfrak{g}_i(t))$ for $i = \{1, 2\}$ can be expanded as

$$\log(\mathfrak{g}_i(t)) = -\log(1 + \mathfrak{d}_i/\mathfrak{c}_i) - \sum_{n=1}^{\infty}\left(\frac{-\mathfrak{d}_i}{\mathfrak{c}_i}\right)^n \frac{z^n}{n} \tag{C.6}$$

Since $\mathfrak{b}_1 < 0 < \kappa$, we see that $\mathfrak{c}_1 < \mathfrak{d}_1 \leq 0$. Since $\eta > 0$ we have $|\mathfrak{d}_2| \leq \max(\frac{\mathfrak{d}_1}{\mathfrak{c}_1}, -\frac{\eta}{\mathfrak{c}_1}) < \mathfrak{c}_2$. Thus, $|\mathfrak{d}_i/\mathfrak{c}_i| < 1$ for $i = 1, 2$. Since $|z| \leq 1$ as well, the expansion in (C.6) is absolutely convergent.

Using the same change of variable, the function $B(t)$ has expansion

$$B(t) = \frac{1-z}{\mathfrak{c}_1 + \mathfrak{d}_1 z} = \frac{1}{\mathfrak{c}_1} + \left(\frac{1}{\mathfrak{c}_1} + \frac{1}{\mathfrak{d}_1}\right) \sum_{n=1}^{\infty} \left(\frac{-\mathfrak{d}_1}{\mathfrak{c}_1}\right)^n z^n \tag{C.7}$$

Again, since $1/(1+x)$ is analytic for $|x| < 1$, this expansion is absolutely convergent.

We combine these results to obtain

$$A(t) + B(t)\lambda_0 = at - \kappa\theta\mathfrak{h}_1 \log(1 + \mathfrak{d}_1/\mathfrak{c}_1) - \zeta\mathfrak{h}_2 \log(1 + \mathfrak{d}_2/\mathfrak{c}_2) + \lambda_0/\mathfrak{c}_1 + \sum_{n=1}^{\infty} q_n \exp(-n\gamma t) \tag{C.8}$$

where $a$ is defined as before and where

$$q_n = \left[\lambda_0 \left(\frac{1}{\mathfrak{c}_1} + \frac{1}{\mathfrak{d}_1}\right) - \frac{\kappa\theta\mathfrak{h}_1}{n}\right] \left(\frac{-\mathfrak{d}_1}{\mathfrak{c}_1}\right)^n - \frac{\zeta\mathfrak{h}_2}{n} \left(\frac{-\mathfrak{d}_2}{\mathfrak{c}_2}\right)^n$$

The expansion in (C.8) is absolutely convergent for $t \geq 0$. Proposition 5 holds with

$$\beta_n = (1 + \mathfrak{d}_1/\mathfrak{c}_1)^{-\kappa\theta\mathfrak{h}_1} (1 + \mathfrak{d}_2/\mathfrak{c}_2)^{-\zeta\mathfrak{h}_2} \exp(\lambda_0/\mathfrak{c}_1) \frac{1}{n!} Y_n(q_1 1!, q_2 2!, \ldots, q_n n!) \tag{C.9}$$

Recurrence rule (C.5) applies in this case as well.

# D Derivatives of the generalized transform

Here we provide analytical expressions for $\Omega_n(t)$. As in the previous appendix, the process $\lambda_t$ is assumed to follow a basic affine process with parameters $(\kappa, \theta, \sigma, \zeta, \eta)$. Recall that

$$\Omega_n(t) = \frac{\partial^n}{\partial u^n} \exp(\breve{\mathfrak{A}}_t(u, -1) + \breve{\mathfrak{B}}_t(u, -1)\lambda_0)\Big|_{u=0}$$

Let $A_j(t)$ and $B_j(t)$ denote the functions

$$\begin{aligned} A_j(t) &= \frac{\partial^j}{\partial u^j} \breve{\mathfrak{A}}_t(u, -1)\Big|_{u=0} \\ B_j(t) &= \frac{\partial^j}{\partial u^j} \breve{\mathfrak{B}}_t(u, -1)\Big|_{u=0} \end{aligned}$$

Then by Faà di Bruno's formula,

$$\Omega_n(t) = S(t) \cdot Y_n(A_1(t) + B_1(t)\lambda_0, A_2(t) + B_2(t)\lambda_0, \ldots, A_n(t) + B_n(t)\lambda_0), \tag{D.1}$$

where $Y_n$ denotes the complete Bell polynomial. Given solutions to the functions $\{A_j(t), B_j(t)\}$, it is straightforward and efficient to calculate the $\Omega_n(t)$ sequentially via recurrence rule (A.6).

The functions $A_1(t)$ and $B_1(t)$ appear to be quite tedious (and the higher order $A_j(t)$ and $B_j(t)$ presumably even more so), as they depend on partial derivatives of $\breve{\mathfrak{a}}_j$, $\breve{\mathfrak{b}}_j$, and so on. Fortunately, these derivatives simplify dramatically when evaluated at $u = 0$. Define

$$\dot{\mathfrak{a}}_i = \frac{\partial}{\partial u} \breve{\mathfrak{a}}_i \Big|_{u=0}$$

and similarly define $\dot{\mathfrak{b}}_i$, $\dot{\mathfrak{c}}_i$, etc. We find

$$\begin{aligned}
\dot{\mathfrak{b}}_i &= \dot{\mathfrak{c}}_i = 0, \quad i \in \{1,2\} \\
\dot{\mathfrak{d}}_1 &= -\kappa \mathfrak{d}_1 - \sigma^2 \\
\dot{\mathfrak{a}}_1 &= \mathfrak{b}_1 \\
\dot{\mathfrak{d}}_2 &= \frac{\dot{\mathfrak{d}}_1 - \eta \dot{\mathfrak{a}}_1}{\mathfrak{c}_1} \\
\dot{\mathfrak{a}}_2 &= \frac{\dot{\mathfrak{d}}_1}{\mathfrak{c}_1}
\end{aligned}$$

An especially useful result is

$$\dot{\mathfrak{h}}_i = \frac{\partial}{\partial u} \breve{\mathfrak{h}}_i \bigg|_{u=0} = 0, \quad i \in \{1,2\}.$$

Last, we can show

$$\begin{aligned}
\dot{\mathfrak{g}}_1(t) &= \frac{\partial}{\partial u} \breve{\mathfrak{g}}_1(t) \bigg|_{u=0} = \frac{1 - \exp(\mathfrak{b}_1 t)}{-\mathfrak{h}_1 \mathfrak{b}_1} \\
\dot{\mathfrak{g}}_2(t) &= \frac{\partial}{\partial u} \breve{\mathfrak{g}}_2(t) \bigg|_{u=0} = \eta \frac{1 - \exp(\mathfrak{b}_2 t)}{-\mathfrak{h}_2 \mathfrak{b}_2}
\end{aligned}$$

We arrive at

$$\begin{aligned}
A_1(t) &= \kappa \theta \mathfrak{h}_1 \frac{\dot{\mathfrak{g}}_1(t)}{\mathfrak{g}_1(t)} + \zeta \mathfrak{h}_2 \frac{\dot{\mathfrak{g}}_2(t)}{\mathfrak{g}_2(t)} \\
B_1(t) &= \frac{\exp(\mathfrak{b}_1 t)}{\mathfrak{c}_1 + \mathfrak{d}_1 \exp(\mathfrak{b}_1 t)} \left( \dot{\mathfrak{a}}_1 - B(t) \dot{\mathfrak{d}}_1 \right)
\end{aligned}$$

Perhaps surprisingly, there are no further complications for $A_j(t)$ and $B_j(t)$ for $j > 1$. Proceeding along the same lines, we find

$$A_j(t) = (j-1)! \left( (-1)^{j+1} \kappa \theta \mathfrak{h}_1 \left( \frac{\dot{\mathfrak{g}}_1(t)}{\mathfrak{g}_1(t)} \right)^j + \zeta \mathfrak{h}_2 \left[ \eta^j - \left( \eta - \frac{\dot{\mathfrak{g}}_2(t)}{\mathfrak{g}_2(t)} \right)^j \right] \right) \tag{D.4a}$$

$$B_j(t) = j! (B(t)/\mathfrak{h}_1)^{j-1} B_1(t) \tag{D.4b}$$

These expressions imply that the cost of computing $\{A_j(t), B_j(t)\}$ does not vary with $j$.

# E Differentiation of the $\Omega_n(t)$ functions

As in the previous appendix, the process $\lambda_t$ is assumed to follow a basic affine process with parameters $(\kappa, \theta, \sigma, \zeta, \eta)$. Let us define

$$g_n(\lambda_t, t) = \exp\left( -\int_0^t \lambda_s \, ds \right) \lambda_t^n$$

so that $\Omega_n(t) = \mathrm{E}\left[ g_n(\lambda_t, t) \right]$. The extended Itô's Lemma (Protter, 1992, Theorem II.32) implies

$$dg_n = \left( \frac{\partial g_n}{\partial t} + \kappa(\theta - \lambda_t) \frac{\partial g_n}{\partial \lambda_t} + \frac{1}{2} \sigma^2 \lambda_t \frac{\partial^2 g_n}{\partial \lambda_t^2} \right) dt + \sigma \sqrt{\lambda_t} dW_t + g_n(\lambda_t, t) - g_n(\lambda_{t^-}, t)$$

The first term is

$$\frac{\partial g_n}{\partial t}+\kappa(\theta-\lambda_t)\frac{\partial g_n}{\partial \lambda_t}+\frac{1}{2}\sigma^2\lambda_t\frac{\partial^2 g_n}{\partial \lambda_t^2}$$
$$=-\exp\left(-\int_0^t \lambda_s\,ds\right)\lambda_t^{n+1}+n\kappa(\theta-\lambda_t)\exp\left(-\int_0^t \lambda_s\,ds\right)\lambda_t^{n-1}+\frac{1}{2}n(n-1)\sigma^2\lambda_t\exp\left(-\int_0^t \lambda_s\,ds\right)\lambda_t^{n-1}$$
$$=-g_{n+1}(\lambda_t,t)+n\kappa\theta g_{n-1}(\lambda_t,t)-n\kappa g_n(\lambda_t,t)+\frac{1}{2}n(n-1)\sigma^2 g_{n-1}(\lambda_t,t)$$
$$=\left(n\kappa\theta+\frac{1}{2}n(n-1)\sigma^2\right)g_{n-1}(\lambda_t,t)-n\kappa g_n(t)-g_{n+1}(\lambda_t,t).$$

Taking expectations,

$$\Omega_n'(t)=\mathrm{E}\left[\frac{d}{dt}g_n(\lambda_t,t)\right]=\left(n\kappa\theta+\frac{1}{2}n(n-1)\sigma^2\right)\mathrm{E}\left[g_{n-1}(\lambda_t,t)\right]-n\kappa\mathrm{E}\left[g_n(\lambda_t,t)\right]-\mathrm{E}\left[g_{n+1}(\lambda_t,t)\right]$$
$$+\sigma\mathrm{E}\left[\sqrt{\lambda_t}dW_t\right]+\mathrm{E}\left[g_n(\lambda_t,t)-g_n(\lambda_{t^-},t)\right]$$
$$=\left(n\kappa\theta+\frac{1}{2}n(n-1)\sigma^2\right)\Omega_{n-1}(t)-n\kappa\Omega_n(t)-\Omega_{n+1}(t)+\zeta\Xi_n(t)$$

where we define

$$\Xi_n(t)\equiv\mathrm{E}\left[g_n(\lambda_t,t)-g_n(\lambda_{t^-},t)|dJ_t>0\right].$$

Note that the $\mathrm{E}\left[\sqrt{\lambda_t}dW_t\right]$ term vanishes because $dW_t$ is independent of $\lambda_t$.

We interpret $\Xi_n(t)$ as the expected jump in $g_n$ conditional on a jump in $J_t$ at time $t$. Let $Z=dJ_t$ be the jump at time $t$. Noting that $Z$ is distributed exponential with parameter $1/\eta$, we have

$$\Xi_n(t)=\mathrm{E}\left[\exp\left(-\int_0^t \lambda_s\,ds\right)\left((\lambda_{t^-}+Z)^n-\lambda_{t^-}^n\right)\right]$$
$$=\mathrm{E}\left[\exp\left(-\int_0^t \lambda_s\,ds\right)\int_0^\infty\left((\lambda_{t^-}+z)^n-\lambda_{t^-}^n\right)(1/\eta)\exp(-z/\eta)\,dz\right]$$

For $n=0$, we have $\Xi_n(t)=0$. Assuming $n>0$, conditioning on $\lambda_{t^-}$ and expanding $(\lambda_{t^-}+z)^n$, the integral is

$$\frac{1}{\eta}\sum_{i=1}^n\binom{n}{i}\lambda_{t^-}^{n-i}\int_0^\infty z^i\exp(-z/\eta)\,dz=\frac{1}{\eta}\sum_{i=1}^n\binom{n}{i}\lambda_{t^-}^{n-i}\eta^{i+1}i!=\sum_{i=1}^n(n)_i\eta^i\lambda_{t^-}^{n-i}$$

where we substitute $\binom{n}{i}i!=(n)_i$. This implies that

$$\Xi_n(t)=\sum_{i=1}^n(n)_i\eta^i\mathrm{E}\left[\exp\left(-\int_0^t \lambda_s\,ds\right)\lambda_{t^-}^{n-i}\right]=\sum_{i=1}^n(n)_i\eta^i\Omega_{n-i}(t)$$

To confirm the recurrence rule, note that

$$\begin{aligned}\Xi_{n+1}(t) - (n+1)\eta\Xi_n(t) &= \sum_{i=1}^{n+1}(n+1)_i\eta^i\Omega_{n+1-i}(t) - (n+1)\eta\sum_{i=1}^{n}(n)_i\eta^i\Omega_{n-i}(t) \\ &= (n+1)\eta\Omega_n(t) + \sum_{i=2}^{n+1}(n+1)_i\eta^i\Omega_{n+1-i}(t) - \eta\sum_{i=1}^{n}(n+1)(n)_i\eta^{i+1}\Omega_{n-i}(t) \\ &= (n+1)\eta\Omega_n(t) + \sum_{i=1}^{n}(n+1)_{i+1}\eta^{i+1}\Omega_{n-i}(t) - \sum_{i=1}^{n}(n+1)_{i+1}\eta^{i+1}\Omega_{n-i}(t) = (n+1)\eta\Omega_n(t).\end{aligned}$$

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