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CVA Need Not Exceed Independent CVA**

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Stochastic Intensity Models of Wrong Way Risk: Wrong Way CVA Need Not Exceed Independent CVA *

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Abstract

Wrong way risk can be incorporated in Credit Value Adjustment (CVA) calculations in a reduced form model. Hull and White [2012] introduced a CVA model that captures wrong way risk by expressing the stochastic intensity of a counterparty's default time in terms of the financial institution's credit exposure to the counterparty. We consider a class of reduced form CVA models that includes the formulation of Hull and White and show that wrong way CVA need not exceed independent CVA. This result is based on some general properties of the model calibration scheme and a formula that we derive for intensity models of dependent CVA (wrong or right way). We support our result with a stylized analytical example as well as more realistic numerical examples based on the Hull and White model. We conclude with a discussion of the implications of our findings for Basel III CVA capital charges, which are predicated on the assumption that wrong way risk increases CVA.

1 Introduction and Summary

Since the 2007–2009 credit crisis, the emphasis on counterparty credit risk by both global and US regulators has increased dramatically. Credit Value Adjustment (CVA) is one of the most important counterparty credit risk measures; according to Basel III,¹ banks are required to hold regulatory capital based on CVA *charges* against each of their counterparties, (see, for instance, Bohme et al. [2011]).

Consider a portfolio of derivative contracts that a financial institution, such as a dealer, holds with a counterparty. CVA is the difference between the portfolio value before and after

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¹Basel III is a global regulatory standard on bank capital adequacy, stress testing and market liquidity risk agreed upon by the members of the Basel Committee on Banking Supervision in 2010-11, and scheduled to be introduced from 2013 until 2018.

adjustment for the risk that the counterparty might default; it is the market price of counterparty credit risk.² CVA is expressed in terms of the dealer’s counterparty credit exposure, V , which is the maximum of zero and the future value of the portfolio. It also depends on the maturity, T , of the longest transaction in the portfolio and the default time, τ , of the counterparty. CVA can be expressed as a risk-neutral expected discounted loss:

$$\text{CVA} = E[(1 - R)D_\tau V_\tau \mathbf{1}\{\tau \leq T\}], \quad (1)$$

where D_t is the stochastic discount factor at time t , $\mathbf{1}\{\cdot\}$ is an indicator function, and R is the financial institution’s recovery rate.³ Hereafter, for notational simplicity, we suppress the dependence of the CVA to the recovery rate, R . A widely adopted assumption is that credit exposure, V , and the counterparty’s default time, τ , are independent. This leads to independent CVA, denoted CVA_I , and it is expressed in terms of the density, f , of τ :

$$\text{CVA}_I = \int_0^T E[D_\tau V_\tau | \tau = t] f(t) dt = \int_0^T E[D_t V_t] f(t) dt, \quad (2)$$

where the last equality follows from the independence of τ and V . In practice, a counterparty’s default time distribution is approximated from counterparty credit spreads observed in the market. Monte Carlo simulation is then used to estimate independent CVA by estimating $E[D_t V_t]$ based on a discrete time grid.

The efficacy of independent CVA is limited since there are important practical cases where credit exposure, V , and the counterparty’s default time, τ , are correlated, (see Chapter 8 of Gregory [2010]). When credit exposure is negatively correlated with a counterparty’s credit quality, the exposure and its associated risk measures are said to be *wrong way*. Wrong way CVA, denoted CVA_W , refers to CVA in the presence of wrong way risk. When the correlation is positive, the exposure and its associated risk measures are said to be *right way*. To simplify the exposition, we concentrate on wrong way CVA. However, there are analogous results for right way CVA. A basic example of wrong way risk occurs when a derivatives dealer takes a long position in a put option on a stock of a company whose fortunes are highly positively correlated with those of its counterparty.

A widely held view among practitioners is that wrong way risk decreases a counterparty’s credit quality, and this, in turn, increases CVA. This is also evident from Basel III CVA capital charges where $\text{CVA}_W \approx \alpha \times \text{CVA}_I$ with $\alpha > 1$. It is difficult to build any practical intuition on the impact of wrong way risk on CVA in the absence of a mathematical model capturing the correlation between credit exposure, V , and the counterparty’s default time, τ , with a well defined calibration scheme using historical data to estimate the model parameters. In this paper, working within the widely used reduced form modeling framework, we show that wrong way risk does not necessarily increase CVA, i.e. CVA_I could exceed CVA_W .

Our starting point is the model introduced by Hull and White [2012] summarized in Section 2. In that model, the logarithm of the counterparty’s default time intensity is an affine function of

²Throughout this paper, we consider the unilateral CVA. See Chapter 7 of Gregory [2010] for discussions on Bilateral CVA.

³A derivation of Formula (1) is in Chapter 7 of Gregory [2010].

the dealer’s exposure to the counterparty. In Section 3, we consider a class of intensity models of CVA that includes the formulation of Hull and White [2012]. We show that the calibration scheme of intensity models imply that the model implied credit quality is to match the market implied credit quality. This holds regardless of how the exogenous relationship between V and τ is specified. Let λ denote the counterparty’s default time stochastic intensity. As shown in Sections 3.1–3.3, this important implication of the calibration scheme gives us a useful expression for CVA_I :

$$CVA_I = \int_0^T E[D_t V_t] E \left[\lambda_t e^{-\int_0^t \lambda_u du} \right] dt.$$

Deriving the following formula for CVA in the presence of wrong way risk,

$$CVA_W = \int_0^T E \left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u du} \right] dt,$$

enables us to directly compare CVA_W and CVA_I and conclude that wrong way CVA need not exceed independent CVA. That is, using reduced form modeling, we derive a formula for CVA_W and a calibration-implied formula for CVA_I so that CVA_W and CVA_I become comparable. We shall emphasize that in the absence of such a framework, i.e., a dependent CVA model with a well defined calibration scheme, no practical comparison can be made between wrong way CVA and independent CVA. In Section 4 we provide numerical examples based on the Hull and White model showing that CVA_I can exceed CVA_W .⁴ We discuss the regulatory implications of our result in Section 5.⁵

2 The Hull and White Stochastic Intensity Model of CVA

Hull and White [2012] incorporate wrong way risk in a CVA model by formulating a counterparty’s default intensity in terms of a dealer’s credit exposure to the counterparty. They assume that the stochastic intensity of a counterparty’s default time, τ , denoted by λ , is given by:

$$\lambda_t = e^{bV_t + a_t} \tag{3}$$

where b is a constant and a_t is a deterministic function of time. The parameter b governs the type and level of dependent risk, and it is calibrated by “subjective judgment” in Hull and White [2012]. A positive value for b indicates wrong way risk and a negative value indicates right way risk.⁶ Let s_t denotes the counterparty’s maturity- t credit spread and let R denotes the recovery

⁴In practice dealer portfolios are complex, and there are almost always collateral and netting agreements associated with positions. However, in order to effectively communicate our main results, we consider uncollateralized contract-level exposure in our numerical examples.

⁵We use the following terms interchangeably in the sequel: counterparty credit exposures and credit exposures; also, stochastic default intensity models, intensity models, and reduced form models.

⁶Hull and White [2012] discuss, but do not implement, an estimation scheme based on historical observations of the exposure V and credit spread of the counterparty.

rate.⁷ Given b , the piecewise constant a_t are sequentially chosen to satisfy:

$$e^{-\frac{ts_t}{1-R}} = E \left[e^{-\int_0^t \lambda_u du} \right], \quad (4)$$

as closely as possible. Hull and White [2012] use the left side of (4) as an approximation of the counterparty's survival probability up to time $t > 0$, i.e., $P(\tau > t)$. The Appendix of Hull and White [2012] details how a_t is sequentially specified using Formula (4); the expectation on the right side is estimated by Monte Carlo simulation after time discretization of the integral of the intensity process, λ .

3 Stochastic Intensity Models of CVA

Motivated by the Hull and White model, we consider intensity models of CVA in which a counterparty's default intensity, λ , is driven by a single risk factor, V . The real-valued process $\{V_t\}_{t \geq 0}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \{G_t\}_{t \geq 0}, P)$, where $\{G_t\}_{t \geq 0}$ denotes the filtration generated by V . To incorporate wrong way risk, the intensity, λ , is defined as an increasing function of exposure, V . In this setting the default time, τ , admits a stochastic intensity, λ . A consequence of this is an expression for survival probabilities, (under technical conditions summarized in Appendix A):

$$P(\tau > t) = E \left[e^{-\int_0^t \lambda_u du} \right], \quad (5)$$

and conditional survival probabilities:

$$P(\tau > t | \tau > s) = E_s \left[e^{-\int_s^t \lambda_u du} \right], \quad (6)$$

where $0 < s < t$ and E_s denotes expectation conditional on all available information at time s .⁸ Also, (5) implies that the density of the default time τ is given by:

$$f_\tau(t) = E \left[\lambda_t e^{-\int_0^t \lambda_u du} \right]. \quad (7)$$

Remark 1 The results of this paper hold when λ is driven by more than one risk factor. This becomes evident from Remark 2 in Appendix A and a common implication of calibration schemes in reduced form models as discussed in Sections 3.1 and 3.3. Defining λ as a function of a single risk factor, V , merely facilitates the communication of our results; it simplifies the notation and resembles the Hull and White model.

⁷This is the recovery rate associated with the credit default swap contract “on” the counterparty, and it may or may not be equal to the recovery rate that appears in the CVA formula. The recovery rate in the CVA formula refers to the fraction of loss that is recovered by the financial institution (a derivatives dealer) if the counterparty defaults.

⁸Clearly, (5) follows from (6) by taking $s = 0$. We have presented them separately to simplify the exposition as each expression is used to calibrate an intensity model to different types of historical data. We discuss this in Section 3.1 and Appendix B.

3.1 Calibration of Stochastic Intensity Models

Many of the reduced form models in the credit literature benefit from the the computational convenience of affine intensity modeling by assuming that λ is an affine function of a given Markov process X , such that the conditional expectation in (6) can be written as:

$$P(\tau > t | \tau > s) = E_s \left[e^{-\int_s^t \lambda(X_u) du} \right] = e^{\alpha(s,t) + \beta(s,t) \cdot X_s}, \quad (8)$$

where coefficients α and β depend only on s and t , $0 < s < t$, (see Duffie and Singleton [2003] and Duffie et al. [2000]). The Markov process X can be multidimensional. However, here, for simplicity, we think of X as a 1-dimensional process, e.g., a square-root diffusion. Suppose that the conditional survival probabilities on the left side of Formula (8) are market implied. For instance, they may be approximated from corporate bond spreads. Given the convenient form of the conditional expectation in (8) and given that X has usually well known distributional properties, statistical estimates of the parameters of X and λ are often based on (approximate) maximum likelihood estimation methods or the Kalman filter. (See Duffie et al. [2000], Appendix B of Duffie and Singleton [2003], and Lando [2004]. Also, Duffie et al. [2003] and Duffie [1999] are examples of papers using an approximate maximum likelihood estimation method and Kalman filter, respectively.)

In CVA stochastic intensity modeling, the unknown parameters of λ are also to be estimated via (5) or (6) assuming that survival probabilities or conditional survival probabilities are market implied. Hull and White [2012] use (5) and approximate survival probabilities based on CDS spreads. Corporate bond spreads can be used to approximate conditional survival probabilities, (see Appendix B). That is, (6) can also be used for the calibration of an intensity model of CVA.

In CVA intensity models considered in this paper, λ is a function of the credit exposure process V , which is the maximum of zero and the value of a derivatives portfolio consisting of possibly thousands of derivatives contracts. So, the stochastic process governing the dynamics of V cannot be assumed as given a priori, and affine intensity modeling cannot be applied here. That is, when the distributional properties of V are not given a priori, the parameters of λ cannot be specified by benefitting from convenient expressions similar to the one on the right side of (8) and using well-known statistical parameter estimation methods. In this sense the term “calibration” as opposed to “statistical estimation” is more suitable for CVA intensity modeling.

We shall emphasize that regardless of the sophistication and the mechanics of statistical estimation or calibration schemes, the parameters of λ are to be estimated or approximated such that the model implied survival probabilities:

$$E \left[e^{-\int_0^t \lambda_u du} \right]$$

match the market implied survival probabilities, or, similarly, the model implied conditional survival probabilities:

$$E_s \left[e^{-\int_s^t \lambda_u du} \right]$$

match the market implied conditional survival probabilities, where $0 < s < t$. That is:

The statistical estimation or calibration scheme of stochastic default intensity models is to ensure that model generated (conditional) survival probabilities match market implied (conditional) survival probabilities.

Hereafter, for simplicity we focus on (5) and survival probabilities. The above observation has important implications for CVA calculations in the presence of wrong way-right way risk. In what follows we further elaborate on this by revisiting the Hull and White calibration scheme.

Consider the Hull and White model again, where $\lambda_t = e^{bV_t + a_t}$. Let $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$ denote a discrete time grid and set $P(\tau > t_i) \equiv p_i$, $i = 1, 2, \dots, n$. Suppose that n market implied survival probabilities p_1, \dots, p_n , approximated based on maturity- t_i CDS spreads with $e^{-\frac{t_1 s t_1}{1-R}}, \dots, e^{-\frac{t_n s t_n}{1-R}}$, are given. Suppose that b is given and the model's unknown parameters are a_1, \dots, a_n , on the above-mentioned time grid; $a_i \equiv a_{t_i}$. The Hull and White calibration scheme sequentially estimates a_i 's by estimating:

$$E \left[e^{-\int_{t_{i-1}}^{t_i} \lambda_u du} \right]$$

with Monte Carlo simulation and making these Monte Carlo estimates equal to p_i , for $i = 1, \dots, n$. For instance, given b and p_1 , the calibration scheme uses:

$$p_1 = E \left[e^{-\int_0^{t_1} e^{bV_u + a_1} du} \right],$$

at its first step to specify a_1 . This is done by replacing the expectation above with its Monte Carlo estimate based on sampling from V and then numerically solving for a_1 . That is, the calibration scheme approximates a_i 's sequentially by making the Hull and White model generated survival probabilities equal to market implied survival probabilities, p_1, \dots, p_n .

3.2 Model Implied Counterparty Credit Quality

Suppose that a counterparty's survival probabilities $P(\tau > t)$, for $t > 0$, are considered to be a measure of its credit quality. Wrong way exposures are defined by Canabarro and Duffie [2003] as "credit exposures that are negatively correlated with the credit quality of the counterparty." In what follows, we show that stochastic intensity models of CVA capture this basic definition. However, reiterating the result of the previous section: the calibration scheme equates a counterparty's model implied credit quality to the counterparty's market implied credit quality. In other words, wrong way risk does not affect a counterparty's credit quality.

In the presence of wrong way risk, the stochastic intensity, λ , of a counterparty's default time, τ , is defined as an increasing function of the credit exposure V . Conditional on a given sample path of the credit exposure process in $[0, t]$, we can write:

$$P(\tau > t | G_t) = e^{-\int_0^t \lambda_{G_t}(V_u) du}. \quad (9)$$

Hereafter, when conditioning on a given sample path of the exposure process in $[0, t]$, we suppress the dependence of λ_{G_t} on G_t , and we refer to the survival probabilities on the left side of (9) as path dependent survival probabilities. Consider two given sample paths, $\{V_u^{(k)} ; u \leq t\}$, $k = 1, 2$, for which:

$$\int_0^t \lambda(V_u^{(1)}) du < \int_0^t \lambda(V_u^{(2)}) du.$$

This implies that the counterparty's credit quality is lower along the second sample path, i.e. counterparty's path dependent survival probability is lower along the second sample path:

$$e^{-\int_0^t \lambda(V_u^{(2)}) du} < e^{-\int_0^t \lambda(V_u^{(1)}) du}.$$

In other words, wrong way risk affects a counterparty's credit quality on a path-wise basis, i.e., it lowers the credit quality along some paths. However, the calibration strategy that uses (5) equates the average of path dependent survival probabilities with the market implied survival probabilities:

$$P(\tau > t) = E[P(\tau > t | G_t)] = \text{Market Implied Time-}t \text{ Survival Probability.}$$

An analogous argument shows that right way risk does not affect the credit quality of the counterparty.

3.3 Wrong Way CVA Need Not Exceed Independent CVA

In Lemma 1 of Appendix A, we derive the following formula for dependent CVA (right or wrong way), which assumes that the stochastic intensity of counterparty's default time, τ , is a function of dealer's credit exposure, V :

$$\text{CVA}_W = \int_0^T E \left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u du} \right] dt. \quad (10)$$

Focusing on wrong way CVA, we now show that CVA_W need not exceed CVA_I in stochastic intensity models of CVA. Our result is based on comparing the wrong way CVA formula with a calibration-implied independent CVA formula introduced below. The calibration-implied expression for independent CVA holds for all intensity models whose calibration scheme uses (5) or (6). We further support our result by constructing a stylized example at the end of this section and our numerical examples of Section 4. Recall that to calculate CVA_I ,

$$\text{CVA}_I = \int_0^T E[D_t V_t] f(t) dt,$$

the probability density function (p.d.f), f , of a counterparty's default time is market implied and approximated from CDS or bond spreads. The calibration scheme of stochastic intensity models equates the market implied (conditional) survival probabilities to the model implied

(conditional) survival probabilities as suggested by Formula (5) and (6). This implies that the market implied p.d.f. of counterparty's default time, $f(t)$, is to match the model implied p.d.f.:

$$E \left[\lambda_t e^{-\int_0^t \lambda_u du} \right],$$

for all $t \in [0, T]$ as also suggested by Formula (7). This gives the useful calibration-implied expression for CVA_I :

$$CVA_I = \int_0^T E[D_t V_t] E \left[\lambda_t e^{-\int_0^t \lambda_u du} \right] dt, \quad (11)$$

which enables us to compare CVA_W and CVA_I directly, regardless of the mechanics and sophistication of the model calibration strategy. Hereafter, for simplicity, assume that the stochastic discount factor D is constant or independent of λ and V . A comparison of the calibration-implied CVA_I (right hand side of Formula (11)) and CVA_W (right hand side of Formula (10)) suggests that wrong way CVA need not exceed independent CVA. Note that since the stochastic default intensity process is defined as an increasing function of the credit exposure process, λ , and V are positively correlated. That is,

$$E[D_t V_t \lambda_t] \geq E[D_t V_t] E[\lambda_t].$$

However, this has no implication for the pair of terms:

$$E \left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u du} \right] \text{ and } E[D_t V_t] E \left[\lambda_t e^{-\int_0^t \lambda_u du} \right], \quad (12)$$

or for the time integrals of those terms.

We end this section by constructing a stylized example for which we analytically prove that $CVA_I \geq CVA_W$ in some parts of the parameter space. In Section 4, we give more realistic numerical examples for which $CVA_I > CVA_W$ in the Hull and White model.

Example 1 Let X denote a $[0, 1]$ uniform random variable. Define the exposure V in the interval $[0, T]$ based on X as follows:

$$V_t = \begin{cases} X & 0 < t \leq t_1 \equiv T/2 \\ nX & t_1 < t \leq t_2 \equiv T, \end{cases}$$

where n is a positive constant. Let λ be the stochastic intensity of a counterparty's default time, τ , and suppose:

$$\int_0^{t_i} \lambda_u du = bK_i + a_i, \quad (13)$$

for $i = 1, 2$ and $K_1 = X$, $K_2 = nX$. In Formula (13), b is a positive constant and the parameters a_1 and a_2 are calibrated to market credit spreads. Note that since the time integral of the stochastic intensity is an increasing function of the exposure, the definition of wrong way risk is

captured in this stylized example. Let p_1 and p_2 denote the market implied survival probabilities of the counterparty by time t_1 and t_2 , respectively. The calibration scheme that uses Formula (5) specifies the unknown parameters a_1 and a_2 based on:

$$p_i = P(\tau > t_i) = E[e^{-bK_i - a_i}], \quad (14)$$

for $i = 1, 2$ and $K_1 = X$ and $K_2 = nX$. We show that for large n :

$$CVA_I \geq CVA_W. \quad (15)$$

The proof is in Appendix C.

3.4 Discussion of Our Results

Our study challenges the premise that wrong way risk always increases CVA and shows that independent CVA can exceed wrong way CVA. Reduced form modeling enables the modeler to exogenously correlate credit exposures and the default time of a counterparty by making the default time's intensity an increasing function of credit exposures. The calibration scheme of any intensity model equates the model implied counterparty's credit quality with the market implied counterparty's credit quality derived from, for instance, CDS prices. This statement has been rephrased by "in intensity models, wrong way risk does not affect a counterparty's credit quality" in Section 3.2 to further emphasize this important implication of the calibration scheme. Using this, we derive a calibration-implied expression for the independent CVA formula to make it directly comparable with dependent CVA, whose formula is derived in the Appendix. See the right side of (11) and (10), respectively. Then, it follows that there is no reason that one should exceed the other.

It is not the purpose of our paper to numerically experiment with a fixed model in order to attach financial interpretations to different parts of the parameter space to formulate a rule prescribing where CVA_W could exceed CVA_I . A different intensity model of CVA, i.e., different functional relation between λ and V , could lead to different numerical results leading to different sets of financial rules and interpretations. It is the purpose of this paper to show that CVA_I can exceed CVA_W for a broad class of reduced form models. Example 1 in the previous section is a stylized setting in which we show that CVA_I exceeds CVA_W in some part of the parameter space.

On the basis of our study, one could argue that the dependence of a counterparty's credit quality on credit exposures is already reflected, for instance, in CDS prices, which are indicators of the credit quality. In fact, when CDS prices are believed to reflect all the information on counterparty's credit quality, one could question the need for dependent CVA, which is then to be compared with independent CVA. After all, in a dependent CVA intensity model, after exogenously fixing a relation between a counterparty's default intensity and credit exposures, one should fit the model to the market implied credit quality, which is also present in the independent CVA formula.

4 Numerical Examples

This section is a summary of our numerical examples based on the Hull and White [2012] model. They demonstrate that independent CVA can exceed wrong way CVA. There are many practical instances where Monte Carlo estimates of CVA_I and CVA_W are close but the former exceeds the latter. We consider contract level exposures for forward type contracts and put options.

In what follows we assume that the risk free rate, r , is constant. That is, the discount factor is $D_t = e^{-rt}$ and independent and wrong way CVA are:

$$CVA_I = \int_0^T D_t E[V_t] f(t) dt \quad \text{and} \quad CVA_W = \int_0^T D_t E \left[V_t \lambda_t e^{-\int_0^t \lambda_u du} \right] dt,$$

where V_t denotes the time $t \geq 0$ value of the derivative contract and T is the maturity of the contract. Also, λ is the stochastic intensity proposed by Hull and White, i.e., $\lambda_t = \exp(bV_t + a_t)$. Assuming that b is given, the piece-wise constant deterministic function a_t is approximated based on counterparty's t -maturity credit spreads, s_t , and (4), (see the details in the Appendix of Hull and White [2012]).

The expected exposures, $E[V_t]$, are with respect to the physical measure in our numerical examples. There is no consensus in counterparty credit risk around choices of measure for CVA calculations, (see Gregory [2009] and Chapters 7 and 9 of Gregory [2010] for discussions on the use of risk-neutral and physical measure in CVA calculations).⁹

Monte Carlo CVA Estimation Monte Carlo estimators of CVA_I and CVA_W , denoted $\hat{\theta}_I$ and $\hat{\theta}_W$, are defined as follows. Consider the time grid, $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$,

$$\hat{\theta}_I = \sum_{i=1}^n D_i \bar{V}_i f(t_i) \Delta_i, \quad \text{and} \quad \hat{\theta}_W = \sum_{i=1}^n D_i \xi_i \Delta_i,$$

where $\Delta_i \equiv t_i - t_{i-1}$, and, $\bar{V}_i = \frac{1}{m} \left(\sum_{j=1}^m V_{ij} \right)$, with V_{ij} being the j th Monte Carlo realization of $V_i \equiv V_{t_i}$. Similarly, ξ_i is the m -simulation-run average of $V_i \lambda_i e^{-\sum_{k=1}^i \lambda_k \tilde{\Delta}_k}$, with $\tilde{\Delta}_k = \tilde{t}_k - \tilde{t}_{k-1}$ being defined based on a finer time grid, $0 \equiv \tilde{t}_0 < \tilde{t}_1 < \dots < \tilde{t}_l \equiv T$, $l > n$.

Let $\{S_t\}_{t \geq 0}$ denote a geometric Brownian motion, $S_t = S_0 e^{X_t}$, where $\{X_t\}_{t \geq 0}$ is a Brownian motion with drift μ and volatility σ . We sample from the risk factor S_t based on the physical measure. Then, given the Monte Carlo realization of S , the valuation is based on the risk-neutral measure.¹⁰ This implies $V_t = e^{-r(T-t)} E[S_T | S_t] = S_t$ for a forward type contract. For the put options, we simply set $V_t = e^{-r(T-t)} E[(K - S_T)^+ | S_t]$.

The credit curve is assumed to be flat at s . So, in the independent case, the default time, τ , is an exponential random variable with mean $1/s$. This leads to the following closed form

⁹Note that in the above setting $\{D_t V_t\}_{t \geq 0}$ is a martingale under the risk-neutral measure. We have chosen the physical measure merely to avoid the trivial case, $V_0 = E[D_t V_t]$ and so $CVA_I = V_0 P(\tau \leq T)$, resulting from $\{D_t V_t\}_{t \geq 0}$ being a martingale.

¹⁰This is the common and well known practice in risk management: sampling from the risk factors based on the physical measure and then risk neutral valuation, (see, for instance, Chapter 9 of Glasserman [2004]).

formula for independent CVA in the forward contract case, $CVA_I = \frac{sS_0}{\alpha}(\exp(\alpha T) - 1)$ with $\alpha = \mu + \frac{\sigma^2}{2} - r - s$.

Numerical Results CVA estimates in the following numerical examples are based on $m = 10^5$ simulation runs.¹¹ We assume a recovery rate of $R = 0$, a constant risk free rate of $r = .01$, and an annualized volatility of 25%. The credit quality of the counterparty is investment grade with a flat spread curve at 100 basis points. The family of forward contracts presented in Exhibit 1 and the family of in-the-money put options analyzed in Exhibit 2 are both examples where independent CVA and wrong way CVA are close, but CVA_I exceeds CVA_W at each maturity. The coefficient $b = .02$ in both Exhibits 1 and 2 indicates a relatively low dependence of stochastic intensity on exposure. Exhibit 3 presents another 20% in-the-money put option example where CVA_W exceeds CVA_I at each maturity; note that the difference is most pronounced for $T = 1$. The coefficient $b = 1$ in Exhibit 3 indicates a relatively higher dependence of intensity on exposure.

T	.1	.2	.4	.6	.8	1
CVA_I	2	4	8	12	16.1	20.1
$\hat{\theta}_W$	1.9	3.7	7.5	10.5	15.5	19.6

Exhibit 1: Forward contract: CVA numbers and estimates are of order 10^{-3} , $m = 10^5$, $b = .02$, $\mu = 0$, $\sigma = .25$, $S_0 = 2$, spread = .01, $\Delta = 5\tilde{\Delta}$, $\tilde{\Delta} = .01$ for $T = 1, .8, .6, .4$, and $\tilde{\Delta} = .001$ for $T = .1, .2$.

T	.1	.2	.4	.6	.8	1
$\hat{\theta}_I$	2	4	8.1	11.5	17.1	21.9
$\hat{\theta}_W$	1.9	3.7	7.6	11.1	16.7	21.6

Exhibit 2: Put option: CVA estimates are of order 10^{-3} , $m = 10^5$, $b = .02$, $\mu = 0$, $\sigma = .25$, $S_0 = 10$, $K = 12$, spread = .01, $\Delta = 5\tilde{\Delta}$, $\tilde{\Delta} = .01$ for $T = 1, .8, .6, .4$, and $\tilde{\Delta} = .001$ for $T = .1, .2$.

We also came across unrealistic cases of put options where CVA_I exceeds CVA_W in a more pronounced way. For instance, consider the case where credit spread is flat at 10^6 basis points, i.e., $s = 100$. This gives $CVA_I = .0169$ and $CVA_W = .0057$ for $T = 1$. That is, independent CVA

¹¹We use MATLAB to produce the results.

T	.1	.2	.4	.6	.8	1
$\hat{\theta}_I$	2	4	8.1	11.5	17.1	21.9
$\hat{\theta}_W$	2.2	4.8	11.27	17.6	29.4	37.9

Exhibit 3: Put option: CVA estimates are of order 10^{-3} , $m = 10^5$, $b = 1$, $\mu = 0$, $\sigma = .25$, $S_0 = 10$, $K = 12$, spread = .01, $\Delta = 5\tilde{\Delta}$, $\tilde{\Delta} = .01$ for $T = 1, .8, .6, .4$, and $\tilde{\Delta} = .001$ for $T = .1, .2$.

is roughly 3 times larger than wrong way CVA.¹² Note that $\hat{\theta}_I$ and $\hat{\theta}_W$ are biased estimators of CVA_I and CVA_W due to the time-discretization. Ideally, the mean square error of these estimators should be estimated. This is computationally extremely expensive in our setting. To get a feel for the statistical efficiency of our estimators, we note that for the forward contract example presented in Exhibit 1, CVA_I is analytically calculated, and Monte Carlo estimates of CVA_I coincide with the exact values. Since Monte Carlo estimation of CVA is computationally intensive, a valuable line of research is to develop efficient Monte Carlo estimators of CVA. (See Ghamami and Zhang [2013] for efficient Monte Carlo independent CVA estimation.)

5 Regulatory Treatment of Wrong Way Risk

Basel III's counterparty credit risk (CCR) regulatory capital charges consist of counterparty default risk (carried over from Basel II) and CVA capital charges for bilateral derivatives transactions (see BCBS [2011]). For centrally-cleared derivatives transactions, the Basel Committee on Banking Supervision (BCBS) has recently devised capital charges on banks for their central counterparty credit risk (see BCBS [2012]). In all these CCR regulatory capital charges, the BCBS assumes that wrong way risk increases different measures of CCR; CVA being one of them. It then approximates a wrong way CCR measure by increasing the independent CCR measure using the so-called α multiplier, which is often set to 1.4. That is, in the case of CVA, wrong way CVA is often approximated by the independent CVA times 1.4. It should be noted that capturing wrong way risk is not the only purpose of the BCBS's α multipliers (see Section 4.2 of Pykhtin and Zhu [2006] on α multipliers and the references there). Similar to the view often held by practitioners in the financial industry, the BCBS's premise in CVA calculations is that wrong way risk increases CVA. Our findings challenge this premise. Our results would be useful when reviewing the methodology underlying CCR capital charges that incorporate dependent risk (wrong or right way).

Historically, BCBS has taken relatively simple and conservative approaches in areas where mathematical modeling becomes challenging – the alpha-multiplier approach to wrong way CVA estimation was to provide simple and conservative wrong way CVA estimates. Financial insti-

¹²The remaining parameters for this unrealistic example are $\sigma = .3$, $b = 2$, $\mu = 0$, $S_0 = 1$, $K = 1.5$. Also, $\tilde{\Delta} = .01$ and $\Delta = .05$

tutions that prove to be sufficiently sophisticated in terms of their quantitative capabilities are usually approved by regulators to use their own internally developed *risk sensitive* models. Our results would be informative for regulators when financial institutions' CVA models are being evaluated to replace the the BCBS's less *risk sensitive* proposed methods.

6 Conclusion

A mathematical model is required to incorporate the dependency between a counterparty's credit quality and credit exposures to compare independent CVA and dependent CVA (wrong or right way). The calibration scheme of the model plays a critical role in quantifying this comparison. In this paper, we focus on stochastic intensity models of CVA that include the formulation of Hull and White [2012]. We derive a formula for CVA and show that the general properties of calibration scheme, regardless of its level of sophistication, imply that dependent CVA may or may not exceed independent CVA. Using the Hull and White model, we generate numerical examples that confirm our result for wrong way and independent CVA. BCBS's regulatory CCR capital charges assume that wrong way risk increases CVA, and CVA_W is approximated by $\alpha \times CVA_I$, where α is often assumed to be 1.4. Our results would be useful when reviewing the regulatory CVA capital charge that incorporate dependent risk (wrong or right way).

Appendix

A Default Times with Stochastic Intensity and the Proof of the Dependent CVA Formula

It is well known that a default time, τ , defined on a filtered probability space, $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, P)$, admits a stochastic intensity, λ , when the process,

$$1\{\tau \leq t\} - \int_0^{t \wedge \tau} \lambda_u du,$$

is a martingale, (where $t \wedge \tau \equiv \min\{t, \tau\}$). To make the martingale property precise the filtration is to be specified. For the general case see Chapter 2 of Bremaud [1981]. In what follows, we do this for our setting. A consequence of the existence of an intensity is the identity:

$$P(\tau > t) = E \left[e^{-\int_0^t \lambda_u du} \right],$$

which is used throughout this paper and in the proof Lemma 1.

Doubly Stochastic Random Times Let τ be a default time on a filtered probability space $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, P)$. Let $\{H_t\}_{t \geq 0}$ denote the filtration generated by the default indicator process $1\{\tau \leq t\}$. Suppose that the distribution of τ depends on additional information denoted by

$\{G_t\}_{t \geq 0}$. Set $F_t \equiv G_t \vee H_t$ where F_t is the smallest σ -algebra that contains G_t and H_t .¹³ The default time, τ , is called *doubly stochastic* when for all $t > 0$,¹⁴

$$P(\tau \leq t | G_\infty) = P(\tau \leq t | G_t),$$

and when conditional on G_t , $\int_0^t \lambda_u du$ is strictly increasing.¹⁵

In our setting $\{G_t\}_{t \geq 0}$ is the filtration generated by the exposure process V . The first condition implies that given the past values, $u \leq t$, of V , the future, $s > t$ does not contain any extra information for predicting the probability that τ occurs before t .¹⁶

The credit exposure process, V , could have jumps due to the the expiration of trades prior to the maturity of the longest instrument in the portfolio. In this case, where V has points of discontinuity, τ may not be doubly stochastic. But, it can be shown that τ still admits a stochastic intensity λ , (see Definition D7 and Theorem D8 of Bremaud [1981]).

Lemma 1. *Consider a real-valued process V defined on the probability space (Ω, \mathcal{F}, P) . Let $\{G_t\}_{t \geq 0}$, denote the filtration generated by V , i.e., $G_t = \sigma\{V_s; 0 \leq s \leq t\}$, the smallest σ -field with respect to which V_s is measurable for every $s \in [0, t]$, and let $G \equiv G_\infty \subset \mathcal{F}$. Let D denote a real-valued process that is adapted to $\{G_t\}_{t \geq 0}$. Let τ denote a counterparty's default time, which admits the stochastic intensity λ that is adapted to $\{G_t\}_{t \geq 0}$. For $t \geq 0$,*

$$P(\tau > t | G) = e^{-\int_0^t \lambda_u du} \quad \text{and} \quad P(\tau > t) = E \left[e^{-\int_0^t \lambda_u du} \right]. \quad (16)$$

Then, the following holds for any given $T \geq 0$:

$$E[D_\tau V_\tau \mathbf{1}\{\tau \leq T\}] = \int_0^T E \left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u du} \right] dt.$$

Proof. *Conditional on G we can write,*

$$\begin{aligned} E[D_\tau V_\tau \mathbf{1}\{\tau \leq T\} | G] &= \int_0^T E[D_\tau V_\tau | G, \tau = t] f_{\tau|G}(t) dt \\ &= \int_0^T D_t V_t f_{\tau|G}(t) dt = \int_0^T D_t V_t \lambda_t e^{-\int_0^t \lambda_u du} dt, \end{aligned}$$

where $f_{\tau|G}$ is the conditional density of τ and is derived based on the left side of (16). Then, the Lemma follows by noting that:

¹³By definition, τ is an H_t -stopping time. Note that τ is also a $F_t \equiv G_t \vee H_t$ -stopping time for any $\{G_t\}_{t \geq 0}$.

¹⁴it represents the first event time of a conditional or doubly stochastic Poisson process.

¹⁵see, for instance, Chapter 9 of McNeil et al. McNeil et al. [2005].

¹⁶Many of the stochastic intensity models in the credit literature work under this doubly stochastic framework, (see, for instance, Duffie and Singleton [2003]).

$$E [D_\tau V_\tau \mathbf{1}\{\tau \leq T\}] = E [E[D_\tau V_\tau \mathbf{1}\{\tau \leq T\} | G]],$$

and

$$E \left[\int_0^T D_t V_t \lambda_t e^{-\int_0^t \lambda_u du} dt \right] = \int_0^T E \left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u du} \right] dt.$$

Remark 2 We would like to emphasize that the dependent CVA formula of Lemma 1 also applies to multi-factor settings. That is, when λ is defined based on more than one risk factor, the proof works by $\{G_t\}_{t \geq 0}$ denoting the filtration generated by all the risk factors.

B Approximating Conditional Survival Probabilities from Zero-Coupon Bond Spreads

Here we use a stylized setting to show how conditional survival probabilities can be approximated from zero-coupon bond spreads. Let $\delta(t, T)$ denote the risk-neutral price of a maturity- T default-free zero-coupon bond at time $t > 0$. It is well known that:

$$\delta(t, T) = E_t[e^{-\int_t^T r_u du}],$$

where r is the short rate process and E_t denotes the risk-neutral expectation conditional on information available by time t , (see, for instance, Bjork [2009]). Let $d(t, T)$ denote the risk neutral price of a maturity- T zero-recovery defaultable zero-coupon bond at time $t > 0$. Reduced form debt pricing for a default time τ with the risk-neutral default intensity process λ gives

$$d(t, T) = E_t[e^{-\int_t^T (r_u + \lambda_u) du}],$$

as shown by Lando [1998]. Note that in a stylized setting where λ and r are independent, conditional survival probabilities are easily obtained from the defaultable and default free bond prices:

$$P(\tau > T | \tau > t) = E_t[e^{-\int_t^T \lambda_u du}] = \frac{d(t, T)}{\delta(t, T)}.$$

More realistic corporate bond reduced form pricing models also allow the modeler to estimate conditional survival probabilities from market data, (see Chapter 6 of Duffie and Singleton [2003] and the references there).

C Proof of the Result of Example 1

Assume zero short rate which gives $D \equiv 1$. First consider CVA_I:

$$\text{CVA}_I = E[V_\tau 1\{\tau \leq T\}] = E[X]P(\tau \in A_1) + E[nX]P(\tau \in A_2),$$

where $A_i = (t_{i-1}, t_i]$, $i = 1, 2$. Note that

$$P(\tau \in A_i) = E[e^{-bK_{i-1}-a_{i-1}}] - E[e^{-bK_i-a_i}],$$

where $K_0 = t_0 = a_0 = 0$, $K_1 = X$, and $K_2 = nX$. Using the right side of the above in the CVA_I formula, we can write

$$\text{CVA}_I = E[X] \left(1 - E[e^{-bX-a_1}]\right) + E[nX] \left(E[e^{-bX-a_1}] - E[e^{-bnX-a_2}]\right) \quad (17)$$

Now, consider CVA_W and recall the proof of Lemma 1:

$$\text{CVA}_W = E[V_\tau 1\{\tau \leq T\}] = E[E[V_\tau 1\{\tau \leq T\} | X]].$$

Consider the conditional expectation on the right side above; further conditioning on the default time gives

$$E[V_\tau 1\{\tau \leq T\} | X] = X \left(1 - e^{-bX-a_1}\right) + nX \left(e^{-bX-a_1} - e^{-bnX-a_2}\right),$$

and so,

$$\text{CVA}_W = E \left[X \left(1 - e^{-bX-a_1}\right) + nX \left(e^{-bX-a_1} - e^{-bnX-a_2}\right) \right]. \quad (18)$$

Using Formula (17), Formula (18), and simple algebraic manipulations, we have

$$\text{CVA}_I - \text{CVA}_W = e^{-a_1}(1-n)\text{Cov}(X, e^{-bX}) + ne^{-a_2}\text{Cov}(X, e^{-bnX}). \quad (19)$$

Simple calculations show

$$n\text{Cov}(X, e^{-bnX}) = -\frac{1}{2b} + \frac{1}{b^2n} - \frac{e^{-bn}}{2b} - \frac{e^{-bn}}{b^2n}.$$

This term converges to a constant as $n \rightarrow \infty$. Note that, using Chebyshev's algebraic inequality, $\text{Cov}(X, e^{-bX}) < 0$ when $b > 0$. So, $\text{CVA}_I \geq \text{CVA}_W$ for large values of n .

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