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Assessing the Effects of the Zero-Interest-Rate Policy through the Lens of a Regime-Switching DSGE Model

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Abstract

Standard dynamic stochastic general equilibrium (DSGE) models assume a Taylor rule and forecast an increase in interest rates immediately after the 2007-2009 economic recession given the predicted output and inflation, contradictory to the extended period of near-zero interest rate policy (ZIRP) conducted by the Federal Reserve. In this paper, I study two methods of modeling the ZIRP in DSGE models: the perfect foresight rational expectations model and the Markov regime-switching model, which I develop in this paper. In this regime-switching model, I assume that, in one regime, the policy follows a Taylor rule, while, in the other regime, it involves a zero interest rate. I also construct the optimal filter to estimate this regime-switching DSGE model with Bayesian methods. I fit these modified DSGE models to the U.S. data from the third quarter of 1987 to the third quarter of 2010, and then, starting from the fourth quarter of 2010, I simulate the U.S. economy forward with and without the ZIRP intervention. I compare the predicted paths of the macro variables, and I find that the ZIRP intervention has a significant effect. The estimated regime-switching model I develop implies a substantial stimulative effect (on average a 0.12% increase in output growth rate and a 0.9% increase in inflation accumulatively over 20 quarters if ZIRP is kept for 6 quarters). The actual path from the fourth quarter of 2010 onward is closer

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to the predicted path derived from the regime-switching model than that generated by the perfect foresight model. The perfect foresight model generates an explosive and spurious rise in inflation. Therefore, the regime-switching model I propose is more appropriate to assess the effectiveness of the ZIRP, which, I find, is indeed effective in stimulating the economy.

JEL codes: E43, E44, E52, E58

Keywords: regime switching, zero interest rate policy, unconventional monetary policy

1 Introduction

In responsze to the 2007-2009 economic recession and the weak recovery that followed, the Federal Reserve has been giving the economy unprecedented support: besides the lending facilities and the large-scale asset purchases, the Federal Reserve decided to keep the target range for the federal funds rate at 0 to 1/4 percent.[1] Bernanke and Reinhart (2004) refer to the commitment to keep interests low (forward guidance) as "unconventional monetary policy," in contrast to conventional monetary policy, which refers to central banks' manipulation of the policy rate (in the United States, the federal funds rate). Standard DSGE models designed to analyze monetary policy and match the macro data well before the crisis must address the challenge of evaluating the Federal Reserve's unconventional policy.

Standard DSGE models, which assume a Taylor rule, often predicts a quick rise of interest rates immediately after a recession.[2] When analyzing the effects of the policy of keeping the interest rates extremely low for an extended period, the standard approach is to estimate a *stochastic* model and then conduct a counterfactual analysis using the perfect foresight rational expectations (hereafter PFRE) solution method (Cúrdia and Woodford, 2011).[3] This method assumes that agents have *perfect foresight* of the path of the future shocks and the interest rates, and rational expectations equilibrium can be solved backwards. As a result, the assumption of perfect foresight for policy analysis inherently conflicts with the assumption of the stochastic model that is used to fit the data. Furthermore, the PFRE model predicts an unrealistic path of macro variables. For example, this model predicts a

[1] The forward guidance has always suggested that the federal funds rate would be low for sometime either date-based or threshold-based. In the January 2014 FOMC meeting, "the Committee also reaffirmed its expectation that the current exceptionally low target range for the federal funds rate of 0 to 1/4 percent will be appropriate at least as long as the unemployment rate remains above 6-1/2 percent, inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee's 2 percent longer-run goal, and longer-term inflation expectations continue to be well anchored."

[2] Reifschneider and Williams (2000), Chung et al. (2011), and Del Negro and Schorfheide (2012).

[3] A detailed description can be found at the online appendix of Chen, Cúrdia and Ferrero (2012).

spurious rise in inflation[4].

How do we reconcile those conflicted assumptions and better predict the distribution of macroeconomic variables? In this work, I propose a Markov regime-switching DSGE model and compare it with the conventional PFRE model. I propose to model the zero interest rate policy (hereafter ZIRP) by a regime-switching monetary policy rule where, in one regime, the policy rates follow a typical Taylor rule, and, in the other regime, the policy rate is set to zero. Also, I solve this regime-switching DSGE model by using the Farmer, Waggoner, and Zha (2011) minimum state variable solution.[5]

In addition, I construct the optimal filters in order to estimate this regime-switching DSGE model with Bayesian methods. I fit the regime-switching and the standard DSGE models to U.S. data from the third quarter of 1987 to the third quarter of 2010. And then, starting from the fourth quarter of 2010, I simulate the U.S. economy forward under two scenarios: the counterfactual scenario with no policy intervention, and under ZIRP for an extended period using either the regime-switching model or PFRE model. To assess the policy's effectiveness, I compare the predicted path of the macro variables (output and inflation) with and without the policy intervention. I find that the ZIRP has a substantial effect and the predicted path of macro variables generated by the regime-switching model is closer to the actual path, while the PFRE model generates an explosive predicted path.

The fundamental difference between these two types of models is how agents' expectations are formulated. In the regime-switching model, at each period agents attach a non-zero probability of exiting the ZIRP regime in the next period despite the Federal Reserve's "extended period" language, because, for example, the simple announcement would be subject

[4] Carlstrom, Fuerst, and Paustian (2012) also make this observation.

[5] One can also use the perturbation method developed by Foerster, A., Rubio-Ramírez, J., Waggoner, D. and Zha, T. (2013). The solutions of those two methods are the same for the parameters in front of the lagged state variables and in front of the shocks. I use Liu, Waggoner, and Zha (2011) to get the regime-switching constant. This approach is different from the regime-switching constant derived from the perturbation method.

to the time inconsistency problem, and is thus not credible. And even after the exit, the agents expect to come back to the ZIRP regime with a non-zero probability in the future. The PFRE assumes that agents believe the Federal Reserve's announcement and have perfect foresight of future interest rates. I argue that the regime-switching model is more appropriate for policy evaluation.

Here, I am looking at this extended period of zero interest rates as a policy choice because the central bank could raise interest rates when output starts growing and the economy improving, as predicted by the Taylor rule.[6] Carlstrom, Fuerst, and Paustian (2012); Cúrdia and Woodford (2010); and Chen, Cúrdia, and Ferrero (2012) also study the effects of a *transient* interest rate peg. Under the assumption of either a deterministic exit or a stochastic exit of the interest rate peg in the previous studies, the policy rate will follow a Taylor rule after the exit and the interest rate peg will never occur again. In the regime-switching model developed in this paper, however, the zero interest rate policy regime is a recurring event. Even at the normal interest rate regime, agents expect to enter a zero interest rate regime in the future with a non-zero probability. Expectations play an important role in the regime-switching model.

An alternative way to look at this persistent period of low interest rates is the zero lower bound (ZLB) problem. A persistent shock drives interest rates below zero for an extended period if we keep following a Taylor rule.[7] A rapidly growing literature on ZLB considers the zero interest rates as a modeling constraint that has to be considered. Global methods include Judd, Maliar, and Maliar (2011); Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012); and Aruoba and Schorfheide (2012). There are also a few short cuts for modeling ZLB, such as Braun and Körber (2011); Adam and Billi (2007); Eggertsson and Woodford (2003); and Christiano, Eichenbaum, and Rebelo (2011). Del

[6] Here the regime-switching is exogenous while ideally it should be endogenous and depend on the macroeconomic condition.

[7] For example a preference shock or a technology shock.

Negro and Schorfheide (2012) describe how to impose zero interest rates via unanticipated or anticipated monetary policy shocks in a DSGE model. Those models focus on modeling how the economy enters an extended period of zero interest rates, while the regime-switching model developed here is more of a model of exiting the zero interest rates.

The rest of the paper proceeds as follows. Section 2 presents a standard DSGE model. Section 3 discusses how to model ZIRP with a regime-switching monetary policy and with the PFRE. Section 4 describes the estimation of the regime-switching model, some basic analysis of parameter estimates, an evaluation of the effects of the ZIRP, and the comparison between the regime-switching model and the PFRE model. Finally, section 5 summarizes the main findings.

2 Models

Except for the monetary policy rule, the model used here is a standard medium-scale DSGE model (Christiano et al., 2005; Smets and Wouters, 2007): Households maximize utility; monopolistic competitive firms hire the labor to produce intermediate goods; and competitive final-goods-producing firms package intermediate goods into a homogeneous consumption good. Finally, the government sets monetary and fiscal policy. To simplify the analysis, I abstract from capital.

2.1 Households

The representative household's objective function is

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s b_{t+s} \left[\frac{\left(\frac{C_{t+s}}{Z_{t+s}}\right)^{1-\sigma}}{1-\sigma} - \frac{L_{t+s}^{1+\nu}}{1+\nu} \right],$$

where β is the discount factor, b_t is a preference shock, C_t is consumption (relative to productivity Z_t, as in An and Schorfheide (2007) to ensure the existence of a balanced growth path with constant relative risk aversion preferences σ), L_t is labor supply, and ν is the inverse of the Frisch elasticity of labor supply.

The time t budget constraint for a household is

$$P_t C_t + B_t \leq R_{t-1} B_{t-1} + W_t L_t + \mathcal{P}_t - T_t, \tag{2.1}$$

where, B_t, are one-period securities purchased at time t that pay a nominal return, R_t, at time $t+1$, P_t is the price of the final consumption good, W_t is the competitive wage, $\mathcal{P}_t$ are the profits distributed by the intermediate goods producers, and T_t are lump-sum taxes.

Let Ξ_t^P represent the Lagrange multiplier for (2.1). The Euler equation for the short-term bonds is

$$\Xi_t^p = \beta \mathbb{E}_t [\Xi_{t+1}^p R_t].$$

2.2 Final Goods Producers

The final good, Y_t, is a composite made of a continuum of intermediate goods indexed by $i \in (0, 1)$

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_f}} di \right]^{1+\lambda_f}. \tag{2.2}$$

The final goods producers buy the intermediate goods on the market, package to Y_t, and sell it to consumers. These firms maximize profits in a perfectly competitive environment. Their problem is:

$$\begin{array}{ll} \max_{Y_t(i)} & P_t Y_t - \int_0^1 P_t(i) Y_t(i) di, \\ \text{s.t.} & Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_f}} di \right]^{1+\lambda_f} . \end{array} \tag{2.3}$$

From the first order conditions:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\frac{1+\lambda_f}{\lambda_f}} Y_t.$$

Combining this condition with the zero profit condition, I obtain the expression for the price of the composite final good:

$$P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\lambda_f}} di\right]^{-\lambda_f}. \tag{2.4}$$

2.3 Intermediate goods producers

Intermediate goods producer i uses the following technology:

$$Y_t(i) = Z_t L_t, \tag{2.5}$$

where Z_t is the technology, and L_t is labor input.

Prices are sticky à la Calvo (1983). Specifically, each firm can readjust prices with a probability $1 - \zeta_p$ in each period. For those firms that cannot adjust prices, $P_t(i)$ will increase at the steady state rate of inflation π. For those firms that can adjust prices, the problem is to choose a price level, $\tilde{P}_t(i)$, that maximizes the sum of the expected discounted profits in all states of the future where the firm cannot adjust the price:

$$\max_{\tilde{P}_t(i)} \mathbb{E}_t \sum_{s=0}^{\infty} \left(\beta\zeta_p\right)^s \Xi_{t+s} \left(\left(\frac{\tilde{P}_t(i)\Pi^s}{P_{t+s}}\right) - w_{z,t+s}\right) \left(\frac{\tilde{P}_t(i)\Pi^s}{P_{t+s}}\right)^{-\frac{1+\lambda_f}{\lambda_f}} Y_{z,t+s},$$

where $\Xi_{t+s} = \Xi^p_{t+s} P_{t+s} Z_{t+s}$, $w_{z,t+s} = \frac{w_{t+s}}{P_{t+s}Z_{t+s}}$, and $Y_{z,t+s} = \frac{Y_{t+s}}{Z_{t+s}}$.

The first order condition for the firm is

$$
\begin{aligned}
0 \;=\; & \tilde{P}_t(i)\mathbb{E}_t \sum_{s=0}^{\infty} \left(\beta\zeta_p\right)^s \Xi_{t+s} \frac{1}{\lambda_f} \left(\frac{\Pi^s}{P_{t+s}}\right)^{-\frac{1}{\lambda_f}} Y_{z,t+s} \\
& -\mathbb{E}_t \sum_{s=0}^{\infty} \left(\beta\zeta_p\right)^s \Xi_{t+s} \frac{1+\lambda_f}{\lambda_f} w_{z,t+s} \left(\frac{\Pi^s}{P_{t+s}}\right)^{-\frac{1+\lambda_f}{\lambda_f}} Y_{z,t+s}.
\end{aligned} \tag{2.6}
$$

Note that all firms that can readjust prices face this identical problem. I will only consider the symmetric equilibrium in which all firms that can readjust prices will choose the same price, so I can drop the i index. From 2.4 it follows that:

$$
P_t = \left[(1-\zeta_p)\tilde{P}_t^{-\frac{1}{\lambda_f}} + \zeta_p \left[\Pi P_{t-1}\right]^{-\frac{1}{\lambda_f}}\right]^{-\lambda_f}. \tag{2.7}
$$

So

$$
1 = (1-\zeta_p)\left(\frac{\tilde{P}_t}{P_t}\right)^{-\frac{1}{\lambda_f}} + \zeta_p \left[\frac{\Pi}{\Pi_t}\right]^{-\frac{1}{\lambda_f}}.
$$

2.4 Government Policies

The monetary policy assumption is taken from Chen et al. (2012). The central bank follows a conventional feedback interest rate rule similar to Taylor (1993), modified to include interest rate smoothing (Clarida et al., 2000) and to use the growth rate of output instead of the output gap (Justiniano et al., 2011):

$$
\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_m} \left[\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} \left(\frac{Y_t/Y_{t-4}}{e^{4\gamma}}\right)^{\phi_y}\right]^{1-\rho_m} e^{\epsilon_{m,t}}, \tag{2.8}
$$

where $\Pi_t \equiv P_t/P_{t-1}$ is the inflation rate, $\rho_m \in (0,1)$, $\phi_\pi > 1$, $\phi_y \geq 0$, and $\epsilon_{m,t}$ is an i.i.d. innovation.[8] In the section (3.1), I will elaborate how to modify the monetary policy rule to

[8] Chen et al. (2012) use the output growth in the Taylor rule, instead of the output gap, to avoid the complication of solving and estimating the system by characterizing the flexible price equilibrium. In

assess ZIRP.

The standard government budget constraint is as follows:

$$B_t = R_{t-1,t} B_{t-1} + P_t G_t - T_t. \tag{2.9}$$

The left-hand side of expression (2.9) is the market value, in nominal terms, of the total amount of bonds issued by the government at time t. The right-hand side is the total deficit at time t, that is, market value plus interest payment of the bonds maturing in that period plus spending G_t net of taxes.

2.5 Exogenous Processes

The model is supposed to be fitted to data on output, inflation, and nominal interest rates. There are four structural shocks in total. The logarithm of the technology follows a random walk with drift:

$$\ln Z_t = \gamma + \ln Z_{t-1} + z_t,$$

where the shock z_t follows a first order autoregressive process (AR(1)):

$$z_t = \rho_z z_t + \epsilon_{z,t}, \epsilon_{z,t} \sim \mathbb{N}\left(0, \sigma^2_{\varepsilon_z}\right).$$

The shock to the discount factor β (intertemporal preference shifter) is also assumed to follow an AR(1) process:

$$\ln b_t = \rho_b \ln b_{t-1} + \epsilon_{b,t}, \epsilon_{b,t} \sim \mathbb{N}\left(0, \sigma^2_{\varepsilon_b}\right).$$

practice, real GDP growth relative to trend is often cited as one of the main indicators of real activity for the conduct of monetary policy.

The government spending is assumed to be an exogenous process:

$$\ln g_t = \rho_g \ln g_{t-1} + \epsilon_{g,t}, \epsilon_{g,t} \sim \mathbb{N}\left(0, \sigma^2_{\varepsilon_g}\right).$$

The monetary policy shock $\epsilon_{m,t}$ is an independent and identically distributed shock.

3 Zero Interest Rate Policy

In this section, I describe two methods of studying the effects of ZIRP in DSGE models: the regime-switching method and the PFRE method. I am going to consider a regime-switching model where, in one regime, the policy rate follows a typical Taylor rule, and, in the other regime, it involves ZIRP. Although the Markov regime switching is imposed to the monetary policy rule before loglinearizing the system, and thus the method begins from first principles rather than adding Markov-switching parameters after linearizing the model, the model is still a *linear* model. Simplifying assumptions in my model may miss some nonlinear interactions between the zero interest rates and the policy functions of the agents, however, I substantially gain tractability. I also construct the optimal filter and likelihood function so that I can fit this model to the macro data including the recent time where the interest rates are maintained near zero for an extended period. This regime-switching model can not only explain the interest rate data, but it also provides a plausible explanation for exiting the zero interest rate policy. It is more of a model for exiting the ZIRP than a model for entering the ZIRP. This regime-switching model offers a tool to conduct forecasts and counterfactual analysis. The other approach to assessing the ZIRP, PFRE, cannot explain the recent episodes of near-zero interest rates. It only asks the counterfactual questions such as what are the effects on the macro variables if the interest rates are pegged at zero for an extended period, and agents have perfect knowledge of this policy experiment? Now I define the regime-switching model more precisely .

3.1 Regime-Switching Policy Rule

Consider a regime-switching policy rule where, in one regime, the federal funds rate follows a Taylor rule while, in the other regime, it sets the interest rates to zero. I will use the Farmer, Waggoner, and Zha (2011) minimum state variable solution method to solve this regime-switching model, and the estimation strategy will be described in section 4. The policy rule is

$$R_t = \left(R_t^*\left(K_t\right)\right)^{1-\rho_R(K_t)} \left[\left(\frac{\pi_t r}{R_t^*\left(K_t\right)}\right)^{\varphi_\pi(K_t)} \left(\frac{Y_t/Y_{t-4}}{e^{4\gamma}}\right)^{\varphi_y(K_t)}\right]^{(1-\rho_R(K_t))} R_{t-1}^{\rho_R(K_t)} \exp\left(\varepsilon_{R,t}\right), \tag{3.1}$$

where all the parameters denoted by (K_t) are regime-dependent. $R_t^*\left(K_t\right)$ are the desired regime-dependent target nominal interest rates. Let $K_t = 1$ denote the normal regime and $K_t = 2$ denote the ZIRP regime. For example, I can set $R_t^*\left(K_t = 1\right) = R_1^* = 1.005$, which corresponds to a target 2% annual interest rate at the normal regime and set $R_t^*\left(K_t = 2\right) = R_2^* = 1.0005$, which corresponds to a target 20 basis points annual interest rate at the second regime. To study the ZIRP, I set

$$\begin{aligned} R_2^* &= 1, \\ \rho_R\left(K_t = 2\right) &= 0, \\ \varphi_\pi\left(K_t = 2\right) &= 0, \\ \varphi_y\left(K_t = 2\right) &= 0, \text{ and} \\ \sigma_{\varepsilon_{R,t}}\left(K_t = 2\right) &= 0. \end{aligned}$$

I define the ergodic mean of the logarithm of the steady state interest rates as

$$\log(R) = \bar{\lambda}_1 \log(R_1^*) + \bar{\lambda}_2 \log(R_2^*),$$

where $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are the ergodic probabilities.

Divide 3.1 by its ergodic mean, R, and thus:

$$\frac{R_t}{R} = \left(\frac{R_t^*}{R}\right)^{(1-\rho_R(K_t))(1-\varphi_\pi(K_t))} \left[\left(\frac{\pi_t}{\pi}\right)^{\varphi_\pi(K_t)} \left(\frac{Y_t/Y_{t-4}}{e^{4\gamma}}\right)^{\varphi_y(K_t)}\right]^{(1-\rho_R(K_t))} \left(\frac{R_{t-1}}{R}\right)^{\rho_R(K_t)} \exp(\varepsilon_{R,t}). \tag{3.2}$$

Loglinearize 3.2 and thus:

$$\begin{aligned} \hat{R}_t &= \rho_R(K_t)\hat{R}_{t-1} + (1-\rho_R(K_t))\left[\varphi_\pi(K_t)\hat{\pi}_t + \varphi_y(K_t)\left(\hat{y}_{z,t}{}^{9} - \hat{y}_{z,t-4} + \sum_{i=0}^{i=3} z_{t-i}\right)\right] \\ &\quad + \varepsilon_{R,t} + (1-\rho_R(K_t))(1-\varphi_\pi(K_t))\hat{R}_t^*(K_t), \end{aligned} \tag{3.3}$$

where the last term represents a regime-switching constant. I am going to apply the trick used by Liu, Waggoner, and Zha (2011). They solve a system where the only regime-switching coefficient is the constant. I can rewrite 3.3 as

$$\begin{aligned} \hat{R}_t &= \rho_R(K_t)\hat{R}_{t-1} + (1-\rho_R(K_t))\left[\varphi_\pi(K_t)\hat{\pi}_t + \varphi_y(K_t)\left(\hat{y}_t - \hat{y}_{t-4} + \sum_{i=0}^{i=3} z_{t-i}\right)\right] + \varepsilon_{R,t} \\ &\quad + (1-\rho_R(K_t))(1-\varphi_\pi(K_t))\left[\log\left(\frac{R_1}{R}\right),\ \log\left(\frac{R_2}{R}\right)\right]\hat{\mathbf{e}}_{s,t}, \end{aligned}$$

where $\hat{\mathbf{e}}_{s,t} = \mathbf{e}_{s,t} - \bar{\mathbf{e}}_s$, and $\bar{\mathbf{e}}_s$ is the ergodic probability. $\mathbf{e}_{s,t}$ is defined as:

$$\mathbf{e}_{s,t} = \begin{bmatrix} \mathbf{1}_{S_t=1} \\ \mathbf{1}_{S_t=2} \end{bmatrix},$$

with $\mathbf{1}\{s_t = j\} = 1$ if $s_t = j$, and 0 otherwise. As shown in Hamilton (1994), the random vector $\mathbf{e}_{s,t}$ follows an AR(1) process:

$$\mathbf{e}_{s,t} = P\mathbf{e}_{s,t-1} + \nu_t, \tag{3.4}$$

where P is the transition matrix of the Markov switching process, and the innovation vector has the property that $\mathbb{E}_{t-1}\nu_t = 0$. In the steady state, $\nu_t = 0$ so that 3.4 defines the ergodic probabilities for the Markov process $\bar{\mathbf{e}}_s$. Schorfheide (2005) also proposes an algorithm to solve DSGE models with a regime-switching constant in the policy rule. One can prove that Schorfheide (2005) and Liu, Waggoner, and Zha (2011) give rise to the same solution.[10]

By adding two extra variables $\mathbf{e}_{s,t}$, I can use Farmer, Waggoner, and Zha's (2011) minimum state variable solution to solve this regime-switching model. The solution of the model can be represented by

$$\begin{aligned} Z_t &= G_t(K_t) Z_{t-1} + R_t(K_t)\varepsilon_t \\ \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} &= \begin{bmatrix} G_{11} & G_{12} \\ 0 & P \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} + \begin{bmatrix} R_{11} & R_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}, \end{aligned}$$

where I can partition the variables Z_t and the shocks ε_t into two parts, respectively. $Z_{2,t}$ are $[\hat{e}_t(1) \;\; \hat{e}_t(2)]'$, $\varepsilon_{2,t}$ are $[v_{1,t} \;\; v_{2,t}]'$, $Z_{1,t}$ are the rest of the states, and $\varepsilon_{1,t}$ are the structural shocks of the DSGE models. I define

$$C(K_t) = G_{12}Z_{2,t} + R_{12}\varepsilon_{2,t}.$$

Notice that $C(K_t)$ are the regime-dependent constants. Finally I can rewrite the system as follows with regime-switching coefficients:

[10] See the appendix for proof.

$$Z_t = C(K_t) + G_t(K_t) Z_{t-1} + R_t(K_t)\varepsilon_t.$$

3.2 Model ZIRP by the PFRE

The solution method of the PFRE model can be found in Cúrdia and Woodford (2011). For a detailed description of the algorithm and an application, please refer to Chen et al. (2012) and its companion online appendix.[11] The basic idea is that agents have perfect foresight of the path of future interest rates and of all shocks until an arbitrary point in time. From this point forward all the shocks are zero, and the solution method is standard, such as in Sims (2002). The system can be solved backwards from this point. The following is a very simple example to illustrate the solution method. Consider the equilibrium system:

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \sigma^{-1}(\hat{\imath}_t - \mathbb{E}_t[\hat{\pi}_{t+1}])$$

$$\hat{\pi}_t = \beta\mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa\hat{y}_t$$

$$\begin{aligned} \hat{\imath}_t &= \phi_\pi\hat{\pi}_t + \nu_t \quad \text{for } t > K \ \nu_t = 0 \\ &= 0 \qquad\qquad\quad \text{for } t = 1, ...K-1, K. \end{aligned}$$

The solution for $t > K$ is:

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{\imath}_t \end{bmatrix} = \begin{bmatrix} \psi_{y\nu} \\ \psi_{\pi\nu} \\ \psi_{i\nu} \end{bmatrix} \nu_t.$$

[11] The appendix can be found at http://onlinelibrary.wiley.com/doi/10.1111/j.1468-0297.2012.02549.x/suppinfo

The system can be broken into the forward-looking and the backward-looking parts. The forward-looking part is:

$$\begin{bmatrix} 1 \ \sigma^{-1} \\ 0 \ \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t \left[\hat{y}_{t+1}\right] \\ \mathbb{E}_t \left[\hat{\pi}_{t+1}\right] \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{\imath}_t \\ \hat{\pi}_t \end{bmatrix}$$

and the backward-looking part is:

$$\begin{bmatrix} 0 & 1 & -\phi_\pi \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{\imath}_t \\ \hat{\pi}_t \end{bmatrix} = \nu_t.$$

At $t = K$, plug in the solution to the forward-looking part and thus:

$$\begin{bmatrix} 1 \ \sigma^{-1} \\ 0 \ \beta \end{bmatrix} \begin{bmatrix} \psi_{y\nu} \mathbb{E}_t \nu_{t+1} \\ \psi_{\pi\nu} \mathbb{E}_t \nu_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{\imath}_t \\ \hat{\pi}_t \end{bmatrix}.$$

Combine this part with the backward-looking part and thus:

$$\begin{bmatrix} 0 & 1 & -\phi_\pi \\ 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{\imath}_t \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 \\ \left(\psi_{y\nu} + \sigma^{-1}\psi_{\pi\nu}\right) \mathbb{E}_t \nu_{t+1} \\ \beta \psi_{\pi\nu} \mathbb{E}_t \nu_{t+1} \end{bmatrix} + \begin{bmatrix} \nu_t \\ 0 \\ 0 \end{bmatrix}.$$

We can solve this system by inverting a matrix. The solution is:

$$\begin{bmatrix} \hat{y}_t \\ \hat{\imath}_t \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\phi_\pi \\ 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ \left(\psi_{y\nu} + \sigma^{-1}\psi_{\pi\nu}\right) \mathbb{E}_t \nu_{t+1} \\ \beta \psi_{\pi\nu} \mathbb{E}_t \nu_{t+1} \end{bmatrix} + \begin{bmatrix} \nu_t \\ 0 \\ 0 \end{bmatrix} \right).$$

We can iterate backwards until the first period.

4 Empirical Analysis

In this section, I compare two approaches to modeling ZIRP in DSGE models. I estimate the DSGE model that either incorporates a regime-switching monetary policy as 3.1 or a typical Taylor rule as 2.8. I extract the filtered states of those estimated DSGE models, and then, starting from the fourth quarter of 2010, I simulate the U.S. economy forward under two scenarios: no intervention and ZIRP for an extended period.[12] I compare the predicted path of macro variables generated from the different models. When I evaluate ZIRP in the DSGE model with the regular Taylor rule, the PFRE method is used to simulate the economy. I will only explicate the estimation strategy of the regime-switching DSGE model. The description of the estimation procedure of the other non-regime-switching model was omitted here. The Bayesian estimation methods for a linearized DSGE model with constant coefficients can be found, for example by An and Schorfheide (2007). Bayesian estimation combines prior information on the parameters with the likelihood function of the model to form the posterior distribution. In the regime-switching model, the optimal filter is no longer the Kalman Filter. I will first illustrate the optimal filter and the likelihood function for this regime-switching model, and then describe data, show estimation results, and make comparisons of simulation results.

[12] Under both scenarios, there are no macro shocks.

4.1 Optimal Filter and Likelihood Function

The regime-switching model is complicated because we have to keep track of the long history of the distribution of the states, and the number of the states grows exponentially.[13] Fortunately, in my application, when updating the states probability, the distribution of the states at each time is degenerated because I observe the interest rates and thus deduce whether or not the economy is at the ZIRP regime in that period. This practice makes inference more efficient.

In this New Keynesian economy, the states are denoted by $\mathbf{S}_t$ and the observables are denoted by $\mathbf{y}_t$. Let K_t denote the Markov regime-switching states and λ_t denote the probability in the ZIRP regime ($K_t = 2$) at time t, thus $K_t = 1$, the normal regime, has probability $1 - \lambda_t$. Let $\hat{R}_t$ denote the log deviation of the regime-switching interest rates from their ergodic mean. Its density function can be written as:

$$P\left(\hat{R}_t\right) = \lambda_t^{\mathbf{1}_{\{R_t=1\}}} \left((1-\lambda_t) f_t\left(\hat{R}_t\right)\right)^{\mathbf{1}_{\{R_t>1\}}},$$

where $f_t\left(\hat{R}_t\right)$ is the conditional density, conditional on at the normal state.[14] That is

$$P\left(\hat{R}_t | R_t > 1\right) = f_t\left(\hat{R}_t\right).$$

Define the Dirac function as

$$\delta_{\tilde{x}}(x) = \begin{cases} 0 \text{ if } x \neq \tilde{x} \\ \infty \text{ if } x = \tilde{x} \end{cases} \text{ and } \int \delta_{\tilde{x}}(x)\, dx = 1.$$

Using the Dirac function, I can express the density of the interest rates as

[13] Even with a two-state Markov regime-switching process, at time t, the number of states is 2^t. Traditionally, we can follow the approximation approaches by Kim and Nelson (1999).

[14] R_t represents gross interest rates.

$$P\left(\hat{R}_t\right) = \lambda_t \delta_1 \left(R_t\right) + \left(1 - \lambda_t\right) f_t \left(\hat{R}_t\right).$$

The transition equations are

$$\mathbf{S}_t \left(K_t\right) = C \left(K_t\right) + G_t \left(K_t\right) \mathbf{S}_{t-1} \left(K_{t-1}\right) + R_t \left(K_t\right) \boldsymbol{\varepsilon}_t,$$

where all the coefficients are regime-dependent and the measurement equations are (no measurement error):

$$\mathbf{y}_t \left(K_t\right) = T\mathbf{S}_t \left(K_t\right).$$

Let $\bar{\lambda}$ denote the ergodic probability of the Markov chain and Σ_k denote the state-dependent variance-covariance matrix of the structural shocks:

$$\Sigma_k = E \left[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' | K_t = k\right].$$

The algorithm of the optimal filter is as follows:

- Initializing at time $t = 1$,the mean of the states:

$$\bar{\mathbf{S}}_1 = \bar{\lambda}_1 \left(I - G\left(K_t = 1\right)\right)^{-1} C\left(K_t = 1\right) + \left(1 - \bar{\lambda}_1\right) \left(I - G\left(K_t = 2\right)\right)^{-1} C\left(K_t = 2\right),$$

and the variance,

$$\bar{P}_1 = \bar{\lambda}_1 X_1 + \left(1 - \bar{\lambda}_1\right) X_2,$$

where X_1 and X_2 solve the discrete Lyapunov matrix equations:

$$G(K_t = 1) X_1 G(K_t = 1)' - X_1 + R(K_t = 1) \Sigma_1 R(K_t = 1) = 0$$

and

$$G(K_t = 2) X_2 G(K_t = 2)' - X_2 + R(K_t = 2) \Sigma_2 R(K_t = 2) = 0,$$

respectively.

- Forecasting $t+1$ given t
  - Transition equation

$$\begin{aligned}
& P\left(S_{t+1}, K_{t+1} | Y^t, \theta\right) \\
= & \int P\left(S_{t+1}, K_{t+1} | S_t, K_t\right) P\left(S_t, K_t | Y^t, \theta\right) d\left(S_t, K_t\right) \\
= & \int P\left(S_{t+1,-\hat{R}_{t+1}}, K_{t+1} | \hat{R}_{t+1}, S_t, K_t\right) P\left(\hat{R}_{t+1}, K_{t+1} | S_t, K_t\right) P\left(S_t, K_t | Y^t, \theta\right) d\left(S_t, K_t\right) \\
= & \int P\left(S_{t+1,-\hat{R}_{t+1}} | K_{t+1}, S_t, K_t\right) P\left(\hat{R}_{t+1} | K_{t+1}, S_t, K_t\right) P\left(K_{t+1} | S_t, K_t\right) P\left(S_t, K_t | Y^t, \theta\right) d\left(S_t, K_t\right) \\
= & \int P\left(S_{t+1,-\hat{R}_{t+1}} | K_{t+1} = 2, S_t, K_t\right) \delta_0\left(\hat{R}_{t+1} = 0\right) P\left(K_{t+1} = 2 | S_t, K_t\right) P\left(S_t, K_t | Y^t, \theta\right) d\left(S_t, K_t\right) \\
& + \int P\left(S_{t+1} | K_{t+1} = 1, S_t, K_t\right) P\left(K_{t+1} = 1 | S_t, K_t\right) P\left(S_t, K_t | Y^t, \theta\right) d\left(S_t, K_t\right),
\end{aligned}$$

where $S_{t+1,-\hat{R}_{t+1}}$ denotes all of the states excluding the interest rates. Since the density of the regime K_{t+1}, conditional on the last period states and regime, $P(K_{t+1}|S_t, K_t)$, is discrete, I can break the integral into two parts when it is in a ZIRP regime, and when it is in the normal regime. Notice that when it is in the ZIRP regime, I do not need to track the distribution of interest rates because it is degenerate.

- – Measurement equation $\Longrightarrow$ likelihood function

$$\begin{aligned}
& P\left(y_{t+1}|Y^{t},\theta\right) \\
= & \int P\left(y_{t+1}|S_{t+1},K_{t+1,}Y^{t},\theta\right)P\left(S_{t+1}|K_{t+1},Y^{t},\theta\right)P\left(K_{t+1}|Y^{t},\theta\right)dS_{t+1}dK_{t+1} \\
= & P\left(K_{t+1}=1|Y^{t},\theta\right)\int P\left(y_{t+1}|S_{t+1},K_{t+1,}Y^{t},\theta\right)P\left(S_{t+1}|K_{t+1,}Y^{t},\theta\right)dS_{t+1} \\
& +P\left(K_{t+1}=2|Y^{t},\theta\right)\int P\left(y_{t+1-\widehat{R}_{t+1}}|S_{t+1},K_{t+1,}Y^{t},\theta\right)P\left(S_{t+1}|K_{t+1,}Y^{t},\theta\right)dS_{t+1}.
\end{aligned}$$

- Updating

  – Updating states

$$\begin{aligned}
& P\left(S_{t+1},K_{t+1}|Y^{t+1},\theta\right) \\
\propto & P\left(y_{t+1}|S_{t+1},K_{t+1,}Y^{t},\theta\right)P\left(S_{t+1},K_{t+1}|Y^{t},\theta\right) \\
\propto & P\left(y_{t+1}|S_{t+1},K_{t+1,}Y^{t},\theta\right)P\left(S_{t+1}|K_{t+1},Y^{t},\theta\right)P\left(K_{t+1}|Y^{t},\theta\right) \\
\propto & P\left(y_{t+1}|S_{t+1},K_{t+1,}Y^{t},\theta\right)P\left(S_{t+1}|K_{t+1},Y^{t},\theta\right)P\left(K_{t+1}=1|Y^{t},\theta\right) \\
& +P\left(y_{t+1-\widehat{R}_{t+1}}|S_{t+1-\widehat{R}_{t+1}},K_{t+1,}Y^{t},\theta\right)P\left(S_{t+1-\widehat{R}_{t+1}}|K_{t+1},Y^{t},\theta\right)P\left(K_{t+1}=2|Y^{t},\theta\right).
\end{aligned}$$

- – Updating states probability

Since I observe the data y_{t+1}, I observe the interest rate. If $R_{t+1}=1$, I deduce that

$$P\left(K_{t+1}=1|Y^{t+1}\right)=0,\text{ and }P\left(K_{t+1}=2|Y^{t+1}\right)=1$$

and *vice versa*. So I do not need to track the long history of the states because when I know the history of Y^t I know for certain the history of the states. The distribution of the states at each time is degenerated. In practice, any quarterly federal funds rate that is smaller than $40bp$[15] is treated as a zero interest rate.

4.2 Data

I use the United States quarterly data from the third quarter of 1987 (1987q3) to the third quarter of 2010 (2010q3) for the following three series: real GDP per capita, core personal consumption expenditures (PCE) deflator, and nominal effective federal funds rate.[16] All data are extracted from the Federal Reserve Economic Data (FRED) maintained by the Federal Reserve Bank of St. Louis. The mapping between these observable variables and the state variables in the DSGE models is

$$\begin{aligned} \Delta Y_t^{obs} &= 100(\gamma + \hat{Y}_{z,t} - \hat{Y}_{z,t-1} + \hat{z}_t), \\ \pi_t^{obs} &= 100(\pi + \hat{\pi}_t), \\ r_t^{obs} &= 100(r + \hat{r}_t), \end{aligned}$$

where all state variables are in deviations from their ergodic steady state values (corresponding to the ergodic steady state R for the policy rate), $\pi \equiv \ln(\Pi)$ and $r \equiv \ln(R)$.

I construct the real GDP per capita series by dividing the nominal GDP series by the population and the GDP deflator. The observable ΔY_t^{obs}, the growth rate of real GDP, corresponds to the first difference in logs of this series, multiplied by 100. The log-difference of the quarterly personal consumption expenditures (PCE) core price index is the measure of inflation. I use the effective federal funds rate as the measure of the nominal short-term

[15] Admittedly, this cut-off choice is arbitrary.

[16] I use an extended sample, starting in 1975q1, to initialize the filter, but the likelihood function itself is evaluated only for the period starting in 1987q3, conditional on the previous sample.

rates.

4.3 Prior Choice

Tables 1 and 2 (columns two to four) summarize the prior distributions of each parameter in the regime-switching DSGE model. I fix the coefficient of relative risk aversion σ at 2. I use Gamma distributions for the prior distributions of the parameters that economic theory suggests must be positive. For those parameters that are defined over the interval $[0, 1]$, I use the Beta distribution. For the standard deviation of the structural shocks, I use the Inverse-Gamma distribution.

The ergodic mean for inflation is centered at 2%, consistent with the Federal Open Market Committee's long-run goal. The steady state annualized growth rate of output is centred at 2.5%. The prior distribution of the discount factor implies the mean of the annualized real interest rate is 2%.

I follow Del Negro and Schorfheide (2008) to choose the priors for the standard parameters in the DSGE models. Table 1 contains two nonstandard parameters (P_{11} and P_{22}) specific to this regime-switching model, which controls the Markov switching probability of staying in the normal regime at time $t+1$ when it is in the normal regime at time t and the Markov switching probability of staying in the ZIRP regime at time $t+1$ when it is in the ZIRP regime at time t. P_{11} is centered at 0.986, which implies an expected duration of staying in the normal regime is 17.86 years. P_{22} is centered at 0.825 at prior, which implies an expected duration of staying in the ZIRP regime is 5.7 quarters, consistent with what is observed in the data by 2010Q3.

The prior for the price rigidity parameter, ζ_p, is centred at 0.5 with a standard deviation of 0.1, as in Smets and Wouters (2007). The interest rate smoothing parameter, ρ_r, is centered at 0.7. The interest rate feedback to output growth, ϕ_y, is centred at 0.4, and the feedback to inflation, ϕ_π, is centred at 1.5 at priors.

All of the structural shocks follow AR(1) processes. Their autocorrelation coefficients are centred at 0.75, with the exception of productivity shocks whose autocorrelation coefficient is centered at 0.4, because this process characterizes the transitory shock to the growth rate of the technology process.

4.4 Parameter Posterior Distribution

In order to obtain the posterior distribution of the parameters, I first obtain the posterior mode by maximizing the likelihood function. The last column of tables 1 and 2 report the posterior mode of each parameter. I then use the random walk Metropolis Hastings algorithm to draw from the posterior distributions. I store those parameter draws and use them for simulation exercises discussed later.

The Markov switching probabilities are well identified because, although the priors are concentrated at their mean, the posterior modes of the transition probabilities are very distinguishable from the prior means. The posterior distributions indicate that the expected duration of staying in the normal regime is 25.45 quarters, and the expected duration of staying in the ZIRP regime is 4.44 quarters. One may argue that data seem to suggest that we have been in the ZIRP regime for at least 20 quarters (from 2009Q1 to 2013Q4). There are two reasons why the estimated duration is substantially shorter than what has happened in actuality. First, the data in my estimation stops at the third quarter of 2010, by which there were only 6 quarters of zero interest rate policy. Second, I treat the 8 quarters from 2002Q4 to 2004Q3 as a ZIRP regime (quarterly FFR is less than 40 basis points) so that we have observations of exiting the ZIRP regime. The time of staying in the ZIRP regime is also short here.

4.5 The Efficacy of the ZIRP in DSGE models

Having estimated the DSGE models, I abstract the filtered states, and, starting from 2010Q4, I simulate the U.S. economy forward for 20 quarters under two scenarios. Under the first scenario, there is no intervention from the central bank, and all of the structural shocks are zero. So, output should gradually go back to its long-term trend, and inflation and interest rates should gradually go back to their steady states. Under the second scenario, the economy is under the intervention of ZIRP by the central bank. One complication of the simulation in the regime-switching DSGE model is that agents have uncertainty over the future states. There are 2^t possible states at time t. To maintain tractability, I combine the states with similar history and only keep track of 16 states at each period.[17] The predicted path of the macro variables is thus the probability weighted average of those 16 states.

When I simulate the U.S. economy under the ZIRP for an extended period, I consider keeping interest rates at zero for four, five, and six quarters at the regime-switching model and keeping interest rates at the 2010Q3 level for four, five, and six quarters in the model where the ZIRP is implemented by the PFRE. In the regime-switching model, at each period, agents *ex ante* always attach a non-zero probability of exiting the ZIRP regime in the next period, and the ZIRP regime is realized for four, five, or six quarters *ex post.* In the PFRE model, agents know that the ZIRP will be kept for four, five, or six quarters. I choose four, five, and six quarters because participants of the Blue Chip Survey of Professional Forecasters (Blue Chip Survey) expected the ZIRP to last four or five quarters at the end of 2010 when LSAP II was implemented. However, since the 2011 October FOMC meeting, the Blue Chip survey suggests that the expected number of quarters until the lift-up of ZIRP is seven or more (top-coded).

Figure 1 and figure 2 show the predicted paths of the output growth rate, inflation,

[17] See Schorfheide (2005) for how this tractability can be achieved.

and the federal fund rate under the ZIRP generated by the regime-switching model and the PFRE model. The red lines in those two figures are the predicted paths of those macro variables without the ZIRP. The blue, green, and magenta lines are the predicted paths with the ZIRP for four, five, and six quarters, respectively. The black dots are the actual observations. All variables are percentage measured quarterly except the output growth rate, which is annualized. Zero interest rate policy is effective in boosting output and inflation. Both of the models considered suggest substantial effects of the ZIRP. The effects are stronger if the ZIRP is kept for a longer period.[18] Also the effects are not monotonic: Although the overall effects are positive, the stimulus to output growth rate is most significant in the near term, and the positive effects gradually revert back.[19] In early periods for inflation, the effects are negative, however, the effects of ZIRP on inflation quickly turn positive and also are long-lasting. Notice that due to the lack of macro shocks, none of the predicted paths should be expected to match the actual data, but the predicted paths generated by the regime-switching model are still close to the actual realizations, which is a desirable feature of the model because it implies that we only need small structural macro shocks to match the observed data.

Figure 3 summarizes the effects of the ZIRP in the regime-switching DSGE models. At each time of the simulated path, I take the difference of the macro variables with and without the ZIRP intervention, and sum up over 20 quarters. This figure plots the total effects. The squares stand for the mean effects and the circles, which are the 90 percent credible sets, reflect the uncertainty of posterior parameter draws. Blue, green, and magenta represent the macro effects of the policy of keeping interest rates at zero for four, five, and six quarters, respectively. The regime-switching DSGE model suggests that keeping interest rates at zero

[18] Notice that in the regime-switching model, the future paths of macro variables are expected, while in the perfect foresight model, there is no uncertainty.

[19] See figure 3.

for four, five, and six quarters on average[20] increases the output growth rate by 0.064%, 0.09%, and 0.12%, respectively[21] and inflation by 0.544%, 0.72%, and 0.90%, respectively[22] over the course of 20 quarters cumulatively. The effects implied by the upper bound of the 90% credible sets can be very significant.

Figure 2 shows that the non-regime-switching model where the ZIRP is implemented by the PFRE suggests much stronger stimulus on output growth rate at peak and on inflation: On average, keeping ZIRP for four, five, and six quarters increases output growth rate by 2.62%, 5.00%, and 9.41% at peak, respectively, though the total cumulative effect gradually reverts back, reaching almost zero after 20 quarters.[23] The effect of the ZIRP on inflation implied by the PFRE model is always large and long-lasting. On average, the cumulative effect on inflation of keeping ZIRP for four, five, and six quarters[24] is 8.53%, 17.92%, and 36.29%, respectively. As mentioned earlier, those two models are fundamentally different in how agents formulate their expectations about future monetary policy. The central bank's "extended period" language is treated as completely credible by the agents in the PFRE model, while the regime-switching model abstracts from the central bank's forward guidance. Notice that the PFRE model generates a spurious rise in inflation. The predicted path generated by the PFRE is explosive while the predicted path generated by the regime-switching model is closer to the actual path; therefore, I argue that the regime-switching DSGE model is better suited to analyzing the effects of ZIRP.

4.6 Robustness

[20] Average over 500 simulations.

[21] Peak at 0.35%, 0.44%, and 0.54%, respectively.

[22] Peak at 0.544%, 0.72%, and 0.90%, respectively.

[23] However, the stimulus on output *level* is positive and long-lasting. On average, keeping ZIRP for four quarters increases output *level* by 25.01% over the course of 20 quarters cumulatively.

[24] In the regime-switching model, zero interest rates are realized ex post. Ex ante, agents always expect to exit zero-interest-rate regime with some non-zero probability.

The advantage of estimating the model using data up to 2010Q3 is that we can compare the predictive distributions of the macro variables derived from the model with the actual observables because we have some realizations of the macro variables of interest. Obviously the Markov transition probability estimated depends on the number of quarters of zero interest rates. Also the expected quarters until the lift-off of ZIRP have changed. A natural question to ask is do the implications of ZIRP change if we estimate the model up to the current data and calibrate the quarters of ZIRP according to current market expectations? I estimate the same model using the data up to the second quarter of 2013, and then simulate the economy forward with and without the ZIRP. ZIRP is realized for seven or eight quarters because since the 2011 October FOMC meeting, the median response of the Blue Chip survey suggests that the expected number of quarters until the lift-off of ZIRP is seven or more.

Figure 4 shows the predicted paths of the output growth rate, inflation, and the federal fund rates with and without the ZIRP generated by the regime-switching model. The red lines are the predicted paths of those macro variables without the ZIRP. The blue and green lines are the predicted paths with the ZIRP for seven and eight quarters, respectively. On average, keeping ZIRP for seven and eight quarters increases output growth rate by 0.105% and 0.137%, respectively, and inflation by 0.666% and 0.794%, respectively, over 20 quarters cumulatively. So the regime-switching model suggests that ZIRP is effective in boosting economic activity.

5 Conclusions

Given the novelty of unconventional monetary policies, it is critical for economists to construct models capable of assessing their effectiveness and guiding policy. This paper develops a new approach to modeling the ZIRP that not only fits the macro data featuring a persistent period of extremely low interest rates, and generates a predicted path closer to the actual path, but also provides a plausible mechanism for modeling the exit of the zero interest rate policy. I find that the Federal Reserve's decision to maintain a zero interest rate for a lengthy period is likely to be effective in boosting economic activity.

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Table 1: Parameter Prior and Posterior Distribution: Structural Parameters.

	Dist	Prior Mean	Std	Posterior Mode
400γ	G	2.5	0.5	2.2951
400π	G	2.0	0.5	1.8648
$400(\beta^{-1}-1)$	G	1.0	0.25	0.9141
ν	G	2.0	0.5	1.8318
ζ_p	B	0.5	0.1	0.8962
ρ_r	B	0.7	0.1	0.7368
ϕ_π	G	1.5	0.25	1.8036
ϕ_y	G	0.4	0.2	0.3878
P_{11}	B	0.986	0.05	0.9607
P_{22}	B	0.825	0.10	0.7749

Table 2: Parameter Prior and Posterior Distribution: Shock Process Parameters.

		Prior		Posterior
	Dist	Mean	Std	Mode
ρ_z	B	0.4	0.2	0.4471
ρ_b	B	0.75	0.1	0.8845
ρ_g	B	0.75	0.1	0.7338
$100\sigma_z$	IG1	0.5	4	0.6779
$100\sigma_b$	IG1	0.5	4	0.9235
$100\sigma_m$	IG1	0.5	4	0.2551
$100\sigma_g$	IG1	0.5	4	0.8033

6 Figures

Fig. 1: Simulation of the U.S. economy forward from 2010Q4 under the ZIRP intervention in the regime switching DSGE model. *The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention. The blue, green, and magenta lines are the predicted paths with the ZIRP for four, five, and six quarters, respectively. The black dots are actual observations.*

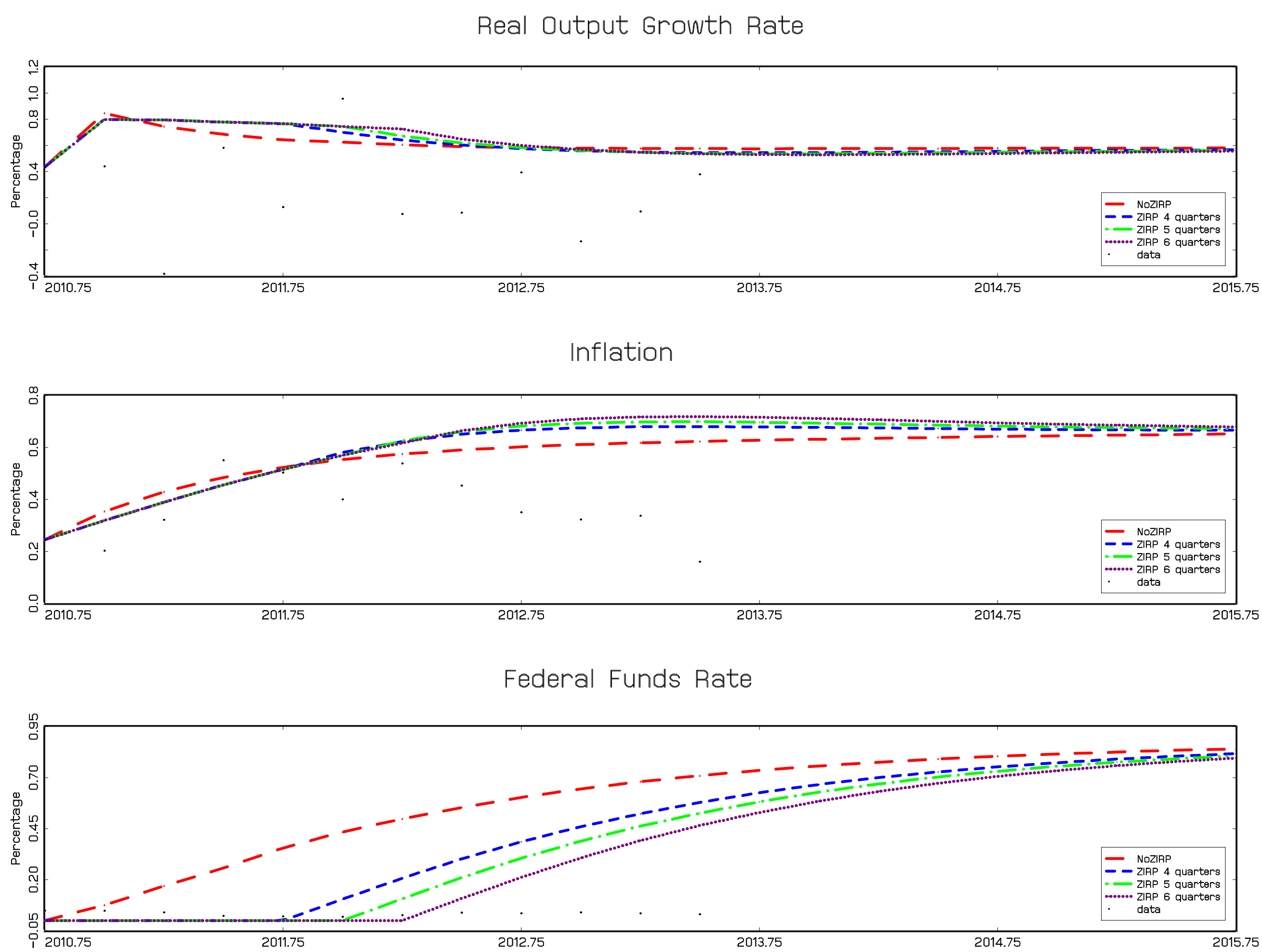


Fig. 2: Simulation of the U.S. economy forward from 2010Q4 under the ZIRP intervention implemented by the PFRE. *The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention generated by the DSGE models. The blue, green, and magenta lines are the predicted paths with the ZIRP for four, five, and six quarters, respectively.*

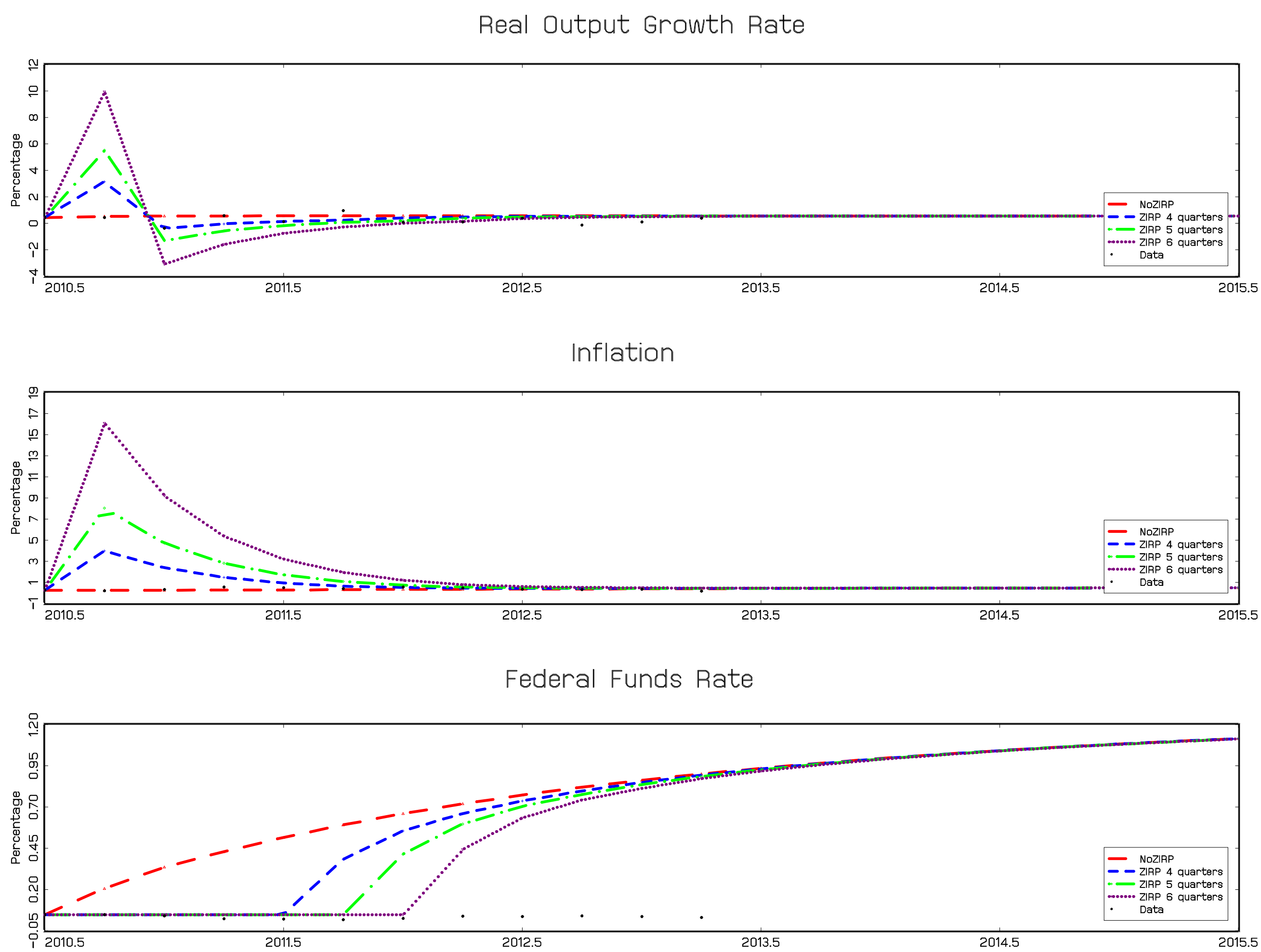


Fig. 3: Summary of macro effects of ZIRP in the regime-switching DSGE model. *The squares stand for mean effects and the circles, which are the 90 credible sets, reflect the uncertainty of posterior parameter draws. Blue, green, and magenta represent the macro effects of the policy of keeping interest rates at zero for four, five, and six quarters, repectively.*

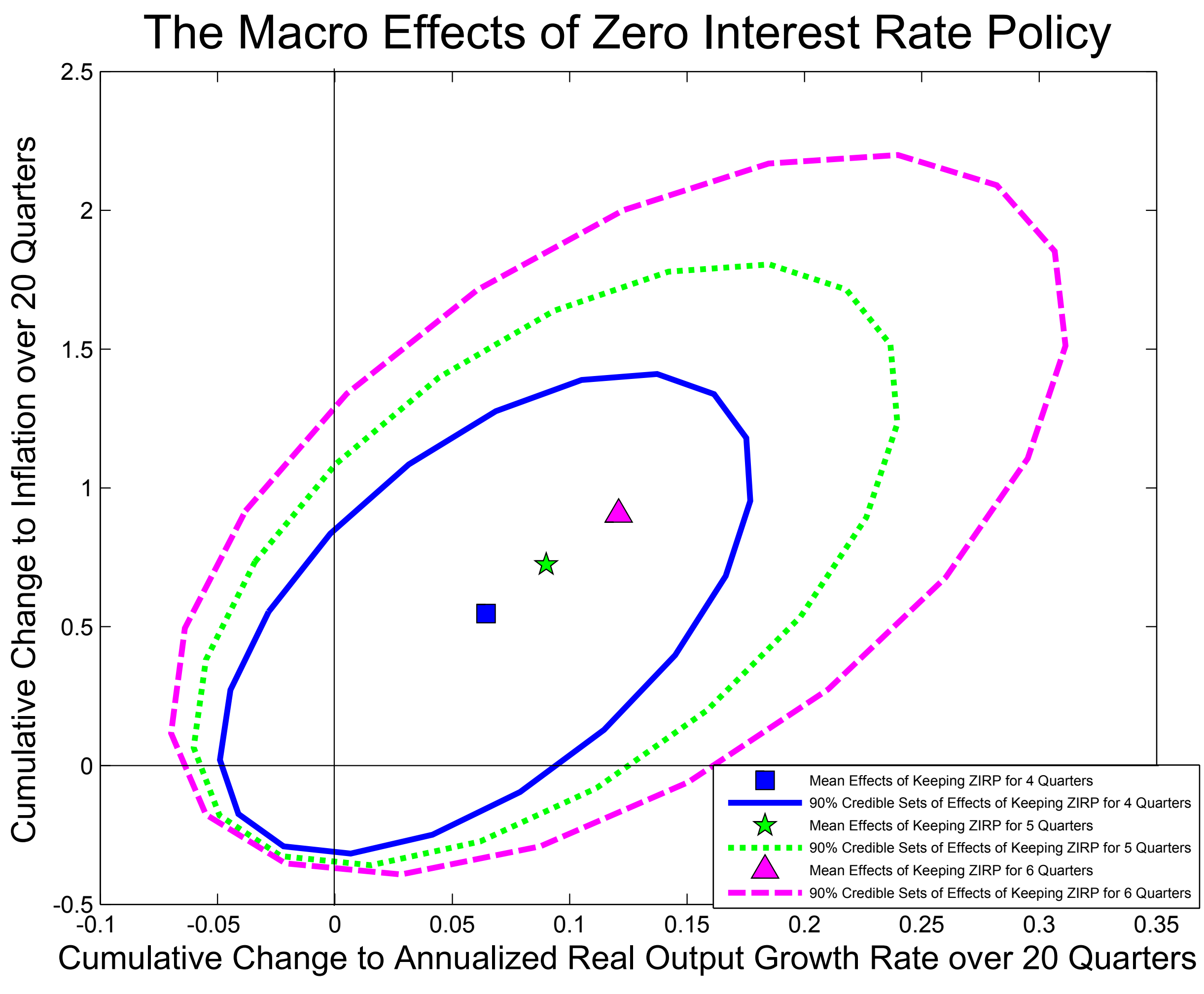
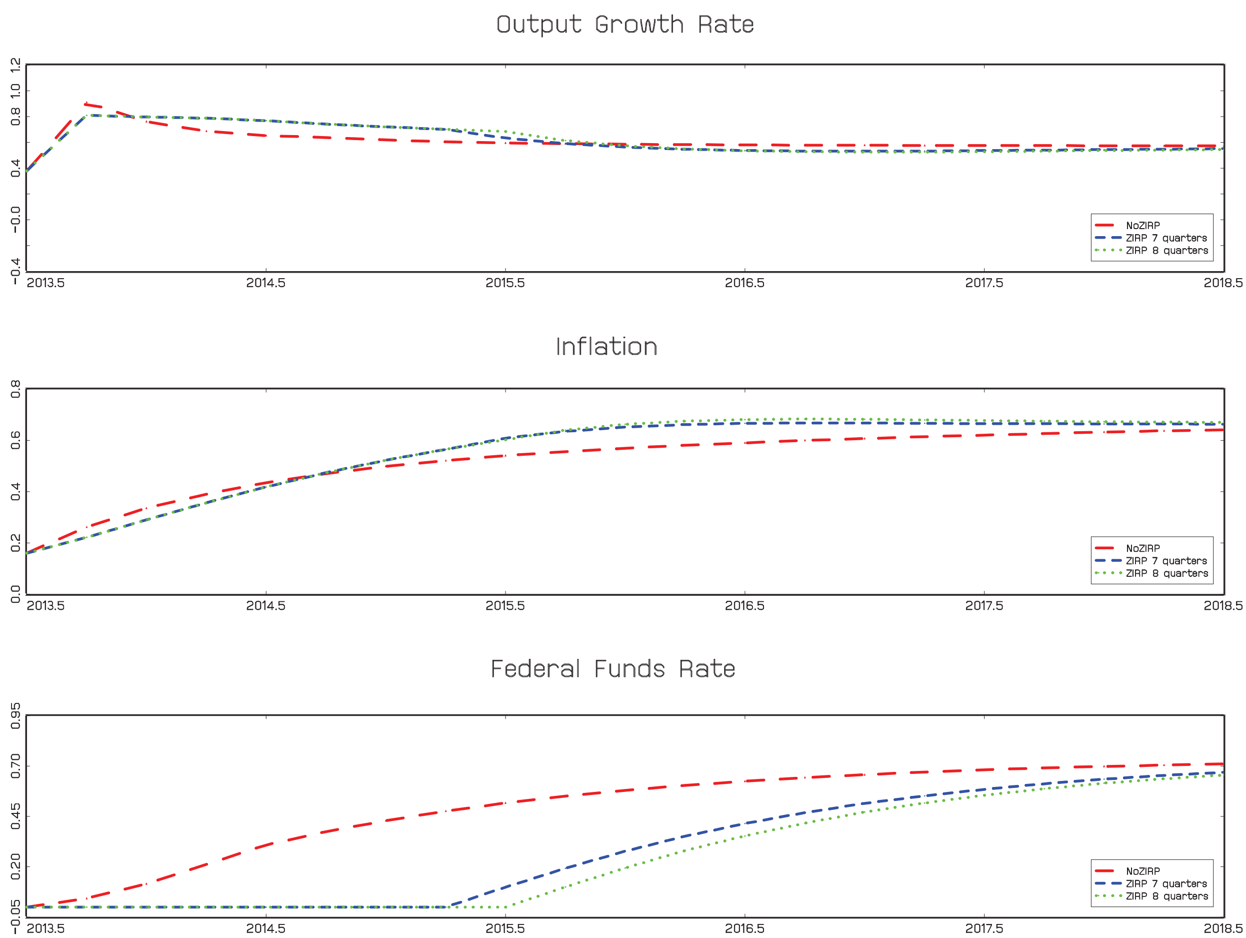


Fig. 4: Simulation of the U.S. economy forward from 2013Q3 under the ZIRP intervention in the regime switching DSGE model. *The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention. The blue and green lines are the predicted paths with the ZIRP for seven and eight quarters, respectively.*



7 Appendix: Proof that Schorfheide (2005) and Liu et al. (2011) give rise to the same solution

This section assumes that the only regime-switching parameter is the target steady state interest rate

Schorfheide (2005):

$$\begin{aligned}
\hat{R}_t &= \rho_R \hat{R}_{t-1} + (1-\rho_R)\varphi_\pi \hat{\pi}_t + (1-\rho_R)\varphi_y \hat{y}_t + \varepsilon_{R,t} + (1-\rho_R)(1-\varphi_\pi)\hat{R}_t^* \\
&= \rho_R \hat{R}_{t-1} + (1-\rho_R)\varphi_\pi \hat{\pi}_t + (1-\rho_R)\varphi_y \hat{y}_t + \varepsilon_{R,t}^*,
\end{aligned}$$

where

$$\varepsilon_{R,t}^* = \varepsilon_{R,t} + (1-\rho_R)(1-\varphi_\pi)\left[\log\left(\frac{R_1}{R}\right),\ \log\left(\frac{R_2}{R}\right)\right]\hat{\mathbf{e}}_{s,t}.$$

Solution by gensys can be written as below, where I assume the first shock is $\varepsilon_{R,t}^*$:

$$\begin{aligned}
y_t &= \Theta_1 y_{t-1} + \Theta_0 z_t + \Theta_y \sum_{s=1}^{\infty} \Theta_f^s \Theta_z E_t z_{t+s} \\
&= \Theta_1 y_{t-1} + \Theta_0 z_t + (1-\rho_R)(1-\varphi_\pi)\Theta_y \sum_{s=1}^{\infty} \Theta_f^s \Theta_z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] P^s.
\end{aligned}$$

So the constant is

$$\begin{aligned}
\Theta_c(K_t) = (1-\rho_R)(1-\varphi_\pi)\Theta_{0\cdot 1} \cdot \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right]\mathbf{e}_{s,t} \\
= (1-\rho_R)(1-\varphi_\pi)\Theta_y \sum_{s=1}^{\infty} \Theta_f^s \Theta_z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] P^s \mathbf{e}_{s,t}.
\end{aligned}$$

Now I will prove that Liu, Waggoner, and Zha (2011) give rise to the same solution.

Assuming the first row of the equilibrium conditions is for the federal funds rate:

$$\left[\begin{array}{cc}\Gamma_0, & \left(\begin{array}{c} -\left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array}\right) \\ \mathbf{0}\ldots, & I_2 \end{array}\right]\left[\begin{array}{c} y_t \\ \hat{\mathbf{e}}_{s,t} \end{array}\right] = \left[\begin{array}{cc} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & P \end{array}\right]\left[\begin{array}{c} y_{t-1} \\ \hat{\mathbf{e}}_{s,t-1} \end{array}\right] +$$

$$\left[\begin{array}{cc} \Psi & \mathbf{0} \\ \mathbf{0} & I_2 \end{array}\right]\left[\begin{array}{c} z_t \\ \nu_t \end{array}\right] + \left[\begin{array}{c} \Pi\eta_t \\ 0 \end{array}\right].$$

Perform QZ decomposition on Γ_0 and Γ_1 and then premultiply both sides by $\left[\begin{array}{cc} Q & \mathbf{0} \\ \mathbf{0} & I_2 \end{array}\right]$:

$$\begin{aligned} & \left[\begin{array}{cc} Q_{nxn} & 0 \\ 0 & I_2 \end{array}\right]\left[\begin{array}{cc} Q'\Lambda Z', & \left(\begin{array}{c} -\left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array}\right)_{nx2} \\ [0]_{2xn} & \ldots, \quad I_2 \end{array}\right]\left[\begin{array}{c} y_t \\ \hat{\mathbf{e}}_{s,t} \end{array}\right] \\ = & \left[\begin{array}{cc} Q_{nxn} & 0 \\ 0 & I_2 \end{array}\right]\left[\begin{array}{cc} Q'\Omega Z' & 0 \\ 0 & P \end{array}\right]\left[\begin{array}{c} y_{t-1} \\ \hat{\mathbf{e}}_{s,t-1} \end{array}\right] + \left[\begin{array}{cc} Q_{nxn} & 0 \\ 0 & I_2 \end{array}\right]\left(\left[\begin{array}{cc} \Psi & 0 \\ 0 & I_2 \end{array}\right]\left[\begin{array}{c} z_t \\ \nu_t \end{array}\right] + \left[\begin{array}{c} \Pi\eta_t \\ 0 \end{array}\right]\right), \end{aligned}$$

and thus:

$$\begin{aligned} & \left[\begin{array}{cc} \Lambda Z' & Q\left(\begin{array}{c} -\left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \;\log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array}\right) \\ 0 & I_2 \end{array}\right]\left[\begin{array}{c} y_t \\ \hat{\mathbf{e}}_{s,t} \end{array}\right] \\ = & \left[\begin{array}{cc} \Omega Z' & 0 \\ 0 & P \end{array}\right]\left[\begin{array}{c} y_{t-1} \\ \hat{\mathbf{e}}_{s,t-1} \end{array}\right] + \left[\begin{array}{cc} Q\Psi & 0 \\ 0 & I_2 \end{array}\right]\left[\begin{array}{c} z_t \\ \nu_t \end{array}\right] + \left[\begin{array}{c} Q\Pi\eta_t \\ 0 \end{array}\right]. \end{aligned} \tag{7.1}$$

Let $w_t = Z'y_t$, and $w_{t-1} = Z'y_{t-1}$. 7.1 becomes:

$$\begin{aligned} & \Lambda w_t + Q\left(\begin{array}{c} -\left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \;\log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array}\right)\hat{\mathbf{e}}_{s,t} \\ = & \;\Omega w_{t-1} + Q\Psi z_t + Q\Pi\eta_t, \end{aligned}$$

and thus:

$$\left[\begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{array}\right]\left[\begin{array}{c} w_1(t) \\ w_2(t) \end{array}\right] - Q\underbrace{\left(\left(\begin{array}{c} (1-\rho_R)(1-\varphi_\pi)\left[\log\left(\frac{R_1}{R}\right),\ \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array}\right)\hat{\mathbf{e}}_{s,t} + \Psi z_t + \Pi\eta_t\right)}_{x_t} \tag{7.2}$$

$$= \left[\begin{array}{cc} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{array}\right]\left[\begin{array}{c} w_1(t-1) \\ w_2(t-1) \end{array}\right].$$

Let $M = \Omega_{22}^{-1}\Lambda_{22}$ and solve forward:

$$\begin{aligned} w_2(t) &= -E_t\left[\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}x_2(t+s)\right] \\ &= -\left[\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}x_2(t+s)\right]. \end{aligned}$$

Replace x_t with their definition and use the fact $E_t\eta_{t+s} = 0$:

$$\begin{aligned} &= -E_t\left[\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\left(\Psi Z_{t+s} + \left(\begin{array}{c} (1-\rho_R)(1-\varphi_\pi)\left[\log\left(\frac{R_1}{R}\right),\ \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array}\right)\hat{\mathbf{e}}_{s,t+s}\right)\right] \\ &= -\left[\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\left(\Psi Z_{t+s} + \left(\begin{array}{c} (1-\rho_R)(1-\varphi_\pi)\left[\log\left(\frac{R_1}{R}\right),\ \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array}\right)\hat{\mathbf{e}}_{s,t+s} + \Pi\eta_{t+s}\right)\right], \end{aligned}$$

and thus:

$$Q_{2\cdot}\Pi\eta_{t+1} = \sum_{s=1}^{\infty} \Omega_{22} M^{s-1} \Omega_{22}^{-1} Q_{2\cdot} \left(\Psi \left(E_{t+1} z_{t+s} - E_t z_{t+s} \right) \right)$$

$$+\sum_{s=1}^{\infty} \Omega_{22} M^{s-1} \Omega_{22}^{-1} Q_{2\cdot} \left(\left(\begin{array}{c} \left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array} \right) \left(E_{t+1} \hat{\mathbf{e}}_{s,t+s} - E_t \hat{\mathbf{e}}_{s,t+s} \right) \right).$$

If the solution is unique:

$$Q_{1\cdot}\Pi = \Phi Q_{2\cdot}\Pi.$$

Premultiplying 7.2 by $[I \ \ -\Phi]$:

$$\begin{aligned} & \left[\begin{array}{cc} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{array} \right] \left[\begin{array}{c} w_1(t) \\ w_2(t) \end{array} \right] \\ & - \left[\begin{array}{c} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{array} \right] \left(\left(\begin{array}{c} \left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \ \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{array} \right) \hat{\mathbf{e}}_{s,t} \right) \\ = \ & \left[\begin{array}{cc} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & 0 \end{array} \right] \left[\begin{array}{c} w_1(t-1) \\ w_2(t-1) \end{array} \right] + \left[\begin{array}{c} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{array} \right] \Psi z_t \\ & - \left[\begin{array}{c} 0 \\ E_t \left[\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} x_2(t+s) \right] \end{array} \right]. \end{aligned}$$

Finally,

$$y_t + Z\begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1}\left(\Lambda_{12}-\Phi\Lambda_{22}\right) \\ 0 & I \end{bmatrix}\begin{bmatrix} Q_{1\cdot}-\Phi Q_{2\cdot} \\ 0 \end{bmatrix}\left(\begin{pmatrix} -\left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \ \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{pmatrix}\hat{\mathbf{e}}_{s,t}\right)$$

$$= Z\begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1}\left(\Lambda_{12}-\Phi\Lambda_{22}\right) \\ 0 & I \end{bmatrix}\begin{bmatrix} \Omega_{11} & \Omega_{12}-\Phi\Omega_{22} \\ 0 & 0 \end{bmatrix} Z' y_{t-1}$$

$$+ Z\begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1}\left(\Lambda_{12}-\Phi\Lambda_{22}\right) \\ 0 & I \end{bmatrix}\begin{bmatrix} Q_{1\cdot}-\Phi Q_{2\cdot} \\ 0 \end{bmatrix}\Psi z_t$$

$$- Z\begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1}\left(\Lambda_{12}-\Phi\Lambda_{22}\right) \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ E_t\left[\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\Psi z_{t+s}\right] \end{bmatrix}$$

$$- Z\begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1}\left(\Lambda_{12}-\Phi\Lambda_{22}\right) \\ 0 & I \end{bmatrix}\cdot$$

$$\begin{bmatrix} 0 \\ E_t\left[\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\begin{pmatrix} \left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{pmatrix}\hat{\mathbf{e}}_{s,t+s}\right] \end{bmatrix}.$$

By simplifying notation, I can rewrite the above equation as:

$$y_t = \Theta_1 y_{t-1} + \Theta_0\left(\Psi z_t + \begin{pmatrix} \left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{pmatrix}\hat{\mathbf{e}}_{s,t}\right)$$

$$+ \Theta_y \sum_{s=1}^{\infty}\Theta_f^{s-1}\Theta_z E_t\left(\Psi z_{t+s} + \begin{pmatrix} \left(1-\rho_R\right)\left(1-\varphi_\pi\right)\left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right)\right] \\ 0 \\ \vdots \end{pmatrix}\hat{\mathbf{e}}_{s,t+s}\right),$$

where

$$\Theta_1 = Z\begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1}\left(\Lambda_{12}-\Phi\Lambda_{22}\right) \\ 0 & I \end{bmatrix}\begin{bmatrix} \Omega_{11} & \Omega_{12}-\Phi\Omega_{22} \\ 0 & 0 \end{bmatrix} Z',$$

$$
\begin{aligned}
\Theta_0 &= Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix}, \\
\Theta_y &= -Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix},
\end{aligned}
$$

$$\Theta_f = M,$$

and

$$\Theta_z = \Omega_{22}^{-1} Q_{2\cdot}.$$

This expression is exactly the same as treating $\hat{\mathbf{e}}_{s,t+s}$ as a shock as in Schorfheide (2005).