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A Robust Capital Asset Pricing Model

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Abstract

We build a market equilibrium theory of asset prices under Knightian uncertainty. Adopting the mean-variance decisionmaking model of Maccheroni, Marinacci, and Ruffino (2013a), we derive explicit demands for assets and formulate a robust version of the two-fund separation theorem. Upon market clearing, all investors hold ambiguous assets in the same relative proportions as the assets' market values. The resulting uncertainty-return tradeoff is a robust security market line in which the ambiguous return on an asset is measured by its beta (systematic ambiguity). A simple example on portfolio performance measurement illustrates the importance of writing ambitious, robust asset-pricing models.

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1 Introduction

The Great Recession of 2008 has prompted calls for a microeconomic theory to predict the behavior of capital markets under Knightian uncertainty.[1] Although many insights can be drawn from the capital asset pricing model (CAPM) under risk, and from its many variations[2], thus far few studies have explored the effects on equilibrium asset prices of uncertainty in investors' preferences. We develop a robust capital asset pricing model (RCAPM) that offers powerful predictions about how to measure the uncertainty-return tradeoff. Also, the model's analytical tractability renders it immediately applicable to capital-budgeting estimations, to the evaluation of professionally managed portfolios, and so on.

Since the CAPM of Treynor (1962), Sharpe (1964), Lintner (1965), and Mossin (1966) is founded on the formal quantitative theory for optimal portfolio selection of Markowitz (1952) and Tobin (1958), we first propose a general solution for the portfolio-selection problem under uncertainty. Our objective function features the second-order approximation of the certainty equivalent of Maccheroni, Marinacci, and Ruffino (2013a), which is the analoge of the Arrow-Pratt approximation under model uncertainty. It also introduces an ambiguity premium that captures variations in returns due to model uncertainty.

First, we find a mean-variance efficient portfolio that depends on the investor's tastes – his aversion to risk and uncertainty – as well as his beliefs over expected returns. Next, we use the optimal solution to derive a robust version of the two-fund separation theorem: here, all mean-variance efficient portfolios come from combining the riskless asset with the mean-variance efficient portfolio made of ambiguous assets only. Last, assuming that all investors are mean-variance optimizers who make decisions according to the same normative model, we derive the set of prices at which everyone's demand is satisfied in equilibrium.

The resulting relationship between asset returns and uncertainty resembles the CAPM security market line under risk. In addition, it displays a robust beta coefficient that measures an asset's systematic ambiguity. Given the results of our model, we discuss the case in which

[1] We use the terms *Knightian uncertainty* (from Knight, 1921) and *ambiguity* interchangeably.

[2] For example, Merton (1973) proposes an intertemporal model for the capital market. Lucas (1978), Breeden (1979), and Grossman and Shiller (1981) derive consumption-based asset-pricing models.

the existence of superior-performance assets (that is, assets whose robust alpha coefficient is greater than zero) allows for the creation of “super-efficient” portfolios.

Our work is related to recent research on optimal portfolio-selection theory under risk and uncertainty. Maccheroni, Marinacci, and Ruffino (2013a) and Gollier (2011) study the effects of higher ambiguity aversion on optimal portfolio rebalancing. They also set the conditions under which ambiguity reinforces (or mitigates) risk. Izhakian and Benninga (2008) find similar results on the separation of risk and ambiguity. Garlappi, Uppal, and Wang (2007) and Chen, Ju, and Miao (2013) contrast the optimal portfolio allocation from the Bayesian and the ambiguity models. Portfolio optimization under uncertainty also helps to explain some puzzling investor behaviors. For example, Epstein and Miao (2003) and Boyle, Garlappi, Uppal, and Wang (2012) provide a rational justification for holding “familiar” assets. Easley and O’Hara (2009) and Illeditsch (2011) tackle poor investor participation in the stock market.

We share some features with the capital asset-pricing models of Chen and Epstein (2002); Collard, Mukerji, Sheppard, and Tallon (2011); Ju and Miao (2012); and Izhakian (2012). In Chen and Epstein’s (2002) model, excess returns also reflect a compensation for risk and a separate compensation for ambiguity. In Collard and others (2011) and Ju and Miao’s (2012) models, the investor’s pessimistic behavior is tied to a variety of dynamic asset-pricing phenomena (equity premium, risk-free rate, and so forth). In Izhakian’s (2012) model, equilibrium prices contain a systematic beta similar to ours. We differ in that his derivation is founded on shadow probability theory, while ours is based on the smooth preferences of Klibanoff, Marinacci, and Mukerji (2005).

The rest of the paper is organized as follows. In Section 2 we introduce the mathematical setup, in Section 3 we derive the mean-variance efficient portfolio and present a robust version of the two-fund separation theorem, in Section 4 we obtain the RCAPM, and in Section 5 we conclude.

2 Theoretical decision framework

The sure amount of money that a von Neumann-Morgenstern expected-utility maximizer with utility u and wealth w considers equivalent to a risky investment h is given by

$$c(w+h,P) = u^{-1}\left(E_P\left(u\left(w+h\right)\right)\right), \tag{1}$$

where P is the probabilistic model that describes the stochastic nature of the problem. The classic approximation of (1) by Arrow (1971) and Pratt (1964)

$$c(w+h,P) \approx w + E_P(h) - \frac{1}{2}\lambda_u(w)\,\sigma_P^2(h), \tag{2}$$

where $\lambda_u = -u''/u'$ denotes the decisionmaker's risk attitude, is widely used in models of investment because it ties the risk premium associated with h to its variance, $\sigma_P^2(h)$. But if a decisionmaker is uncertain about the true probabilistic model P and instead adopts alternative models Q, then $c(w+h,Q)$ becomes a variable amount of money that depends on Q. The smooth characterization of (1) under ambiguity is the certainty equivalent of Klibanoff, Marinacci and Mukerji (2005):

$$\begin{aligned} C(w+h) &= v^{-1}\left(E_\mu\left(v\left(c\left(w+h\right)\right)\right)\right) \\ &= v^{-1}\left(E_\mu\left(v\left(u^{-1}\left(E\left(u\left(w+h\right)\right)\right)\right)\right)\right), \end{aligned} \tag{3}$$

where μ denotes the decisionmaker's prior probability on the space of possible models Q, and v is his attitude toward model uncertainty. In Maccheroni, Marinacci, and Ruffino (2013a) we derive a second-order approximation of (3):

$$C(w+h) \approx w + E_{\bar{Q}}(h) - \frac{1}{2}\lambda_u(w)\,\sigma_{\bar{Q}}^2(h) - \frac{1}{2}\left(\lambda_v(w) - \lambda_u(w)\right)\sigma_\mu^2\left(E\left(h\right)\right), \tag{4}$$

where $\lambda_v = -v''/v'$ is the decisionmaker's ambiguity attitude, $\bar{Q}$ is the reduced probability induced by the prior μ, and $E(h)$ is the random variable that associates the expected value

$E_Q(h)$ to each model Q.

The last term in (4) – the *ambiguity premium* – is new relative to (2).[3] Specifically, the ambiguity premium changes the certainty equivalent through the decisionmaker's aversion to ambiguity λ_v as well as the variance of the return $E(h)$. Hence, this parsimonious extension of the mean-variance model under risk is fully determined by three parameters: λ_u, λ_v, and μ. Higher values of λ_u and λ_v indicate stronger negative attitudes toward risk and ambiguity, respectively. Higher values of $\sigma_\mu^2(E(h))$ indicate poorer information on outcomes and models. In the special case of $\sigma_\mu^2(E(h)) = 0$ (that is, where μ is a trivial measure), there is no source of model uncertainty and (4) reduces to (2). Last, the variance decomposition between state and model uncertainty,

$$\sigma_{\bar{Q}}^2(h) = E_\mu\left(\sigma^2(h)\right) + \sigma_\mu^2(E(h)), \tag{5}$$

allows us to rearrange (4) by the Arrow-Pratt coefficients of u and v as follows:

$$C(w+h) \approx w + E_{\bar{Q}}(h) - \frac{\lambda_u(w)}{2} E_\mu\left(\sigma^2(h)\right) - \frac{\lambda_v(w)}{2} \sigma_\mu^2(E(h)).^4 \tag{6}$$

In (6), risk aversion and ambiguity aversion determine the decisionmaker's response to the average variance, $E_\mu(\sigma^2(h))$, and the variance of averages, $\sigma_\mu^2(E(h))$, respectively.

Next, we set $\lambda_u(w) = \lambda$ and $\lambda_v(w) - \lambda_u(w) = \theta$. Then, a decisionmaker is risk averse when $\lambda > 0$ and ambiguity averse when $\theta > 0$.[5] Last, we assume that

- The ratio of θ to λ is equal for all investors.

- $\bar{Q}$ is equal to the baseline probability P.[6]

[3] Nau (2006), Izhakian and Benninga (2008), and Jewitt and Mukerji (2011) obtain approximations for the ambiguity premium on the basis of special assumptions.

[4] This formulation shows that, when the indexes u and v are sufficiently smooth, both state and model uncertainty have at most a second order effect on the evaluation. In Maccheroni, Marinacci, and Ruffino (2013b) we study in detail orders of risk aversion and of model uncertainty aversion in the smooth ambiguity model.

[5] Ambiguity neutrality is modeled as $\theta = 0$.

[6] Under (ii), the certainty equivalent (4) is always finite.

In Section 3 we apply the "enhanced" Arrow-Pratt approximation to the mean-variance model of optimal portfolio-selection theory.

3 Mean-variance portfolio theory

We allow for an arbitrary number of investors who make portfolio decisions based on their prior probability μ on the space of possible probabilistic models of their end-of-period wealth.[7] For the normative results that follow, investors need not agree on the prospects of the various investments; thus, in general, beliefs are not homogenous. The market is formed of n ambiguous assets $(\sigma_\mu^2(E(r_i)) > 0,\ i = 1, ..., n)$ with expected rate of return r_i and a risk-less asset, whose return r_f is known with certainty. Denote by $\mathbf{r}$ the vector of returns on the first n assets and by $\widehat{\mathbf{w}}$ the mean-variance efficient portfolio with expected return

$$r_{\widehat{\mathbf{w}}} = r_f + \widehat{\mathbf{w}} \cdot (\mathbf{r} - r_f \mathbf{1}), \tag{7}$$

where $\mathbf{1}$ is the n-dimensional unit vector. We assume a friction-less market environment in which assets are traded in the absence of transaction costs, of spreads between the borrowing and the lending rates, and of short sale restrictions. From (4) and (7), $\widehat{\mathbf{w}}$ must be the solution to the portfolio problem:

$$\max_{\mathbf{w}\in\mathbb{R}^n} C(r_{\mathbf{w}}) = \max_{\mathbf{w}\in\mathbb{R}^n} \left(E_P(r_{\mathbf{w}}) - \frac{\lambda}{2}\sigma_P^2(r_{\mathbf{w}}) - \frac{\theta}{2}\sigma_\mu^2(E(r_{\mathbf{w}})) \right). \tag{8}$$

To deliver the optimality condition, set

$$\begin{aligned}
\mathrm{E}_P[\mathbf{r} - r_f\mathbf{1}] &= [E_P(r_1 - r_f), ..., E_P(r_n - r_f)]^\intercal, \\
\mathbf{\Sigma_P}[\mathbf{r}] &= \sigma_P(r_i, r_j)_{i,j=1}^n, \\
\mathbf{\Sigma}_\mu[\mathrm{E}[\mathbf{r}]] &= \sigma_\mu(E(r_i), E(r_j))_{i,j=1}^n, \\
\mathbf{\Xi} &= \lambda\mathbf{\Sigma_P}[\mathbf{r}] + \theta\mathbf{\Sigma}_\mu[\mathrm{E}[\mathbf{r}]].
\end{aligned}$$

[7] Investors are myopic in the sense that they focus on their wealth only one decision-period ahead of their current decision.

Hence, from (8), we have that

$$\max_{\mathbf{w}\in\mathbb{R}^n} C\left(r_{\mathbf{w}}\right) = \max_{\mathbf{w}\in\mathbb{R}^n} \left(\mathbf{w}\cdot \mathrm{E}_P\left[\mathbf{r} - r_f \mathbf{1}\right] - \frac{1}{2}\mathbf{w}^{\intercal}\Xi\mathbf{w}\right).$$

The first-order condition for a maximum is:

$$\mathrm{E}_P\left[\mathbf{r} - r_f\mathbf{1}\right] - \Xi\widehat{\mathbf{w}} = \mathbf{0},$$

which can be solved by matrix inversion assuming that Ξ is positive-definite. The mean-variance efficient portfolio is:

$$\widehat{\mathbf{w}} = \left[\lambda\mathbf{\Sigma}_{\mathbf{P}}\left[\mathbf{r}\right] + \theta\mathbf{\Sigma}_{\mu}\left[\mathrm{E}\left[\mathbf{r}\right]\right]\right]^{-1}\mathrm{E}_P\left[\mathbf{r} - r_f\mathbf{1}\right], \tag{9}$$

where $\mathbf{\Sigma}_{\mathbf{P}}\left[\mathbf{r}\right]$ and $\mathbf{\Sigma}_{\mu}\left[\mathrm{E}\left[\mathbf{r}\right]\right]$ are the variance-covariance matrixes of returns and expected excess returns under P and μ, respectively, and $\mathrm{E}_P\left[\mathbf{r} - r_f\mathbf{1}\right]$ is the vector of expected excess returns under P.

The optimal solution has the merit to naturally adjust the classic risk model to reflect investors' uncertainty over expected returns, $\mathbf{\Sigma}_{\mu}\left[\mathrm{E}\left[\mathbf{r}\right]\right]$. In fact, if investors are either ambiguity-neutral ($\theta = 0$) or approximately unambiguous ($\mathbf{\Sigma}_{\mu}\left[\mathrm{E}\left[\mathbf{r}\right]\right] = 0$), the Markowitz-Tobin derivation readily obtains.[8] The form of (9) is especially convenient because it allows for a direct application of the ample research on mean-variance preferences developed for problems involving risk to the analysis of problems involving ambiguity. In particular, provided that information on r_f, $\mathrm{E}_P\left[\mathbf{r}\right]$, $\mathbf{\Sigma}_{\mathbf{P}}\left[\mathbf{r}\right]$, and $\mathbf{\Sigma}_{\mu}\left[\mathrm{E}\left[\mathbf{r}\right]\right]$ is available, all mean-variance investors use the normative model (8) to select the optimal combination of ambiguous assets with the risk-free asset. The resulting allocation is a smooth function of the taste parameters λ and θ, which is particularly well suited for comparative statics analysis. Maccheroni, Marinacci and Ruffino (2013a) exhaustively map the conditions under which higher ambiguity aversion (or higher ambiguity in expectations) lowers an investor's optimal exposures to the

[8] Approximately unambiguous prospects are defined by Maccheroni, Marinacci, and Ruffino (2013a), p. 1086.

ambiguous assets, spurring severe "flights-to-quality" and investments "in the familiar."[9,10]

Now assume that investors share the same prior probability μ on the space of possible probabilistic models. Then, for $\mathrm{E}_P[\mathbf{r} - r_f\mathbf{1}] > \mathbf{0},$ the allocations to ambiguous assets in (9) have the same relative proportions, independent of wealth, risk aversion, or ambiguity aversion, as long as θ is a fixed proportion of λ equal for all investors. We have that

$$\frac{\widehat{w_i}}{\widehat{w_j}} = \frac{\sum_{l=1}^{n} \xi_{l,i} E_P\left(r_l - r_f\right)}{\sum_{l=1}^{n} \xi_{l,j} E_P\left(r_l - r_f\right)}, \qquad \forall i, j = 1, ..., n,$$

where $\xi_{i,j}$ is defined as the i, j element of the inverse of Ξ. That is, $\Xi^{-1} = \left[\xi_{i,j}\right]_{i,j=1}^{n}$.

We define the *optimal combination of ambiguous assets (OCAA)* as the mean-variance efficient portfolio that contains ambiguous assets only. Labeling $\widehat{w_i^{OCAA}}$ the fraction of OCAA made up by i, we have that:

- $\widehat{w_i^{OCAA}}$ solves (8).
- $\sum_{i=1}^{n} \widehat{w_i^{OCAA}} = 1.$

Thus,

$$\widehat{w_i^{OCAA}} = \frac{\sum_{l=1}^{n} \xi_{l,i} E_P\left(r_l - r_f\right)}{\sum_{i=1}^{n}\sum_{l=1}^{n} \xi_{l,i} E_P\left(r_l - r_f\right)}, \qquad \forall i = 1, ..., n. \tag{10}$$

Theorem 1 *Denote by π the fraction of an investor's mean-variance efficient portfolio (9) that is allocated to the OCAA portfolio. From (10), it follows that the fraction of the investor's portfolio allocated to asset i is $\widehat{w_i} = \pi \widehat{w_i^{OCAA}}$.*

In other words, the mean-variance efficient portfolio with expected return $r_{\widehat{\mathbf{w}}}$ can be constructed from mixing the optimal combination of ambiguous assets with the risk-less asset – a *robust two-fund separation theorem.* In particular

$$r_{\widehat{\mathbf{w}}} = r_f + \boldsymbol{\pi} \cdot \left(\mathbf{r}_{\widehat{\mathbf{w}^{OCAA}}} - r_f\mathbf{1}\right), \tag{11}$$

[9] Allocation strategies driven by an investor's geographical or professional proximity to a particular stock are generally conceptualized in the term *familiarity.* See, among others, Huberman (2001).

[10] Gollier (2011) makes a similar point in a static two-asset portfolio problem with one safe asset and one uncertain one.

where $\mathbf{r}_{\mathbf{w}^{\widehat{OCAA}}}$ is the expected return on OCAA.

4 Equilibrium prices of capital assets

In this section we propose a positive asset-pricing theory that formalizes the relationship between asset returns and uncertainty, assuming that investors follow the mean-variance norm (9). First, we derive the set of prices that clears the market. Then, we discuss superior portfolio performance measurement when the RCAPM fails.

Theorem (1) characterizes an investor's demand function with respect to the mean-variance efficient portfolio that contains ambiguous assets only. Thus, the relative proportion of i to j is given by

$$\frac{\widehat{w}_i}{\widehat{w}_j} = \frac{\widehat{w_i^{OCAA}}}{\widehat{w_j^{OCAA}}}, \qquad \forall i, j = 1, ..., n. \tag{12}$$

The equilibrium implication of (12) is summarized in Theorem (2).

Theorem 2 *In equilibrium the relative proportions* $\frac{\widehat{w_i^{OCAA}}}{\widehat{w_j^{OCAA}}}$, $\forall i, j = 1, ..., n$ *must equal the relative proportions of the asset values in the market.*

That is, only if the market portfolio is mean-variance efficient (equal to the optimal combination of ambiguous assets) is it feasible for all investors to hold ambiguous assets in the same relative proportions as OCAA. Denoting by r_M the return on the market portfolio, the equilibrium expected return on asset i is given by

$$E_P(r_i) = r_f + \beta_i^A E_P(r_M - r_f), \tag{13}$$ [11]

with

$$\beta_i^A = \frac{\lambda \sigma_P(r_i, r_M) + \theta \sigma_\mu(E(r_i), E(r_M))}{\lambda \sigma_P^2(r_M) + \theta \sigma_\mu^2(E(r_M))}. \tag{14}$$

[11] The equilibrium expected return $E_P(r_i)$ obtains from (9), by combining asset i with the market portfolio (of which i is a part) and requiring a pure investment in M.

The *Security Market Line* (13) features a *robust beta*, β_i^A, that measures the marginal contribution of asset i to the ambiguity of the optimal portfolio M. Borrowing from the terminology of the CAPM under risk, we say that β_i^A measures the *systematic ambiguity* of asset i.

Observe that if there exists an asset j whose expected return violates (13), then the market portfolio is not the optimal combination of ambiguous assets. Instead, one can create OCAA by combining j with the market portfolio. Define α_j^A to be the deviation of asset j from the expected return profile (13). We decompose the excess return on asset j by means of the ordinary least square coefficients to obtain

$$\alpha_j^A = E_P\left(r_j - r_f\right) - \frac{\sigma_P\left(r_j, r_M\right)}{\sigma_P^2\left(r_M\right)} E_P\left(r_M - r_f\right).^{12} \tag{15}$$

Here, α_j^A is the expected value of the residual for the regression of $(r_j - r_f)$ on $(r_M - r_f)$ – that is, the expected value of the portion of $(r_j - r_f)$ that is uncorrelated with $(r_M - r_f)$. Then, from (13) and (14), it follows that

$$\begin{aligned} \alpha_j^A &= \beta_j^A E_P\left(r_M - r_f\right) - \frac{\sigma_P\left(r_j, r_M\right)}{\sigma_P^2\left(r_M\right)} E_P\left(r_M - r_f\right) \\ &= \frac{\theta\left[\sigma_\mu\left(E\left(r_j\right), E\left(r_M\right)\right)\sigma_P^2\left(r_M\right) - \sigma_P\left(r_j, r_M\right)\sigma_\mu^2\left(E\left(r_M\right)\right)\right]}{\sigma_P^2\left(r_M\right)\left[\lambda\sigma_P^2\left(r_M\right) + \theta\sigma_\mu^2\left(E\left(r_M\right)\right)\right]} E_P\left(r_M - r_f\right). \end{aligned} \tag{16}$$

It is easy to show that the sign of α_j^A is:

$$\operatorname{sgn}\alpha_j^A = \operatorname{sgn}\left[\frac{\sigma_\mu\left(E\left(r_j\right), E\left(r_M\right)\right)}{\sigma_\mu^2\left(E\left(r_M\right)\right)} - \frac{\sigma_P\left(r_j, r_M\right)}{\sigma_P^2\left(r_M\right)}\right], \tag{17}$$

where $\frac{\sigma_\mu(E(r_j),E(r_M))}{\sigma_\mu^2(E(r_M))}$ and $\frac{\sigma_P(r_j,r_M)}{\sigma_P^2(r_M)}$ are the "pure" marginal contributions of holding asset j to the risk and the ambiguity of the market portfolio, respectively. In brief, asset j is:

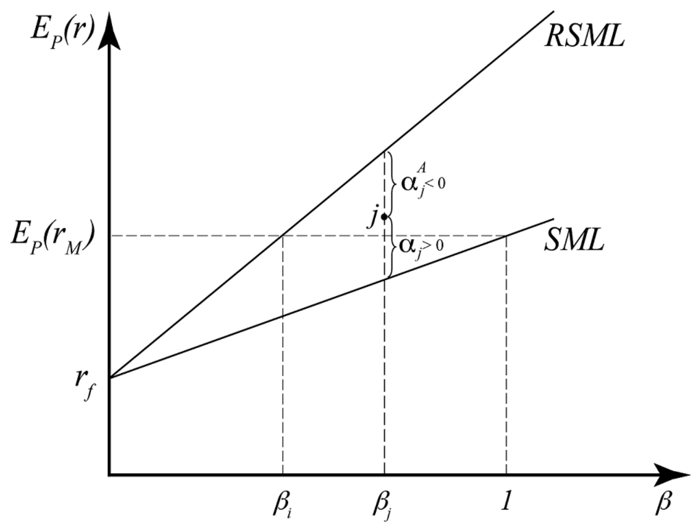
- Underpriced if $\frac{\sigma_\mu(E(r_j),E(r_M))}{\sigma_\mu^2(E(r_M))} > \frac{\sigma_P(r_j,r_M)}{\sigma_P^2(r_M)}$.
- Overpriced if $\frac{\sigma_\mu(E(r_j),E(r_M))}{\sigma_\mu^2(E(r_M))} < \frac{\sigma_P(r_j,r_M)}{\sigma_P^2(r_M)}$.

[12] The ratio $\frac{\sigma_P(r_j,r_M)}{\sigma_P^2(r_M)}$ on the right-hand side of (15) measures the systematic risk of asset j.

- Fairly priced if $\frac{\sigma_\mu(E(r_j),E(r_M))}{\sigma^2_\mu(E(r_M))} = \frac{\sigma_P(r_j,r_M)}{\sigma^2_P(r_M)}$.

Finally, we remark that the original capital asset pricing model under risk is nested in our equilibrium model of the capital market under uncertainty. In fact, setting $\sigma_\mu\left(E\left(r_i\right), E\left(r_j\right)\right) = 0$, $\forall i, j = 1, ..., n$, we have that $E_P\left(r_j\right) = r_f + \frac{\sigma_P(r_j,r_M)}{\sigma^2_P(r_M)} E_P\left(r_M - r_f\right)$ and $\alpha^A_j = 0$. Figure 1 summarizes our results.

Figure 1
Classic and Robust CAPM



Notes:
(1) $\beta_j = \frac{\sigma_P\left(r_j,r_M\right)}{\sigma^2_P(r_M)} \quad \forall j = 1, ..., n$.
(2) $\beta_i = \frac{\sigma_P(r_i,r_M)}{\sigma^2_P(r_M)} = 1 - \frac{\theta\left[\sigma_\mu(E(r_i),E(r_M))\sigma^2_P(r_M) - \sigma_P(r_i,r_M)\sigma^2_\mu(E(r_M))\right]}{\sigma^2_P(r_M)\left[\lambda\sigma^2_P(r_M)+\theta\sigma^2_\mu(E(r_M))\right]}$.
(3) $RSML$ has been drawn assuming $\frac{\sigma_\mu(E(r_i),E(r_M))}{\sigma^2_\mu(E(r_M))} > \frac{\sigma_P(r_i,r_M)}{\sigma^2_P(r_M)} \quad \forall i = 1, ..., n$.

The menu of equilibrium expected returns if the CAPM holds is given by the Security Market Line (*SML*). If the RCAPM holds, and ambiguity reinforces risk, the robust Security Market Line (*RSML*) is steeper than *SML*, reflecting higher returns due to model uncertainty. Instead, asset j is created to violate both *SML* and *RSML*. Note that if an investor makes decisions according to the CAPM, he identifies j as underpriced ($\alpha_j > 0$) when in fact, properly accounting for model uncertainty, j is overpriced ($\alpha^A_j < 0$). Then, the portfolio constructed by combining j with the market portfolio is not a superior performer as the investor believes, but an inferior one.

Finally, if ambiguity mitigates risk, $RSML$ is flatter than SML and assets command lower returns for the same level of risk. This result is important in light of compelling evidence that the U.S. stock market SML is much flatter than the CAPM-implied SML. For example, Black, Jensen, and Scholes (1972) analyze the returns of portfolios formed by ranking U.S. stocks on β values. They find that between 1926 and 1966 low-beta stocks earned higher average returns than the CAPM predicts, and vice versa. Even after accounting for measurement errors, the CAPM relation between expected returns and β is so weak that the model must be rejected. By contrast, the joint effect of risk and uncertainty on individual decisionmaking allows the RCAPM to predict a proportional relation between expected returns and β – one that is consistent with the above-mentioned empirical evidence.

5 Conclusions

We extend the capital asset pricing model by including Knightian uncertainty about asset returns. First, we derive the mean-variance efficient portfolio using the certainty equivalent approximation of Maccheroni, Marinacci, and Ruffino (2013a). The optimal portfolio holdings decrease with ambiguity aversion and perceived ambiguity – a phenomenon usually referred to as flight to quality. Next, we define the optimal combination of ambiguous assets and propose a robust version of the two-fund separation theorem: we show that all mean-variance efficient portfolios come from combining the optimal combination of ambiguous assets with the risk-free asset. Last, we find the robust security market line and compare it with the security market line under risk. In particular, we show that the interaction of risk and ambiguity can predict a robust security market line whose flatter slope fits the data. We argue that this result is important to address known empirical failures of the capital asset pricing model.

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