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**Debt Deflation Effects of Monetary Policy**

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# Debt Deflation Effects of Monetary Policy <sup>\*</sup>

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## Abstract

This paper assesses the role that monetary policy plays in the decision to default using a General Equilibrium model with collateralized loans, trade in fiat money and production. Long-term nominal loans are backed by collateral, the value of which depends on monetary policy. The decision to default is endogenous and depends on the relative value of the collateral to face value of the loan. Default results in foreclosure, higher borrowing costs, inefficient investment and a decrease in total output. We show that pre-crisis contractionary monetary policy interacts with Fisherian debt-deflation dynamics and can increase the probability that a crisis occurs.

**Keywords:** Default, Collateral, Debt-deflation

**JEL Classification:** E4; E5; G0; G1; G2

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# 1 Introduction

The financial crisis of 2007-2008 has renewed the interest in the ability of monetary policy to mitigate the adverse consequences that financial frictions can have on real economic activity. Mishkin (2009) and Gertler and Karadi (2011) argue that accommodative monetary policy is helpful during financial crisis episodes. This paper takes a step back and examines whether pre-crisis contractionary monetary policy can increase the likelihood that a crisis occurs in the future and, if yes, what are its real effects.

Our model can succinctly nest competing visions of the causes of the Great Depression (and of similar episodes) where debt-deflation dynamics act as an amplification mechanism.<sup>1</sup> On the one hand, Friedman and Schwartz (1963) find a high positive correlation between money supply and output and conclude that the decline in the money stock before the Great Depression was a substantial factor for the subsequent deflation and decline in GDP. On the other hand, Bernanke (1983) establishes that the Great Depression can be better explained when one explicitly models the financial frictions, which can impede the supply of credit to the real economy and, thus, GDP growth. Our analysis suggests that monetary forces are capable of inducing debt-deflation dynamics, but only when they exacerbate the underlying financial frictions, which in our model lead to default. Thus, we propose a “debt-deflation” channel of monetary policy.

We examine the effects of monetary policy on total output within a framework of fully flexible prices. The underlying friction is that we allow agents to (endogenously) default on their long-term loan obligations. Thus, there is a need for collateral to back these loans. In all other respects, we maintain all the structural characteristics of General Equilibrium analysis, i.e. optimizing behavior, perfectly competitive markets and rational expectations.

We show how an adverse monetary shock in the present can lead to over-in-debtedness

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<sup>1</sup>The origin of this view can be traced back to Fisher (1933). His analysis is based on two fundamental conditions, over-indebtedness and deflation. He argued that over-indebtedness can precipitate deflation in future periods and subsequently liquidation of collateralized debt and bankruptcy, which can lead to fire sales suppressing the value of the collateral even further. Hence, the initial deflationary pressures are exacerbated and they precipitate to even higher default, and, ultimately, to lower output.

and future deflation that in some state of the world can result in default, debt liquidation, reallocation of capital and finally reduction in GDP. Market incompleteness is central to our analysis, since agents cannot write comprehensive contracts and hedge the possibility of default. We consider a two period economy populated by entrepreneurs, who both consume and produce, and show under what conditions the system can move to a state which is characterized by defaults on collateralized loan obligations. Agents engage into long-term borrowing to buy the productive assets, which they pledge as collateral to secure their loan. The decision to default is endogenous and depends on the difference between the value of the collateral and the loan as in Geanakoplos (2003). We introduce money to emphasize how a nominal shock, and not only a productivity shock, can lead to financial fragility and a reduction in GDP.

Our result can be summarized as follows. Consider the case where debt is fully collateralized and entrepreneurs never default, since the nominal value of their contractual obligation is less than the value of their pledged collateral. Assume also that there is an adverse future state of the economy whereby the value of their debt is equal to the value of their collateral, and, thus, the borrower is indifferent between default and fully repaying his loan. In other words, he is on-the-verge of defaulting. Any further adverse shock in the economy that reduces further the value of his collateral will inevitably provide him with an incentive to default, since the benefit from defaulting will exceed its cost. In such a situation, the impact of the real economy becomes evident. When the entrepreneur defaults he loses the capital asset he has pledged as collateral and, therefore, his production will decrease. Subsequently, he needs to attract new capital in the market under more stringent financial conditions. The upshot of the argument is that this process may lead to productive inefficiency due to capital reallocation to firms that are not debt or liquidity constrained, yet their marginal product of capital is lower. We refer the reader to Gilchrist et al. (2013) for an empirical assessment of the magnitude of the loss in aggregate resources due to such misallocation and for a review of the related literature.

Our work relates to the strand of literature that argues that the financial crisis and in particular defaults on financial contracts can lead to economic recessions. Bernanke and Gertler (1989) and Bernanke et al. (1999) model a credit constraint, arising from costly state verification, whereby the firm is only able to obtain collateralized loans and the amount of credit to the firm shrinks in the presence of deflationary pressures on the prices of its assets. This introduces an external finance premium, which increases with a decrease in the relative price of capital. In turn, an increase in the cost of capital will result in a decrease in the marginal product and a reduction in GDP. They show that GDP and investment do not only depend on the fundamentals and productivity, but also on the soundness of the firms' financial situation, which is an important source of financial instability. We argue in this paper that informational asymmetries are not the crucial element for the financial situation of firms to result in GDP contraction. Instead, the possibility of positive default and asset liquidation provide the genesis of a chain reaction that weakens the stability of the financial system and results into lower production.

Our approach is also related to the work on the debt deflation theory of Sudden Stops (Mendoza (2006), Mendoza (2010), and Mendoza and Smith (2006)). These papers introduce collateral constraints similar to Kiyotaki and Moore (1997) in an RBC model of a Small Open Economy to show that when debt is sufficiently high, an adverse productivity shock triggers the constraints and results in a fire-sales spiral, falling prices and a reduction in output. Our results point to the same direction, though contrary to them we consider a monetary economy with nominal contracts and focus on monetary shocks, which have not been thoroughly studied in the literature. In addition, they do not allow for the possibility of default. The latter is crucial for our analysis, since it is the reason that capital gets reallocated to result in inefficient production. Due to fully flexible nominal prices, monetary policy only affects the price level in the final period and not the total output in the absence of default.<sup>2</sup> However, default makes credit conditions more adverse and capital is

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<sup>2</sup>Eggertsson and Krugman (2012) show that when prices are sticky, deleveraging and deflation will still affect output due to a reduction in aggregate demand.

not allocated efficiently.

We contribute to the aforementioned papers by studying the effect of nominal loan contracts on the propagation of shocks and output. Importantly, Bernanke et al. (1999) focus on real contracts and argue that the modeling of nominal ones is an important step for future research. In our work, nominal long-term loans play a crucial role, since their face value is invariant to deflationary pressures, while the value of collateral that backs them is not.

Our framework is rich enough to analyze productivity shocks as the cause of debt deflation. A number of papers model fire-sales due to adverse productivity or funding shocks to capture debt-deflationary effects on asset prices leading to loss spirals and financial instability.<sup>3</sup> However, we choose to focus on the monetary channel, since it is the least explored in the literature. In our model, default on the collateralized loan due to a fall in the value of collateral, exacerbated the debt-deflation dynamics leading to further price declines. Agents do not face additional borrowing constraints and the drop in output is due to an inefficient reallocation of capital. The introduction of funding constraints as in the papers above, would exacerbate the channel that we describe.

The rest of the paper proceeds as follows. Section 2 presents the model, while section 3 discusses the equilibrium. Section 4 describes how monetary policy can cause default and how the latter results in higher borrowing costs, capital reallocation and lower output. Section 5 concludes. The proofs are relegated to the Appendix.

## **2 The Model**

We build a general equilibrium model where two types of agents interact to produce a consumption good. Agents are considered to be entrepreneurs, who both produce and consume the same good. Production happens through the utilization of another capital

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<sup>3</sup>For example, Shleifer and Vishny (1992), Gromb and Vayanos (2002), Fostel and Geanakoplos (2008), Lorenzoni (2008), Brunnermeier and Pedersen (2009), Adrian and Shin (2009), Korinek (2011).

good, from which agents derive no utility at any point in time. Nonetheless, its price will be always positive in the beginning of every period, since it is essential for the production of the consumption good from which agents derive utility. An important consequence of default is the reallocation of resources. The agent that defaults loses the pledged collateral, which is put for sale in the market. Heterogeneity is an important factor, since it is the reallocation of collateral that results in lower aggregate output. In a general equilibrium framework, the market for the capital good clears and all capital will be used for production. Total output depends on the efficient use of capital, which means any additional unit should end up to the agent with the higher marginal productivity. However, in the presence of financing frictions capital may end up with the least productive agent, since he may not face these frictions.

We assume that production takes time and receipts from the sale of goods are not immediately available. This creates the need for a short term funding market, which bridges the gap between expenditures and receipts from sales. Implicitly, agents cannot directly trade the capital good against the subsequent production of the consumption good and they cannot not write their own IOUs to facilitate these transactions. Instead, they need to hold money, which is modeled as credit from the central bank, to transact in the capital and consumption goods' markets. The transaction demand for money motive naturally emerges from the cash-in-advance constraint. Since capital is a durable good, in view of the inherent moral hazard problem of honoring long-term debt obligation, agents are required to pledge the capital they purchase as collateral. Finally, the introduction of uncertainty is crucial, since under certainty there would be no default. Without loss of generality, we allow for default in only some realizations in the future.

The possibility of default on the contractual obligations that an agent undertakes underscores the necessity for our cash-in-advance constraints. The interplay of liquidity and default justifies fiat money as the stipulated mean of exchange. Otherwise, the mere presence of a monetary sector without any possibility of endogenous default or any other fric-

tion in equilibrium would become a veil without affecting real trade and, eventually, final equilibrium allocation. Indeed, cash-in-advance constraints are the minimal institutional arrangement to capture a fundamental aspect of liquidity and how it interacts with default to affect the real economy.

To sum up, our minimal modeling characteristics are agents' heterogeneity, consumption and a durable capital good, a collateralized long-term loan and short-term loan markets, flexible prices, a monetary economy, uncertainty and incomplete markets. Even though complexity increases with the introduction of these characteristics, we are able to solve the model in closed-form and derive analytical results for our thesis. We now describe the model in a more rigorous manner.

## 2.1 The Economy

We consider a two-period monetary general equilibrium model with production, where agents know the present ( $t = 0$ ) but face an uncertain future ( $t = 1$ ), when nature chooses one of the states of the world  $s \in S = \{1, 2\}$  with probability  $\pi_s$ . State 1 and 2 are otherwise the same except that there is a lower short-term money supply by the central bank in state 2 than in state 1<sup>4</sup>. Let  $S^* = \{0\} \cup S$  be the set of all states. There are two goods in the economy. Good 1 is a commodity and is perishable. Good 2 is a capital good and is durable. Two heterogeneous agents,  $a$  and  $b$  trade these two goods. Agent  $a$  has an endowment  $e \in R_{++}$  of the capital good at  $t = 0$ , while the poor agent  $b$  has zero endowment of the capital good at every point in time. Agents are not endowed with the commodity good, but rather use capital to produce it. Agents obtain utility from consuming the commodity, while the capital has no consumption value and is only used for production. Let  $x_{s^*}^h$  be the consumption of commodity in state  $s^*$  by agent  $h \in H$ . For the purpose of finding a closed

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<sup>4</sup>We can consider this as a monetary shock. This is the only source of uncertainty in the model. Alternatively, we could have distinguished the two states via a productivity shock. What matters is that there is some fundamental uncertainty between the two states, thus our results would be qualitatively the same under a productive shock as well.

form solution, we assume a logarithmic utility function  $v(x_{s^*}^h) = \ln(x_{s^*}^h) : R_+ \rightarrow R, \forall s^* \in S^*, h \in H$ . Let  $y_{s^*2}^h$  be the capital good held by agent  $h$  at the end of state  $s^*$ . Both agents have Cobb-Douglas production functions given by  $F_{s^*}^h(y_{s^*2}^h) = A_{s^*}^h (y_{s^*2}^h)^\sigma : R_+ \rightarrow R, \forall s^* \in S^*, h \in H$ , where  $A_{s^*}^h$  is the total factor productivity and  $\sigma$  is the output elasticity of capital. Production takes place within each period. Without loss of generality, we let both states occur with equal probability (i.e.  $\pi_1 = \pi_2 = 1/2$ ), and assume  $A_{s^*}^h = 1, \forall s^* \in S^*, h \in H$ ,  $\sigma = 0.3$  and  $e = 2$ .

## 2.2 Money, Short-term Money Markets, and Money Storage

Money in our model is the stipulated means of exchange and a store of value. We introduce it through cash-in-advance constraints, such that an agent can purchase either the capital or the commodity in the relevant markets only by paying in money. Although money is fiat and has no intrinsic (consumption) value, it has value because it is essential for the conduct of transactions in the goods' markets. Agents cannot print their own money and they have to borrow it from the Central Bank, which intervenes directly in the short-term and long-term money markets. In particular, when the central bank purchases intra-period bonds within each state of the world, it injects a quantity of money  $M_{s^*}, \forall s^* \in S^*$  into the system. Moreover, when the central bank extends a collateralized loan at  $t = 0$ , it injects a quantity of money  $\bar{m}$  into the system<sup>5</sup>. Money exits the system when agents repay their short-term and long-term loan to the central bank. At the end of period 2 all money will exit the system, since it has no consumption value for any agent.

For  $s^* \in S^*$ , let  $\mu_{s^*}^h$  be the amount of fiat money that agent  $h$  chooses to owe in the short-term money market and  $r_{s^*}$  be the short-term interest rate. From market clearing, we have

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<sup>5</sup>Collateralized long-term loan extension is not an unusual function of modern central banks especially in the aftermath of the 2007 financial crisis. Alternatively, one could think of government sponsored institutions, which extend collateralized loans, e.g. Freddie Mac or Fannie Mae in the case of mortgages. Abstracting from a competitive optimizing banking sector allows us to focus on the effects of credit extension/money supply by the central bank on default and output. However, by doing so we cannot derive conclusions about financial fragility and the possibility of credit crunches, which issues are kept for further research.

that  $1 + r_{s^*} = \sum_{h \in H} \mu_{s^*}^h / M_{s^*}$ . Thus, the ratio of *nominal* value of loans over the central bank's credit extension determines the gross nominal interest rate. The amount of fiat money that each agent  $h$  borrows is  $\mu_{s^*}^h / (1 + r_{s^*})$ . Agents do not default in the short-term money markets. since there is no uncertainty about their production within each period and their short-term borrowing can be fully collateralized.

The only way for agents to transfer money across periods is through a money storage technology, potentially offered by the central bank. Agent  $a$  may store  $d$  amount of money in the beginning of  $t = 0$  so that he will be able to use it at  $t = 1$ .<sup>6</sup> We assume that the money storage technology is only available at the beginning of  $t = 0$ , not in the end of  $t = 0$ .

## 2.3 Commodity and capital good markets

Denote by  $p_{s^*1}$  the price of the commodity and  $p_{s^*2}$  the price of capital in  $s^* \in S^*$ . These are taken as given by both agents to maintain price-taking behavior. Let  $b_{s^*1}^h$  and  $b_{s^*2}^h$ ,  $\forall h \in H$ , be the amount of fiat money spent by agent  $h$  to trade in the commodity and capital goods' markets in state  $s^* \in S^*$ . In addition, let  $q_{s^*1}^h$  and  $q_{s^*2}^h$  be the amount of commodity and capital offered for sale in state  $s^* \in S^*$  by  $h$ . In equilibrium, at positive levels of trade,  $0 < p_{s^*1} = \sum_{h \in H} b_{s^*1}^h / \sum_{h \in H} q_{s^*1}^h < \infty$ , and  $0 < p_{s^*2} = \sum_{h \in H} b_{s^*2}^h / \sum_{h \in H} q_{s^*2}^h < \infty$ . Note that agents cannot sell commodities or capital goods they do not own.

The amount of capital good held by agent  $a$  at the end of  $t = 0$  is  $y_{02}^a = e - q_{02}^a$ , while in state  $s$  it is  $y_{s2}^a = e - q_{02}^a - q_{s2}^a$ .<sup>7</sup> The amount of capital good held by agent  $b$  at the end of  $t = 0$  is  $y_{02}^b = b_{02}^b / p_{02}$ , while in state  $s$  agent  $b$ 's final holdings depend on whether he defaults on the collateralized loan or not, which is discussed in the following section.

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<sup>6</sup>In our model only agent  $a$  stores money intertemporally. However, the arrangement described here applies to agent  $b$  as well; the same goes for the next section where we only describe agent  $b$  taking out a collateralized loan.

<sup>7</sup>We have modeled agent  $a$  selling the capital good in both periods. In the initial period, this is always true since he is the only one endowed with it. However, it may well be the case that he buys back some capital in the second period. If this was the case  $q_{s2}^a$  would be negative and the cash-in-advance constraints would need to be adjusted accordingly. Note that this does not affect the results of our thesis.

As mentioned, all transactions are intermediated through the use of fiat money, i.e. the proceeds from commodity sales in state  $s^*$  cannot be used to purchase the capital good directly, and vice versa. This institutional arrangement is a fundamental feature of a model that captures the importance of liquidity constraints and generates a transaction demand for fiat money. We have chosen to introduce money in our model through cash-in-advance constraints as it is methodologically convenient and captures the way goods prices are determined through the Quantity Theory of Money (QTM), whereby both prices and quantities are affected when monetary variables change.<sup>8</sup> Cash-in-advance constraints should be viewed as liquidity constraints that distinguish goods from liquid wealth.

An alternative way to introduce a demand for money, is by incorporating money balances in the utility and production function. Stein (2012) considers such a model where banks engage in money creation and show that this can lead to financial instability due to fire-sales. When banks try to retain the riskless character of their IOUs, they will need to liquidate a part of their portfolio in bad realizations. Although in his model prices are flexible, monetary policy can play a role through controlling money creation. The reason is that money enters as an input in the objective function of both households and firms function. In our framework, the only role for money aggregates is to determine the price level of goods through the QTM. A change in the quantity of money will have **no** real effect on output in the final period if agents choose not to default on their long-term obligations. The only effect would be an adjustment in prices, since prices are fully flexible. However, the money stock in the initial period affects the investment decision by agent b. This is another financing friction due to the fact that the long-term loan needs to be backed by collateral. Given the scarcity of collateral, a change in  $M_0$  will affect investment decisions. Monetary policy has real effects, when deflationary pressures due to a lower money supply induces

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<sup>8</sup>The methodology is close to Dubey and Geanakoplos (2006), Tsomocos (2003) and Goodhart et al. (2006), who introduce cash-in-advance constraints to examine the interaction between liquidity and default and analyze financial stability. However, only Goodhart et al. (2010) examine the interaction between money and collateral values in the case of mortgages.

agents to default after a certain point, which results in a reallocation of resources.<sup>9</sup> We call this the **debt deflation channel** of monetary policy, which is described in detail in section 4. We discuss the endogenous decision of agents to default in the following section. Recapitulatingly, our debt deflation channel is initiated via positive default, thus emphasizing the important interconnection of liquidity and default. Consequently, the externality induced by positive default leads to inefficient capital allocation and investment in the economy.

## 2.4 Default and Collateralized Loan

In the initial period, agent  $b$  finances his investment in the capital good both through short-term and collateralized borrowing. When he borrows from the collateralized loan market,<sup>10</sup> he pledges the capital purchased as collateral. In the second period, the borrower either delivers in full the amount of the collateralized loan or defaults. In the case of default, the collateral pledged is foreclosed and is put for sale in the secondary capital market. The receipts are transferred to the central bank and determine the effective return on the collateralized loan.

Formally, at  $t = 0$ , agent  $b$  takes out a collateralized loan to finance the purchase of the capital good. The interest rate is  $\bar{r}$  and he promises to payback  $\bar{\mu}$  in the next period. The collateralized loan extension is therefore  $\bar{m} = \bar{\mu}/(1 + \bar{r})$ , since the credit extension is  $\bar{m}$ . He spends  $b_{02}^b \leq \bar{\mu}/(1 + \bar{r}) + \mu_0^b/(1 + r_0)$ <sup>11</sup> amount of money to purchase  $b_{02}^b/p_{02}$  amount of the capital good, which he then pledges as collateral. We denote by  $C$  the

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<sup>9</sup>An advantage of our model is that it yields closed-form results. Hence, we are able to identify clearly the propagation mechanism and present the unfolding of events, through which monetary policy affects the decision to default and subsequently the allocation of capital and total output. To that extent we do not engage in a detailed discussion of optimal monetary policy, but rather propose default as an additional channel for affecting aggregate output.

<sup>10</sup>As mentioned the price of the capital good will be higher than the the proceeds for goods sales, since it is durable and can be used for production in the second period as well. Thus, agent  $b$  will partially finance his capital good's purchases through short-term borrowing or equivalently his income from goods sales within the same period, and partially through a long-term loan agreement.

<sup>11</sup>The ratio  $\frac{\mu_0^b/(1 + r_0)}{b_{02}^b}$  determines the margin on the collateralized loan, i.e. how much individual resources agent  $b$  has to utilize to purchase the capital good. The lower the margin, the easier for the agent to purchase capital by using it as collateral.

amount of collateral pledged in terms of units of the capital good, *i.e.*  $C = b_{02}^b/p_{02}$ . Thus the collateralized loan is defined by both the interest rate and the collateral requirement. At  $t = 1$ , the agent will deliver  $\min(\bar{\mu}, p_{s2}C)$ . If  $p_{s2}C \geq \bar{\mu}$ , then agent  $b$  does not default on the collateralized loan and delivers the full amount  $\bar{\mu}$ . This is not a naive assumption. Due to our General Equilibrium framework every contract is priced in equilibrium. When equilibrium prices are such that the value of the collateral in the future is less than the amount the agent has to repay, he would rather default, purchase the same amount of capital from the secondary market and be better off.<sup>12</sup> Default is an endogenous decision stemming from utility optimization. Only when equilibrium prices are such that the value of the collateral is higher than the nominal value of the loan will the agent repay fully. This is the debt deflation channel through which monetary policy and money supply matter for the determination of asset prices and they affect the decision to default and aggregate output, which is analyzed thoroughly in section 4.

Moreover, agent  $b$  spends an additional amount of money  $b_{s2}^b$  in the capital market at  $t=1$ , which brings his final capital good's holdings to  $y_{s2}^b = b_{02}^b/p_{02} + b_{s2}^b/p_{s2}$ .<sup>13</sup> When  $p_{s2}C < \bar{\mu}$ , the borrower will give up the collateral  $C$ , which is then sold on the market for  $p_{s2}C$ . He will then spend  $b_{s2}^b$  to purchase the capital good and his holding is  $y_{s2}^b = b_{s2}^b/p_{s2}$ .

## 2.5 Time-structure of the markets

At  $t = 0$ , the short-term (intra-period) money and collateralized loan markets open. Then commodity and capital good markets meet. Agents produce within the period. Settlements of short-term loans occur at the end of each period. Finally, consumption takes place. The same market activities take place at  $t = 1$  in all the states and in addition agent  $b$  repays the

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<sup>12</sup>An implicit assumption is that the agent is not further penalized for defaulting apart from losing the capital good he owns. Given that there is additional punishment, the wedge between the loan and the collateral value has to be higher for him to default. Such an assumption only adds complexity and does not alter the mechanism through which money supply affects the decision to default.

<sup>13</sup>In principle, the agent may choose to sell some of the capital good he owns in the second period. In this case,  $b_{s2}^b$  is negative and the cash-in-advance constraints need to be adjusted accordingly. Again this does not affect the results of our thesis.

collateralized loans or alternatively defaults and the pledged collateral is foreclosed.

## 2.6 Budget sets

Denote the macro variables which are determined in equilibrium, and which every agent regards as fixed, by  $\boldsymbol{\eta} = (\boldsymbol{p}, \boldsymbol{r}, \bar{r}) \in \mathbb{R}_+^{2S^*} \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+$ . Denote  $\boldsymbol{\sigma}^a \in \sum_a(\boldsymbol{\eta})$ , where  $\boldsymbol{\sigma}^a = (\boldsymbol{b}^a, \boldsymbol{q}^a, \boldsymbol{\mu}^a, d) \in \mathbb{R}_+^{S^*} \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+$  and  $\boldsymbol{\sigma}^b \in \sum_b(\boldsymbol{\eta})$ , where  $\boldsymbol{\sigma}^b = (\boldsymbol{b}^b, \boldsymbol{q}^b, \boldsymbol{\mu}^b, \bar{\mu}) \in \mathbb{R}_+^{S^*} \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+$  the vectors of agent  $a$  and  $b$ 's market decisions. Agent  $a$ 's optimization problem is as follows

$$\begin{aligned} \max_{\boldsymbol{\sigma}^a \in \sum_a} \Pi^a &= \ln(x_0^a) + \sum_{s \in S} \pi_s \ln(x_s^a) \\ s.t. B^a(\boldsymbol{\eta}) &= \{\boldsymbol{\sigma}^a \in \sum_a(\boldsymbol{\eta}) : (01^a) - (s2^a)\} \end{aligned}$$

where:

$$\begin{aligned} (01^a) \quad & b_{01}^a + d \leq \frac{\mu_0^a}{1 + r_0} \\ (02^a) \quad & \mu_0^a \leq p_{02} q_{02}^a \\ (s1^a) \quad & b_{s1}^a \leq \frac{\mu_s^a}{1 + r_s} + d \\ (s2^a) \quad & \mu_s^a \leq p_{s2} q_{s2}^a \end{aligned}$$

(01<sup>a</sup>) says that in the beginning of  $t = 0$ , agent  $a$  borrows short-term to purchase commodities and deposits the rest. (02<sup>a</sup>) says that in the end of  $t = 0$ , agent  $a$  repays the short-term loan using the proceeds of capital sales. (s1<sup>a</sup>) says that in the beginning of each state  $s \in S$ , agent  $a$  uses the deposits and short-term borrowing to purchase the commodity. (s2<sup>a</sup>) says that in the end of each state  $s \in S$ , agent  $a$  repays the short-term loan using the proceeds of capital sales. The capital owned by agent  $a$  in the end of each period is  $y_{02}^a = e - q_{02}^a$  and  $y_{s2}^a = e - q_{02}^a - q_{s2}^a$  respectively, as discussed in section 2.3. Note

that agent  $a$  cannot sell more of the capital good than what he initially owns, *i.e.*  $q_{02}^a < e$ ,  $q_{s2}^a < y_{02}^a$ .  $x_{s^*}^a = (y_{s^*2}^a)^\sigma + b_{s^*}^a/p_{s^*}$  is agent  $a$ 's consumption, which is equal to what he produces plus the (net) purchases of the commodity.

Agent  $b$ 's optimization problem is as follows:

$$\begin{aligned} \max_{\boldsymbol{\sigma}^b \in \Sigma_b} \Pi^b &= \ln(x_0^b) + \sum_{s \in S} \pi_s \ln(x_s^b) \\ \text{s.t. } B^b(\boldsymbol{\eta}) &= \{\boldsymbol{\sigma}^b \in \Sigma_b(\boldsymbol{\eta}) : (01^b) - (s2^b)\} \end{aligned}$$

where:

$$\begin{aligned} (01^b) \quad b_{02}^b &\leq \frac{\mu_0^b}{1+r_0} + \frac{\bar{\mu}}{1+\bar{r}} \\ (02^b) \quad \mu_0^b &\leq p_{01}q_{01}^b \\ (03^b) \quad C &= \frac{b_{02}^b}{p_{02}} \\ (s1^b) \quad \bar{\mu} + b_{s2}^b &\leq \frac{\mu_s^b}{1+r_s} \quad \text{if } b \text{ does not default in state } s \\ (s1^b) \quad b_{s2}^b &\leq \frac{\mu_s^b}{1+r_s} \quad \text{if } b \text{ defaults in state } s \\ (s2^b) \quad \mu_s^b &\leq p_{s1}q_{s1}^b \end{aligned}$$

(01<sup>b</sup>) says that in the beginning of  $t = 0$ , agent  $b$  enters both a short-term and a collateralized loan to purchase the capital good. (02<sup>b</sup>) says that in the end of  $t = 0$ , agent  $b$  repays the short-term loan using the proceeds of commodity sales. (03<sup>b</sup>) says that agent  $b$  puts all the capital good it bought as collateral for the intertemporal loan. (s1<sup>b</sup>) says that in the beginning of each state  $s \in S$ , agent  $b$  borrows short-term to purchase more of the capital good and also to repay the collateralized loan if he chooses not to default. If he chooses to default, he does not repay the collateralized loan and uses the money borrowed short-term only to purchase capital, since the capital he owned has been seized and put for sale. (s2<sup>b</sup>) says that in the end of each state  $s \in S$ , agent  $b$  repays the short-term loan using

the proceeds of the commodity sales.  $x_{s^*}^b = (y_{s^*2}^b)^\sigma - q_{s^*}^b$  is agent  $b$ 's consumption, which is equal to the amount of the commodity he produces minus what he sells to repay his short-term loan. The capital owned by agent  $b$  in the end of each period is  $y_{02}^b = b_{02}^b/p_{02}$  at  $t = 0$ ,  $y_{s2}^b = b_{02}^b/p_{02} + b_{s2}^b/p_{s2}$  in state  $s$  if  $b$  does not default and  $y_{s2}^b = b_{s2}^b/p_{s2}$  in state  $s$  if  $b$  chooses to default.

### 3 Equilibrium

We say that  $(\boldsymbol{\eta}, (\boldsymbol{\sigma}^h)_{h \in H})$  is a Monetary Collateral Equilibrium (MCE) for the economy

$E\{\mathfrak{v}, e, F; \mathbf{M}, \bar{m}\}$ , iff:

- (i)  $p_{s^*1} = \frac{b_{s^*1}^a}{q_{s^*1}^b}, \quad \forall s^* \in S^*$
- (ii)  $p_{02} = \frac{b_{02}^b}{q_{02}^a}$
- (ii')  $p_{s2} = \frac{b_{s2}^b}{q_{s2}^a}$  if  $b$  does not default in state  $s \in S$
- (ii'')  $p_{s2} = \frac{b_{s2}^b}{q_{s2}^a + C}$  if  $b$  defaults in state  $s \in S$
- (iii)  $1 + \bar{r} = \frac{\bar{\mu}}{\bar{m}}$
- (iv)  $1 + r_{s^*} = \frac{\sum_{h \in H} \mu_{s^*}^h}{M_{s^*}} \quad \forall s^* \in S^*$
- (v)  $\boldsymbol{\sigma}^h \in \underset{\boldsymbol{\sigma}^h \in B^h(\boldsymbol{\eta})}{\operatorname{argmax}} \Pi^h$

Condition (i) says that the commodity market clears. Conditions (ii), (ii') and (ii'') say that the capital good markets clear for all  $s^* \in S^*$ . Condition (iii) says that the collateralized loan market clears. Condition (iv) says that the short-term money markets clear. Condition (v) says that both agents optimize. In sum, all markets clear, expectations are rational, i.e. future prices and interest rates are correctly anticipated, and agents optimize given their budget sets.

Since agent  $b$  is not endowed with capital and agent  $a$  has a decreasing returns to scale production function, there will always exist gains from trade. Thus, agent  $a$  will sell part of his capital endowment to  $b$  and subsequently buy back some of  $b$ 's output. We refer the interested reader to Geanakoplos and Zame (2013) for details about the existence of equilibrium.

Hereafter, we analyze two types of equilibria: an equilibrium where there is no default in any state, and an equilibrium where there is default on the collateralized loan in one state in the second period. We show that there is a threshold such that the former equilibrium obtains if the money supply at  $t=0$  is higher than the threshold, while the latter obtains otherwise. We consider that the budget constraints of agents are always binding in equilibrium, i.e., agents do not hold idle cash within each period. To guarantee this, we postulate that there is an infinitely small cost when agents hold money within each period, which could be rationalized as the fee of maintaining checking accounts. Hence, agents will use all the borrowed funds to trade in the goods' markets and the budget constraints  $B^h(\boldsymbol{\eta})$  are always binding.

## 4 Debt Deflation Channel of Monetary Policy

The main objective of this paper is to characterize the **debt deflation channel** of monetary policy. As already mentioned, we want to examine the way that the money stock matters for the aggregate output level. Given that we have abstracted from any other financial frictions apart from default and since prices are flexible, for the most part monetary policy will only affect the general price level, while production will be efficient if there is no default. The money supply in the initial period affects the allocation of capital and total output at  $t=0$ , while the latter is maximized at  $t=1$  given that no default occurs in any state of the world. This is the conclusion of proposition 4.5. To reach this conclusion we solve for the agents' optimal production decision in proposition 4.3 and show that there is a wedge between the

marginal productivity of the two agents, which is a function of the short-term interest rate. Proposition 4.1 models the term-structure of interest rates and proposition 4.4 solves for the interest rates when there is no default in equilibrium.

We aim to treat debt-deflation as a monetary phenomenon. We show that a lower circulation of money in the first period leads to debt-deflation in the second period. We also show how default leads to an inefficient allocation of capital. Our analysis proceeds in the following three steps. First, we want to examine the relationship between monetary policy in  $t = 0$  and agent  $b$ 's decision to default (Proposition 4). Will a contractionary money supply in the initial period lead to default in the next period? Second, we want to see how the money supply and default lead to a change in interest rates (Proposition 4.6). Due to cash-in-advance constraints, interest rates are the "price" for liquidity, and they play an important role in the allocation of the capital good. Finally, we want to study the effects of interest rate variations on total production in  $t = 1$  due to monetary policy change and subsequent default by agents (Proposition 4.7).

Fiat money is the stipulated means of exchange, and it is exchanged for the acquisition of capital and commodities, while receipts from sales are used to pay back loans and possibly transfer wealth from one period to the other. However, we maintain all the structural characteristics of rational expectations modeling and since money does not enter into the utility function, agents will not hold money idle in the end. All available liquidity will be channeled in the capital and commodity markets at  $t = 1$ . This means that all the central bank money supply (*i.e.*  $M_0, M_1, M_2, \bar{m}$ ) would exit the system via short-term and collateralized loan repayments. This is captured by proposition 4.1. However, at  $t = 0$  due to missing financial markets agents may opt to hold precautionary savings, to hedge against future uncertainty.

**Proposition 4.1. *Term Structure of Interest Rates.*** *At  $t = 0$ , the aggregate money that exits the system is equal to the short-term loan repayment at  $t = 0$  plus any precautionary saving, while the aggregate money that enters the system is equal to the collateralized*

loan extension by the central bank plus the short-term loan credit extension. At  $t = 1$ , the aggregate money that exits the system is equal to the repayment on the short-term and collateralized loans, while the aggregate money that enters the system is equal to the precautionary savings plus the short-term loan extension. Thus,

$$(4.11) \quad M_0 r_0 + d = \bar{m}$$

$$(4.12) \quad M_1 r_1 + \min[p_{12}C, \bar{\mu}] = d$$

$$(4.13) \quad M_2 r_2 + \min[p_{22}C, \bar{\mu}] = d$$

The above proposition shows that the liquidity provision by the central bank and the default decision by agent  $b$  may produce an intricate relationship among interest rates. Two important aims of our paper are to examine how liquidity and default affect interest rates, and how aggregate output fluctuates with interest rate levels.<sup>14</sup>

Nevertheless, our thesis suggests that not only the interest rate, but also the quantity of money are important for the determination of the price and output levels. The quantity theory of money (Proposition 4.2) provides the intuition for the result. Reducing the quantity of money at  $t=0$  does not only affect prices, but also quantities sold, since it has an effect on the ability of the poor in capital agent to leverage up and purchase capital (unlike the representative agent's *sell-all* assumption). This, in turn, affects the price of capital in the second period, since the quantity sold will depend on the stock of the durable good that agents hold from the previous period.

**Proposition 4.2. Quantity Theory of Money Proposition.** *In a MCE, the aggregate income at  $t = 0$ , namely the value of all capital good and commodity sales, is equal to the sum of total short-term credit and collateralized loan extension provided by the central bank minus the precautionary savings. In state  $s$  at  $t = 1$ , if agent  $b$  does not default, aggregate income*

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<sup>14</sup>We consider liquidity to be the ability to borrow in the short-term loan markets. When the interest rate is higher, it is more costly to borrow money and liquidity is lower.

*equals the sum of total short-term central bank money supply and of precautionary savings minus the collateralized loan repayment. If agent  $b$  defaults, aggregate income equals the sum of total short-term central bank money supply and of precautionary savings. The QTM holds for each point in time. In particular, period 0,*

$$p_{01}q_{01}^b + p_{02}q_{02}^a = M_0 + \bar{m} - d$$

*period 1,*

*if agent  $b$  does not default in state  $s$ :*

$$p_{s1}q_{s1}^b + p_{s2}q_{s2}^a = M_s + d - \bar{\mu}$$

*if agent  $b$  defaults in state  $s$ :*

$$p_{s1}q_{s1}^b + p_{s2}(q_{s2}^a + C) = M_s + d$$

## **4.1 Interest Rates and Production**

We first show how individual production varies with the interest rate level (Lemma 4.1) and finally how the latter affects the allocation of capital and aggregate output (Proposition 4.3). We then distinguish between the default and no default cases. Proposition 4.4 solves for the interest rate in the case of no default, whereas proposition 4.6 corresponds to the case where default is present in equilibrium. When there is no default, production will be efficient/optimal in the last period (Proposition 4.5). As already discussed, production will not be optimal at  $t=0$  and will depend on the available liquidity at that point in time, i.e.  $M_0$ . We show later that monetary policy in the initial period can increase aggregate output, but at the same time affect prices as well. It is the credit friction of collateralized loans that allows this relationship to exist. When agent  $b$  chooses to default, capital gets reallocated and production ceases to be optimal even in the last period. This inefficiency of

default is shown in proposition 4.7. The inefficiency stems from a change in interest rates, which creates a wedge between buying and selling capital (Proposition 4.6). In section 4.2 we show how contractionary monetary policy can create debt deflationary pressures in the value of collateral, which result in default in the last period and a reduction in aggregate output.

In the following lemma, we formally examine the impact of money stock on production via interest rate changes. The agent who demands the capital good will purchase it from the agent who is rich in it, financing his purchase partly with short-term credit. A change in the price of short-term credit will have an impact on the trade of capital goods, thus it will affect the allocation of the capital good and output.

**Lemma 4.1. *Relative prices, allocations and short-term interest rates.*** *For agent b who borrows in the short-term money market, purchases capital goods and sells commodities, we have:*

at  $t = 0$

$$(4.11^*) \quad \frac{\left[ \frac{1}{x_0^b} B_0 \sigma(y_{02}^b)^{\sigma-1} + \pi_1 \frac{1}{x_1^b} B_1 \sigma(y_{12}^b)^{\sigma-1} + \pi_2 \frac{1}{x_2^b} B_2 \sigma(y_{22}^b)^{\sigma-1} \right]}{\frac{1}{x_0^b}} = \frac{p_{02}(1+r_0)}{p_{01}}$$

at  $t = 1, \forall s \in S$

$$(4.12^*) \quad \frac{\frac{1}{x_s^b} B_s \sigma(y_{s2}^b)^{\sigma-1}}{\frac{1}{x_s^b}} = \frac{p_{s2}(1+r_s)}{p_{s1}}$$

*For agent a who borrows in the short-term money market, purchases commodities and sells capital goods, we have:*

at  $t = 0$

$$(4.13^*) \quad \frac{\left[ \frac{1}{x_0^a} A_0 \sigma(y_{02}^a)^{\sigma-1} + \pi_1 \frac{1}{x_1^a} A_1 \sigma(y_{12}^a)^{\sigma-1} + \pi_2 \frac{1}{x_2^a} A_2 \sigma(y_{22}^a)^{\sigma-1} \right]}{\frac{1}{x_0^a}} = \frac{p_{02}}{p_{01}(1+r_0)}$$

at  $t = 1, \forall s \in S$

$$(4.14^*) \quad \frac{\frac{1}{x_s^a} A_s \sigma (y_{s2}^a)^{\sigma-1}}{\frac{1}{x_s^a}} = \frac{p_{s2}}{p_{s1}(1+r_s)}$$

Equation (4.11\*) shows the trade-off between purchasing capital goods and selling commodities. The numerator of the LHS is the marginal utility of agent  $b$  from the use of the durable capital to produce commodities. The denominator of the LHS is the marginal utility of his consumption. The RHS is the relative price of the capital good and commodity, including the interest rate wedge, since the purchase of the capital good is financed by short-term borrowing and thus is costly. The same discussion follows for the other three equations.

The above lemma 4.1 shows that interest rates have intricate effects on the allocation of commodity and capital good, as well as production and final consumption. We are particularly interested in how interest rate variation affects the allocation of capital good and total production, which is examined in the following proposition.

**Proposition 4.3. *Interest Rate's Redistribution Effect on Capital Good.*** *At  $t = 1$ , there is an interest rate wedge between the marginal productivity of agent  $a$  and agent  $b$ .*

$$\frac{B_s \sigma (y_{s2}^b)^{\sigma-1}}{A_s \sigma (y_{s2}^a)^{\sigma-1}} = (1+r_s)^2$$

*The change of  $r_s, s \in S$  is positively related to the change of  $y_{s2}^a$  and negatively related to the change of  $y_{s2}^b$ .*

Both agents produce using the capital good. Agent  $a$ , who is rich in it, does not need to purchase any capital and thus avoids the financing cost. Agent  $b$ , who purchases capital, borrows short-term and has to pay the financing cost. The interest rate acts as a wedge between the marginal productivities of the two agents. In other words, there is a financing premium. When the interest rate increases, it is more expensive for agent  $b$  to purchase the capital good. An increase in the marginal productivity of agent  $b$  is needed to compensate for the higher financing cost, otherwise it would not be profitable to purchase an additional

unit. Due to a concave production function, this results in a lower capital input for agent  $b$ . Since the total amount of the capital good is fixed in the economy, agent  $a$  will hold more of it after an increase in the interest rate. Proposition 4.3 shows that an increase in the interest rate in state  $s$  will redistribute capital from the (initial) buyer to the (initial) seller due to the interest rate wedge between the marginal productivity of the seller and the buyer.

Hence, the level of the interest rate determines the allocation of capital. When agent  $b$  does not default on his obligations, all interest rates are zero as shown in the following proposition.

**Proposition 4.4. *Interest Rates under no default.*** *When agent  $b$  does not default on the collateralized loan, the interest rates on short-term loans and the collateralized loan are all equal to zero, i.e.  $r_{s^*} = 0, \forall s^* \in S^*$  and  $\bar{r} = 0$ .*

This proposition says that, if there is no default, all interest rates and the collateralized loan rate are zero, even if the central bank alters the money supply. This is contradictory with reality where money supply has an inverse relationship with interest rate. The intuition is as follows. In the end of  $t = 0$  both agents will repay all their short-term debts in full. In the end of  $t = 1$  both agents will repay all their short-term loans and the collateralized loan fully. The total amount of repayment in the two periods, including principal and interest, is  $M_0(1 + r_0) + M_s(1 + r_s) + \bar{m}(1 + \bar{r})$ . The total amount of money available for them to repay (i.e. all the money available in the system) is equal to total amount of money supply injected by the central bank, i.e.  $M_0 + M_s + \bar{m}$ . In the absence of default, only when all interest rates and the collateralized loan rate are zero will the money available be sufficient for agents to fulfill their obligations. Since they do not have monetary endowment themselves, the only way possible to repay each loan is to pay back an amount exactly equal to what they borrowed.

Given that short-term interest rates are zero, we can conclude from proposition 4.3 that production is efficient and total output is maximized in the last period.

**Proposition 4.5. *Optimal Production in the Absence of Default.*** *Assume  $A_s = B_s$ . If agent*

*b* does not default on the collateralized loan, the production in the economy is optimized at  $t = 1$ .

Due to cash-in-advance constraints, agent *b* who is short in capital in period  $t = 1$  needs to borrow short-term to finance additional purchases of capital. When there is no default and the interest rate is zero, there is no financing cost and the economy allocates capital efficiently. In other words, there is no interest rate wedge between the marginal productivities in the last period. The marginal productivities of the two agents are therefore the same, which results in optimal production and maximum aggregate output. Otherwise, it is always welfare improving to transfer some of the capital from one agent to the other. This is not the case in the initial period regardless of the zero interest rate wedge. As it will be more obvious in section 4.2, the money stock at  $t=0$  affects the quantity of the capital good sold due to the financing friction introduced by the need for collateral.

We now turn to the determination of the interest rates under the presence of default and show the inefficiency in production that default yields.

**Proposition 4.6. *Interest Rates Under Default.*** *Consider an equilibrium in which agent *b* defaults on the collateralized loan in state 2, but not in state 1. Then, the short-term interest rate in state 2 is positive, i.e.  $r_2 = \frac{\bar{m} - p_{22}C}{M_2} > 0$ , while the short-term interest rates at  $t = 0$  and in state 1, and the collateralized loan rate are all equal to zero, i.e.  $r_0 = 0$ ,  $r_1 = 1$  and  $\bar{r} = 0$ .*

We can see that when agent *b* defaults in state 2 (and does not do so in state 1), the short-term interest rate is no longer zero. Agent *b* defaults and the collateral is foreclosed and sold. The proceeds go to the central bank as a form of repayment. However, this repayment is not in full, so there is some money left in the system. As we discussed above, in the end all money will exit the system, hence the extra money left in the system will exit as an additional interest payment for the short-term credit provided by the central bank. The intuition is that when agent *b* decides to default, the central bank cannot do

anything except foreclosing the collateral, which is less valuable than the full payment of collateralized loan. To compensate for the money lost in the collateralized loan extension, the central bank will charge a positive interest rate on the short-term credit as a penalty for default. We now show the inefficiency that a positive interest rate brings in production due to default.

**Proposition 4.7. *Suboptimal Production in the Presence of Default.*** *When agent  $b$  defaults on the collateralized loan, production in the economy is not efficient.*

We can see that after default, although all the capital good is still fully utilized, it is not allocated in an optimal way. Due to a positive financing cost, capital is no longer allocated efficiently. The positive interest rate acts as a wedge between the two agents' marginal productivities, so that agent  $a$  has a lower productivity than agent  $b$ , or agent  $b$  ends up holding less capital good than agent  $a$ . It is welfare improving to transfer some capital from agent  $a$  to agent  $b$ , since  $b$  has a higher marginal productivity. The total production in the economy is reduced due to the inefficiency that default brings along.

## 4.2 Contractionary Monetary Policy and Default

In this section we study the endogenous decision to default and examine when agent  $b$  decides to default on the collateralized loan. It is a market consensus that agent will default and surrender the collateral when the value of collateral is lower than the value of loan. The following proposition offers economic intuition on agent's decision on default.

**Proposition 4.8. *Marginal Decision of Default*** *Agent  $b$  will marginally default on the collateralized loan when the marginal gain from default equals to the marginal loss from default. Formally, we have:*

$$\frac{(1+r_s)(\bar{\mu}-b_{s2}^b)}{p_{s1}} \frac{1}{x_{s1}^b} = (C - \frac{b_{s2}^b}{p_{s2}}) B_s \sigma (y_{s2}^b)^{\sigma-1} \frac{1}{x_{s1}^b}$$

The LHS is the marginal gain from default. If the agent  $b$  defaults on the mortgage loan, then it does not pay  $\bar{\mu}$  and it will spend  $b_{s2}^b$  to purchase some capital good after foreclosure. So agent  $b$  will end up having an increment of money amount equals to  $(\bar{\mu} - b_{s2}^b)$ . Since cash-in-advance constraint is assumed, agent  $b$  needs to borrow  $(\bar{\mu} - b_{s2}^b)$  less amount of short-term credit to repay the inter-temporal loan. This means that it will sell  $(1 + r_s)(\bar{\mu} - b_{s2}^b)/p_{s1}$  less amount of commodity to repay the loan. This means that it will have an incremental utility of  $(1/x_{s1}^b)(1 + r_s)(\bar{\mu} - b_{s2}^b)/p_{s1}$ .

The RHS is the marginal loss from default. In default, agent  $b$  losses the collateral  $C$  and then buys back some  $b_{s2}^b/p_{s2}$  amount of capital. However, he cannot buy back all of them, so he has  $(C - b_{s2}^b/p_{s2})$  amount less of capital, evaluated at the marginal utility obtained from the commodity produced by capital good  $B_s \sigma (y_{s2}^b)^{\sigma-1} (1/x_{s1}^b)$ .

One of the purpose of this paper is to study the relationship between the money supply and the default condition. The following provides a formal study on this topic. To simplify the proof, from now on we assume  $A_s = B_s$ ,  $e = 2$ ,  $\sigma = 0.3$ , and  $M_{s^*} > \bar{m}, \forall s^* \in S^*$ . We first derive the necessary conditions for agent  $b$  to default (lemma 4.2) and then show how contractionary monetary policy can lead to this condition. Given the production inefficiency that default brings along (Proposition 4.7), we prove the existence of a suboptimal equilibrium due to debt deflationary pressures in proposition 4.9.

**Lemma 4.2. Default Condition.** *Consider an equilibrium where agent  $b$  does not default on the collateralized loan. We say that he is on the verge of defaulting if  $q_{02}^a = \frac{2\bar{m}}{M_2 + \bar{m}}$  and will start defaulting if  $q_{02}^a < \frac{2\bar{m}}{M_2 + \bar{m}}$ .*

This lemma provides the equilibrium solution for the default condition that an agent will default on the collateralized loan if the collateral is less valuable than the amount of loan. It says that when the capital good sold by agent  $a$  in  $t = 0$  (equivalent to the capital good purchased by agent  $b$  in  $t = 0$ ) is lower than a certain threshold specified by the fundamentals of the economy, then agent  $b$  will default on the collateralized loan. Answering the question whether monetary policy has an impact on the default decision

is equivalent to seeing whether there is a money supply such that  $q_{02}^a$  is smaller than this threshold. This is examined in the following proposition.

**Proposition 4.9. Debt deflation channel of Monetary Policy.** *Consider an equilibrium where agent  $b$  does not default on the collateralized loan. Then,  $\frac{\partial q_{02}^a}{\partial M_0} > 0$ . Also,  $\exists M_0^*$ , such that  $q_{02}^a = \frac{2\bar{m}}{M_2 + \bar{m}} > \frac{2\bar{m}}{M_1 + \bar{m}}$  and for  $M_0 < M_0^*$  agent  $b$  starts defaulting in state 2. Finally, default occurs due to debt deflationary pressures on the price of the collateral, since  $p_{22} = b_{22}^b / (1 - y_{02}^b) = (M_2 - \bar{m}) / (2(1 - q_{02}^a))$  and  $\frac{\partial p_{22}}{\partial M_0} > 0$ .*

This proposition shows that in an initial equilibrium where agent  $b$  does not default in either state, there is a positive relation between the money supply at  $t = 0$  and  $q_{02}^a$ . When the money supply at  $t = 0$  is reduced,  $q_{02}^a$  goes down as well. Also, there is a certain money supply  $M_0^*$  at which  $q_{02}^a$  reaches the default threshold in state 2, but not in state 1 where there is a relatively higher money supply. In another words, agent  $b$  is on the verge of default in state 2. This proposition says that a contractionary monetary policy in  $t = 0$  will lead agent  $b$  into default in state 2.

Since  $\frac{\partial q_{02}^a}{\partial M_0} > 0$ , we can see that when the central bank reduces the money supply in period  $t = 0$ , agent  $b$  purchases less capital. However, agent  $b$  will still borrow the same amount of collateralized loan,  $\bar{m}$ , extended by the central bank. Thus, the same amount of collateralized loan is backed by less capital, or equivalently leverage is higher or the margin is lower. Moreover, we can see that with a lower  $q_{02}^a$ , the price of the capital good in state  $s$ ,  $p_{s2} = b_{s2}^b / (1 - y_{02}^b) = (M_2 - \bar{m}) / (2(1 - q_{02}^a))$ , is lower. To sum up, a lower money supply in  $t = 0$  leads to a lower  $q_{02}^a$  and a lower  $p_{s2}$ . Since the default decision in state  $s$  is given by  $p_{s2}C < \bar{m}$ , which is equivalent to  $p_{s2}q_{02}^a < \bar{m}$ , we can see that a lower  $M_0$  will drive agent  $b$  closer to default. In fact, lemma 4.2 points out that when  $q_{02}^a$  is reduced to a certain point, it will lead agent  $b$  into default in state  $s$ . Proposition 4.9 shows that when the money supply in  $t = 0$  is lower, agent  $b$  is closer to default. When the money supply is reduced to  $M_0^*$ , agent  $b$  is on the verge of defaulting in state 2, since  $q_{02}^a = 2\bar{m} / (M_2 + \bar{m})$ , but agent  $b$  will still be away from default in state 1, since  $q_{02}^a > 2\bar{m} / (M_1 + \bar{m})$ . When the central bank

reduces the money supply even more, then agent  $b$  will start defaulting in state 2.

The above lemma 4.2 and proposition 4.9 show debt-deflation and default as monetary phenomena: a lower circulation of money in the first period leads to debt-deflation in the second period. We proxy the circulation of money with money supply. The debt-deflation here means relative deflation, i.e. a lower ratio of collateral value to the corresponding loan value. It shows that a decreasing money supply by the central bank in the first period leads to a lower ratio of collateral value to the corresponding loan value in the second period monotonically, i.e. there is a positive correlation between the money supply at  $t = 0$  and the ratio of collateral value to loan value at  $t = 1$ . The lower the money supply, the lower the ratio of collateral value to loan value. We coined this term "relative deflation." The lemma 4.2 points out the condition for default. When a money supply is reduced to a certain point, the ratio of collateral value to the loan value is equal to one. If the money supply is reduced further, the value of the collateral is less than the loan value, and the agent finds it profitable to default on the loan repayment. This is what we call a **debt deflation channel** of monetary policy, since in the presence of default capital gets reallocated and aggregate output decreases.

## 5 Conclusion

We build a monetary general equilibrium model with collateral and production and have a formal treatment of the Fisher debt-deflation effects of monetary policy. We see that the usual propositions in a monetary general equilibrium model hold in this model, namely the quantity theory of money and the term structure of interest rate. Since this is a model with production, we also show that money and interest rates have an effect on total production (real output). One important result of this model is that interest rate as the cost of financing has a redistribution effect on investment. When the interest rate is higher, then the capital good will be redistributed from more productive agents to less productive ones.

We argue that Fisherian debt-deflation can be explained as a monetary phenomenon. We examined how a negative shock in money supply in the initial period can lead to default in the second period through over-indebtedness and deflation. It is straightforward that a reduction in the money supply in the second period after the shock hits would result in debt-deflation dynamics and default. On the contrary, we focus on the pre-crisis money supply, presumably when the economy is on a stable path, and we advocate that future default and collateral prices are not independent of current monetary policy. Following Fisher, the two dominant diseases for debt-deflation is too-much debt (in our case high leverage) and subsequent deflation. We show that when the central bank reduces the short-term money supply in the first period, the leverage ratio in that period increases: the agent still borrows the same amount of collateralized loan while put less amount of capital good as collateral. Furthermore, when the initial money supply is reduced, we find that the price of the collateral (i.e. the capital good) is lower in the second period. The higher leverage and deflation are the lower ratio of collateral value to loan value becomes in the second period and this brings the agent closer to default. In fact, we find when the money supply in the initial period is lower than a threshold level, agents will default. If initially the agent does not default and the money supply is close to the threshold, then a small negative money supply shock creates relative deflation and generates default. This suggests that considerations about the price of durables used as collateral should be included in the determination of policy apart from inflation and GDP growth. However, this does not have to be a continuous target, but rather a binary objective monitoring the incentive of agents to default on collateralized loans.

One would imagine that if an economy is at its potential output, then it would not matter significantly whether or not there is default. However, this turns out not to be the case. In our model, the other important result is that after default, the interest rate in state 2 increases significantly, which results in a redistribution of capital good from the more productive firm to the less productive one. The production in state 2 is reduced and deviates from optimal

production. These variations in interest rate, investment and output do not have significant impact in an equilibrium without default. Note that agent's default creates an externality to the economy by driving the short-term interest rate up that finally results into output contraction.

The upshot is that, given all the production factors are fully utilized after debt-deflation, we still manage to show the reduction in production and the misallocation of resources. That is, we allow agent  $b$  to bid for the capital good in the market. Alternatively, had we put  $b$  into bankruptcy and forbid him from any further activities, then all the production factor will be in the hand of agent  $a$  and the adverse effect on total production will be even worse. This is where we differentiate ourselves from Fisher's debt deflation theory. Recall, in his 1933 paper, Fisher considered the extreme case when defaulters get into bankruptcy after debt-deflation. This naturally leads to lower production since agents that default stop producing altogether. However, in our model, we manage to show inefficiency of the debt-deflation without forcing defaulters into bankruptcy. Here, all the production factors are still in use. The externality is that they are not used as optimally as previously. Indeed, due to the higher financing cost, the poor in capital agent produces less than the initially richer. Thus, deflation favors in a sense the "creditor" and harms the "debtor".

In sum, Fisher's debt deflation argument crucially depends on both liquidity and default as it is shown in proposition 4.9. It is precisely the interplay of liquidity and default that activates the default channel that distorts optimal capital investments.

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## Appendix

### Proof to proposition 4.1

*Proof.* From the binding budget constraints  $(01^a)$ ,  $(02^a)$ ,  $(01^b)$ ,  $(02^b)$  and market clear conditions  $p_{01} = b_{01}^a/q_{01}^b$ ,  $p_{02} = b_{02}^b/q_{02}^a$ ,  $1 + \bar{r} = \bar{\mu}/\bar{m}$  and  $1 + r_0 = (\mu_0^a + \mu_0^b)/M_0$ , we have  $M_0 r_0 + d = \bar{m}$ ; In state  $s \in S$ , if  $\bar{\mu} \leq p_{s2}C$ , then agent  $b$  does not default; from the binding budget constraints  $(s1^a)$ ,  $(s2^a)$ ,  $(s1^b)$ ,  $(s2^b)$  and market clear conditions  $p_{s1} = b_{s1}^a/q_{s1}^b$ ,  $p_{s2} = b_{s2}^b/q_{s2}^a$  and  $1 + r_s = (\mu_s^a + \mu_s^b)/M_s$ , we have  $M_s r_s + \bar{\mu} = d$ ; if  $p_{s2}C < \bar{\mu}$ , then agent  $b$  defaults; from the binding budget constraints  $(s1^a)$ ,  $(s2^a)$ ,  $(s1^b)$ ,  $(s2^b)$  and market clear conditions  $p_{s1} = b_{s1}^a/q_{s1}^b$ ,  $p_{s2} = b_{s2}^b/(q_{s2}^a + C)$  and  $1 + r_s = (\mu_s^a + \mu_s^b)/M_s$ , we have  $M_s r_s + p_{s2}C = d$ . To sum up, we have  $M_s r_s + \min[p_{s2}C, \bar{\mu}] = d$ .  $\square$

### Proof to proposition 4.2

*Proof.* The equation for  $t = 0$  comes from combining the binding equation  $(01^a)$  and  $(01^b)$  and market clear conditions  $p_{01} = b_{01}^a/q_{01}^b$ ,  $p_{02} = b_{02}^b/q_{02}^a$ ,  $1 + \bar{r} = \bar{\mu}/\bar{m}$  and  $1 + r_0 = (\mu_0^a + \mu_0^b)/M_0$ . The proof for the other two equations are on the same line.  $\square$

### Proof to lemma 4.1

*Proof.* Equation (4.11\*) comes from combining the first order conditions of agent  $b$ 's optimization problem w.r.t.  $b_{02}^b, \mu_0^b$  and  $q_{01}^b$ , we have :

$$\lambda_{01}^b = \frac{1}{p_{02}} \left[ \frac{1}{x_0^b} B_0 \sigma(y_{02}^b)^{1-\sigma} + \pi_1 \frac{1}{x_1^b} B_1 \sigma(y_{12}^b)^{1-\sigma} + \pi_2 \frac{1}{x_2^b} B_2 \sigma(y_{22}^b)^{1-\sigma} \right], \frac{1}{x_0^b} = \lambda_{02}^b p_{01}, \frac{\lambda_{01}^b}{1+r_0} = \lambda_{02}^b \text{ and } .$$

Likewise, we can get equations (4.12\*), (4.13\*) and (4.14\*) from other first order equations.  $\square$

### Proof to proposition 4.3

*Proof.* From the equations (4.12\*) and (4.14\*), we have  $(B_s/A_s)(y_{s2}^b/y_{s2}^a)^{1-\sigma} = (1+r_s)^2$ . Because  $B_s/A_s$  is fixed and positive, when  $r_s$  increases, we have  $y_{s2}^b/y_{s2}^a$  reduces. Because  $y_{s2}^b + y_{s2}^c = e$ , we have  $y_{s2}^b$  increases and  $y_{s2}^a$  decreases.  $\square$

### Proof to proposition 4.4

*Proof.* Given that the short-term interest rates  $r_{s^*} \geq 0, \forall s^* \in S^*$  and the collateralized loan rate  $\bar{r} \geq 0$ . When agent  $b$  does not default on the collateralized loan in any state  $s \in S$ , then there must be  $p_{s2}C \geq \bar{\mu}$ , with market clearing condition  $1 + \bar{r} = \bar{\mu}/\bar{m}$ , equations (4.12) and (4.13) become

$$(4.41) \quad M_1 r_1 + \bar{m} + \bar{m} \bar{r} = d$$

$$(4.42) \quad M_2 r_2 + \bar{m} + \bar{m} \bar{r} = d$$

From (4.11), we can see  $d \leq \bar{m}$  since otherwise  $M_0 r_0 < 0$ . If  $d < \bar{m}$ , then from (4.41), we have  $\bar{m} \bar{r} + M_1 r_1 = d - \bar{m} < 0$ , which contradicts with the fact that  $\bar{r} \geq 0$  and  $r_1 \geq 0$ . So only  $\bar{d} = \bar{m}$  is possible. Hence we have  $M_0 r_0 = 0$  and  $M_1 r_1 + \bar{m} \bar{r} = 0$ . With the nonnegative interest rates, we can see that  $r_0 = 0, r_1 = 0$  and  $\bar{r} = 0$ . The proof for  $r_2 = 0$  follows the same line.  $\square$

### Proof to proposition 4.5

*Proof.* When agent  $b$  does not default, from proposition 4.3, we know that  $(B_s/A_s)(y_{s2}^b/y_{s2}^a)^{1-\sigma} = 1$ . We can see that  $(y_{s2}^b/y_{s2}^a)^{1-\sigma} = A_s/B_s$ . If the interest rate  $r_s$  is not zero, let  $\hat{y}_{s2}^b$  and

$y_{s2}^{\hat{a}}$  be the capital good owned by agent  $b$  and  $a$  in state  $s$  respectively, then we have  $(y_{s2}^{\hat{b}}/y_{s2}^{\hat{a}})^{1-\sigma} = (A_s/B_s)(1+r_s)^2$ . We have  $(y_{s2}^{\hat{b}}/y_{s2}^{\hat{a}})^{1-\sigma} > (y_{s2}^b/y_{s2}^a)^{1-\sigma}$ . Thus we can see  $(y_{s2}^{\hat{b}}/y_{s2}^{\hat{a}}) < (y_{s2}^b/y_{s2}^a)$ . Since  $A_s = B_s$ ,  $y_{s2}^a + y_{s2}^b = e$  and  $y_{s2}^{\hat{a}} + y_{s2}^{\hat{b}} = e$ , we have  $y_{s2}^{\hat{b}} < y_{s2}^b = e/2$  and  $y_{s2}^{\hat{a}} > y_{s2}^a = e/2$ . So when interest rate is positive, the capital good owned by agent  $b$  is lower than the capital good owned by agent  $a$ , and the productivity of agent  $b$  is higher than the productivity of agent  $a$ . We can always distribute some capital good from agent  $a$  to agent  $b$  to achieve higher total production. When the interest rate is zero, we can see that monetary policy, any of  $M_{s^*}, \forall s^* \in S^*$  or  $\bar{m}$  or the combination of the above, has no impact of the allocation of capital good in  $t = 1$ . The capital good is evenly allocated to the two agents. Since the production function is concave, we can see that evenly distributed capitals good leads to the maximum production in the economy at  $t = 1$ .  $\square$

#### **Proof to proposition 4.6**

*Proof.* The proof for  $r_0 = 0$ ,  $r_1 = 0$  and  $\bar{r} = 0$  follow the proposition 4.4. Since agent  $b$  defaults on the collateralized loan, with the market clear conditions  $1 + \bar{r} = \bar{\mu}/\bar{m}$ , we have  $p_{s2}C < \bar{\mu}^b = \bar{m}(1 + \bar{r}) = \bar{m}$ . From (4.13), we have  $M_2r_2 = \bar{d} - p_{s2}C > \bar{d} - \bar{m} = 0$ , so  $r_2 > 0$ . Q.E.D.  $\square$

#### **Proof to proposition 4.7**

*Proof.* Follows the proof for proposition 4.5.  $\square$

#### **Proof to proposition 4.8**

*Proof.* Since we know agent  $b$  is on the verge of default when  $\bar{\mu} = CP_{s2}$ . Subtracting both sides with  $b_{s2}^b$ , we have  $\bar{\mu} - b_{s2}^b = CP_{s2} - b_{s2}^b$ . Combining the above equation and equation (4.12\*), we get the proof.  $\square$

#### **Proof to lemma 4.2**

*Proof.* When agent  $b$  does not default, from proposition 4.4, we have  $r_{s^*} = 0, \forall s^* \in S^*$  and  $\bar{r} = 0$ . From proposition 4.5, we have  $y_{s2}^a = y_{s2}^b = 1$ . The capital good sold by agent  $a$  in state  $s \in S$  is  $q_{s2}^a = \frac{b_{s2}^b}{p_{s2}} = y_{s2}^b - y_{02}^b = 1 - y_{02}^b$ . And with the binding budget constraints,  $(s1^a)$ ,  $(s2^a)$ ,  $(s1^b)$  and  $(s2^b)$  and market clearing conditions:  $\frac{b_{s2}^b}{p_{s2}} = q_{s2}^a$ ,  $\frac{b_{s1}^a}{p_{s1}} = q_{s1}^b$ ,  $\frac{\bar{\mu}^b}{1 + \bar{r}} = \bar{m}$  and  $\frac{\mu_s^a + \mu_s^b}{1 + r_s} = M_s$ , we have  $b_{s2}^b = \mu_s^a$ ,  $b_{s1}^a = \mu_s^b$ ,  $\mu_s^b = \frac{M_s + \bar{m}}{2}$ ,  $\mu_s^a = \frac{M_s - \bar{m}}{2}$ . In state  $s$ , agent  $b$  will default on the collateralized loan in state  $s$  when  $p_{s2}C < \bar{\mu}$ . From the market clearing condition  $\frac{b_{02}^b}{p_{02}} = q_{02}^a$ , we have  $C = \frac{b_{02}^b}{p_{02}} = q_{02}^a$ . Hence  $p_{s2}C = \frac{b_{s2}^b}{q_{s2}^a} q_{02}^a = \frac{M_s - \bar{m}}{1 - y_{02}^b} q_{02}^a = \frac{M_s - \bar{m}}{1 - q_{02}^a} q_{02}^a < \bar{m}$ , which is equivalent to  $q_{02}^a \leq \frac{2\bar{m}}{M_s + \bar{m}}$ . Q.E.D.

We also derive here the following results which will be important later in the paper:

Since  $y_{s2}^a = y_{s2}^b = 1$ ,  $\sigma = 0.3$ , and  $B_s = 1$ , from equation (4.12\*), we have  $\frac{p_{s2}}{p_{s1}} = 0.3(y_{s2})^{-0.7} = 0.3, \quad \forall s \in S, .$

Since  $p_{s2} = (M_s - \bar{m})/[2(1 - q_{02}^a)]$ , we have  $p_{s1} = p_{s2}/(p_{s2}/p_{s1}) = \frac{M_s - \bar{m}}{0.6(1 - q_{02}^a)}$

$$q_{s1}^b = \frac{\mu_s^b}{p_{s1}} = 0.3(1 - q_{02}^a) \frac{M_s + \bar{m}}{M_s - \bar{m}}$$

$$x_s^a = (y_{s2}^a)^{0.3} + q_{s1}^b = 1 + 0.3(1 - q_{02}^a) \frac{M_s + \bar{m}}{M_s - \bar{m}}$$

$$x_s^b = (y_{s2}^b)^{0.3} + q_{s1}^b = 1 - 0.3(1 - q_{02}^a) \frac{M_s + \bar{m}}{M_s - \bar{m}} \quad \square$$

### Proof to proposition 4.9

*Proof.* Step1: from the first order conditions of agent  $b$ 's optimization problem w.r.t.  $\bar{\mu}$ ,

$b_{02}^b, \mu_0^b$  and  $q_{01}^b$ , we have:

$$\frac{\lambda_{01}^b}{1 + \bar{r}} = \lambda_{03}^b + \lambda_{05}^b,$$

$$\lambda_{01}^b = \frac{1}{p_{02}} \left[ \frac{1}{x_0^b} B_0 \sigma (y_{02}^b)^{\sigma-1} + \pi_1 \frac{1}{x_1^b} B_1 \sigma (y_{12}^b)^{\sigma-1} + \pi_2 \frac{1}{x_2^b} B_2 \sigma (y_{22}^b)^{\sigma-1} \right]$$

$$\frac{\lambda_{01}^b}{1 + r_0} = \lambda_{02}^b$$

$$\lambda_{02}^b p_{01} = \frac{1}{x_0^b}$$

From the first order conditions of agent  $b$ 's optimization problem w.r.t.  $\mu_1^b$ ,  $q_1^b$ ,  $\mu_2^b$  and  $q_2^b$ ,

we have

$$\frac{\lambda_{03}^b}{(1+r_1)} = \lambda_{04}^b$$

$$\lambda_{04}^b p_{11} = \pi_1 \frac{1}{x_1^b}$$

$$\frac{\lambda_{05}^b}{(1+r_2)} = \lambda_{06}^b$$

$$\lambda_{06}^b p_{21} = \pi_2 \frac{1}{x_2^b}$$

Since  $r_{s^*} = 0, \forall s^* \in S^*$ ,  $\pi_1 = \pi_2 = \frac{1}{2}$ ,  $\sigma = 0.3$ , and  $B_{s^*} = 1, \forall s^* \in S^*$  We have

$$\frac{1}{p_{01}} \frac{1}{x_0^b} = \frac{1}{2} \frac{1}{p_{11}} \frac{1}{x_1^b} + \frac{1}{2} \frac{1}{p_{21}} \frac{1}{x_2^b}$$

$$\frac{1}{p_{01}} \frac{1}{x_0^b} = 0.3 \frac{1}{p_{02}} \left[ \frac{1}{x_0^b} (q_{02}^a)^{-0.7} + \frac{1}{2} \frac{1}{x_1^b} + \frac{1}{2} \frac{1}{x_2^b} \right]$$

Let

$$k = \frac{1}{2} \frac{1}{p_{11}} \frac{1}{x_1^b} + \frac{1}{2} \frac{1}{p_{21}} \frac{1}{x_2^b}$$

$$t = 0.3 \left( \frac{1}{2} \frac{1}{x_1^b} + \frac{1}{2} \frac{1}{x_2^b} \right)$$

The above two equations become:

$$(4.91) \quad \frac{1}{p_{01}} \frac{1}{x_0^b} = k$$

$$(4.92) \quad \frac{1}{p_{01}} \frac{1}{x_0^b} = \frac{1}{p_{02}} \left[ 0.3 \frac{1}{x_0^b} (q_{02}^a)^{-0.7} + t \right]$$

Step2: from the results that  $r_{s^*} = 0, \forall s^* \in S^*$  and  $\bar{r} = 0$ , and the budget constraints and

market clearing conditions:  $\mu_0^a = p_{02} q_{02}^a$ ,  $p_{02} = \frac{b_{02}^b}{q_{02}^a}$ ,  $b_{02}^b = \frac{\mu_0^b}{1+r_0} + \frac{\bar{\mu}}{1+\bar{r}}$ ,  $1+\bar{r} = \frac{\bar{\mu}}{\bar{m}}$  and

$1+r_0 = \frac{\mu_0^a + \mu_0^b}{M_0}$ , we have  $\mu_0^a = \frac{M_0 + \bar{m}}{2}$  and  $\mu_0^b = \frac{M_0 - \bar{m}}{2}$ . We have  $p_{02}^a = \frac{\mu_0^a}{q_{02}^a} = \frac{(M_0 + \bar{m})}{2q_{02}^a}$ ,

$p_{01} = \frac{\mu_0^b}{q_{01}^b} = \frac{(M_0 - \bar{m})}{2q_{01}^b}$  and  $x_0^b = (q_{02}^a)^{0.3} - q_{01}^b$ .

Step3: substitute  $p_{01}$ ,  $p_{02}$  and  $x_0^b$  into the two equations (4.91) and (4.92) in step 1. We

have

$$q_{01}^b = \frac{k(M_0 - \bar{m})(q_{02}^a)^{0.3}}{2 + k(M_0 - \bar{m})}$$

$$q_{01}^b = \frac{0.3(q_{02}^a)^{0.3} + (q_{02}^a)^{1.3}t}{(M_0 + \bar{m}) + q_{02}^a t(M_0 - \bar{m})}(M_0 - \bar{m})$$

Combine the above two, we have

$$(4.93)k[0.7M_0 + 1.3\bar{m}] = 0.6 + 2tq_{02}^a$$

$$\text{Let } J = [0.7M_0 + 1.3\bar{m}]$$

Step4: substitute  $k$  and  $t$  into the equation (4.93) and simplify, and let  $h = 1 - q_{02}^a$ , we have a quadratic equations with one unknown,

$$a_1h^2 - a_2h + a_3 = 0$$

$a_1, a_2$  and  $a_3$  are all exogenous variables, where:

$$\begin{aligned} a_1 &= \left( (M_1 + \bar{m})(M_2 + \bar{m})0.3^2 + \frac{1}{2}(M_1 - \bar{m})(M_2 + \bar{m})0.3 + \frac{1}{2}(M_2 - \bar{m})(M_1 + \bar{m})0.3 + \right. \\ &\quad \left. + J\frac{1}{2}(M_2 + \bar{m})0.3 + J\frac{1}{2}(M_1 + \bar{m})0.3 \right) \\ a_2 &= \left( (M_1 + \bar{m})(M_2 - \bar{m})0.3 \left( 1 + \frac{1}{2} \right) + (M_1 - \bar{m})(M_2 + \bar{m})0.3 \left( 1 + \frac{1}{2} \right) + (M_1 - \bar{m}) \right. \\ &\quad \left. (M_2 - \bar{m}) + J\frac{1}{2}(M_2 - \bar{m}) + J\frac{1}{2}(M_1 - \bar{m}) \right) \\ a_3 &= 2(M_1 - \bar{m})(M_2 - \bar{m}) \end{aligned}$$

Step5: We resort to Mathematica to solve the above equations and have two roots  $h_1$  and  $h_2$ . We want to check that there exists an  $M_0^*$  where  $0 < q_{02}^a = \frac{2\bar{m}}{M_s + \bar{m}} < 1$ , which is equivalent to  $0 < h = 1 - q_{02}^a = \frac{(-\bar{m} + M_2)}{(\bar{m} + M_2)} < 1$ . And  $\frac{\partial q_{02}^a}{\partial M_0} > 0$ , which is equivalent to  $\frac{\partial h}{\partial M_0} < 0$ .

Under the restriction  $\bar{m} > 0, M_0 - \bar{m} > 0, M_1 - \bar{m} > 0, M_2 - \bar{m} > 0$ , and  $M_1 > M_2$ . First, we take  $h_1$  and verify that  $h_1$  is positive. And there exists  $\{\bar{m}, M_1, M_0, M_2\}$  such that  $h_1 < 1$ . We then also verify that there exist  $\{\bar{m}, M_1, M_0, M_2\}$  such that  $0 < h_1 = \frac{(-\bar{m} + M_2)}{(\bar{m} + M_2)} < 1$ . Then we also verify that  $\frac{\partial h_1}{\partial M_0} < 0$ .

Under the same restriction, we can also verify that  $h_2$  is positive and there exists  $\{\bar{m}, M_1, M_0, M_2\}$  such that  $h_2 < 1$ . However, there does not exist  $\{M_1, \bar{m}, M_0, M_2\}$  such

that  $h_2 = \frac{(-\bar{m} + M_2)}{(\bar{m} + M_2)}$ . What whatever  $h_2$  might be, agent  $b$  will never be on the verge of default.

□