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NO END TO THE "AGE OF ZINC": THE LENGTH OF THE OPTIMAL PROGRAM WHEN DEPLETION AFFECTS EXTRACTION COSTS

by

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NO END TO THE "AGE OF ZINC":[1/]
THE LENGTH OF THE OPTIMAL PROGRAM WHEN DEPLETION AFFECTS EXTRACTION COSTS

by Stephen W. Salant[*]

The literature on the optimal mining of an exhaustible resource usually assumes that costs of extraction depend only on the amount extracted during the current period. The depletion effect--the dependence of the marginal cost curve on the amount already extracted--is either ignored or captured inadequately by assuming that shifts in the marginal cost curve occur exogenously over time regardless of the extractor's previous behavior. Under these conventional cost assumptions, extraction optimally ceases in finite time except in one "neither believable nor important" case, noted by Solow, where the demand curve never cuts the price axis.

In this note it is assumed that the position of the marginal cost of extraction curve depends on the depth reached in the mine. Hence, subsequent shifts in the marginal cost curve are under the extractor's control. Furthermore, it is assumed that beyond some depth, the ore is too expensive to extract. Under these more realistic[2/] assumptions, it is found that the optimal policy always involves extracting positive amounts forever. The stock is drawn down asymptotically to the level beyond which extraction is uneconomical.

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*/ The views expressed herein are solely those of the author and do not necessarily represent the views of the Federal Reserve System.

1/ "The Age of Oil or Zinc or Whatever It Is will have to come to an end. (There is a limiting case, of course, in which demand goes asymptotically to zero as price rises to infinity and the resource is exhausted only asymptotically. But it is neither believable nor important.)" Robert M. Solow in "The Economics of Resources or the Resources of Economics" (AER, May 1974, p. 3).

2/ As Gordon has observed: "Natural gas dissolved in crude oil exerts pressure that forces the oil to the surface. As production proceeds, the pressure drops and the well produces less. Similarly, metal mining costs may rise because the firm must turn to lower grade ores or deeper or more distant deposits ... ." See Richard L. Gordon, "A Reinterpretation of the Pure Theory of Exhaustion," JPE 1967, p. 278.

The purpose of this note is to provide both a rigorous proof and an intuitive explanation for this novel result. Since the optimal program in the case analyzed does not completely exhaust the stock, a question is raised as to what fundamental characteristic of an "exhaustible" resource distinguishes it from a "producible" good. This question is answered at the conclusion of the note.

For simplicity, the extractor is taken to be a monopolist making decisions at discrete intervals. The optimal program would also require infinite time, however, if the extractor were instead a planner or if decisions were made continuously.3/

Formally, the monopolist's problem4/ is to choose $\{q_1, q_2, \ldots\}$ in order to

$$\text{maximize} \sum_{i=0}^{\infty} \beta^i \{R(q_i) - C(q_i, S_i)\} \tag{1}$$

subject to:

$$S_i \geq q_i \geq 0$$

$$S_0 = \bar{I} < \infty$$

$$S_{i+1} = S_i - q_i,$$

where S_i is the stock remaining in the mine when the i^{th} period is entered, q_i is the amount sold then, β is a discount factor, and $\bar{I}$ is the total ore initially in the mine.

3/ In continuous time, the monopolist's problem would be to

$$\max \int_0^{\infty} \{R(q) - C(q,S)\} e^{-rt} dt$$

subject to: $q \geq 0$, $S \geq 0$, $S(0) = \bar{I}$, $\dot{S} = -q$. To prove the basic result, it must be shown that the optimal extraction function, $q(t)$, is strictly positive for all t. See the appendix.

4/ For the case of the planner, the net benefit function, $U(q,S)$, replaces the net profit function of the monopolist in (1).

It has been shown[5/] that a maximum always exists for such a problem provided that

1) the net profit function, R-C, is continuous and bounded from above.

2) $0 \leq \beta < 1$.

It is assumed that these very weak restrictions are satisfied by the problem under consideration.

Additional assumptions must be made about the net profit function to generate the result that the optimal extraction program requires infinite time. For this result, it is sufficient that the net profit function be twice continuously differentiable and satisfy the following requirements:

A1: $R(0) - C(0,S) = 0$

A2: $\exists S^* \varepsilon [0,\bar{I})$ such that $R_q(0) - C_q(0,S^*) = 0$

A3: $\frac{\partial}{\partial S}\{R_q(q) - C_q(q,S)\} \equiv -C_{qs} > 0$[6/]

A4: $\frac{\partial}{\partial q}\{R_q(q) - C_q(q,S)\} < C_{qs}(q,S) < 0, \forall q,S.$

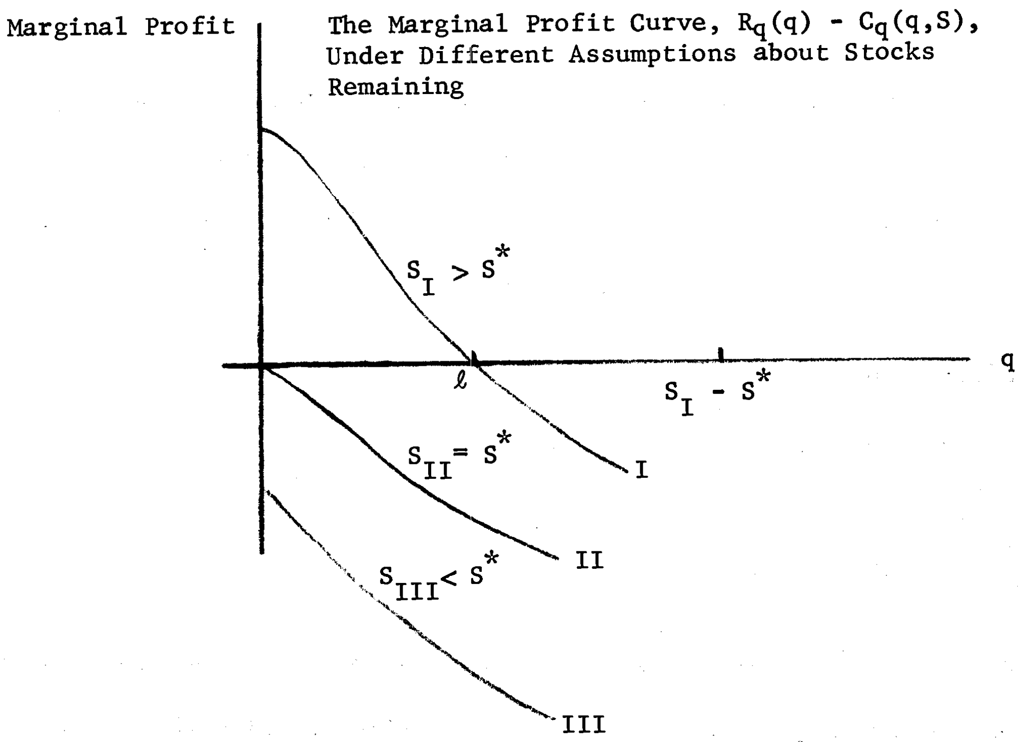
These assumptions[7/] are reasonable. Since the revenue from selling (and the cost of extracting) nothing is zero, the first assumption is plausible. The second simply defines the level beyond which extraction is too costly. The third asserts the existence of a depletion effect which shifts the marginal cost curve up as the stock remaining falls. The last is slightly stronger than the standard assumption that marginal profits are a downward sloping function of the rate of extraction although, as indicated below, the weaker formulation is sufficient to prove the basic result in continuous time.

5/ See, for example, Samuel Karlin, "The Structure of Dynamic Programming Models", Naval Research Logistics Quarterly (1955), p. 285-293.

6/ A1-A3 imply that the net profit function is not concave in both variables. See "A Note on Theoretical Issues of Resource Depletion", by Tracy R. Lewis, California Institute of Technology.

7/ Notice that A1-A4 constitute restrictions on the net profit function rather than on either the revenue or cost function individually. Since A1-A4 can be translated into restrictions on the net benefit function (U(q,S)), it is clear that the basic result also applies to the planning problem.

Assumptions A2-A4 constitute restrictions on the marginal profit curve. It must be downward sloping, must shift down if the stock remaining is reduced, and must pass through the origin when the amount remaining is S^*. These properties are illustrated in the following diagram:



In addition, the marginal profit of entering with more than S^* units remaining and extracting the entire excess $(S-S^*)$ in a single period is negative.[8] This property arises because the size of the immediate effect on marginal profits of depletion is assumed (A4) small[9] relative to the effect of extraction. The

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8/ This can be proved by considering the function

$$f(x) = R_q(x) - C_q(x, S^* + x).$$

f(x) indicates the marginal profit of selling down to S^* in a single period. f(x) passes through the origin (A2) and is strictly decreasing (A4).

9/ Since this limitation on C_{qs} is completely unnecessary in continuous time, and since other analyses have neglected altogether a magnitude assumed here merely to be small relative to something else, I feel this limitation is legitimate.

property arises in continuous time even without this limitation. For, if extraction at an "epsilon" rate once S^* is reached generates negative marginal profits, so must extracting at a much larger rate on the instant before S^* is reached.

Since going all the way to S^* in one step causes losses at the margin, there must be some largest step[10/], $\ell(S) < S-S^*$, beyond which extraction in a single period from a specified level would generate negative marginal profits. In the diagram above, for example, the most that can be extracted in one period from the level on which Curve I is based is represented by that curve's horizontal intercept.

The properties of the marginal profit curves summarized in the diagram above facilitate the proof about the length of the optimal program. Since the optimal program exists, it must be either finite or infinite. It will be shown that <u>any</u> proposed finite program which is feasible can be dominated. Every finite program must have the following form:

$$\{q_1, q_2, q_3, \ldots, q_t, 0, 0, 0, \ldots\}$$

where t is defined as the last period of strictly positive sales.

Consider any such program. When sales cease, the stock remaining (S_{t+1}) must be (1) greater than, (2) less than, or (3) equal to the level (S^*) dividing profitable from unprofitable ore. In the first case, profitable ore is left in the ground (see Curve I in the diagram). Such a program can be dominated by another, identical to the first except that extraction continues in period t+1 ($\hat{q}_{t+1} = \ell(S_{t+1})$). In the second case, unprofitable ore is extracted in the last

10/ Algebraically, the largest step is defined by

$$R_q(\ell,S) - C_q(\ell,S) = 0, \; S > S^*.$$

period ($q_t > \max(0, S_t - S^*)$). As can be verified from the diagram, such a program can be dominated by another, identical to the first except that no unprofitable ore is mined in the last period ($\hat{q}_t = \max(0, S_t - S^*)$). In the third case, so much of the resource is extracted in the last period ($q_t > \ell(S_t)$) that losses occur on the margin. As is indicated by Curve I, such a program can be dominated by a different one where the amount extracted in the last period is reduced ($\hat{q}_t = \ell(S_t)$). Thus, regardless of what finite path one might propose, it can always be dominated. Since an optimal program exists and it cannot be finite, it must requite infinite time.

In the conventional literature, depletion effects are ignored and extraction requires infinite time only in the unbelievable case where no upper bound exists on the price. Once depletion effects are introduced, however, extraction always requires infinite time[11/] even when a "choke" price exists above which demand is zero. However, the price paths in these two cases differ radically. In the conventional case with no depletion effect, the price ultimately dominates marginal cost and eventually rises in percentage terms at the rate of interest. In contrast,

11/ If assumptions A1-A4 hold, the optimal program will satisfy the following conditions:

$$q_i > 0, \qquad i = 0,1,2, \ldots \infty$$

$$\sum_{i=0}^{\infty} q_i = \bar{I} - S^*.$$

In addition, no profitable arbitrage must be possible along the optimal path. Selling one more unit today and one less unit tomorrow would permit the extractor to enter the future with his stock depleted by the same amount; such a maneuver must generate no additional profits along the optimal path. Hence,

$$R_q(q_i) - C_q(q_i, S_i) - \beta[-C_s(q_{i+1}, S_i - q_i)] = \beta\{R_q(q_{i+1}) - C_q(q_{i+1}, S_i - q_i)\}.$$

The left-hand side is the current marginal gain from selling a unit more today and includes the (discounted) increase in cost tomorrow of drawing down the stock today by an additional unit; the right-hand side is the (discounted) loss of selling one unit less tomorrow.

Appending our initial condition and our "growth" equation,

$$S_0 = \bar{I}$$

$$S_{i+1} = S_i - q_i,$$

these equations constitute the necessary conditions for the optimal program.

when a depletion effect is present, the gap between price and marginal cost all but disappears and eventually price creeps upward forever at a negligible and declining rate.

The model presented in this note provides an alternative view of the basic distinction between an exhaustible resource and an ordinary producible. The usual distinction is that an extractor faces a constraint on the total amount he can produce. However, this distinction has no force if--as in the case we have considered--the constraint is not binding. For then, it could be removed altogether without altering the optimal program. A more fruitful distinction between an exhaustible resource and the usual[12/] producible good is that, for a resource, current costs of production depend on previous production.

A monopolistic producer with no constraint on total production would behave exactly like our extractor if his costs of production depended in a gradual way on past output and satisfied A1-A4. Similarly, he would behave exactly like Solow's extractor if his costs of production were unaffected by previous output until it reached some critical level beyond which the entire marginal cost curve abruptly jumped above the choke price.

12/ An unusual producible good would be one which becomes more difficut to produce the greater has been cumulative past production. This might result from a machine or laborer wearing out or from "unlearning by doing". The economics of such a good would be the same as that for an exhaustible resource.

Appendix

Proof that Assumptions A1-A4 Imply Extraction Forever in Continuous Time

Pick $\{q\}$ to max $\int_0^\infty \{R(q) - C(q,S)\}e^{-rt}dt$

subject to $q \geq 0$, $S \geq 0$, $S(0) = \bar{I} > S^*$

where $\dot{S} = -q$.

Any optimal path must satisfy

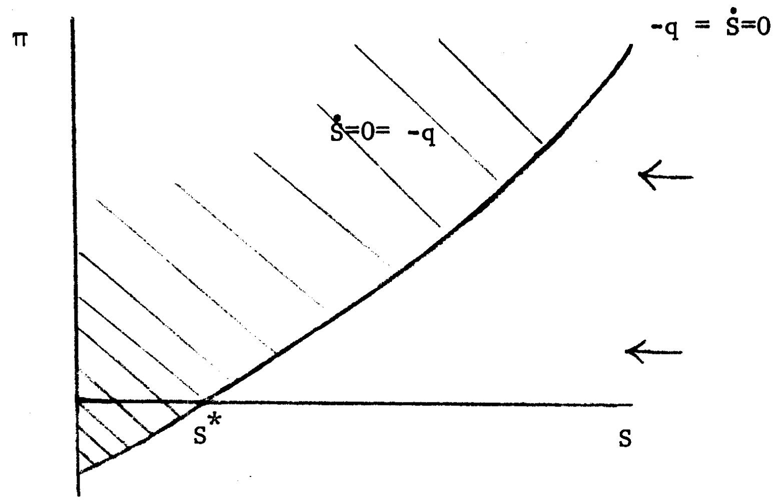
1. $R_q(q) - C_q(q,S) - \pi \leq 0$ with equality for $q > 0$
2. $\dot{\pi} = r\pi + C_s(q,S)$
3. $\dot{S} = -q$

for a continuous function π.

The locus of points in S,π space s.t $q = k > 0$ has slope $-C_{qs}$. By A3, $C_{qs} < 0$. Hence, the iso - "q" curves will be upward sloping.

A3 and A4 imply that $R_{qq} - C_{qq}(q,S) < 0$. This in turn implies that as we move up any vertical line in S,π, space q decreases. Eventually we hit the boundary (and then the region) where $q = 0$.

Since $\dot{S} = -q$, $\dot{S} < 0$ outside this region; $\dot{S} = 0$ inside or on the boundary of this region:



Since $R_q(0) - C_q(0,S^*) = 0$, the point $S^*,0$ is on the $q = 0$ boundary. (We used A2 here).

Consider the $\dot{\pi}$ equation (2):

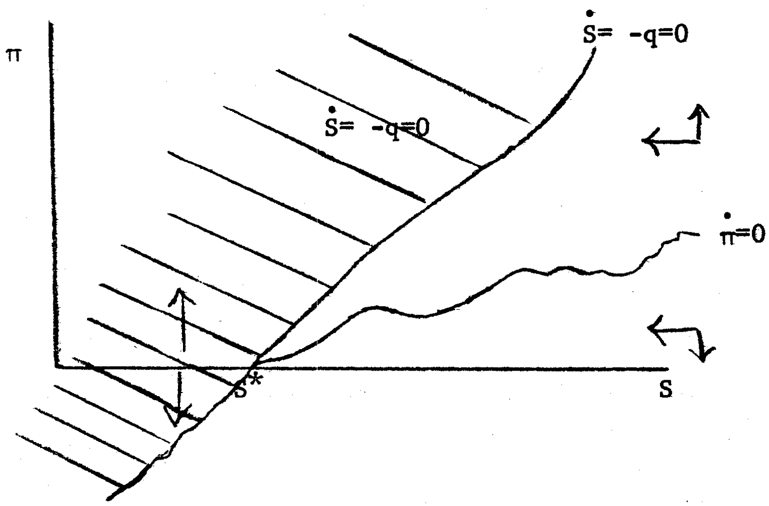
Since $C(0,S) = 0$ (A1)

$C_s(0,S) = 0.$

This implies that $\dot{\pi} = r\pi$ everywhere that $q(\pi,S) = 0$. In the $\dot{S} = 0$ region where $\pi > 0$, $\dot{\pi} > 0$. In the $\dot{S} = 0$ region where $\pi < 0$, $\dot{\pi} < 0$. Finally, in the $\dot{S} = 0$ region where $\pi = 0$ (the S axis to the left of S^*), $\dot{\pi} = 0$. The only points where $\dot{\pi} = 0$ and $\dot{S} = 0$ lie on the S axis to the left of (including) $S^*,0$.

The locus of points s.t $\dot{\pi} = 0$ lies on the S axis to the left of S^* and then rises initially more slowly than the $\dot{S} = 0$ boundary. The rest of its shape depends on properties of the cost function neither assumed nor needed for the result. The $\dot{\pi} = 0$ locus never touches the $\dot{S} = 0$ region to the right of S^*, as we have seen above. If we assume that $C_S < 0$ for $q > 0$, $\dot{\pi} < 0$ along the S axis to the right of $S^*,0$. Hence, the $\dot{\pi} = 0$ locus would never cut the S- axis to the right of S^*. As we move vertically up any line, π increases and q decreases. Therefore, $\dot{\pi}$ rises. It is negative below the $\dot{\pi} = 0$ locus and positive above the locus.

The phase diagram has the following appearance:



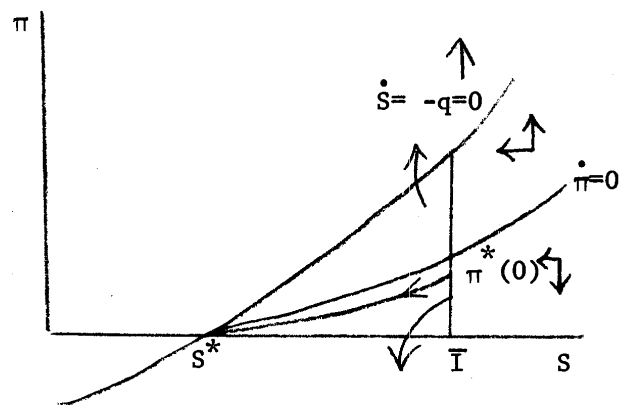
We now incorporate the initial condition and consider trajectories implied by the dynamic system.

Since $S(0) = \bar{I} > S^*$, the initial condition can be represented by a vertical line cutting the S-axis at S(0) (to the right of S^*).

If we set $\pi(0)$ very high, we move up and to the left, crossing the $\dot{S} = 0$ line vertically and continuing vertically in that region at some $S > S^*$. All such paths are sub-optimal since they involve stopping extraction with $S > S^*$. They can be dominated by any path where a little more is sold at the end.

If we set $\pi(0)$ very low, we move down and to the left, cutting the S axis in finite time and moving down to the left. Such paths involve $\pi < 0$ and are sub-optimal. (Motion would continue beyond S^*, which is sub-optimal).

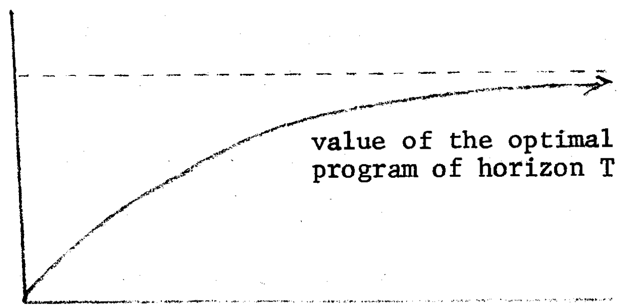
There is, however, one $\pi^*(0)$ from which we can travel toward $S^*,0$. We can find it by working backward from the neighborhood of $S^*,0$. In so doing, we may cross the $\dot{\pi} = 0$ line (horizontally). (This depends on the shape of the $\dot{\pi} = 0$ locus, but causes no problem since $\dot{\pi} = 0$ is bounded away from $\dot{S} = 0$ to the right of S^*.) Eventually, we hit $S = \bar{I}$ at $\pi^*(0)$. This trajectory is the only one which satisfies the necessary conditions and cannot be dominated. Since the problem has a maximum, this is the optimal trajectory:



If we reached $S^*,0$ in finite time, we could start from S^*,) and reach $\bar{I},\pi^*$ by running the system backwards. However, since motion is non-existent at $S^*,0$ we cannot reach $\bar{I},\pi^*$ from there in finite time.

Reaching $S^*,0$ along the optimal path requires infinite time since motion stops at $S^*,0$ and is very slow in its neighborhood.

Perhaps an easier way to see matters is to plot the value of the optimal program of length T against T:



It is increasing and fails to reach a maximum for any finite T.

NOTE: In continuous time, we never need the condition that $R_{qq} - C_{qq} < C_{qs}$. We require merely that both sides be negative.

Acknowledgments

This paper grew out of several interesting discussions with my brother, Michael Salant, who has worked on other properties of models with depletion effects. Discussions with John Bryant, Steve Salop, and Dale Henderson of the Federal Reserve Board have also been very useful.